On Continuous Time Trading of a Small Investor in a Limit Order Market

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to my lovely parents

Abstract

We provide a mathematical framework to model continuous time trading in limit order markets of a small investor whose transactions have no impact on order book dynamics. The investor can continuously place market and limit orders. A market order is executed immediately at the best currently available price, whereas a limit order is stored until it is executed at its limit price or canceled. The limit orders can be chosen from a continuum of limit prices.

In this framework we show how elementary strategies (hold limit orders with only finitely many different limit prices and rebalance at most finitely often) can be extended in a suitable way to general continuous time strategies containing orders with infinitely many different limit prices. The general limit buy order strategies are predictable processes with values in the set of nonincreasing demand functions (not necessarily left- or right-continuous in the price variable). It turns out that this family of strategies is closed and any element can be approximated by a sequence of elementary strategies.

Furthermore, we study Merton's portfolio optimization problem in a specific instance of this framework. Assuming that the risky asset evolves according to a geometric Brownian motion, a proportional bid-ask spread, and Poisson execution times for the limit orders of the small investor, we show that the optimal strategy consists in using market orders to keep the proportion of wealth invested in the risky asset within certain boundaries, similar to the result for proportional transaction costs, while within these boundaries limit orders are used to profit from the bid-ask spread.

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Contents

Chapter 1

Introduction

The main objective of this thesis is to introduce a mathematical framework of trading in a limit order market, when the investor is small. In a *limit order market* an investor can use e.g. a market buy order to immediately buy the desired amount of shares at the best-ask price (or at an average price higher than the best-ask price if the order is large enough to eat into the order book). Alternatively, he can use a limit buy order to specify the limit price which he is willing to pay per share and wait until another market participant matches his order. The trading of a *small* investor does not change the dynamics of the order book, i.e. his trading opportunities are exogenously given.

This assumption of a small investor is prevalent in many models in mathematical finance, the most prominent example being the model of Black and Scholes [BS73], where the trading opportunities of the investor in risky assets are modeled by a geometric Brownian motion, which is not influenced by whatever amounts the investor buys or sells. Furthermore, the well-established mathematical theory of trading in a market with proportional transaction costs (see e.g. Kabanov [Kab99], Kabanov, Rásonyi, and Stricker [KRS02], and Schachermayer [Sch04] for earlier works or Kabanov and Safarian [KS09a] for a comprehensive account) also assumes that the best-bid and the best-ask price processes are exogenously given.

In the research on limit order markets this is not necessarily the case. On the contrary, one reason to study limit order markets in the first place is as a means to understand the price impact of a large trader whose orders eat into the book. This is done for example in Obizhaeva and Wang [OW05], Alfonsi and Schied [AS10], Alfonsi, Fruth, and Schied [AFS10] and Predoiu, Shaikhet, and Shreve [PSS11] to name a few recent articles. This line of research goes into the direction of large trader models such as Bank and Baum [BB04].

Other stochastic models of order book dynamics are not focused on larger traders per se, still the dynamics of the whole order book is typically considered (such as Luckock [Luc03], Cont, Stoikov, and Talreja [CST10], or Höschler [Hös11]). Of course to model the whole dynamics in a completely general way, without restrictions to make the model tractable, would be a gargantuan task. Therefore, more or less concrete assumptions about the underlying order flow have to be made.

If we now come back to our intention to develop a framework for the trading in a limit order book of a *small* investor, a first important assumption along the lines of the Black-Scholes model or the proportional transaction costs theory is that the orders of the small investor will *never* eat into the order book or even change the state of the order book in any other way. The intricate details of the order book are only relevant to the small investor inasmuch as they translate into the dynamics of the best-bid and the best-ask price processes and insofar as they trigger any executions of his limit orders. There are articles about a small investor trading in a limit order market, e.g. Guilbaud and Pham [PG11], but they do not focus on developing a general framework but rather start with a more specific model and then solve an optimization problem for example.

Another observation, already made by Šmid [Šmi07], is that under certain circumstances a limit order can be replaced by a market order. If a small investor e.g. places a limit buy order with limit price p^B lower than the current best-ask price and afterwards the best-ask price process $\overline{S}$ hits $[0,p^B]$ at some stopping time $T^{\overline{S}}$, then of course the limit buy order should be executed. But unless this happens due to a jump of $\overline{S}$ below p^B, the investor can just use a market order to buy at the same price immediately after $T^{\overline{S}}$ as long as the best-bid and best-ask price processes are assumed to be right-continuous.

In *Chapter 2* we use the ideas described in the previous two paragraphs to *construct a mathematical framework to model continuous time trading in limit order markets of a small investor.* The exogenously given trading opportunities are described by a quadruple consisting of the best-bid and best-ask price processes $\underline{S}$ and $\overline{S}$ as well as two integer-valued random measures μ and ν which describe which limit orders are executed at what time. To lead to a sensible model, the integer-valued random measures will have to satisfy certain properties to be in line with the best-bid and the best-ask price processes but cannot be derived from them.

The investor can continuously place market and limit orders. A market order is executed immediately at the best currently available price, whereas a limit order is stored until it is executed at its limit price or canceled. The limit orders can be chosen from a continuum of limit prices.

We show how elementary strategies ("hold limit orders with only finitely many different limit prices and rebalance at most finitely often") can be extended in a suitable way to general continuous time strategies containing orders with infinitely many different limit prices. The general limit buy order strategies are predictable processes with values in the set of nonincreasing demand functions (not necessarily left- or right-continuous in the price variable). *It turns out that the family of strategies is closed and any element can be approximated by a sequence of elementary strategies.*

In *Chapter 3* we deal with a specific instance of the framework introduced in Chapter 2 to study a portfolio optimization problem in a limit order market. We assume that the best-bid price follows a geometric Brownian motion and that the bid-ask spread is proportional to the size of the best-bid price. Furthermore, we assume that the times at which limit buy orders and limit sell orders of the small investor are executed can be described by two independent Poisson processes with constant rates.

Merton [Mer69, Mer71] solved a portfolio problem for a continuous time frictionless market consisting of one risky asset and one riskless asset. For the proportional transaction costs model, the problem was first studied by Magill and Constantinides [MC76]. They were able to gain important insights into the structure of the solution using stochastic control theory, but had to rely on heuristic arguments to some extend. Later on, Davis and Norman [DN90] were able to solve the problem in a rigorous way. Shreve and Soner [SS94] further extended the results of Davis and Norman by applying the theory of viscosity solutions to Hamilton-Jacobi-Bellman equations to the problem.

While the aforementioned articles all rely on stochastic control theory, recently Kallsen and Muhle-Karbe [KMK10] have successfully applied martingale methods to solve this portfolio problem under proportional transaction costs. This approach has been introduced in a seminal article of Jouini and Kallal [JK95]. They were able to show that the question whether a market with proportional transaction costs is arbitrage free or not can be answered by determining whether a related fictitious frictionless market is arbitrage free. The price process in this fictitious frictionless market is called a *shadow price* and evolves within the bid-ask spread of the original market with proportional transaction costs.

Building on the ideas developed in [KMK10] we show that *the optimal strategy in the limit* *order market described above consists in using market orders only to keep the proportion of wealth invested in the risky asset within certain boundaries, similar to the result for proportional transaction costs, while within these boundaries limit orders are used to profit from the bid-ask spread.* Although the given best-bid and best-ask price processes are geometric Brownian motions, the resulting shadow price process possesses jumps.

In *Chapter 4* we discuss the *extension of the elementary stochastic Itô-integral w.r.t. an optional semimartingale.* The paths of an optional semimartingale possess limits from the left and from the right, but may have double jumps. We find a mathematically tractable domain of general integrands. The simple integrands are embedded into this domain. Then, *we characterize the integral as the unique continuous and linear extension of the elementary integral and show closedness of the set of integrals.* Thus, our integral possesses desirable properties to model dynamic trading gains in mathematical finance when security price processes follow optional semimartingales.

A brief overview

Let us recapitulate the contents of this thesis.

Chapter 2 is basically a more detailed version of the preprint Kühn and Stroh [KS11]. We introduce a mathematical framework to model continuous time trading in a limit order market of a small investor. The framework can be seen as an extension to the well-known proportional transaction costs model. We show that the family of general strategies is closed and any element can be approximated by a sequence of elementary strategies.

Chapter 3 is based on the article Kühn and Stroh [KS10]. We study Merton's portfolio optimization problem in a specific instance of the framework introduced in Chapter 2. By means of a shadow price approach, we show that the optimal strategy consists in using market orders to keep the proportion of wealth invested in the risky asset within certain boundaries, while within these boundaries limit orders are used to profit from the bid-ask spread.

Chapter 4 stems from the note Kühn and Stroh [KS09b]. We discuss the extension of the elementary stochastic Itô-integral w.r.t. an optional semimartingale and find a mathematically tractable domain of general integrands. The integral is characterized as the unique continuous and linear extension of the elementary integral. Furthermore closedness of the set of integrals is shown.

The reader is advised that as a general rule any item introduced in a chapter is only valid in the particular chapter it is introduced in.

Chapter 2

Continuous time trading of a small investor in a limit order market

2.1 Introduction

In today's electronic markets the predominant market structure is the limit order market (or continuous double auction) where traders can continuously place market and limit orders. By the enormous increase of trading speed and a reduction of immediate order execution costs, there is a huge demand for sophisticated mathematical models of high-frequency trading that take the precise price formation mechanism into account and allow for the computation of optimal trading strategies. This chapter provides a mathematical background to model self-financing continuous time portfolio processes for a "small" trader in a limit oder market with a continuum of limit prices. Under the assumption that the order sizes of the investor are small compared to the orders in the book, trading solely with market orders corresponds to models with proportional transaction costs. These models and their arbitrage theory are very well developed and we can apply some of these results. However, the modeling of limit order execution is more complex. The trader can submit limit orders at different prices and orders may be stored in the order book waiting for execution.

The aim of this chapter is not to *explain* the evolution of the order book or the transaction price as e.g. in the models by Cont, Stoikov, and Talreja [CST10], Cvitanić and Kirilenko [CK10], Osterrieder [Ost07], Luckock [Luc03], and Roşu [Roş09]. We rather model the trading opportunities of *one* investor *given* the order book dynamics. In contrast to Alfonsi and Schied [AS10] and Predoiu, Shaikhet, and Shreve [PSS11] among others who consider the price impact of market orders and the order book resilience, we assume that the trader under consideration is small. Models with both market and limit orders have already been considered in Guilbaud and Pham [PG11] and Kühn and Stroh [KS10] among others. But, in these models only special limit order prices are permitted, especially the current best-bid price or one tick above it (for buy orders) and the best-ask price or one tick below it (for sell orders). As the best-bid and the best-ask may move continuously in time, [PG11] and [KS10] call for a verification in a more general framework that these strategies can be approximated by strategies with piecewise constant limit prices (see Example 2.38).

More importantly, in our model orders with arbitrary limit prices can be placed in the book. Thus, the model can e.g. be used to analyze the trade-off between the risks and the rewards connected with the placement of limit orders with different limit prices. In a time interval during which the fundamental value exhibits only minor fluctuations, it is quite profitable to place limit orders with limit prices close to the best-bid and the best-ask price to profit from the trades of liquidity driven investors. But should a sudden change of the fundamental value of the financial instrument occur, this would lead to quite unfavorable trades. By placing a limit buy order for example, the trader takes a similar risk as the issuer of a short-term put option on the fundamental value. To see this, assume that the small trader placed a limit buy order with a limit price slightly below the current best-bid price and a new information about the fundamental value appears suddenly. If the information is positive, the price goes up and the limit buy order is not executed. If the information is negative, the limit buy order is executed at its limit price and the new bid-ask-spread may be far below this price. Depending on the stochastic model, both limit prices near and far below the best-bid price may be a good choice.

Market makers are faced with the same risk. To avoid this risk there are so-called *immediate-or-cancel orders*. They must be executed completely or partially once they come onto the market. Those portions which cannot be executed are deleted immediately. Similarly, a *good-for-day order* is automatically canceled at the end of the day if it is not yet executed. In a recent paper Cvitanić and Kirilenko [CK10] show that a high frequency “machine trader” makes positive expected profits by using immediate-or-cancel orders for “sniping” out human orders.

In addition, we allow for the placement of limit orders in the inner of the spread which are executed with a higher probability.

The chapter is organized as follows. In Subsection 2.1.1 we provide a motivation of the order execution mechanism behind our model. For the convenience of the reader we briefly introduce random measures in Section 2.2. They are needed to introduce the model formally, which is done in Section 2.3. The main results, Theorem 2.17 and Theorem 2.22, are presented in Section 2.4. Their proofs can be found in Section 2.5 and Section 2.6. In Section 2.7 some examples are given. In particular, we show how a sequence of limit orders can turn into a market order when passing to the limit. The chapter ends with a conclusion in Section 2.8 and an appendix in Section 2.9.

Note that this chapter corresponds to the preprint [KS11].

2.1.1 A motivation of the execution mechanism

The basic assumption is that the investor is small. Only trades of other market participants, called exogenous orders in what follows, change the state of the order book, whereas the impact of the orders of the small investor is neglected. Being small also implies that there are no partial executions of his limit orders. A single limit order of the small investor with limit price L is either completely executed or not.

One building block of the model are the exogenous best-bid and best-ask price processes (not including the orders placed by the small trader). They are modeled by the càdlàg stochastic processes $\underline{S}$ and $\overline{S}$ with $\underline{S} < \overline{S}$. Market buy orders are immediately executed at $\overline{S}$ and market sell orders at $\underline{S}$. Let t be the point in time at which the exogenous order arrives. Since $\overline{S}$ and $\underline{S}$ are càdlàg, $\underline{S}_t$ and $\overline{S}_t$ are interpreted as the prices immediately after the order execution (or cancelation) at time t and $\underline{S}_{t-}$ and $\overline{S}_{t-}$ are the prices immediately before this event. Let us discuss some typical "events" driven by the actions of the other (exogenous) market participants to get an idea of what our model should cover.

(i) Market buy order arrives: The best-bid price is certainly unchanged, but the best-ask price may or may not jump upwards, depending on whether the market buy order eats into the book or not, i.e. $\underline{S}_t = \underline{S}_{t-}$ and $\overline{S}_t \geq \overline{S}_{t-}$. In addition, all limit sell orders of the small investor with limit price smaller (or equal) some x with $x \in [\underline{S}_t, \overline{S}_t]$ are executed.

(ii) Market sell order arrives: The best-ask price is certainly unchanged, but the best-bid price may or may not jump downwards, depending on whether the market sell order eats into the book or not, i.e. $\overline{S}_t = \overline{S}_{t-}$ and $\underline{S}_t \leq \underline{S}_{t-}$. In addition, all limit buy orders of the small investor with limit price higher (or equal) some x with $x \in [\underline{S}_t, \overline{S}_t]$ are executed.

(iii) Limit buy order with limit price L arrives:

(a) $L \leq \underline{S}_{t-}$. Nothing changes, i.e. $\underline{S}_t = \underline{S}_{t-}$ and $\overline{S}_t = \overline{S}_{t-}$.

(b) $\underline{S}_{t-} < L < \overline{S}_{t-}$. The best-bid price increases to L, while the best-ask price does not change, i.e. $\underline{S}_t = L$ and $\overline{S}_t = \overline{S}_{t-}$. In addition, all limit sell orders of the small trader with limit price smaller or equal L are executed. Note that $\overline{S}$ is the best-ask price without the small trader's orders.

(c) $L \geq \overline{S}_{t-}$. The same impact as in (i).

(iv) Limit sell order with limit price L arrives:

(a) $L \geq \overline{S}_{t-}$. Nothing changes, i.e. $\underline{S}_t = \underline{S}_{t-}$ and $\overline{S}_t = \overline{S}_{t-}$.

(b) $\underline{S}_{t-} < L < \overline{S}_{t-}$. The best-ask price decreases to L, while the best-bid price does not change, i.e. $\overline{S}_t = L$ and $\underline{S}_t = \underline{S}_{t-}$. In addition, all limit buy orders of the small trader with limit price higher or equal L are executed.

(c) $L \leq \underline{S}_{t-}$. The same impact as in (ii).

(v) Limit buy order is canceled: The best-ask price does not change, but depending on whether the canceled limit order is the only one at the best-bid price, the best-bid price may move downwards, i.e. $\overline{S}_t = \overline{S}_{t-}$ and $\underline{S}_t \leq \underline{S}_{t-}$.

(vi) Limit sell order is canceled: The best-bid price does not change, but depending on whether the canceled limit order is the only one at the best-ask price, the best-ask price may move upwards, i.e. $\underline{S}_t = \underline{S}_{t-}$ and $\overline{S}_t \geq \overline{S}_{t-}$.

It is important to note that the execution mechanism is not determined solely from the best-bid price and the best-ask price processes. Namely, a downward jump of the best-bid price from $\underline{S}_{t-}$ to $\underline{S}_t$ may or may not execute a limit buy order of the small investor with limit price $\underline{S}_t < L < \underline{S}_{t-}$. This depends whether the downward jump is triggered by a large exogenous market sell order eating into the book (as in (ii)) or by a cancelation of a limit buy order in the book (as in (v)). Therefore, we introduce two integer-valued random measures that model the execution of the limit orders of the small investor explicitly. They have to be in line with the processes $\underline{S}$ and $\overline{S}$, but they cannot be derived from them. This is in contrast to the model of Smid [Šmi07], where limit buy (sell) orders are only executed when the best-ask (bid) process hits the limit price. In the model considered in Osterrieder [Ost07] the execution of limit orders is triggered by an exogenous transaction price process.

2.2 Notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space satisfying the usual conditions. Denote by $\mathcal{O}$ (resp. by $\mathcal{P}$) the optional σ-algebra (resp. the predictable σ-algebra) on $\Omega \times [0,T]$ and remember that $\mathcal{P} \subset \mathcal{O} \subset \mathcal{F} \otimes \mathcal{B}([0,T])$.

Most of the following definitions are from Chapter XI in [HWY92]. As they are the building blocks for our model, we quote them here rather completely for the convenience of the reader. In the following let $\mathbb{R}_+ = [0,\infty)$, $\overline{\mathbb{R}}_+ = [0,\infty]$, and $\overline{\mathbb{R}} = [-\infty,\infty]$.

Definition 2.1. *Define*

$$\begin{aligned} (\widetilde{\Omega}, \widetilde{\mathcal{F}}) &:= (\Omega \times [0,T] \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+)), \\ \widetilde{\mathcal{O}} &:= \mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+), \qquad \widetilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+). \end{aligned}$$

We call $\widetilde{\mathcal{O}}$ the optional *σ-field in $\widetilde{\Omega}$ and $\widetilde{\mathcal{P}}$ the* predictable *σ-field in $\widetilde{\Omega}$.*

Note that $\widetilde{\mathcal{P}} \subset \widetilde{\mathcal{O}} \subset \widetilde{\mathcal{F}}$ follows from $\mathcal{P} \subset \mathcal{O} \subset \mathcal{F} \otimes \mathcal{B}([0,T])$.

Definition 2.2. *An extended real function $\mu : \Omega \times \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+) \to \overline{\mathbb{R}}_+$ is called a* random measure *if*

(i) $\mu(\omega,\cdot)$ is a σ-finite measure on $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+)$ for all $\omega \in \Omega$ and

(ii) $\mu(\cdot,\hat{B})$ is a random variable on $(\Omega,\mathcal{F})$ for all $\hat{B} \in \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+)$.

Definition 2.3. *For any $\widetilde{B} \in \widetilde{\mathcal{F}}$ define*

$$\begin{aligned} M_\mu(\widetilde{B}) &:= E\left[\int_{[0,T]\times\mathbb{R}_+} 1_{\widetilde{B}}(t,x)\mu(dt,dx)\right] \\ &= \int_\Omega \int_{[0,T]\times\mathbb{R}_+} 1_{\widetilde{B}}(\omega,t,x)\mu(\omega,dt,dx)P(d\omega). \end{aligned}$$

Note that M_μ is a measure on $\widetilde{\mathcal{F}}$. It is called the *measure generated by μ*. μ is said to be *integrable* if M_μ is a finite measure, i.e. $M_\mu(\widetilde{\Omega}) < \infty$. μ is said to be *optionally* (resp. *predictably*) *σ-integrable*, if the restriction of M_μ on $\widetilde{\mathcal{O}}$ (resp. $\widetilde{\mathcal{P}}$) is a σ-finite measure.

Definition 2.4. *For every $\widetilde{\mathcal{F}}/\mathcal{B}(\overline{\mathbb{R}})$-measurable function H denote by M the set of ω such that*

$$\int_{[0,T]\times\mathbb{R}_+} |H(\omega,s,x)|\mu(\omega,ds,dx) < \infty.$$

We define the integral

$$\int_{[0,t]\times\mathbb{R}_+} H(\omega,s,x)\mu(\omega,ds,dx) := \begin{cases} \int_{[0,t]\times\mathbb{R}_+} H(\omega,s,x)\mu(\omega,ds,dx) & \omega\in M, \\ \infty & \omega\in M^c, \end{cases} \qquad \forall(\omega,t)\in\Omega\times[0,T],$$

and call such an H μ-integrable *if $P(M)=1$.*

A random measure μ is said to be *optional* (resp. *predictable*), if for any $\widetilde{\mathcal{O}}$-measurable, μ-integrable function H (resp. $\widetilde{\mathcal{P}}$-measurable, μ-integrable function H), $\int_{[0,\cdot]\times\mathbb{R}_+} H d\mu$ is an optional (resp. predictable) process.

Definition 2.5. *A random measure μ is called an* integer-valued random measure *if*

(i) μ takes only values in $\mathbb{N}_0\cup\{\infty\}$,

(ii) $\mu(\omega,\{t\}\times\mathbb{R}_+)\le 1$ for all $\omega\in\Omega$, $t\ge 0$,

(iii) μ is optional and optionally σ-integrable.

Definition and Proposition 2.6. *Let $(X,\mathcal{G})$ be a measurable space and let $f: X\to\overline{\mathbb{R}}$ be a $\mathcal{G}/\mathcal{B}(\overline{\mathbb{R}})$-measurable function. Then the following subsets of $X\times\overline{\mathbb{R}}$ are $\mathcal{G}\otimes\mathcal{B}(\overline{\mathbb{R}})$-measurable:*

$$\begin{aligned} \textit{supergraph(f)} &:= \{(x,y)\in X\times\overline{\mathbb{R}} : f(x)<y\}, \\ \textit{subgraph(f)} &:= \{(x,y)\in X\times\overline{\mathbb{R}} : f(x)>y\}, \\ \textit{graph(f)} &:= \{(x,y)\in X\times\overline{\mathbb{R}} : f(x)=y\}. \end{aligned}$$

Proof. This is well known and easy to prove. For example

$$\{(x,y)\in X\times\overline{\mathbb{R}} : f(x)<y\} = \bigcup_{q\in\mathbb{Q}\cup\{-\infty\}\cup\{\infty\}} \{x\in X : f(x)\le q\}\times(q,\infty].$$

The result holds essentially because $\mathbb{Q}$ is dense in $\mathbb{R}$, compare Theorem 4.45 in [AB06]. $\square$

2.3 Model of a small investor trading in a limit order market

Let $\underline{S}$ and $\overline{S}$ be two adapted càdlàg processes with values in $\mathbb{R}_+$ s.t. $0<\inf_{s\in[0,T]}\underline{S}_s(\omega)\le\underline{S}_t(\omega)<\overline{S}_t(\omega)$ for all $(\omega,t)\in\Omega\times[0,T]$. One may interpret $\underline{S}$ as the best-bid price and $\overline{S}$ as the best-ask price without the orders of the small investor. The condition that $0<\inf_{s\in[0,T]}\underline{S}_s(\omega)$ has to hold tells us that there is always at least one exogenous limit buy order in the order book (by the càdlàg assumption we also have $\sup_{s\in[0,T]}\overline{S}_s(\omega)<\infty$, i.e. there is also always one exogenous limit sell order in the order book).

Let μ,ν be two integer-valued random measures. The random measure μ models when and up to which price the limit buy orders of the small trader are executed. The random measure ν models when and up to which price the limit sell orders of the small trader are executed. *Let the following assumption hold for the rest of this chapter.*

Assumption 2.7. *(i) For all $(\omega,t,x)\in\widetilde{\Omega}$ it holds that*

$$\begin{aligned}\mu(\omega,\{t\}\times\{x\})=1 \quad &\Rightarrow \quad \underline{S}_t(\omega)\le x\le \overline{S}_t(\omega),\\ \nu(\omega,\{t\}\times\{x\})=1 \quad &\Rightarrow \quad \underline{S}_t(\omega)\le x\le \overline{S}_t(\omega).\end{aligned}$$

(ii) For all $(\omega,t)\in\Omega\times[0,T]$ it holds that

$$\begin{aligned}\Delta\overline{S}_t(\omega)<0 \quad &\Rightarrow \quad \exists x\in[\underline{S}_t(\omega),\overline{S}_t(\omega)] \text{ with } \mu(\omega,\{t\}\times\{x\})=1,\\ \Delta\underline{S}_t(\omega)>0 \quad &\Rightarrow \quad \exists x\in[\underline{S}_t(\omega),\overline{S}_t(\omega)] \text{ with } \nu(\omega,\{t\}\times\{x\})=1.\end{aligned}$$

(iii) For all $\omega\in\Omega$ we have that

$$\begin{aligned}\mu(\omega,\{(t,x)\in[0,T]\times\mathbb{R}_+ : x<\overline{S}_t(\omega)\}) \quad &< \quad \infty,\\ \nu(\omega,\{(t,x)\in[0,T]\times\mathbb{R}_+ : x>\underline{S}_t(\omega)\}) \quad &< \quad \infty.\end{aligned}$$

(iv) For all $\omega\in\Omega$ we have that

$$\begin{aligned}\mu(\omega,\{t\}\times\{\overline{S}_t(\omega)\})=1 \quad &\Rightarrow \quad \Delta\overline{S}_t(\omega)<0,\\ \nu(\omega,\{t\}\times\{\underline{S}_t(\omega)\})=1 \quad &\Rightarrow \quad \Delta\underline{S}_t(\omega)>0.\end{aligned}$$

(v) There does not exist a pair $(\omega,t)\in\Omega\times[0,T]$ with

$$\mu(\omega,\{t\}\times[0,\overline{S}_t(\omega)))=1 \qquad \text{and} \qquad \nu(\omega,\{t\}\times(\underline{S}_t(\omega),\infty])=1.$$

(vi) For all $\omega\in\Omega$ we have that

$$\mu(\omega,\{0\}\times\mathbb{R}_+)=\nu(\omega,\{0\}\times\mathbb{R}_+)=0.$$

Remark 2.8. *For any càdlàg processes $\underline{S}$ and $\overline{S}$ with $\underline{S}<\overline{S}$ there exist random measures μ and ν satisfying Assumption 2.7.*

Let us discuss Assumption 2.7. (i) and (ii) are justified by the considerations in Subsection 2.1.1. As $\underline{S}_t$ stands for the highest remaining exogenous limit buy order in the book, clearly no limit buy order of the small investor with a limit price strictly below $\underline{S}_t$ can be executed at time t. Similarly, it would not make sense that a limit buy order of the small investor with a limit price strictly higher than $\overline{S}_t$ persists, because $\overline{S}_t$ denotes the lowest limit price of outstanding exogenous limit sell orders. The first part of Assumption 2.7 (ii) means that a downward jump of the best-ask entails that at least all limit buy orders of the small investor with limit prices larger or equal the best-ask after the jump are executed.

Assumption 2.7 (iii) says that there are only finitely many executions of limit orders of the small investor up to time T leading to a better trade than using market orders. This assumption is made as in reasonable models with continual execution of limit orders (at favorable prices) the small investor could make risk-less gains by placing simultaneously a limit buy order near to $\underline{S}$ and a limit sell order near to $\overline{S}$. Note however that there can be countably many executions of limit buy orders by (small) downward jumps of $\overline{S}$ (this is the reason why we do not restrict to finite random measures). These executions do not lead to arbitrage as the buyer has to pay at least the new best-ask price which is the price he has to pay when using a market order (cf. also (iv)). Condition (v) is needed to exclude simultaneous limit buy and sell order executions at similar prices which could cancel each other out and thus they would possibly not enter in the portfolio process. Assumption 2.7 (vi) is made w.l.o.g. and only to keep the notation simpler. In this regard it is similar to Assumption 2.2. in [CS06] and could be discarded by starting the model at time -1 and demanding that on $[-1,0)$ the filtration as well as the best-bid and the best-ask price are constant (compare Remark 4.2 in [CS06]).

Now we define the set of general continuous time strategies and the self-financing condition for the small trader. Later on, we embed real-world trading strategies into this strategy set which can be implemented by finitely many operations.

Definition 2.9. *Denote by $\mathcal{L}^B$ the family of $\widetilde{\mathcal{P}}/\mathcal{B}(\overline{\mathbb{R}}_+)$-measurable functions $L^B : \widetilde{\Omega} \to \overline{\mathbb{R}}_+$, which satisfy*

(i) $x \mapsto L^B(\omega,t,x)$ is monotonically decreasing, for all $(\omega,t) \in \Omega \times [0,T]$,

(ii) $L^B(\omega,t,x) = 0$ for all $(\omega,t) \in \Omega \times [0,T]$ and $x \geq \overline{S}_{t-}(\omega)$,

(iii) L^B is μ-integrable.

Similarly, let $\mathcal{L}^S$ be the family of $\widetilde{\mathcal{P}}/\mathcal{B}(\overline{\mathbb{R}}_+)$-measurable functions $L^S : \widetilde{\Omega} \to \overline{\mathbb{R}}_+$, which satisfy *(iv)* $x \mapsto L^S(\omega,t,x)$ *is monotonically increasing, for all* $(\omega,t) \in \Omega \times [0,T]$,

(v) $L^S(\omega,t,x) = 0$ *for all* $(\omega,t) \in \Omega \times [0,T]$ *and* $x \leq \underline{S}_{t-}(\omega)$,

(vi) L^S *is* ν*-integrable.*

$L^B(\omega,t,x)$ is the sum of outstanding limit buy orders of the small investor with limit price x or higher, which could possibly be executed at time t. The orders are placed (resp. updated) with the information $\mathcal{F}_{t-}$, i.e. in general without the knowledge of the order flow at time t. This reflects the fact that a limit order has to be in the book in advance before it can be executed by a market order. Condition (i) is self-explanatory. A limit buy order of the small trader placed at $\overline{S}_-$ or above would be executed immediately at $\overline{S}$, hence such an order would in effect be a market order. Thus, condition (ii) separates limit from market orders and is no restriction, see Subsection 2.4.1 for the relation to real-world trading strategies.

Definition 2.10. *Let* M^B, M^S *be real-valued predictable increasing processes with* $M_0^B = M_0^S = 0$ *and let* $L^B \in \mathcal{L}^B$ *and let* $L^S \in \mathcal{L}^S$*. We call a quadruple* $\mathfrak{S} = (M^B, M^S, L^B, L^S)$ *a* trading strategy*.*

At several places in this chapter we have to define integrals w.r.t. processes of finite variation which are neither left- nor right-continuous. Let X be a process of finite variation. It follows that X is làglàd, i.e. it possesses left and right limits, but it can have double jumps. Let $\Delta X_t := X_t - X_{t-}$ denote the jump at time t and let $\Delta^+ X_t := X_{t+} - X_t$ denote the jump immediately after time t. For a càdlàg integrand Y we define the integral $(Y_-, Y) \bullet X$ by

$$(Y_-, Y) \bullet X_t := (Y_- \bullet X^r)_t + \sum_{0 \leq s < t} Y_s \Delta^+ X_s, \quad t \geq 0, \tag{2.1}$$

where $X_t^r := X_t - \sum_{0 \leq s < t} \Delta^+ X_s$. The first term on the right-hand side of (2.1) is just a standard Lebesgue-Stieltjes integral. As the notation indicates, the left jumps of X are weighted with Y_- and the right jumps with Y. If Y is continuous we use the shorter notation $Y \bullet X$ for the integral defined in (2.1). Note that the notations are consistent with the common integral w.r.t. càdlàg integrators.

Definition 2.11. *For a given initial endowment* $(\eta^0, \eta^1) \in \mathbb{R}^2$ *we define the (self-financing)* portfolio process $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S}))$ *associated with the trading strategy* $\mathfrak{S}$ *by*

$$\begin{aligned}
\varphi_t^0(\mathfrak{S}) &:= \eta^0 - \int_0^t (\overline{S}_{s-}, \overline{S}_s) dM_s^B + \int_0^t (\underline{S}_{s-}, \underline{S}_s) dM_s^S \\
&+ \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty y L^B(s,dy) \mu(ds,dx) + \int_{[0,t)\times\mathbb{R}_+} \int_0^x y L^S(s,dy) \nu(ds,dx), \\
\varphi_t^1(\mathfrak{S}) &:= \eta^1 + M_t^B - M_t^S + \int_{[0,t)\times\mathbb{R}_+} L^B(s,x)\mu(ds,dx) - \int_{[0,t)\times\mathbb{R}_+} L^S(s,x)\nu(ds,dx).
\end{aligned}$$

For $L \in \{L^B, L^S\}$, the integral $\int yL(s,dy)$ is defined by

$$\int_a^b yL(s,dy) := \int_{(a,b)} yL^c(s,dy) + \sum_{a<y\leq b} y\Delta^- L(s,y) + \sum_{a\leq y<b} y\Delta^+ L(s,y),$$

where L^c denotes the continuous part and $\Delta^- L(s,y)$ resp. $\Delta^+ L(s,y)$ the jumps of the function $y \mapsto L(s,y)$.

Definition 2.12. *For any constant $a > 0$ a trading strategy $\mathfrak{S}$ is called* admissible *with threshold a if its associated portfolio process $\left(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S})\right)$ satisfies*

$$\varphi^0(\mathfrak{S}) + a + \underline{S}\left(\varphi^1(\mathfrak{S}) + a\right) 1_{\{\varphi^1(\mathfrak{S})+a\geq 0\}} + \overline{S}\left(\varphi^1(\mathfrak{S}) + a\right) 1_{\{\varphi^1(\mathfrak{S})+a<0\}} \geq 0. \tag{2.2}$$

This can be interpreted that given strategy $\mathfrak{S}$, if at all times we have a additional units of cash and a additional shares in our portfolio, then we are always able to close our position in the stock using market orders without going into debt. Note that if a strategy is admissible with threshold a as defined above, then its portfolio process is also admissible with threshold a in the sense of Campi and Schachermayer [CS06]. We will make use of this later on, when we prove the closedness result. Note however, that a portfolio process in our model does not have to be self-financing in the sense of [CS06], because the changes in the portfolio process generated by limit order executions clearly do not have to be self-financing in a proportional transaction costs model.

Before we present the main results, let us mention the following representations for the integer-valued random measures μ and ν, which we will use repeatedly.

Remark 2.13. *The integer-valued random measures μ and ν can be written as*

$$\mu(dt,dx) = \sum_{i=1}^{\infty} \delta_{(\tau_i, Y_i)}(dt,dx),$$

where δ_x denotes the Dirac measure on x, $(\tau_i)_{i\in\mathbb{N}}$ is a sequence of stopping times with disjoint graphs, and Y_i are $\mathcal{F}_{\tau_i}$-measurable random variables, and

$$\nu(dt,dx) = \sum_{i=1}^{\infty} \delta_{(\sigma_i, Z_i)}(dt,dx),$$

where $(\sigma_i)_{i\in\mathbb{N}}$ is a sequence of stopping times with disjoint graphs and Z_i are $\mathcal{F}_{\sigma_i}$-measurable random variables (this is a consequence of Theorem 11.13 in [HWY92]).

2.4 Main results

In this section we are going to present the main results of this chapter, Theorem 2.17 and Theorem 2.22, which in a way justify the model described in the previous section. We start by looking at what happens when the small investor places a single limit buy order.

2.4.1 Approximation of general strategies

Elementary or real-world trading strategies are trading strategies that can be implemented by finitely many operations. Executed real-world limit orders are not automatically renewed and in addition, the best-ask (bid) price can pass *continuously* through the limit price of a buy (sell) order placed by the small trader. This entails an execution as no buy (sell) order with limit price higher (smaller) than the best-ask (bid) can persist in the book. This “continuous execution” cannot be triggered by the σ-finite random measures μ and ν and has to be modeled separately.

Suppose at a stopping time T_1^B we place a limit buy order $\widehat{L}^B := (\theta^B, p^B, T_1^B, T_2^B)$ of size $\theta^B \in L_+^0(\mathcal{F}_{T_1^B})$ and limit price $p^B \in L_+^0(\mathcal{F}_{T_1^B})$ with $p^B < \overline{S}_{T_1^B}$ and if the order is not executed up to stopping time $T_2^B \geq T_1^B$ we cancel it. Define the stopping times

$$\begin{aligned} T^{\overline{S}} &:= \inf\{t \in (T_1^B, T_2^B] : \overline{S}_t \leq p^B\}, \\ T^{\mu} &:= \inf\{\tau_i \in (\tau_j)_{j\in\mathbb{N}} : T_1^B < \tau_i \leq T_2^B, Y_i \leq p^B\}, \\ T^* &:= T^{\overline{S}} \wedge T^{\mu}. \end{aligned}$$

T^* models the time at which the limit buy order is executed. If at all, the trade takes place at price p^B. The portfolio process of the limit buy order $\widehat{L}^B$ is defined as

$$\begin{aligned} \varphi_t^0(\widehat{L}^B) &:= -\theta^B p^B 1_{]\!]T^*,T]\!]}(t), \\ \varphi_t^1(\widehat{L}^B) &:= \theta^B 1_{]\!]T^*,T]\!]}(t). \end{aligned} \tag{2.3}$$

In the following, we show that any real-world trading strategy can be replicated by a general strategy $\mathfrak{S} = (M^B, M^L, L^B, L^S)$ satisfying $L^B = 0$ on $[\overline{S}_-, \infty)$ and $L^S = 0$ on $(-\infty, \underline{S}_-]$. Thus, on the level of general strategies the limit buy (sell) order is taken out before the best-ask (bid) passes and a “continuous execution” does not appear.

Assumption 2.14. *For all $(\omega, t) \in \Omega \times [0,T]$ we have that*

$$\mu(\omega, \{t\} \times \{\overline{S}_{t-}(\omega)\}) = 1 \quad \Longrightarrow \quad \Delta\overline{S}_t(\omega) \leq 0,$$

$$\nu(\omega, \{t\} \times \{\underline{S}_{t-}(\omega)\}) = 1 \quad \Longrightarrow \quad \Delta\underline{S}_t(\omega) \geq 0.$$

Proposition 2.15. *The quadruple* $\mathfrak{S} = (M^B, 0, L^B, 0)$ *with*

$$\begin{aligned} M^B_t(\omega) &:= \theta^B(\omega) 1_{]\!]T^*,T]\!]}(\omega,t) 1_{\{T^{\overline{S}} < T^\mu\}}(\omega), \qquad (2.4) \\ L^B(\omega,t,x) &:= \theta^B(\omega) 1_{]\!]T^B_1, T^* \wedge T^B_2]\!]}(\omega,t) \left(1_{\{x \le p^B(\omega), \overline{S}_{t-}(\omega) > p^B(\omega)\}} + 1_{\{x < p^B(\omega), \overline{S}_{t-}(\omega) = p^B(\omega)\}} \right) \end{aligned}$$

is a trading strategy in the sense of Definition 2.10. Under Assumption 2.14 it leads to the portfolio process given in (2.3).

Proof. The result follows from plugging the elements of trading strategy given in (2.4) into the equations of the portfolio process, given in Definition 2.11, and comparing the result with (2.3). As this is elementary, but somewhat tedious due to the various indicator functions in the definition, we leave it to the reader to go through each possible case. □

The intuition behind the embedding (2.4) is to separate the execution of the limit buy order triggered by $\overline{S}$ hitting the limit price p^B at no jump time (i.e. $T^{\overline{S}} < T^\mu$) from all other possible executions of the limit buy order (including the case that $\overline{S}$ jumps into $[0, p^B]$) and treat this "continuous execution" by market buy orders at the same price instead of limit orders, whereas all the other executions of limit buy orders are modeled by L^B and μ. A limit buy order which is triggered by $\overline{S}$ hitting p^B at no jump time is superfluous. The asset can be purchased instead by a market order placed at the hitting time paying also the best-ask price. Note that Assumption 2.7 (ii) also plays an important role in this argument, as it implies that all "noncontinuous executions" of the limit buy order due to the best-ask price $\overline{S}$ are covered by μ. By these considerations, we gain much in tractability as we do not have to deal with the "continuous executions" of the limit orders. The analysis of real-world limit sell orders is completely analog and thus omitted.

Due to the observation made in Proposition 2.15, the real-world limit buy order $\widehat{L}^B$ can be identified with the trading strategy $\mathfrak{S}$ from (2.4). This leads to the following definition.

Definition 2.16 (Real-world strategies)**.** *A trading strategy* $\mathfrak{S}$ *is called a* real-world buying strategy*, if it can be written as a finite conical combination of trading strategies of the form* $(M^{B,1}, 0, L^B, 0)$*, where the pair* $(M^{B,1}, L^B)$ *is defined in (2.4), and trading strategies of the form* $(M^{B,2}, 0, 0, 0)$*, where* $M^{B,2} = \theta^1 1_{]\!]T_1,T_2]\!]} + \theta^2 1_{[\![T_3]\!]}$*, where* T_1, T_2 *are* $[0,T]$*-valued stopping times,* T_3 *is a* $[0,T]$*-valued predictable stopping time, and* $\theta^1 \in L^0_+(\mathcal{F}_{T_1})$*,* $\theta^2 \in L^0_+(\mathcal{F}_{T_3})$*. A* real-world selling strategy *is defined correspondingly. A trading strategy* $\mathfrak{S}$ *is called a* real-world trading strategy *if it can be written as the sum of a real-world buying strategy and a real-world selling strategy.*

For $\mathcal{F}\otimes\mathcal{B}([0,T])$-measurable real-valued processes X and Y let

$$d_{\mathrm{up}}(X,Y) := \quad E\left(1\wedge \sup_{t\in[0,T]}|X_t - Y_t|\right).$$

d_{up} metricizes the convergence "uniformly in probability", cf. e.g. Section II.4 in [Pro04].

Theorem 2.17 (Approximation by real-world trading strategies)**.** *For any $\varepsilon > 0$ and any trading strategy $\mathfrak{S}$ there exists a real-world trading strategy $\mathfrak{S}^{\varepsilon}$ s.t.*

$$d_{up}\left(\varphi^0(\mathfrak{S}^{\varepsilon}),\varphi^0(\mathfrak{S})\right) < \varepsilon \qquad \textit{and} \qquad d_{up}\left(\varphi^1(\mathfrak{S}^{\varepsilon}),\varphi^1(\mathfrak{S})\right) < \varepsilon.$$

In other words, the portfolio processes that can be generated by real-world trading strategies are dense w.r.t. the convergence „uniformly in probability" in the set of all portfolio processes.

Theorem 2.17 has two different aspects. It shows that we can approximate the portfolio process resulting from strategies with possibly infinitely many limit prices and continuously varying order sizes by placing only finitely many orders. This is of the same flavor as the fact that (under certain assumptions) the stochastic integral of a predictable process can be approximated by the stochastic integrals of simple predictable processes. But furthermore Theorem 2.17 also vindicates the slightly counterintuitive limit order execution mechanism of our model, which somehow "ignores continuous execution" of limit orders, because it implies that if Assumption 2.14 holds then any portfolio process of a trading strategy in our model can be approximated arbitrarily close by the completely intuitive portfolio process given in (2.3), which does include "continuous execution".

2.4.2 Closedness of the strategy set

The possibility of approximating the portfolio processes in our model with real-world trading strategies alone would not make the model particularly useful, if the set of trading strategies would not be closed in some sense. To proceed towards the closedness result, let us first recall the concept of a strictly consistent price process.

Definition 2.18. *An adapted $(0,\infty)$-valued process $\widetilde{S} = (\widetilde{S}_t)_{t\in[0,T]}$ is called a* strictly consistent price process *for the risky asset if there exists a probability measure $\widetilde{P} \sim P$ s.t. $\widetilde{S}$ is a càdlàg $\widetilde{P}$-martingale with*

$$\widetilde{S}_t \in (\underline{S}_t,\overline{S}_t),\ \forall t\in[0,T] \quad \textit{and} \quad \widetilde{S}_{t-} \in (\underline{S}_{t-},\overline{S}_{t-}),\ \forall t\in(0,T], \qquad P\textit{-a.s.}.$$

It is important to note that the existence of such an $\widetilde{S}$ as given in Definition 2.18 is equivalent to the existence of a strictly consistent price process in the sense of Definition 2.3 in Campi and Schachermayer [CS06], since we are going to use a result from this article in the proof of Theorem 2.22 (see the proof of Lemma 2.31).

We have made the assumption that μ is an integer-valued random measure, hence M_μ is a σ-finite measure on $\widetilde{\mathcal{O}}$. Consequently, there exists a pairwise disjoint sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{O}$ s.t. $0 < M_\mu(A_n) < \infty$ and $\bigcup_{n\in\mathbb{N}} A_n = \widetilde{\Omega}$, which can be used to construct a probability measure $\check{M}_\mu$ equivalent to M_μ. For any $A \in \mathcal{F}$ let

$$\check{M}_\mu(A) := \sum_{n\in\mathbb{N}} \frac{M_\mu(A\cap A_n)}{2^n M_\mu(A_n)}.$$

Let $\check{M}_\nu$ be defined similarly.

Definition 2.19. *Define the measures $\widehat{M}_\mu$ and $\widehat{M}_\nu$ on $\mathcal{F}\otimes\mathcal{B}([0,T])$ by*

$$\begin{aligned}\widehat{M}_\mu(A) &:= \check{M}_\mu(A\times\mathbb{R}_+), \qquad A\in\mathcal{F}\otimes\mathcal{B}([0,T]),\\ \widehat{M}_\nu(A) &:= \check{M}_\nu(A\times\mathbb{R}_+), \qquad A\in\mathcal{F}\otimes\mathcal{B}([0,T]).\end{aligned}$$

Definition 2.20. *Define the following sets of stochastic processes*

$$\begin{aligned}\mathcal{P}_1 &:= \{X \text{ is a } [0,\infty]\text{-valued predictable process with } P(X_{\tau_i}\le Y_i)=1\ \forall i\in\mathbb{N}\},\\ \mathcal{P}_2 &:= \{X \text{ is a } [0,\infty]\text{-valued predictable process with } P(X_{\sigma_i}\ge Z_i)=1\ \forall i\in\mathbb{N}\},\end{aligned}$$

where $(\tau_i, Y_i)_{i\in\mathbb{N}}$ and $(\sigma_i, Z_i)_{i\in\mathbb{N}}$ are the representations of μ resp. ν from Remark 2.13. Let $\underline{X}$ be the essential supremum of the functions in $\mathcal{P}_1$ taken w.r.t. the predictable σ-algebra on $\Omega\times[0,T]$ and the measure $\widehat{M}_\mu$ from Definition 2.19. Accordingly, let $\overline{X}$ be the essential infimum of the functions in $\mathcal{P}_2$ taken w.r.t. the predictable σ-algebra and the measure $\widehat{M}_\nu$ defined as in Definition 2.19.

Assumption 2.21.

$$P(\underline{X}_{\tau_i} = Y_i) = 0 \quad \textit{and} \quad P(\overline{X}_{\sigma_i} = Z_i) = 0 \quad \forall i\in\mathbb{N}.$$

$\underline{X}$ resp. $\overline{X}$ can be interpreted as the highest (resp. smallest) predictable limit price below (above) which a limit buy (resp. sell) order is *not* executed for sure. Assumption 2.21 says that at these boundary limit prices execution is also not possible.

Theorem 2.22 (Closedness of the strategy set)**.** *Let Assumption 2.21 be satisfied and suppose that there exists a strictly consistent price process for the risky asset in the sense of Definition 2.18. In addition, assume that* $\underline{S}$ *and* $\overline{S}$ *are semimartingales. Let* $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ *be an admissible sequence of trading strategies with the same threshold level* a *and the same initial endowment* (η^0,η^1) *for all* n*. If the sequence of associated portfolio processes* $\Big((\varphi^0(\mathfrak{S}^n),\varphi^1(\mathfrak{S}^n))\Big)_{n\in\mathbb{N}}$ *is a Cauchy sequence w.r.t. the convergence "uniformly in probability", then there exists an admissible trading strategy* $\mathfrak{S}$ *with threshold level* a *and initial endowment* (η^0,η^1) *s.t.* $\Big((\varphi^0(\mathfrak{S}^n),\varphi^1(\mathfrak{S}^n))\Big)_{n\in\mathbb{N}}$ *converges uniformly in probability to the associated portfolio process* $(\varphi^0(\mathfrak{S}),\varphi^1(\mathfrak{S}))$ *of* $\mathfrak{S}$*.*

Before we begin to prove Theorem 2.17 and Theorem 2.22, let us first discuss why Assumption 2.21 is made.

Example 2.23. *Let* $\underline{S}\equiv 100$ *and* $\overline{S}\equiv 101$*, i.e. the best-bid price and the best-ask price are constant. Furthermore, let* X *be a random variable with values in* $\mathbb{R}_+$ *with distribution* $0.5\delta_{100}+0.5\lambda[100,101]$*, where* δ_{100} *denotes a Dirac measure and* λ *denotes the uniform distribution. Consider the usual augmentation of the filtration generated by the stochastic process* $X1_{[t_0,T]}$*, with* $t_0\in(0,T)$*. Define* $\mu:=\delta_{(t_0,X)}$ *and let* ν *be without any mass, i.e. at time* t_0 *limit buy orders with a limit price of* X *or higher are executed, whereas no limit sell orders are executed at all. The initial endowment is supposed to be one unit of cash and no shares, i.e.* $(\eta^{0,n},\eta^{1,n})=(1,0)$*. Now consider the sequence of strategies* $(L^{B,n},L^{S,n},M^{B,n},M^{S,n})_{n\in\mathbb{N}}$ *with* $L^{S,n}\equiv 0$ *and* $M^{B,n}\equiv 0$ *for all* n *and define (the deterministic)*

$$L^{B,n}(\omega,t,x):=\begin{cases} n & \text{if } x\le 100+e^{-n},\\ -\ln(x-100) & \text{if } 100+e^{-n}<x<101,\\ 0 & \text{if } x\ge 101,\end{cases}\qquad \forall(\omega,t)\in\Omega\times[0,T],n\in\mathbb{N}.$$

Basically, for $L^{B,n}$ *we restrict the function* $-\ln(x-100)$ *at level* n*. In Figure 2.1 the part of the graph of* $L^{B,n}$ *at which this restriction is effective is colored, whereas the graph is black where* $L^{B,n}$ *is equal to* $-\ln(x-100)$*. We abbreviate* $L^{B,n}(\omega,t,x)$ *to* $L^{B,n}(x)$*, in what follows. Let* $M_t^{S,n}:=n1_{\{X=100\}}1_{(t_0,T]}$*, which is clearly predictable. Thus, the* n*-th strategy consists in buying* $L^{B,n}(X)$ *shares via limit order and selling the same amount via market order iff* $X=100$*. Let us have a look at what happens to* $(\varphi_t^{0,n},\varphi_t^{1,n})$ *as* n *goes to infinity. The only time of interest is*

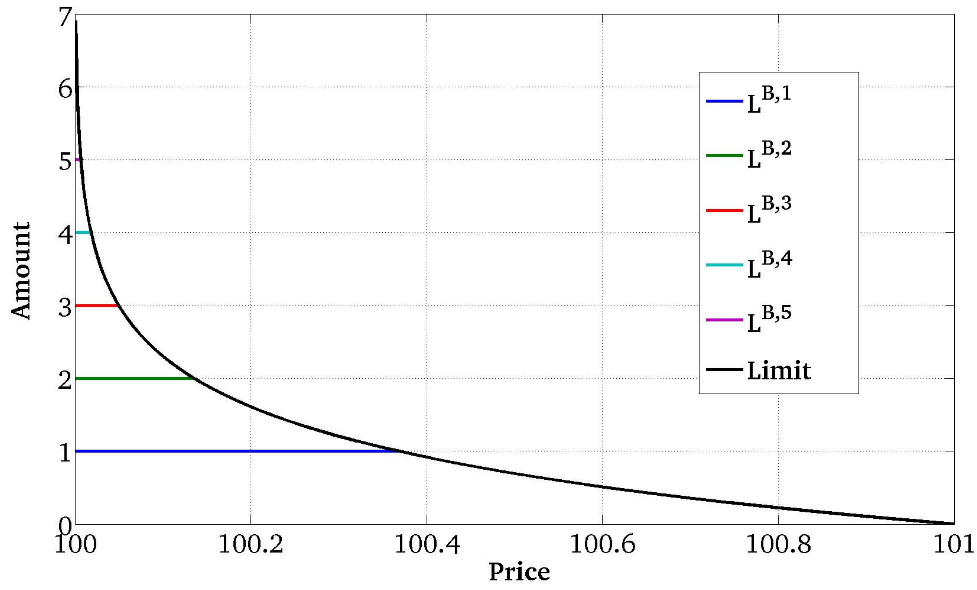

Figure 2.1: Illustration of the limit buy orders $L^{B,n}$ in Example 2.23.

of course the instant from t_0 to t_0+. We can write the change in the cash position as follows

$$\begin{aligned}
\Delta^+\varphi^{0,n}_{t_0} &= \int_X^{101} x L^{B,n}(dx) + 100n1_{\{X=100\}} \\
&= \int_{X\vee(100+e^{-n})}^{101} \frac{-x}{x-100}dx + 100n1_{\{X=100\}} \\
&= [-x-100\ln(x-100)]_{X\vee(100+e^{-n})}^{101} + 100n1_{\{X=100\}} \\
&= -101 + X\vee(100+e^{-n}) + 100\ln(X\vee(100+e^{-n})-100) + 100n1_{\{X=100\}}.
\end{aligned}$$

It is straightforward to check that each trading strategy $(\varphi^{0,n},\varphi^{1,n})$ is admissible with threshold 0*. Furthermore, uniformly in probability*

$$\begin{aligned}
&\lim_{n\to\infty}(\varphi^{0,n},\varphi^{1,n}) \\
&= (1,0)1_{[0,t_0]} + \Big(\Big(X-101+100\ln(X-100), -\ln(X-100)\Big)1_{\{X>100\}} + (0,0)1_{\{X=100\}}\Big)1_{(t_0,T]} \\
&=: (\psi^0,\psi^1).
\end{aligned}$$

(ψ^0,ψ^1) *even satisfies inequality (2.2), i.e. it would be admissible with threshold* 0*, if it were a portfolio process. The problem here is to find a limit trading strategy $\mathfrak{S}$ s.t. $(\varphi^0(\mathfrak{S}),\varphi^1(\mathfrak{S})) = (\psi^0,\psi^1)$. On the one hand to buy the correct amount of shares at the right prices on $\{X>100\}$, $L^B(t_0,\cdot)$ would have to be of the form $-\ln(x-100)1_{\{100<x\le 101\}}$ on the event $\{X>100\}$, which implies $L^B(t_0,100)=\infty$ by the monotonicity requirement (i) in Definition 2.9. On the other hand*

L^B has to be μ-integrable, i.e. on the event $\{X > 100\}^c$ it must not be the case that $L^B(t_0, 100) = \infty$. This is impossible to achieve with a predictable L^B. Indeed, for any $\widetilde{\mathcal{P}}$-measurable L^B the stochastic process $L^B(\cdot, \cdot, 100)$ is predictable and thus $L^B(t_0, 100)$ has to be an $\mathcal{F}_{t_0-}$-measurable random variable. But in our example we have by construction only $\mathcal{F}_{t_0-} = \sigma(\mathcal{N})$, where $\mathcal{N}$ denotes all P-null sets. So clearly the value of $L^B(t_0, 100)$ cannot depend on $\{X > 100\}$, as this is not a null set.

2.5 Proof of Theorem 2.17: Approximation of general strategies

Step 1: Let $\mathfrak{S} = (M^B, M^S, L^B, L^S)$. By linearity of the portfolio process $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S}))$ in $\mathfrak{S}$, it is sufficient to approximate $(M^B, 0, 0, 0), (0, M^S, 0, 0), (0, 0, L^B, 0)$, and $(0, 0, 0, L^S)$ separately. The assertion for $(M^B, 0, 0, 0)$ and $(0, M^S, 0, 0)$ holds by Theorem A.10 in Denis, Guasoni, and Rásonyi [DGR11]. Note that for denseness w.r.t. d_{up} it is not necessary that $\underline{S}$ and $\overline{S}$ are locally bounded as any càdlàg process is prelocally bounded. It remains to prove the assertion for $(0, 0, L^B, 0)$. The proof for $(0, 0, 0, L^S)$ is analog.

Step 2: $\mathfrak{S} = (0, 0, L, 0)$. Note that to keep the notation shorter, in the rest of this section we write L instead of L^B. The only exception pertains to Lemma 2.29, which is also used outside of this section.

Definition 2.24. *Denote by $(x_k)_{k \in \mathbb{N}}$ a sequence running through $\mathbb{Q}_+$. We define the finite measures $\widetilde{M}_\mu$ and $\widetilde{M}_\nu$ on $\widetilde{\mathcal{F}}$. For any $A \in \widetilde{\mathcal{F}}$ let*

$$\begin{aligned}
\widetilde{M}_\mu(A) &:= \frac{1}{2\check{M}_\mu(\widetilde{\Omega})} \left(\check{M}_\mu(A) + \widehat{M}_\mu \otimes \sum_{k=1}^{\infty} 2^{-k} \delta_{x_k} \left(A \cap \textit{supergraph}(\underline{X}) \right) \right), \\
\widetilde{M}_\nu(A) &:= \frac{1}{2\check{M}_\nu(\widetilde{\Omega})} \left(\check{M}_\nu(A) + \widehat{M}_\nu \otimes \sum_{k=1}^{\infty} 2^{-k} \delta_{x_k} \left(A \cap \textit{subgraph}(\overline{X}) \right) \right).
\end{aligned}$$

Note that by construction it holds that $\check{M}_\mu \ll \widetilde{M}_\mu$ and because of $M_\mu \sim \check{M}_\mu$ we also have $M_\mu \ll \widetilde{M}_\mu$. The previous definition will be used repeatedly throughout the rest of this chapter. A key property of $\widetilde{M}_\mu$ is that when a sequence of functions $(L^{B,n})_{n \in \mathbb{N}} \subset \mathcal{L}^B$ converges $\widetilde{M}_\mu$-a.e. to a function $L^B \in \mathcal{L}^B$ the integrals containing $L^{B,n}$ found in the cash component of the portfolio process converge as well. The exact formulation of this property is given in Lemma 2.29.

In the following for any $\delta > 0$ denote by $\overline{S}^\delta$ the canonical simple predictable process constructed on page 57 in [Pro04] that satisfies

$$P\Big(\sup_{t \in [0,T]} |\overline{S}^\delta_t - \overline{S}_{t-}| > \delta \Big) < \delta.$$

In addition, denote by τ^δ the first time that $\overline{S}^\delta$ departs farther than δ from $\overline{S}_-$, i.e.

$$\tau^\delta := \inf\{t > 0 : |\overline{S}^\delta_t - \overline{S}_{t-}| > \delta\},$$

and note that at time τ^δ the processes are still not more than δ apart as they are both left-continuous.

Definition 2.25 (δ-cut off)**.** *For any $\delta > 0$ and L in $\mathcal{L}^B$ let us denote by L^δ the function defined by*

$$L^\delta(\omega,t,x) := L(\omega,t,x)1_{\{x\le\overline{S}^\delta_t(\omega)-3\delta\}}1_{[\![0,\tau^\delta\wedge T]\!]}(\omega,t).$$

Lemma 2.26. *For any $\delta > 0$, we have that $L^\delta \in \mathcal{L}^B$. Furthermore, there exists a sequence $(\delta_i)_{i\in\mathbb{N}} \subset \mathbb{R}_+ \setminus \{0\}$ with $\delta_i \to 0$ for $i \to \infty$ s.t. $(L^{\delta_i})_{i\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. to L.*

Proof. As $\overline{S}^\delta$ and $1_{[\![0,\tau^\delta]\!]}$ are predictable, L^δ is $\widetilde{\mathcal{P}}$-measurable. Integrability follows immediately from $L^\delta \le L$ and the other requirements for L^δ being in $\mathcal{L}^B$ are also obviously satisfied.

Put $\delta_i := 2^{-i}$. By the lemma of Borel-Cantelli the events $\{\sup_{t\in[0,T]} |\overline{S}^{2^{-i}}_t - \overline{S}_{t-}| > 2^{-i}\}$, $i = 1,2,\dots$ occur only finitely often on a set N^c with $P(N^c) = 1$. Thus, for any $\omega \in N^c$ there exists an $i_0(\omega)$ s.t. $|\overline{S}^{2^{-i}}_t(\omega) - \overline{S}_{t-}(\omega)| \le 2^{-i}$ for all $i \ge i_0(\omega)$, $t \in [0,T]$ and hence $\tau^{2^{-i}}(\omega) = \infty$. Consequently, for all $\omega \in N^c$, $t \in [0,T]$, and $x < \overline{S}_{t-}(\omega)$ we have that

$$1_{\{x\le\overline{S}^{2^{-i}}_t(\omega)-3\cdot 2^{-i}\}}1_{[\![0,\tau^{2^{-i}}\wedge T]\!]}(\omega,t) = 1 \quad \text{and thus} \quad L^{2^{-i}}(\omega,t,x) = L(\omega,t,x)$$

for $i \ge i_0(\omega) \vee \Big(1 - \log_2(1/4) - \log_2(\overline{S}_{t-}(\omega) - x)\Big)$. For $\omega \in N^c$, $t \in [0,T]$, and $x \ge \overline{S}_{t-}(\omega)$ we obtain

$$1_{\{x\le\overline{S}^{2^{-i}}_t(\omega)-3\cdot 2^{-i}\}} = 0 \qquad \text{for } i \ge i_0(\omega).$$

By assumption $L \in \mathcal{L}^B$ and thus $L(\omega,t,x) = 0$ if $x \ge \overline{S}_{t-}(\omega)$. Therefore, $L^{2^{-i}}$ converges to L pointwise on $N^c \times [0,T] \times \mathbb{R}_+$ and thus $\widetilde{M}_\mu$-a.e. □

We proceed by discretizing L^δ in the price variable. Fix any $m \in \mathbb{N}$ and divide $(0,m]$ into dyadic intervals $((l-1)2^{-m}, l2^{-m}]$ for $l = 1,\dots,m2^m$. Now we want to approximate $x \mapsto L^\delta(\omega,t,x)$ by a left-continuous step function $L^{\delta,m}$, which is constant between two points of the dyadic grid. For each interval we check if there exists a point x in this interval s.t. $L^\delta(\omega,t,x-) > L^\delta(\omega,t,x+)$. If this is the case, we fix the price $x^*_{l,m}(\omega,t)$ for which this "jump" is the largest and let our function take the value of $L^\delta(\omega,t,x^*_{l,m}(\omega,t))$ for the whole interval. When the largest jump is attained at different prices (which can only be finitely many), we take the smallest of these prices. If there is no "jump", we just set $x^*_{l,m}(\omega,t) = (l-1)2^{-m}$, i.e. for the interval we take the value of L^δ at the left boundary. It is advisable to have a look at Figure 2.2 to grasp the basic idea of the definitions below which are complicated by technical problems. In particular, the formal definition has to ensure that $L^{\delta,m}(\omega,t,x)$ is only infinite if $L^\delta(\omega,t,x)$ is infinite. For any $\delta > 0$, $m \in \mathbb{N}$, and $l \in \{1,\dots,m2^m\}$ we define

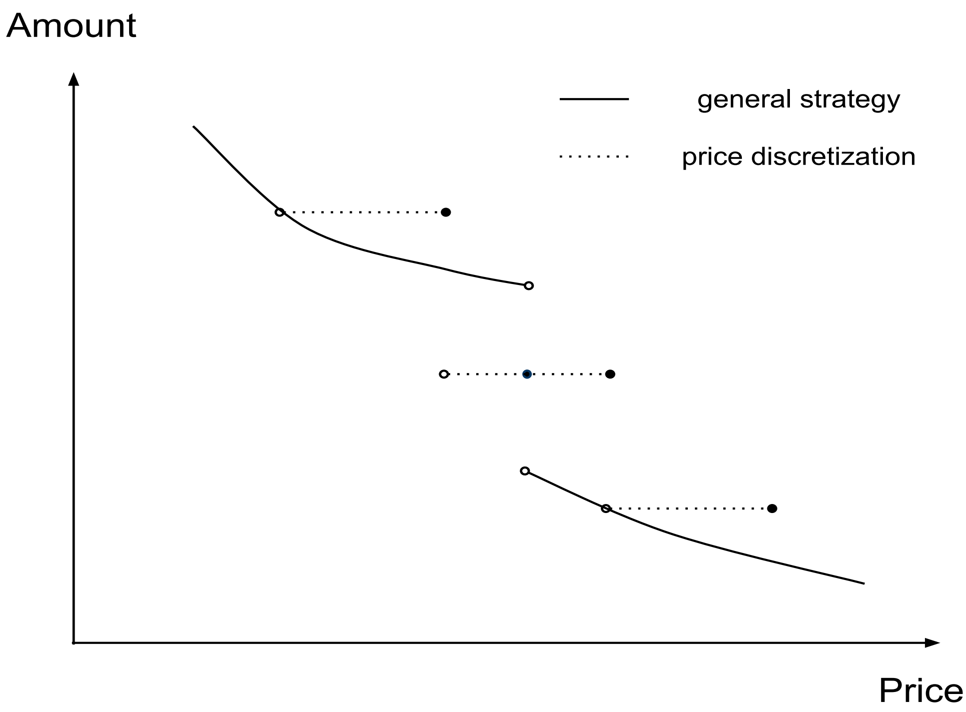

Figure 2.2: Illustration how L^δ is approximated by $L^{\delta,m}$.

$$x^*_{l,m}(\omega,t) := \begin{cases} \min\Big\{\operatorname{argmax}_{x\in((l-1)2^{-m},l2^{-m}]}\Big(L^\delta(\omega,t,x-) - L^\delta(\omega,t,x+)\Big)\Big\} \\ \qquad\qquad \text{if } L^\delta(\omega,t,(l-1)2^{-m}) < \infty \text{ and } \sup_x\Big(L^\delta(\omega,t,x-) - L^\delta(\omega,t,x+)\Big) > 0, \\ (l-1)2^{-m} \\ \qquad\qquad \text{if } L^\delta(\omega,t,(l-1)2^{-m}) < \infty \text{ and } \sup_x\Big(L^\delta(\omega,t,x-) - L^\delta(\omega,t,x+)\Big) = 0, \\ \inf\{x \in \mathbb{R}_+ : L^\delta(\omega,t,x) < \infty\} \\ \qquad\qquad \text{if } L^\delta(\omega,t,(l-1)2^{-m}) = \infty \text{ and } L^\delta(\omega,t,l2^{-m}) < \infty, \\ (l-1)2^{-m} \\ \qquad\qquad \text{if } L^\delta(\omega,t,(l-1)2^{-m}) = \infty \text{ and } L^\delta(\omega,t,l2^{-m}) = \infty. \end{cases}$$

Definition 2.27 (1/m-price discretization)**.** *Let $\delta > 0$ and $m \in \mathbb{N}$. For any $l \in \{1,\dots,m2^m\}$ we define*

$$L^{\delta,m}(\omega,t,x) := \sum_{l=1}^{m2^m} \theta_t^{\delta,l,m}(\omega) 1_{\{(l-1)2^{-m}<x\le l2^{-m}\}} + L^\delta(\omega,t,0)1_{\{x=0\}},$$

where

$$\theta_t^{\delta,l,m}(\omega) := \begin{cases} L^\delta(\omega,t,x^*_{l,m}(\omega,t)), & \text{if } L^\delta(\omega,t,x^*_{l,m}(\omega,t)) < \infty, \\ L^\delta(\omega,t,l2^{-m}), & \text{otherwise.} \end{cases}$$

Lemma 2.28. *For any $\delta > 0$ and $m \ge [-\log_2(\delta)]+1 =: m_0$, we have that $L^{\delta,m} \in \mathcal{L}^B$. Furthermore, $\sup_{m\in\{m_0,m_0+1,\dots\}} L^{\delta,m}$ is μ-integrable and $(L^{\delta,m})_{m\in\{m_0,m_0+1,\dots\}}$ converges to L^δ $\widetilde{M}_\mu$-a.e..*

Note that [x] denotes the largest natural number smaller or equal to x, i.e. $[x] := \max\{k \in \mathbb{N}_0 : k \le x\}$.

Proof. Step 1: By Lemma 2.26, L^δ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+)$-measurable. In addition, we observe that for all $l = 1,\dots,m2^m$ the process $(\omega,t) \mapsto x^*_{l,m}(\omega,t)$ is predictable. This is the case because the location of the largest jump $L^\delta(\omega,t,x-) - L^\delta(\omega,t,x+)$ for $x \in ((l-1)2^m, l2^m]$ can be expressed by suprema and pointwise limits of distances between elements of $\{(\omega,t) \mapsto L^\delta(\omega,t,q) : q \in \mathbb{Q}_+\}$ (the detailed proof which makes use of the monotonicity of $x \mapsto L^\delta(\omega,t,x)$ is straightforward but somewhat tedious and left to the reader). Consequently, $(\omega,t) \mapsto L^\delta(\omega,t,x^*_{l,m}(\omega,t))$ is a composition of the $\mathcal{P}/(\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+))$-measurable function $(\omega,t) \mapsto (\omega,t,x^*_{l,m}(\omega,t))$ and the $(\mathcal{P}\otimes\mathcal{B}(\mathbb{R}_+))/\mathcal{B}(\overline{\mathbb{R}}_+)$-measurable function $(\omega,t,x) \mapsto L^\delta(\omega,t,x)$ and thus $\mathcal{P}/\mathcal{B}(\overline{\mathbb{R}}_+)$-measurable, i.e. predictable.

The monotonicity of $L^{\delta,m}$ follows immediately from the monotonicity of $x \mapsto L^\delta(\omega,t,x)$. Moreover, by construction of $L^{\delta,m}$, the largest x for which $L^{\delta,m}(\omega,t,x) > 0$ holds, can only exceed the largest x for which $L^\delta(\omega,t,x) > 0$ holds by at most 2^{-m}. Thus, we have that

$$L^{\delta,m} = 0 \quad \text{on} \quad \{(\omega,t,x) \in \widetilde{\Omega} : x > \overline{S}_{t-}(\omega) - \delta\} \quad \forall m \ge m_0. \tag{2.5}$$

Consequently, part (ii) of Definition 2.9 is satisfied.

Step 2: Let us now show that $\sup_{m\in\{m_0,m_0+1,\dots\}} L^{\delta,m}$ is μ-integrable. Let $(\omega,t,x) \in \widetilde{\Omega}$ such that $L^\delta(\omega,t,x) < \infty$.

Case 1: $L^\delta(x,t,x-) < \infty$, i.e. there exists $\varepsilon > 0$ s.t. $L^\delta(x,t,x-\varepsilon) < \infty$. We have that $x^*_{l_m,m}(\omega,t) \ge x - \varepsilon$ for all m up to finitely many (where l_m satisfy $(l_m - 1)2^{-m} < x \le l_m 2^{-m}$). In addition, we have that $L^{\delta,m}(\omega,t,x) < \infty$ for any $m \in \mathbb{N}$.

Case 2: $L^\delta(x,t,x-) = \infty$. Then, $x^*_{l_m,m}(\omega,t) = x$ for all $m \in \mathbb{N}$.

Thus, in both cases we arrive at

$$\sup_{m\in\{m_0,m_0+1,\dots\}} L^{\delta,m}(\omega,t,x)<\infty. \tag{2.6}$$

Together this implies that (2.6) holds M_μ-a.e. as $\{L^\delta=\infty\}$ is a M_μ-null set. In addition, we have that $\mu(\omega,[0,T]\times(\overline{S}_{t-}(\omega)-\delta,\infty))<\infty$ (Assumption 2.7(iii) combined with the fact that $\overline{S}$ is càdlàg). Due to (2.5), this already implies that $\sup_{m\in\{m_0,m_0+1,\dots\}} L^{\delta,m}<\infty$ is μ-integrable.

Step 3: Let us now deal with the *convergence* part of the Lemma. Fix a (ω,t,x) with $L^\delta(\omega,t,x)<\infty$.

Case 1: $L^\delta(\omega,t,x-)=L^\delta(\omega,t,x+)<\infty$.

For any $\varepsilon>0$ there exists a constant $c_\varepsilon(\omega,t,x)>0$ s.t. for all $y\in(x-c_\varepsilon,x+c_\varepsilon)$ it holds that $|L^\delta(\omega,t,y)-L^\delta(\omega,t,x)|<\varepsilon$. Thus, for all m large enough s.t. $((l_m-1)2^{-m},l_m2^{-m}]\subset(x-c_\varepsilon,x+c_\varepsilon)$ we have that $|L^{\delta,m}(\omega,t,x)-L^\delta(\omega,t,x)|<\varepsilon$.

Case 2: $L^\delta(\omega,t,x-)>L^\delta(\omega,t,x+)$.

Clearly, this implies $x^*_{l_m,m}(\omega,t)=x$ for all m large enough, thus $L^\delta(\omega,t,x)=L^{\delta,m}(\omega,t,x)$ holds for all m large enough.

The case differentiation above yields the convergence for all (ω,t,x) s.t. $L^\delta(\omega,t,x)<\infty$. It remains to show that $\{(\omega,t,x)\in\Omega\times[0,T]\times\mathbb{R}_+ : L^\delta(\omega,t,x)=\infty\}$ is a $\widetilde{M}_\mu$-null set. It is clear that the set is a M_μ-null set as L^δ is μ-integrable. However, we still have to verify that for all $q\in\mathbb{Q}_+$

$$\left(\widehat{M}_\mu\otimes\delta_q\right)\left(\{L^\delta=\infty\}\cap\text{supergraph}(\underline{X})\right)=0,$$

i.e. $\widehat{M}_\mu(A_q)=0$, where $A_q:=\{(\omega,t)\in\Omega\times[0,T]:\underline{X}(\omega,t)<q \text{ and } L^\delta(\omega,t,q)=\infty\}$.

Assume that $\widehat{M}_\mu(A_q)>0$. Then the predictable process $\widetilde{X}_t(\omega):=q1_{A_q}(\omega,t)+\underline{X}_t(\omega)1_{A_q^c}(\omega,t)$ is *not* another version (besides $\underline{X}$) of the essential supremum introduced in Definition 2.20. Consequently, there exists an $i\in\mathbb{N}$ with

$$P\Big(\{\omega\in\Omega:Y_i(\omega)<q\text{ and }(\omega,\tau_i(\omega))\in A_q\}\Big)=P\left(Y_i<\widetilde{X}_{\tau_i}\right)>0,$$

which would imply by the monotonicity of $y\mapsto L^\delta(\omega,t,y)$ that $P(\{\omega\in\Omega:L^\delta(\omega,\tau_i(\omega),Y_i(\omega))=\infty\})>0$. But this is a contradiction to the μ-integrability of L^δ. □

Let us recapitulate what we have achieved so far. In the previous two definitions and appendant lemmas, we have first approximated L by L^δ and then L^δ by $L^{\delta,m}$. In each case, we have shown that for $\delta\to0$ resp. $m\to\infty$ these approximations "work $\widetilde{M}_\mu$-a.e.". By the following lemma as well as Proposition 2.42 this implies that if we choose δ small enough and then m large enough, we can approximate the portfolio process of $\mathfrak{S} = (0,0,L,0)$ by the portfolio process of $\mathfrak{S}^{\delta,m} = (0,0,L^{\delta,m},0)$ arbitrarily well w.r.t. to d_{up}.

Lemma 2.29. *Let $(L^{B,n})_{n\in\mathbb{N}} \subset \mathcal{L}^B$ and $L^B \in \mathcal{L}^B$. Furthermore, assume that $(L^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. towards L^B and that $\sup_{n\in\mathbb{N}} L^{B,n}$ is μ-integrable. Then for $n\to\infty$*

$$\int_{[0,\cdot)\times\mathbb{R}_+}\int_x^\infty yL^{B,n}(s,dy)\mu(ds,dx) \to \int_{[0,\cdot)\times\mathbb{R}_+}\int_x^\infty yL^{B}(s,dy)\mu(ds,dx),$$

uniformly in probability.

Similarly, let $(L^{S,n})_{n\in\mathbb{N}} \subset \mathcal{L}^S$ and $L^S \in \mathcal{L}^S$. Furthermore, assume that $(L^{S,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\nu$-a.e. towards L^S and that $\sup_{n\in\mathbb{N}} L^{S,n}$ is ν-integrable. Then for $n\to\infty$

$$\int_{[0,\cdot)\times\mathbb{R}_+}\int_0^x yL^{S,n}(s,dy)\nu(ds,dx) \to \int_{[0,\cdot)\times\mathbb{R}_+}\int_0^x yL^{S}(s,dy)\nu(ds,dx),$$

uniformly in probability.

Note that the convergence has to hold $\widetilde{M}_\mu$-a.e.. It is not sufficient to assume convergence only M_μ-a.e..

Proof. We only prove the first part of the lemma, because the proof of the second part is completely analog. Let $\widetilde{N}$ be a $\widetilde{M}_\mu$-null set s.t. $(L^{B,n})_{n\in\mathbb{N}}$ converges pointwise towards L^B on $\widetilde{N}^c$.

Step 1: Let us show that

$$H^n(\omega,t,x) := \int_x^\infty yL^{B,n}(\omega,t,dy) = \int_x^{\overline{S}_{t-}(\omega)} yL^{B,n}(\omega,t,dy)$$

converges pointwise to

$$H(\omega,t,x) := \int_x^\infty yL^{B}(\omega,t,dy) = \int_x^{\overline{S}_{t-}(\omega)} yL^{B}(\omega,t,dy)$$

for all $(\omega,t,x)\in N^c$, where

$$\begin{aligned} N \ := \ & \widetilde{N}\cup \text{subgraph}(\underline{\mathrm{X}}) \cup \bigcup_{q\in\mathbb{Q}_+} \{(\omega,t,x)\in\widetilde{\Omega} : (\omega,t,q)\in \widetilde{N}\cap \text{supergraph}(\underline{X})\} \\ & \cup \{L^B = \infty\}. \end{aligned}$$

Fix any $(\omega,t,x)\in N^c$. For any $\varepsilon>0$ choose $K\in\mathbb{N}$ and $y_1<\ldots<y_K$ in $\mathbb{Q}_+$ s.t. $x =: y_0 < y_1$, $y_K \geq \overline{S}_{t-}(\omega)$ and $y_i - y_{i-1} < \varepsilon$ for all $i\in\{1,\ldots,K\}$. As $(\omega,t,x)\notin \text{subgraph}(\underline{\mathrm{X}})$, and $y_i > x$ for

$i \geq 1$, we have that $(\omega, t, y_i) \in \text{supergraph}(\underline{\mathrm{X}})$ for $i \geq 1$. Thus, for all $i \in \{0, 1, \dots, K\}$ we get $L^{B,n}(\omega, t, y_i) \to L^B(\omega, t, y_i)$ as $n \to \infty$. For any $\widetilde{L}^B \in \mathcal{L}^B$ we have that

$$\begin{aligned}
\sum_{i=1}^{K} y_{i-1} \left(\widetilde{L}^B(\omega,t,y_i) - \widetilde{L}^B(\omega,t,y_{i-1}) \right) &\geq \int_x^{\overline{S}_{t-}(\omega)} y \widetilde{L}^B(\omega,t,dy) \\
&= \int_{y_0}^{y_K} y \widetilde{L}^B(\omega,t,dy) \\
&\geq \sum_{i=1}^{K} y_i \left(\widetilde{L}^B(\omega,t,y_i) - \widetilde{L}^B(\omega,t,y_{i-1}) \right)
\end{aligned}$$

(Note that $y \mapsto \widetilde{L}^B(\omega, t, y)$ is decreasing). By $L^{B,n}(\omega, t, y_i) \to L^B(\omega, t, y_i)$ for all $i = 0, \dots, K$ as $n \to \infty$ this implies

$$\begin{aligned}
&\liminf_{n\to\infty} \int_x^{\overline{S}_{t-}(\omega)} y L^{B,n}(\omega,t,dy) \\
\geq\ & \sum_{i=1}^{K} y_i \left(L^B(\omega,t,y_i) - L^B(\omega,t,y_{i-1}) \right) \\
\geq\ & \sum_{i=1}^{K} (y_{i-1} + \varepsilon) \left(L^B(\omega,t,y_i) - L^B(\omega,t,y_{i-1}) \right) \\
\geq\ & -\varepsilon L^B(\omega,t,y_0) + \sum_{i=1}^{K} y_{i-1} \left(L^B(\omega,t,y_i) - L^B(\omega,t,y_{i-1}) \right) \\
\geq\ & -\varepsilon L^B(\omega,t,y_0) + \sum_{i=1}^{K} \int_{y_{i-1}}^{y_i} y L^B(\omega,t,dy) \\
=\ & -\varepsilon L^B(\omega,t,x) + \int_x^{\overline{S}_{t-}(\omega)} y L^B(\omega,t,dy).
\end{aligned}$$

Since ε can be chosen arbitrarily small and $L^B(\omega, t, x) < \infty$ by construction of N this yields

$$\liminf_{n\to\infty} H^n(\omega,t,x) \geq H(\omega,t,x).$$

Analogously, we obtain that $\limsup_{n\to\infty} H^n(\omega, t, x) \leq H(\omega, t, x)$ and thus

$$H^n(\omega,t,x) \to H(\omega,t,x) \quad \forall (\omega,t,x) \in N^c.$$

Step 2: Let us show that $M_\mu(N) = 0$. By $\widetilde{M_\mu}(\widetilde{N}) = 0$, we have that $M_\mu(\widetilde{N}) = 0$ and $\widehat{M_\mu}(\{(\omega,t) \in \Omega \times [0,T] : \exists q \in \mathbb{Q}_+ \text{ s.t. } (\omega,t,q) \in (\widetilde{N} \cap \text{supergraph}(\underline{X})) = 0$. In addition, we use $M_\mu(\text{subgraph}(\underline{X})) = 0$, $M_\mu\left(\{\sup_{n\in\mathbb{N}} L^{B,n} = \infty\}\right) = 0$, and $M_\mu\left(\{L^B = \infty\}\right) = 0$ to arrive at

$$\begin{aligned}
M_\mu(N) &\leq M_\mu(\widetilde{N}) + M_\mu(\text{subgraph}(\underline{X})) \\
&\quad + M_\mu(\{(\omega,t,x) \in \widetilde{\Omega} : \exists q \in \mathbb{Q}_+ \text{ s.t. } (\omega,t,q) \in \widetilde{N} \cap \text{supergraph}(\underline{X})\}) \\
&\quad + M_\mu\left(\{L^B = \infty\}\right) \\
&= 0 + 0 + 0 + 0 = 0.
\end{aligned}$$

Now note that M_μ-a.e. we have

$$0 \geq H^n(\omega,t,x) = \int_x^{\overline{S}_{t-}(\omega)} yL^{B,n}(\omega,t,dy) \geq -\sup_{t\in[0,T]} \overline{S}_{t-}(\omega) \sup_{n\in\mathbb{N}} L^{B,n}(\omega,t,x) > -\infty,$$

i.e. $(H^n)_{n\in\mathbb{N}}$ is dominated by $\sup_{t\in[0,T]} \overline{S}_{t-} \sup_{n\in\mathbb{N}} L^{B,n}$, which is clearly μ-integrable since $\sup_{n\in\mathbb{N}} L^{B,n}$ is μ-integrable by assumption. Thus, an application of Proposition 2.42 completes the proof.

□

So far, we have already shown that $L^{\delta,m}$ is a good approximation of L w.r.t. the portfolio processes. The problem with $L^{\delta,m}$ is, that while it is already discrete in the price variable, this is not the case for the time variable. Hence, it is not clear how to approximate the portfolio process of the trading strategy $(0,0,L^{\delta,m},0)$ with the portfolio process of a real-world trading strategy. The following theorem tackles this problem. It tells us that we can approximate $L^{\delta,m}$, and thus also L, by a function $\widehat{L}$ which is discrete not only in the price variable, but also in the time variable. Furthermore, it is still equal to zero "to the right of $\overline{S}_- - \delta$" and thus "continuous execution" is not a problem when we want to approximate $\widehat{L}$ with real-world trading strategies.

Theorem 2.30. *For any $\varepsilon > 0$ and any $L \in \mathcal{L}^B$ there exist $A^\varepsilon \in \mathcal{F}$, $\delta > 0$, $m \in \mathbb{N}$, and nonnegative* simple *predictable processes $\widehat{\xi}^0, \widehat{\xi}^1, \ldots, \widehat{\xi}^{m2^m}$ s.t. $P(A^\varepsilon) \geq 1 - \varepsilon$ and*

$$\widehat{\xi}^l = 0, \qquad on \quad \{(\omega,t) \in \Omega \times [0,T] : \overline{S}_{t-}(\omega) - \delta \leq l2^{-m}\} \cup \Omega \times \{0\}.$$

Furthermore, for

$$\widehat{L}(\omega,t,x) := \sum_{l=0}^{m2^m} \widehat{\xi}^l_t(\omega) 1_{\{x \leq l2^{-m}\}}$$

we have that $\widehat{L} \in \mathcal{L}^B$ and for every $\omega \in A^\varepsilon$

$$\sup_{t\in[0,T]} \left| \int_{[0,t)\times\mathbb{R}_+} L(\omega,s,x)\mu(\omega,ds,dx) - \int_{[0,t)\times\mathbb{R}_+} \widehat{L}(\omega,s,x)\mu(\omega,ds,dx) \right| < \varepsilon$$

and

$$\sup_{t\in[0,T]} \left| \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL(\omega,s,dy)\mu(\omega,ds,dx) - \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty y\widehat{L}(\omega,s,dy)\mu(\omega,ds,dx) \right| < \varepsilon.$$

Proof. Step 1: Let $\varepsilon > 0$. By Lemma 2.26, Lemma 2.28, Proposition 2.42, Lemma 2.29, and the fact that the up-convergence is metrizable, it is possible to choose at first a $\delta > 0$ small enough and afterwards an $m \in \mathbb{N}$ large enough s.t. there exists a set $U \in \mathcal{F}$ s.t. $P(U) \geq 1 - \frac{\varepsilon}{3}$ and for all $\omega \in U$

$$\sup_{t\in[0,T]} \left| \int_{[0,t)\times\mathbb{R}_+} L(\omega,s,x)\mu(\omega,ds,dx) - \int_{[0,t)\times\mathbb{R}_+} L^{\delta,m}(\omega,s,x)\mu(\omega,ds,dx) \right| < \frac{\varepsilon}{2} \quad \text{and} \tag{2.7}$$

$$\sup_{t\in[0,T]} \left| \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL(\omega,s,dy)\mu(\omega,ds,dx) - \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL^{\delta,m}(\omega,s,dy)\mu(\omega,ds,dx) \right| < \frac{\varepsilon}{2}$$

holds. Furthermore, if we choose δ at least as small as $\frac{\varepsilon}{3}$ by the definition of $\overline{S}^\delta$ there exists a set $V \in \mathcal{F}$ s.t. $P(V) \geq 1 - \frac{\varepsilon}{3}$ and for all $\omega \in V$

$$\sup_{t\in[0,T]} |\overline{S}^\delta_t(\omega) - \overline{S}_{t-}(\omega)| \leq \delta \quad \text{and} \quad \tau^\delta(\omega) = \infty. \tag{2.8}$$

Finally, m can be chosen large enough s.t. $m > -\log_2(\delta)$.

Step 2: For any $\delta > 0$ we decompose μ into the executions triggered by the jumps of $\overline{S}$ with sizes lying in $[-\delta, 0)$ and the rest. More precisely, let

$$\mu = \mu^{1,\delta} + \mu^{2,\delta} \tag{2.9}$$

with $\mu^{1,\delta} \perp \mu^{2,\delta}$ and $\mu^{1,\delta}(\{t\} \times \{x\}) = 1$ iff $x = \overline{S}_t$ and $\Delta\overline{S}_t \in [-\delta, 0)$. Note that by (i), (iii), and (iv) of Assumption 2.7 and as $\overline{S}$ is càdlàg, $\mu^{2,\delta}$ is a finite random measure. By contrast, $\mu^{1,\delta}$ is in general infinite. Orders with limit prices below $\overline{S}_- - \delta$ cannot be executed by $\mu^{1,\delta}$.

Define $\xi^l_t(\omega) := \theta^{\delta,l,m}_t(\omega) - \theta^{\delta,l+1,m}_t(\omega)$ for all $l = 1, \ldots, m2^m - 1$, $\xi^{m2^m}_t(\omega) := \theta^{\delta,m2^m,m}_t(\omega)$, and $\xi^0_t(\omega) := L^{\delta,m}(\omega,t,0) - \theta^{\delta,1,m}_t(\omega)$, where $\theta^{\delta,l,m}$, $l = 1, \ldots, m2^m$ are introduced in Definition 2.27. In addition define

$$A^l_t(\omega) := \mu^{2,\delta}(\omega, [0,t] \times [0, l2^{-m}])$$

for $l = 0, \ldots, m2^m$. Observe that we can use these processes to specify a representation of the shares bought and the cash payments resulting from strategy $L^{\delta,m}$ by

$$\begin{aligned} \int_{[0,t)\times\mathbb{R}_+} L^{\delta,m}(s,x)\mu(ds,dx) &= \int_{[0,t)\times\mathbb{R}_+} L^{\delta,m}(s,x)\mu^{2,\delta}(ds,dx) \\ &= \sum_{l=0}^{m2^m} \xi^l \bullet A^l_{t-} \qquad \text{and} \end{aligned}$$

$$\begin{aligned} -\int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL^{\delta,m}(s,dy)\mu(ds,dx) &= -\int_{[0,t)\times\mathbb{R}_+} \int_x^\infty yL^{\delta,m}(s,dy)\mu^{2,\delta}(ds,dx) \\ &= \sum_{l=0}^{m2^m} l2^{-m}\xi^l \bullet A^l_{t-}. \end{aligned}$$

By Assumption 2.7 (vi), we have that $A^l_0 = 0$. Thus, different conventions for the integral w.r.t. A at 0 do not matter. Note that we can replace μ by $\mu^{2,\delta}$ as by construction we have that $L^{\delta,m} = 0$ on $[\overline{S}_- - \delta, \infty)$. Since $L^{\delta,m}$ is μ-integrable, the integrability of any ξ^l w.r.t. A^l is satisfied. As $\mu^{2,\delta}(\omega,\cdot)$ is a finite measure for any $\omega \in \Omega$, there exists a probability measure $Q \sim P$ s.t.

$$E_Q\left[A^l_{T-}\right] < \infty \quad \text{and} \quad E_Q\left[\int_0^{T-} \xi^l \, dA^l\right] < \infty.$$

Then it is well-known (and provable by the monotone class theorem, compare Theorem IV.2 and Theorem IV.14 in [Pro04]) that the predictable process ξ^l can be approximated by a simple predictable process $\widetilde{\xi}^l$ in the sense that

$$E_Q\left[\int_0^{T-} |\xi^l - \widetilde{\xi}^l| dA^l\right] \tag{2.10}$$

gets arbitrarily small. As ξ^l is nonnegative, $\widetilde{\xi}^l$ can be chosen to be nonnegative as well. Since $L^1(Q)$-convergence implies convergence in Q- resp. P-probability, $\widetilde{\xi}^l$ can be chosen s.t. on a set $U^l \in \mathcal{F}$ with $P(U^l) \geq 1 - \frac{\varepsilon}{3(m2^m+1)}$ it holds that

$$\int_0^{T-} |\xi^l - \widetilde{\xi}^l| dA^l < \frac{\varepsilon}{2m(m2^m+1)}. \tag{2.11}$$

Define the process

$$\widehat{\xi}^l := \widetilde{\xi}^l 1_{\{\overline{S}^\delta - 2\delta > l2^{-m}\}} 1_{]\!]0,\tau^\delta]\!]}$$

which is simple predictable as $\overline{S}^\delta$ and $\widetilde{\xi}^l$ are simple predictable. By construction of $L^{\delta,m}$, we also have for ξ^l that $\xi^l = 0$ on $\{\overline{S}^\delta - 2\delta \leq l2^{-m}\} \cup]\!]\tau^\delta, T]\!]$. Furthermore, by Assumption 2.7 (vi) we know that $A^l_0 = 0$, i.e. whether $\xi^l_0 = 0$ as well or not does not matter. Thus, (2.11) implies

$$\int_0^{T-} |\xi^l - \widehat{\xi}^l| dA^l < \frac{\varepsilon}{2m(m2^m+1)} \tag{2.12}$$

on U^l. In addition, we have

$$\widehat{\xi}^l = 0 \qquad \text{on } \{(\omega,t) \in \Omega \times [0,T] : \overline{S}_{t-}(\omega) - \delta \leq l2^{-m}\} \cup \Omega \times \{0\}. \tag{2.13}$$

Now (2.12) clearly implies that on U^l it holds that

$$\sup_{t\in[0,T]} |\xi^l \bullet A^l_{t-} - \widehat{\xi}^l \bullet A^l_{t-}| < \frac{\varepsilon}{2m(m2^m+1)},$$

and because $l2^{-m} \leq m$ we also have

$$\sup_{t\in[0,T]} |l2^{-m}\xi^l \bullet A^l_{t-} - l2^{-m}\widehat{\xi}^l \bullet A^l_{t-}| < \frac{\varepsilon}{2(m2^m+1)}$$

on U^l. Hence on $\bigcap_{l=0}^{m2^m} U^l$ we arrive at

$$\sup_{t\in[0,T]}\left|\int_{[0,t)\times\mathbb{R}_+} L^{\delta,m}(s,x)\mu(ds,dx)-\int_{[0,t)\times\mathbb{R}_+}\widehat{L}(s,x)\mu(ds,dx)\right|<\frac{\varepsilon}{2m}\qquad\text{and}\tag{2.14}$$
$$\sup_{t\in[0,T]}\left|\int_{[0,t)\times\mathbb{R}_+}\int_x^\infty yL^{\delta,m}(s,dy)\mu(ds,dx)-\int_{[0,t)\times\mathbb{R}_+}\int_x^\infty y\widehat{L}(s,dy)\mu(ds,dx)\right| < \frac{\varepsilon}{2}.$$

Now we only have to make certain that (2.7), (2.8), (2.13), and (2.14) all hold on the same set A^ε, which is easily achieved by setting

$$A^\varepsilon := U\cap V\cap\left(\bigcap_{l=0}^{m2^m} U^l\right).$$

$\square$

To finish this section let us show how Theorem 2.17 follows from Theorem 2.30. In Theorem 2.30 the processes $\widehat{\xi}^0,\widehat{\xi}^1,\dots,\widehat{\xi}^{m2^m}$ were introduced. As any such simple predictable process $\widehat{\xi}^l$ starts with value 0, it can be written as a finite sum of terms of the form $\widehat{\xi}^{l,i}1_{]\!]T_1^{l,i},T_2^{l,i}]\!]}$, where $T_1^{l,i},T_2^{l,i}$ are stopping times with $T_1^{l,i}<T_2^{l,i}$ and $\widehat{\xi}^{l,i}$ is $\mathcal{F}_{T_1^{l,i}}$-measurable. In addition, any finite conical combination of real-world trading strategies is again a real-world trading strategy and of course the mapping $\mathfrak{S}\mapsto\varphi(\mathfrak{S})$ is linear. Consequently, to finish the proof of Theorem 2.17, it is sufficient to show that we can approximate the trading strategy $(0,0,\widehat{\xi}^{l,i}1_{]\!]T_1^{l,i},T_2^{l,i}]\!]}1_{\{x\le l2^{-m}\}},0)$ arbitrarily well with a real-world trading strategy (w.r.t. their respective portfolio processes and d_{up}).

We define the sequence of stopping times

$$\begin{aligned}\overline{\tau}_0 &:= T_1^{l,i},\\ \overline{\tau}_j &:= \inf\left\{t>\overline{\tau}_{j-1}:\mu^{2,\delta}\left(\{t\}\times[0,l2^{-m}]\right)>0\right\},\end{aligned}$$

where $\mu^{2,\delta}$ refers to the finite measure defined in the proof of Theorem 2.30. Thus, we get $P(\overline{\tau}_j\ge T_2^{l,i})\uparrow 1$ as $j\to\infty$. Furthermore, note that for a limit buy order given by $(\widehat{\xi}^{l,i},l2^{-m},\overline{\tau}_{j-1}\wedge T_2^{l,i},T_2^{l,i})$, the appendant stopping time T^* describing the execution time of the order, as defined in Section 2.4.1, satisfies $T^*=\overline{\tau}_j$ on $\{\widehat{\xi}^{l,i}>0,\overline{\tau}_j\le T_2^{l,i}\}$, since $\widehat{\xi}^{l,i}1_{]\!]T_1^{l,i},T_2^{l,i}]\!]}=0$ on $\{(\omega,t)\in\Omega\times[0,T]:\overline{S}_{t-}(\omega)-\delta\le l2^{-m}\}$. If we let

$$M^K:=\{\omega\in\Omega:\overline{\tau}_K(\omega)\ge T_2^{l,i}\}$$

for any $K \in \mathbb{N}$ then on $M^K \times [0,T] \times \mathbb{R}_+$ we get

$$\begin{aligned}
\widehat{\xi}^{l,i} 1_{\rrbracket T_1^{l,i}, T_2^{l,i} \rrbracket} 1_{\{x \leq l2^{-m}\}} &= \sum_{j=1}^{K} \widehat{\xi}^{l,i} 1_{\rrbracket \overline{\tau}_{j-1} \wedge T_2^{l,i}, \overline{\tau}_j \wedge T_2^{l,i} \rrbracket} 1_{\{x \leq l2^{-m}\}} \\
&= \sum_{j=1}^{K} \widehat{\xi}^{l,i} 1_{\rrbracket \overline{\tau}_{j-1} \wedge T_2^{l,i}, \overline{\tau}_j \wedge T_2^{l,i} \rrbracket} \left(1_{\{x \leq l2^{-m}, \overline{S}_{t-} > l2^{-m}\}} + 1_{\{x < l2^{-m}, \overline{S}_{t-} = l2^{-m}\}} \right),
\end{aligned} \tag{2.15}$$

where we have again used $\widehat{\xi}^{l,i} 1_{\rrbracket T_1^{l,i}, T_2^{l,i} \rrbracket} = 0$ on $\{(\omega,t) \in \Omega \times [0,T] : \overline{S}_{t-}(\omega) - \delta \leq l2^{-m}\}$ for the second equality. Therefore, for all $\omega \in M^K$ the path of the portfolio process associated with a trading strategy consisting only of the term on the lhs of (2.15) is identical to the path of the portfolio process associated with a trading strategy consisting only of the term on the lower rhs of (2.15). Hence, for any $\varepsilon > 0$ if we choose $K_\varepsilon \in \mathbb{N}$ large enough s.t. $P(M^{K_\varepsilon}) \geq 1 - \varepsilon$ this clearly implies that

$$\begin{aligned}
d_{\mathrm{up}} \Bigg(& \varphi^0 \left(\left(0, 0, \widehat{\xi}^{l,i} 1_{\rrbracket T_1^{l,i}, T_2^{l,i} \rrbracket} 1_{\{x \leq l2^{-m}\}}, 0 \right) \right), \\
& \varphi^0 \left(\left(0, 0, \sum_{j=1}^{K_\varepsilon} \widehat{\xi}^{l,i} 1_{\rrbracket \overline{\tau}_{j-1} \wedge T_2^{l,i}, \overline{\tau}_j \wedge T_2^{l,i} \rrbracket} \left(1_{\{x \leq l2^{-m}, \overline{S}_{t-} > l2^{-m}\}} + 1_{\{x < l2^{-m}, \overline{S}_{t-} = l2^{-m}\}} \right), 0 \right) \right) \Bigg) < \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
d_{\mathrm{up}} \Bigg(& \varphi^1 \left(\left(0, 0, \widehat{\xi}^{l,i} 1_{\rrbracket T_1^{l,i}, T_2^{l,i} \rrbracket} 1_{\{x \leq l2^{-m}\}}, 0 \right) \right), \\
& \varphi^1 \left(\left(0, 0, \sum_{j=1}^{K_\varepsilon} \widehat{\xi}^{l,i} 1_{\rrbracket \overline{\tau}_{j-1} \wedge T_2^{l,i}, \overline{\tau}_j \wedge T_2^{l,i} \rrbracket} \left(1_{\{x \leq l2^{-m}, \overline{S}_{t-} > l2^{-m}\}} + 1_{\{x < l2^{-m}, \overline{S}_{t-} = l2^{-m}\}} \right), 0 \right) \right) \Bigg) < \varepsilon
\end{aligned}$$

Thus, Theorem 2.17 is proven.

2.6 Proof of Theorem 2.22: Closedness of the strategy set

In the whole section let the assumptions of Theorem 2.22 hold and let $(\varphi^{0,n}, \varphi^{1,n})_{n \in \mathbb{N}}$ with $\varphi^{0,n} := \varphi^0(\mathfrak{S}^n)$ and $\varphi^{1,n} = \varphi^1(\mathfrak{S}^n)$ be an up-Cauchy sequence where $(\mathfrak{S}^n)_{n \in \mathbb{N}}$ is an a-admissible sequence of trading strategies.

Since the space of làdlàg functions (also called regulated functions) mapping from $[0,T]$ to $\mathbb{R}$ is complete w.r.t. the supremum norm, there exist predictable làdlàg processes ψ^0 and ψ^1 s.t. $(\varphi^{0,n})_{n \in \mathbb{N}}$ converges uniformly in probability to ψ^0 and $(\varphi^{1,n})_{n \in \mathbb{N}}$ converges uniformly in probability to ψ^1. By going to a subsequence of $(\mathfrak{S}^n)_{n \in \mathbb{N}}$ we can assume w.l.o.g. that $(\varphi^{0,n}, \varphi^{1,n})_{n \in \mathbb{N}}$ even converges (component wise) P-a.s. uniformly on $[0,T]$ to (ψ^0, ψ^1).

Lemma 2.31. *Let $\widehat{\tau}^0 := 0$ and for $k,n \in \mathbb{N}$ define the stopping times*

$$\begin{aligned}\widehat{\tau}^{k,n} &:= \inf\{t>0 : |\varphi_t^{0,n}|>k\} \wedge \inf\{t>0 : |\varphi_t^{1,n}|>k\} \\ &\quad \wedge \inf\{t>0 : \int_{[0,t]\times\mathbb{R}_+} 1_{[\underline{S}_s,\overline{S}_s)}(x)\mu(ds,dx) + \int_{[0,t]\times\mathbb{R}_+} 1_{(\underline{S}_s,\overline{S}_s]}(x)\nu(ds,dx) > k\}, \\ \widehat{\tau}^k &:= \inf_{n\in\mathbb{N}} \widehat{\tau}^{k,n} \wedge T.\end{aligned}$$

There exists a probability measure Q equivalent to P s.t. for all $k \in \mathbb{N}$ there exists a constant $K_k > 0$ s.t.

$$E_Q\left[\operatorname{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \operatorname{var}(\varphi^{1,n})_{\widehat{\tau}^k}\right] \le K_k, \qquad \forall n \in \mathbb{N}.$$

Furthermore, $(\widehat{\tau}^k)_{k\in\mathbb{N}}$ is an increasing sequence of stopping times with $P(\widehat{\tau}^k = T) \to 1$ for $k \to \infty$, i.e. it is localizing.

Proof. Fix any $k \in \mathbb{N}$. Note that $(\widehat{\tau}^k)_{k\in\mathbb{N}}$ is indeed a sequence of stopping times, as $(\varphi^{0,n})_{n\in\mathbb{N}}, (\varphi^{1,n})_{n\in\mathbb{N}}, \underline{S}, \overline{S}, \mu$ and ν are optional. Let $\widehat{\sigma}^0 := 0$ and for $i = 1,2,\dots$ let

$$\widehat{\sigma}^i := \inf\left\{t > \widehat{\sigma}^{i-1} : \int_{[0,t]\times\mathbb{R}_+} 1_{[\underline{S}_s,\overline{S}_s)}(x)\mu(ds,dx) + \int_{[0,t]\times\mathbb{R}_+} 1_{(\underline{S}_s,\overline{S}_s]}(x)\nu(ds,dx) \ge i\right\},$$

which are also stopping times by the reasons given for $\widehat{\tau}^k$. Note that for $i > k$ it follows that $\widehat{\sigma}^i(\omega) \ge \widehat{\tau}^k(\omega)$. Furthermore, from the definition of $\widehat{\tau}^k$ we see that $|\Delta^+(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k})_t|, |\Delta^+(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})_t| \le 2k$ for all $t \in [0,T]$ since $|\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k}|, |\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k}| \le k$ on $[\![0,\widehat{\tau}^k[\![$ and $\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k}, \varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k}$ are constant on $[\![\widehat{\tau}^k, T]\!]$. Thus,

$$\sum_{i=0}^{\infty}\left(\Delta^+\operatorname{var}\left(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k}\right)_{\widehat{\sigma}^i} + \Delta^+\operatorname{var}\left(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k}\right)_{\widehat{\sigma}^i}\right) \le 4k(k+1). \tag{2.16}$$

For any $(\varphi^{0,n}, \varphi^{1,n})$ and each $i = 1,2,\dots,k+1$ we define a self-financing, admissible portfolio process in the sense of Campi and Schachermayer (see [CS06] for details) with initial endowment $\varphi_0^{0,n,i} = k$, $\varphi_0^{1,n,i} = k$ and threshold level a by

$$\begin{aligned}\varphi^{0,n,i} &:= k1_{[\![0,\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k]\!]} + \varphi^{0,n}1_{]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k,\widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]} - a1_{]\!]\widehat{\sigma}^i\wedge\widehat{\tau}^k,T]\!]} \\ \varphi^{1,n,i} &:= k1_{[\![0,\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k]\!]} + \varphi^{1,n}1_{]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k,\widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]} - a1_{]\!]\widehat{\sigma}^i\wedge\widehat{\tau}^k,T]\!]}.\end{aligned}$$

By construction $(\varphi^{0,n,i}, \varphi^{1,n,i}) = (\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k}, \varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})$ on $]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k, \widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]$. Thus, $(\varphi^{0,n,i}, \varphi^{1,n,i})$ is certainly a-admissible. If $\widehat{\sigma}^{i-1} < \widehat{\tau}^k$ than the change of the portfolio from (k,k) to $(\varphi^{0,n}_{\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k}, \varphi^{1,n}_{\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k})$ is self-financing due to the first row in the definition of $\widehat{\tau}^{k,n}$. If $\widehat{\sigma}^{i-1} \ge \widehat{\tau}^k$ then $]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k, \widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]$ is empty and the change to $(-a,-a)$ is clearly self-financing. Furthermore, on $]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k, \widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]$ no favorable executions of limit orders can influence the portfolio process (remember that a limit order executed at stopping time $\widehat{\sigma}^i$ only shows up in the portfolio process immediately after $\widehat{\sigma}^i$). While there may be executions of limit orders on $]\!]\widehat{\sigma}^{i-1}\wedge\widehat{\tau}^k,\widehat{\sigma}^i\wedge\widehat{\tau}^k]\!]$, the prices paid by the small investor are at most as favorable as in the model with proportional transaction costs. If e.g. a limit buy order of size $\theta^B(\omega)$ with limit price $p^B(\omega)$ is executed at time $T^*(\omega)$ with $\widehat{\sigma}^{i-1}(\omega) < T^*(\omega) < \widehat{\sigma}^i(\omega)$ we know by construction that $\overline{S}_{T^*}(\omega) \le p^B(\omega)$. Hence, the investor would be at least as well of just buying amount $\theta^B(\omega)$ at time $T^*(\omega)$ with a market order at price $\overline{S}_{T^*}(\omega)$. Thus, $(\varphi^{0,n,i},\varphi^{1,n,i})$ is indeed a self-financing portfolio process in the sense of [CS06] (in which it is allowed to "throw away" assets). More precisely, if we translate $\{\underline{S},\overline{S}\}$ into the càdlàg bid-ask process

$$\Pi := \begin{pmatrix} 1 & \overline{S} \\ \frac{1}{\underline{S}} & 1 \end{pmatrix}$$

used in [CS06], then $\widehat{V}^{n,i} := (\varphi^{0,n,i},\varphi^{1,n,i})$ is a self-financing, admissible portfolio process with threshold a in the sense of Definition 2.7 in [CS06].

Right from the definition of $(\varphi^{0,n,i},\varphi^{1,n,i})$ it follows that for all $n\in\mathbb{N}$ and $i=1,2,\dots,k+1$

$$\begin{aligned}
\mathrm{var}(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+} &= \mathrm{var}(\varphi^{0,n,i})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{0,n,i})_{\widehat{\sigma}^{i-1}+},\\
\mathrm{var}(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+} &= \mathrm{var}(\varphi^{1,n,i})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{1,n,i})_{\widehat{\sigma}^{i-1}+}.
\end{aligned}$$

Remember that by $\widetilde{P}$ we denote the measure, which makes the strictly consistent price process a martingale. By Lemma 3.2 in [CS06] there exist a probability measure $Q\sim P$ and a constant $C>0$ such that for all $k,n\in\mathbb{N}$ and all $i=1,\dots,k+1$

$$\begin{aligned}
E_Q[\mathrm{var}(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+}] &\le E_Q[\mathrm{var}(\varphi^{0,n,i})_T] \le C(k+a),\\
E_Q[\mathrm{var}(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+}] &\le E_Q[\mathrm{var}(\varphi^{1,n,i})_T] \le C(k+a).
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&E_Q\left[\sum_{i=1}^{\infty}\Big(\mathrm{var}(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{0,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+}\Big) + \sum_{i=1}^{\infty}\Big(\mathrm{var}(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^i} - \mathrm{var}(\varphi^{1,n}_{\cdot\wedge\widehat{\tau}^k})_{\widehat{\sigma}^{i-1}+}\Big)\right]\\
\le\ & (k+1)2C(k+a). \qquad (2.17)
\end{aligned}$$

By combining (2.16) and (2.17) the first part of the lemma is proven.

Concerning the localizing sequence we immediately see that $(\widehat{\tau}^k)_{k\in\mathbb{N}}$ is increasing from its definition. As discussed at the beginning of the section, there exists a set $N\in\mathcal{F}$ s.t. $P(N)=0$ and s.t. $\big(\varphi^{0,n}(\omega)\big)_{n\in\mathbb{N}}$ converges towards $\psi^0(\omega)$ uniformly on $[0,T]$ for all $\omega\in N^C$. Fix any

$\omega \in N^C$. Remember that any làdlàg function is bounded on a compact interval. Thus, there exists a $n_0(\omega)$ such that for all $n \in \mathbb{N}$ we have

$$\sup_{t\in[0,T]} |\varphi_t^{0,n}(\omega)| \leq \left(\bigvee_{j=1}^{n_0(\omega)} \sup_{t\in[0,T]} |\varphi_t^{0,j}(\omega)| \right) \vee \left(\sup_{t\in[0,T]} |\psi_t^0(\omega)| + 1 \right) < \infty.$$

Hence,

$$P\left(\sup_{t\in[0,T]} |\varphi_t^{0,n}| \leq k, \forall n \in \mathbb{N} \right) \uparrow P(N^C) = 1 \qquad \text{as } k \to \infty.$$

Similar arguments yield $P\left(\sup_{t\in[0,T]} |\varphi_t^{1,n}| \leq k, \forall n \in \mathbb{N}\right) \uparrow 1$. By Assumption 2.7 (iii) we also have

$$P\left(\int_{[0,t]\times\mathbb{R}_+} 1_{[\underline{S}_s,\overline{S}_s)}(x)\mu(ds,dx) + \int_{[0,t]\times\mathbb{R}_+} 1_{(\underline{S}_s,\overline{S}_s]}(x)\nu(ds,dx) \leq k \right) \uparrow 1 \qquad \text{as } k \to \infty.$$

Therefore, we arrive at $P(\widehat{\tau}^k = T) \to 1$ as $k \to \infty$. □

For proportional transaction costs the statement above holds for arbitrary families of portfolio processes, even without stopping. The basic idea is that any trade costs a little bit of wealth and by the martingale in between the best-bid and the best-ask price in expectation no money can be made from directional trades. Hence, the amount of trading has to be limited in expectation. This does not apply directly to the case with limit orders, because the execution of a limit order at a better price than what is available via market orders may increase wealth, thus supplying the trader with additional wealth to spend on market orders. We deal with this problem in the lemma above by using stopping times in such a way that the gains by limit orders up to the stopping time are limited. By Assumption 2.7 there are only finitely many instances at which a trade at a more favorable price than what is available via market orders can be made. The assumption that $(\varphi^{0,n}, \varphi^{1,n})_{n\in\mathbb{N}}$ converges towards (ψ^0, ψ^1) bounds the other terms found in the definition of the stopping times, hence together they yield that $(\widehat{\tau}^k)_{k\in\mathbb{N}}$ is localizing.

Lemma 2.32. *We have*

$$\begin{aligned} M_\mu\left(\{(\omega,t,x) \in \widetilde{\Omega} : \sup_{n\in\mathbb{N}} L^{B,n}(\omega,t,x) = \infty\} \right) &= 0, \\ M_\nu\left(\{(\omega,t,x) \in \widetilde{\Omega} : \sup_{n\in\mathbb{N}} L^{S,n}(\omega,t,x) = \infty\} \right) &= 0, \end{aligned}$$

i.e. $(L^{B,n})_{n\in\mathbb{N}}$ *is* M_μ*-a.e. bounded and* $(L^{S,n})_{n\in\mathbb{N}}$ *is* M_ν*-a.e. bounded.*

Proof. We only deal with $(L^{B,n})_{n\in\mathbb{N}}$ as the assertion regarding $(L^{S,n})_{n\in\mathbb{N}}$ can be proved similarly. Each $L^{B,n}$ is μ-integrable and hence it holds that $L^{B,n} < \infty$ M_μ-almost everywhere. Thus, we can ignore the beginning of the sequence. Define the set A by

$$A := \{(\omega,t,x) \in \widetilde{\Omega} : \limsup_{n\to\infty} L^{B,n}(\omega,t,x) = \infty\}.$$

A is $\widetilde{\mathcal{P}}$-measurable, because the lim sup of measurable functions is measurable. Furthermore, for any $\varepsilon > 0$ let

$$\begin{aligned} B_\varepsilon &:= \text{supergraph}(\underline{X}+\varepsilon) \\ &= \{(\omega,t,x) \in \widetilde{\Omega} : \underline{X}(\omega,t)+\varepsilon < x\}. \end{aligned}$$

By Lemma 2.6 B_ε is $\widetilde{\mathcal{P}}$-measurable as well. This implies that for any $q \in \mathbb{Q}_+$ the q-section $(A\cap B_\varepsilon)_q$ of the set $A\cap B_\varepsilon$ is $\mathcal{P}$-measurable. Note that by the monotonicity of the functions $L^{B,n}(\omega,t,\cdot)$ it holds that $(\omega,t,x) \in A$ implies $(\omega,t,y) \in A$ for all $y < x$, i.e. the nonempty (ω,t)-sections of A are either of the form $[0,a(\omega,t)) \subset \mathbb{R}_+$ or of the form $[0,a(\omega,t)] \subset \mathbb{R}_+$. A similar property holds for B_ε. Directly from its definition, it follows that the (ω,t)-sections of B_ε are always of the form $(\underline{X}(\omega,t)+\varepsilon,\infty) \subset \mathbb{R}_+$. Hence the equality in the second row of the following holds

$$\begin{aligned} \widehat{N}_\varepsilon &:= \{(\omega,t) \in \Omega\times[0,T] : \exists x \in \mathbb{R}_+ : (\omega,t,x) \in A\cap B_\varepsilon\} \\ &= \{(\omega,t) \in \Omega\times[0,T] : \exists x \in \mathbb{Q}_+ : (\omega,t,x) \in A\cap B_\varepsilon\} \\ &= \bigcup_{q\in\mathbb{Q}_+} \{(\omega,t) \in \Omega\times[0,T] : (\omega,t,q) \in A\cap B_\varepsilon\} = \bigcup_{q\in\mathbb{Q}_+} (A\cap B_\varepsilon)_q, \end{aligned}$$

and thus $\widehat{N}_\varepsilon$ is $\mathcal{P}$-measurable.

Suppose there exists an $\varepsilon > 0$ s.t. $\widehat{M}_\mu(\widehat{N}_\varepsilon) > 0$. For now keep this ε fixed. Define

$$N_\varepsilon := \{(\omega,t,x) \in \widetilde{\Omega} : (\omega,t) \in \widehat{N}_\varepsilon, \underline{X}(\omega,t) \le x \le \underline{X}(\omega,t) + \frac{\varepsilon}{2}\}.$$

By

$$N_\varepsilon^c = \widehat{N}_\varepsilon^c \times \mathbb{R}_+ \cup \{(\omega,t,x) \in \widetilde{\Omega} : x < \underline{X}(\omega,t)\} \cup \{(\omega,t,x) : \underline{X}(\omega,t) + \frac{\varepsilon}{2} < x\}$$

and Lemma 2.6 we get $N_\varepsilon \in \widetilde{\mathcal{P}}$. Now if $M_\mu(N_\varepsilon) = 0$ would hold, we could define a predictable process $Z := \underline{X}1_{\widehat{N}_\varepsilon^c} + (\underline{X}+\frac{\varepsilon}{2})1_{\widehat{N}_\varepsilon}$ with $M_\mu(\text{subgraph(Z)}) = 0$ but $Z \ge \underline{X}$ with $Z > \underline{X}$ on a set with positive weight $\widehat{M}_\mu(\widehat{N}_\varepsilon) > 0$, which is a contradiction to the definition of $\underline{X}$. Hence if $\widehat{M}_\mu(\widehat{N}_\varepsilon) > 0$ then also $M_\mu(N_\varepsilon) > 0$.

Referring to the discussion at the beginning of this section we have P-a.s.

$$\lim_{n\to\infty}\sup_{t\in[0,T)} |\Delta^+\varphi_t^{0,n}(\omega) - \Delta^+\psi_t^0(\omega)| \;=\; 0, \tag{2.18}$$

$$\lim_{n\to\infty}\sup_{t\in[0,T)} |\Delta^+\varphi_t^{1,n}(\omega) - \Delta^+\psi_t^1(\omega)| \;=\; 0. \tag{2.19}$$

Now (2.19) implies for all (τ_i, Y_i) (introduced in Remark 2.13) P-a.s.

$$\lim_{n\to\infty}\left(\Delta^+M^{B,n}_{\tau_i\wedge T} - \Delta^+M^{S,n}_{\tau_i\wedge T} + L^{B,n}(\tau_i,Y_i)1_{\{\tau_i\le T\}} - \sum_{j=1}^{\infty} L^{S,n}(\sigma_j,Z_j)1_{\{\tau_i=\sigma_j\le T\}}\right) = \Delta^+\psi^1_{\tau_i\wedge T}.$$

Thus for all (τ_i, Y_i) it holds that the event

$$\Bigg\{\omega\in\Omega : \lim_{n\to\infty}\Bigg(\Delta^+M^{B,n}_{\tau_i\wedge T}(\omega) - \Delta^+M^{S,n}_{\tau_i\wedge T}(\omega) + L^{B,n}(\tau_i,Y_i)(\omega)1_{\{\tau_i\le T\}}(\omega)$$
$$-\sum_{j=1}^{\infty} L^{S,n}(\sigma_j,Z_j)1_{\{\tau_i=\sigma_j\le T\}}(\omega)\Bigg) = \Delta^+\psi^1_{\tau_i\wedge T}(\omega)\Bigg\}^c$$

has probability zero and by Proposition 2.41 this yields that M_μ-a.e. for $n\to\infty$ we have

$$\Delta^+M^{B,n}_t(\omega) - \Delta^+M^{S,n}_t(\omega) + L^{B,n}(\omega,t,x) - \int_{\{t\}\times\mathbb{R}_+} L^{S,n}(\omega,s,z)\nu(\omega,ds,dz) \to \Delta^+\psi^1_t(\omega). \tag{2.20}$$

By Assumption 2.7 (i) we have $\underline{S}_t(\omega)\le x\le \overline{S}_t(\omega)$ for M_μ-a.a. $(\omega,t,x)\in\widetilde{\Omega}$. Combining this with Assumption 2.7 (v) implies that limit sell orders can M_μ-a.e. only be executed if $x=\overline{S}_t(\omega)$. By Assumption 2.7 (i) for ν, in the latter case no limit sell order with limit price above x is executed. Thus, we have M_μ-a.e.

$$\begin{aligned}
&\Delta^+\varphi^{0,n}_t(\omega)\\
=\;& -\overline{S}_t(\omega)\Delta^+M^{B,n}_t(\omega) + \underline{S}_t(\omega)\Delta^+M^{S,n}_t(\omega) + \int_x^\infty yL^{B,n}(\omega,t,dy)\\
&+\int_{\{t\}\times\mathbb{R}_+}\int_0^z yL^{S,n}(\omega,s,dy)\nu(\omega,ds,dz)\\
=\;& -\overline{S}_t(\omega)\Delta^+M^{B,n}_t(\omega) + \underline{S}_t(\omega)\Delta^+M^{S,n}_t(\omega) + \int_x^{x+\varepsilon/2} yL^{B,n}(\omega,t,dy)\\
&+\int_{x+\varepsilon/2}^\infty yL^{B,n}(\omega,t,dy) + \int_{\{t\}\times\mathbb{R}_+}\int_0^z yL^{S,n}(\omega,s,dy)\nu(\omega,ds,dz)\\
\le\;& -x\Delta^+M^{B,n}_t(\omega) + x\Delta^+M^{S,n}_t(\omega) - x\left(L^{B,n}(\omega,t,x) - L^{B,n}(\omega,t,x+\frac{\varepsilon}{2})\right)\\
&-\left(x+\frac{\varepsilon}{2}\right)L^{B,n}(\omega,t,x+\frac{\varepsilon}{2}) + x\int_{\{t\}\times\mathbb{R}_+} L^{S,n}(\omega,s,z)\nu(\omega,ds,dz)\\
=\;& x\left(-\Delta^+M^{B,n}_t(\omega) + \Delta^+M^{S,n}_t(\omega) - L^{B,n}(\omega,t,x) + \int_{\{t\}\times\mathbb{R}_+} L^{S,n}(\omega,s,z)\nu(\omega,ds,dz)\right)\\
&-\frac{\varepsilon}{2}L^{B,n}(\omega,t,x+\frac{\varepsilon}{2}).
\end{aligned}$$

Now (2.20) implies that for M_μ-a.a. $(\omega,t,x)\in N_\varepsilon$ the first term converges to $-x\Delta^+\psi^1_t(\omega)$ whereas for the second term we get

$$\liminf_{n\to\infty}\left(-\frac{\varepsilon}{2}L^{B,n}(\omega,t,x+\frac{\varepsilon}{2})\right)=-\infty$$

because for $(\omega,t,x)\in N_\varepsilon$ it always holds that $(\omega,t,x+\varepsilon/2)\in A$. Hence, M_μ-a.e. on N_ε (a set with measure $M_\mu(N_\varepsilon)>0$) this yields

$$\liminf_{n\to\infty}\Delta^+\varphi^{0,n}_t(\omega)=-\infty\neq\Delta^+\psi^0_t(\omega),$$

which is a contradiction to (2.18) (using Proposition 2.41 as above). Thus for all $\varepsilon>0$ it has to hold that $\widehat{M}_\mu(\widehat{N}_\varepsilon)=0$.

Therefore

$$M_\mu(A\cap B_\varepsilon)\le M_\mu(\widehat{N}_\varepsilon\times\mathbb{R}_+)=\widehat{M}_\mu(\widehat{N}_\varepsilon)=0.$$

Note that for $r\to\infty$ we get supergraph$(\underline{X}+\frac{1}{r})\uparrow$ supergraph$(\underline{X})$ and thus $A\cap$supergraph$(\underline{X}+\frac{1}{r})\uparrow A\cap$supergraph$(\underline{X})$, which yields

$$M_\mu(A\cap\text{supergraph}(\underline{X}))=\lim_{r\to\infty}M_\mu(A\cap B_{\frac{1}{r}})=0,$$

and hence

$$\begin{aligned}M_\mu(A) &= M_\mu(A\cap\text{supergraph}(\underline{X}))\\ &+ M_\mu(A\cap\text{subgraph}(\underline{X}))\\ &+ M_\mu(A\cap\text{graph}(\underline{X}))\\ &= 0,\end{aligned}$$

where the second term on the right hand side of the equation is equal to 0 by Proposition 2.40 and the last term is equal to 0 by Assumption 2.21. □

Using the two previous lemmas we are now able to show that the total number of purchased shares and the total number of sold shares (up to the stopping time $\widehat{\tau}^k$) are bounded in expectation under a probability measure $\widetilde{Q}$ equivalent to P.

Lemma 2.33. *There exists a probability measure $\widetilde{Q}$ equivalent to P s.t. for any stopping time $\widehat{\tau}^k$ as defined in Lemma 2.31 there exists a constant $\widetilde{K}_k$ s.t. for all $n\in\mathbb{N}$*

$$\begin{aligned}E_{\widetilde{Q}}\left[M^{B,n}_{\widehat{\tau}^k}+\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+}L^{B,n}(s,x)\mu(ds,dx)\right] &< \widetilde{K}_k,\\ E_{\widetilde{Q}}\left[M^{S,n}_{\widehat{\tau}^k}+\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+}L^{S,n}(s,x)\nu(ds,dx)\right] &< \widetilde{K}_k.\end{aligned}$$

Proof. For any $A \in \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}_+)$ define $\widetilde{\mu}(A) := \mu(A \cap \{(t,x) \in [0,T] \times \mathbb{R}_+ : x < \overline{S}_t\})$, $\mu^{\overline{S}}(A) = \mu(A \cap \{(t,x) \in [0,T] \times \mathbb{R}_+ : x = \overline{S}_t\})$. Clearly $\mu^{\overline{S}} \perp \widetilde{\mu}$ and by Assumption 2.7 (i) we furthermore know that $\mu = \widetilde{\mu} + \mu^{\overline{S}}$. Let $\widetilde{\nu}$ and $\nu^{\underline{S}}$ be defined similarly. Note that by Assumption 2.7 (iii) we get that $\widetilde{\mu}$ and $\widetilde{\nu}$ are P-a.s. finite measures.

An important observation regarding $\mu^{\overline{S}}$ and $\nu^{\underline{S}}$ is that the limit order executions that are triggered by these measures are at most as favorable to the investor as trading by market orders. This yields

$$\begin{aligned} &\int_{\{t\}\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{\overline{S}}(ds,dx) \\ \leq\quad & \frac{-\underline{S}_t}{\overline{S}_t - \underline{S}_t}\int_{\{t\}\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{\overline{S}}(ds,dx) - \frac{1}{\overline{S}_t - \underline{S}_t}\int_{\{t\}\times\mathbb{R}_+}\int_x^\infty yL^{B,n}(s,dy)\mu^{\overline{S}}(ds,dx) \end{aligned} \tag{2.21}$$

and

$$\frac{-\underline{S}_t}{\overline{S}_t - \underline{S}_t}\int_{\{t\}\times\mathbb{R}_+} L^{S,n}(s,x)\nu^{\underline{S}}(ds,dx) + \frac{1}{\overline{S}_t - \underline{S}_t}\int_{\{t\}\times\mathbb{R}_+}\int_0^x yL^{S,n}(s,dy)\nu^{\underline{S}}(ds,dx) \leq 0. \tag{2.22}$$

By rearranging the equations of the portfolio process to eliminate $\Delta^+ M_t^{S,n}$ we get

$$\begin{aligned} &\left(\overline{S}_t - \underline{S}_t\right)\Delta^+ M_t^{B,n} - \underline{S}_t\int_{\{t\}\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{\overline{S}}(ds,dx) - \int_{\{t\}\times\mathbb{R}_+}\int_x^\infty yL^{B,n}(s,dy)\mu^{\overline{S}}(ds,dx) \\ =\ & -\Delta^+\varphi_t^{0,n} - \underline{S}_t\Delta^+\varphi_t^{1,n} + \underline{S}_t\int_{\{t\}\times\mathbb{R}_+} L^{B,n}(s,x)\widetilde{\mu}(ds,dx) - \underline{S}_t\int_{\{t\}\times\mathbb{R}_+} L^{S,n}(s,x)\widetilde{\nu}(ds,dx) \\ &+\int_{\{t\}\times\mathbb{R}_+}\int_x^\infty yL^{B,n}(s,dy)\widetilde{\mu}(ds,dx) + \int_{\{t\}\times\mathbb{R}_+}\int_0^x yL^{S,n}(s,dy)\widetilde{\nu}(ds,dx) \\ &-\underline{S}_t\int_{\{t\}\times\mathbb{R}_+} L^{S,n}(s,x)\nu^{\underline{S}}(ds,dx) + \int_{\{t\}\times\mathbb{R}_+}\int_0^x yL^{S,n}(s,dy)\nu^{\underline{S}}(ds,dx). \end{aligned}$$

The lhs of this equation is an upper bound to the lhs of (2.23) by (2.21). The rhs of this equation is a lower bound to the rhs of (2.23) due to (2.22). Hence, the following inequality holds

$$\begin{aligned} &\Delta^+ M_t^{B,n} + \int_{\{t\}\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{\overline{S}}(ds,dx) \\ \leq\ & \frac{\Delta^+\mathrm{var}(\varphi^{0,n})_t + \underline{S}_t\left(\Delta^+\mathrm{var}(\varphi^{1,n})_t + \int_{\{t\}\times\mathbb{R}_+} L^{B,n}(s,x)\widetilde{\mu}(ds,dx)\right)}{\overline{S}_t - \underline{S}_t} \\ +\ & \frac{\int_{\{t\}\times\mathbb{R}_+}\int_0^x yL^{S,n}(s,dy)\widetilde{\nu}(ds,dx)}{\overline{S}_t - \underline{S}_t}. \end{aligned} \tag{2.23}$$

By a similar rearrangement of (the càdlàg part of) the portfolio process to get rid of (the càdlàg part of) $M^{S,n}$ as above (and by the associativity of the stochastic integral) we get the following inequality for the càdlàg part of $M^{B,n}$

$$M_{\widehat{\tau}^k}^{B,n} - \sum_{t<\widehat{\tau}^k}\Delta^+ M_t^{B,n} \ \leq\ \frac{\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \sup_{t\in[0,T]}\underline{S}_t\mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k}}{\inf_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)}.$$

Combining the càdlàg part with the right jumps, for all $n \in \mathbb{N}$ we get

$$\begin{aligned}
&M^{B,n}_{\widehat{\tau}^k} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx) \\
\leq\ & \frac{2\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + 2\sup_{t\in[0,T]}\underline{S}_t\mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + \sup_{t\in[0,T]}\underline{S}_t\int_{[0,T)\times\mathbb{R}_+}\sup_{n\in\mathbb{N}} L^{B,n}(s,x)\widetilde{\mu}(ds,dx)}{\inf_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)} \\
+\ & \frac{\sup_{t\in[0,T]}\overline{S}_t\int_{[0,T)\times\mathbb{R}_+}\sup_{n\in\mathbb{N}} L^{S,n}(s,x)\widetilde{\nu}(ds,dx)}{\inf_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)} \\
+\ & \int_{[0,T)\times\mathbb{R}_+}\sup_{n\in\mathbb{N}} L^{B,n}(s,x)\widetilde{\mu}(ds,dx).
\end{aligned}$$

By Lemma 2.32 we know that $\sup_{n\in\mathbb{N}} L^{B,n}$ is M_μ-a.e. finite and that $\sup_{n\in\mathbb{N}} L^{S,n}$ is M_ν-a.e. finite. Hence, because $\widetilde{\mu}$ and $\widetilde{\nu}$ have a.s. only finite mass, we conclude there exist a.s. finite, nonnegative random variables A, B and C s.t. for all $n \in \mathbb{N}$

$$\begin{aligned}
M^{B,n}_{\widehat{\tau}^k} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx) &\leq A\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + B\mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + C \\
&\leq (A+B+C)\left(\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + 1\right).
\end{aligned}$$

Similarly, we can show that there exist a.s. finite, nonnegative random variables D, E and F s.t. for all $n \in \mathbb{N}$

$$\begin{aligned}
M^{S,n}_{\widehat{\tau}^k} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{S,n}(s,x)\nu(ds,dx) &\leq D\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + E\mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + F \\
&\leq (D+E+F)\left(\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + 1\right).
\end{aligned}$$

By Lemma 2.31 we know that there exists a measure $Q \sim P$ (independent of k) and a constant $K_k > 0$ s.t. for all $n \in \mathbb{N}$

$$E_Q\left[\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + 1\right] \leq K_k + 1.$$

Because $Z := (A+B+C+D+E+F)$ is a.s. finite we can change the measure with density

$$\frac{d\widetilde{Q}}{dQ} = E_Q\left[(Z\vee 1)^{-1}\right]^{-1}(Z\vee 1)^{-1},$$

which satisfies $\frac{d\widetilde{Q}}{dQ}Z \leq E_Q\left[(Z\vee 1)^{-1}\right]^{-1} =: K'$. This yields for all $n \in \mathbb{N}$

$$\begin{aligned}
&E_{\widetilde{Q}}\left[\left(M^{B,n}_{\widehat{\tau}^k} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx)\right)\vee\left(M^{S,n}_{\widehat{\tau}^k} + \int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} L^{S,n}(s,x)\nu(ds,dx)\right)\right] \\
&\leq E_{\widetilde{Q}}\left[Z\left(\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + 1\right)\right] \\
&= E_Q\left[\frac{d\widetilde{Q}}{dQ}Z\left(\mathrm{var}(\varphi^{0,n})_{\widehat{\tau}^k} + \mathrm{var}(\varphi^{1,n})_{\widehat{\tau}^k} + 1\right)\right] \\
&\leq K'(K_k+1) =: \widetilde{K}_k.
\end{aligned}$$

□

Given the sequence of limit buy orders $(L^{B,n})_{n\in\mathbb{N}}$, later on we will use a Komlós-like result to gain a limit (of convex combinations) L^B. The following lemma will then be used to assure that the limit function provided by the Komlós-like result can be replaced by a proper limit buy order, i.e. by an element of $\mathcal{L}^B$.

Lemma 2.34. *If there exists a $\widetilde{\mathcal{P}}$-measurable function $L^B : \widetilde{\Omega} \to \overline{\mathbb{R}}_+$ s.t. $(L^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. towards L^B then there exists a $\widehat{L}^B \in \mathcal{L}^B$ s.t. $L^B = \widehat{L}^B$ holds $\widetilde{M}_\mu$-a.e..*

Similarly, if there exists a $\widetilde{\mathcal{P}}$-measurable function $L^S : \widetilde{\Omega} \to \overline{\mathbb{R}}_+$ s.t. $(L^{S,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\nu$-a.e. towards L^S then there exists a $\widehat{L}^S \in \mathcal{L}^S$ s.t. $L^S = \widehat{L}^S$ holds $\widetilde{M}_\nu$-a.e..

Proof. In the following we only deal with L^B, as the second part of the lemma can be shown analogously.

We start by constructing a $\widetilde{\mathcal{P}}$-measurable $\widehat{L}^B$ which is $\widetilde{M}_\mu$-a.e. equal to L^B and monotonically decreasing in x for all $(\omega,t) \in \Omega\times[0,T]$. Because $(L^{B,n})_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}^B$, for all $n\in\mathbb{N}$ and $(\omega,t)\in\Omega\times[0,T]$ we know that $x \mapsto L^{B,n}(\omega,t,x)$ is monotonically decreasing. Denote by $\widetilde{N}$ a $\widetilde{M}_\mu$-null set such that $(L^{B,n})_{n\in\mathbb{N}}$ converges pointwise to L^B on $\widetilde{N}^c$, by $(x_k)_{k\in\mathbb{N}}$ a sequence running through $\mathbb{Q}_+$, and by $\left(\widetilde{N}\cap\text{supergraph}(\underline{X})\right)_{x_k}$ the x_k-section of $\widetilde{N}\cap\text{supergraph}(\underline{X})$, i.e. the set $\left\{(\omega,t)\in\Omega\times[0,T] : (\omega,t,x_k)\in\widetilde{N}\cap\text{supergraph}(\underline{X})\right\}\in\mathcal{P}$. Define

$$\overline{N} := \bigcup_{k=1}^{\infty}\left(\widetilde{N}\cap\text{supergraph}(\underline{X})\right)_{x_k}$$

and note that from $\left(\widetilde{N}\cap\text{supergraph}(\underline{X})\right)_{x_k}\times\{x_k\}\subset\widetilde{N}\cap\text{supergraph}(\underline{X})$ and

$$\left(\left(\widetilde{N}\cap\text{supergraph}(\underline{X})\right)_{x_k}\times\{x_k\}\right)\cap\text{supergraph}(\underline{X}) = \left(\widetilde{N}\cap\text{supergraph}(\underline{X})\right)_{x_k}\times\{x_k\}$$

it follows that $\widehat{M}_\mu\left(\left(\widetilde{N}\cap\text{supergraph}(\underline{X})\right)_{x_k}\right) = 0$ and thus also $\widehat{M}_\mu(\overline{N}) = 0$, i.e. $\check{M}_\mu(\overline{N}\times\mathbb{R}_+) = 0$. Hence, we arrive at $\widetilde{M}_\mu(\overline{N}\times\mathbb{R}_+) = 0$. By definition of $\overline{N}$, for all $(\omega,t,x)\in\left(\overline{N}\times\mathbb{R}_+\right)^c\cap$ supergraph$(\underline{X})\cap(\Omega\times[0,T]\times\mathbb{Q}_+)$ we know that $\left(L^{B,n}(\omega,t,x)\right)_{n\in\mathbb{N}}$ converges to $L^B(\omega,t,x)$. We proceed by defining the function $\widehat{L}^B$ by

$$\begin{aligned}\widehat{L}^B(\omega,t,x) &:= \infty 1_{\{x\le\underline{X}_t(\omega)\}}1_{\overline{N}^c}(\omega,t)\\ &+ \text{median}\left\{L^B(\omega,t,x), \sup_{x<x_k}L^B(\omega,t,x_k), \inf_{x_k<x}L^B(\omega,t,x_k)\right\}1_{\{x>\underline{X}_t(\omega)\}}1_{\overline{N}^c}(\omega,t),\end{aligned}$$

which is obviously $\widetilde{\mathcal{P}}$-measurable (remember Definition and Proposition 2.6). Given two points (ω,t,x) and (ω,t,y) with $x<y$ for which both $L^{B,n}(\omega,t,x)\to L^B(\omega,t,x)$ and $L^{B,n}(\omega,t,y)\to L^B(\omega,t,y)$ hold for $n\to\infty$ we get

$$L^B(\omega,t,x)=\lim_{n\to\infty}L^{B,n}(\omega,t,x)\geq\lim_{n\to\infty}L^{B,n}(\omega,t,y)=L^B(\omega,t,y). \tag{2.24}$$

Hence, for $(\omega,t,x)\in\widetilde{N}^c\cap(\overline{N}\times\mathbb{R}_+)^c\cap\text{supergraph}(\underline{X})$ it holds that

$$\sup_{x<x_k}L^B(\omega,t,x_k)\leq L^B(\omega,t,x)\leq\inf_{x_k<x}L^B(\omega,t,x_k),$$

and thus $\widehat{L}^B=L^B$ on $\widetilde{N}^c\cap(\overline{N}\times\mathbb{R}_+)^c\cap\text{supergraph}(\underline{X})$. By Assumption 2.21, Proposition 2.40 and the construction of $\widetilde{M}_\mu$ we already know that $\text{subgraph}(\underline{X})\cup\text{graph}(\underline{X})$ is a $\widetilde{M}_\mu$-null set. Hence, the set of points on which we set $\widehat{L}^B$ to the value ∞ is in any case not relevant for the question whether $L^B=\widehat{L}^B$ $\widetilde{M}_\mu$-a.e. or not (though it does play a role to assure monotonicity of course, which is supposed to hold for *all* $(\omega,t)\in\Omega\times[0,T]$). Furthermore, we have seen above that $\widetilde{M}_\mu(\overline{N}\times\mathbb{R}_+)=0$ and consequently $L^B=\widehat{L}^B$ holds $\widetilde{M}_\mu$-almost everywhere.

Let us verify that $x\mapsto\widehat{L}^B(\omega,t,x)$ is indeed monotonically decreasing for all $(\omega,t)\in\Omega\times[0,T]$, which is part (i) of Definition 2.9. By definition of $\widehat{L}^B$ via median and since $\sup_{x<x_k}L^B(\omega,t,x_k)\leq\inf_{x_k<x}L^B(\omega,t,x_k)$ on $\left(\overline{N}\times\mathbb{R}_+\right)^c\cap\text{supergraph}(\underline{X})$ by (2.24) we get

$$\sup_{x<x_k}L^B(\omega,t,x_k)\leq\widehat{L}^B(\omega,t,x)\leq\inf_{x_k<x}L^B(\omega,t,x_k).$$

This yields for all $(\omega,t,x),(\omega,t,y)\in\left(\overline{N}\times\mathbb{R}_+\right)^c\cap\text{supergraph}(\underline{X})$ with $x<y$ that

$$\widehat{L}^B(\omega,t,x)\geq\sup_{x<x_k}L^B(\omega,t,x_k)\geq\inf_{x_k<y}L^B(\omega,t,x_k)\geq\widehat{L}^B(\omega,t,y).$$

Moreover, for all $(\omega,t)\in\overline{N}^c$ we have that $\widehat{L}^B(\omega,t,x)=\infty$ for all $x\leq\underline{X}_t(\omega)$. Therefore, the monotonicity of $x\mapsto\widehat{L}^B(\omega,t,x)$ on $\mathbb{R}_+$ is established for all $(\omega,t)\in\overline{N}^c$. For $(\omega,t)\in\overline{N}$ we have $\widehat{L}^B(\omega,t,\cdot)\equiv0$ and the monotonicity is trivially satisfied.

We proceed by checking that part (iii) of Definition 2.9 holds, i.e. that $\widehat{L}^B$ is μ-integrable. By the $\widetilde{M}_\mu$-a.e. convergence of $(L^{B,n})_{n\in\mathbb{N}}$ to $\widehat{L}^B$, Fatou's lemma, and Lemma 2.33 there exist a measure $\widetilde{Q}\sim P$ s.t. for all $k\in\mathbb{N}$ there exists a constant $\widetilde{K}_k>0$ s.t.

$$\begin{aligned}&E_{\widetilde{Q}}\left[\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+}\widehat{L}^B(t,x)\mu(dt,dx)\right]\\ \leq\ &\liminf_{n\to\infty}E_{\widetilde{Q}}\left[\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+}L^{B,n}(t,x)\mu(dt,dx)\right]\\ <\ &\widetilde{K}_k,\end{aligned}$$

where $\widehat{\tau}^k$ refers to the stopping time defined in Lemma 2.31. Hence, for all $k \in \mathbb{N}$

$$\int_{[0,\widehat{\tau}^k)\times\mathbb{R}_+} \widehat{L}^B(t,x)\mu(dt,dx) < \infty, \qquad P\text{-a.s..}$$

Because $(\widehat{\tau}^k)_{k\in\mathbb{N}}$ is a localizing sequence, for almost all ω there exists a finite $k_0(\omega)$ s.t. $\widehat{\tau}^{k_0}(\omega) = T$. Therefore,

$$\int_{[0,T)\times\mathbb{R}_+} \widehat{L}^B(t,x)\mu(dt,dx) < \infty, \qquad P\text{-a.s.}$$

and the μ-integrability of $\widehat{L}^B$ is verified.

Regarding Definition 2.9 (ii), i.e. if $\widehat{L}^B(\omega,t,x) = 0$ for all $(\omega,t) \in \Omega\times[0,T]$ and $x \geq \overline{S}_{t-}(\omega)$, note that we may assume that L^B satisfies this property without loss of generality, because for all $n \in \mathbb{N}$ we have $L^{B,n}(\omega,t,x)1_{\{x<\overline{S}_{t-}(\omega)\}} = L^{B,n}(\omega,t,x)$. As $(L^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. to L^B it follows that $(L^{B,n})_{n\in\mathbb{N}}$ also converges $\widetilde{M}_\mu$-a.e. to $L^B(\omega,t,x)1_{\{x<\overline{S}_{t-}(\omega)\}}$. Now, if $\widehat{L}^B$ as defined above does not satisfy this property as well, we just replace it by $\widehat{L}^B(\omega,t,x)1_{\{x<\overline{S}_{t-}(\omega)\}}$, which is still $\widetilde{M}_\mu$-a.e. equal to L^B by the assumption just made, i.e. that $L^B(\omega,t,x) = L^B(\omega,t,x)1_{\{x<\overline{S}_{t-}(\omega)\}}$. Note that if Definition 2.9 (i) and (iii) hold for $\widehat{L}^B$ this is still true for $\widehat{L}^B(\omega,t,x)1_{\{x<\overline{S}_{t-}(\omega)\}}$, so we do not invalidate any of the facts established above. □

We have already seen in Lemma 2.33 that the total number of purchased shares is locally bounded in expectation under a probability measure $\widetilde{Q}$ equivalent to P. Hence, the total number of purchased shares up to time T is stochastically bounded. The following lemma uses this fact to show that the stochastic processes describing the total number of purchased shares are a Cauchy sequence.

Lemma 2.35. *Both the total number of purchased shares* $\left(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx)\right)_{n\in\mathbb{N}}$ *and the total number of sold shares* $\left(M^{S,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{S,n}(s,x)\nu(ds,dx)\right)_{n\in\mathbb{N}}$ *are Cauchy sequences w.r.t. the convergence "uniformly in probability" (up).*

Proof. Of course it is sufficient to prove only the first part of the assertion as the second one is completely analog. Assume that $(\varphi^{0,n},\varphi^{1,n})_{n\in\mathbb{N}}$ is an up-Cauchy sequence.

Step 1: Let us consider the corresponding discounted wealth processes if stock positions are evaluated at the best-bid price $\underline{S}$ and the numeraire is the spread $\overline{S}-\underline{S}$, i.e.

$$\widehat{V}^n := \frac{\varphi^{0,n}}{\overline{S}-\underline{S}} + \frac{\varphi^{1,n}\underline{S}}{\overline{S}-\underline{S}}.$$

The stock evaluation and the choice of the numeraire simplify the calculations. Namely, sales by market orders do not change the wealth process and the purchase of one share by a market order reduces the discounted wealth by one unit. Note that $(\widehat{V}^n)_{n\in\mathbb{N}}$ is again up-Cauchy and the processes $\frac{1}{\overline{S}-\underline{S}}$ and $\frac{\underline{S}}{\overline{S}-\underline{S}}$ are again semimartingales by $P\Big(\inf\{\overline{S}_t-\underline{S}_t \mid t\in[0,T]\}>0\Big)=1$ and Itô's formula. By Definition 2.11 and Lemma 8.2 in [MK09] we obtain

$$\begin{aligned}
&\widehat{V}^n = \widehat{V}_0^n + \varphi^{0,n}\bullet\left(\frac{1}{\overline{S}-\underline{S}}\right) + \varphi^{1,n}\bullet\left(\frac{\underline{S}}{\overline{S}-\underline{S}}\right) - M^{B,n}\\
&+\int_{[0,\cdot)\times\mathbb{R}_+}\int_{[x,\overline{S}_{s-})}\frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s}L^{B,n}(s,dy)\mu(ds,dx)+\int_{[0,\cdot)\times\mathbb{R}_+}\int_{(\underline{S}_{s-},x]}\frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s}L^{S,n}(s,dy)\nu(ds,dx).
\end{aligned}$$

Note that $L^{B,n}(s,x)=0$ for $x\geq\overline{S}_{s-}$ and $L^{S,n}(s,x)=0$ for $x\leq\underline{S}_{s-}$. Let $\mu=\mu^{1,\delta}+\mu^{2,\delta}$ be the decomposition from (2.9). In the following, executed limit buy orders with limit price near to the best-ask are charged at the best-ask. The process $A^{\delta,n}$ is the corresponding error term and formally defined by

$$\begin{aligned}
&\int_{[0,t)\times\mathbb{R}_+}\int_{[x,\overline{S}_{s-})}\frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s}L^{B,n}(s,dy)\mu(ds,dx)\\
&=\int_{[0,t)\times\mathbb{R}_+}\int_{[\overline{S}_{s-}-\delta,\overline{S}_{s-})}\frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s}L^{B,n}(s,dy)\mu^{1,\delta}(ds,dx)\\
&\quad+\int_{[0,t)\times\mathbb{R}_+}\int_{[x,\overline{S}_{s-})}\frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s}L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx)\\
&=-\int_{[0,t)\times\mathbb{R}_+}L^{B,n}(s,x)\,\mu^{1,\delta}(ds,dx)+\int_{[0,t)\times\mathbb{R}_+}\int_{[x,\overline{S}_{s-})}\frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s}L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx)+A_t^{\delta,n}.
\end{aligned}$$

$A^{\delta,n}$ is nonincreasing and

$$|A_T^{\delta,n}|\leq\frac{\delta\int_{[0,T)\times\mathbb{R}_+}L^{B,n}(s,x)\mu(ds,dx)}{\inf\{\overline{S}_t-\underline{S}_t \mid t\in[0,T]\}}. \tag{2.25}$$

Analogously, we define $\nu^{1,\delta},\nu^{2,\delta}$ by $\nu=\nu^{1,\delta}+\nu^{2,\delta}$, $\nu^{1,\delta}\perp\nu^{2,\delta}$, and $\nu^{1,\delta}(\{t\}\times\{x\})=1$ iff $x=\underline{S}_t$ and $\Delta\underline{S}_t\in(0,\delta]$. Again, $\nu^{2,\delta}$ is a finite random measure. The process $B^{\delta,n}$ is the error term when limit sell orders with limit price near to the best-bid are charged at the best-bid. Formally, it is defined by

$$\begin{aligned}
&\int_{[0,t)\times\mathbb{R}_+}\int_{(\underline{S}_{s-},x]}\frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s}L^{S,n}(s,dy)\nu(ds,dx)\\
&=\int_{[0,t)\times\mathbb{R}_+}\int_{(\underline{S}_{s-},x]}\frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s}L^{S,n}(s,dy)\nu^{2,\delta}(ds,dx)+B_t^{\delta,n}.
\end{aligned}$$

$B^{\delta,n}$ is nonincreasing and

$$|B_T^{\delta,n}|\leq\frac{\delta\int_{[0,T)\times\mathbb{R}_+}L^{S,n}(s,x)\nu(ds,dx)}{\inf\{\overline{S}_t-\underline{S}_t \mid t\in[0,T]\}}. \tag{2.26}$$

We arrive at

$$\begin{aligned}\widehat{V}^n \quad = \quad & \widehat{V}_0^n + \varphi^{0,n} \bullet \left(\frac{1}{\overline{S}-\underline{S}}\right) + \varphi^{1,n} \bullet \left(\frac{\underline{S}}{\overline{S}-\underline{S}}\right) - M^{B,n} - \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx) \\ & + \int_{[0,\cdot)\times\mathbb{R}_+} \int_{[x,\overline{S}_{s-})} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx) \\ & + \int_{[0,\cdot)\times\mathbb{R}_+} \int_{(\underline{S}_{s-},x]} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{S,n}(s,dy)\nu^{2,\delta}(ds,dx) + A^{\delta,n} + B^{\delta,n}\end{aligned}$$

and thus

$$\begin{aligned}& M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx) \\ & = -\widehat{V}^n + \widehat{V}_0^n + \varphi^{0,n} \bullet \left(\frac{1}{\overline{S}-\underline{S}}\right) + \varphi^{1,n} \bullet \left(\frac{\underline{S}}{\overline{S}-\underline{S}}\right) + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{2,\delta}(ds,dx) \\ & + \int_{[0,\cdot)\times\mathbb{R}_+} \int_{[x,\overline{S}_{s-})} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx) \\ & + \int_{[0,\cdot)\times\mathbb{R}_+} \int_{(\underline{S}_{s-},x]} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} L^{S,n}(s,dy)\nu^{2,\delta}(ds,dx) + A^{\delta,n} + B^{\delta,n}. \end{aligned} \tag{2.27}$$

Step 2: Now let $\varepsilon > 0$. As $\widehat{V}^n$, $\widehat{V}_0^n$, $\varphi^{0,n} \bullet \left(\frac{1}{\overline{S}-\underline{S}}\right)$, and $\varphi^{1,n} \bullet \left(\frac{\underline{S}}{\overline{S}-\underline{S}}\right)$ are up-Cauchy sequences, there exists a $n_1 \in \mathbb{N}$ s.t.

$$\begin{aligned}& P\left(\left|-(\widehat{V}_t^n - \widehat{V}_t^m) + (\widehat{V}_0^n - \widehat{V}_0^m) + (\varphi^{0,n} - \varphi^{0,m}) \bullet \left(\frac{1}{\overline{S}-\underline{S}}\right)_t + (\varphi^{1,n} - \varphi^{1,m}) \bullet \left(\frac{\underline{S}}{\overline{S}-\underline{S}}\right)_t\right|\right. \\ & \left.\le \frac{\varepsilon}{4},\ \forall t \in [0,T]\right) \ge 1 - \frac{\varepsilon}{4}, \quad \forall n,m \ge n_1. \end{aligned} \tag{2.28}$$

By Lemma 2.33 the sequences $\left(\int_{[0,T]\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx)\right)_{n\in\mathbb{N}}$ and $\left(\int_{[0,T]\times\mathbb{R}_+} L^{S,n}(s,x)\nu(ds,dx)\right)_{n\in\mathbb{N}}$ are stochastically bounded. Thus, by (2.25) and (2.26), there exists a $\delta > 0$ s.t.

$$P\left(|A_T^{\delta,n} + B_T^{\delta,n}| \le \frac{\varepsilon}{4}\right) \ge 1 - \frac{\varepsilon}{4}, \quad \forall n \in \mathbb{N}. \tag{2.29}$$

We fix this δ. As $\mu^{2,\delta}$ and $\nu^{2,\delta}$ are finite random measures the remaining terms on the rhs of (2.27) are up-Cauchy sequences by Lemma 2.32 and Lemma 2.29. Thus there exists a $n_2 \in \mathbb{N}$ s.t.

$$\begin{aligned}& P\left(\left|\int_{[0,t)\times\mathbb{R}_+} \left(L^{B,n}(s,x) - L^{B,m}(s,x)\right)\mu^{2,\delta}(ds,dx)\right.\right. \\ & + \int_{[0,t)\times\mathbb{R}_+} \int_{[x,\overline{S}_{s-})} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} (L^{B,n} - L^{B,m})(s,dy)\mu^{2,\delta}(ds,dx) \\ & \left.\left.+ \int_{[0,t)\times\mathbb{R}_+} \int_{(\underline{S}_{s-},x]} \frac{y-\underline{S}_s}{\overline{S}_s-\underline{S}_s} (L^{S,n} - L^{S,m})(s,dy)\nu^{2,\delta}(ds,dx)\right| \le \frac{\varepsilon}{4},\ \forall t \in [0,T]\right) \\ & \ge 1 - \frac{\varepsilon}{4},\end{aligned}$$

for all $n,m \geq n_2$. Combining this with (2.28), (2.29), and (2.27), we arrive at

$$P\left(\left|M_t^{B,n} + \int_{[0,t)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx) - M_t^{B,m}\right.\right.$$
$$\left.\left. - \int_{[0,t)\times\mathbb{R}_+} L^{B,m}(s,x)\mu(ds,dx)\right| \leq \varepsilon,\ \forall t \in [0,T]\right) \geq 1-\varepsilon$$

for all $n,m \geq n_1 \vee n_2$. Thus $(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L_s^{B,n} d\mu_s)_{n\in\mathbb{N}}$ is a Cauchy sequence w.r.t. the convergence "uniformly in probability". □

Proof of Theorem 2.22. Our goal is to find a *limit strategy* $\mathfrak{S} = (M^B, M^S, L^B, L^S)$ which satisfies $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S})) = (\psi^0, \psi^1)$, where (ψ^0, ψ^1) is the predictable làdlàg limit process of $(\varphi^{0,n}, \varphi^{1,n})$ introduced at the beginning of this section. Let us deal with the limit orders first. We apply Lemma 9.8.1 (which is a Komlós-like theorem) and Remark 9.8.2 in Delbaen and Schachermayer [DS06] twice (first w.r.t. the limit buy orders and measure $\widetilde{M}_\mu$, then w.r.t. the limit sell orders and measure $\widetilde{M}_\nu$, where we build convex combinations of the convex combinations chosen for the limit buy orders), which yields that there exist $\widetilde{\mathcal{P}}$-measurable $\overline{\mathbb{R}}_+$-valued functions L^B and L^S and a sequence of (finite) convex combinations $\widehat{\mathfrak{S}}^n \in \operatorname{conv}(\mathfrak{S}^n, \mathfrak{S}^{n+1}, \ldots)$ such that $(\widehat{L}^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. to L^B and $(\widehat{L}^{S,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\nu$-a.e. to L^S. Note that by a convex combination of strategies $\mathfrak{S}^n$ we mean a quadruple $(\widehat{M}^{B,n}, \widehat{M}^{S,n}, \widehat{L}^{B,n}, \widehat{L}^{S,n})$ where $\widehat{M}^{B,n} \in \operatorname{conv}\{M^{B,n}, M^{B,n+1}, \ldots\}$ and so forth, where we use the same weights for $\widehat{M}^{B,n}, \widehat{M}^{S,n}, \widehat{L}^{B,n}$, and $\widehat{L}^{S,n}$. The associated portfolio process of a finite convex combination of trading strategies is just the convex combination of the respective associated portfolio processes. This is due to the linearity of the various integrals in Definition 2.11. Hence, a convex combination of a-admissible trading strategies is again a-admissible. Since the convex combinations were taken of trading strategies for which $\left(\varphi^0(\mathfrak{S}^n), \varphi^1(\mathfrak{S}^n)\right)_{n\in\mathbb{N}}$ converges P-a.s. uniformly on $[0,T]$ to (ψ^0, ψ^1) this also holds for $(\widehat{\mathfrak{S}}^n)_{n\in\mathbb{N}}$. Thus, we can assume that w.l.o.g. already the original sequence $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ satisfies

$$L^{B,n} \to L^B, \qquad \widetilde{M}_\mu\text{-a.e.} \qquad \text{and} \qquad L^{S,n} \to L^B, \qquad \widetilde{M}_\nu\text{-a.e.},$$

as $n \to \infty$. Then, we may apply Lemma 2.34 and obtain that w.l.o.g. $L^B \in \mathcal{L}^B$ and $L^S \in \mathcal{L}^S$. Given L^B and L^S we are now in the position to present the market order part of our guess for a limit strategy. By Lemma 2.35 there exist predictable increasing processes Y and Z s.t.

$$Y = \lim_{n\to\infty}\left(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu(ds,dx)\right), \tag{2.30}$$
$$Z = \lim_{n\to\infty}\left(M^{S,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{S,n}(s,x)\nu(ds,dx)\right),$$

w.r.t. the convergence "uniformly in probability". This suggests the following definition

$$
\begin{aligned}
M_t^B &:= Y_t - \int_{[0,t)\times\mathbb{R}_+} L^B(s,x)\mu(ds,dx),\\
M_t^S &:= Z_t - \int_{[0,t)\times\mathbb{R}_+} L^S(s,x)\nu(ds,dx).
\end{aligned}
$$

So now we have found a candidate $\mathfrak{S} := (M^B, M^S, L^B, L^S)$ for our limit strategy. To make sure that $\mathfrak{S}$ satisfies Definition 2.10 we need to verify that M^B and M^S are predictable increasing processes. Because Y and Z are predictable, the predictability is immediate from the fact that for any t we integrate only up to $t-$ in the definition above.

We do need to check though, that M^B and M^S are increasing. To avoid repeating ourselves, we only examine M^B. Remember that $(L^{B,n})_{n\in\mathbb{N}}$ converges $\widetilde{M}_\mu$-a.e. to L^B. Thus, P-a.e. $\left(L^{B,n}(\omega)\right)_{n\in\mathbb{N}}$ converges μ_ω-a.e. to $L^B(\omega)$. In addition, the convergence in (2.30) holds P-a.s. uniformly on $[0,T]$ for a subsequence. Let $A \in \mathcal{F}$ be the combined exceptional null set and $\omega \in A^c$. Now let $0 \le t_1 \le t_2 \le T$ and an application of Fatou's lemma yields that

$$
\begin{aligned}
M_{t_2}^B(\omega) - M_{t_1}^B(\omega) &\ge \liminf_{n\to\infty}\left(M_{t_2}^{B,n}(\omega) - M_{t_1}^{B,n}(\omega) + \int_{[t_1,t_2)\times\mathbb{R}_+} L^{B,n}(\omega,s,x)\mu(\omega,ds,dx)\right)\\
&\quad - \int_{[t_1,t_2)\times\mathbb{R}_+} L^B(\omega,s,x)\mu(\omega,ds,dx)\\
&\ge \liminf_{n\to\infty}\left(\int_{[t_1,t_2)\times\mathbb{R}_+} L^{B,n}(\omega,s,x)\mu(\omega,ds,dx)\right)\\
&\quad - \int_{[t_1,t_2)\times\mathbb{R}_+} L^B(\omega,s,x)\mu(\omega,ds,dx)\\
&\ge 0.
\end{aligned}
$$

Therefore, the candidate $\mathfrak{S} = (M^B, M^S, L^B, L^S)$ for our limit strategy is a valid trading strategy in the sense of Definition 2.10.

All that is left is to check whether it yields the right portfolio process, i.e. $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S})) = (\psi^0, \psi^1)$, and that $\mathfrak{S}$ is a-admissible. We defer the question of admissibility for the moment, because it follows easily when we can be sure that the trading strategy $\mathfrak{S}$ has the associated portfolio process we are looking for. Right from the definition of Y and Z and Lemma 2.35 we get that $\psi^1 = \eta^1 + Y + Z = \varphi^1(\mathfrak{S})$, so we only have to verify that $\varphi^0(\mathfrak{S}) = \psi^0$.

Main step: Let us show that $\varphi^{0,n} \to \varphi^0$ uniformly in probability where $\varphi^0 := \varphi(\mathfrak{S})$. If we are able to show the convergence for the buy and the sell order terms separately, we are done. The idea is to account executed limit buy orders with limit prices close to the best-ask as market orders (in the limit they can indeed turn into market orders as Example 2.36 shows, by contrast, executed limit orders "away" from the best-ask price remain limit orders in the limit strategy as there are only finitely many execution times). For $\delta > 0$ let $\mu = \mu^{1,\delta} + \mu^{2,\delta}$ be the decomposition from (2.9). We have

$$\begin{aligned}&\left| \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B,n}(s,dy)\mu^{1,\delta}(ds,dx) - \int_{[0,\cdot)\times\mathbb{R}_+} \overline{S}_{s-} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx)\right| \\ &\leq \delta \int_{[0,T)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx), \quad \forall n \in \mathbb{N}. \end{aligned} \tag{2.31}$$

Let $\varepsilon > 0$. It follows from Lemma 2.33 that $\left(\int_{[0,T)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx)\right)_{n\in\mathbb{N}}$ is P-stochastically bounded. Together with $P(\int_{[0,T)\times\mathbb{R}_+} L^{B}(s,x)\mu^{1,\delta}(ds,dx) < \infty) = 1$, we derive the existence of a $\delta > 0$ s.t.

$$d_{\text{up}}\left(\int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B,n}(s,dy)\mu^{1,\delta}(ds,dx), (\overline{S}_-,\overline{S}) \bullet \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx)\right) \leq \frac{\varepsilon}{4}$$

for all $n \in \mathbb{N}$ and

$$d_{\text{up}}\left(\int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B}(s,dy)\mu^{1,\delta}(ds,dx), (\overline{S}_-,\overline{S}) \bullet \int_{[0,\cdot)\times\mathbb{R}_+} L^{B}(s,x)\mu^{1,\delta}(ds,dx)\right) \leq \frac{\varepsilon}{4}.$$

We fix this δ. By Lemma 2.32 and Lemma 2.29 applied to $\mu^{2,\delta}$, there exists a $n_1 \in \mathbb{N}$ with

$$d_{\text{up}}\left(\int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B,n}(s,dy)\mu^{2,\delta}(ds,dx), \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B}(s,dy)\mu^{2,\delta}(ds,dx)\right) \leq \frac{\varepsilon}{4}, \quad \forall n \geq n_1.$$

By Lemma 2.32 and Proposition 2.42 applied to $\mu^{2,\delta}$, we know that $\left(\int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{2,\delta}(ds,dx)\right)_{n\in\mathbb{N}}$ converges to $\int_{[0,\cdot)\times\mathbb{R}_+} L^{B}(s,x)\mu^{2,\delta}(ds,dx)$ uniformly in probability. This implies by definition of M^B that $\left(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx)\right)_{n\in\mathbb{N}}$ converges to $M^B + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B}(s,x)\mu^{1,\delta}(ds,dx)$ uniformly in probability. From Lemma 2.33 and Proposition 2.43 it follows the existence of a n_2 s.t.

$$\begin{aligned} d_{\text{up}}\Bigg(&(\overline{S}_-,\overline{S}) \bullet \left(M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B,n}(s,x)\mu^{1,\delta}(ds,dx)\right), \\ &(\overline{S}_-,\overline{S}) \bullet \left(M^{B} + \int_{[0,\cdot)\times\mathbb{R}_+} L^{B}(s,x)\mu^{1,\delta}(ds,dx)\right)\Bigg) \leq \frac{\varepsilon}{4}. \end{aligned}$$

Finally, we obtain by the triangle inequality

$$\begin{aligned} d_{\text{up}}\Bigg(&(\overline{S}_-,\overline{S}) \bullet M^{B,n} + \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B,n}(s,dy)\mu(ds,dx), \\ &(\overline{S}_-,\overline{S}) \bullet M^{B} + \int_{[0,\cdot)\times\mathbb{R}_+} \int_x^{\overline{S}_{s-}} y L^{B}(s,dy)\mu(ds,dx)\Bigg) \leq \varepsilon. \end{aligned}$$

As the corresponding result holds for the sell orders we obtain that $\varphi^{0,n} \to \varphi^0$ uniformly in probability.

To conclude the proof, we only have to verify that $\mathfrak{S}$ is an a-admissible trading strategy, i.e. that its portfolio process satisfies the inequality in Definition 2.2. This follows from the fact that ψ is the limit (P-a.s. uniformly on $[0,T]$) of the sequence of a-admissible portfolio processes $(\varphi(\mathfrak{S}^n))_{n\in\mathbb{N}}$. To see this, note that while for any sequence $(a_n)_n \subset \mathbb{R}$ that converges towards $a \in \mathbb{R}$ we cannot be sure that $(1_{\{a_n \geq 0\}})_{n\in\mathbb{N}}$ converges towards $(1_{\{a\geq 0\}})_{n\in\mathbb{N}}$ we nevertheless know that for any $\alpha, \beta \in \mathbb{R}$

$$\alpha a_n 1_{\{a_n \geq 0\}} + \beta a_n 1_{\{a_n < 0\}} \to \alpha a 1_{\{a\geq 0\}} + \beta a 1_{\{a<0\}}, \qquad \text{as } n \to \infty,$$

which is all we need.

□

2.7 Examples

We give an example of a sequence of limit buy order strategies whose portfolio processes converge to the portfolio process of a market buy order strategy. An inspection of the proof of Theorem 2.17 reveals that this phenomenon cannot occur if the execution measures μ and ν are finite.

Example 2.36. *Assume that X is a Lévy process with infinitely many downward jumps, i.e. $\lim_{n\to\infty} \overline{\mu}((-\infty,-1/n]) = \overline{\mu}((-\infty,0)) = \infty$, where $\overline{\mu}$ is the Lévy measure of X. Now let us suppose that the best-ask price $\overline{S}$ is modeled as exponential-Lévy, i.e. $\overline{S}_t = \overline{S}_0 \exp(X_t)$. Consider the limit buy order strategies satisfying*

$$L^{B,n}(t,x) \;=\; (t-\varphi^{1,n}_t)1_{\left\{x\leq \overline{S}_{t-}-\frac{1}{n}\overline{S}_{t-}\right\}} \quad \textit{where } \varphi^{1,n}_0 = \varphi^{0,n}_0 = 0,$$

i.e. limit prices are slightly below the best-ask price and directly after a successful execution at time t the total number of bought assets is t (that is $\varphi^{1,n}_t = t$). $L^{B,n}$ and $\varphi^{1,n}$ are obviously well-defined with $0 \leq \varphi^{1,n}_t \leq t$ as for every n and every path there are only finitely many executions.

Let us show that the associated portfolio processes $(\varphi^{0,n}, \varphi^{1,n})_{n\in\mathbb{N}}$ converge to $\varphi^0_t := -\int_0^t \overline{S}_s\, ds$ and $\varphi^1_t := t$ uniformly in probability. (φ^0, φ^1) is generated by the market buy order strategy $M^B_t = t$. Let $Z_1, \ldots, Z_m$ be i.i.d. exponential random variables with parameter 1. A well-known limit result for maxima tells us that

$$P\left(Z_k \leq \ln(m) + x,\ k = 1,\ldots,m\right) \;\to\; \exp(-\exp(-x)), \quad m\to\infty, \quad \forall x \in \mathbb{R}. \tag{2.32}$$

The interarrival times of jumps $\Delta X \le \ln(1-1/n)$ *are i.i.d. exponentially distributed random variables with parameter* $\overline{\mu}\Big((-\infty,\ln(1-1/n)]\Big)$ *and thus we have for any* $\varepsilon > 0$

$$
\begin{aligned}
&P(\text{“time between two successive executions of } L^{B,n} \text{ is always smaller than } \varepsilon\text{”}) \\
\ge\ & P\left(Z_k < \varepsilon\overline{\mu}\Big((-\infty,\ln(1-1/n)]\Big),\ k=1,\dots,\left[2T\overline{\mu}\Big((-\infty,\ln(1-1/n)]\Big)\right]\right) \qquad (2.33)\\
&- P\left(\sum_{k=1}^{[2T\overline{\mu}((-\infty,\ln(1-1/n)])]} Z_k < T\overline{\mu}\Big((-\infty,\ln(1-1/n)]\Big)\right),
\end{aligned}
$$

where $[x] := \max\{k \in \mathbb{N}_0 : k \le x\}$. *Put* $m := \left[2T\overline{\mu}\Big((-\infty,\ln(1-1/n)]\Big)\right]$. *By (2.33), (2.32), the law of large numbers, and the fact that* $\left(\overline{\mu}\Big((-\infty,\ln(1-1/n)]\Big)\right)_{n\in\mathbb{N}}$ *converges faster to infinity than* $\left(\ln\left(2T\overline{\mu}\Big((-\infty,\ln(1-1/n)]\Big)\right)\right)_{n\in\mathbb{N}}$, *it follows that for any* $\varepsilon > 0$ *there exists an* n_0 *s.t. for all* $n \ge n_0$

$$
\begin{aligned}
&P\Big(|\varphi_t^{1,n} - \varphi_t^1| \le \varepsilon, \quad \forall t\in[0,T]\Big) \qquad (2.34)\\
\ge\ & P(\text{“time between two successive executions of } L^{B,n} \text{ is always smaller than } \varepsilon\text{”}) \ge 1-\varepsilon.
\end{aligned}
$$

It remains to show that $(\varphi^{0,n})_{n\in\mathbb{N}}$ *converges to* φ^0. *We have that* $\operatorname{var}(\varphi^{1,n})_T \le T$ *for all* $n \in \mathbb{N}$. *Consequently, we can apply Proposition 2.43 and conclude that* $((\overline{S}_-,\overline{S}) \bullet \varphi^{1,n})_{n\in\mathbb{N}}$ *converges to* $(\overline{S}_-,\overline{S}) \bullet \varphi^1 = -\varphi^0$ *uniformly in probability.*

Let $\varepsilon > 0$. *Due to the up-convergence there exists an* $n_0 \in \mathbb{N}$ *s.t.*

$$
P\Big(|(\overline{S}_-,\overline{S}) \bullet \varphi_t^{1,n} + \varphi_t^0| \le \varepsilon/3, \quad \forall t\in[0,T]\Big) \ \ge\ 1-\varepsilon/3, \quad \forall n \ge n_0. \qquad (2.35)
$$

For any $\delta > 0$ *and* $t \in [0,T]$ *we have that*

$$
\begin{aligned}
\left|\varphi_t^{0,n} + (\overline{S}_-,\overline{S}) \bullet \varphi_t^{1,n}\right| &= \left|\int_{[0,t)\times\mathbb{R}_+}\int_x^\infty (y-\overline{S}_s) L^{B,n}(s,dy)\mu(ds,dx)\right| \\
&\le \left(\delta \vee \frac{1}{n}\right)\int_{[0,T]\times\mathbb{R}_+}\int_x^\infty L^{B,n}(s,dy)\mu(ds,dx) \\
&\quad + \sup_{s\in[0,T]}|\overline{S}_s| \sup_{s\in[0,T]}|\varphi_s^{1,n} - \varphi_s^1|\mu^{2,\delta}([0,T]\times\mathbb{R}_+) =: I(n) + II(n),
\end{aligned}
$$

where $\mu^{2,\delta}$ *is defined after equation (2.9). As* $\int_{[0,T]\times\mathbb{R}_+}\int_{[x,\infty)} L^{B,n}(s,dy)\mu(ds,dx) \le T$ *for all* $n \in \mathbb{N}$, *we choose* $\delta := \varepsilon/(3T)$ *to obtain* $P(I(n) \le \varepsilon/3) = 1$ *for all* $n \ge (3T)/\varepsilon =: n_1$. *We fix this* δ *and observe that* $P(\mu^{2,\delta}([0,T]\times\mathbb{R}_+) < \infty) = 1$. *Thus, by (2.34) applied to some appropriate* $\widetilde{\varepsilon} > 0$, *there exists an* $n_2 \in \mathbb{N}$ *s.t.* $P(II(n) \le \varepsilon/3) \ge 1 - (2\varepsilon)/3$ *for all* $n \ge n_2$. *Combining this* *with (2.35) we arrive at*

$$P(|\varphi_t^{0,n} - \varphi_t^0| \le \varepsilon, \quad \forall t \in [0,T]) \ \ge\ 1 - \varepsilon, \quad \forall n \ge n_0 \vee n_1 \vee n_2.$$

Example 2.37. *A somehow artificial but instructive example for a sequence of market order strategies is*

$$M_t^{B,n} \ := \ \int_0^t \sum_{k=1}^{2^n-2} 1_{((2k-1)2^{-n}T, 2k2^{-n}T]}(s)\, ds,$$
$$M_t^{S,n} \ := \ \int_0^t \sum_{k=1}^{2^n-2} 1_{(2k2^{-n}T, (2k+1)2^{-n}T]}(s)\, ds,$$

i.e. the investor buys and resells faster and faster, but the accumulated number of trades is constant in n. Both $M_t^{B,n}$ and $M_t^{S,n}$ converge to $t/2$ uniformly in time, but not in the variation norm. To obtain the right limit it is obviously important to consider the sequences $(M^{B,n})_{n\in\mathbb{N}}$ and $(M^{S,n})_{n\in\mathbb{N}}$ separately. The difference $M^{B,n} - M^{S,n}$ would converge to 0 and the transaction costs would asymptotically disappear.

Example 2.38. *In Chapter 3 a model with continuous best-bid and best-ask price processes is considered in which limit buy orders can only be placed at the current best-bid $\underline{S}_t$ and limit sell orders only at the current best-ask-price $\overline{S}_t$. As $\underline{S}$ and $\overline{S}$ move continuously in time, it calls for a verification that the strategies can be approximated by real-world trading strategies with piecewise constant limit prices (and order sizes).*

To do this, let us embed the model from Chapter 3 into the more general framework of the current chapter. The best-bid $\underline{S}$ and the best-ask $\overline{S}$ are certainly càdlàg processes satisfying the conditions at the beginning of Section 2.3. Denote by τ_i the i-th jump time of the Poisson process N^1. Then $\underline{S}_{\tau_i}$ is clearly $\mathcal{F}_{\tau_i}$-measurable and we define the random measures μ by

$$\mu(dt,dx) \ := \ \sum_{i=1}^{\infty} \delta_{(\tau_i, \underline{S}_{\tau_i})}(dt,dx).$$

If we define ν similarly using N^2 it is easy to see that Assumption 2.7 is satisfied (formally we have to exclude the null set in Ω on which there exists a point in time s.t. the two independent processes $\Delta N_t^1 > 0$ and $\Delta N_t^2 > 0$ for (v) to hold).

Restricting to limit buy orders this yields strategies of the form

$$L^B(\omega,t,x) \ := \ \widetilde{L}_t^B(\omega) 1_{[0,\underline{S}_t(\omega)]}(x),$$

where the nonnegative predictable process specifying the size of the limit buy order at the best-bid from Chapter 3 is now denoted by $\widetilde{L}^B$. Now Theorem 2.17 assures that the trading strategies considered in Chapter 3 can indeed be approximated by real-world trading strategies.

2.8 Conclusion

We provide a mathematical framework to model continuous time trading in limit order markets of a small investor. The model possesses the desirable properties that it is closed under the up-convergence of the portfolio process and any strategy can be approximated by elementary strategies. An interesting observation is that if the best-bid or the best-ask price process possesses infinitely many jumps on compact time intervals, then a sequence of limit order strategies can turn into a market order strategy when tending to the limit.

2.9 Appendix

Remark 2.39. *Note that for $\mathcal{P}_1$ and $\mathcal{P}_2$ from Definition 2.20 we have*

$$\begin{aligned}\mathcal{P}_1 &= \{X \textit{ is a } [0,\infty]\textit{-valued predictable process with } M_\mu(\textit{subgraph}(X)) = 0\},\\ \mathcal{P}_2 &= \{X \textit{ is a } [0,\infty]\textit{-valued predictable process with } M_\nu(\textit{supergraph}(X)) = 0\},\end{aligned}$$

and that Assumption 2.21 is the same as assuming $M_\mu(\textit{graph}(\underline{X})) = 0$ and $M_\nu(\textit{graph}(\overline{X})) = 0$.

Proposition 2.40. *$M_\mu(\textit{subgraph}(\underline{X})) = 0$ and $M_\nu(\textit{supergraph}(\overline{X})) = 0$.*

Proof. We only consider the first part of the statement, as the second part can be shown analogously. Clearly for $X_1, X_2 \in \mathcal{P}_1$ we have

$$\text{subgraph}(X_1 \vee X_2) = \text{subgraph}(X_1) \cup \text{subgraph}(X_2)$$

and thus $M_\mu(\text{subgraph}(X_1 \vee X_2)) = 0$, i.e. $\mathcal{P}_1$ is closed under pairwise maximization. By Theorem A.3 in [KS98] there exists an increasing sequence $(X_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{P}_1$ s.t. $\underline{X} = \lim_{n\to\infty} X_n$ M_μ-a.e. and thus by

$$\{(\omega,t,x) \in \widetilde{\Omega} : \lim_{n\to\infty} X_n(\omega,t) > x\} = \{(\omega,t,x) \in \widetilde{\Omega} : \exists n \in \mathbb{N} s.t. X_n(\omega,t) > x\} = \bigcup_{n=1}^{\infty} \text{subgraph}(X_n)$$

we arrive at

$$M_\mu\left(\text{subgraph}(\underline{X}) \Delta \bigcup_{n=1}^{\infty} \text{subgraph}(X_n)\right) = 0.$$

Because $(X_n)_{n\in\mathbb{N}}$ is an increasing sequence we also have $\text{subgraph}(X_i) \subset \text{subgraph}(X_j)$ for $i \le j$ and therefore

$$M_\mu(\text{subgraph}(\underline{X})) = M_\mu\left(\bigcup_{n=1}^{\infty} \text{subgraph}(X_n)\right) = \lim_{n\to\infty} M_\mu(\text{subgraph}(X_n)) = 0.$$

□

The following two propositions are rather straightforward. Still, we did not find them in the literature in the right formulation for us to apply. Hence, we give short proofs for the convenience of the reader, but do not make any claim of originality.

Proposition 2.41. *Let μ be an integer-valued random measure and let $A \in \widetilde{\mathcal{F}}$. Then $M_\mu(A) = 0$ if and only if*

$$P\left(\{\omega \in \Omega : (\tau_i(\omega), Y_i(\omega)) \in A_\omega\}\right) \;=\; 0, \qquad \forall i \in \mathbb{N}.$$

Proof. By monotone convergence

$$\begin{aligned} M_\mu(A) \;&= E\left[\int_{[0,T]\times\mathbb{R}_+} 1_A(t,x)\mu(dt,dx)\right] \\ &= E\left[\int_{[0,T]\times\mathbb{R}_+} 1_A(t,x)\sum_{i=1}^{\infty}\delta_{(\tau_i,Y_i)}(dt,dx)\right] \\ &= E\left[\sum_{i=1}^{\infty} 1_A(\tau_i, Y_i)\right] \\ &= \sum_{i=1}^{\infty} P\left(\{\omega \in \Omega : (\tau_i(\omega), Y_i(\omega)) \in A_\omega\}\right). \end{aligned}$$

□

Proposition 2.42. *Let $(H^n)_{n\in\mathbb{N}}$ be a sequence of $\overline{\mathbb{R}}$-valued and $\widetilde{\mathcal{F}}$-measurable functions that converges M_μ-a.e. to an $\overline{\mathbb{R}}$-valued and $\widetilde{\mathcal{F}}$-measurable function H. Suppose there exists an $\overline{\mathbb{R}}$-valued and $\widetilde{\mathcal{F}}$-measurable function K, which is μ-integrable and dominates $(H^n)_{n\in\mathbb{N}}$, i.e. $|H^n| \le K$ M_μ-a.e. for all $n \in \mathbb{N}$. Then $(H^n)_{n\in\mathbb{N}}$ and H are μ-integrable and $(\int_{[0,\cdot)\times\mathbb{R}_+} H^n d\mu)_{n\in\mathbb{N}}$ converges to $\int_{[0,\cdot)\times\mathbb{R}_+} H d\mu$ uniformly in probability.*

Proof. Let $N \in \widetilde{\mathcal{F}}$ with $M_\mu(N) = 0$ and $H^n \to H$, $|H^n| \le K$ on $\widetilde{\Omega} \setminus N$. By Fubini's theorem for transition kernels we obtain that $\mu(\omega, N_\omega) = 0$ for P-a.a. $\omega \in \Omega$. By dominated convergence we obtain that

$$\int_{[0,T]\times\mathbb{R}_+} |H^n(\omega,s,x) - H(\omega,s,x)|\, \mu(\omega, ds, dx) \;\to 0\;, \quad n \to \infty,$$

for all $\omega \in \Omega$ with $\mu(\omega, N_\omega) = 0$ and $\int K d\mu(\omega,\cdot) < \infty$. As K is assumed to be μ-integrable, we have that $P(\int K d\mu < \infty) = 1$ and thus $(\int_{[0,\cdot)\times\mathbb{R}_+} H^n d\mu)_{n\in\mathbb{N}}$ converges to $\int_{[0,\cdot)\times\mathbb{R}_+} H d\mu$ uniformly in probability. □

The following proposition is a variant of Theorem A.9 iii) in [DGR11]. We start the proof similarly, but later have to deviate to account for the different assumptions.

Proposition 2.43. *Let S be an adapted real-valued càdlàg process and let $(A^n)_{n\in\mathbb{N}}$ be a sequence of predictable real-valued processes of finite variation for which there exists a measure $Q \sim P$ and a constant $K > 0$ s.t.*

$$E_Q\left[\operatorname{var}(A^n)_T\right] \;\leq\; K, \qquad \forall n \in \mathbb{N},$$

and let A be a predictable real-valued process of finite variation s.t.

$$\sup_{t\in[0,T]} |A^n_t - A_t| \;\rightarrow\; 0, \qquad \textit{in probability},$$

as n goes to infinity. Then

$$\sup_{t\in[0,T]} |((S_-,S)\bullet A^n)_t - ((S_-,S)\bullet A)_t| \;\rightarrow\; 0, \qquad \textit{in probability},$$

as n goes to infinity.

Proof. The well-known equivalence between convergence in probability of a sequence of random variables to zero and that any subsequence of this sequence contains a subsubsequence converging almost surely to zero implies implies that we may assume w.l.o.g. that for $n \to \infty$

$$\sup_{t\in[0,T]} |A^n_t - A| \to 0, \qquad P\text{-a.s..}$$

By the assumption that the limit A is itself of finite variation, it follows from Proposition A.1 ii) in [DGR11] and Fatou's lemma that $E_Q[\operatorname{var}(A)_T] \leq K$ holds. Thus, it is sufficient to prove the result for $A \equiv 0$. Furthermore, by a stopping argument, we may suppose that there exist constants $C_1 > 0$ and $C_2 > 0$ s.t. $\sup_{t\in[0,T]} |A^n_t| \leq C_1$ for all $n \in \mathbb{N}$ and $\sup_{t\in[0,T]} |S_t| \leq C_2$.

By Theorem A.9 ii) in [DGR11] it holds for all $t \in [0,T]$ and $n \in \mathbb{N}$ that

$$\sup_{t\in[0,T]} |((S_-,S)\bullet A^n)_t| \;\leq\; \operatorname{var}(A^n)_T \sup_{t\in[0,T]} |S_t|. \tag{2.36}$$

For any $m \in \mathbb{N}$ define the sequence of stopping times $T^m_0, T^m_1, \ldots$ by

$$T^m_0 \;:=\; 0 \qquad \text{and} \qquad T^m_{i+1} \;:=\; \inf\left\{t > T^m_i : |S_t - S_{T^m_i}| > \frac{1}{m}\right\},$$

and let

$$S^m \;:=\; \sum_{i=0}^{\infty} S_{T^m_i} 1_{[\![T^m_i, T^m_{i+1}[\![}.$$

For every m by construction of S^m it is possible to find a constant $\alpha_m \in \mathbb{N}$ and a set $B_m \in \mathcal{F}$ such that S^m consist only of α_m steps or less on B_m and it holds that $Q(B^c_m) \leq 2^{-m}$. Now (2.36) and the linearity of the integral w.r.t. the integrand yield for any $m_0 \in \mathbb{N}$ and any $m \geq m_0$

$$
\begin{aligned}
& E_Q\left[\sup_{t\in[0,T]} |((S_-,S)\bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right] \\
\leq\ & E_Q\left[\sup_{t\in[0,T]} |((S_- - S_-^m, S - S^m)\bullet A^n)_t + ((S_-^m, S^m)\bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right] \\
\leq\ & \frac{1}{m} E_Q\left[\operatorname{var}(A^n)_T\right] + E_Q\left[\sup_{t\in[0,T]} |((S_-^m, S^m)\bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right].
\end{aligned}
$$

By assumption of the Lemma, by construction of S^m, α_m, and B_m, as well as by the assumptions at the beginning of the proof this implies for all $m \geq m_0$ that

$$
E_Q\left[\sup_{t\in[0,T]} |((S_-,S)\bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right] \ \leq\ \frac{1}{m} K + 2C_2 \alpha_m E_Q\left[\sup_{t\in[0,T]} |A_t^n| 1_{\bigcap_{m\geq m_0} B_m}\right].
$$

For any fixed m the second term on the right-hand side of the equation goes to zero as n goes to infinity, by dominated convergence. Therefore, for any $\varepsilon > 0$ and any $m_0 \in \mathbb{N}$ there exists a $n_0(\varepsilon, m_0) \in \mathbb{N}$ such that for all $n \geq n_0(\varepsilon, m_0)$ it holds that

$$
E_Q\left[\sup_{t\in[0,T]} |((S_-,S)\bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m}\right] \ \leq\ \varepsilon.
$$

Hence for any $m_0 \in \mathbb{N}$ we have that

$$
\sup_{t\in[0,T]} |((S_-,S)\bullet A^n)_t| 1_{\bigcap_{m\geq m_0} B_m} \ \to\ 0, \qquad \text{in probability}
$$

as n goes to infinity. Note that by a Borel-Cantelli argument we have that $Q(\limsup_m B_m^c) = 0$, which implies that $(\bigcap_{m\geq m_0} B_m)_{m_0\in\mathbb{N}}$ is an increasing sequence with $P(\liminf_m B_m) = 1$. Thus, it also holds that

$$
\sup_{t\in[0,T]} |((S_-,S)\bullet A^n)_t| \ \to\ 0, \qquad \text{in probability}
$$

as n goes to infinity. $\square$

Chapter 3

Optimal portfolios of a small investor in a limit order market

3.1 Introduction

A *portfolio problem* in mathematical finance is the optimization problem of an investor possessing a given initial endowment of assets who has to decide how many shares of each asset to hold at each time instant in order to maximize his expected utility from consumption (see [Kor97]). To change the asset allocation of his portfolio or finance consumption, the investor can buy or sell assets at the market. Merton [Mer69, Mer71] solved the portfolio problem for a continuous time frictionless market consisting of one risky asset and one riskless asset. When the price process of the risky asset is modeled as a geometric Brownian motion (GBM), Merton was able to show that the investor's optimal strategy consists of keeping the fraction of wealth invested in the risky asset constant. Due to the fluctuations of the GBM this leads to incessant trading.

The assumption that investors can purchase and sell arbitrary amounts of the risky asset at a fixed price per share is quite unrealistic in a less liquid market which possesses a significant bid-ask spread. In today's electronic markets the predominant market structure is the limit order market, where traders can continuously place market and limit orders, and change or delete them as long as they are not executed. When a trader wants to buy shares for example, he has a basic choice to make. He can either place a market buy order or he can submit a limit buy order, with the limit specifying the maximum price he would be willing to pay per share. If he uses a market order his order is executed immediately, but he is paying at least the best-ask price (the lowest limit of all unexecuted limit sell orders), and an even higher average price if the order size is large. By using a limit buy order with a limit lower than the current best-ask price he pays less, but he cannot be sure if and when the order is executed by an incoming sell order matching his limit.

We introduce a new model for continuous-time trading using both market and limit orders. This allows us to analyze e.g. the *trade-off between rebalancing the portfolio quickly and trading at favorable prices*. To obtain a mathematically tractable model we keep some idealized assumptions of the frictionless market model resp. the model with proportional transaction costs. E.g. we assume that the investor under consideration is small, i.e. the size of his orders is sufficiently small to be absorbed by the orders in the order book. The best-ask and the best-bid price processes solely result from the behavior of the other market participants and can thus be given *exogenously*. Furthermore, we assume that the investor's limit orders are small enough to be executed against any arising market order whose arrival times are also exogenously given and modeled as Poisson times. We also assume that limit orders can be submitted and taken out of the order book for free.

The model tries to close a gap between the market microstructure literature which lacks analytical tractability when it comes to dynamic trading and the literature on portfolio optimization under idealized assumptions with powerful closed-form and duality results.

In the economic literature on limit order markets (see e.g. the survey by Parlour and Seppi [PS08] for an overview) the incentive to trade quickly (and therefore submit market orders) is usually modeled exogenously by a preference for immediacy. This is e.g. the case in the multi-period equilibrium models of Foucault, Kadan, and Kandel [FKK05] and Roşu [Roş09], which model the limit order market as a stochastic sequential game. Even in research concerning the optimal behavior of a single agent, this exogenous motivation to trade is common. Consider e.g. Harris [Har98], which deals with optimal order submission strategies for certain stylized trading problems, e.g. for a *risk-neutral* trader who has to sell one share before some deadline. By contrast, in our model the trading decision is directly derived from the maximization of expected utility from a consumption stream (thus from "first principles"), i.e. the incentive to trade quickly is explained. Furthermore, in Harris [Har98] the order size is discarded and the focus is on the selection of the right limit price at each point in time. In our work the limit prices used by the small investor are effectively reduced to selling at the best-ask and buying at the best-bid, but in view of the agent's underlying portfolio problem, the size of these limit orders is a key question. *There is a trade-off between placing large limit orders to profit from the* *spread and taking too much risk by the resulting large positions (usually called* inventory risk *in the literature on market making).*

In Section 3.2 we introduce the market model on a quite general level. In Section 3.3 we specify stochastic processes for which we study the problem of maximizing expected logarithmic utility from consumption over an infinite horizon. Namely, we let the best-bid and best-ask price processes be geometric Brownian motions and the spread be proportional to them. Market orders of the other traders arise according to two independent Poisson processes with constant rates. In Section 3.3 we also provide some intuition on how we obtain a promising candidate for an optimal strategy and connect it to the solution of a suitable free boundary problem. In Section 3.4 we prove the existence of a solution of this free boundary problem. The verification that the constructed solution is indeed optimal is done in Section 3.5.

The optimal strategy consists in placing the minimal amount of market orders which is necessary to keep the proportion of wealth invested in the risky asset within certain boundaries – similar to the result of Davis and Norman [DN90] for transaction costs – while within these boundaries limit orders are used to hit one of the boundaries when at a Poisson time trading is possible at a favorable price (i.e. the investor adjusts the sizes of his limit orders continuously in such a way that the proportion invested in the risky asset jumps to one of the boundaries whenever a limit order is executed by an incoming exogenous market order). By the latter the investor profits from the bid-ask spread. Thus, although the structure of the solution looks at first glance quite similar to the case with proportional transaction costs, a key incentive of the investor is now to capitalize on the spread by placing limit orders. Whereas the investor generally tries to avoid using market orders, he is always willing to trade using limit orders. In a sense, trading with limit orders corresponds to negative proportional transactions costs.

We derive the optimal trading strategy by showing the existence of a shadow price process of the asset – similar to the work of Kallsen and Muhle-Karbe [KMK10] with proportional transaction costs. A shadow price process $\widetilde{S}$ for the risky asset has to satisfy the following two properties. Firstly, in a fictitious frictionless market without spread and with price process $\widetilde{S}$ *any* transaction feasible in the original market can be implemented at better or equal prices. Secondly, there is an *optimal* trading strategy in the fictitious market which can also be realized in the original market leading to the same trading gains.

The main difference of the shadow price process in our model compared to [KMK10] is that it possesses jumps – namely at the Poisson arrival times of the exogenous market orders.

Note that the contents of this chapter have already been published in [KS10].

3.2 The model

3.2.1 Trading of a small investor in a limit order market

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space satisfying the usual conditions. Regarding conventions and notation we mostly follow Jacod and Shiryaev [JS02]. For a process X with left and right limits (also called làglàd) let $\Delta X_t := X_t - X_{t-}$ denote the jump at time t and let $\Delta^+ X_t := X_{t+} - X_t$ denote the jump immediately after time t. If we write $X = Y$ for two stochastic processes X and Y, we mean equality up to indistinguishability.

We model the best-bid price $\underline{S}$ and the best-ask price $\overline{S}$ as two *continuous*, adapted, exogenously given stochastic processes such that $\underline{S} \leq \overline{S}$. The continuity of $\underline{S}$ and $\overline{S}$ will play a key role in the reduction of the dimension of the strategy set. The arrivals of market sell orders and market buy orders by the other traders are modeled exogenously by counting processes N^1 and N^2 (as defined e.g. in [Pro04], Section 1.3).

In our model (formally introduced in Definition 3.2) the investor may submit market buy and sell orders which are immediately executed at price $\overline{S}$ and $\underline{S}$, resp. In addition, he may submit limit buy and sell orders. The limit buy price is restricted to $\underline{S}$ and these orders are executed at the jump times of N^1 at price $\underline{S}$. Accordingly, the limit sell price is restricted to $\overline{S}$ and the limit sell orders are executed at the jump times of N^2 at the price $\overline{S}$.

This restriction is an immense reduction of the dimensionality of the problem, as we do not consider limit orders at arbitrary limit prices. It can be justified by the following considerations. A superior limit order strategy of the small investor is to place a limit buy order at a *"marginally" higher price than the current best-bid price* $\underline{S}$ (of course this necessitates to update the limit price according to the movements of the best-bid price, which could in practice be approximately realized as long as the submission and deletion of orders is for free). Then, the limit buy order is executed as soon as the next limit sell order by the other traders arrives (i.e. at the next jump time of N^1). As $\underline{S}$ is continuous there is no reason to submit a limit buy order at a limit price strictly lower than the current best-bid price. Namely, such an order could not be executed before $\underline{S}$ hits the lower limit buy price of the order. As this appears at a *predictable* stopping time it is sufficient to place the order at this stopping time and take the current best-bid price as the limit price. On the other hand, a limit buy order with limit price in $(\underline{S}, \overline{S})$ is executed at the same time as a buy order with limit price $\underline{S}$ (resp. "marginally" higher than $\underline{S}$), but at a higher price than $\underline{S}$ (assuming that market sell orders of the other traders still arise according to N^1).

Thus, in our model it is implicitly assumed that the small investor does not influence the best-ask price or the best-bid price and his orders are small enough to be executed against any market order arising at $\Delta N^1 = 1$ and $\Delta N^2 = 1$. Furthermore, the market orders arising at $\Delta N^1 = 1$, $\Delta N^2 = 1$ (although being large in comparison to the size of the orders of the small investor) are sufficiently small to be absorbed by the orders in the book, i.e. a jump of N^1 or N^2 does not cause a movement of $\underline{S}$ and $\overline{S}$.

With the considerations above we are in the quite fortunate situation that the quadruple $(\underline{S},\overline{S},N^1,N^2)$ *is sufficient to model the trading opportunities of the small investor.* Thus, our mathematical model can be build on these processes alone (rather than on the dynamics of the whole order book).

A possible economic interpretation is that $\underline{S}$ and $\overline{S}$ move as nonaggressive traders update their limit prices with varying fundamentals whereas N^1 and N^2 model immediate supply and demand for the asset.

Remark 3.1. *Note that the investor in our model does not play the role of a market maker who, however, also wants to profit from the spread. The market maker can influence the spread and he is* forced *to trade with arising market orders.*

Definition 3.2. *Let M^B, M^S, L^B and L^S be predictable processes. Furthermore, let M^B and M^S be non-decreasing with $M_0^B = M_0^S = 0$ and L^B and L^S non-negative. Let c be an optional process. A quintuple $\mathfrak{S} = (M^B, M^S, L^B, L^S, c)$ is called a* strategy*. For $\eta^0, \eta^1 \in \mathbb{R}$ we define the* portfolio process $(\varphi^0,\varphi^1)(\mathfrak{S},\eta^0,\eta^1)$ associated with strategy $\mathfrak{S}$ and initial endowment (η^0,η^1) *to be*

$$\begin{aligned} \varphi_t^0 &:= \eta^0 - \int_0^t c_s ds - \int_0^t \overline{S}_s \, dM_s^B + \int_0^t \underline{S}_s \, dM_s^S \\ &\quad - \int_0^{t-} L_s^B \underline{S}_s \, dN_s^1 + \int_0^{t-} L_s^S \overline{S}_s \, dN_s^2 \\ \varphi_t^1 &:= \eta^1 + M_t^B - M_t^S + \int_0^{t-} L_s^B \, dN_s^1 - \int_0^{t-} L_s^S \, dN_s^2. \end{aligned} \tag{3.1}$$

φ^0 is the number of risk-free assets and φ^1 the number of risky assets. For simplicity, we assume there is a risk-free interest rate, which is equal to zero. The interpretation is that *aggregated* market buy or sell orders up to time t are modeled with M_t^B and M_t^S, whereas L_t^B

(resp. L_t^S) specifies the size of a limit buy order with limit price $\underline{S}$ (resp. the size of a limit sell order with limit price $\overline{S}$), i.e. the amount that is bought or sold favorably if an exogenous market sell or buy order arrives at time t. L^B and L^S can be arbitrary predictable processes which is justified under the condition that submission and deletion of orders which are not yet executed is for free. Finally, c_t is interpreted as the rate of consumption at time t.

Note that integrating w.r.t. the processes M^B and M^S which are of finite variation and therefore have left and right limits is a trivial case of integrating w.r.t. optional semimartingales (as discussed e.g. in [Gal85] and [KS09b]). For a càdlàg process Y we define the integral $\int (Y_-, Y) dM^B$ by

$$\int_0^t (Y_{s-}, Y_s) dM_s^B := \int_0^t Y_{s-} d(M_s^B)^r + \sum_{0 \leq s < t} Y_s \Delta^+ M_s^B, \quad t \geq 0, \tag{3.2}$$

where $(M^B)_t^r := M_t^B - \sum_{0 \leq s < t} \Delta^+ M_s^B$. The first term on the right-hand side of (3.2) is just a standard Lebesgue-Stieltjes integral. For a continuous integrand Y, as e.g. in (3.1), we set $\int Y dM^B := \int (Y, Y) dM^B$ (which is consistent with the integral w.r.t. càdlàg integrators).

In (3.1) the integrals w.r.t. N^1 and N^2 are only up to time $t-$, a limit order triggered by $\Delta N_t^i = 1$ is not yet included in φ_t. The integrals w.r.t. M^B and M^S are up to time t, but note that by (3.2) just the orders ΔM_t^B and ΔM_t^S (corresponding to trades at time $t-$) are already included in φ_t at time t, whereas the orders $\Delta^+ M_t^B$ and $\Delta^+ M_t^S$ (corresponding to trades at time t) are only included in φ_t right after time t. Hence, (3.1) goes conform to the usual interpretation of φ_t as the holdings at time $t-$ (and the amount invested in the jump at time t) and for $\underline{S} = \overline{S}$ it coincides with the self-financing condition in frictionless markets (up to the restriction to finite variation strategies).

3.2.2 The Merton problem in a limit order market

Given initial endowment (η^0, η^1) a strategy $\mathfrak{S}$ is called *admissible* if its associated portfolio process $(\varphi^0, \varphi^1)(\mathfrak{S}, \eta^0, \eta^1)$ satisfies

$$\varphi_t^0 + 1_{\{\varphi_t^1 \geq 0\}} \underline{S}_t \varphi_t^1 + 1_{\{\varphi_t^1 < 0\}} \overline{S}_t \varphi_t^1 \geq 0, \quad \forall t \geq 0. \tag{3.3}$$

Thus, a strategy is considered admissible if at any time a market order can be used to liquidate the position in the risky asset resulting in a non-negative amount held in the risk-free asset. Let $\mathcal{A}(\eta^0, \eta^1)$ denote the *set of admissible strategies* for initial endowment (η^0, η^1).

Now the *value function* V for the optimization problem of an investor with initial endowment η^0 in the risk-free asset and η^1 in the risky asset and logarithmic utility function who wants to maximize expected utility from consumption can be written as

$$V(\eta^0,\eta^1) := \sup_{\mathfrak{S}\in\mathcal{A}(\eta^0,\eta^1)} \mathcal{J}(\mathfrak{S}) := \sup_{\mathfrak{S}\in\mathcal{A}(\eta^0,\eta^1)} E\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right), \tag{3.4}$$

where $\delta > 0$ models the time preference. Note that due to the spread the optimization problem is not myopic.

3.2.3 Fictitious markets and shadow prices

To solve (3.4) we consider – similar to [KMK10] – a fictitious frictionless market comprising of the same two assets as above. In this frictionless market the discounted price process of the risky asset is modeled as a real-valued semimartingale $\widetilde{S}$. Any amount of the risky asset can be bought or sold instantly at price $\widetilde{S}$.

Let (ψ^0,ψ^1) be a two-dimensional predictable process, integrable w.r.t. to the two-dimensional semimartingale $(1,\widetilde{S})$, i.e. $(\psi^0,\psi^1)\in L((1,\widetilde{S}))$ in the notation of [JS02]. Suppose c is an optional process. We call $\widetilde{\mathfrak{S}} = (\psi^0,\psi^1,c)$ a *self-financing* strategy with initial endowment (η^0,η^1) if it satisfies

$$\psi_t^0 + \psi_t^1\widetilde{S}_t = \eta^0 + \eta^1\widetilde{S}_0 + \int_0^t \psi_s^1 d\widetilde{S}_s - \int_0^t c_s ds.$$

A self-financing strategy $\widetilde{\mathfrak{S}}$ is called *admissible* if

$$\psi_t^0 + \psi_t^1\widetilde{S}_t \geq 0, \quad \forall t\geq 0.$$

Denote by $\widetilde{\mathcal{A}}(\eta^0,\eta^1)$ the set of all admissible strategies given initial endowment (η^0,η^1). Again, we introduce a value function $\widetilde{V}$ by

$$\widetilde{V}(\eta^0,\eta^1) := \sup_{\widetilde{\mathfrak{S}}\in\widetilde{\mathcal{A}}(\eta^0,\eta^1)} \widetilde{\mathcal{J}}(\widetilde{\mathfrak{S}}) := \sup_{\widetilde{\mathfrak{S}}\in\widetilde{\mathcal{A}}(\eta^0,\eta^1)} E\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right).$$

Note that because the spread is zero, for another initial endowment (ζ^0,ζ^1) we have $V(\eta^0,\eta^1) = V(\zeta^0,\zeta^1)$ if $\eta^0+\eta^1\widetilde{S}_0 = \zeta^0+\zeta^1\widetilde{S}_0$. Nonetheless, to keep the notation for the frictionless market close to the notation for the limit order market we write $\widetilde{V}(\eta^0,\eta^1)$.

Definition 3.3. *We call the real-valued semimartingale $\widetilde{S}$ a* shadow price process *of the risky asset if it satisfies for all $t\geq 0$:*

$$\underline{S}_t \leq \widetilde{S}_t \leq \overline{S}_t, \qquad \widetilde{S}_t = \begin{cases} \underline{S}_t & \textit{if} \quad \Delta N_t^1 = 1 \\ \overline{S}_t & \textit{if} \quad \Delta N_t^2 = 1 \end{cases} \tag{3.5}$$

and if there exists a strategy $\mathfrak{S} = (M^B, M^S, L^B, L^S, c) \in \mathcal{A}(\eta^0, \eta^1)$ *in the limit order market such that for the associated portfolio process* (φ^0, φ^1) *we have* $\widetilde{\mathfrak{S}} = (\varphi^0, \varphi^1, c) \in \widetilde{\mathcal{A}}(\eta^0, \eta^1)$ *and* $\widetilde{\mathcal{J}}(\widetilde{\mathfrak{S}}) = \widetilde{V}(\eta^0, \eta^1)$ *in the frictionless market with* $\widetilde{S}$ *as the discounted price process of the risky asset, i.e. the associated portfolio process of* $\mathfrak{S}$ *paired with the consumption rate* c *of* $\mathfrak{S}$ *has to be an optimal strategy in the frictionless market.*

The concept of a shadow price process consists of two parts. Firstly, trading in the frictionless market at prices given by the shadow price process should be at least as favorable as in the market with frictions. The investor can use a market order *at any time* to buy the risky asset at price $\overline{S}$. Hence, we have to require $\widetilde{S}_t \leq \overline{S}_t$ *for all* $t \geq 0$ to make sure that he never has to pay more than in the market with frictions. Analogously, to take care of the market sell orders, we demand $\underline{S} \leq \widetilde{S}_t$ for all $t \geq 0$. In a market with proportional transaction costs this would be sufficient, but in our limit order market the investor can also buy at $\underline{S}$ *whenever an exogenous market sell order arrives.* Thus, we have to require $\widetilde{S}_t \leq \underline{S}_t$ *whenever* $\Delta N^1_t = 1$. Accordingly, to cover the opportunities to sell at $\overline{S}$ using limit sell orders, we need to demand $\widetilde{S}_t \geq \overline{S}_t$ whenever $\Delta N^2_t = 1$. Combining these four requirements, we arrive at condition (3.5). Secondly, the maximal utility which can be achieved by trading at the shadow price must not be higher than by trading in the market with frictions. This is ensured by the second part of the definition. Note that for a shadow price to exist, N^1 and N^2 must not jump simultaneously at any time at which $\underline{S} < \overline{S}$ holds, otherwise (3.5) cannot be satisfied.

The following lemma is a reformulation of Lemma 2.2 in [KMK10]. We quote it for convenience of the reader.

Lemma 3.4. *(Kallsen and Muhle-Karbe [KMK10]) Let* S *be a real-valued semimartingale and let* $\varphi \in L(S)$ *be a finite variation process (not necessarily right-continuous). Then their product* φS *can be written as*

$$\begin{aligned} \varphi_t S_t &= \varphi_0 S_0 + \int_0^t \varphi_s dS_s + \int_0^t (S_{s-}, S_s) d\varphi_s \\ &= \varphi_0 S_0 + \int_0^t \varphi_s dS_s + \int_0^t S_{s-} d\varphi^r_s + \sum_{0 \leq s < t} S_s \Delta^+ \varphi_s. \end{aligned}$$

Proposition 3.5. *If* $\widetilde{S}$ *is a shadow price process and* $\mathfrak{S}$ *is a strategy in the limit order market corresponding to an optimal strategy* $\widetilde{\mathfrak{S}}$ *in the frictionless market as in Definition 3.3, then* $\mathfrak{S}$ *is an optimal strategy in the limit order market, i.e.* $\mathcal{J}(\mathfrak{S}) = V(\eta^0, \eta^1)$.

Proof. Step 1. We begin by showing $V(\eta^0,\eta^1) \le \widetilde{V}(\eta^0,\eta^1)$. Let $\mathfrak{S} \in \mathcal{A}(\eta^0,\eta^1)$ with corresponding portfolio process (φ^0,φ^1). Define

$$\begin{aligned}\psi_t^0 \quad := \quad & \eta^0 - \int_0^t c_s ds - \int_0^t (\widetilde{S}_{s-},\widetilde{S}_s) dM_s^B + \int_0^t (\widetilde{S}_{s-},\widetilde{S}_s) dM_s^S \\ & - \int_0^{t-} L_s^B \widetilde{S}_s dN_s^1 + \int_0^{t-} L_s^S \widetilde{S}_s dN_s^2\end{aligned}$$

and $\psi^1 := \varphi^1$. Applying Lemma 3.4 we get

$$\psi_t^1 \widetilde{S}_t = \eta^1 \widetilde{S}_0 + \int_0^t \psi_s^1 d\widetilde{S}_s + \int_0^t (\widetilde{S}_{s-},\widetilde{S}_s) d\psi_s^1.$$

This equation is equivalent to

$$\psi_t^0 + \psi_t^1 \widetilde{S}_t - \eta^0 - \eta^1 \widetilde{S}_0 - \int_0^t \psi_s^1 d\widetilde{S}_s + \int_0^t c_s ds = \psi_t^0 + \int_0^t (\widetilde{S}_{s-},\widetilde{S}_s) d\psi_s^1 - \eta^0 + \int_0^t c_s ds. \qquad (3.6)$$

By definition of ψ^0 and ψ^1 and associativity of the integral the term on the right side is equal to 0. Hence (3.6) implies that (ψ^0,ψ^1,c) is a self-financing strategy in the frictionless market. Furthermore, by (3.5) and (3.3) we get

$$\psi_t^0 + \psi_t^1 \widetilde{S}_t \ge \varphi_t^0 + \varphi_t^1 \widetilde{S}_t \ge \varphi_t^0 + 1_{\{\varphi_t^1 \ge 0\}} \varphi_t^1 \underline{S}_t + 1_{\{\varphi_t^1 < 0\}} \varphi_t^1 \overline{S}_t \ge 0.$$

Thus for every $\mathfrak{S} \in \mathcal{A}(\eta^0,\eta^1)$ we have an admissible strategy $\widetilde{\mathfrak{S}} = (\psi^0,\psi^1,c) \in \widetilde{\mathcal{A}}(\eta^0,\eta^1)$ with the same consumption rate.

Step 2. By the definition of a shadow price there is a strategy $\mathfrak{S} = (M^B,M^S,L^B,L^S,c)$ in the limit order market with associated portfolio process (φ^0,φ^1) such that $\widetilde{\mathfrak{S}} = (\varphi^0,\varphi^1,c)$ is an optimal strategy in the frictionless market, i.e.

$$\mathcal{J}(\mathfrak{S}) = \widetilde{\mathcal{J}}(\widetilde{\mathfrak{S}}) = \widetilde{V}(\eta^0,\eta^1).$$

By Step 1 this implies $\mathcal{J}(\mathfrak{S}) = V(\eta^0,\eta^1)$, hence $\mathfrak{S}$ is optimal. $\square$

3.3 Heuristic derivation of a candidate for a shadow price process

The model of a small investor trading in a limit order market makes sense in the generality introduced above. Still, to get enough tractability to be able to construct a shadow price process we reduce the complexity by restricting ourselves to a more concrete case. From now on we model the spread as proportional to the best-bid price, which is modeled as a standard geometric Brownian motion with starting value $\underline{s}$, i.e.

$$d\underline{S}_t = \underline{S}_t(\mu dt + \sigma dW_t), \qquad \underline{S}_0 = \underline{s}, \tag{3.1}$$

with $\mu, \sigma \in \mathbb{R}_+ \setminus \{0\}$. The size of the spread is modeled with a constant $\lambda > 0$. Similarly to [KMK10] define

$$\overline{C} := \log(1+\lambda) \qquad \text{and} \qquad \overline{S} := \underline{S}e^{\overline{C}} = \underline{S}(1+\lambda).$$

Let $\alpha_1, \alpha_2 \in \mathbb{R}_+$. The arrival of exogenous market orders is modeled as two independent time-homogenous Poisson processes N^1 and N^2 with rates α_1 and α_2. These memoryless and stationary arrival times, the time-independent coefficients in the dynamics of the best bid price, the proportional spread, and the infinite horizon of the optimization problem (3.4) will lead to a time-homogenous structure of the solution.

For $\alpha_1 = \alpha_2 = 0$ the model reduces to the model with proportional transaction costs as e.g. in [DN90], [KMK10] or [SS94]. For $\lambda = 0$ and by allowing to trade *only* at the jump times of the Poisson process we would arrive at an illiquidity model introduced by Rogers and Zane [RZ02] which is widely investigated in the literature, see e.g. Matsumoto [Mat06] who studies optimal portfolios w.r.t. terminal wealth in this model. Pham and Tankov [PT08] recently introduced a related illiquidity model under which the price of the risky asset cannot even be observed apart from the Poisson times at which trading is possible.

We will show (under certain restrictions to the parameters $\mu, \sigma, \lambda, \alpha_1, \alpha_2$, see Proposition 3.6) that it is optimal to control the portfolio as follows. There exist $\pi_{\min}, \pi_{\max} \in \mathbb{R}_+$ with $0 < \pi_{\min} < \pi_{\max}$ such that the proportion of wealth invested in the risky asset (measured in terms of the best bid price) is kept in the interval $[\pi_{\min}, \pi_{\max}]$ by using **market orders**, i.e.

$$\pi_{\min} \leq \frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} \leq \pi_{\max}, \quad \forall t > 0 \tag{3.2}$$

(Note that, as $\underline{S}$ and $\overline{S}$ only differ in a constant factor, the structure of the solution would remain unaffected if wealth was measured in terms of the best-ask price instead of the best-bid price – only the numbers $\pi_{\min}$ and $\pi_{\max}$ would change). To keep the proportion within this interval, as is the case with proportional transaction costs, M^B and M^S will have local time at the boundary. In the inner they are constant. Furthermore, at all times two **limit orders** are kept in the order book such that

$$\frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} = \pi_{\max}, \quad \text{after limit buy order is executed with limit } \underline{S}_t \tag{3.3}$$

$$\frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} = \pi_{\min}, \quad \text{after limit sell order is executed with limit } \overline{S}_t. \tag{3.4}$$

To follow this strategy both limit *prices* and limit order *sizes* have to be permanently adjusted. The former to stay at $\underline{S}$ and $\overline{S}$, resp. The latter as after a successful execution of a limit order the *proportion of wealth* invested in the risky asset and not the number of risk assets is time-homogenous. Finally, optimal consumption is proportional to wealth measured w.r.t. the shadow price.

In this section we provide some intuition on how to use the *guessed* properties of the optimal strategy described in (3.2), (3.3), and (3.4) to find a promising candidate for a shadow price process by combining some properties a shadow price process should satisfy. Later, in Section 3.5, we construct a semimartingale that satisfies these properties by using solutions of a suitable free boundary problem and a related Skorohod problem. This semimartingale is then *verified* to be indeed a shadow price process of the risky asset.

The definition of a shadow price process suggests that if for example market order sales become worthwhile, $\widetilde{S}$ approaches $\underline{S}$ as in [KMK10]. Moreover, by (3.5) if an exogenous market buy order arises (i.e. the asset can be sold expensively), the shadow price has to jump to $\overline{S}$. Consider a $[0,\overline{C}]$-valued Markov process which satisfies

$$dC_t = \widetilde{\mu}(C_{t-})dt + \widetilde{\sigma}(C_{t-})dW_t - C_{t-}dN_t^1 + (\overline{C} - C_{t-})dN_t^2,$$

where the real functions $\widetilde{\mu}$ and $\widetilde{\sigma}$ are not yet specified, but are assumed to be sufficiently nice for a solution C of the stochastic differential equation to exist. As an ansatz for the shadow price $\widetilde{S}$ we use $\widetilde{S} := \underline{S}\exp(C)$. C is similar to the process in [KMK10] apart from its jumps. From Itô's formula we get

$$\begin{aligned} d\widetilde{S}_t &= \widetilde{S}_{t-}\left[\left(\mu + \frac{\widetilde{\sigma}(C_{t-})^2}{2} + \sigma\widetilde{\sigma}(C_{t-}) + \widetilde{\mu}(C_{t-})\right)dt + (\sigma + \widetilde{\sigma}(C_{t-}))\,dW_t \right.\\ &+ \left.\left(e^{-C_{t-}\Delta N_t^1 + (\overline{C} - C_{t-})\Delta N_t^2} - 1\right)\right]. \end{aligned}$$

For $\widetilde{S}$ to be a shadow price process, we have to be able to find a strategy which is optimal in the frictionless market with price process $\widetilde{S}$, but can also be carried out in the limit order market at the same prices. Fortunately, optimal behavior in the frictionless market is well understood for logarithmic utility. The plan is to choose the dynamics of $\widetilde{S}$ in such a way, that the portfolio process of the suspected optimal strategy described in (3.2), (3.3), and (3.4) is an optimal strategy in the frictionless market. To do this, we can use a theorem by Goll and Kallsen [GK00] (Theorem 3.1) which gives a sufficient condition for a strategy in a frictionless markets to be optimal. It says that if the triple $(\widetilde{b},\widetilde{c},\widetilde{F})$ is the differential semimartingale characteristics of the special semimartingale $\widetilde{S}$ (w.r.t. to the predictable increasing process $I(\omega,t) := t$ and "truncation function" $h(x) = x$, see e.g. [JS02] (Proposition II.2.9)) and if the equation

$$\widetilde{b}_t - \widetilde{c}_t H_t + \int \left(\frac{x}{1+H_t x} - x \right) \widetilde{F}_t(dx) = 0$$

was fulfilled $(P \otimes I)$-a.e on $\Omega \times [0,\infty)$ by $H := \varphi^1/\widetilde{V}_-$, then H would be optimal. Using that N^1 and N^2 are independent and thus

$$\Delta N^1 \Delta N^2 = 0 \quad \text{and} \quad e^{-C_-\Delta N^1 + (\overline{C}-C_-)\Delta N^2} - 1 = e^{-C_-\Delta N^1} - 1 + e^{(\overline{C}-C_-)\Delta N^2} - 1$$

up to evanescence, the characteristic triple of $\widetilde{S}$ becomes

$$\begin{aligned}
\widetilde{b}_t &= \widetilde{S}_{t-}\left(\mu + \frac{\widetilde{\sigma}(C_{t-})^2}{2} + \sigma\widetilde{\sigma}(C_{t-}) + \widetilde{\mu}(C_{t-})\right) + \int x\widetilde{F}_t(dx) \\
\widetilde{c}_t &= \left(\widetilde{S}_{t-}(\sigma + \widetilde{\sigma}(C_{t-}))\right)^2 \\
\widetilde{F}_t(\omega,dx) &= \alpha_1 \delta_{x_1(\omega,t)}(dx) + \alpha_2 \delta_{x_2(\omega,t)}(dx),
\end{aligned}$$

with

$$x_1(\omega,t) := \widetilde{S}_{t-}(\omega)(e^{-C_{t-}(\omega)} - 1), \quad x_2(\omega,t) := \widetilde{S}_{t-}(\omega)(e^{\overline{C}-C_{t-}(\omega)} - 1).$$

Denote by $\widetilde{\pi}_t := H_t \widetilde{S}_{t-}$ the optimal fraction invested in the risky asset, measured in terms of the shadow price. Even though we cannot write down $\widetilde{\pi}_t$ explicitly, we know that a $\widetilde{\pi}$ is optimal, if it satisfies

$$\begin{aligned}
F(C_{t-},\widetilde{\pi}_t) &:= \mu + \frac{\widetilde{\sigma}(C_{t-})^2}{2} + \sigma\widetilde{\sigma}(C_{t-}) + \widetilde{\mu}(C_{t-}) - \widetilde{\pi}_t(\sigma + \widetilde{\sigma}(C_{t-}))^2 \\
&+ \alpha_1(e^{-C_{t-}} - 1)\left(\frac{1}{1+\widetilde{\pi}_t(e^{-C_{t-}} - 1)}\right) \\
&+ \alpha_2(e^{\overline{C}-C_{t-}} - 1)\left(\frac{1}{1+\widetilde{\pi}_t(e^{\overline{C}-C_{t-}} - 1)}\right) \\
&= 0.
\end{aligned} \tag{3.5}$$

Consider the stopping time

$$\tau := \inf\left\{t > 0 : C_t \in \{0, \overline{C}\}\right\}.$$

As long as $\underline{S} < \widetilde{S} < \overline{S}$, it should be optimal in the frictionless market to keep the number of shares in the risky asset constant, i.e. there is no trading. Thus, on $]\!]0,\tau[\![$ we should have that

$$d\log(\varphi_t^0) = \frac{-c_t}{\varphi_t^0}dt = \frac{-\delta\widetilde{V}_t}{\widetilde{V}_t - \widetilde{\pi}_t\widetilde{V}_t}dt = \frac{-\delta}{1-\widetilde{\pi}_t}dt,$$

where (φ^0, φ^1) are the optimal amounts of securities. The second equality holds as optimal consumption is given by $c = \delta\widetilde{V}$ (again by Theorem 3.1 in [GK00]). Using the same approach to simplify the calculations as in [KMK10] we introduce

$$\beta := \log\left(\frac{\widetilde{\pi}}{1-\widetilde{\pi}}\right) = \log\left(\frac{\varphi^1\widetilde{S}}{\varphi^0}\right).$$

On $]\!]0,\tau[\![$ we have $C = C_-$, hence the dynamics of β_t on $]\!]0,\tau[\![$ can be written as

$$\begin{aligned} d\beta_t &= d\log(\varphi_t^1) + d\log(\widetilde{S}_t) - d\log(\varphi_t^0) \\ &= \left(\mu - \frac{\sigma^2}{2} + \widetilde{\mu}(C_t) + \frac{\delta}{1-\widetilde{\pi}(C_t)}\right)dt + (\sigma + \widetilde{\sigma}(C_t))dW_t. \end{aligned} \tag{3.6}$$

Furthermore, $\widetilde{\pi}$ is a function of C_- implicitly given by optimality equation (3.5). On $]\!]0,\tau[\![$ we can even write $\beta = f(C)$ for some function f which, however, depends on the functions $\widetilde{\mu}$ and $\widetilde{\sigma}$ that are not yet specified. Assume that $f \in C^2$. By Itô's formula we get

$$d\beta_t = \left(f'(C_t)\widetilde{\mu}(C_t) + f''(C_t)\frac{\widetilde{\sigma}(C_t)^2}{2}\right)dt + f'(C_t)\widetilde{\sigma}(C_t)dW_t. \tag{3.7}$$

By comparing the factors of (3.6) and (3.7) we can write down $\widetilde{\mu}$ and $\widetilde{\sigma}$ as functions of f, μ and σ:

$$\begin{aligned} \widetilde{\sigma} &= \frac{\sigma}{f'-1} \\ \widetilde{\mu} &= \left(\mu - \frac{\sigma^2}{2} + \frac{\delta(1+e^{-f})}{e^{-f}} - \frac{\sigma^2}{2}\frac{f''}{(f'-1)^2}\right)\frac{1}{f'-1}. \end{aligned}$$

Note that to get rid of $\widetilde{\pi}_t$ we have used that from $f(C) = \beta = \log\left(\frac{\widetilde{\pi}}{1-\widetilde{\pi}}\right)$ follows $\widetilde{\pi} = \frac{1}{1+e^{-f(C)}}$.

Now that we have expressions for $\widetilde{\mu}$ and $\widetilde{\sigma}$ we can insert them into the optimality equation (3.5) to get an ODE similar to the one in [KMK10]. The ODE in our case is

$$\begin{aligned} & \mu + \frac{1}{2}\left(\frac{\sigma}{f'(x)-1}\right)^2 + \frac{\sigma^2}{f'(x)-1} \\ + & \left(\mu - \frac{\sigma^2}{2} + \frac{\delta(1+e^{-f(x)})}{e^{-f(x)}} - \frac{\sigma^2}{2}\frac{f''(x)}{(f'(x)-1)^2}\right)\frac{1}{f'(x)-1} \\ - & \frac{(\sigma + \frac{\sigma}{f'(x)-1})^2}{1+e^{-f(x)}} + \alpha_1(e^{-x}-1)\left(\frac{1}{1+\frac{e^{-x}-1}{1+e^{-f(x)}}}\right) + \alpha_2(e^{\overline{C}-x}-1)\left(\frac{1}{1+\frac{e^{\overline{C}-x}-1}{1+e^{-f(x)}}}\right) \\ = & \ 0. \end{aligned} \tag{3.8}$$

Remember that apart from a possible bulk trade at time 0 in our suspected optimal strategy the aggregated market buy and sell orders are local times. This implies that the fraction invested in the risky asset also has a local time component, and hence the same is true for β. Thus a smooth function f with $\beta = f(C)$ has to possess an exploding first derivative as in C no local time appears (the ansatz that C resp. $\widetilde{S}$ has no local time makes sense, as it is well known that a local time component in the discounted price process would imply arbitrage, see e.g. Appendix B in [KS98] or [JP05] for an introduction to the problematics). To avoid an explosion, we turn the problem around by considering C as a function of β, i.e. $C = g(\beta) := f^{-1}(\beta)$. Defining

$$B(y,z) := \alpha_1(e^{-z}-1)\left(\frac{1}{1+\frac{e^{-z}-1}{1+e^{-y}}}\right) + \alpha_2(e^{\overline{C}-z}-1)\left(\frac{1}{1+\frac{e^{\overline{C}-z}-1}{1+e^{-y}}}\right), \tag{3.9}$$

we can invert ODE (3.8) and get

$$\begin{aligned} g''(y) &= -\frac{2}{\sigma^2}B(y,g(y)) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}} \\ &+ \left(\frac{6}{\sigma^2}B(y,g(y)) + \frac{4\mu}{\sigma^2} - \frac{2}{1+e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1+e^y)\right)g'(y) \\ &+ \left(-\frac{6}{\sigma^2}B(y,g(y)) - \frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^y)\right)(g'(y))^2 \\ &+ \left(\frac{2}{\sigma^2}B(y,g(y)) - \frac{2\delta}{\sigma^2}(1+e^y)\right)(g'(y))^3. \end{aligned} \tag{3.10}$$

Note that this equation without the term B is the same ODE as in [KMK10]. We need to take care that the local time in β does not show up in C but since local time only occurs at $\underline{\beta}$ and $\overline{\beta}$ by choosing the right boundary conditions for g' this can be avoided easily. Namely, g' has to vanish at the boundaries. Similar to [KMK10] we arrive at the boundary conditions

$$g(\underline{\beta}) = \overline{C}, \qquad g(\overline{\beta}) = 0, \qquad g'(\underline{\beta}) = g'(\overline{\beta}) = 0, \tag{3.11}$$

where $\underline{\beta}$ and $\overline{\beta}$ have to be chosen. Indeed, an application of Itô's formula shows that these boundary conditions for g' imply that C does not have a local time component.

3.4 Existence of a solution to the free boundary problem

Proposition 3.6. *Let $\alpha_1 < \mu\frac{1+\lambda}{\lambda}$, $\alpha_2 < (\sigma^2-\mu)\frac{1+\lambda}{\lambda}$, and $\delta > \alpha_2\lambda$. Then the free boundary problem (3.10)/(3.11) admits a solution $(g,\underline{\beta},\overline{\beta})$ such that $g : [\underline{\beta},\overline{\beta}] \to [0,\overline{C}]$ and g is strictly decreasing.*

The first two parameter restrictions can be interpreted economically quite well, whereas the last restriction is a technical condition, which is sufficient for the existence of a shadow price.

As $\alpha_1, \alpha_2 \geq 0$ the first two parameter restrictions imply that

$$0 < \mu < \sigma^2. \tag{3.1}$$

In the case with proportional transaction costs, (3.1) guarantees that $0 < \pi_{\min} < \pi_{\max} < 1$, i.e. the optimal strategy entails neither leveraging nor shorting of the risky asset. This is not the case when the opportunity to trade at more favorable prices using limit orders exists. Namely, short selling the stock by a limit order and liquidating this position again after the successful execution of a limit buy order leads to some additional expected return whose rate is for small λ roughly $\alpha_1 \lambda$ (note that α_1 is the rate of the arrival times which allow to buy the stock cheaply back, the expected return is earned as long as the investor holds a short position). Thus $\alpha_1 < \mu \frac{1+\lambda}{\lambda}$ guarantees that short selling is not worthwhile. Analogously long positions that are build up with limit buy orders yield an additional expected return with approximative rate $\alpha_2 \lambda$. Thus $\alpha_2 < (\sigma^2 - \mu)\frac{1+\lambda}{\lambda}$ becomes necessary to exclude leveraging. Summing up, the first two conditions are necessary to avoid leveraging and short selling.

Proof. Define for $y, z \in \mathbb{R}$

$$\widetilde{B}(y,z) := \begin{cases} B(y,z) & \text{if } z \in [0, \overline{C}], \\ \alpha_2 \left(e^{\overline{C}} - 1\right)\left(1 + \frac{e^{\overline{C}}-1}{1+e^{-y}}\right)^{-1} & \text{if } z < 0, \\ \alpha_1 \left(e^{-\overline{C}} - 1\right)\left(1 + \frac{e^{-\overline{C}}-1}{1+e^{-y}}\right)^{-1} & \text{if } z > \overline{C}. \end{cases}$$

Note that $\widetilde{B}(y,z)$ is decreasing in y and z. Furthermore, for all $y, z \in \mathbb{R}$ we have

$$-\frac{\alpha_1 \lambda}{1+\lambda} < \widetilde{B}(y,z) < \alpha_2 \lambda.$$

Instead of dealing with the original free boundary problem (3.10)/(3.11), we now replace (3.10) with

$$\begin{aligned} g''(y) &= -\frac{2}{\sigma^2}\widetilde{B}(y,g(y)) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}} \\ &+ \left(\frac{6}{\sigma^2}\widetilde{B}(y,g(y)) + \frac{4\mu}{\sigma^2} - \frac{2}{1+e^{-y}} - 1 - \frac{2\delta}{\sigma^2}(1+e^y)\right) g'(y) \\ &+ \left(-\frac{6}{\sigma^2}\widetilde{B}(y,g(y)) - \frac{2\mu}{\sigma^2} + 1 + \frac{4\delta}{\sigma^2}(1+e^y)\right)(g'(y))^2 \\ &+ \left(\frac{2}{\sigma^2}\widetilde{B}(y,g(y)) - \frac{2\delta}{\sigma^2}(1+e^y)\right)(g'(y))^3, \end{aligned} \tag{3.2}$$

whereas the boundary condition (3.11) stays the same. We will see that the change from B to $\widetilde{B}$ guarantees that functions satisfying the ODE do not explode, because the impact of $g(y)$ on $g''(y)$ remains bounded, even when $g(y)$ leaves $[0,\overline{C}]$. Note that if we show the existence of a solution $g : [\underline{\beta},\overline{\beta}] \to [0,\overline{C}]$ to this modified free boundary problem, we have also shown the existence of a solution to the original free boundary problem, since $B(y,z) = \widetilde{B}(y,z)$ on $\mathbb{R} \times [0,\overline{C}]$. Denote by y_0 the unique root of the function

$$H(y) := \frac{-\alpha_1}{\sigma^2}\left(e^{-\overline{C}}-1\right)\left(1+\frac{e^{-\overline{C}}-1}{1+e^{-y}}\right)^{-1} - \frac{\mu}{\sigma^2} + \frac{1}{1+e^{-y}}.$$

Such an y_0 exists. Indeed, we have assumed $\alpha_1 < \mu\frac{1+\lambda}{\lambda}$, which implies $\frac{\alpha_1\lambda-\mu-\mu\lambda}{\sigma^2+\sigma^2\lambda} < 0$. Thus, as $\overline{C} = \log(1+\lambda)$, it follows $\lim_{y\to-\infty} H(y) < 0$. $e^{-\overline{C}} - 1 < 0$ and $\mu < \sigma^2$ imply $\lim_{y\to\infty} H(y) > 0$. Since H is continuous, the intermediate value theorem implies the existence of a y_0, which is unique since H is strictly increasing.

For any $\Delta > 0$ let $\underline{\beta}_\Delta := y_0 - \Delta$. For any choice of $\Delta > 0$ the initial value problem given by (3.2) with initial conditions $g(\underline{\beta}_\Delta) = \overline{C}$ and $g'(\underline{\beta}_\Delta) = 0$ admits a unique local solution g_Δ. Because $\delta - \alpha_2\lambda > 0$, we can define a real number $\underline{M} < 0$ by

$$\underline{M} := \min\left\{-\sqrt[3]{\frac{3(\alpha_2\lambda+\mu)}{\delta-\alpha_2\lambda}}, -\sqrt{\frac{3(3\alpha_2\lambda+2\mu)}{\delta-\alpha_2\lambda}}, -\frac{3\alpha_2\lambda+\mu}{\delta-\alpha_2\lambda}\right\}.$$

For $g'_\Delta(y) < \underline{M}$ we have $g''_\Delta(y) > 0$. Similarly, define a real number $\overline{M} > 0$ by

$$\overline{M} := \max\left\{\sqrt[3]{\frac{3\left(\frac{\alpha_1\lambda}{1+\lambda}+\sigma^2\right)}{\delta-\alpha_2\lambda}}, \sqrt{\frac{3(3\alpha_2\lambda+2\mu)}{\delta-\alpha_2\lambda}}, \frac{3\left(\frac{6\alpha_1\lambda}{1+\lambda}+\sigma^2+4\delta\right)}{2(\delta-\alpha_2\lambda)}\right\}.$$

For $g'_\Delta(y) > \overline{M}$ we have $g''_\Delta(y) < 0$. Hence, $g'_\Delta(y) \in [\underline{M},\overline{M}]$ for all $y \geq \underline{\beta}_\Delta$ and the maximal interval of existence for g_Δ is $\mathbb{R}$. Note that $\underline{M},\overline{M}$ do not depend on the choice of Δ.

By $\alpha_2 < (\sigma^2-\mu)\frac{1+\lambda}{\lambda}$, there exist $y^\star \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$-\frac{2}{\sigma^2}\widetilde{B}(y,z) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}} > \varepsilon$$

for all $y \geq y^\star, z \in \mathbb{R}$ (this can be proved analogously to the existence of y_0). Combining this with (3.2) shows that there even exists an $\overline{y}_\Delta$ such that $g''_\Delta(y) > \varepsilon$ for $g'_\Delta(y) \leq 0$ and $y \geq \overline{y}_\Delta$. Thus, g'_Δ has at least another root larger than $\underline{\beta}_\Delta$, i.e.

$$\overline{\beta}_\Delta := \min\{y > \underline{\beta}_\Delta : g'_\Delta(y) = 0\} < \infty.$$

Hence, by definition g_Δ is decreasing on $[\underline{\beta}_\Delta,\overline{\beta}_\Delta]$. The remainder of the proof consists in showing that $g_\Delta(\overline{\beta}_\Delta) \to \overline{C}$ for $\Delta \to 0$, $g_\Delta(\overline{\beta}_\Delta) \to -\infty$ for $\Delta \to \infty$ and that $\Delta \mapsto g_\Delta(\overline{\beta}_\Delta)$ is a continuous mapping. Then, by the intermediate value theorem, there exists a Δ such that g_Δ is a solution to the free boundary problem (3.2)/(3.11).

Step 1. We prove that $g_\Delta(\overline{\beta}_\Delta) \to \overline{C}$ for $\Delta \to 0$. The boundedness of $(\Delta, y) \mapsto g'_\Delta(y)$ together with (3.2) implies that $|g''_\Delta(y)|$ is bounded by a constant M'' on $[y_0 - 1, y_0 + 1]$. For $\Delta < 1$ and $y \in [y_0 - 1, y_0 + 1]$ we get $|g'_\Delta(y)| \leq (y - y_0 + \Delta)M''$. Hence, by (3.2), $g_\Delta(y) \to \overline{C}$ for $\Delta \to 0$ and $y \to y_0$, the continuity of $\widetilde{B}$, and the definition of y_0 we have that

$$\sup_{y \in [y_0 - \Delta, y_0 + \widetilde{\Delta}]} |g''_\Delta(y)| \to 0 \qquad \text{for } \Delta, \widetilde{\Delta} \downarrow 0. \tag{3.3}$$

Firstly, by (3.3) the last three summands in (3.2) are of order $o(y - y_0 + \Delta)$ for $(\Delta, y) \to (0, y_0)$. Let us rewrite the first summand of (3.2) as

$$\begin{aligned} & -\frac{2}{\sigma^2}\widetilde{B}(y, g_\Delta(y)) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}} \\ = \; & \left(-\frac{2}{\sigma^2}\widetilde{B}(y, g_\Delta(y)) + \frac{2}{\sigma^2}\widetilde{B}(y, \overline{C})\right) + \left(-\frac{2}{\sigma^2}\widetilde{B}(y, \overline{C}) - \frac{2\mu}{\sigma^2} + \frac{2}{1+e^{-y}}\right). \end{aligned} \tag{3.4}$$

Secondly, because of $g'_\Delta(y_0 - \Delta) = 0$, a first order Taylor expansion of the first summand in (3.4) at $y_0 - \Delta$ shows that

$$\begin{aligned} & -\frac{2}{\sigma^2}\widetilde{B}(y, g_\Delta(y)) + \frac{2}{\sigma^2}\widetilde{B}(y, \overline{C}) \\ = \; & \frac{1}{2}\left(g''_\Delta(\xi_\Delta)\partial_2 \widetilde{B}(y, g_\Delta(\xi_\Delta)) + (g'_\Delta(\xi_\Delta))^2 \partial_{22}\widetilde{B}(y, g_\Delta(\xi_\Delta))\right)(y - y_0 + \Delta)^2, \end{aligned}$$

for $\xi_\Delta \in [y_0 - \Delta, y]$, i.e. this term is also of order $o(y - y_0 + \Delta)$ for $(\Delta, y) \to (0, y_0)$.

Thirdly, a first order Taylor expansion of the second summand in (3.4) at y_0 shows that the term can be written as $a(y - y_0) + o(y - y_0)$, where $a := -\frac{2}{\sigma^2}\partial_1 \widetilde{B}(y_0, \overline{C})) + \frac{2e^{-y_0}}{(1+e^{-y_0})^2} > 0$. Combining the three points above it follows that

$$g''_\Delta(y) = a(y - y_0) + o(y - y_0) + o(y - y_0 + \Delta), \quad \text{for } (\Delta, y) \to (0, y_0).$$

Thus, for any constant $K > 0$ we can choose Δ small enough that $g''_\Delta(y) > \frac{a}{2}\Delta$ on $y \in [y_0 + \Delta, y_0 + (K+1)\Delta]$. Hence,

$$\overline{\beta}_\Delta - \underline{\beta}_\Delta < 2\Delta + \frac{4\Delta \sup_{y \in [y_0 - \Delta, y_0 + \Delta]} |g''_\Delta(y)|}{a\Delta} \to 0, \qquad \text{for } \Delta \to 0.$$

Since $(y, \Delta) \mapsto g'_\Delta$ is bounded it follows that $g_\Delta(\overline{\beta}_\Delta) \to \overline{C}$ for $\Delta \to 0$.

Step 2. We prove that $g_\Delta(\overline{\beta}_\Delta) \to -\infty$ for $\Delta \to \infty$. Remember that the definition of y_0 and the strict monotonicity of H imply $H(y^\star)<0$ for any $y^\star<y_0$. Let

$$\begin{aligned}\widetilde{M}(y^\star) \quad := \quad \max\Bigg\{ & \frac{\frac{1}{3}H(y^\star)}{\frac{6}{\sigma^2}\frac{\alpha_1\lambda}{1+\lambda}+3+\frac{2\delta}{\sigma^2}(1+e^{y^\star})}, \\ & -\sqrt{\frac{-\frac{1}{3}H(y^\star)}{\frac{6}{\sigma^2}\frac{\alpha_1\lambda}{1+\lambda}+1+\frac{4\delta}{\sigma^2}(1+e^{y^\star})}}, -\sqrt[3]{\frac{-\frac{1}{3}H(y^\star)}{\frac{2}{\sigma^2}\frac{\alpha_1\lambda}{1+\lambda}+\frac{2\delta}{\sigma^2}(1+e^{y^\star})}}\Bigg\}<0.\end{aligned}$$

For $y\le y^\star$ and $0\ge g'_\Delta(y)>\widetilde{M}(y^\star)$ we have that $g''_\Delta(y)<H(y^\star)<0$. By $g''_\Delta(\underline{\beta}_\Delta)<0$, this yields $g'_\Delta(y)<0$ for $y\le y^\star$ and also $g'_\Delta(y)\le\widetilde{M}(y^\star)$ for $y\in[y_0-\Delta+\frac{\widetilde{M}(y^\star)}{H(y^\star)},y^\star]$. Therefore, $g_\Delta(\overline{\beta}_\Delta)\to-\infty$ as $\Delta\to\infty$.

Step 3. We prove that $\Delta\mapsto g_\Delta(\overline{\beta}_\Delta)$ is continuous. By Theorem 2.1 in [Har64] and because for every choice of $\Delta\in(0,\infty)$ the maximal interval of existence of g_Δ is $\mathbb{R}$, it follows that the general solution $(g,g')(\Delta,y):=(g_\Delta(y),g'_\Delta(y)):(0,\infty)\times\mathbb{R}\to\mathbb{R}^2$ is continuous. Thus, (g_Δ,g'_Δ) converges to $\left(g_{\Delta_0},g'_{\Delta_0}\right)$ uniformly on compacts as $\Delta\to\Delta_0$.

Therefore, it is sufficient to show that $\Delta\to\Delta_0$ implies $\overline{\beta}_\Delta\to\overline{\beta}_{\Delta_0}$. Fix $\Delta_0\in(0,\infty)$. To verify that $\liminf_{\Delta\to\Delta_0}\overline{\beta}_\Delta\ge\overline{\beta}_{\Delta_0}$ note that by Step 2 we have $g'_\Delta(y)<0$ for all $\Delta>0$, $y<y_0$ (as $y^\star$ was chosen arbitrary). In addition, given an $\varepsilon>0$, g'_{Δ_0} is strictly separated from $[0,\infty)$ on $[y_0,\overline{\beta}_{\Delta_0}-\varepsilon]$. By the uniform convergence on compacts of g'_Δ to g'_{Δ_0}, it follows that $\liminf_{\Delta\to\Delta_0}\overline{\beta}_\Delta\ge\overline{\beta}_{\Delta_0}$.

By the continuity of g''_{Δ_0} we have $g''_{\Delta_0}(\overline{\beta}_{\Delta_0})\ge0$. In the case that $g''_{\Delta_0}(\overline{\beta}_{\Delta_0})>0$, a first order Taylor expansion of g'_{Δ_0} at $\overline{\beta}_{\Delta_0}$ shows that $g'_{\Delta_0}(y)>0$ immediately after $\overline{\beta}_{\Delta_0}$. Otherwise, i.e. if $g''_{\Delta_0}(\overline{\beta}_{\Delta_0})=0$, the same fact follows from a second order Taylor expansion of g'_{Δ_0} at $\overline{\beta}_{\Delta_0}$, because for $g'_{\Delta_0}(\overline{\beta}_{\Delta_0})=g''_{\Delta_0}(\overline{\beta}_{\Delta_0})=0$ we have $g'''_{\Delta_0}(\overline{\beta}_{\Delta_0})=-\frac{2}{\sigma^2}\partial_1\widetilde{B}(\overline{\beta}_{\Delta_0},g_{\Delta_0}(\overline{\beta}_{\Delta_0}))+\frac{2\exp(-\overline{\beta}_{\Delta_0})}{(1+\exp(-\overline{\beta}_{\Delta_0}))^2}>0$. Here the definition of $\widetilde{B}$ requires us to assume $g_{\Delta_0}(\overline{\beta}_{\Delta_0})\neq0$ to ensure the differentiability of g''_{Δ_0} at $\overline{\beta}_{\Delta_0}$, but this is not problematic, because otherwise $(g_{\Delta_0},\underline{\beta}_{\Delta_0},\overline{\beta}_{\Delta_0})$ would already be a solution to the free boundary problem. Thus, there exists an $\varepsilon_0>0$ such that $g'_{\Delta_0}(\overline{\beta}_{\Delta_0}+\varepsilon)>0$ for any $\varepsilon\in(0,\varepsilon_0)$. This implies that $\limsup_{\Delta\to\Delta_0}\overline{\beta}_\Delta\le\overline{\beta}_{\Delta_0}$ and altogether continuity. □

3.5 Proof of the existence of a shadow price

Throughout the section we assume that the assumptions of Proposition 3.6 are satisfied so that the free boundary problem specified in (3.10) and (3.11) has a solution $(g,\underline{\beta},\overline{\beta})$ with $g:[\underline{\beta},\overline{\beta}]\to[0,\overline{C}]$ strictly decreasing.

Lemma 3.7. *Let* $\beta_0 \in [\underline{\beta}, \overline{\beta}]$ *and*

$$a(y) := \left(\mu - \frac{\sigma^2}{2} + \delta(1+e^y) + \frac{\sigma^2 g''(y)}{2(1-g'(y))^2}\right)\frac{1}{1-g'(y)}, \qquad b(y) := \frac{\sigma}{1-g'(y)}$$

for $y \in [\underline{\beta}, \overline{\beta}]$. *Then there exists a unique solution* (β, Ψ) *to the following stochastic variational inequality*

(i) β *is càdlàg and takes values in* $[\underline{\beta}, \overline{\beta}]$. Ψ *is continuous and of finite variation with starting value* $\Psi_0 = 0$,

(ii)

$$\begin{aligned}\beta_t &= \beta_0 + \int_0^t a(\beta_{s-})ds + \int_0^t b(\beta_{s-})dW_s \\ &\quad + \sum_{s\le t}\left((\overline{\beta} - \beta_{s-})\Delta N_s^1 + (\underline{\beta} - \beta_{s-})\Delta N_s^2\right) + \Psi_t,\end{aligned} \tag{3.1}$$

(iii) for every progressively measurable process z *which has càdlàg paths and takes values in* $[\underline{\beta}, \overline{\beta}]$, *we have*

$$\int_0^t (\beta_s - z_s)\, d\Psi_s \le 0, \qquad \forall t \le 0. \tag{3.2}$$

Proof. We want to apply Theorem 1 in [MR85], which guarantees existence and uniqueness of reflected diffusion processes with jumps in convex domains under certain conditions. Thus we only need to verify that the conditions of the theorem are fulfilled in our setting.

Firstly, $(\underline{\beta}, \overline{\beta})$ is trivially bounded and convex. Secondly, the jump term in (3.1) ensures that all jumps from $[\underline{\beta}, \overline{\beta}]$ are inside $[\underline{\beta}, \overline{\beta}]$. All that is left is to check a Lipschitz-type condition. Note that if g is a solution to ODE (3.10) on $[\underline{\beta}, \overline{\beta}]$ the functions g, g' and g'' are continuous and therefore bounded on the compact set $[\underline{\beta}, \overline{\beta}]$. Furthermore, as we know that $g' \le 0$ on $[\underline{\beta}, \overline{\beta}]$, the derivative b' of b is bounded on $[\underline{\beta}, \overline{\beta}]$. In addition, this also implies that B defined in (3.9) is bounded on $[\underline{\beta}, \overline{\beta}]$ as well, and the same is true for $\partial_1 B$ and $\partial_2 B$. Thus also g''' is bounded on $[\underline{\beta}, \overline{\beta}]$ (using that the solution g of the free boundary problem (3.10)/(3.11) can be extended to some neighborhood of $\underline{\beta}$ and $\overline{\beta}$, resp.) This implies that even the derivative a' of a is bounded on $[\underline{\beta}, \overline{\beta}]$. $\square$

Remark 3.8. *Since* Ψ *is of finite variation there exist two non-decreasing processes* $\overline{\Psi}$ *and* $\underline{\Psi}$ *such that* $\Psi = \overline{\Psi} - \underline{\Psi}$ *and* $\mathrm{Var}(\Psi) = \overline{\Psi} + \underline{\Psi}$. *Furthermore, (3.2) implies that* $\overline{\Psi}$ *increases only on* $\{\beta = \underline{\beta}\}$ *(resp. on* $\{\beta_- = \underline{\beta}\}$*)and* $\underline{\Psi}$ *increases only on* $\{\beta = \overline{\beta}\}$ *(resp. on* $\{\beta_- = \overline{\beta}\}$*).*

Lemma 3.9. *For $\beta_0 \in [\underline{\beta}, \overline{\beta}]$ let $a(\cdot), b(\cdot)$ and the process β be from Lemma 3.7. Then $C := g(\beta)$ is a $[0, \overline{C}]$-valued semimartingale with*

$$\begin{aligned} dC_t &= \left(g'(\beta_{t-})a(\beta_{t-}) + \frac{1}{2} g''(\beta_{t-}) b(\beta_{t-})^2 \right) dt + g'(\beta_{t-}) b(\beta_{t-}) dW_t \\ &- g(\beta_{t-}) dN_t^1 + (\overline{C} - g(\beta_{t-})) dN_t^2 \end{aligned}$$

and $\widetilde{S} := \underline{S} e^C$ satisfies

$$\begin{aligned} d\widetilde{S}_t &= \widetilde{S}_{t-} \left(g'(\beta_{t-})a(\beta_{t-}) + \frac{1}{2} g''(\beta_{t-}) b(\beta_{t-})^2 + \frac{1}{2} \left(g'(\beta_{t-}) b(\beta_{t-}) \right)^2 + \mu + \sigma g'(\beta_{t-}) b(\beta_{t-}) \right) dt \\ &+ \widetilde{S}_{t-} \left(g'(\beta_{t-}) b(\beta_{t-}) + \sigma \right) dW_t \\ &+ \widetilde{S}_{t-} \left(\exp\{ -g(\beta_{t-}) \Delta N_t^1 + (\overline{C} - g(\beta_{t-})) \Delta N_t^2 \} - 1 \right). \end{aligned}$$

Proof. Since $g'(\underline{\beta}) = g'(\overline{\beta}) = 0$ the result follows by Itô's lemma, the integration by parts formula and Remark 3.8. □

Lemma 3.10. *$\widetilde{S}$ is a special semimartingale. The differential semimartingale characteristics of $\widetilde{S}$ w.r.t I and "truncation function" $h(x) = x$ are*

$$\begin{aligned} \widetilde{b}_t &= \widetilde{S}_{t-} \left(-B(\beta_{t-}, g(\beta_{t-})) + \frac{1}{1 + e^{-\beta_{t-}}} \left(\frac{\sigma}{1 - g'(\beta_{t-})} \right)^2 \right) + \int x \widetilde{F}_t(dx) \\ \widetilde{c}_t &= \widetilde{S}_{t-}^2 \left(\frac{\sigma}{1 - g'(\beta_{t-})} \right)^2 \\ \widetilde{F}_t(\omega, dx) &= \alpha_1 \delta_{x_1(\omega,t)}(dx) + \alpha_2 \delta_{x_2(\omega,t)}(dx), \end{aligned}$$

with

$$x_1(\omega, t) := \widetilde{S}_{t-}(\omega)(e^{-C_{t-}(\omega)} - 1), \quad x_2(\omega, t) := \widetilde{S}_{t-}(\omega)(e^{\overline{C} - C_{t-}(\omega)} - 1).$$

Proof. With the definition of $a(\cdot)$ and $b(\cdot)$ in Lemma 3.7 and ODE (3.10) we get

$$\begin{aligned} g'(\beta_{t-})a(\beta_{t-}) = &- \frac{\sigma^2}{2} \frac{g'(\beta_{t-})}{(1 - g'(\beta_{t-}))^2} + g'(\beta_{t-}) \delta (1 + e^{\beta_{t-}}) - g'(\beta_{t-}) B(\beta_{t-}, g(\beta_{t-})) \\ &+ \frac{\sigma^2}{1 + e^{-\beta_{t-}}} \frac{g'(\beta_{t-})}{(1 - g'(\beta_{t-}))^2}, \\ \frac{1}{2} g''(\beta_{t-}) b(\beta_{t-})^2 = &- B(\beta_{t-}, g(\beta_{t-}))(1 - g'(\beta_{t-})) - \mu + \frac{\sigma^2}{1 + e^{-\beta_{t-}}} \frac{1}{1 - g'(\beta_{t-})} \\ &- \frac{\sigma^2}{2} \frac{g'(\beta_{t-})}{1 - g'(\beta_{t-})} - g'(\beta_{t-}) \delta (1 + e^{\beta_{t-}}), \\ \frac{1}{2} \left(g'(\beta_{t-}) b(\beta_{t-}) \right)^2 = &\frac{\sigma^2}{2} \left(\frac{g'(\beta_{t-})}{1 - g'(\beta_{t-})} \right)^2, \\ \sigma g'(\beta_{t-}) b(\beta_{t-})) = &\sigma^2 \frac{g'(\beta_{t-})}{1 - g'(\beta_{t-})}. \end{aligned}$$

The result now follows from Lemma 3.9. □

Proposition 3.11. *Given initial endowment* (η^0,η^1)*, let* β_0 *be defined by*

$$\beta_0 := \begin{cases} \underline{\beta} & \text{if} \quad \frac{\eta^1\overline{s}}{\eta^0+\eta^1\overline{s}} < \frac{1}{1+e^{-\underline{\beta}}}, \qquad (\overline{s} := \overline{S}_0) \\ \overline{\beta} & \text{if} \quad \frac{\eta^1\underline{s}}{\eta^0+\eta^1\underline{s}} > \frac{1}{1+e^{-\overline{\beta}}}, \end{cases}$$

or else, let β_0 *be the solution of*

$$\frac{\eta^1 e^{g(y)}\underline{s}}{\eta^0+\eta^1 e^{g(y)}\underline{s}} = \frac{1}{1+e^{-y}}.$$

Given the reflected jump-diffusion β *starting in* β_0 *as is Lemma 3.7 and the resulting* $\widetilde{S}$ *of Lemma 3.9 let*

$$\begin{aligned} \widetilde{V}_t &:= (\eta^0+\eta^1\widetilde{S}_0)\mathcal{E}\left(\int_0^\cdot \frac{1}{(1+e^{-\beta_{s-}})\widetilde{S}_{s-}} d\widetilde{S}_s - \int_0^\cdot \delta ds\right)_t, \qquad t\geq 0, \\ c_t &:= \delta\widetilde{V}_t, \qquad t \geq 0, \\ \varphi_t^1 &:= \frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}\widetilde{V}_{t-}, \qquad \varphi_t^0 := \widetilde{V}_{t-} - \varphi_t^1\widetilde{S}_{t-}, \qquad t>0, \end{aligned}$$

and let $\varphi_0^0 := \eta^0$ *and* $\varphi_0^1 := \eta^1$*. Then* $\widetilde{V}_t = \eta^0+\eta^1\widetilde{S}_0 + \int_0^t \varphi_s^1 d\widetilde{S}_s - \int_0^t c_s ds$ *and* (φ^0,φ^1,c) *is an optimal strategy for initial endowment* (η^0,η^1) *in the frictionless market with price process* $\widetilde{S}$*.*

Proof. Given the semimartingale characteristics in Lemma 3.10 we need to check that $H_t := \frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}$ solves the optimality equation of Goll and Kallsen ([GK00], Theorem 3.1), i.e. that $(P\otimes I)$-a.e.

$$\widetilde{b}_t - \widetilde{c}_t H_t + \int\left(\frac{x}{1+H_t x} - x\right)\widetilde{F}_t(dx) = 0$$

holds. Of course the choice of H_0 is irrelevant for optimality.

Moreover, note that for $t>0$ the term $-\widetilde{S}_{t-}B(\beta_{t-},g(\beta_{t-})) + \int x\widetilde{F}_t(dx)$ in $\widetilde{b}_t$ and the integral term in the optimality equation cancel each other. The key to seeing this is

$$\begin{aligned} \int\left(\frac{x}{1+H_t x}\right)\widetilde{F}_t(dx) &= \int\left(\frac{x}{1+H_t x}\right)\alpha_1\delta_{x_1}(dx) + \int\left(\frac{x}{1+H_t x}\right)\alpha_2\delta_{x_2}(dx) \\ &= \frac{\alpha_1\widetilde{S}_{t-}\left(e^{-g(\beta_{t-})}-1\right)}{1+\frac{\widetilde{S}_{t-}\left(e^{-g(\beta_{t-})}-1\right)}{\left(1+e^{-\beta_{t-}}\right)\widetilde{S}_{t-}}} + \frac{\alpha_2\widetilde{S}_{t-}\left(e^{\overline{C}-g(\beta_{t-})}-1\right)}{1+\frac{\widetilde{S}_{t-}\left(e^{\overline{C}-g(\beta_{t-})}-1\right)}{\left(1+e^{-\beta_{t-}}\right)\widetilde{S}_{t-}}} \\ &= \alpha_1\widetilde{S}_{t-}\left(e^{-g(\beta_{t-})}-1\right)\left(\frac{1}{1+\frac{\left(e^{-g(\beta_{t-})}-1\right)}{1+e^{-\beta_{t-}}}}\right) \\ &+ \alpha_2\widetilde{S}_{t-}\left(e^{\overline{C}-g(\beta_{t-})}-1\right)\left(\frac{1}{1+\frac{\left(e^{\overline{C}-g(\beta_{t-})}-1\right)}{1+e^{-\beta_{t-}}}}\right) \\ &= \widetilde{S}_{t-}B(\beta_{t-},g(\beta_{t-})), \end{aligned}$$

where the second equality follows from the definition of x_1 and x_2 (in Lemma 3.10) and the definition of H. Thus the specified strategy is optimal in the frictionless market. $\square$

Lemma 3.12. *There exist two deterministic functions $F^1 : [\underline{\beta},\overline{\beta}] \to [0,\infty)$ and $F^2 : [\underline{\beta},\overline{\beta}] \to (-\infty,0]$ such that for $t > 0$*

$$\varphi_t^1 - \varphi_0^1 = \int_0^t \frac{\varphi_s^1 e^{-\beta_{s-}}}{1+e^{-\beta_{s-}}} d\Psi_s + \sum_{0<s<t} \varphi_s^1 (e^{F^1(\beta_-)} - 1)\Delta N_s^1 + \sum_{0<s<t} \varphi_s^1 (e^{F^2(\beta_-)} - 1)\Delta N_s^2. \qquad (3.3)$$

Remark 3.13. *As we will see in the proof of Theorem 3.14, Lemma 3.12 can be interpreted in the following way. The first summand on the right-hand side of (3.3) tells us that market orders are only used when the proportion invested in the risky asset is at the boundary. The last two summands imply that the sizes of the limit orders divided by the current holdings in the stock are deterministic functions of the current fraction of wealth invested in the stock (in terms of the shadow price).*

Proof. By Proposition 3.11 φ^1 is càglàd. Therefore, it is sufficient to show that (3.3) holds for the right-continuous versions of the processes on both sides of the equation.

After taking the logarithm of φ_+^1 we can write its dynamics as

$$d\log\varphi_{t+}^1 = d\log\widetilde{V}_t - d\log\widetilde{S}_t - d\log(1+e^{-\beta_t}).$$

By Itô's formula and Proposition 3.11 we have that

$$\begin{aligned}
d\log\widetilde{V}_t &= \frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}} d\widetilde{S}_t - \delta dt - \frac{1}{2}\left(\frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}\right)^2 d[\widetilde{S},\widetilde{S}]_t^c \\
&+ \log\left(1+\frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}\Delta\widetilde{S}_t\right) - \frac{1}{(1+e^{-\beta_{t-}})\widetilde{S}_{t-}}\Delta\widetilde{S}_t \\
&= \left[\frac{1}{1+e^{-\beta_{t-}}}\Big(g'(\beta_{t-})a(\beta_{t-}) + \frac{1}{2}g''(\beta_{t-})b(\beta_{t-})^2 + \frac{1}{2}(g'(\beta_{t-})b(\beta_{t-}))^2\right. \\
&+ \left.\mu + \sigma g'(\beta_{t-})b(\beta_{t-})\Big) - \delta - \frac{1}{2}\frac{(g'(\beta_{t-})b(\beta_{t-})+\sigma)^2}{(1+e^{-\beta_{t-}})^2}\right] dt \\
&+ \frac{g'(\beta_{t-})b(\beta_{t-})+\sigma}{1+e^{-\beta_{t-}}} dW_t \\
&+ \log\left(1+\frac{\exp\{-g(\beta_{t-})\Delta N_t^1 + (\overline{C}-g(\beta_{t-}))\Delta N_t^2\}-1}{1+e^{-\beta_{t-}}}\right).
\end{aligned} \qquad (3.4)$$

Because $\widetilde{S}$ is defined as $\underline{S}\exp(C)$ we get

$$\begin{aligned}
-d\log\widetilde{S}_t &= \left(\frac{\sigma^2}{2}-\mu\right)dt-\sigma dW_t-dC_t \\
&= \left(\frac{\sigma^2}{2}-\mu-g'(\beta_{t-})a(\beta_{t-})-\frac{1}{2}g''(\beta_{t-})b(\beta_{t-})^2\right)dt \\
&- \left(g'(\beta_{t-})b(\beta_{t-})+\sigma\right)dW_t \\
&+ g(\beta_{t-})\Delta N_t^1-\left(\overline{C}-g(\beta_{t-})\right)\Delta N_t^2.
\end{aligned}$$

Using the properties of β from Lemma 3.7, another application of Itô's formula yields

$$\begin{aligned}
-d\log(1+e^{-\beta_t}) &= \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}d\beta_t-\frac{1}{2}\frac{e^{-\beta_{t-}}}{(1+e^{-\beta_{t-}})^2}d[\beta,\beta]_t^C \\
&- \left(\log(1+e^{-\beta_t})-\log(1+e^{-\beta_{t-}})\right)-\frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}\Delta\beta_t \\
&= \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}\left(a(\beta_{t-})-\frac{1}{2}\frac{e^{-\beta_{t-}}}{(1+e^{-\beta_{t-}})^2}b(\beta_{t-})^2\right)dt \\
&+ \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}b(\beta_{t-})dW_t \\
&+ \frac{e^{-\beta_{t-}}}{1+e^{-\beta_{t-}}}\left(d\overline{\Psi}_t-d\underline{\Psi}_t\right) \\
&- \left(\log(1+e^{-\overline{\beta}})-\log(1+e^{-\beta_{t-}})\right)\Delta N_t^1 \\
&- \left(\log(1+e^{-\underline{\beta}})-\log(1+e^{-\beta_{t-}})\right)\Delta N_t^2.
\end{aligned}$$

Plugging in ODE (3.10) for g'' and summing up we see that all dt-terms and all dW-terms of the process $\log\varphi_+^1$ cancel out. Define

$$\begin{aligned}
F^1(x) &:= \log\left(1+\frac{\exp\{-g(x)\}-1}{1+e^{-x}}\right)+g(x)-\log\left(\frac{1+e^{-\overline{\beta}}}{1+e^{-x}}\right), \\
F^2(x) &:= \log\left(1+\frac{\exp\{(\overline{C}-g(x))\}-1}{1+e^{-x}}\right)-\left(\overline{C}-g(x)\right)-\log\left(\frac{1+e^{-\underline{\beta}}}{1+e^{-x}}\right).
\end{aligned} \tag{3.5}$$

Itô's formula applied to the semimartingale $\log(\varphi_+^1)$ and the C^2-function $x\mapsto\exp(x)$ shows that (3.3) holds for the right-continuous versions. To finish the proof note that $F^1(x)\geq 0$ for all $x\in[\underline{\beta},\overline{\beta}]$ follows from $g\geq 0$. $F^2(x)\leq 0$ for all $x\in[\underline{\beta},\overline{\beta}]$ follows analogously, now making use of $\overline{C}-g\geq 0$. □

Theorem 3.14. *$\widetilde{S}$ is a shadow price process. An optimal strategy $\mathfrak{S}$ in the limit order market* *is given by*

$$M_t^B = 1_{\{t>0\}} \left(\frac{\eta^0 + \eta^1 \underline{s}(1+\lambda)}{\left(1+\exp(-\underline{\beta})\right)\underline{s}(1+\lambda)} - \eta^1 \right)^+ + \int_0^t 1_{\{\beta_- = \underline{\beta}\}} \frac{\varphi^1 e^{-\underline{\beta}}}{1+e^{-\underline{\beta}}} d\Psi,$$

$$M_t^S = 1_{\{t>0\}} \left(\frac{\eta^0 + \eta^1 \underline{s}}{\left(1+\exp(-\overline{\beta})\right)\underline{s}} - \eta^1 \right)^- - \int_0^t 1_{\{\beta_- = \overline{\beta}\}} \frac{\varphi^1 e^{-\overline{\beta}}}{1+e^{-\overline{\beta}}} d\Psi,$$

$$L_t^B = \varphi_t^1 (e^{F^1(\beta_{t-})} - 1), \quad L_t^S = -\varphi_t^1 (e^{F^2(\beta_{t-})} - 1),$$

and $c_t = \delta \widetilde{V}_t$*, where* F^1*,* F^2 *are defined in (3.5) and* $\underline{s} = \underline{S}_0$*. The strategy yields finite expected utility.*

Remark 3.15. *Theorem 3.14 can be interpreted as follows.* M^B *is the minimal amount of risky assets the investor has to buy by market orders to prevent that the fraction of wealth invested in the risky asset leaves the acceptable interval at the lower boundary (the first summand of* M^B *puts the fraction on the lower boundary if it starts below the interval at time zero). Analogously,* M^S *is the minimal amount of risky assets the investor has to sell by market orders to prevent that the fraction of wealth invested in the risky asset leaves the interval at the upper boundary. Mathematically these minimal trades correspond to the local time of the two dimensional wealth process at the boundaries of the cone illustrated in Fig. 3.4.*

The choice of L^B *(resp.* L^S*) ensures that after a successful execution of the limit buy order (resp. the limit sell order) the fraction of wealth invested in the risky asset jumps on the upper boundary (resp. the lower boundary) of the interval. As* $L^B > 0$ *and* $L^S > 0$ *apart from the time at which the wealth process is at the boundary (which has Lebesgue measure zero) the investor is always willing to trade both with limit buy and with limit sell orders. However, the order sizes depend on how far away the wealth process is from the boundaries and they have to be adjusted continuously with the movements of the process* $(\beta_t)_{t\geq 0}$*.*

Remark 3.16. *An important detail in model (3.1) is that a limit order has to be in the book already at* $\Delta N^i = 1$ *to be executed against the arising market order. This market mechanism is reflected in the condition that the limit order sizes* L^B *and* L^S *have to be predictable. By contrast, in the frictionless market with price process* $\widetilde{S}$ *the buying decision at a time* τ *at which* $\widetilde{S}_\tau = \underline{S}_\tau$*, may depend on all new information available at time* τ *(Note that by the standard convention in frictionless market models a simple purchase at time* τ *only affects the simple trading strategy on* (τ, ∞)*, i.e. the value of the strategy at* τ *itself is not affected. Thus the latter is no contradiction* *to the fact that the strategy in the frictionless market with price process $\widetilde{S}$ is predictable as well. See also the discussion after Definition 3.2). However, as the jumps of $\widetilde{S}$ always land on one of the two continuous processes $\underline{S}$ or $\overline{S}$, and limit orders are submitted* contingent *that they can be executed, it turns out that this subtle distinction does not matter.*

of Theorem 3.14. By construction of $\widetilde{S}$ (3.5) is clearly satisfied. All we have to do is to construct an admissible strategy $\mathfrak{S} = (M^B, M^S, L^B, L^S, c)$ in the limit order market such that the associated portfolio process of $\mathfrak{S}$ as defined in (3.1) is equal to the optimal strategy in the frictionless market $(\varphi^0, \varphi^1, c)$ from Proposition 3.11.

By Lemma 3.12 φ^1 is of finite variation, hence we can write it as the difference of two increasing càglàd processes Z^1 and Z^2, i.e. $\varphi^1 = \eta^1 + Z^1 - Z^2$. Since the sum $\sum_{s<t} \Delta^+ Z^i_s$ clearly converges, we can define the continuous component $(Z^i)^c_t := Z^i_t - \sum_{s<t} \Delta^+ Z^i_s$ of Z^i for $i \in \{1,2\}$. Note that $(Z^i)^c$ indeed has continuous paths since Z^i has càglàd paths.

Now let $M^B_t := \Delta^+ Z^1_0 1_{\{t>0\}} + (Z^1)^c_t$ and $M^S_t := \Delta^+ Z^2_0 1_{\{t>0\}} + (Z^2)^c_t$. Clearly, M^B and M^S are non-decreasing predictable processes. Again by Lemma 3.12 and by Remark 3.8 we have

$$\int_0^{\cdot} 1_{\{\widetilde{S} \neq \overline{S}\}} dM^B = \int_0^{\cdot} 1_{\{\widetilde{S} \neq \underline{S}\}} dM^S = 0. \tag{3.6}$$

Thus, we have $\int_0^{\cdot} \overline{S} dM^B = \int_0^{\cdot} \widetilde{S} dM^B$ and $\int_0^{\cdot} \underline{S} dM^S = \int_0^{\cdot} \widetilde{S} dM^S$. Furthermore, let $L^B_t := \varphi^1_t (e^{F^1(\beta_{t-})} - 1)$ and $L^S_t := -\varphi^1_t (e^{F^2(\beta_{t-})} - 1)$. L^B and L^S are predictable and by Lemma 3.12 we have $\Delta^+ Z^1_t = L^B_t \Delta N^1_t$ and $\Delta^+ Z^2_t = L^S_t \Delta N^2_t$ for $t > 0$. Therefore, this construction of $\mathfrak{S}$ satisfies

$$\varphi^1_t = \eta^1 + M^B_t - M^S_t + \int_0^{t-} L^B dN^1 - \int_0^{t-} L^S dN^2, \qquad \forall t \geq 0.$$

Define

$$\begin{aligned} \psi^0_t \;\; := \;\; & \eta^0 - \int_0^t c_s ds - \int_0^t \overline{S}_s dM^B_s + \int_0^t \underline{S}_s dM^S_s \\ & - \int_0^{t-} L^B_s \underline{S}_s dN^1_s + \int_0^{t-} L^S_s \overline{S}_s dN^2_s, \end{aligned}$$

where c is from Proposition 3.11. By (3.6), $\underline{S} = \widetilde{S}$ on $\Delta N^1 = 1$ resp. $\overline{S} = \widetilde{S}$ on $\Delta N^2 = 1$ and Lemma 3.4, we have that (ψ^0, φ^1, c) is self-financing in the frictionless market. Thus, $\psi^0 = \varphi^0$ implying that (φ^0, φ^1) is indeed the associated portfolio process of $\mathfrak{S}$. From their definitions in Proposition 3.11 it can be seen that $\varphi^1 > 0$ and $\varphi^0 > 0$. Thus $(\varphi^0, \varphi^1, c)$ is clearly admissible.

The last term in (3.4) consists of dt-, dW_t-, dN^1_t-, and dN^2_t-integrals with bounded integrands. Together with the Poisson-distribution of N^1_t and N^2_t, the fact that c_t is proportional to $\widetilde{V}_t$, and $\delta > 0$, this yields that the discounted logarithmic utility from consumption is integrable. □

In Theorem 3.14 the optimal strategy in the limit order market is expressed in terms of the shadow price process resp. the wealth process based on the shadow price. In the following proposition we want to the characterize M^B, M^S, L^B, and L^S by the fraction of wealth invested in the risky asset *based on the best-bid price* $\underline{S}$. This verifies our guess (3.2)-(3.4). The optimal consumption rate is still expressed in terms of the wealth process based on the shadow price. We consider a reflected SDE – similar to that in Lemma 3.7.

Proposition 3.17. *Let $\beta' := \log\left((\varphi^1_+\underline{S})/\varphi^0_+\right)$, where (φ^0,φ^1) is the optimal strategy from Proposition 3.11. Define $\beta'_{\min} := \underline{\beta} - \log(1+\lambda)$ and $\beta'_{\max} := \overline{\beta}$. Assume that $\beta'_0 \in [\beta'_{\min}, \beta'_{\max}]$. Let*

$$c(y) := \mu - \frac{\sigma^2}{2} + \delta(1+\exp(h(y))), \quad y \in [\beta'_{\min}, \beta'_{\max}], \tag{3.7}$$

where $h : [\beta'_{\min}, \beta'_{\max}] \to [\underline{\beta}, \overline{\beta}]$ is the inverse of $\mathrm{Id} - g$ (the inverse exists as $g' \leq 0$). Let Ψ be the local time from Lemma 3.7. Then, given β'_0, (β', Ψ) is the unique solution to the following stochastic variational inequality

(i) β' is càdlàg and takes values in $[\beta'_{\min}, \beta'_{\max}]$. Ψ is continuous and of finite variation with starting value $\Psi_0 = 0$,

(ii)

$$\beta'_t \;=\; \beta'_0 + \int_0^t c(\beta'_{s-})ds + \sigma W_t + \sum_{s\leq t}\Big((\beta'_{\max} - \beta'_{s-})\Delta N^1_s + (\beta'_{\min} - \beta'_{s-})\Delta N^2_s\Big) + \Psi_t,$$

(iii) for every progressively measurable process z which has càdlàg paths and takes values in $[\beta'_{\min}, \beta'_{\max}]$, we have

$$\int_0^t (\beta'_s - z_s)\, d\Psi_s \leq 0, \qquad \forall t \geq 0.$$

Remark 3.18. *The function h in (3.7) converts the process β' which is based on the valuation of portfolio positions by $(1, \underline{S})$ into the process β which is based on $(1, \widetilde{S})$. This conversion is needed as the optimal consumption rate is proportional to the wealth based on the shadow price.*

of Proposition 3.17. At first note that by construction of the shadow price process

$$\left\{\beta_- = \underline{\beta}\right\} = \{\beta'_- = \beta'_{\min}\} \quad \text{and} \quad \left\{\beta_- = \overline{\beta}\right\} = \{\beta'_- = \beta'_{\max}\}.$$

Thus, $(P \otimes I)\left(\beta'_- \in \{\beta'_{\min}, \beta'_{\max}\}\right) = 0$ (i.e. dt-terms and dW_t-terms acting solely on this set vanish). In addition, $(P \otimes N^i)\left(\beta'_- \in \{\beta'_{\min}, \beta'_{\max}\}\right) = (P \otimes I)\left(\beta'_- \in \{\beta'_{\min}, \beta'_{\max}\}\right) = 0$ for $i = 1, 2$. By $\beta' = \log(\varphi^1) + \log(\underline{S}) - \log(\varphi^0)$, this implies that

$$\int_0^t 1_{\{\beta'_- \in \{\beta'_{\min}, \beta'_{\max}\}\}}\, d\beta' = \int_0^t 1_{\{\beta'_- \in \{\beta'_{\min}, \beta'_{\max}\}\}}\, d\beta = \int_0^t 1_{\{\beta'_- \in \{\beta'_{\min}, \beta'_{\max}\}\}}\, d\Psi,$$

where the latter equation follows by Lemma 3.7. As we have $\beta = \overline{\beta}$, $\underline{S} = \widetilde{S}$ on $\Delta N^1 = 1$ and $\beta = \underline{\beta}$, $\underline{S} = \frac{\widetilde{S}}{1+\lambda}$ on $\Delta N^2 = 1$, it follows from the definition of β', $\beta'_{\min}$, and $\beta'_{\max}$ that

$$\beta' = \beta'_{\max} \quad \text{on} \quad \Delta N^1 = 1 \quad \text{and} \quad \beta' = \beta'_{\min} \quad \text{on} \quad \Delta N^2 = 1. \tag{3.8}$$

By (3.8) and Itô's formula we obtain

$$\begin{aligned}\int_0^t 1_{\{\beta'_{\min} < \beta'_- < \beta'_{\max}\}}\, d\beta' \quad &= \quad \int_0^t 1_{\{\beta'_{\min} < \beta'_- < \beta'_{\max}\}} a(\beta'_-) dI + \sigma \int_0^t 1_{\{\beta'_{\min} < \beta'_- < \beta'_{\max}\}}\, dW \\ &\quad + \sum_{s \le t} \Big((\beta'_{\max} - \beta'_{s-}) \Delta N^1_s + (\beta'_{\min} - \beta'_{s-}) \Delta N^2_s \Big).\end{aligned}$$

As β' stays by construction in $[\beta'_{\min}, \beta'_{\max}]$ we have that (β', Ψ) is the solution of (i)-(iii). □

3.6 An illustration of the optimal strategy

Let us fix parameters for the model such that the assumptions of Proposition 3.6 are satisfied:

$$\mu = 0.05, \qquad \sigma = 0.4, \qquad \lambda = 0.01, \qquad \alpha_1 = 1, \qquad \alpha_2 = 1, \qquad \delta = 0.1.$$

With these parameters specified, the free boundary problem consisting of (3.2) and (3.11) can be solved numerically. The approach used is based on the idea behind the proof of Proposition 3.6. It can be roughly described as follows. First a value x for $\underline{\beta}$ is assumed, then a computer program for numerical calculations is used to solve the initial value problem consisting of (3.2) and the initial conditions $g(x) = \log(1+\lambda)$ and $g'(x) = 0$. Then the smallest $y > x$ with $g'(y) = 0$ is determined. Now if $g(y) < 0$ we choose a larger x in the next iteration, if $g(y) > 0$ we choose a smaller x, and if $g(x) = 0$ the algorithm stops and we have found our boundary $\{\underline{\beta}, \overline{\beta}\} = \{x, y\}$.

When the boundary $\{\underline{\beta}, \overline{\beta}\}$ is now known, we can calculate the boundary for the fraction of wealth invested in the risky asset (here measured in the shadow price) by

$$\pi_{\min} = \frac{\exp(\underline{\beta})}{1 + \exp(\underline{\beta})}, \qquad \pi_{\max} = \frac{\exp(\overline{\beta})}{1 + \exp(\overline{\beta})}.$$

For our example this yields $\pi_{\min} = 0.206$ and $\pi_{\max} = 0.412$. In addition, in Table 3.1 we have calculated $\pi_{\min}$ and $\pi_{\max}$ for various values of α to illustrate the effects of a change in the arrival rate of exogenous market orders. We see that $\pi_{\min}$ and $\pi_{\max}$ are close to the boundaries in the proportional transaction costs model, when α is small.

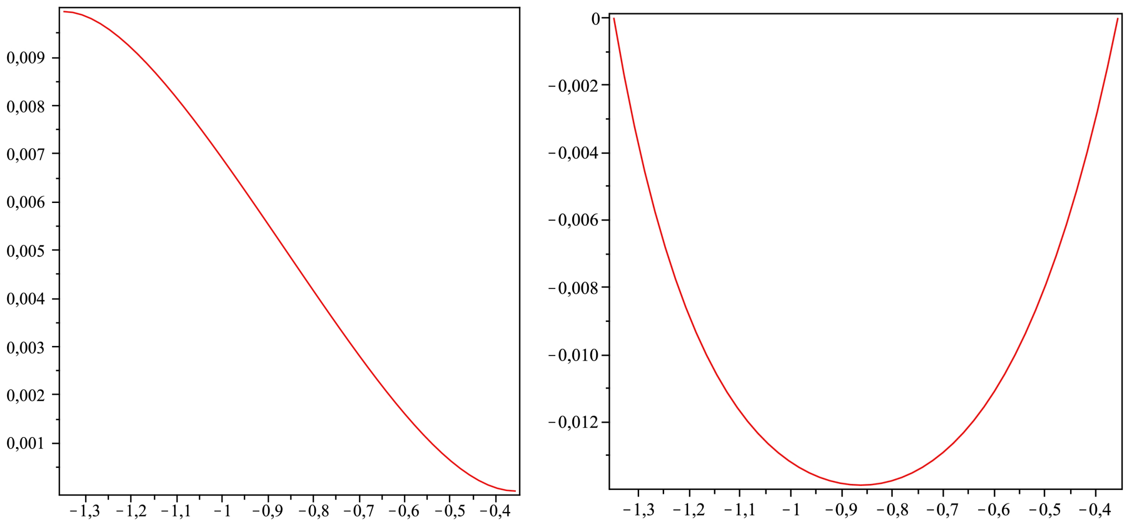

Figure 3.1: The function $C = g(\beta)$ and its derivative $g'(\beta)$

α	π_{min}	π_{max}
0	0.231	0.368
0.01	0.231	0.368
0.1	0.229	0.371
0.5	0.221	0.388
1	0.206	0.412
2	0.163	0.467
3	0.112	0.525
4	0.058	0.583

Table 3.1: Optimal boundaries for different α

The numerical solution to the free boundary problem can furthermore be used to simulate paths of various quantities. Fig. 3.2, Fig. 3.3, and Fig. 3.4 are the result of this procedure for the parameters given above and illustrate the structure of the solution.

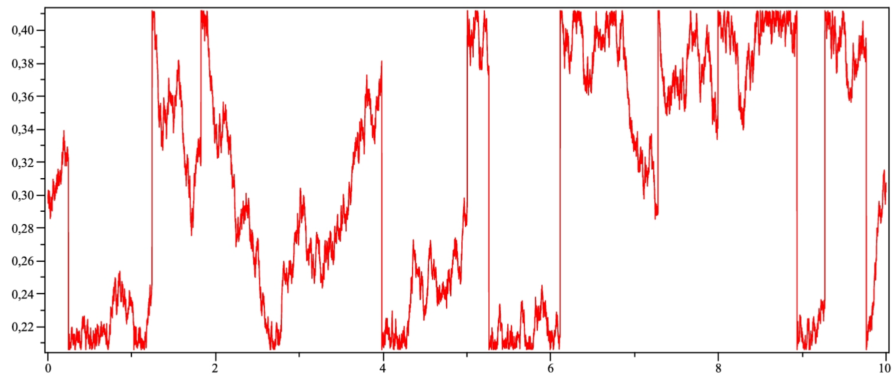

Figure 3.2: Optimal fraction $\widetilde{\pi}$ invested in stock (with local time at the boundaries)

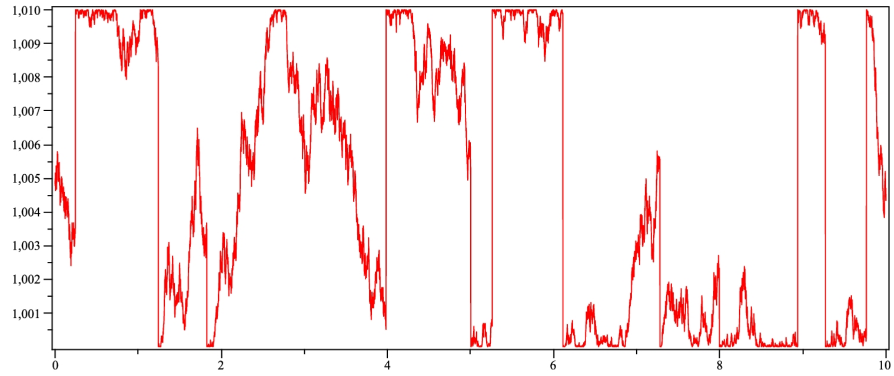

Figure 3.3: Shadow factor $\exp(C)$ (without local time)

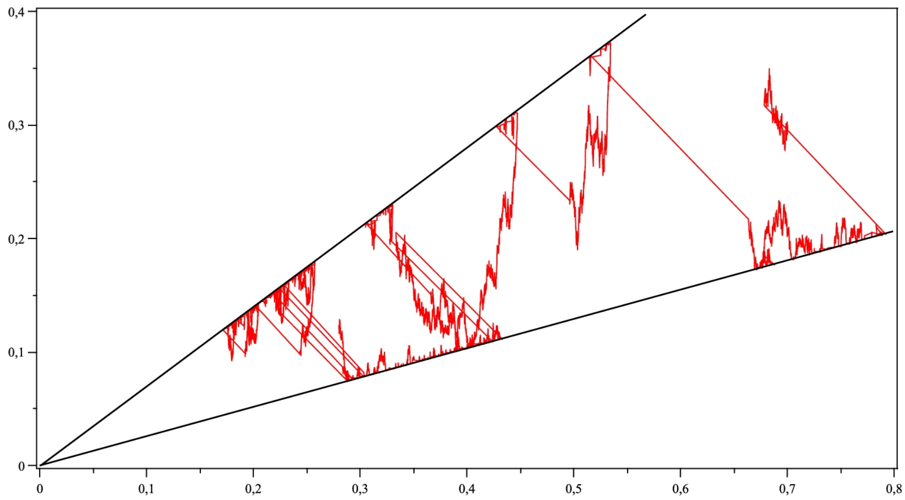

Figure 3.4: Wealth in bond φ^0, liquidation wealth in stock $\varphi^1 \underline{S}$

3.7 Conclusion

We introduced a simple, analytically tractable model for continuous-time trading in limit order markets. Although our mathematical results heavily rely on the quite idealized assumptions of the model, especially on the assumption that the considered investor is "small", i.e. his trades do not affect the dynamics of the order book, we think that in more complex situations the structure of the optimal strategy is still economically meaningful.

The investor tries to profit from the bid-ask spread by permanently holding both limit buy and limit sell orders in the book. After a successful execution of the limit buy order at the lower bid-price he holds a large stock position in his portfolio which is quite speculative. But, ideally he is able to liquidate the position quite shortly afterwards by the execution of the limit sell order at the higher ask-price. To limit the *inventory risk* he takes by this strategy the fraction of wealth he invests in the risky stock is always kept in a bounded interval (using market orders whenever the fraction is at the boundary of the interval). Thus the model carries the flavor of a market model with negative transaction costs, but which is arbitrage-free as favorable trades can only be realized at Poisson times.

Consider for example the case that the investor's limit orders are *not* small compared to the incoming market orders from other traders. Then, his wealth process does not always jump on the boundary of the cone (cf. Fig. 3.4), as incoming market orders may not be large enough to cover the full order size of his limit orders. But, still it seems to be worthwhile for the investor to place, say, limit buy orders as long as the fraction of wealth invested in stocks does not surpass a certain threshold. Under this scenario the threshold might be approached by several successive partial executions of these limit buy orders.

Furthermore, if the investor's *market* orders were not small enough to be filled by the orders placed at the best-bid resp. the best-ask price, such a large market order would eat into the book and would therefore be executed against various limit orders with *different* limit prices at a *single* point in time. Hence, a shadow price could obviously not exist.

In this spirit we see this chapter also as an impetus to solve more complicated portfolio optimization problems in continuous-time limit order markets (most probably in less explicit form).

Chapter 4

Stochastic integration w.r.t. optional semimartingales

4.1 Introduction

In this chapter we discuss the extension of the elementary stochastic Itô-integral in a general framework where the integrator is an optional semimartingale. The paths of an optional semimartingale possess limits from the left and from the right, but may have double jumps. Such processes have been studied extensively by Lenglart [Len80] and Galtchouk [Gal77, Gal81, Gal82, Gal85].

It turns out that the extension of the elementary integral to all predictable integrands is too small. Namely, the set of integrals for (suitably integrable) predictable integrands is still not closed (even w.r.t. the uniform convergence). This is of course in contrast to the standard framework with a càdlàg integrator, cf. [DM82].

Galtchouk [Gal81] has introduced a stochastic integral w.r.t. an optional martingale with a larger domain. But the integral of [Gal81] is not the *unique* (continuous and linear) extension of the elementary integral. There are stochastic integrals that can in no way be approximated by elementary integrals. This is an undesirable feature in some applications, e.g. if one wants to model trading gains from dynamic strategies by the integral. As real-world investment strategies are of course piecewise constant, it would not make sense to optimize over a set of integrals including some elements that cannot be approximated by elementary integrals.

In this chapter we introduce a mathematically tractable domain of integrands which is somehow between the small set of predictable integrands and the large domain in [Gal81]. The latter is a two-dimensional product space of predictable and optional processes.

The simple strategies are embedded into our domain. Then, in the usual manner, we characterize the integral defined on this domain as the unique continuous and linear extension of the elementary integral and show its closedness. In mathematical finance closedness of the set of achievable trading gains guarantees that the supremum in a portfolio optimization problem is attained and in “complete markets” derivatives can be replicated and not only be approximated by gains from dynamic trading in the underlying securities.

In addition, this chapter may also provide another abstract view to the extension of the elementary integral and the identification of $1_{]\!]\tau_1,\tau_2]\!]} \bullet X_t$ with $X_{t\wedge\tau_2} - X_{t\wedge\tau_1}$ in the usual situation of a càdlàg integrator X.

Note that the contents of this chapter have already been published in [KS09b].

4.2 Notation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a complete filtered probability space, where the family $(\mathcal{F}_t)_{t\in[0,T]}$ is not necessarily right-continuous. $\mathcal{P}$ and $\mathcal{O}$ denote the predictable resp. the optional σ-algebra on $\Omega \times [0,T]$, i.e. $\mathcal{P}$ is generated by all left-continuous adapted processes and $\mathcal{O}$ is generated by all càdlàg adapted processes (considered as mappings on $\Omega \times [0,T]$). If X and Y are two optional processes and we write $X = Y$, we mean equality up to indistinguishability.

The following definitions are from [Gal85]. Adjusted to our finite time horizon setting, we repeat them here for convenience of the reader. We add a localization procedure based on stopping which preserves the martingale property of a process. The results of Galtchouk that we use still hold when localization is done in the way chosen here.

Definition 4.1. *A stochastic process $X = (X_t)_{t\in[0,T]}$ is called an* optional martingale *(resp.* square integrable optional martingale*), and we write $X \in \mathscr{M}$ (resp. $X \in \mathscr{M}^2$), if X is an optional process and there exists an $\mathcal{F}_T$-measurable random variable $\widetilde{X}$ with $E[|\widetilde{X}|] < \infty$ (resp. $E[\widetilde{X}^2] < \infty$) such that $X_\tau = E[\widetilde{X}|\mathcal{F}_\tau]$ a.s. for every $[0,T]$-valued stopping time τ.*

Galtchouk has shown in [Gal77] that for any $\mathcal{F}_T$-measurable integrable random variable Z there exists an optional martingale $(X_t)_{t\in[0,T]}$ with terminal value $X_T = Z$. Almost all paths of X possess limits from the left and the right (see e.g. Theorem 4 in Appendix I of [DM82]). Thus if one considers general filtrations, optional martingales emerge quite naturally. For a làglàd process X we denote $\Delta^- X_t := X_t - X_{t-}$ and $\Delta^+ X_t := X_{t+} - X_t$.

Definition 4.2. *Denote by $\mathcal{T}$ (resp. $\mathcal{T}_+$) the set of all $[0,T]\cup\{+\infty\}$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-stopping times (resp. $(\mathcal{F}_{t+})_{t\in[0,T]}$-stopping times). Let $\mathscr{C}$ be a class of stochastic processes. A stochastic process X with right-hand limits is in the localized class of $\mathscr{C}$, and we write $X\in\mathscr{C}_{loc}$ if there exists an increasing sequence $(\tau_n,\sigma_n)_{n\in\mathbb{N}}\subset\mathcal{T}\times\mathcal{T}_+$ such that $\lim_{n\to\infty}P(\tau_n\wedge\sigma_n=T)=1$ and the stopped processes $X^{(\tau_n,\sigma_n)}$ defined by*

$$X_t^{(\tau_n,\sigma_n)} := X_t 1_{\{t\le\tau_n\wedge\sigma_n\}} + X_{\tau_n}1_{\{t>\tau_n,\ \tau_n\le\sigma_n\}} + X_{\sigma_n+}1_{\{t>\sigma_n,\ \tau_n>\sigma_n\}}$$

are in $\mathscr{C}$ for all n.

Definition 4.3. *Let $\mathscr{V}$ denote the set of adapted finite variation processes (that is P-a.a. paths are of finite variation) with $A_0=0$. We say that $A\in\mathscr{V}$ is in $\mathscr{A}$ if $E[\sum_{0\le s<T}|\Delta^+A_s| + \int_{[0,T]}|dA_s^r|]<\infty$.*

Galtchouk has shown that it is possible to uniquely decompose a local martingale M into a càdlàg part M^r and an orthogonal part M^g, i.e. $M^g\widetilde{M}$ is a local martingale for any càdlàg martingale $\widetilde{M}$. M^g possesses càglàd paths (see Theorem 4.10 in [Gal81] for details). Furthermore, any $A\in\mathscr{V}$ can obviously be decomposed uniquely into a càglàd part $A^g:=\sum_{0\le s<t}\Delta^+A_s$ and a càdlàg part $A^r:=A-A^g$. Note however that for processes which are both local martingales and of finite variation the decompositions usually differ.

Definition 4.4. *A stochastic process X is called* strongly predictable *if its trajectories have right limits, $(X_t)_{t\in[0,T]}$ is $\mathcal{P}$-measurable, and $(X_{t+})_{t\in[0,T]}$ is $\mathcal{O}$-measurable.*

Definition 4.5. *A stochastic process X is called an* optional semimartingale *if it can be written as*

$$X = X_0+M+A,\qquad M\in\mathscr{M}_{loc},\qquad A\in\mathscr{V},\qquad M_0=0. \tag{4.1}$$

A semimartingale X is called special if there exists a representation (4.1) with a strongly predictable process $A\in\mathscr{A}_{loc}$.

Note that any optional semimartingale has limits from the left and the right, i.e. almost all paths are làglàd (again by [DM82] this assertion holds for the local martingale component; for the finite variation component the assertion is trivial).

4.3 Results

Suppose X is the (for simplicity deterministic) evolution of a stock price given by $X_t := t - 1_{[t_0]}(t)+1_{]t_0,T]}(t)$, where $t_0\in(0,T)$ is the time of a double jump. $]t_0,T]$ denotes an interval on $\mathbb{R}$

whereas for τ_1, τ_2 stopping times $]\!]\tau_1,\tau_2]\!] := \{(\omega,t) \in \Omega \times [0,T] \mid \tau_1(\omega) < t \leq \tau_2(\omega)\}$ is a stochastic interval. Now consider the strategies A^n where we buy one unit of the stock at time $t_0 - 1/n$ and sell it at time t_0. The (negative) trading gain would be $1/n - 1$, and as $n \to \infty$ the trading loss would go to 1 and occur exactly at time t_0. Other possible strategies B^n would be to buy one unit of the stock at time t_0 and sell it at time $t_0 + 1/n$. The trading gain would be $2 + 1/n$, which would converge to a trading gain of 2 also occurring at time t_0. If we wanted the set of trading strategies to be closed, for the two sequences of trading strategies there should be limit trading strategies $\widetilde{A}$ and $\widetilde{B}$ reproducing the limit trading gain such that it occurred exactly at time t_0. If we wanted to use one-dimensional processes to specify our trading strategy, we would run into a dilemma because something like $1_{[t_0]}$ would have to represent both $\widetilde{A}$ and $\widetilde{B}$, but this is clearly impossible since the trading gains from $\widetilde{A}$ and $\widetilde{B}$ are completely different.

Put differently, since the process has double jumps, there might be a left jump $\Delta^- X_t$ and a right jump $\Delta^+ X_t$ at the same time. Using a one-dimensional integrand, an investor cannot differentiate between what should be invested in the left jump and what should be invested in the right jump, because at each point in time he only has a single value of the integrand at his disposal. For example, in the considerations above, the limit strategy $\widetilde{A}$ would have to invest 1 in $\Delta^- X_{t_0}$ but 0 in $\Delta^+ X_{t_0}$.

This explains why Galtchouk [Gal81] introduced two-dimensional integrands (H,G) where H is a $\mathcal{P}$-measurable process and G is an $\mathcal{O}$-measurable process. Unfortunately, *this* expansion of the set of integrands to two dimensions leads to a new problem. The integrals of *these* two-dimensional integrands can in general no longer be approximated by integrals of simple predictable integrands as the following example shows.

Example 4.6. *Consider the process $M = M^r + M^g$, where M^r is a compensated Poisson process with jump rate 1 and jump size 1 (so it is càdlàg), and M^g is the left-continuous modification of a compensated Poisson process with jump rate 1 and jump size -1, i.e. $M^r_t = N_t - t$ and $M^g_t = -\widetilde{N}_{t-} + t$ where N and $\widetilde{N}$ are Poisson processes. Assume that N and $\widetilde{N}$ are independent of each other and let $(\mathcal{F}_t)_{t\in[0,T]}$ be the (not right-continuous) natural filtration of (M^r, M^g). If we consider the integrand $(H,G)_t \equiv (2,1)$, the integral $Y := (H,G) \bullet M = H \bullet M^r + G \bullet M^g$ is an optional martingale linearly decreasing with rate -1 (if no jump occurs), $\Delta^- Y$ jumps of size 2 and $\Delta^+ Y$ jumps of size -1. Clearly Y cannot be approximated by any sequence $Z^n \bullet M$, where (Z^n) is a sequence of simple predictable integrands because $Z^n \bullet M_1 = 0$ if no jump occurs up to time 1. Furthermore, it is impossible to approximate the left jumps of Y (which are of size 2)* *and the right jumps of Y (with size -1) by the same process $Z^n \bullet M$. This is because the jumps of M cannot be anticipated.*

For two sets A,B we define $A\Delta B := (A\setminus B)\cup(B\setminus A)$. Let $\widetilde{\Omega} := \Omega\times[0,T]$. Define a collection $\mathcal{A}$ of subsets of $\{1,2\}\times\widetilde{\Omega}$ by

$$\begin{aligned}\mathcal{A} \quad := \quad \Big\{(\{1\}\times A)\cup(\{2\}\times B) \quad &: \quad (A,B)\in\mathcal{P}\times\mathcal{O} \text{ with} \\ & A\Delta B = \bigcup_{n\in\mathbb{N}} [\![\tau_n]\!] \text{ for some } (\tau_n)_{n\in\mathbb{N}}\subset\mathcal{T}\Big\},\end{aligned} \tag{4.2}$$

i.e. the symmetric difference $A\Delta B$ has to be a thin set. Note that τ is $[0,T]\cup\{+\infty\}$-valued, but $[\![\tau]\!] = \{(\omega,t)\in\Omega\times[0,T] \mid \tau(\omega)=t\}$. Our general integrands will be $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable functions.

Proposition 4.7. *$\mathcal{A}$ is a σ-field.*

Proof. Obvious as $\mathcal{P}$ and $\mathcal{O}$ are σ-fields and countable unions of thin sets are thin sets. □

An immediate observation is that if H is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable, then $H^1 := H(1,\cdot,\cdot)$ is a predictable process and $H^2 := H(2,\cdot,\cdot)$ is an optional process. Furthermore, H^1 and H^2 differ only at countably many $(\mathcal{F}_t)_{t\in[0,T]}$-stopping times (as H can be approximated pointwise by simple functions).

Proposition 4.8. *Define the set*

$$\mathcal{C} := \{\{1\}\times\widetilde{A}\times\{0\} : \widetilde{A}\in\mathcal{F}_0\}\cup\{\{1\}\times]\!]\tau_1,\tau_2]\!]\cup\{2\}\times[\![\tau_1,\tau_2[\![: \tau_1,\tau_2\in\mathcal{T},\tau_1\le\tau_2\}.$$

Then $\sigma(\mathcal{C}) = \mathcal{A}$.

Proof. $\sigma(\mathcal{C})\subset\mathcal{A}$ holds by $\mathcal{C}\subset\mathcal{A}$. Since $\bigcap_{i=1}^{\infty}(\{1\}\times]\!]\tau,\tau+\frac{1}{n}]\!]\cup\{2\}\times[\![\tau,\tau+\frac{1}{n}[\![)\in\sigma(\mathcal{C})$, we have that $\{1\}\times\emptyset\cup\{2\}\times[\![\tau]\!]\in\sigma(\mathcal{C})$ for any $\tau\in\mathcal{T}$. Therefore also $\{1\}\times]\!]\tau_1,\tau_2]\!]\cup\{2\}\times]\!]\tau_1,\tau_2]\!]\in\sigma(\mathcal{C})$ for all $\tau_1,\tau_2\in\mathcal{T}$. Because $\mathcal{P}$ is generated by the family of sets $\{\widetilde{A}\times\{0\} : \widetilde{A}\in\mathcal{F}_0\}\cup\{]\!]\tau_1,\tau_2]\!] : \tau_1,\tau_2\in\mathcal{T}\}$ and since $\widetilde{A}\times\{0\}$ is the graph of a stopping time, we have $\{1\}\times A\cup\{2\}\times A\in\sigma(\mathcal{C})$ for any $A\in\mathcal{P}$. Now let $F\in\mathcal{A}$, i.e. $F=\{1\}\times A\cup\{2\}\times B$, where $A\in\mathcal{P}$, $B\in\mathcal{O}$. $A\setminus B$ and $B\setminus A$ are both thin sets by Theorem 3.19 in [HWY92], thus there exist two sequences of stopping times $(\tau_i)_{i\in\mathbb{N}}$ and $(\nu_j)_{j\in\mathbb{N}}$ such that $B = (A\setminus\bigcup[\![\tau_i]\!])\cup(\bigcup[\![\nu_j]\!])$. Therefore $F\in\sigma(\mathcal{C})$ as required. □

Consider simple integrands of the form

$$H = Z^0 1_{\{1\}\times\widetilde{A}\times\{0\}\cup\{2\}\times\widetilde{A}\times\{0\}} + \sum_{i=1}^{n} Z^i 1_{\{1\}\times]\!]\tau_i,\tau_{i+1}]\!]\cup\{2\}\times[\![\tau_i,\tau_{i+1}[\![}, \tag{4.3}$$

where $\tau_i \in \mathcal{T}$, $\tau_1 \leq \tau_2 \ldots \leq \tau_{n+1}$, Z^0 is $\mathcal{F}_0$-measurable, and each Z^i is a $\mathcal{F}_{\tau_i}$-measurable random variable. Let $\mathcal{E}$ denote the class of simple integrands. Note that the simple integrands are indeed $\mathcal{A}$-measurable, and that there is a one-to-one correspondence between the simple integrands defined in (4.3) and the usual one-dimensional simple predictable integrands. By Proposition 4.8 $\mathcal{E}$ generates the σ-field $\mathcal{A}$ on $\{1,2\} \times \widetilde{\Omega}$. We call simple integrands simple $\mathcal{A}$-measurable.

We now define for $H \in \mathcal{E}$ the elementary stochastic integral in the usual way by

$$(H \bullet X)_t := \sum_{i=1}^{n} Z^i (X_{\tau_{i+1} \wedge t} - X_{\tau_i \wedge t}), \qquad t \in [0,T].$$

Remark 4.9. *The second summand in (4.3) can be motivated as follows: To obtain $Z^i(X_{\tau_{i+1}} - X_{\tau_i})$ one weights the right jump of X at τ_i already with Z^i whereas the left jumps are weighted with Z^i only immediately after τ_i.*

The next theorem shows that the elementary integral possesses a unique continuous and linear extension to general integrands defined as $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable functions.

Theorem 4.10. *Suppose X is an optional semimartingale. The mapping $H \mapsto H \bullet X$ on $\mathcal{E}$ has a unique extension (also denoted $H \mapsto H \bullet X$) to all locally bounded $\mathcal{A}$-measurable processes $H : \{1,2\} \times \widetilde{\Omega} \to \mathbb{R}$ such that*

(i) $H \mapsto H \bullet X$ is linear;

(ii) if an $\mathcal{A}$-measurable sequence $(H^n)_{n\in\mathbb{N}}$ converges pointwise to H and $|H^n| \leq K$, where K is a locally bounded $\mathcal{A}$-measurable process, then $\sup_{s\in[0,T]} |(H^n \bullet X)_s - (H \bullet X)_s|$ converges in probability to 0.

Proof. Step 1 (uniqueness). Let $H \bullet X$ and $H \circ X$ be two extensions satisfying (i) and (ii). Then (i) and (ii) imply that $\mathcal{G} := \{F \in \mathcal{A} : 1_F \bullet X = 1_F \circ X\}$ is a Dynkin system. Since $\mathcal{C} \subset \mathcal{G}$ and $\mathcal{C}$ is a $\cap$-stable generator of $\mathcal{A}$, by a Dynkin argument we have $\mathcal{A} = \mathcal{G}$. A locally bounded $\mathcal{A}$-measurable process H can be approximated pointwise by the sequence $(H^n)_{n\in\mathbb{N}}$, where

$$H^n := \sum_{k=-n^2}^{n^2} \frac{k}{n} 1_{\{\frac{k-1}{n} < H \leq \frac{k}{n}\}}.$$

Because of the linearity requirement (i) we know that $H^n \bullet X = H^n \circ X$ for all n. In addition it is true that $|H^n| \leq |H| + 1$. Thus from (ii) follows $H \bullet X = H \circ X$ and the uniqueness of the extension is established.

Step 2 (existence). Let $X = X_0 + M + A$ with $M \in \mathscr{M}_{loc}$ and $A \in \mathscr{V}$ be any decomposition of X. Consider the integral (once again denoted by $H \mapsto H \bullet X$)

$$H \bullet X := H^1 \bullet M^r + H^1 \bullet A^r + H^2 \bullet M^g + H^2 \bullet A^g, \tag{4.4}$$

which is by Galtchouk defined for any locally bounded $H^1 \in \mathcal{P}$ and $H^2 \in \mathcal{O}$, thus in particular when H is locally bounded and $\mathcal{A}$-measurable. Note that (4.4) generally depends on the decomposition of the optional semimartingale into a local martingale and a process of finite variation (Thus in Galtchouk $H^1 \bullet M^r + H^2 \bullet M^g$ and $H^1 \bullet A^r + H^2 \bullet A^g$ are seen as separate integrals. But, later on by the uniqueness of the extension it will turn out that for $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable integrands the choice of the decomposition is not relevant).

If H is a simple integrand this integral is equal to our definition of the simple integral, i.e. it is an extension. From the standard theory (see e.g. [DM82], chapter VIII) we know that the first half of the right-hand side of (4.4) fulfils properties (i) and (ii). For the left-continuous parts $H^2 \bullet M^g$ and $H^2 \bullet A^g$ the same line of argument holds true: M^g can be decomposed into a locally square integrable martingale and a local martingale of finite variation (by considering the process $\sum_{0 \le s \le \cdot} \Delta^+ M_s 1_{\{|\Delta^+ M_s| > 1\}} \in \mathscr{A}_{loc}$ and using the existence of strongly predictable càglàd compensators, see Lemma 1.10 in [Gal85]). Because a version of Doob's inequality still holds for optional square-integrable martingales (see Appendix I in [DM82] on how to prove such inequalities using the optional section-theorem, which still holds under nonusual conditions), the usual arguments for the càdlàg case can be reproduced for the locally square integrable part. The martingale part of finite variation is treated like $(H^2 \bullet A^g)_t = \int_{[0,t[} H^2_s dA^g_{s+}$ which is a Lebesgue-Stieltjes integral. Thus it is known that it is linear and has the continuity property. □

Remark 4.11. *We have shown that it is possible to extend the integral in a unique way from all simple $\mathcal{A}$-measurable integrands (which are in a one-to-one correspondence with the (one-dimensional) simple predictable integrands) to all locally bounded $\mathcal{A}$-measurable integrands. Note that the elementary integral does not depend on the decomposition in (4.4). In Galtchouk's framework [Gal85] the integral is extended uniquely from all two-dimensional simple $\mathcal{P} \otimes \mathcal{O}$-measurable integrands to all locally bounded $\mathcal{P} \otimes \mathcal{O}$-measurable integrands. What cannot be done is to extend the integral* uniquely *from one-dimensional simple predictable integrands to all locally bounded $\mathcal{P} \otimes \mathcal{O}$-measurable integrands. To see this note that besides $H \bullet X := H^1 \bullet M^r + H^1 \bullet A^r + H^2 \bullet M^g + H^2 \bullet A^g$ the mapping $H \circ X := H \bullet X + H^1 \bullet I - H^2 \bullet I$, where $I_t(\omega) := t$, is also a continuous and linear extension of the elementary integral. But generally for $\mathcal{P} \otimes \mathcal{O}$-measurable* *integrands $H \bullet X$ and $H \circ X$ are different. Confer this with Example 4.6.*

Any special semimartingale Y for which the canonical decomposition $Y_0 + N + B$ satisfies $N \in \mathscr{M}^2$ and $B \in \mathscr{A}$, can be considered an element of the Banach space $\mathscr{M}^2 \oplus \mathscr{A}$, where the norm is given by $E[N_T^2]^{1/2} + E[\mathrm{Var}(B)_T]$. Now we show a closedness property for the set of integrands for which the integrals are in $\mathscr{M}^2 \oplus \mathscr{A}$. At first we define analogously to the standard theory the set of general integrands (cf. definition III.6.17 in [JS02]).

Definition 4.12. *We say that a $\mathcal{A}$-measurable process $H = (H^1, H^2)$ is integrable w.r.t. an optional semimartingale X if there exists a decomposition $X = X_0 + M + A$ with $M \in \mathscr{M}^2_{loc}$ and $A \in \mathscr{V}$ such that*

$$(H^1)^2 \bullet [M^r, M^r] \in \mathscr{A}_{loc}, \quad (H^2)^2 \bullet [M^g, M^g] \in \mathscr{A}_{loc}$$

and the Lebesgue-Stieltjes integrals $|H^1| \bullet \mathrm{Var}(A^r)$, $|H^2| \bullet \mathrm{Var}(A^g)$ are finite-valued. We denote by $L(X)$ the set of these processes.

Let $H \in L(X)$. By Theorem 4.10 the integral $\left(H1_{\{|H| \le n\}}\right) \bullet X = \left(H^1 1_{\{|H| \le n\}}\right) \bullet M^r + \left(H^1 1_{\{|H| \le n\}}\right) \bullet A^r + \left(H^2 1_{\{|H| \le n\}}\right) \bullet M^g + \left(H^2 1_{\{|H| \le n\}}\right) \bullet A^g$ is well-defined (i.e. it does not depend on the decomposition $X = X_0 + M + A$). By Theorem I.4.40 and Lemma III.6.15 in [JS02] and Theorem 7.3 in [Gal81] all four integrals converge uniformly in probability against the corresponding integrals without truncation. Thus $H \bullet X$ is also well-defined.

Theorem 4.13. *Let X be a special semimartingale. If $(H^n)_{n \in \mathbb{N}} \subset L(X)$ such that $(H^n \bullet X)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathscr{M}^2 \oplus \mathscr{A}$, then there exists a $H \in L(X)$ such that $H^n \bullet X \to H \bullet X$ in $\mathscr{M}^2 \oplus \mathscr{A}$.*

Proof. Step 1. We start by showing that for all n the canonical decomposition of $H^n \bullet X$ can be written as $H^n \bullet M + H^n \bullet A$, where $X = X_0 + M + A$ is the canonical decomposition of X. The reasoning is similar to the proof of Lemma III.3 in [M8́0], but we present it here for the convenience of the reader. Some facts about (strongly predictable) compensators are used; they can be found in the appendix. Let n be fixed. There exists a decomposition $X = N + B$ such that $(H^n \bullet N) \in \mathscr{M}^2_{loc}$ and $(H^n \bullet B) \in \mathscr{V}$. Since $H^n \bullet X$ is in $\mathscr{M}^2 \oplus \mathscr{A}$, we have by Lemma 4.2 in [Gal85] that $H^n \bullet B \in \mathscr{A}_{loc}$. As X is special, we have with the same argument that $B \in \mathscr{A}_{loc}$. Again by Lemma 4.2 in [Gal85], $H^n \bullet X$ is special and hence it possesses a canonical decomposition $L + D$. By Proposition 4.18 the unique compensators of B and $H^n \bullet B$ are given by A and D.

But since B and $H^n \bullet B$ are both in $\mathscr{A}_{loc}$, by Proposition 4.20 $(H^n \bullet B)^p = H^n \bullet B^p = H^n \bullet A$, i.e. the compensator of $H^n \bullet B$ is $H^n \bullet A$. Thus $D = H^n \bullet A$, which in turn implies $L = H^n \bullet M$.

Step 2. For any local martingale M, we define a nonnegative measure m on $(\{1,2\} \times \widetilde{\Omega}, \mathcal{A})$ by

$$m(F) := E[1_B \bullet [M^r, M^r]_T + 1_C \bullet [M^g, M^g]_T], \quad \forall F = \{1\} \times B \cup \{2\} \times C \in \mathcal{A}.$$

Similarly, for $A \in \mathscr{A}_{loc}$ let

$$n(F) := E[1_B \bullet \mathrm{Var}(A^r)_T + 1_C \bullet \mathrm{Var}(A^g)_T].$$

By the decomposition of M (resp. A) into a right- and a left-continuous part we ensure that m (resp. n) is a measure. Note that m and n are in general not σ-finite. Let $H \bullet M \in \mathscr{M}^2$; then we have that

$$\begin{aligned} E[(H \bullet M)_T^2] &= E[(H^1 \bullet M^r + H^2 \bullet M^g)_T^2] \\ &= E[(H^1 \bullet M^r)_T^2 + (H^2 \bullet M^g)_T^2 + 2(H^1 \bullet M^r)_T (H^2 \bullet M^g)_T] \\ &= E[(H^1 \bullet M^r)_T^2 + (H^2 \bullet M^g)_T^2] \\ &= E[(H^1)^2 \bullet [M^r, M^r]_T + (H^2)^2 \bullet [M^g, M^g]_T] \\ &= \int_0^T (H)^2 dm. \end{aligned} \tag{4.5}$$

The crucial third equality follows because $H^1 \bullet M^r$ and $H^2 \bullet M^g$ are orthogonal optional martingales, which is due to fact that

$$[H^1 \bullet M^r, H^2 \bullet M^g] = H^2 \bullet [(H^1 \bullet M^r)^g, M^g] = [0, M^g] = 0$$

(see [Gal81], Theorem 7.11). The fourth equality is valid since there are Itô isometries for both the standard stochastic integral and the optional stochastic integral w.r.t. to a càglàd optional martingale (see [Gal81], Section 7).

Let us verify an isometry property for the integrable variation part. Note that for the finite variation part A, the process A^g is just the sum of the jumps $\Delta^+ A$. The total variation can thus be split into two parts by

$$\mathrm{Var}(A)_t = \int_0^t |dA_s^r| + \sum_{s<t} |\Delta^+ A_s| = \mathrm{Var}(A^r) + \mathrm{Var}(A^g),$$

and the following isometry holds for any $A \in \mathscr{A}$

$$\begin{aligned} E[\mathrm{Var}(H \bullet A)_T] &= E[\mathrm{Var}(H^1 \bullet A^r + H^2 \bullet A^g)_T] \\ &= E[\mathrm{Var}(H^1 \bullet A^r)_T + \mathrm{Var}(H^2 \bullet A^g)_T] \\ &= E[|H^1| \bullet \mathrm{Var}(A^r)_T + |H^2| \bullet \mathrm{Var}(A^g)_T] \\ &= \int_0^T |H| dn. \end{aligned} \tag{4.6}$$

By (4.5) and (4.6), $\left(L^2(\{1,2\} \times \widetilde{\Omega}, \mathcal{A}, m) \cap L^1(\{1,2\} \times \widetilde{\Omega}, \mathcal{A}, n)\right) \subset L(X)$ and $H \mapsto H \bullet X$ is an isometry mapping from $L^2(\{1,2\} \times \widetilde{\Omega}, \mathcal{A}, m) \cap L^1(\{1,2\} \times \widetilde{\Omega}, \mathcal{A}, n)$ to $\mathscr{M}^2 \oplus \mathscr{A}$ (surjective onto the subspace of $\mathscr{M}^2 \oplus \mathscr{A}$ whose elements can be represented by stochastic integrals). As $L^2(\{1,2\} \times \widetilde{\Omega}, \mathcal{A}, m) \cap L^1(\{1,2\} \times \widetilde{\Omega}, \mathcal{A}, n)$ is a complete vector space this implies the assertion. □

Remark 4.14. *Suppose for any $(\mathcal{F}_t)_{t\in[0,T]}$-stopping time τ we have $P(\Delta^+ X_\tau \neq 0, \tau < T) = 0$ (we call such a process* quasi-right-continuous*). Then for any locally bounded $\mathcal{A}$-measurable process H the stochastic integrals $H \bullet X = H^1 \bullet X^r + H^2 \bullet X^g$ and $H^1 \bullet X^r + H^1 \bullet X^g$ are indistinguishable. To see this, note that we only have to check that $(H^1 - H^2) \bullet X^g = 0$. Now $H^1 - H^2$ is equal to 0 on the complement of a thin set and according to the condition above there are a.s. no jumps of X^g on this thin set. Thus if X is quasi-right-continuous, the set of locally bounded predictable integrands is adequate, as in the usual right-continuous setting.*

Remark 4.15. *In mathematical finance a similar problem arises in the standard model with càdlàg-price processes when portfolio adjustments cause transaction costs. At time t the value of a portfolio may change due to a jump of the asset prices between $t-$ and t. In addition, any portfolio adjustments (which may be seen as taking place at time $t-$) reduce the wealth of the investor (in contrast to the model without transaction costs). Thus, the wealth process may have double jumps. However, the portfolio holdings in each asset can still be represented by a one-dimensional process, cf. [CS06].*

4.4 Appendix

Lemma 4.16. *Suppose $A \in \mathscr{V}$. Then A is strongly predictable if and only if $(A^r_t)_{t\in[0,T]}$ is predictable and $(A^g_{t+})_{t\in[0,T]}$ is optional.*

Proof. Obvious, as $A_t = A^r_t + A^g_t = A^r_t + A^g_{t-}$ and $A_{t+} = A^r_{t+} + A^g_{t+} = A^r_t + A^g_{t+}$. □

Lemma 4.17. *Let $A \in \mathscr{V}$ be strongly predictable and $H = (H^1, H^2)$ be an $\mathcal{A}$-measurable function s.t. $H^1 \bullet A^r$ and $H^2 \bullet A^g$ exist. Then $H \bullet A$ is strongly predictable.*

Proof. By Lemma 4.16 and $H \bullet A = H^1 \bullet A^r + H^2 \bullet A^g$ we only have to check that $(H^1 \bullet A^r_t)$ is predictable and $(H^2 \bullet A^g_{t+})$ is optional. Since H^1 is predictable and again by Lemma 4.16 (A^r_t) is also predictable, Proposition I.3.5 in [JS02] ensures that $H^1 \bullet A^r$ is predictable, too. Once more by Lemma 4.16 $(A^g_{t+})_{t\in[0,T]}$ is optional, thus $\Delta^+ A_s$ is $\mathcal{F}_t$ measurable for all $s \le t$. As $H^2 1_{\Omega\times[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t])$-measurable, by Fubini's theorem for transition measures this implies that $(H^2 \bullet A^g)_{t+} = \sum_{0\le s\le t} H^2_s \Delta^+ A_s$ is $\mathcal{F}_t$-measurable and therefore optional. □

Proposition 4.18. *Let $A \in \mathscr{A}_{loc}$. There exists a process, called the compensator of A and denoted by A^p, which is unique up to indistinguishability, and which is characterized by being a strongly predictable process of $\mathscr{A}_{loc}$ such that $A - A^p$ is a local martingale.*

Proof. $A \in \mathscr{A}_{loc}$ implies $A^r, A^g \in \mathscr{A}_{loc}$. By Theorem I.3.18 in [JS02], there exists a unique predictable càdlàg process $(A^r)^p$ such that $A^r - (A^r)^p \in \mathscr{M}_{loc}$ (formally we apply the theorem to A^r under the right-continuous filtration $(\mathcal{F}_{t+})_{t\in[0,T]}$ and use that the $(\mathcal{F}_{t+})_{t\in[0,T]}$-predictable processes coincide with the $(\mathcal{F}_t)_{t\in[0,T]}$-predictable processes). By Lemma 1.10 in [Gal85], there exists a unique strongly predictable càglàd process $(A^g)^p$ such that $A^g - (A^g)^p \in \mathscr{M}_{loc}$. The process $A^p := (A^r)^p + (A^g)^p$ is strongly predictable and $A - A^p \in \mathscr{M}_{loc}$. If two strongly predictable processes B and C are compensators of A, $B - C$ is in $\mathscr{M}_{loc} \cap \mathscr{A}_{loc}$, i.e. $B - C = 0$ (since as in the standard model, using Theorem 3.5 in [Gal82] it can be shown that if $X \in \mathscr{M}_{loc} \cap \mathscr{A}_{loc}$, then $X = 0$.) □

Proposition 4.19. *Let $A \in \mathscr{A}^+_{loc}$. The compensator A^p can then be characterized as being a strongly predictable process in $\mathscr{A}^+_{loc}$ meeting any of the two following equivalent statements*

(i) $E[A^p_\tau] = E[A_\tau]$ *for all* $\tau \in \mathcal{T}$;

(ii) $E[(H \bullet A^p)_T] = E[(H \bullet A)_T]$ *for all nonnegative $\mathcal{A}$-measurable processes H.*

Proof. The proof is similar to the proof of Theorem I.3.17 in [JS02]. Just note that (ii) implies (i) because $H := 1_{\{1,2\}\times[\![0,\tau]\!]}$ is $\mathcal{A}$-measurable. (i) implies for all $\tau \in \mathcal{T}$ that

$$\begin{aligned} E[(1_{\{1\}\times[\![0,\tau]\!]\cup\{2\}\times[\![0,\tau[\![} \bullet A^p)_T] &= E[(A^p)^r_\tau + (A^p)^g_\tau] \\ &= E[A^r_\tau + A^g_\tau] \\ &= E[(1_{\{1\}\times[\![0,\tau]\!]\cup\{2\}\times[\![0,\tau[\![} \bullet A)_T]. \end{aligned}$$

Since $\mathcal{A}$ is also generated by $\{\{1\}\times\widetilde{A}\times\{0\}:\widetilde{A}\in\mathcal{F}_0\}\cup\{\{1\}\times[\![0,\tau]\!]\cup\{2\}\times[\![0,\tau[\![:\tau\in\mathcal{T}\}$ and because $A_0=A^p_0=0$, we have (ii) by monotone convergence and a monotone class argument. $\square$

Proposition 4.20. *Let $A\in\mathscr{A}_{loc}$. For each $\mathcal{A}$-measurable process H such that $H\bullet A\in\mathscr{A}_{loc}$, we have that $H\bullet A^p\in\mathscr{A}_{loc}$ and $H\bullet A^p=(H\bullet A)^p$, and in particular $H\bullet A-H\bullet A^p$ is a local martingale.*

Proof. The proof of the second half of Theorem I.3.18 in [JS02] can be reproduced without any major changes (using Proposition 4.19 and Lemma 4.17). Note that the associativity of the integral used in the proof holds because

$$\begin{aligned}
H\bullet(G\bullet A) &= H^1\bullet(G\bullet A)^r+H^2\bullet(G\bullet A)^g\\
&= H^1\bullet(G^1\bullet A^r+G^2\bullet A^g)^r+H^2\bullet(G^1\bullet A^r+G^2\bullet A^g)^g\\
&= H^1\bullet(G^1\bullet A^r)+H^2\bullet(G^2\bullet A^g)\\
&= (H^1G^1)\bullet A^r+(H^2G^2)\bullet A^g\\
&= (HG)^1\bullet A^r+(HG)^2\bullet A^g=(HG)\bullet A,
\end{aligned}$$

where the crucial third equality is true because for any $A\in\mathscr{V}$ we obviously have $(A^r)^g=(A^g)^r=0$. The fourth equality follows from the associativity of the one-dimensional Lebesgue-Stieltjes integral. $\square$

Deutsche Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Modellierung der Handelsmöglichkeiten eines kleinen Investors in einem Limitordermarkt. In einem *Limitordermarkt* kann bei jeder Order zusätzlich zur Anzahl ein Limitpreis angegeben werden. Wird ein Limitpreis spezifiziert, so spricht man von einer *Limitorder*, ansonsten von einer *Marketorder*. Die Angabe eines Limitpreises führt dazu, dass eine Order im Normalfall nicht sofort ausgeführt werden kann. Diese unausgeführten Limitorders werden in einer elektronischen Datenbank der Börse gespeichert, dem sogenannten *Orderbuch*.

Möchte ein Investor nun in einem Limitordermarkt etwa Wertpapiere erwerben, so kann er dies entweder sofort durch eine Market-Kauforder, oder indem er eine Limit-Kauforder nutzt. Im Fall der Market-Kauforder spezifiziert der Investor lediglich, welche Anzahl an Wertpapieren er zu kaufen beabsichtigt. Der Preis, den der Investor pro Aktie zahlt, entspricht dem Best-Ask-Preis, d.h. dem Limitpreis der niedrigsten im Orderbuch enthaltenen Limit-Verkaufsorders (sollte es sich um eine große Order handeln, kann es auch passieren, dass die Order zu einem Durchschnittspreis oberhalb des Best-Ask-Preises ausgeführt wird). Sofern die sofortige Ausführung der Order nicht zwingend erforderlich ist, kann der Investor eine Limit-Kauforder mit einem Limit-Preis unterhalb des aktuellen Best-Ask-Preises nutzen. Diese Limit-Kauforder wird dann solange im Orderbuch gespeichert, bis sie gegen eine Verkaufsorder eines anderen Investors ausgeführt werden kann, oder bis der Investor die Limit-Kauforder storniert.

Unter der Annahme eines *kleinen* Investors versteht man die idealisierte Hypothese, dass die Anzahl der vom Investor gekauften und verkauften Wertpapiere gering genug ist, sodass sie die Dynamik des Orderbuchs nicht beeinflusst, d.h. die Handelsmöglichkeiten des Investors können als exogen gegeben modelliert werden, ohne dass etwa spieltheoretische Erwägungen in Betracht gezogen werden müssen. Das bekannteste Beispiel für die Modellierung eines Marktes mit kleinem Investor ist das Black-Scholes-Modell, in dem der Preis einer Aktie als exogen vorgegebene geometrische Brownsche Bewegung modelliert wird. Unabhängig davon, welche Käufe und Verkäufe der Investor tätigt, bleibt die Dynamik der Aktie stets dieselbe.

Handel eines kleinen Investors in Limitordermärkten

In *Kapitel 2* wird ein mathematischer Rahmen vorgestellt, um die Handelsgewinne eines kleinen Investors in einem Limitordermarkt zu modellieren.

Es seien $\underline{S}$ und $\overline{S}$ zwei adaptierte càdlàg-Prozesse mit $0 < \inf_{s\in[0,T]} \underline{S}_s(\omega) \leq \underline{S}_t(\omega) < \overline{S}_t(\omega)$ für alle $(\omega,t) \in \Omega \times [0,T]$. Im Folgenden kann man $\underline{S}$ als den Best-Bid- und $\overline{S}$ als den Best-Ask-Preis ansehen. Der kleine Investor kann jederzeit mit Marketorders zum Preis $\overline{S}$ kaufen und zum Preis $\underline{S}$ verkaufen. Zusätzlich seien die beiden ganzzahligen Zufallsmaße μ und ν gegeben. Mit diesen beiden Zufallsmaßen werden die Ausführungen der Limit-Kauf- bzw. Limit-Verkaufsorders des kleinen Investors modelliert. Die beiden Maße müssen zusätzliche Annahmen erfüllen, damit diese Modellierung sinnvoll ist. Die Details finden sich in Annahme 2.7.

Nun zur Modellierung der Handelsstrategien. Es werden zunächst die allgemeinen Limitorderstrategien und dann die Handelsstrategien insgesamt eingeführt.

Definition 1. *Mit $\mathcal{L}^B$ sei die Familie von $\widetilde{\mathcal{P}}/\mathcal{B}(\overline{\mathbb{R}}_+)$-messbaren Funktionen $L^B : \Omega \times [0,T] \times \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ bezeichnet, für die gilt*

(i) $x \mapsto L^B(\omega,t,x)$ ist monoton fallend, für alle $(\omega,t) \in \Omega \times [0,T]$,

(ii) $L^B(\omega,t,x) = 0$ für alle $(\omega,t) \in \Omega \times [0,T]$ und $x \geq \overline{S}_{t-}(\omega)$,

(iii) L^B ist μ-integrierbar.

Hierbei bezeichnet $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$ die Produkt-σ-Algebra aus der vorhersehbaren σ-Algebra und der Borel-σ-Algebra $\mathcal{B}(\mathbb{R}_+)$. $L^B(\omega,t,x)$ kann als die Summe der unausgeführten Limit-Kauforders des kleinen Investors mit Limitpreis x oder höher interpretiert werden. $\mathcal{L}^S$ als Familie der Limit-Verkaufsorders ist analog definiert.

Definition 2. *Es seien M^B, M^S vorhersehbare, nicht-fallende Prozesse mit $M_0^B = M_0^S = 0$ und es gelte des Weiteren $L^B \in \mathcal{L}^B$ und $L^S \in \mathcal{L}^S$. Wir bezeichen das Quadrupel $\mathfrak{S} = (M^B, M^S, L^B, L^S)$ als* Handelsstrategie.

Hierbei können M_t^B und M_t^S als die aggregierten Marketorderkäufe und -verkäufe bis inklusive Zeitpunkt t angesehen werden. Da die Strategien nun vollständig eingeführt sind, kann der Ausführungsmechanismus des Modells dargestellt werden.

Definition 3. *Gegeben eine Handelsstrategie $\mathfrak{S}$ definieren wir den zugehörigen (selbstfinanzierenden)* Vermögensprozess (φ^0, φ^1) *bei Anfangsvermögen* $(\eta^0, \eta^1) \in \mathbb{R}^2$ *durch*

$$\begin{aligned}
\varphi_t^0(\mathfrak{S}) &:= \eta^0 - \int_0^t (\overline{S}_{s-}, \overline{S}_s) dM_s^B + \int_0^t (\underline{S}_{s-}, \underline{S}_s) dM_s^S \\
&\quad + \int_{[0,t)\times\mathbb{R}_+} \int_x^\infty y L^B(s,dy)\mu(ds,dx) + \int_{[0,t)\times\mathbb{R}_+} \int_0^x y L^S(s,dy)\nu(ds,dx), \\
\varphi_t^1(\mathfrak{S}) &:= \eta^1 + M_t^B - M_t^S + \int_{[0,t)\times\mathbb{R}_+} L^B(s,x)\mu(ds,dx) - \int_{[0,t)\times\mathbb{R}_+} L^S(s,x)\nu(ds,dx).
\end{aligned}$$

Dabei gibt φ^0 das Geldvermögen und φ^1 die Zahl der gehaltenen Aktien an. Die Definition des Vermögensprozesses lässt sich direkt nutzen, um ein Zulässigkeitskriterium zu formulieren, das Verdoppelungsstrategien ausschließt.

Definition 4. *Für $a > 0$ bezeichnen wir eine Handelsstrategie $\mathfrak{S}$ als* zulässig *mit Schranke a sofern der zugehörige Vermögensprozess $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S}))$ die Ungleichung*

$$\varphi^0(\mathfrak{S}) + a + \underline{S}\left(\varphi^1(\mathfrak{S}) + a\right) 1_{\{\varphi^1(\mathfrak{S})+a\geq 0\}} + \overline{S}\left(\varphi^1(\mathfrak{S}) + a\right) 1_{\{\varphi^1(\mathfrak{S})+a<0\}} \geq 0$$

erfüllt.

Damit sind die Grundzüge des Modells beschrieben. Nun zu den beiden Hauptresultaten des Kapitels. Bevor das erste Resultat präsentiert werden kann, bedarf es noch einer Erklärung, was unter einer realistischen Strategie zu verstehen ist. Angenommen der kleine Investor platziert zu einer Stoppzeit T_1^B eine einzelne Limit-Kauforder $\widehat{L}^B := (\theta^B, p^B, T_1^B, T_2^B)$ von Größe $\theta^B \in L_+^0(\mathcal{F}_{T_1^B})$ und Preis $p^B \in L_+^0(\mathcal{F}_{T_1^B})$ mit $p^B < \overline{S}_{T_1^B}$. Wenn die Order bis zu der Stoppzeit $T_2^B \geq T_1^B$ nicht ausgeführt ist, lässt er sie streichen. Man betrachte die Stoppzeiten

$$\begin{aligned}
T^{\overline{S}} &:= \inf\left\{t \in (T_1^B, T_2^B] : \overline{S}_t \leq p^B\right\}, \\
T^\mu &:= \inf\left\{t \in (T_1^B, T_2^B] : \mu\left((T_1^B, t] \times [0, p^B]\right) > 0\right\}, \\
T^* &:= T^{\overline{S}} \wedge T^\mu.
\end{aligned}$$

T^* beschreibt den Zeitpunkt, an dem die Limit-Kauforder ausgeführt wird. Falls der Kauf stattfindet, so stets zum Preis p^B. Für solch eine elementare, einzelne Limit-Kauforder ist anschaulich klar, dass der zugehörige Vermögensprozess folgendermaßen aussehen muss:

$$\begin{aligned}
\varphi_t^0(\widehat{L}^B) &:= -\theta^B p^B 1_{]\!]T^*,T]\!]}(t), \qquad \text{(A)} \\
\varphi_t^1(\widehat{L}^B) &:= \theta^B 1_{]\!]T^*,T]\!]}(t).
\end{aligned}$$

Für eine einzelne Limit-Verkaufsorder ergibt sich analog ein ähnlicher Vermögensprozess. Unter einer *realistischen Strategie* versteht man nun eine endliche konische Kombination aus solch einzelnen Limit-Kauf- und Limit-Verkaufsorders sowie aus endlich vielen Market-Kauf- und Market-Verkaufsorders (d.h. M^B und M^S sind im Wesentlichen elementar vorhersehbare Prozesse). Für eine vollständige formale Definition einer realistischen Strategie siehe Abschnitt 2.4.1. Dort wird auch erklärt, wie sich unter der moderaten zusätzlichen Annahme 2.14 der durch (A) beschriebene Vermögensprozess stets durch eine allgemeine Handelsstrategie aus Definition 2 replizieren lässt. Umgekehrt ist dies natürlich erst einmal nicht der Fall. Eine allgemeine Handelsstrategie mit unendlich vielen möglichen Limitpreisen und zeitstetigen Veränderungen der Ordermengen kann nicht eins zu eins durch eine realistische Strategie dargestellt werden. Wie der folgende Satz zeigt, lässt sie sich aber beliebig gut approximieren. Für zwei $\mathcal{F}\otimes\mathcal{B}([0,T])$-messbare, reellwertige stochastische Prozesse X und Y sei

$$d_{\mathrm{up}}(X,Y) := \quad E\left(1\wedge \sup_{t\in[0,T]}|X_t-Y_t|\right),$$

d.h. d_{up} metrisiert die Konvergenz „gleichmäßig in Wahrscheinlichkeit“.

Satz 5 (Approximation durch realistische Strategien)**.** *Annahme 2.14 sei erfüllt. Für $\varepsilon>0$ und eine beliebige Handelsstrategie $\mathfrak{S}$ existiert stets eine realistische Strategie $\mathfrak{S}^\varepsilon$, sodass*

$$d_{up}\left(\varphi^0(\mathfrak{S}^\varepsilon),\varphi^0(\mathfrak{S})\right)<\varepsilon \qquad \textit{and} \qquad d_{up}\left(\varphi^1(\mathfrak{S}^\varepsilon),\varphi^1(\mathfrak{S})\right)<\varepsilon.$$

Dieses Ergebnis rechtfertigt in gewissem Sinne auch die gewählte, eher abstrake Konstruktion des Modells, für die vielleicht nicht unbedingt auf den ersten Blick klar ist, dass sie mit dem intuitiven Verständnis, wie ein Limitordermarkt funktioniert, harmoniert. Die zweite wichtige Eigenschaft des Modells besteht darin, dass die Familie der allgemeinen Handelsstrategien abgeschlossen ist. Ähnlich wie im Fall proportionaler Transaktionskosten (siehe [CS06]) bedarf es dafür eines Prozesses im Bid-Ask-Spread, der sicherstellt, dass die Variation einer Folge von Handelsstrategien nicht explodiert. Man beachte, dass im englischen Original des obigen Theorems in der Arbeit Annahme 2.14 nicht gefordert wird. Dies hängt damit zusammen, dass dort realistische Strategien direkt im Rahmen der allgemeinen Handelsstrategien formuliert sind.

Definition 6. *Ein adaptierter $(0,\infty)$-wertiger stochastischer Prozess $\widetilde{S}=(\widetilde{S}_t)_{t\in[0,T]}$ wird als strictly consistent price process* (SCPP) *für die Aktie bezeichnet, sofern ein W-Maß $\widetilde{P}\sim P$ existiert s.d. $\widetilde{S}$ ein càdlàg $\widetilde{P}$-Martingal ist für das gilt*

$$\widetilde{S}_t\in(\underline{S}_t,\overline{S}_t),\ \forall t\in[0,T] \quad \textit{and} \quad \widetilde{S}_{t-}\in(\underline{S}_{t-},\overline{S}_{t-}),\ \forall t\in(0,T].$$

Neben der Existenz eines SCPP wird für das folgende Resultat noch die technische Annahme 2.21 zu den Ausführungsmaßen μ und ν benötigt (siehe Abschnitt 2.4.2).

Satz 7 (Abgeschlossenheit der Familie der Strategien). *Es sei Annahme 2.21 erfüllt und es existiere ein SCPP für die Aktie. Außerdem seien $\underline{S}$ und $\overline{S}$ Semimartingale. Sei $(\mathfrak{S}^n)_{n\in\mathbb{N}}$ eine Folge von zulässigen Handelsstrategien jeweils mit Schranke a und Anfangsvermögen (η^0,η^1). Wenn die Folge der zugehörigen Vermögensprozesse $\left((\varphi^0(\mathfrak{S}^n),\varphi^1(\mathfrak{S}^n))\right)_{n\in\mathbb{N}}$ eine Cauchy-Folge bzgl. d_{up} ist, dann existiert eine zulässige Handelsstrategie $\mathfrak{S}$ mit Schranke a und Anfangskapital (η^0,η^1), sodass $\left((\varphi^0(\mathfrak{S}^n),\varphi^1(\mathfrak{S}^n))\right)_{n\in\mathbb{N}}$ gleichmäßig in Wahrscheinlichkeit gegen $(\varphi^0(\mathfrak{S}),\varphi^1(\mathfrak{S}))$ konvergiert.*

Optimale Portfolios eines kleinen Investors in Limitordermärkten

Im Rahmen des in Kapitel 2 eingeführten Modells wird in *Kapitel 3* ein Portfolio-Optimierungsproblem für einen kleinen Investor in einem Limitordermarkt analysiert. Der Investor hat ein bestimmtes Anfangsvermögen gegeben und kann im Limitordermarkt handeln, um die Aufteilung seines Vermögens zwischen Bankkonto und Aktie zu verändern. Sein Ziel ist es dabei, den Erwartungswert seines Nutzens aus zukünftigem Konsum zu maximieren.

Um das Optimierungsproblem lösen zu können, wird eine Reihe von Annahmen getroffen, die aus dem abstrakten Modellrahmen aus Kapitel 2 ein handhabbares Modell machen. Der Zeithorizont ist von nun an unendlich und der Best-Bid $\underline{S}$ folgt einer geometrischen Brownschen Bewegung. Der Spread ist proportional zum Wert von $\underline{S}$. Dies wird durch $\overline{S} := \underline{S}(1+\lambda)$ erreicht, wobei $\lambda > 0$ eine Konstante ist. Da Best-Bid und Best-Ask in diesem Modell stetige Pfade besitzen, ist es für den kleinen Investor niemals sinnvoll, eine Limit-Kauforder mit Limitpreis unterhalb des Best-Bid $\underline{S}$ zu setzen und ebenso wird er niemals eine Limit-Verkaufsorder mit Limitpreis größer als der Best-Ask $\overline{S}$ setzen. Als weitere Vereinfachung wird angenommen, dass jegliche Limitorder-Ausführung des kleinen Investors auf eine exogene Marketorder zurückzuführen ist. Daher ist es für den kleinen Investor auch nie angebracht, eine Limit-Kauforder mit einem Limitpreis größer als $\underline{S}$ zu platzieren, da jede seiner Limit-Kauforders mit einem Limitpreis größer oder gleich $\underline{S}$ ohnehin durch eine exogene Market-Verkaufsorder ausgeführt wird. Das Eintreffen der exogenen Marketorders wird durch zwei unabhängige Poisson-Prozesse N^1 und N^2 mit konstanten Raten α_1 und α_2 beschrieben. Unter zusätzlicher Berücksichtigung einer

Konsumrate, modelliert durch einen adaptierten Prozess c, ergibt sich für den Vermögensprozess

$$\begin{aligned}
\varphi_t^0 &= \eta^0 - \int_0^t c_s ds - \int_0^t \overline{S}_s\, dM_s^B + \int_0^t \underline{S}_s\, dM_s^S - \int_0^{t-} L_s^B \underline{S}_s\, dN_s^1 + \int_0^{t-} L_s^S \overline{S}_s\, dN_s^2 \\
\varphi_t^1 &= \eta^1 + M_t^B - M_t^S + \int_0^{t-} L_s^B\, dN_s^1 - \int_0^{t-} L_s^S\, dN_s^2,
\end{aligned}$$

hierbei bezeichen L^B und L^S nur noch reellwertige, vorhersehbare Prozesse, welche die Ordergrößen der Limit-Kauforder mit Limitpreis $\underline{S}$ und der Limit-Verkaufsorder mit Limitpreis $\overline{S}$ darstellen.

Es bezeichne $\mathcal{A}(\eta^0,\eta^1)$ die *Familie der zulässigen Strategien* mit Schranke 0 und Anfangsvermögen (η^0,η^1). Die Wertfunktion V für das Optimierungsproblem des kleinen Investors mit Anfangsvermögen (η^0,η^1) und logarithmischer Nutzenfunktion lässt sich nun schreiben als

$$V(\eta^0,\eta^1) := \sup_{\mathfrak{S}\in\mathcal{A}(\eta^0,\eta^1)} \mathcal{J}(\mathfrak{S}) := \sup_{\mathfrak{S}\in\mathcal{A}(\eta^0,\eta^1)} E\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right), \tag{B}$$

wobei der Parameter $\delta > 0$ als Zeitpräferenz interpretiert werden kann.

Um das Problem (B) zu lösen, betrachten wir – analog zu [KMK10] – einen fiktiven friktionslosen Markt, in dem der Kurs der Aktie als ein Semimartingal $\widetilde{S}$ modelliert ist. D.h. jede beliebige Anzahl an Aktien kann in diesem friktionslosen Markt zum Preis $\widetilde{S}$ gekauft und verkauft werden. In Abschnitt 3.2.3 ist der friktionslose Markt genauer beschrieben, insb. die Selbstfinanzierungsbedingung und ein Zulässigkeitskriterium.

Nun bezeichne $\widetilde{\mathcal{A}}(\eta^0,\eta^1)$ die Familie der zulässigen Strategien mit Schranke 0 und Anfangsvermögen (η^0,η^1) *im friktionslosen Markt.* Die Wertfunktion im friktionslosen Markt $\widetilde{V}$ ist dann gegeben durch

$$\widetilde{V}(\eta^0,\eta^1) := \sup_{\widetilde{\mathfrak{S}}\in\widetilde{\mathcal{A}}(\eta^0,\eta^1)} \widetilde{\mathcal{J}}(\widetilde{\mathfrak{S}}) := \sup_{\widetilde{\mathfrak{S}}\in\widetilde{\mathcal{A}}(\eta^0,\eta^1)} E\left(\int_0^\infty e^{-\delta t}\log(c_t)dt\right).$$

Es folgt die zentrale Definition des Kapitels.

Definition 8. *Ein reellwertiges Semimartingal $\widetilde{S}$ wird als* Schattenpreis *für die Aktie bezeichnet sofern für alle $t \geq 0$ gilt*

$$\underline{S}_t \leq \widetilde{S}_t \leq \overline{S}_t, \qquad \widetilde{S}_t = \begin{cases} \underline{S}_t & \text{if} \quad \Delta N_t^1 = 1 \\ \overline{S}_t & \text{if} \quad \Delta N_t^2 = 1 \end{cases}$$

und sofern eine Handelsstrategie $\mathfrak{S} = (M^B, M^S, L^B, L^S, c) \in \mathcal{A}(\eta^0,\eta^1)$ im Limitordermarkt-Modell existiert, sodass im friktionslosen Modell gilt $\widetilde{\mathfrak{S}} = (\varphi^0,\varphi^1,c) \in \widetilde{\mathcal{A}}(\eta^0,\eta^1)$ und $\widetilde{\mathcal{J}}(\widetilde{\mathfrak{S}}) =$

$\widetilde{V}(\eta^0, \eta^1)$, wobei $\widetilde{S}$ den Preis der Aktie modelliert. D.h. der Vermögensprozess $(\varphi^0(\mathfrak{S}), \varphi^1(\mathfrak{S}))$ von Strategie $\mathfrak{S}$ kombiniert mit der Konsumrate c aus $\mathfrak{S}$ muss eine optimale Strategie im friktionslosen Markt sein.

Das Konzept des Schattenpreises besteht somit aus zwei Teilen. Einerseits kann im friktionslosen Markt jede Transaktion aus dem Limitordermarkt zum gleichen oder sogar besseren Preis getätigt werden. Andererseits ist die *optimale* Strategie im friktionslosen Markt auch eine zulässige Handelsstrategie im Limitordermarkt. Es lässt sich leicht zeigen (siehe Proposition 3.5), dass aus der Definition des Schattenpreises direkt folgt, dass die in der Definition erwähnte Strategie $\mathfrak{S}$ eine optimale Strategie im Limitordermarkt sein muss. Die Lösung des Optimierungsproblems ist damit eine direkte Konsequenz des folgenden Satzes:

Satz 9. *Es existiert ein Schattenpreis $\widetilde{S}$.*

Um die Existenz eines Schattenpreises zu zeigen, bedarf es nach Definition 8 auch einer (optimalen) Handelsstrategie. Es erweist sich, dass das durch diese Strategie beschriebene nutzenmaximierende Verhalten des kleinen Investors im Limitordermarkt wie folgt aussieht. Es existieren zwei Konstanten $\pi_{\min}, \pi_{\max} \in \mathbb{R}_+$ mit $0 < \pi_{\min} < \pi_{\max}$, sodass der Anteil des Vermögens, der in die Aktie investiert ist, durch den Einsatz von Marketorders im Intervall $[\pi_{\min}, \pi_{\max}]$ gehalten wird, d.h.

$$\pi_{\min} \leq \frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} \leq \pi_{\max}, \quad \forall t > 0.$$

Solange sich der Anteil im Inneren dieses Intervalls befindet, werden keinerlei Marketorder-Käufe oder -Verkäufe getätigt. Darüber hinaus platziert der kleine Investor zu jedem Zeitpunkt zwei Limitorders, sodass

$$\begin{aligned} \frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} &= \pi_{\max}, \quad \text{nach Ausführung der Limit-Kauforder mit Limitpreis } \underline{S}_t \\ \frac{\varphi_t^1 \underline{S}_t}{\varphi_t^0 + \varphi_t^1 \underline{S}_t} &= \pi_{\min}, \quad \text{nach Ausführung der Limit-Verkaufsorder mit Limitpreis } \overline{S}_t. \end{aligned}$$

Die Größe der Limitorders L^B and L^S wird folglich permanent so angepasst, dass eine Ausführung zu einem Sprung des Anteils an Vermögen in der Aktie auf $\pi_{\min}$ oder $\pi_{\max}$ führt. Der optimale Konsum erweist sich als proportional zum Vermögen, wobei hier jedoch die Aktie mit dem Schattenpreis anstelle des Best-Bid zu bewerten ist.

Stochastische Integration bzgl. optionaler Semimartingale

In *Kapitel 4* wird die Erweiterung des elementaren stochastischen Itô-Integrals für den Fall diskutiert, dass die Integratoren optionale Semimartingale sind. Eine genaue Bestimmung eines optionalen Semimartingals findet sich in Definition 4.1. Der wesentliche Unterschied zu einem herkömmlichen càdlàg-Semimartingal besteht darin, dass die Pfade eines optionalen Semimartingals Doppelsprünge aufweisen können, d.h. es handelt sich um làdlàg-Pfade.

Es stellt sich heraus, dass in diesem Fall die Erweiterung der Elementarintegrale auf alle vorhersehbaren Integranden zu klein ist. Im Gegensatz zum Standardfall mit càdlàg-Integratoren ist die entstehende Familie von Integralen nicht abgeschlossen.

Galtchouk [Gal81] hat bereits ein stochastisches Integral für optionale Martingale als Integratoren eingeführt (siehe 4.1 für die genaue Definition; wieder handelt es sich im Wesentlichen um ein Martingal, was aber auch Doppelsprünge aufweisen kann). Hier tritt in gewisser Weise jedoch das gegensätzliche Problem auf. Die Klasse der in [Gal81] gewählten Integranden ist zu groß. Es handelt sich nicht mehr um eine *eindeutige* stetige und lineare Fortsetzung des Integrationsoperators von elementar vorhersehbaren Prozessen. Manche stochastischen Integrale können nicht mehr durch Elementarintegrale approximiert werden.

Blendet man den ökonomischen Gehalt der Modellierung in Kapitel 2 einmal aus, geht es in Kapitel 4 mathematisch gesehen um ganz ähnliche Fragen. In Kapitel 2 war der wesentliche Punkt, eine Familie von Strategien (hier Integranden) zu finden, sodass die entstehenden Vermögensprozesse (hier Integrale) einerseits abgeschlossen sind und andererseits dennoch durch die Vermögensprozesse von realistischen Strategien (hier Elementarintegrale) approximiert werden können.

In Kapitel 4 wird eine Klasse von Integranden eingeführt, die in gewissem Sinne zwischen der zu kleinen Familie der vorhersehbaren Integranden und der zu großen Familie von Integranden aus [Gal81] anzusiedeln ist. Die Familie der Integranden wird charakterisiert durch Messbarkeit bzgl. einer σ-Algebra $\mathcal{A}$ auf $\{1,2\} \times \Omega \times [0,T]$. Die Familie $\mathcal{E}$ der elementar vorhersehbaren Prozesse lässt sich in diese Familie einbetten.

Das stochastische Integral für allgemeine Integranden kann nun als die eindeutige, stetige und lineare Fortsetzung des Elementarintegrals charakterisiert werden.

Satz 10. *Sei X ein optionales Semimartingal. Die Abbildung $H \mapsto H \bullet X$ lässt sich von $\mathcal{E}$ eindeutig fortsetzen (die Fortsetzung wird ebenfalls mit $H \mapsto H \bullet X$ bezeichnet) auf alle lokal* *beschränkten $\mathcal{A}$-messbaren Prozesse $H : \{1,2\} \times \Omega \times [0,T] \to \mathbb{R}$, sodass*

(i) *$H \mapsto H \bullet X$ linear ist;*

(ii) *wenn eine $\mathcal{A}$-messbare Folge $(H^n)_{n\in\mathbb{N}}$ punktweise gegen H konvergiert und für alle $n \in \mathbb{N}$ gilt $|H^n| \leq K$, wobei K ein lokal beschränkter $\mathcal{A}$-messbarer Prozess ist, dann konvergiert $\sup_{s\in[0,T]} |(H^n \bullet X)_s - (H \bullet X)_s|$ in Wahrscheinlichkeit gegen 0.*

Wie im càdlàg-Fall ist das Integral noch etwas über die lokal beschränkten Prozesse hinaus erweiterbar. Im Folgenden bezeichnet $L(X)$ diese Familie der zulässigen Integranden (siehe Proposition 4.12). Angenommen ein optionales Semimartingal lässt sich als Summe eines quadratintegrierbaren optionalen Martingals $N \in \mathcal{M}^2$ und eines Prozesses $B \in \mathcal{A}$ von integrierbarer Variation schreiben. Mit der Norm gegeben durch $E[N_T^2]^{1/2} + E[\mathrm{Var}(B)_T]$ kann es dann als Element des Banachraums $\mathcal{M}^2 \oplus \mathcal{A}$ angesehen werden. Mit den oben eingeführten Begriffen ist es nun möglich, das Abgeschlossenheitsresultat aus Kapitel 4 darzustellen.

Satz 11. *Sei X ein optionales Spezialsemimartingal. Wenn für $(H^n)_{n\in\mathbb{N}} \subset L(X)$ die Folge der zugehörigen Integrale $(H^n \bullet X)_{n\in\mathbb{N}}$ eine Cauchy-Folge in $\mathscr{M}^2 \oplus \mathscr{A}$ ist, dann existiert ein Integrand $H \in L(X)$, sodass $H^n \bullet X \to H \bullet X$ in $\mathscr{M}^2 \oplus \mathscr{A}$.*

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