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Proof that it is not always optimal to locate bonds in a tax-deferred account

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Abstract

The tax codes in many countries allow for special tax advantages for investments in special retirement plans. Probably the most important advantage to these plans is that profits usually remain untaxed. This paper deals with the question, which assets are preferable in a tax-deferred account (TDA). Contrary to the conventional wisdom that one should prefer bonds in the TDA, it is shown that especially in early years, stocks can be the preferred asset to hold in the TDA for an investor maximizing final wealth, given a certain asset allocation. The higher the performance of stocks compared to bonds, the higher the tax burden put on stocks compared to bonds. Simultaneously, the longer the remaining investment horizon, the larger the relative outperformance of the optimal asset location strategy compared to the myopic strategy of locating bonds in the TDA. An algorithm is provided to determine the investment strategy that maximizes (expected) funds at the end of a given investment horizon when there is an analytical solution.

1 Introduction

Investing for retirement is arguably one of the most important financial decisions households face. To motivate individuals to voluntarily save for retirement, the governments in many countries established preferential tax treatments for retirement saving within special retirement-accounts. Following the typology developed by Dilnot (1995), the engagement in retirement plans are connected with three basic transactions: the contributions made to the plan, income on assets that accrues in the plan, and the payments of benefits. Each of them can be subject to special tax treatments. In practice, there are three main tax regimes. One of these is taxed-deferred retirement accounts (TDAs), which allow for income tax exclusion of contributions up to a certain limit and for earning profits on assets inside the plan on a tax-exempt basis. At retirement, payouts from the retirement assets are taxed as ordinary income at the personal tax rate. Prominent examples where taxation occurs according to the rules of TDAs are 401(k) occupational pension schemes in the USA, registered retirement saving plans (RRSP) in Canada, and Riester private pension plans in Germany. A second regime is that of taxed exempt accounts (TEA), whereby contributions are funded with after-income tax-dollars, and neither the profits on assets nor the payouts are taxable. Both Roth-IRAs as described in the US tax-code and contributions in endowment life insurance policies in Germany are examples where taxation occurs according to the rules of TEAs. Finally, households can put their retirement savings into limitless conventional taxable accounts (TA), whereby contributions result from after tax-income and provide taxation for investment profits, but exempt payouts. Private savings in non-tax sheltered bank accounts, direct holdings of common stocks, and bonds are examples where taxation occurs according to the rules of TAs.

In practice, the different tax-regimes exist simultaneously in many countries. Therefore, as Shoven (1999) pointed out, households have to decide not only about asset allocation, i.e. how to spread wealth across different investment categories to diversify risk, but also about the asset location, i.e. whether to place the different assets inside or outside a tax sheltered account. The objective of the asset location decisions, which is the focus of our analysis, is to locate assets in such a way that final wealth at the end of a given investment horizon is maximized.

Neglecting the inherent illiquidity associated with holding assets in TDAs,[1] for an investor whose income tax-rate at the payout phase is less than or equal to the tax-rate during the asset accumulation phase, the preferred location is the TDA. As long as the funds remain below

[1] Usually if assets are withdrawn before retirement from a TDA, some kind of penalty tax must be paid. Therefore, individuals that face a high risk of having to withdraw funds early might prefer not to invest into a TDA.

the contribution limits, it is usually optimal to fully locate retirement savings inside a TDA. However, when contributions exceed this upper bound, the investor must decide which assets to hold inside the TDA. As the TDA allows earning pre-tax-returns, the investor should take the specific tax-treatment of the assets into account. In many countries, the profits from assets held in TAs are subject to different tax treatment. Most important, interest on bonds and dividends paid from stocks are often subject to other tax-rates than realized capital gains. In the US, as well as in Germany, capital gains are (after a certain holding period) taxed at a lower rate than interest and dividends.

Therefore, the investment returns from stocks, which are comprised mainly from capital gains, are taxed more lightly than the interest income from bonds. This is why some authors (e.g. Shoven & Sialm (2003)) argue that the preferred location for bonds should be the TDA and stocks should be located inside the TDA only if no bonds are held in the TA. Although this line of argumentation is quite intuitive and has been shown to be true for a one-period investment horizon (see Dammon *et al.* (2004) and Shoven & Sialm (2003)), it is, as Poterba *et al.* (2001) pointed out, in general not correct for a multi-period investment horizon. However, the analysis of Poterba *et al.* (2001) is based on the historical return data for actual mutual funds over a certain period, but does not provide general results based on a theoretical analysis. This paper fills the lack in theoretical research by exploring the asset location problem, and shows that within a formal multi-period-model, holding stocks in the TDA in early years can result in higher expected total wealth at the end of a given investment horizon than holding bonds in the TDA for the entire investment horizon.

This paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces the model. In section 4, some analytical results are derived. Section 5 provides numerical evidence for the relevance of the problem. Section 6 concludes.

2 Related literature

The pioneering work in the field of investing in an environment with tax-deferred accounts was done by Tepper & Affleck (1974), Black (1980), and Tepper (1981), who show that companies should hold bonds in their defined-benefit pension plans to take full advantage of the preferred tax treatment of bonds. Shoven & Sialm (1998) and Shoven (1999) introduce the question of asset location to households' investment decisions in the context of saving for retirement. They compare simulated distributions of wealth levels at retirement for several portfolio locations and allocations and conclude that due to the tax-inefficiency of many equity funds, these funds should be located in the TDA, as equity funds that are especially actively managed put a high tax-burden on investors. Shoven & Sialm (2003) provide numerical evidence that the preferred location for tax-efficient stock portfolios is the TA and the preferred location for tax-inefficient stock portfolios is the TDA. Taxable bonds have a preferred location in the TDA, and tax-exempt bonds should be held in the TA. In general, they conclude that heavily taxed assets should be located in the tax-deferred environment. The point that personal income tax has a powerful effect on investor demand for portfolio assets after adjusting for the effects of net worth age, sex, and the ratio of human to nonhuman capital has already been made by Feldstein (1976). Dammon *et al.* (2004) provide numerical evidence that there is a strong locational preference for taxable bonds in the TDA and for equity in the TA. They further conclude that the proportion of total wealth allocated to equity is inversely related to the fraction of total wealth in the tax-deferred account. Nevertheless, it can be optimal to hold some bonds in the TA in case of an income shock and low equity price levels, so as to avoid having to pay a penalty tax for early withdrawal. According to an empirical study by Gale & Scholz (1994), most investors contributing to individual retirement accounts (IRAs) are either older than 59 years of age or have large amounts of non-IRA assets and therefore face a low risk of having to withdraw early.

In a theoretical paper which considers the opportunity of investing in both a TEA and a TA, Huang (2003) proves that for a one-period investment horizon without short-selling constraints, it is always optimal to hold bonds in the TEA. This is because without short-selling constraints, one can construct a perfect hedge in the TA for any assets held in the TEA. In this case, the opportunity to invest into a TEA is equivalent to a tax-subsidy. Garlappi & Huang (2004) develop a model with short-selling constraints that computes simultaneously both the optimal asset location and asset allocation. Nevertheless, they assume that future tax-subsidies are driven by a stochastic process that does not depend on the composition of the TDA in past years. They also take bonds to be the preferred asset for the TDA. They do not, however, take the effect of growing wealth in the TDA on tax-subsidies into account.

The investment strategies that are employed in practice by households investing in TDAs and TAs are analyzed in Bodie & Crane (1997), Poterba & Samwick (2001), and Barber & Odean (2003). These authors conclude that investors do not seem to take full advantage of the potential benefits of optimal asset location, even though welfare losses from choosing the wrong contribution rate can be substantial (Gomes *et al.* (2006)). Bergstresser & Poterba (2002) and Amromin (2002) report similar results and also note that many investors have substantially more equity than bonds in their TDAs than in their TAs. Even though this seems initially to be quite unreasonable, as stocks tend to be taxed less heavily than bonds, we show that holding stocks in the TDA (especially in early years) can result in higher expected final wealth than always holding bonds in the TDA.

3 The Model

Taxable accounts (TAs) are defined as accounts for which payments are taxed at the moment of contribution at the personal tax rate.[2] Profits are taxable at tax rates that may differ for gains and dividends/interest, whereas withdrawals are tax-exempt. Tax-exempt accounts (TEAs) are TAs in which profits are tax-exempt. Tax-deferred accounts (TDAs) are TEAs where the taxation at the personal tax rate does not occur at the moment of contribution, but rather at the moment of withdrawal. These definitions are summarized in the following Table:

Taxation of	TA	TEA	TDA
Contribution	X	X	
Profits	X		
Withdrawal			X

We consider a stylized investor who maximizes final after tax wealth over an n-period investment horizon. The investment opportunity set is constant with two financial assets, S and B. The characteristics of asset S can be regarded as similar to stocks, and the characteristics of B similar to bonds. Capital gains, dividends and interest are assumed to be predetermined fraction of the assets' values held over the last period. The assets can be kept in two accounts: a taxable account (TA) and, up to a contribution limit, a tax deferred account (TDA). Within each account, the portfolio can be rebalanced without creating a tax liability. The results readily generalize to a world with TEAs and TAs (see Proposition 4.6). In the TDA, no taxes are withheld on investment returns. Contributions are from before tax salaries, and the portfolio can be rebalanced without creating a tax liability. At the end of the investment horizon, withdrawals from the TDA are taxed according to the personal income tax-rate $\tau_{e,n}$. We assume that the investor optimizes his asset location given his asset allocation.

Let $\tau_{e,t}$ $(0 \leq \tau_{e,t} < 1)$ be the personal tax rate of the investor in period t. Then contributions into the TA are taxed at a rate $\tau_{e,0}$ and payouts from the TDA are taxed at rate $\tau_{e,t}$. Let $g_{S,t}$ and $g_{B,t}$ $(-1 \leq g_{S,t}, g_{B,t})$ be the relative capital gain of asset S or B, respectively, which is taxed

[2] In practice, contributions are made from taxed income. This is in accordance with this definition if contributions are made right at the moment of earning the income.

in the TA at a rate of $\tau_{g,t}$ ($0 \leq \tau_{g,t} < 1$) in period $0 \leq t \leq n$.[3] Let $d_{S,t}$ and $d_{B,t}$ ($d_{S,t}, d_{B,t} \geq 0$) be the relative dividend or interest of asset S or B, respectively, which is taxed at a rate of $0 \leq \tau_{d,t} < 1$ in the TA in period t. Then the pre-tax returns $r_{S,t}$ and $r_{B,t}$ of asset S and B in period t are defined as $r_{S,t} := g_{S,t} + d_{S,t}$ and $r_{B,t} := g_{B,t} + d_{B,t}$. For deterministic $g_{S,t}$ and $d_{S,t}$, $\tau_{g,t}$ and $\tau_{d,t}$, the after-tax return for assets in the TA, can be represented as $r_{S,t}(1 - \tau_{S,t})$ and $r_{B,t}(1 - \tau_{B,t})$ for some average tax-rate on gains and dividends on stocks $\tau_{S,t} \in [0, 1]$ and some average tax-rate on bonds $\tau_{B,t} \in [0, 1]$. These are given by

$$\tau_{S,t} := \frac{d_{S,t}\tau_d + g_{S,t}\tau_g}{d_{S,t} + g_{S,t}} \qquad \text{for } d_{S,t} + g_{S,t} \neq 0$$
$$\tau_{B,t} := \frac{d_{B,t}\tau_d + g_{B,t}\tau_g}{d_{B,t} + g_{B,t}} \qquad \text{for } d_{B,t} + g_{B,t} \neq 0$$

If $d_{S,t} + g_{S,t} = 0$ or $d_{B,t} + g_{B,t} = 0$, $\tau_{S,t}$ or $\tau_{B,t}$, respectively, can be chosen arbitrarily from $[0, 1]$. As argued, bonds' returns tend to be taxed at a higher tax-rate than stocks' returns. We therefore assume $\tau_{S,t} < \tau_{B,t}$. By $a_{in,t}$. We denote the weight of asset S in the TDA in period t and by $a_{out,t}$, we denote the weight of asset S in the TA in period t. $V_{out,0}$ denotes the wealth of the investor in the TDA at the beginning of the investment horizon. By $V_{in,0}$, we denote the wealth of the investor in the TA at the beginning of the investment horizon, which can be effectively considered the investor's by already taking the forthcoming taxation at the time of withdrawal into account.[4] The wealth $V_{out,t}$ in the TA and the wealth $V_{in,t}$ in the TDA (taking the forthcoming income taxation already into account) for $t \geq 1$ is then given by

$$V_{in,t} := V_{in,t-1}\Big(a_{in,t}(1 + r_{S,t}) + (1 - a_{in,t})(1 + r_{B,t})\Big)$$
$$V_{out,t} := V_{out,t-1}\left(a_{out,t}\Big(1 + r_{S,t}(1 - \tau_{S,t})\Big) + (1 - a_{out,t})\Big(1 + r_{B,t}(1 - \tau_{B,t})\Big)\right)$$

The investor's total wealth V_t at time t is given by $V_t := V_{in,t} + V_{out,t}$. The investor seeks to maximize his total wealth until the end of the investment horizon at time n by searching the optimal asset location strategy represented by $a_{in,t}$ and $a_{out,t}$ for a given asset allocation. That means the investor can only decide where to hold the assets, not which assets to hold.

[3]This assumption following Gomes *et al.* (2006) implies that all gains, whether realized or not, are taxable. This is only an approximation to reality and most resembles mutual funds gains, which are taxable the moment they occur inside the fund. For studies on the impact of tax-timing on asset location, see e.g. Dammon *et al.* (2004) or Garlappi & Huang (2004).

[4]It is thereby implicitly assumed that the investor takes into account the fact that at withdrawal at the end of period n, only the fraction $1 - \tau_{e,n}$ of the wealth in the TDA is effectively his and can be used for consumption.

4 Analytical results

In case of a one-period investment horizon ($n = 1$) and an investor who has the opportunity to invest into a TEA and a TA, it has already been shown that the asset that puts a higher relative tax-burden on the investor should be located in the TEA.[5] A similar conclusion is true for an investor who has the opportunity to invest into a TDA and a TA.[6]

Proposition 4.1 (Dominance of bonds in the TDA for one-period investment horizon). *Let $r_{S,1}\tau_{S,1} < r_{B,1}\tau_{B,1}$, $\tau_{e,1} \geq \tau_{e,0}$ and $r_{S,1} > r_{B,1}$. An investor who wants to invest one dollar in S and one dollar in B for a single period, during which only one dollar may be invested in the TDA, should locate B in the TDA and S in the TA.*

According to Proposition 4.1, the strategy of locating B in the TDA and S in the TA is optimal for a one-year investment horizon if the conditions of Proposition 4.1 are met. This strategy is not necessarily true for a longer investment horizon, as models with more than two periods allow for an endogenous wealth accumulation in the TDA, which changes the tax-exempt basis. As Proposition 4.2 shows, there exist cases in which S is the preferred asset to hold in the TDA.

Proposition 4.2 (Existence of investment horizon for which locating stocks in the TDA dominates locating bonds in the TDA). *Consider a buy-and-hold investor who wants to invest one dollar in S and one dollar in B for n years. Only one dollar may be invested in a TDA. If*

$$\forall\ 1 \leq t : (1 + r_{S,t} > 1 + g_{B,t} > 1) \wedge (r_{S,t} > r_{S,t}(1 - \tau_{S,t})),$$
$$\prod_{t=1}^{n} \Big(1 + r_{S,t}\Big)(1 - \tau_{e,n}) > 1 \quad \text{for some } n \in \mathbb{N}$$

as well as for technical reasons

$$\sup_t \left\{ \frac{1 + r_{B,t}}{1 + r_{S,t}} \right\} \leq q_1 < 1 \quad \text{for some } q_1 > 0,$$
$$\sup_t \left\{ \frac{1 + r_{S,t}(1 - \tau_{S,t})}{1 + r_{S,t}} \right\} \leq q_2 < 1 \quad \text{for some } q_2 > 0$$

and

$$\inf_t \left\{ \frac{1 + r_{B,t}(1 - \tau_{B,t})}{1 + r_{S,t}} \right\} \leq q_3 < 1 \quad \text{for some } q_3 > 0$$

[5]See e.g. Shoven & Sialm (2003) or Dammon *et al.* (2004).
[6]The proofs for all Lemmas and Propositions are given in the appendix.

hold, then $\exists\, n \in \mathbb{N}$, such that locating S in the TDA dominates locating B in the TDA.

Proposition 4.2 even holds if $r_{S,t}\tau_{S,t} < r_{B,t}\tau_{B,t}$. This means that even if there is a lower relative tax burden on S, there is some $n \in \mathbb{N}$ such that the strategy to locate S in the TDA is better than the myopic strategy of locating B in the TDA. The strategy of always locating B in the TDA is a myopic short-term location strategy as according to Proposition 4.1, as it always maximizes total wealth in the next period, but does not take the following ones into account. When considering investment decisions with TDAs and TAs, the following two factors have to be taken into account concerning the location decision:

- Let the **growth-effect** be defined as the increase of the tax-exempt basis (wealth in the TDA) caused by the returns of assets held in the TDA.
- Let the **tax-effect** be defined as the taxes saved in a certain period by locating an asset in the TDA.

As the proof in the appendix shows, the result of Proposition 4.2 is due to the assumption that the pre-tax growth rate of asset S is higher than the growth rate of any other asset. For an investment horizon of infinite length the growth of the portfolio only depends on the growth of the asset with the highest growth rate. For calculating avoided taxes, both the absolute gains and the tax rates matter. As the absolute gains are the product of the relative gains and the wealth in the TDA, the wealth has a crucial impact on avoided taxes. It therefore may make sense to locate stocks in the TDA in early years and thereby pay more taxes in these years (lower tax-effect) if taxes saved in forthcoming years due to increased wealth in the TDA outweigh those paid in early years (positive future tax-effects due to the preceding growth effect).

Proposition 4.3 states that under mind assumptions, an investor who wishes to hold a certain fraction x_t given by

$$x_t := \frac{V_{in,t-1}a_{in,t} + V_{out,t-1}a_{out,t}}{V_{t-1}} \tag{1}$$

of his total wealth in period $1 \le t \le n$ in asset S and who chooses the investment strategy that maximizes his (expected) final wealth should never hold a mixture of S and B in his TDA.

Proposition 4.3 (Algorithm to determine which asset to prefer in the TDA). *Let $a^z_{in,t} \in \{0,1\}$ denote the value of $a_{in,t}$ which maximizes V_n on condition that $a_{in,t} \in \{0,1\}$.*

$\forall x_t \in (0,1)$ *(in which* x_t *is given by equation* (1)*): if*

$$\forall\, 1 \leq t \leq n : V_{in,t-1} \leq x_t V_{t-1} \quad \textit{for } a^z_{in,t} = 1 \tag{2}$$
$$\textit{and } V_{out,t-1} \geq x_t V_{t-1} \quad \textit{for } a^z_{in,t} = 0$$

holds, the problem of finding the optimal asset location in period $1 \leq t \leq n$ *is reduced to the question of whether asset* S *or asset* B *should be held in the TDA in that period.*

Lemmas (4.1) and (4.2) deal with the question how the problem of finding the optimal asset location looks if condition (2) is no longer fulfilled.

Lemma 4.1. *Let assumptions be as in Proposition 4.3, but assume*

$$\exists\, 1 \leq t \leq n : \big(V_{in,t-1} > x_t V_{t-1}\big) \wedge (a^z_{in,t} = 1)$$

Then, the problem of finding the optimal asset location in period t *is reduced to the question of whether asset* S *is held in the TDA or in the TA during period* t*. In the case(s) when* $V_{in,t-1} > x_t V_{t-1}$ *and* $a^z_{in,t} = 1$*, a positive amount of* B *should still be held in the TDA during period* t*.*

Lemma 4.2. *Let assumptions be as in Proposition 4.3, but assume*

$$\exists\, 1 \leq t \leq n : \big(V_{out,t-1} < x_t V_{t-1}\big) \wedge (a^z_{in,t} = 0)$$

Then, the problem of finding the optimal asset location during period t *is reduced to the question whether asset* S *is held in the TDA or in the TA in period* t*. In the case(s) where* $V_{out,t-1} < x_t V_{t-1}$ *and* $a^z_{in,t} = 0$*, there is still a positive amount of* S *to be held in the TDA in period* t*.*

As $a^z_{in,t} = 1$ and $a^z_{in,t} = 0$ cannot be true at the same time, in period t, the conditions of Lemma 4.1 and the prerequisites of Lemma 4.2 cannot be fulfilled simultaneously. If the conditions of Lemma 4.1 or 4.2 are fulfilled, there is no longer an analytical solution (except for the case that one of the two Lemmas is used for the very last period) and numerical methods have to be used to find a solution.

The comparison of the strategy to invest exclusively S in the TDA and that of investing exclusively B in the TDA has shown that especially for long investment horizons, it can make sense to locate asset S in the TDA. It has also been shown that for a one-period investment horizon, the strategy to locate B in the TDA dominates the location of S in the TDA. It is further known from Proposition 4.3 that the entire TDA (after applying the contractive mapping from the proofs for Lemma 4.1 or 4.2 if necessary) should only contain either S or B in period t. Let $a^*_{in,t}$ denote the optimal weight of stocks in the TDA in period t. In general, $a^*_{in,t}$ depends on $a^*_{in,t+1}$, $a^*_{in,t+2}$, $\ldots$, $a^*_{in,n}$, as the question of whether it is optimal to pay more taxes at the moment and exploit the effect of higher wealth in the TDA depends on how much taxes can be saved in upcoming periods, which in turn depends on the asset location in these periods. However, these weights in turn depend on $V_{in,t}$, $V_{out,t}$, $V_{in,t+1}$, $V_{out,t+1}$, $\ldots$, $V_{in,n-1}$, and $V_{out,n-1}$, which in turn depend on $a^*_{in,t}$. Therefore, $a^*_{in,1}$, $a^*_{in,2}$, $\ldots$, $a^*_{in,n}$ cannot be determined consecutively and the solution to the general case needs to be approximated with the help of numerical optimization methods. Nevertheless, the problem can be solved without numerical methods if $\forall\, 1 \leq t \leq n$: $V_{in,t-1}$ is of a magnitude such that from $a^z_{in,t} = 1$, it follows that $a^*_{in,t} = 1$ is true, and also from $a^z_{in,t} = 0$ it follows that $a^*_{in,t} = 0$ is true, and hence only the favored asset is located in the TDA. These conditions are met if equations (2) hold. Proposition 4.4 provides an algorithm to solve the problem of optimal asset location in case equations (2) hold.

Proposition 4.4 (If equations (2) hold, only one asset should be held in the TDA). *Let $a^*_{in,t}$ denote the value of $a_{in,t}$ that maximizes V_n on condition that $a_{out,t} \in [0,1]\,\forall\, 1 \leq t \leq n$. Let $g_{S,t}$, $g_{B,t}$, $d_{S,t}$, $d_{B,t}$, $\tau_{g,t}$, $\tau_{d,t}$, $\tau_{e,t}$ and x_t be given for all $1 \leq t \leq n$ and let equation (2) hold. Then it holds that the $a^*_{in,t} \in \{0,1\}$ and $a^*_{in,t}$ that maximize total wealth at the end of period n can be determined for all $1 \leq t \leq n$ without numerical search routines.*

Conditions (2) are equivalent to

$$\begin{aligned} V_{in,t-1} &\leq \frac{x_t}{1-x_t} V_{out,t-1} \quad \text{for } a^z_{in,t} = 1 \\ V_{in,t-1} &\leq \frac{1-x_t}{x_t} V_{out,t-1} \quad \text{for } a^z_{in,t} = 0 \end{aligned}$$

$\forall\, 1 \leq t \leq n$. These equations show that condition (2) holds if $V_{in,t-1}$ is small relative to $V_{out,t-1}$ and x_t is not too close to either 0 or 1.

Proposition 4.5 (Generalization to saving plans). *Let denotation and assumptions be chosen as in Propositions 4.3 and 4.4. Assume that in period t, contributions to the TDA are allowed up to a maximum contribution limit of $C_t \geq 0\,\forall\, 1 \leq t \leq n$. Let the empty product be defined as one. Then the problem of determining the asset location that maximizes total wealth at the end of the investment horizon can be computed with the algorithm derived in Proposition 4.4.*

Proposition 4.3 and Lemmas 4.1 and 4.2 generalize accordingly.

Proposition 4.6. *All the results derived for TDAs hold for TEAs as well.*

If gains and dividends are assumed to be stochastic, the algorithm given in the proof of Proposition 4.4 can still be applied to maximize expected final wealth if consistent estimators for the products that are essential within the computation, can be determined. However, the analysis of optimal asset location strategies in a stochastic setting is beyond the scope of this paper.

5 Numerical results

In this section, numerical evidence is offered to show that suboptimal asset location strategies can lead to substantial differences in the capital at the end of the investment horizon. It is assumed that the prerequisites of Proposition 4.4 are fulfilled. As shown in Proposition 4.5, the optimal asset location for an investor who can make contributions each period is the same as the optimal asset location of an investor who only contributes once. It is therefore sufficient to analyze a single investment. To focus solely on the fact that growth in capital in the TDA is tax-exempt, it is assumed that $\tau_{e,0} = \tau_{e,n} = 0$. We consider an investor who initially has 1,000 dollars in his TDA and 10,000 dollars in his TA. Table 1 shows the total wealth at the end of investment horizons of different lengths and different parameter constellations for the strategy to invest S exclusively in the TDA, B exclusively in the TDA (myopic location strategy), and the optimal location strategy. Only those cases that fulfill the prerequisites of Proposition 4.4 have been analyzed so as to avoid the problem of running into numerical instability. To highlight certain effects that changes in parameters have on the relative superiority of the optimal strategy compared to the strategy to invest only B in the TDA, let $\tau_d := \tau_{d,t} = \tau_{d,t+1} \; \forall \; 1 \leq t \leq n-1$, $\tau_g := \tau_{g,t} = \tau_{g,t+1} \; \forall \; 1 \leq t \leq n-1$, and g_S, d_S, g_B, d_B and x be defined accordingly. In the numerical example, $\tau_d = 0.3$, $\tau_g = 0.15$, $g_S = 0.09$, $d_S = 0$, $g_B = 0$, $d_B = 0.06$, $x = 0.5$ or $x = 1-$age were chosen for the base-case scenario. The investment horizon h for the base-case scenario was assumed to start at age 20 and continue until age 65. Let $V_n(S)$ be the total wealth at the end of the investment horizon when only S is invested in the TDA, $V_n(B)$ be the total wealth at the end of the investment horizon when only B is invested in the TDA, and $V_n(opt)$ be the total wealth at the end of the investment horizon of the optimal asset location strategy computed with the algorithm provided in Proposition 4.4.

The impact of the growth- and the tax-effect is illustrated in Figure 1. One can see that the total wealth when choosing the optimal location strategy is lower than when always locating asset B in the TDA. This suggests at first glance that the location decision cannot have been the optimal location decision, since total wealth is less at some points in time. Total wealth

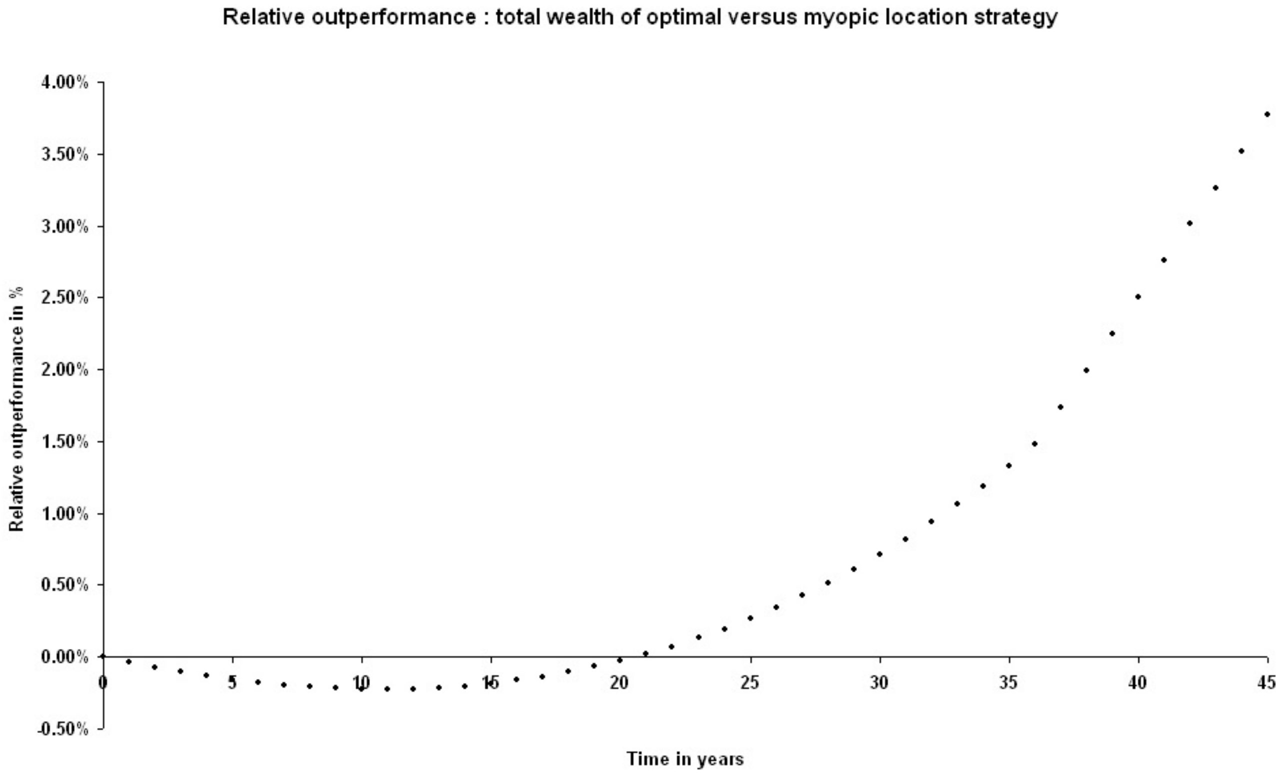

Figure 1: Illustration of the long-run power of the growth-effect

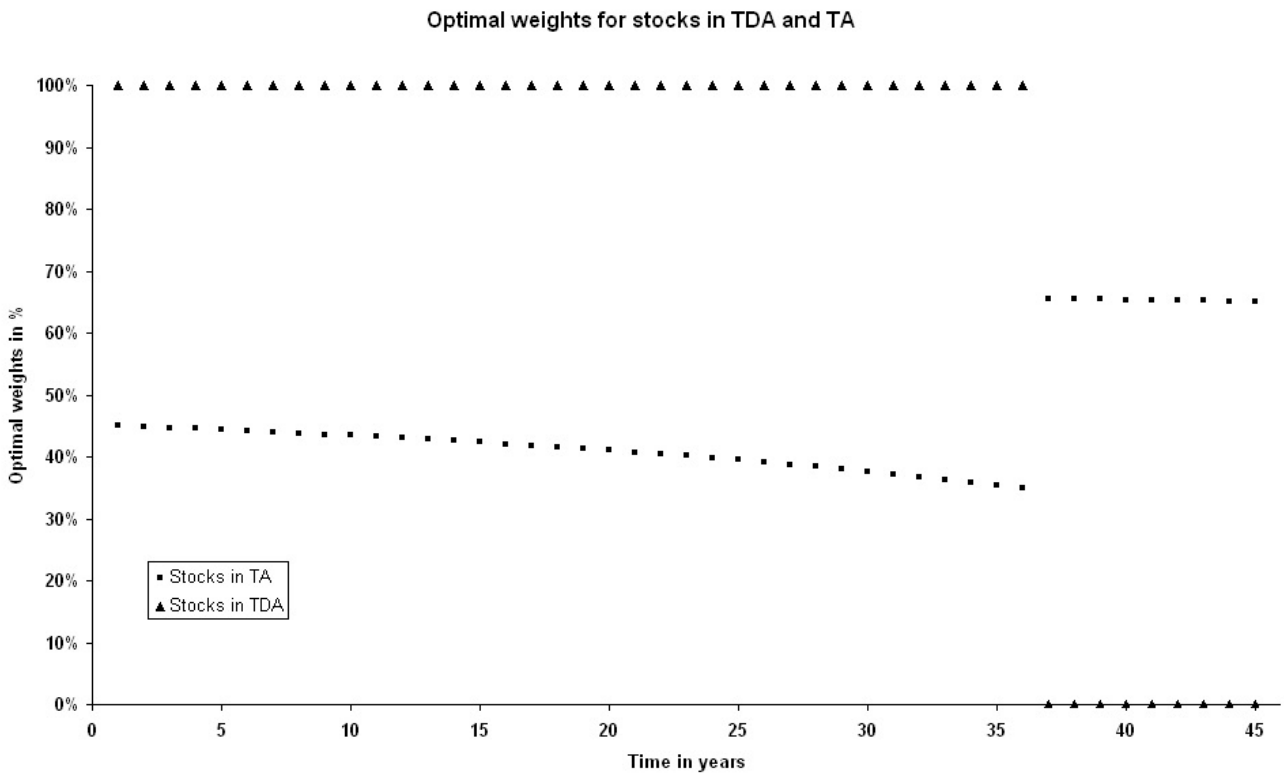

Figure 2: Fraction of stocks in the TDA and TA in the course of time

is the sum of capital in the TDA and capital in the TA. As the capital in the TDA allows to earn future gains tax-free, one dollar in the TDA is worth more than one dollar in the TA.[7] As the optimal location strategy invests into asset S in early years, which has a higher return than asset B, the capital in the TDA for the early strategy is greater than the capital invested in the TDA when always locating asset B there. As the tax-effect of asset B outweighs the tax-effect of asset S, locating asset B in the TDA is optimal for short investment horizons. In the base-case in figure 1, the investment horizon is 45 years and the growth-effect takes much more time to show significant effects. As it allows one to earn more future returns tax-free, it has an impact on future returns, while the tax-effect only has an impact on the present return. Investing in asset S in the TDA allows for a more increased tax-exempt basis than would an investment in asset B in the TDA. Since by assumption the contributions to the TDA are exogenously determined, it is not possible to increase the tax-exempt basis by further investments into the TDA. This can only be done by exploiting the growth-effect in the TDA, which is stronger for asset S. Figure 1 shows that starting at a certain point in time (about 11 periods in the example), the order of magnitude at which the growth-effect outweighs the tax-effect increases exponentially with the number of years passed. The kink in period 36 is due to the fact that from period 37 on, the investor only holds asset B in the TDA because for the remaining periods, the taxes that one can expect to save in the future due to the stronger growth-effect of asset S no longer outweigh the additional taxes one can save during that period by locating asset B in the TDA. In the example in Figure 1, the optimal strategy outperforms the strategy of always holding asset B in the TDA by less than 4%, a figure which seems minor for a 45-year investment horizon. This is because the parameters were chosen in a way such that the total wealth in early years was significantly less when riding the optimal strategy than when locating B in the TDA all the time. It is not difficult to construct examples in which the optimal location strategy outperforms the strategy of only locating B in the TDA by more than 4%. For example, by choosing $\tau_g = 0.2$, $\tau_d = 0.5$, $g_S = 0.07$, $d_S = 0.01$, $g_B = 0.0$, $d_B = 0.04$ and leaving the other parameters unchanged, one finds that the optimal strategy outperforms the other by 10.47% for $x = 0.50$. Nevertheless, in that example, the tax-effect in early years is very small and hardly observable in a graph.

Figure 2 shows the optimal weights for stocks in the TDA and the TA. As shown in Proposition 4.3, the optimal weights for stocks in the TDA are always either zero or one. In the numerical example, the optimal switch from asset S to asset B in the TDA is at the end of period 36. The weights in the TA always have to be chosen so that the weight of stocks is equal to x_t in each

[7]Dammon *et al.* (2004) call this value the "shadow price"of the dollar.

period. In the early years, the weights one has to assign to stocks in the TA in the numerical example decrease because the growth-rates in the TDA are bigger than the growth-rates in the TA. To balance the total portfolio in a way such that x remains constant at $x = 0.5$, the fraction of stocks held in the TA must decrease. From period 36 to period 37 the slope of the graph increases sharply, as from that point in time, the optimal asset to hold in the TDA is no longer stocks, but bonds.

Table 1 only contains cases in which it holds that

$$r_S > r_B$$
$$\text{and } r_S \tau_S < r_B \tau_B$$

This is to rule out those cases in which one asset shows both a higher growth-effect and a higher tax-effect. In these cases, the strategy to hold that asset in the TDA in each period would be better than any strategy in which the other asset would be held in the TDA for at least one period. The numerical results in Table 1 suggest ten different effects on the superiority of the optimal strategy to the myopic strategy, measured by the relative outperformance of the optimal strategy: An increase in τ_d leads to a decrease in the superiority of the optimal strategy because the higher tax burden put on B increases the tax-effect while the growth-effect remains constant. An increase in τ_g results in a decrease in the superiority of the optimal strategy as well, as this implies that S is taxed less heavily relative to B, and therefore should be located in the TDA more often. An increase in g_S results in a higher growth-effect and a higher tax-effect (if $\tau_g \neq 0$) for asset S and therefore increases the superiority of the optimal strategy. An increase in d_S also results in a higher growth-effect and a higher tax-effect (if $\tau_d \neq 0$) for asset S and therefore increases the superiority of the optimal strategy as well. An increase in g_B increases the growth- and the tax-effect of B, while the effects of S remain unchanged. This tends to make B the preferred asset to hold in the TDA earlier and therefore results in a decreasing superiority of the optimal strategy. An increase in d_B increases the growth- and tax-effect of B as well, and the same argument as is used in the case of an increase in g_B applies. A change in the structure of returns of asset S from gains to dividends results in an increasing superiority of the optimal strategy as the tax-effect of asset S increases, while the growth-effect remains the same. A change in the structure of returns of asset B from dividends to gains also results in an increasing superiority of the optimal strategy as the tax-effect of B decreases, while the growth-effect of S remains the same. An increase of x_t results in a decreasing superiority as an increasing fraction of S to hold results in an increasing fraction of

Numerical results

Effect of	$V_{in,0}$	$V_{out,0}$	τ_d	τ_g	g_S	d_S	g_B	d_B	x	h	$V_n(opt)$	$V_n(S)$	$V_n(B)$	Outperf. opt,B	Switch $S \to B$
base-case	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.06	0.5	20-65	162,939	162,027	157,020	3.77	36
g_S	1,000	10,000	0.30	0.15	0.10	0.00	0.00	0.06	0.5	20-65	199,542	199,016	186,942	6.74	40
d_S	1,000	10,000	0.30	0.15	0.09	0.01	0.00	0.06	0.5	20-65	195,732	195,571	181,276	7.97	42
$g_S \to d_S$	1,000	10,000	0.30	0.15	0.08	0.01	0.00	0.06	0.5	20-65	159,363	158,926	152,261	4.66	39
g_B	1,000	10,000	0.30	0.15	0.09	0.00	0.01	0.06	0.5	20-65	194,109	192,109	190,733	1.77	29
d_B	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.07	0.5	20-65	189,204	186,414	186,169	1.63	27
$d_B \to g_B$	1,000	10,000	0.30	0.15	0.09	0.00	0.01	0.05	0.5	20-65	167,436	166,964	161,052	3.96	38
τ_d	1,000	10,000	0.35	0.15	0.09	0.00	0.00	0.06	0.5	20-65	154,608	152,597	149,319	3.54	32
τ_g	1,000	10,000	0.30	0.10	0.09	0.00	0.00	0.06	0.5	20-65	175,246	171,988	172,211	1.76	27
x	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.06	0.4	20-65	142,068	141,171	136,290	4.24	36
x	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.06	0.6	20-65	186,967	186,041	180,905	3.35	36
base-case	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.06	1-la	20-65	178,486	177,593	172,736	3.33	36
g_S	1,000	10,000	0.30	0.15	0.10	0.00	0.00	0.06	1-la	20-65	222,827	222,309	210,598	5.81	40
d_S	1,000	10,000	0.30	0.15	0.09	0.01	0.00	0.06	1-la	20-65	217,429	217,269	203,360	6.92	42
$g_S \to d_S$	1,000	10,000	0.30	0.15	0.08	0.01	0.00	0.06	1-la	20-65	173,733	173,301	166,799	4.16	39
g_B	1,000	10,000	0.30	0.15	0.09	0.00	0.01	0.06	1-la	20-65	207,921	205,964	204,663	1.59	29
d_B	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.07	1-la	20-65	203,370	200,646	200,472	1.45	27
$d_B \to g_B$	1,000	10,000	0.30	0.15	0.09	0.00	0.01	0.05	1-la	20-65	182,762	182,297	176,535	3.53	38
τ_d	1,000	10,000	0.35	0.15	0.09	0.00	0.00	0.06	1-la	20-65	170,501	168,547	165,420	3.07	32
τ_g	1,000	10,000	0.30	0.10	0.09	0.00	0.00	0.06	1-la	20-65	194,670	191,520	191,846	1.47	27
h	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.06	0.5	30-65	88,534	88,147	86,992	1.77	26
h	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.06	0.5	40-65	48,480	48,317	48,189	0.60	16
h	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.06	0.5	50-65	26,711	26,642	26,691	0.07	6
h	1,000	10,000	0.30	0.15	0.09	0.00	0.00	0.06	0.5	60-65	14,782	14,759	14,782	0.00	0

$V_{in,0}$ is the amount held in the TDA at the beginning of the investment horizon, and $V_{out,0}$ is the amount held in the TA at the beginning of the investment horizon. $h = a - b$ is the investment horizon from age a to age b. A value for x of 1-la means that x_t is $\frac{100-\text{investor's age}}{100}$. τ_d is defined as follows: $\tau_d := \tau_{d,t} = \tau_{d,t+1} \; \forall \, 1 \leq t \leq n-1$. τ_g, g_S, d_S, g_B, d_B and x are defined accordingly. $V_n(S)$ denotes the total wealth at the end of the investment horizon, when holding only asset S in the TDA, $V_n(B)$ denotes the total wealth at the end of the investment horizon when holding only asset B in the TDA and $V_n(opt)$ denotes the total wealth at the end of the investment horizon when driving the optimal location strategy computed with the algorithm derived in Proposition 4.4. Outperf. opt,B is the outperformance of $V_n(opt)$ over $V_n(B)$ in percent: $\left(\frac{V_n(opt)}{V_n(B)} - 1\right) \cdot 100\%$ Switch $S \to B$ denotes the last period in which the optimal investment strategy is to hold asset S in the TDA. After that period, it becomes optimal to keep asset B in the TDA.

Table 1: Numerical results

S in the TA. This leads to a stronger growth in the TA and implies that the effect caused by the growth in the TDA has a lower impact on total wealth. A decrease in the length of the investment horizon results in a decrease in the superiority of the optimal strategy, since the power of the growth-effect depends on the length of the remaining investment horizon. As the time increases in which the growth-effect can be exploited, the superiority of locating asset S in the TDA becomes stronger. A shorter investment horizon tends to 'cut off' those years in which asset S is the preferred asset to hold in the TDA. These findings can be summarized as follows: the higher the outperformance of S compared to B, the higher the tax burden put on S compared to B. Similarly, the longer the remaining investment horizon is, the larger the superiority of the optimal strategy is.

6 Conclusion

Conventional wisdom suggests that individual investors should place heavily taxed bonds in their tax-deferred accounts (TDA), and more lightly taxed equities in the taxable accounts (TA) to maximize long run wealth accumulation. This paper has shown that this is generally not true. The intuition behind that finding is that it can be optimal to hold stocks in the TDA in early years (and maybe pay more taxes in these years) if, since stocks usually display a higher growth rate than bonds, the total wealth in the TDA is growing faster, and if in upcoming years more assets can be placed in the TDA, which in turn leads to lower tax-payments in these years. The option to locate stocks in the TDA always outperforms the option of locating bonds in the TDA if the expected future tax saving outweighs the effect of the present tax burden that results from the decision to locate stocks instead of bonds in the TDA.

The results derived here depend on the assumption that gains and losses are taxed at the source. Yet, in many tax systems around the world gains are only taxable (if ever) when they are realized. This offers the investor the opportunity of riding tax timing strategies like those described in Constantinides (1983) and Constantinides (1984). According to these strategies, losses should be realized the moment they occur to obtain a tax advantage and then to earn the interest/gains on that tax advantage. Gains should be deferred as long as possible to avoid having to pay the taxes on them and to thereby earn the interest/gains on the avoided taxes. The more volatile an asset, the larger the gains one can exploit with this strategy. In a tax system where gains are taxable at the moment they are realized, it is possible that the benefits from the tax-timing strategy of holding S in the TA outweigh the effect of S being the strongest growing asset. As in practice, many tax systems only allow one to offset negative gains against positive gains of other assets, but not with other income (or only up to a limited amount), the potential advantage of locating a volatile asset in the TA should be quite small as only tax-effects of positive gains can be exploited. By assuming $\tau_{g,t}$ to be the tax-rate of the not modelled optimal tax realization strategy of the capital gains, which is referred to as the "effective tax rate"by Constantinides (1983), the problem of tax-timing can be entirely circumvented from a theoretical perspective. Nevertheless, this implies that for the maximization problem, this optimal tax-rate has to be given.

The assumption that contribution limits are exploited each year is crucial. As an investment in a TDA dominates an investment in a TA when returns are non-negative, an investor should always invest into the TDA if he has not exploited his contribution limits as long as he can be sure of never having to withdraw early (and thereby face the penalty tax). The assumption that the investor does not face the risk of early withdrawals from a TDA is quite strong. It is made so that it is possible to fully concentrate on the asset location effect and to suppress potential flexibility aspects.

The assumption that the investment strategy of holding a certain fraction of one's wealth in asset S and the remainder in asset B is made to expose the impact of asset location. It does not fully take the interdependence of asset location and asset allocation into account, which can result in inefficient portfolios. It is implicitly assumed that x_t is a convenient risk-measure and for a given x_t, the tax-optimized strategy can be found.[8] For a rational investor, only after-tax returns should matter, as only these generate growth in capital or can be used for consumption without decreasing one's capital stock. Taxation has three impacts on returns. The first effect is that after-tax returns shrink towards zero for nonzero tax rates. The second is that volatilities decrease due to shrinking after-tax returns. The third effect is that correlations between different assets may change if these assets are taxed differently. This implies that a prudent investor should take the impact of asset location on asset allocation into account. On the other hand, asset allocation also has an impact on asset location, as one can only locate to an account that one has actually chosen to hold. Therefore, prudent investors need to determine asset location and asset allocation simultaneously. However, the assumption of an investor making the decision of asset allocation and asset location consecutively will probably be quite similar to decisions made by a vast majority of the population that does not have expert knowledge in the field of finance.

Formal conditions for those cases in which a solution can be determined without numerical

[8] This assumption ignores the impact of the asset location decision on volatility of the total wealth at the end of the investment horizon.

optimization routines have been given in Proposition 4.4 and proved in the appendix. This algorithm allows, for example, the computation of location strategies that would have been optimal in the past for values of x_t and $V_{in,t}$ und $V_{out,t}$ that fulfill condition (2). The numerical results presented in Table 1 have shown that asset location has a significant impact on total wealth at the end of the investment horizon. They have further helped identify different factors that have an impact on the relative outperformance of the optimal location strategy compared to the myopic strategy following conventional wisdom that only bonds should be located in the TDA. The higher the performance of S compared to B, the higher taxes on S compared to B and the longer the remaining investment horizon, the larger the superiority of the optimal strategy.

The question of optimal asset location was analyzed, assuming the asset allocation as given. It is straightforward to show that prudent investors have to determine the asset location and asset allocation that maximize their utility functions simultaneously. Furthermore, returns were assumed to not be stochastic in order to facilitate the analysis and allow an intuitive interpretation of the obtained results. To allow for stochastic returns and to model the interrelation of the asset location and asset allocation decision simultaneously are interesting avenues of further research.

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Appendix

This appendix contains the proofs of the Lemmas and Propositions in the text.

Proof of Proposition 4.1. If one invests the dollar in the TDA in B and the dollar in the TA in S, total wealth at the end of period 1 is

$$(1+r_{B,1})(1-\tau_{e,1})+\Big(1+r_{S,1}(1-\tau_{S,1})\Big)(1-\tau_{e,0})$$

If one invests the dollar in the TDA in S and the dollar in the TA in B, total wealth at the end of period 1 is

$$(1+r_{S,1})(1-\tau_{e,1})+\Big(1+r_{B,1}(1-\tau_{B,1})\Big)(1-\tau_{e,0})$$

The difference in the funds at the end of period 1 is then

$$\begin{aligned}&(1+r_{B,1})(1-\tau_{e,1})+\Big(1+r_{S,1}(1-\tau_{S,1})\Big)(1-\tau_{e,0})\\&-(1+r_{S,1})(1-\tau_{e,1})-\Big(1+r_{B,1}(1-\tau_{B,1})\Big)(1-\tau_{e,0})\\=&(\tau_{e,1}-\tau_{e,0})(-r_{B,1}+r_{S,1})+(-r_{S,1}\tau_{S,1}+r_{B,1}\tau_{B,1})(1-\tau_{e,0})\\>&0\end{aligned}$$

□

Proof of Proposition 4.2. Let $n\in\mathbb{N}$ be chosen to be so large, that $\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)(1-\tau_{e,n})>1$. It then holds that

$$\begin{aligned}&\prod_{t=1}^{n}(1+r_{S,t})(1-\tau_{e,n})+\prod_{t=1}^{n}\Big(1+r_{B,t}(1-\tau_{B,t})\Big)(1-\tau_{e,0})\\&-\prod_{t=1}^{n}(1+r_{B,t})(1-\tau_{e,n})+\prod_{t=1}^{n}\Big(1+r_{S,t}(1-\tau_{S,t})\Big)(1-\tau_{e,0})\\>&1+\frac{\prod_{t=1}^{n}\Big(1+r_{B,t}(1-\tau_{B,t})\Big)(1-\tau_{e,0})}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)(1-\tau_{e,n})}\\&-\frac{\prod_{t=1}^{n}\Big(1+r_{B,t}\Big)}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)}-\frac{\prod_{t=1}^{n}\Big(1+r_{S,t}(1-\tau_{S,t})\Big)(1-\tau_{e,0})}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)(1-\tau_{e,n})}\end{aligned}\tag{3}$$

As $r_{S,t}(1-\tau_{S,t}) < r_{S,t}$ and $r_{B,t}(1-\tau_{B,t}) < r_{B,t} < r_{S,t}$ for all $1 \leq i \leq n$ it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\left\{1+\frac{\prod_{t=1}^{n}\Big(1+r_{B,t}(1-\tau_{B,t})\Big)(1-\tau_{e,0})}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)(1-\tau_{e,n})}\right.\\
&\left.-\frac{\prod_{t=1}^{n}\Big(1+r_{B,t}\Big)}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)}-\frac{\prod_{t=1}^{n}\Big(1+r_{S,t}(1-\tau_{S,t})\Big)(1-\tau_{e,0})}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)(1-\tau_{e,n})}\right\}\\
=&1+\lim_{n\to\infty}\frac{\prod_{t=1}^{n}\Big(1+r_{B,t}(1-\tau_{B,t})\Big)(1-\tau_{e,0})}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)(1-\tau_{e,n})}\\
&-\lim_{n\to\infty}\frac{\prod_{t=1}^{n}\Big(1+r_{B,t}\Big)}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)}-\lim_{n\to\infty}\frac{\prod_{t=1}^{n}\Big(1+r_{S,t}(1-\tau_{S,t})\Big)(1-\tau_{e,0})}{\prod_{t=1}^{n}\Big(1+r_{S,t}\Big)(1-\tau_{e,n})}\\
&\geq 1+\lim_{n\to\infty} q_3^n\frac{1-\tau_{e,0}}{1-\tau_{e,n}}-\lim_{n\to\infty} q_1^n-\lim_{n\to\infty} q_2^n\frac{1-\tau_{e,0}}{1-\tau_{e,n}}\\
=&1+0-0-0\\
=&1>0
\end{aligned}$$

□

For the proof of Proposition 4.3, the following technical Lemma is needed:

Lemma 6.1. *Let $a^a_{out,i}$ $(i \geq t)$ denote the value of $a_{out,i}$ if $a_{in,t} = 1$, and let $a^b_{out,i}$ denote the value of $a_{out,i}$ if $a_{in,t} = 0$. Let $a^*_{in,t}$ denote the value of $a_{in,t}$ when following the optimal investment strategy. Let $S_{V,k}$, $S_{N,k}$, $B_{V,k}$, $B_{N,k}$, C_k and D_k be defined as follows:*

$$\begin{aligned}
S_{V,k} &:= 1+r_{S,t}\\
S_{N,k} &:= 1+r_{S,t}(1-\tau_{S,t})\\
B_{V,k} &:= 1+r_{B,t}\\
B_{N,k} &:= 1+r_{B,t}(1-\tau_{B,t})\\
C_k &:= \prod_{\substack{i=1\\ i\neq k}}^{n}\Big(a^*_{in,i}S_{V,i}+(1-a^*_{in,i})B_{V,i}\Big)(1-\tau_{e,n})\\
D_k &:= \prod_{\substack{i=1\\ i\neq k}}^{n}\Big(a^*_{out,i}S_{N,k}+(1-a^*_{out,i})B_{N,k}\Big)(1-\tau_{e,0})
\end{aligned}$$

Let $a^z_{out,t}$ be the value for $a_{out,t}$ that fulfills equation (1) for given $a^z_{in,t}$. Further assume

$$\begin{aligned}\forall\, 1 \leq t \leq n : & V_{in,t-1} \leq x_t V_{t-1} \quad \textit{if } a^z_{in,t} = 1 \\ & \textit{and } V_{out,t-1} \geq x_t V_{t-1} \quad \textit{if } a^z_{in,t} = 0\end{aligned}$$

Then it holds for given $1 \leq k \leq n$ that there is no $0 < a_{in,k} < 1$ such that

$$\begin{aligned}&C_k \cdot (a_{in,k} S_{V,k} + (1 - a_{in,k}) B_{V,k}) (1 - \tau_{e,n}) + D_k \cdot (a_{out,k} S_{N,k} + (1 - a_{out,k}) B_{N,k}) (1 - \tau_{e,0}) \\ >&C_k \cdot \left(a^z_{in,k} S_{V,k} + (1 - a^z_{in,k}) B_{V,k}\right) (1 - \tau_{e,n}) + D_k \cdot \left(a^z_{out,k} S_{N,k} + (1 - a^z_{out,k}) B_{N,k}\right) (1 - \tau_{e,0})\end{aligned}$$

Proof of Lemma 6.1. Assume

$$\begin{aligned}&\exists\, 0 < a_{in,k} < 1 : \\ &C_k \cdot (a_{in,k} S_{V,k} + (1 - a_{in,k}) B_{V,k}) (1 - \tau_{e,n}) + D_k \cdot (a_{out,k} S_{N,k} + (1 - a_{out,k}) B_{N,k}) (1 - \tau_{e,0}) \\ >&C_k \cdot \left(a^z_{in,k} S_{V,k} + (1 - a^z_{in,k}) B_{V,k}\right) (1 - \tau_{e,n}) + D_k \cdot \left(a^z_{out,k} S_{N,k} + (1 - a^z_{out,k}) B_{N,k}\right) (1 - \tau_{e,0})\end{aligned}$$

Consider three investment strategies:

1. $a_{in,k} \in (0,1)$, $a_{in,t} := a^*_{in,t}$ for $k \neq t$.
2. $a_{in,k} = 1$, $a_{in,t} := a^*_{in,t}$ for $k \neq t$.
3. $a_{in,k} = 0$, $a_{in,t} := a^*_{in,t}$ for $k \neq t$.

Let $V_n(j)$ denote the value V_n takes when choosing strategy j. It then holds that

$$\begin{aligned}V_n(1) &= C_k\Big(a_{in,k} S_{V,k} + (1 - a_{in,k}) B_{V,k}\Big) + D_k (a_{out,k} S_{N,k} + (1 - a_{out,k}) B_{N,k}) \\ V_n(2) &= C_k S_{V,k} + D_k (a^a_{out,k} S_{N,k} + (1 - a^a_{out,k}) B_{N,k}) \\ V_n(3) &= C_k B_{V,k} + D_k (a^b_{out,k} S_{N,k} + (1 - a^b_{out,k}) B_{N,k})\end{aligned} \tag{4}$$

in which $a^a_{out,k}$ and $a^b_{out,k}$ denote the weights of $a_{out,k}$, which follow from the choice of $a^z_{in,k} = 1$ and $a^z_{in,k} = 0$, respectively. From (1) it follows that

$$\begin{aligned}a_{out,t} &= \frac{x_t (V_{in,t-1} + V_{out,t-1}) - V_{in,t-1} a_{in,t}}{V_{out,t-1}} \\ &= \frac{(x_t - a_{in,t}) V_{in,t-1}}{V_{out,t-1}} + x_t\end{aligned} \tag{5}$$

For $a^*_{in,t} = 1$ it holds that

$$V_{in,t-1} \leq x_t V_{t-1}$$
$$\Leftrightarrow V_{in,t-1} \leq \frac{x_t}{1-x_t} V_{out,t-1}$$

and therefore

$$\begin{aligned} a_{out,t} &= \frac{(x_t - 1)V_{in,t-1}}{V_{out,t-1}} + x_t \quad (< 1) \\ &\geq \frac{(x_t - 1)\frac{x_t}{1-x_t}V_{out,t-1}}{V_{out,t-1}} + x_t \\ &= -x_t + x_t = 0 \end{aligned}$$

For $a^*_{in,t} = 0$ it holds that

$$V_{out,t-1} \geq x_t V_{t-1}$$
$$\Leftrightarrow V_{out,t-1} \geq \frac{x_t}{1-x_t} V_{in,t-1}$$

and therefore

$$\begin{aligned} a_{out,t} &= \frac{(x_t - 0)V_{in,t-1}}{V_{out,t-1}} + x_t \quad (> 0) \\ &\leq \frac{x_t V_{in,t-1}}{\frac{x_t}{1-x_t}V_{in,t-1}} + x_t \\ &= 1 - x_t + x_t = 1 \end{aligned}$$

which shows that $a_{out,t} \in [0, 1]$. Plugging (5) into (4), one obtains

$$\begin{aligned} V_n(1) =& C_k\big(a_{in,k}S_{V,k} + (1 - a_{in,k})B_{V,k}\big) + D_k\left(\left(\frac{(x_k - a_{in,k})V_{in,k-1}}{V_{out,k-1}} + x_k\right)(S_{N,k} - B_{N,k}) + B_{N,k}\right) \\ V_n(2) =& C_k S_{V,k} + D_k\left(\left(\frac{(x_k - 1)V_{in,k-1}}{V_{out,k-1}} + x_k\right)(S_{N,k} - B_{N,k}) + B_{N,k}\right) \\ V_n(3) =& C_k B_{V,k} + D_k\left(\left(\frac{x_k V_{in,k-1}}{V_{out,k-1}} + x_k\right)(S_{N,k} - B_{N,k}) + B_{N,k}\right) \end{aligned}$$

By assumption $\exists\, 0 < a_{in,k} < 1 : \Big(V_n(1) > V_n(2)\Big) \wedge \Big(V_n(1) > V_n(3)\Big)$. It therefore has to hold, that

$$
\begin{aligned}
V_n(1)-V_n(2) =& C_k\Big(a_{in,k}S_{V,k}+(1-a_{in,k})B_{V,k}\Big)+D_k\Bigg(\left(\frac{(x_k-a_{in,k})V_{in,k-1}}{V_{out,k-1}}+x_k\right)(S_{N,k}-B_{N,k})+B_{N,k}\Bigg)\\
&-\left[C_kS_{V,k}+D_k\Bigg(\left(\frac{(x_k-1)V_{in,k-1}}{V_{out,k-1}}+x_k\right)(S_{N,k}-B_{N,k})+B_{N,k}\Bigg)\right]\\
&= C_k(a_{in,k}-1)(S_{V,k}-B_{V,k})+D_k(1-a_{in,k})\frac{V_{in,k-1}}{V_{out,k-1}}(S_{N,k}-B_{N,k})>0 \qquad (6)
\end{aligned}
$$

and

$$
\begin{aligned}
V_n(1)-V_n(3) =& C_k\Big(a_{in,k}S_{V,k}+(1-a_{in,k})B_{V,k}\Big)+D_k\Bigg(\left(\frac{(x_k-a_{in,k})V_{in,k-1}}{V_{out,k-1}}+x_k\right)(S_{N,k}-B_{N,k})+B_{N,k}\Bigg)\\
&-\left[C_kB_{V,k}+D_k\Bigg(\left(\frac{x_kV_{in,k-1}}{V_{out,k-1}}+x_k\right)(S_{N,k}-B_{N,k})+B_{N,k}\Bigg)\right]\\
&= C_ka_{in,k}(S_{V,k}-B_{V,k})+D_k\left(-\frac{a_{in,k}V_{in,k-1}}{V_{in,k-1}}\right)(S_{N,k}-B_{N,k})>0 \qquad (7)
\end{aligned}
$$

From (6) it follows that

$$
\begin{aligned}
S_{V,k}-B_{V,k} &< -\frac{D_k}{C_k}\cdot\frac{(1-a_{in,k})V_{in,k-1}}{V_{out,k-1}}\cdot\frac{S_{N,k}-B_{N,k}}{a_{in,k}-1}\\
&=\frac{D_k}{C_k}\cdot\frac{V_{in,k-1}}{V_{out,k-1}}(S_{N,k}-B_{N,k})
\end{aligned}
$$

And with (7)

$$
\begin{aligned}
&C_ka_{in,k}(S_{V,k}-B_{V,k})+D_k\left(-a_{in,k}\frac{V_{in,k-1}}{V_{out,k-1}}\right)(S_{N,k}-B_{N,k})\\
<&C_ka_{in,k}\frac{D_k}{C_k}\cdot\frac{V_{in,k-1}}{V_{out,k-1}}(S_{N,k}-B_{N,k})-D_ka_{in,k}\cdot\frac{V_{in,k-1}}{V_{out,k-1}}(S_{N,k}-B_{N,k})\\
=&0
\end{aligned}
$$

which proves the Lemma by contradiction. □

Proof of Proposition 4.3. By assumption, it holds that

$$
\begin{aligned}
\forall\, 1\le t\le n :& V_{in,t-1}\le x_tV_{t-1} \quad \text{if } a^z_{in,t}=1\\
\text{and } \forall\, 1\le t\le n :& V_{out,t-1}\ge x_tV_{t-1} \quad \text{if } a^z_{in,t}=0
\end{aligned}
$$

From Lemma 6.1, it follows directly that $a_{in,t}\in\{0,1\}\;\forall\,1\le t\le n$ and the problem of finding the optimal asset location in period $1 \leq t \leq n$ is equivalent to answering the question whether to hold S or B in period t in the TDA. $\square$

Proof of Lemma 4.1. In those cases where $V_{in,t-1} \leq x_t V_{t-1}$ holds, Proposition 4.3 can be applied. In the other cases one can make use of the fact that $V_{in,t-1} > x_t V_{t-1}$ is equivalent to $V_{in,t-1} > \frac{x_t}{1-x_t} V_{out,t-1}$. Define a contractive mapping $f : \mathbb{R} \to \mathbb{R}, V_{in,t-1} \to f(V_{in,t-1}) := qV_{in,t-1}$, in which $q \in [0,1]$ is to be chosen in such a way that $qV_{in,t-1} = \frac{x_t}{1-x_t} V_{out,t-1}$. Proposition 4.3 can then be applied to the contracted problem which results in locating the amount of $qV_{in,t-1}$ of asset S in the TDA. No decision has been made so far about the amount of $(1-q)V_{in,t-1}$. As only asset B is still to be located and an investment of B in the TA is strictly dominated by an investment of B in the TDA, the optimal strategy locates B to the remaining amount of $(1-q)V_{in,t-1}$. $\square$

Proof of Lemma 4.2. In those cases where $V_{out,t-1} \geq x_t V_{t-1}$ Proposition 4.3 can be applied. In the other cases one can make use of the fact that $V_{out,t-1} < x_t V_{t-1}$ is equivalent to $V_{out,t-1} < \frac{x_t}{1-x_t} V_{in,t-1}$. Define a contractive mapping $f : \mathbb{R} \to \mathbb{R}, V_{in,t-1} \to f(V_{in,t-1}) := qV_{in,t-1}$, in which $q \in [0,1]$ is to be chosen in such a way that $qV_{in,t-1} = \frac{1-x_t}{x_t} V_{out,t-1}$. Proposition 4.3 can then be applied to the contracted problem which results in locating the amount of $qV_{in,t-1}$ of asset B in the TDA. No decision has been made so far about the amount of $(1-q)V_{in,t-1}$ that can still be invested in the TDA. As there is only asset S remaining to be located and an investment of S in the TDA strictly dominates an investment of S in the TA, the optimal strategy locates S to the remaining amount of $(1-q)V_{in,t-1}$. $\square$

Proof of Proposition 4.4. Whether $a^*_{in,t} = 1$ or $a^*_{in,t} = 0$ only depends on the length of the remaining investment horizon and the future returns for S and B for all $1 \leq t \leq n$. This implies that $a^*_{in,t}$ is independent of $V_{in,t-1}$ and $V_{out,t-1}$ as long as (2) holds. Only an answer to the question of whether to locate S or B in the TDA has to be found. Ignoring the constraint that $a_{out,t} \in [0,1]\ \forall\, 1 \leq t \leq n$, a solution to the location problem can be found by choosing $V_{in,t-1} = V_{out,t-1} = 1\ \forall 1 \leq t \leq n$. A solution is easily computed with the algorithm given in this proof. As (2) holds by assumption, the $a^*_{in,t}$ of the unconstrained problem are the same as in the constrained problem. For given $a^*_{in,i}$, $t+1 \leq i \leq n$, the optimal choice for $t < n$ can be determined[9] by checking if

$$\begin{aligned}
&S_{V,t}\prod_{i=t+1}^{n}\Big(a^*_{in,i}S_{V,i}+(1-a^*_{in,i})B_{V,i}\Big)(1-\tau_{e,n})\\
&\quad+\Big(a^a_{out,t}S_{N,t}+(1-a^a_{out,t})B_{N,t}\Big)\prod_{i=t+1}^{n}\Big(a^a_{out,i}S_{N,i}+(1-a^a_{out,i})B_{N,i}\Big)(1-\tau_{e,0})\\
\geq&B_{V,t}\prod_{i=t+1}^{n}\Big(a^*_{in,i}S_{V,i}+(1-a^*_{in,i})B_{V,i}\Big)(1-\tau_{e,n})\\
&\quad+\Big(a^b_{out,t}S_{N,t}+(1-a^b_{out,t})B_{N,t}\Big)\prod_{i=t+1}^{n}\Big(a^b_{out,i}S_{N,i}+(1-a^b_{out,i})B_{N,i}\Big)(1-\tau_{e,0})
\end{aligned}$$

If true, $a^*_{in,t}=1$, else $a^*_{in,t}=0$. If this step is done for all $1\leq t\leq n-1$, beginning in $n-1$ and going down to 1, the optimal asset location is obtained. With Proposition 4.3, conditions (2) assure that $a^*_{in,t}\in\{0,1\}\ \forall\, 1\leq t\leq n$. $\square$

Proof of Proposition 4.5. From

$$\begin{aligned}
V_{in,n}=&\sum_{t=0}^{n}C_t\prod_{i=t+1}^{n}\Big(a_{in,i}S_{V,i}+(1-a_{in,i})B_{V,i}\Big)(1-\tau_{e,n})\\
=&C_0\prod_{i=1}^{n}\Big(a_{in,i}S_{V,i}+(1-a_{in,i})B_{V,i}\Big)(1-\tau_{e,n})\\
&+C_1\prod_{i=2}^{n}\Big(a_{in,i}S_{V,i}+(1-a_{in,i})B_{V,i}\Big)(1-\tau_{e,n})\\
&+\dots\\
&+C_{n-1}\prod_{i=n}^{n}\Big(a_{in,i}S_{V,i}+(1-a_{in,i})B_{V,i}\Big)(1-\tau_{e,n})\\
&+C_n(1-\tau_{e,n})
\end{aligned}$$

[9] $a^*_{in,n}=0$ follows directly from Proposition 4.1.

and

$$\begin{aligned}
V_{out,n} =& \sum_{t=0}^{n} C_t(1-\tau_{e,t}) \prod_{i=t+1}^{n} \Big(a_{out,i}S_{N,i} + (1-a_{out,i})B_{N,i} \Big) \\
=& C_0(1-\tau_{e,0}) \prod_{i=1}^{n} \Big(a_{out,i}S_{N,i} + (1-a_{out,i})B_{N,i} \Big) \\
&+ C_1(1-\tau_{e,1}) \prod_{i=2}^{n} \Big(a_{out,i}S_{N,i} + (1-a_{out,i})B_{N,i} \Big) \\
&+ \dots \\
&+ C_{n-1}(1-\tau_{e,n-1}) \prod_{i=n}^{n} \Big(a_{out,i}S_{N,i} + (1-a_{out,i})B_{N,i} \Big) \\
&+ C_n(1-\tau_{e,n})
\end{aligned}$$

it follows that

$$\begin{aligned}
V_n =& V_{in,n} + V_{out,n} \\
=& C_0 \left[\prod_{i=1}^{n} \Big(a_{in,i}S_{V,i} + (1-a_{in,i})B_{V,i} \Big)(1-\tau_{e,n}) + \prod_{i=1}^{n} \Big(a_{out,i}S_{N,i} + (1-a_{out,i})B_{N,i} \Big)(1-\tau_{e,0}) \right] \\
&+ C_1 \left[\prod_{i=2}^{n} \Big(a_{in,i}S_{V,i} + (1-a_{in,i})B_{V,i} \Big)(1-\tau_{e,n}) + \prod_{i=2}^{n} \Big(a_{out,i}S_{N,i} + (1-a_{out,i})B_{N,i} \Big)(1-\tau_{e,1}) \right] \\
&\dots \\
& C_{n-1} \left[\Big(a_{in,n}S_{V,n} + (1-a_{in,n})B_{V,n} \Big)(1-\tau_{e,n}) + \Big(a_{out,n}S_{N,n} + (1-a_{out,n})B_{N,n} \Big)(1-\tau_{e,n-1}) \right] \\
&+ 2C_n(1-\tau_{e,n})
\end{aligned}$$

Contributions at different points in time can therefore be considered investments in different TDAs with different times until maturity and different contributions C_t $(1 \leq t \leq n)$. As $g_{S,t}$, $d_{S,t}$, $g_{B,t}$, $d_{B,t}$, $\tau_{g,t}$ and $\tau_{d,t}$ are identical for each $1 \leq t \leq n$ in each of these different TDAs, the periods in which it is optimal to hold S in the TDA and the periods in which it is optimal to hold B in the TDA are the same for all of these problems. As the assets in the TDA are but a linear combination of all the assets in the TDAs that only allow for one contribution (fictional TDAs), it follows that the optimal strategy that can be determined for the fictional TDAs holds as well for the TDA, which solves the problem. □

Proof of Proposition 4.6. From the definition of the TDA and TEA this follows directly by letting $\tau_{e,n} := \tau_{e,0}$. □