On the
Geometry, Topology
and
Approximation
of
Amoebas

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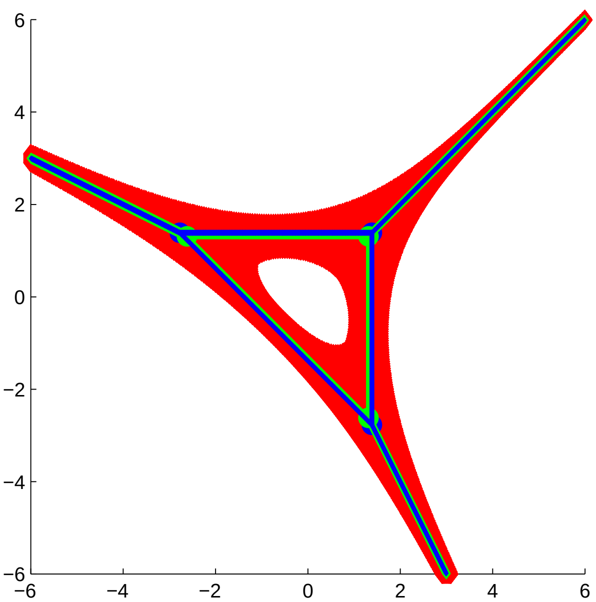
Amoebas are fords between the shores of discrete and continuous mathematics;
a synthesis of discrete, tropical and algebraic geometry,
of complex analysis and algebraic topology.
Its tentacles reach into combinatorics and even applied topics.



They inhabit huge parts of the mathematical ocean

—

hidden below the surface, where they are not discovered at a first glance.

Deutsche Zusammenfassung

Amöben sind eine mathematische Entität im Grenzgebiet zwischen *algebraischer* und *tropischer Geometrie*, die diese beiden Gebiete der Mathematik in natürlicher Weise verbindet.

Der Terminus "*algebraische Geometrie*" wird gegenwärtig in derart vielfältiger Weise verwendet, dass eine für jeden akzeptable Definition nicht leicht zu finden ist. Hier – und dies ist sicherlich zumindest *eine* häufig vertretene Auffassung – verstehen wir die algebraische Geometrie als das Gebiet der Mathematik, das sich der Untersuchung *algebraischer Varietäten*, d.h. der Nullstellenmengen polynomieller Gleichungssysteme, widmet.

Für unsere Zwecke betrachten wir Laurent-Polynome in n Variablen mit komplexen Koeffizienten, deren Varietäten wir auf den algebraischen Torus, d.h. nicht-null Einträge, beschränken. Demzufolge haben wir einerseits ein algebraisches Objekt in Form eines Polynoms bzw. eines polynomiellen Gleichungssystems und andererseits ein geometrisches Objekt, nämlich eine Varietät in Form einer (glatten) komplexen $(n-1)$ Mannigfaltigkeit im algebraischen Torus $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \{0\}$. Ziel ist es, die Beziehung zwischen diesen beiden Objekten zu studieren und zu verstehen.

Bekanntermaßen ist dies ein sehr schwieriges Problem. Deshalb ist es naheliegend, Vereinfachungen dieses Problems zu betrachten – beispielsweise Projektionen der ursprünglichen Varietät.

Die komplexen Zahlen besitzen zwei natürliche Zerlegungen, nämlich einerseits in Real- und Imaginärteil und andererseits in Absolutbetrag und Winkel. Letztere motiviert die Definition von *Amöben* in kanonischer Weise, denn die Amöbe $\mathcal{A}(f)$ eines (Laurent) Polynoms f ist gerade das Bild der zu f gehörigen Varietät unter folgender Log-Abbildung:

$$\text{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|).$$

D.h., die Amöbe ist die Menge der (komponentenweise logarithmierten) Absolutbeträge aller Elemente der Varietät $\mathcal{V}(f)$. In analoger Weise definiert man die *Coamöbe* $\text{co}\mathcal{A}(f)$ als das Bild der Varietät $\mathcal{V}(f)$ unter der Arg-Abbildung:

$$\text{Arg} : (\mathbb{C}^*)^n \to (S^1)^n, \quad (z_1, \ldots, z_n) \mapsto (\arg(z_1), \ldots, \arg(z_n)),$$

d.h., als die Menge aller Winkel (Argumente) der Elemente in $\mathcal{V}(f)$. Ergo können Coamöben als natürliche duale Objekte von Amöben verstanden werden.

Schwerpunkt der *tropischen Geometrie* ist das Studium n-variater tropischer Laurent-Polynome $\text{trop}(f)$, d.h. von Laurent-Polynomen, die über dem tropischen Semiring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ definiert sind. Hierbei bezeichnet "$\oplus$" das klassische Maximum und "$\odot$" die klassische Addition. Die *tropische Varietät* $\mathcal{T}(\text{trop}(f))$ eines derartigen tropischen Polynoms ist definiert als die Menge aller Punkte im $\mathbb{R}^n$, an denen das Maximum mindestens zweimal, d.h. von mindestens zwei tropischen Monomen, angenommen wird. Die so definierte Menge ist ein polyedrischer Komplex (s. Abb. 2.4).

Tropische Geometrie ist ein Gebiet der Mathematik, das etwa seit Beginn des neuen Jahrtausends eine rasante Entwicklung erfahren hat. Der vornehmliche Grund hierfür liegt darin, dass einerseits die untersuchten Objekte stückweise linear und insofern gut handhabbar sind, andererseits aber beim Prozess der Tropikalisierung die ursrüngliche Struktur in überraschend hohem Maße erhalten bleibt und demzufolge viele klassische Theoreme auch "im Tropischen" gelten (als allgemeine Referenz zu tropischer Geometrie s. etwa [**10, 22, 38, 42, 74**]).

Amöben können als Brücke zwischen der klassischen und der tropischen Welt interpretiert werden – maßgeblich aus zwei Gründen: Erstens, betrachtet man Amöben in einer Folge, die, vereinfacht gesagt, durch Konvergenz der Basis eines Logarithmus gegen ∞ gegeben ist, so konvergieren die Amöben gegen eine tropische Hyperfläche. Zweitens postuliert ein Kernresultat der Amöbentheorie zu jeder Amöbe $\mathcal{A}(f)$ die Existenz einer bestimmten tropischen Hyperfläche, genannt "*Gerüst*" (im engl. "*spine*"), die Deformationsretrakt von $\mathcal{A}(f)$ ist. Das bedeutet, die Homotopie jeder Amöbe ist tropisch beschreibbar (s. Kapitel 2, Abschnitt 3 für weitere Details).

Überraschenderweise wurden Amöben (multivariater Polynome) erst vor 28 Jahren erstmals von Gelfand, Kapranov und Zelevinsky in [**23**] definiert. Ihre ursprüngliche Motivation hierfür lag weder in der tropischen Geometrie (die zu dieser Zeit noch nicht existierte), noch in der klassischen algebraischen Geometrie, sondern vielmehr darin, strukturelle Eigenschaften von Polynomen, die im zugehörigen *Newton Polytop* (i.e., die konvexe Hülle aller Exponentenvektoren) verborgen liegen, besser zu verstehen. Hierfür geben sie Amöben als ein Beispiel an (siehe [**23**, Kapitel 6, S. 195]; s. außerdem Theorem 2.15 und anstehende Erläuterungen) und beweisen einige elementare Eigenschaften. Für Coamöben lässt sich nicht mit Sicherheit sagen, wann diese zum ersten Mal definiert wurden. Vermutlich wurden sie zum ersten Mal 2004 von Passare während eines Vortrages erwähnt (s. z.B. [**54**]).

Amöbentheorie begann sich insbesondere seit Anfang des neuen Jahrtausend rasch zu entwickeln. Zentral hierfür waren strukturelle Resultate von Passare et al. auf Grundlage komplexanalytischer Methoden (insbes. [**20**]), Mikhalkins wegweisende Resultate bzgl. reell algebraischer Kurven, die u.a. auf Amöbentheorie beruhen ([**41**]) und Resultate von Kapranov, Maslov, Mikhalkin, Viro et al. zur Verbindung zwischen Amöben und tropischer Geometrie (s. Kapitel 2, Abschnitt 3; für Details s. auch [**40, 88**]). Außerdem Rullgårds Dissertation 2002/03 ([**77**]), in der das Gerüst von Amöben eingeführt und eine systematische Untersuchung von Konfigurationsräumen begonnen wird.

Seitdem lieferten verschiedene Autoren mannigfache Resultate mit unterschiedlichen Fokussen (beispielsweise [**29, 60, 61, 64, 70, 72, 85**]) und die Untersuchung von Coamöben schritt voran (z.B. [**21, 48, 49, 50, 53, 54**]). Darüberhinaus sind inzwischen Anknüpfungspunkte in verschiedene, andere Gebiete der Mathematik bekannt (etwa dynamische Systeme [**16**], die Berechnung unendlicher Reihen [**58**] oder statistische Thermodynamik [**59**]).

In dieser Dissertation lösen wir eine Reihe von Problemen innerhalb der bzw. mit Bezug zur Amöbentheorie. Diese lassen sich grob in vier Hauptthemengebiete gliedern.

(1) Die kombinatorischen Strukturen, die durch die Trägermengen von Polynomen und ihren Amöben induziert werden.
(2) Der Rand von Amöben.
(3) Die geometrische und topologische Struktur von Amöben in Abhängigkeit der Koeffizienten und Exponenten eines (die Amöbe induzierenden) Laurent-Polynoms.
(4) Die Berechnung und Approximation von Amöben.

Die Geometrie und Topologie von Amöben. Hinter der Untersuchung der Geometrie und Topologie von Amöben verbirgt sich gewöhnlich die Frage, wie die Existenz von Komplementkomponenten einer Amöbe $\mathcal{A}(f)$ von der Wahl der Koeffizienten von f abhängt. Hierbei nimmt man an, dass die Menge $A \subset \mathbb{Z}^n$ der Exponenten von f fixiert ist. D.h., man betrachtet alle Amöben von Polynomen innerhalb eines (durch A) fixierten *Konfigurationsraumes* $\mathbb{C}^A$. Dieser Ansatz wurde bereits von Gelfand, Kapranov und Zelevinsky verwendet und später erfolgreich von Rullgård und anderen übernommen.

In der Terminologie von Konfigurationsräumen besteht die Untersuchung geometrischer und topologischer Eigenschaften von Amöben im Wesentlichen aus der Untersuchung von Mengen $U_\alpha^A \subseteq \mathbb{C}^A$, bestehend aus allen Polynomen, deren Amöben über eine spezifische Komplementkomponente verfügen (präziser gesprochen: über eine Komplementkomponente mit einer bestimmten *Ordnung* $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$. Siehe Kapitel 2, Abschnitt 2 für weitere Details). Im Falle linearer Polynome ist diese Beziehung vollständig verstanden ([**20**]). Außerdem lieferte Rullgård eine Reihe allgemeiner, struktureller Resultate ([**76, 77**]; siehe auch Kapitel 2, Abschnitt 4). Jenseits dieser Ergebnisse sind allerdings, abgesehen von einem einzigen, sehr speziellen Beispiel von Passare und Rullgård (siehe [**63, 77**]), für *keine* (spezielle) Klasse von Polynomen konkrete Eigenschaften jedweder Art bewiesen oder auch nur vermutet. Die Problematik ist seit langem bekannt, aber dennoch es gab in diesem Gebiet keinerlei nennenswerte Fortschritte innerhalb der letzten zehn Jahre.

Insbesondere drei Probleme bezüglich des Konfigurationsraumes können als zentral erachtet werden:

(1) Wo liegen (scharfe) Schranken für die Koeffizienten eines Polynoms f (mit fixierten Exponenten), so dass die zugehörige Amöbe über eine Komplementkomponente mit einer bestimmten Ordnung α verfügt (d.h. $f \in U_\alpha^A$).
(2) Sind die Mengen U_α^A zusammenhängend?
(3) Unter welchen Bedingungen gilt $U_\alpha^A \neq \emptyset$, falls vorausgesetzt wird, dass $\alpha \in (\text{conv}(A) \cap \mathbb{Z}^n) \setminus A$.

Das erste Problem (formal: Problem 2.25) wurde bereits von Gelfand, Kapranov und Zelevinksy als "das" kanonische, offene und schwierige Problem bzgl. Amöben charakterisiert (genauer gesagt, ist es das einzige Problem, dass sie bereits mit der Definition von Amöben und den Grundlagen der zugehörigen Theorie erkennen und benennen; siehe [**23**, Kap. 6, Bem. 1.10, S. 198]). Das zweite Problem wurde von Rullgård als offene Frage in seiner Dissertation gestellt ([**77**, S. 39]; hier formal: Problem 2.22). Rullgård beweist in seiner Dissertation, dass das Komplement $(U_\alpha^A)^c$ jeder Menge U_α^A sogar zusammenhängend entlang jedes Schnittes mit einer beliebigen projektiven, komplexen Geraden ist. Doch obwohl die von ihm gestellte Frage insofern eine sehr natürliche ist und eine positive Antwort überzeugend erscheint, ist die Frage vollkommen offen, mit Ausnahme des linearen Falles, der trivial ist, da hier $U_\alpha^A = \mathbb{C}^A$ für jedes $\alpha \in \mathrm{conv}(A) \cap \mathbb{Z}^n$ gilt. Das dritte Problem schließlich führt Rullgårds Liste offener Probleme der Amöbentheorie innerhalb seiner Dissertation an ([**77**, S. 60]; hier formal: Problem 2.20).

Wir lösen das erste Problem für alle n-variaten Polynome, deren Newton Polytop ein Simplex ist und die darüber hinaus genau über $n + 2$ Monome verfügen, wobei der Exponent des $(n + 2)$-ten Monoms im Inneren des Newton Polytops liegen muss. Derartige Amöben können als "Amöben mit minimal abhängigem Träger" charakterisiert werden (siehe etwa [**4, 68**]). Die hierdurch beschriebene Menge von Polynomen ist überraschend reichhaltig und (Träger-)Mengen dieser Form wurden in einer Reihe anderer Zusammenhänge untersucht (s. z.B. [**1, 73**]). Für Polynome innerhalb dieser Klasse mit $n \geq 2$ zeigen wir, dass die zugehörige Amöbe über höchstens eine beschränkte Komplementkomponente verfügt (Theorem 4.1). Für die Existenz dieser beschränkten Komplementkomponente liefern wir untere und obere Schranken. Desweiteren beweisen wir, dass die obere Schranke optimal ist, in dem Sinne, dass sie unter bestimmten Extremalbedingungen scharf wird (s. Theoreme 4.8, 4.10 and 4.13).

Darüber hinaus können wir sogar eine vollständige, explizite Beschreibung der untersuchten Menge U_α^A angeben, falls zusätzlich der innere Gitterpunkt genau dem Schwerpunkt des Simplexes entspricht, das das Newton Polytop darstellt. Wir zeigen, dass ihr Komplement lokal (innerhalb des Konfigurationsraumes $\mathbb{C}^A$) exakt der Fläche entspricht, die durch die Trajektorie einer bestimmten (evtl. rotierten) Hypozykloide berandet wird (Theorem 4.20). Dieses Resultat löst nicht nur das erste der oben genannten Probleme, sondern erlaubt es uns zudem den Zusammenhang der Mengen U_y^A in dieser Klasse zu beweisen (Korollar 4.25). Ferner stellt es eine starke Verallgemeinerung des einen, bekannten Beispiels von Passare und Rullgård zur Struktur von Konfigurationsräumen dar, das weiter oben erwähnt wurde.

Die im zweiten Problem gestellte Frage können wir außerdem positiv für alle univariaten Polynome beantworten, deren Träger A genau mit $\mathrm{conv}(A) \cap \mathbb{Z}$ übereinstimmt (Theorem 3.12; wir nennen derartige Polynome *minimal dünnbesetzt* – s. Kapitel 3, Abschnitte 2 und 3).

Jenseits dieser Resultate untersuchen wir außerdem den univariaten Fall von Polynomen mit minimal abhängigem Träger, d.h. *Trinome* von der Form $z^s + p + qz^{-t}$ mit $p, q \in \mathbb{C}$. Die Frage, wie die beiden Koeffizienten zu wählen sind, derart dass die Nullstellen des Trinoms bestimmte Eigenschaften aufweisen, ist ein klassisches Problem, dessen Ursprünge in das späte neunzehnte und frühe zwanzigste Jahrhundert zurückreichen (s. beispielsweise [**8, 33, 45**]). Ein typisches Problem, dem wir uns hier widmen, ist die Frage, wie die Koeffizienten zu wählen sind, derart dass eine ganz bestimmte Anzahl von Nullstellen höchstens über einen bestimmten Absolutbetrag verfügen. Diese Frage wurde algebraisch im Jahre 1908 von Bohl beantwortet ([**8**]) – die geometrischen und topologischen Strukturen im zugehörigen Konfigurationsraum sind jedoch, obwohl inzwischen über einhundert Jahre vergangen sind, weiterhin vollkommen unbekannt.

Wir übersetzen diese Frage in die Terminologie von Amöben und zeigen, dass hinter den algebraischen Eigenschaften eine reichhaltige Geometrie verborgen liegt. Wir beweisen, dass ein Trinom über eine Nullstelle mit Betrag $|z^*| \in \mathbb{R}_{>0}$ verfügt dann und nur dann, wenn der Koeffizient p auf der Trajektorie einer bestimmten, explizit berechenbaren (evtl. rotierten) Hypotrochoide liegt, deren Parameter von s, t, q und $|z^*|$ abhängen (Theorem 4.32). Darüber hinaus zeigen wir, dass ein Trinom zwei Nullstellen des gleichen Betrages besitzt dann und nur dann, wenn p auf einem explizit berechenbaren 1-Fächer liegt, der genau durch die nodalen Singularitäten bestimmter Hypotrochoiden induziert wird (Theorem 4.40).

Topologisch hat dieses Resultat auf der Seite von Amöben zur Folge, dass für Trinome die Mengen U_α^A für $\alpha \neq 0$ auf eine $(s+t)$-blättrige Überlagerung der S^1 deformationsretrahiert werden können (Theorem 4.51). Dies erlaubt uns nicht nur die Probleme (1) und (2) für Trinome zu lösen, sondern liefert außerdem die Fundamentalgruppe für die Mengen U_α^A und beweist damit insbesondere, dass diese nicht einfach zusammenhängend sind, was bisher für keine Klasse von Polynomen gezeigt werden konnte.

Desweiteren hat Theorem 4.40 Konsequenzen für Problem (3). Rullgård liefert in seiner Dissertation eine notwendige und (davon verschiedene) hinreichende Bedingung dafür, dass $U_\alpha^A \neq \emptyset$ wobei $\alpha \in (\text{conv}(A) \cap \mathbb{Z}^n) \setminus A$ (Theorem 2.19; siehe auch [**77**, Theorem 11]). Leider weist, wie wir zeigen, sein Beweis bzgl. der hinreichenden Bedingung eine kleine Lücke auf, da er einen nicht trivialen Beweisschritt lediglich mit einer Heuristik begründet. Mit oben genanntem Theorem über Trinome können wir besagte Lücke schließen (Theorem 4.43) und desweiteren das, soweit mir bekannt, erste explizite Beispiel einer Amöbe eines multivariaten Polynoms konstruieren, die über eine Komplementkomponente verfügt, deren Ordnung nicht im Träger des definierenden Polynoms enthalten ist (Beispiel 4.44; s. außerdem Abb. 4.10).

Kombinatorische Aspekte und Dünnbesetztheit. Ein weiteres Themengebiet, dem wir uns in dieser Dissertation widmen, lässt sich folgendermaßen motivieren: Einerseits hat es sich als sehr brauchbar erwiesen, den Konfigurationsraum $\mathbb{C}^A$ von Amöben als die Menge aller Polynome mit Träger $A \subset \mathbb{Z}^n$ und Koeffizienten in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ zu definieren. Andererseits entsteht für jede Folge $(c_r)_{r \in \mathbb{N}} \in \mathbb{C}^*$ eines Koeffizienten mit $\lim_{r \to \infty} c_r = 0$ im Limes keine (neue) Komplementkomponente in der zugehörigen Amöbe, die nicht bereits für Koeffizienten innerhalb der Folge existierte. Deshalb ergibt es Sinn, neben dem gewöhnlichen auch einen *augmentierten Konfigurationsraum* $\mathbb{C}_\Diamond^A$ zu betrachten, in dem Koeffizienten den Wert 0 annehmen dürfen, solange durch das Verschwinden des zugehörigen Monoms das Newton Polytop nicht variiert (Letzteres garantiert, dass das "*logarithmic limit set*", d.h., bildlich gesprochen, "die Richtung der Tentakel" erhalten bleibt).

Wir zeigen, dass für jedes Gitterpolytop P die Menge aller Konfigurationsräume $\mathbb{C}^A$ mit $\text{conv}(A) = P$ einen boolschen Verband $L(P)$ bzgl. einer Relation $\sqsubseteq$ bildet, die durch mengentheoretische Inklusion auf den Mengen $A \subset \mathbb{Z}^n$ induziert wird (Theorem 3.2). Diese Verbandsstruktur löst obigen Konflikt elegant, da wir zeigen können, dass jeder augmentierte Konfigurationsraum $\mathbb{C}_\Diamond^A$ genau mit der Menge $\bigcup_{\mathbb{C}^B \sqsubseteq \mathbb{C}^A} \mathbb{C}^B$ übereinstimmt, d.h., gerade die Vereinigung über alle Elemente des Ordnungsideals $\mathcal{O}(\{\mathbb{C}^A\})$ von $\mathbb{C}^A$ bzgl. $L(P)$ ist (Korollar 3.3).

Ferner respektiert der Verband die Struktur der Mengen $U_\alpha^A \subseteq \mathbb{C}^A$ derart, dass gilt: wenn $U_\alpha^A = \emptyset$, dann $U_\alpha^B = \emptyset$ für alle $\mathbb{C}^B \sqsubseteq \mathbb{C}^A$, d.h., für alle Elemente des Ordnungsideals $\mathcal{O}(\{\mathbb{C}^A\})$ von $\mathbb{C}^A$. Diese Beobachtung liefert insbesondere eine (unabhängige) Motivation für folgende prominente Frage (s. hier Problem 3.4) von Passare und Rullgård ([**62**]; s. auch [**66**]):

(4) Haben maximal dünnbesetzte Polynome solide Amöben?

Hierbei heißt eine Amöbe "solide", falls jede Komplementkomponente der Amöbe (bzgl. ihrer Ordnung) zu einer Ecke im Newton Polytop korrespondiert. Ein Polynom heißt "maximal dünnbesetzt", falls der Exponent jedes seiner Monome eine Ecke des zugehörigen Newton Polytops ist. Der Konfigurationsraum, der die maximal dünnbesetzten Polynome bzgl. eines Gitterpolytops P enthält, stellt genau das minimale Element des boolschen Verbandes $L(P)$ dar. Insofern ist die obige Frage tatsächlich durch unser Theorem 3.6 (re-)motiviert, da es, vereinfacht gesagt, impliziert, dass falls ein U_α^A in irgendeinem Konfigurationsraum leer ist, dann auch im Konfigurationsraum der "zugehörigen" maximal dünnbesetzten Polynome.

Das Problem (4) wurde in der Vergangenheit bereits von Nisse behandelt [**52**]. Wir lösen das Problem hier nicht vollständig, liefern allerdings unabhängige, weitgehend elementare Beweise für reichhaltige Klassen von Polynomen (Theoreme 3.9 und 3.10).

Der Rand von Amöben. Da Amöben abgeschlossene Mengen sind, stellt die Charakterisierung ihres Randes ein evidentes Problem dar. Offensichtlich kann ein Punkt nur dann ein Randpunkt sein, wenn er Bild eines kritischen Punktes unter der Log-Abbildung, eingeschränkt auf die zugehörige Varietät, ist. Die Menge all dieser Bilder kritischer Punkte nennen wir die *Contour* der Amöbe. Mikhalkin konnte zeigen (s. [**41, 43**]), dass die Punkte in $\mathcal{V}(f)$, die kritisch unter der Log-Abb. sind, genau übereinstimmen mit der Menge der Punkte $S(f)$, die ein reelles Bild haben unter der logarithmischen Gauß-Abbildung

$$\gamma : \mathcal{V}(f) \to \mathbb{P}_{\mathbb{C}}^{n-1}, \quad (z_1, \ldots, z_n) \mapsto \left(z_1 \cdot \frac{\partial f}{\partial z_1}(\mathbf{z}) : \cdots : z_n \cdot \frac{\partial f}{\partial z_n}(\mathbf{z}) \right).$$

Die logarithmische Gauß-Abbildung ist eine Komposition des komplexen Logarithmus und der gewöhnlichen Gauß-Abbildung, die jeden Punkt einer (nicht singulären) Varietät auf den (projektiven) Normalenvektor seines korrespondierenden Tangentialraumes abbildet. Anders ausgedrückt bedeutet dies, dass ein Punkt $\mathbf{w} \in \mathbb{R}^n$ nur im Rand $\partial\mathcal{A}(f)$ einer Amöbe $\mathcal{A}(f)$ liegen kann, falls innerhalb des Schnittes seiner Faser $\mathbb{F}_{\mathbf{w}}$ (bzgl. der Log-Abb.) und der Varietät $\mathcal{V}(f)$ ein Punkt existiert, der in der Menge $S(f)$ der kritischen Punkte unter der logarithmischen Gauß-Abbildung enthalten ist (s. Korollar 3.14).

Unglücklicherweise ist die Contour im Allgemeinen eine strikte Obermenge des Randes einer Amöbe (s. z.B. [**66**]) und es ist bislang vollkommen unklar, wodurch die Mengen voneinander unterschieden werden können. Wir liefern ein Kriterium zur Unterscheidung, indem wir zeigen, dass ein Punkt $\mathbf{w} \in \mathbb{R}^n$ nur dann Randpunkt einer Amöbe $\mathcal{A}(f)$ sein kann, falls *jeder* Punkt $\mathbf{z}$ im Schnitt seiner Faser $\mathbb{F}_{\mathbf{w}}$ und der Varietät $\mathcal{V}(f)$ in $S(f)$ enthalten ist (Theorem 3.15).

Approximation von Amöben. Der Startpunkt für die Berechnung und Approximation von Amöben und Coamöben, an dem wir uns orientieren, ist Purbhoos Artikel [**70**]. Die Problematik wurde erstmals von Theobald in [**85**] behandelt, wo er insbesondere Amöben durch die Berechnung ihrer Contour (s.o.) approximiert. Heutzutage gilt es jedoch als kanonischer Ansatz, folgendes *Membership Problem* (s. Problem 2.26) effizient zu lösen:

(5) Sei f ein multivariates Laurent-Polynom und $\mathbf{w} \in \mathbb{R}^n$. Entscheide, ob $\mathbf{w}$ in der Amöbe $\mathcal{A}(f)$ von f enthalten ist.

Für Coamöben lässt sich das Problem analog formulieren (s. Problem 2.33). Purbhoo präsentierte eine erste Lösung des obigen Problems (5). Er liefert ein Zertifikat, das er als "*Lopsidedness-Bedingung*" bezeichnet, dafür, dass ein Punkt im Komplement einer Amöbe enthalten ist, wobei er sogar die Ordnung der Komplementkomponente anhand des Zertifikates bestimmen kann. Mit zusätzlicher Hilfe einer Relaxierung des gegebenen Polynoms, basierend auf iterierten Resultanten, ist es möglich die zugehörige Amöbe bis auf eine ε-Umgebung des Randes zu approximieren (s. Theorem 2.28 sowie [**70**]). Der Grad des relaxierten Polynoms wächst hierbei exponentiell in der Anzahl der Relaxierungsschritte.

Darüber hinaus hat sich die Lopsidedness-Bedingung als genuines Beweisinstrument struktureller Aussagen über Amöben erwiesen. Wir verwenden es beispielsweise, um explizite Wege in Konfigurationsräumen zu konstruieren (Theorem 4.24) oder um die Lage reeller Nullstellen von reellen Trinomen zu charakterisieren (Theorem 4.39).

Dennoch ist Purbhoos Ansatz nicht frei von Schwierigkeiten, die eine weitere Untersuchung von Problem (5) nahelegen. Erstens lässt sich Purbhoos Resultat lediglich auf Amöben und nicht auf Coamöben anwenden. Zweitens ist sein Zertifikat kein *algebraisches Zertifikat* im strikten Sinne, d.h. es basiert nicht auf einer polynomiellen Ungleichung, die anhand des gegebenen Polynoms und Punktes $\mathbf{w} \in \mathbb{R}^n$ bestimmbar ist. Drittens existiert kein kanonischer Ansatz, um Purbhoos Resultat zu implementieren. Zwar gibt es keine generellen Hindernisse für eine Implementierung, aber sein Algorithmus lässt sich nicht unmittelbar mit existierenden Berechnungsansätzen oder etablierter Software verknüpfen.

Wir lösen hier diese Probleme durch einen alternativen, Zertifikat-basierten Ansatz auf Grundlage *semidefiniter Programmierung* (*semidefinite programming* – SDP) und *Summen von Quadraten* (*sums of squares* – SOS). *Semidefinite Programmierung* ist eine Verallgemeinerung linearer Optimierung. Der Unterschied besteht darin, dass nicht über den positiven Orthanten, sondern über den Kegel der semidefiniten Matrizen optimiert wird und die Nebenbedingung nicht durch lineare Ungleichungen, sondern durch lineare Matrixungleichungen gegeben sind (s. beispielsweise [**6, 35, 39**]). Ein reelles Polynom f (vom Grad $2d$) heißt *Summe von Quadraten*, falls es sich als Summe $f = \sum_{j=1}^{r} s_j^2$ schreiben lässt, wobei die s_j reelle Polynome (vom Grad d) sind.

Unser Ansatz besteht darin, ein gegebenes n-variates, komplexes Polynom in zwei $2n$-variate, reelle Polynome f^{re} und f^{im} zu transformieren, indem wir jede Variable z_j

durch $x_j + i \cdot y_j$ ersetzen und anschließend den Real- und Imaginärteil von f bestimmen. Auf diese Art und Weise lässt sich das Membership Problem als reelles, polynomielles Gleichungssystem reformulieren. Der reelle Nullstellensatz (Theorem 5.3; s. z.B. [**7**]) garantiert nun, dass dieses System keine Lösung besitzt dann und nur dann, wenn ein Zertifikat der Form $G + H + 1 = 0$ existiert, wobei G ein Polynom im durch die Nebenbedingungen erzeugten Ideal und H eine Summe von Quadraten ist. Durch dieses Zertifikat erhalten wir, sowohl für Amöben als auch für Coamöben, ein algebraisches Zertifikat für das Membership Problem (Korollar 5.4 und Theorem 5.8).

Ein Standardresultat der reell algebraischen Geometrie besagt, dass ein Polynom Summe von Quadraten ist genau dann, wenn eine bestimmte positiv semidefinite Matrix existiert. Aufgrund dessen kann ein Zertifikat für das Membership Problem durch Nachweis der Unlösbarkeit eines bestimmten semidefiniten Optimierungsproblems erbracht werden. Und da sowohl für die Übersetzung von SOS nach SDP als auch für die Lösung von SDPs Standardsoftware existiert, können wir unseren Ansatz einfach implementieren. Wir verwenden die Softwarepakete SOSTOOLS und SEDUMI und präsentieren eine Reihe von Beispielen – sowohl für die Approximation von Amöben, als auch für verwandte Probleme (s. Kapitel 5, Abschnitt 4 für weitere Details).

Da die drei Hauptprobleme, die bei Purbhoos Ansatz bestehen bleiben, durch unsere Methode gelöst werden (bzw. sich dort nicht als Problem stellen), bleibt lediglich die Frage bestehen, wie gut unser Ansatz (komplexitätstheoretsisch) im Vergleich zu Purbhoos ist. Mit der Antwort liefern wir unser zentrales, theoretisches Resultat dieses Bereiches (Korollar 5.17), indem wir zeigen, dass unser SDP-basierter Ansatz dieselbe Komplexität hat wie Purbhoos Lopsidedness-basierter Ansatz. Genauer gesagt zeigen wir: Angenommen, es existiert ein Zertifikat vom Grad höchstens d dafür, dass ein Punkt $\mathbf{w} \in \mathbb{R}^n$ nicht in einer gegebenen Amöbe liegt, das von Purbhoos Ansatz gefunden wird. Dann liefert unser Ansatz ein Zertifikat dafür, dass $\mathbf{w} \in \mathbb{R}^n$ nicht in jener Amöbe liegt, und dieses Zertifikat ist vom Grad höchstens $2d$.

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Contents

CHAPTER 1

Introduction

Amoebas are a mathematical structure appearing at the border between algebraic and tropical geometry and connecting these two topics in a natural way.

In contemporary mathematics the term "*algebraic geometry*" is used so widespreadly that it is hard, to give a definition of it, which everybody can agree with. Here – and I think this is at least *one* common point of view on algebraic geometry – we understand algebraic geometry as the investigation of *algebraic varieties*, i.e., the zero set of a system of polynomial equations.

For our purposes we will investigate Laurent polynomials in n variables with complex coefficients. We restrict ourselves to such polynomials, which only vanish for non-zero entries. Hence, on the one hand, we have an algebraic object in the form of a polynomial resp. a system of polynomials depending of coefficients in $\mathbb{C}$ and exponents in $\mathbb{Z}^n$. On the other hand, we have a geometrical object in the form of a variety given by a (smooth) complex $(n-1)$-manifold in the complex torus $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \{0\}$. The relation between these two objects is what we want to understand.

This is known to be hard. Thus, it is a convincing strategy to try to understand simplifications of this problem – for example to try to understand the same relation for a projection of the given variety.

The two most natural decomposition of complex numbers are either into real and imaginary part or into absolute value and argument. This makes *amoebas* very genuine objects of interests, since an amoeba $\mathcal{A}(f)$ of a (Laurent) polynomial f is nothing else than the image of the corresponding variety $\mathcal{V}(f)$ under the Log-map

$$\text{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n, \quad (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|),$$

i.e., the set of (componentwise logarithmized) absolute values of all elements in the corresponding variety $\mathcal{V}(f)$. Analogously, the *coamoeba* $\text{co}\mathcal{A}(f)$ is the image of the variety $\mathcal{V}(f)$ under the Arg-map

$$\text{Arg} : (\mathbb{C}^*)^n \to (S^1)^n, \quad (z_1, \ldots, z_n) \mapsto (\arg(z_1), \ldots, \arg(z_n)),$$

i.e., the set of arguments of all elements of $\mathcal{V}(f)$ and hence can be considered as the genuine dual object of the amoeba.

In *tropical geometry* we investigate n-variate Laurent polynomials $\text{trop}(f)$ over the tropical semi-ring $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, where "$\oplus$" denotes the usual maximum and "$\odot$" denotes the usual "+". The *tropical variety* $\mathcal{T}(\text{trop}(f))$ of such a tropical polynomial is defined as the set of all points in $\mathbb{R}^n$ where the maximum is attained at least by two monomials. This object is a polyhedral complex, i.e., a discrete geometrical object (see Figure 2.4). Tropical geometry has been an emerging field in mathematics within, say, the last 12 years since, roughly spoken, on the one hand, the investigated objects are piecewise linear and thus nice to handle. On the other hand, a tropicalization keeps an surprisingly high amount of the original structure such that very many classical results also hold tropically (as general references see e.g., [**10, 22, 38, 42, 74**]).

Indeed, amoebas are a bridge connecting the classical with the tropical world. Firstly, if one lets for a given (classical) variety $\mathcal{V}(f)$ the the basis t of the logarithm in the Log-map converge to ∞, then the corresponding amoeba $\mathcal{A}(f)$ will converge against a tropical hypersurface. Secondly, a central result of amoeba theory states that for every amoeba $\mathcal{A}(f)$ there exists a particular tropical hypersurface, the *spine*, which is a deformation retract of $\mathcal{A}(f)$. I.e., the homotopy of every amoeba can be described tropically (see Chapter 2, Section 3 for further details).

1. Historical Background

Surprisingly, amoebas (of multivariate polynomials) were defined only 28 years ago by Gelfand, Kapranov and Zelevinsky in [**23**]. Gelfand, Kapranov and Zelevinsky write that their intention is to show strong structural properties hidden in the *Newton polytope* (i.e., the convex hull of all exponent vectors) of a given Laurent polynomial and they introduce amoebas as one example (see [**23**, Chapter 6, p. 195]; what they basically refer to is the connection between the amoebas' *tentacles* and the Newton polytope – see also Theorems 2.15 and the explanation behind). Furthermore, they realized that the set of complement components of an amoeba $\mathcal{A}(f)$ is in bijective correspondence with all possible Laurent series expansions of the Laurent polynomial f (see Theorem 2.2), which makes amoebas interesting from the viewpoint of complex analysis. Coamoebas were, according to the literature (and personal conversation), even defined only seven years ago by Passare motivated by his studies of amoebas (see e.g., [**54**]).

After Gelfand's, Kapranov's and Zelevinsky's initial results amoeba theory started to develop strongly roundabout at the beginning of 21st century. Passare et. al. gave structural results on amoebas mainly obtained by the usage of complex analytical methods (in particular [**20**]). More or less at the same time Mikhalkin used amoebas to achieve seminal results on real algebraic curves ([**41**]), mainly using topological methods, and Kapranov, Maslov, Mikhalkin, Viro et. al. layed the foundations to reveal the connection between amoebas and the emerging field of tropical geometry by figuring out that amoebas can be understood as the bridge connecting the "classical world" with the "tropical world" in the sense as described above (see Chapter 2, Section 3 for details, see also [**40, 88**]).

A second large step in amoeba theory was made in 2002/03. Theobald used results by Mikhalkin to initialize research on computation and approximation of amoebas ([**85**]). Furthermore, Rullgård finished his PhD-thesis (mainly) on amoebas ([**77**]; under supervision of Passare). He and his coauthors presented a bundle of outstanding results on amoebas. In particular, he strengthened the connection between amoebas and tropical geometry by proving that the homotopy of amoebas is encoded in a certain tropical hypersurface, which he denotes as "*spine*" (see Chapter 2, Section 3; see also [**63**]), and he was the first one who systematically investigated the *configuration space* of amoebas (see Chapter 2, Section 4; see also [**76**]). His results lead to many contemporary open key questions on amoebas and as well to many problems investigated in this thesis.

In the following years advances were made in diverse directions (e.g., [**29, 60, 61, 64, 72**]) and the investigation of coamoebas began and emerged (see e.g., [**21, 48, 49, 50, 53, 54**]). One last paper, which, since it is central for this thesis, should be mentioned at this point is [**70**] by Purbhoo. In this paper a first certificate based approximation process of amoebas is yielded. But, the key condition called "*lopsidedness*", which is used by Purbhoo, additionally plays an important structural role for amoebas and will be used in many different contexts in this thesis.

Until today amoeba theory did not only make big advantages itself and is considered as one of the foundations of tropical geometry, but people also recognized that amoebas appear in various fields of mathematics – concerning problems, which seem not to be connected to amoebas at all at the first glance (e.g., dynamical systems [**16**], the computation of infinite series [**58**] or statistical thermodynamics [**59**]).

2. Investigated Problems and Main Results

In this thesis we concentrate on amoeba related problems, which can roughly be divided into four topics.

(1) The relation between combinatorial aspects of the support set of exponents of a polynomial and its amoeba.
(2) The boundary of amoebas.
(3) The geometrical and topological structure of amoebas in dependence of the exponents and coefficients of a given initial Laurent polynomial.
(4) The computation and approximation of amoebas.

Notice that we re-discuss all main solutions and give an overview about the most interesting resp. urgent problems in amoeba theory in the final Chapter 6.

2.1. The Geometrical and Topological Structure of Amoebas. When one is interested in geometrical and topological questions about amoebas, the usual setting is to fix an arbitrary set of exponent vectors $A \subset \mathbb{Z}^n$ and ask about the correspondence between the coefficients and the structure of the amoeba, which means here in particular the existence of certain complement components. The set of all polynomials with this set of exponent vectors A forms the *configuration space* $\mathbb{C}^A$. This approach was already used by Gelfand, Kapranov and Zelevinsky and has later proved its effectivity in the works of Rullgård et.al..

In terms of the configuration space the investigation of geometrical and topological properties of an amoeba transforms basically in the investigation of sets $U_\alpha^A \subseteq \mathbb{C}^A$ containing all polynomials whose amoebas have a specific complement component (i.e., a complement component of *order* $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$, see Chapter 2, Section 2 for further details). For linear polynomials this correspondence is well understood ([**20**]) and a couple of structural properties were proven by Rullgård ([**76, 77**]; see also Chapter 2, Section 4). But besides that *no* facts are known (except one specific example by Passare and Rullgård; see [**63, 77**]) and *no* progress was made at all within the last ten years.

In particular, three problems on configuration spaces can be marked as central.

(1) Give bounds on the coefficients of a polynomial f (with fixed exponents) such that its amoeba has a complement of a given order α (i.e., $f \in U_\alpha^A$).
(2) Are the sets U_α^A connected?
(3) Under which conditions is $U_\alpha^A \neq \emptyset$ if $\alpha \in (\text{conv}(A) \cap \mathbb{Z}^n) \setminus A$?

The first problem (here formally introduced as Problem 2.25) was already marked as "the“ canonical open problem on amoebas by Gelfand, Kapranov and Zelevinsky (i.e., the only open problem on amoebas they formulated together with the foundations of amoeba theory; see [**23**, Chapter 6, Remark 1.10, p. 198]). The second problem was marked as open question by Rullgård ([**77**, p. 39]; here formally introduced as Problem 2.22). But although it is, indeed, a very natural question, since Rullgård proved connectivity of the complement $(U_\alpha^A)^c$ of every set U_α^A intersected with an arbitrary complex line in $\mathbb{C}^A$ (see [**77**]; see also Theorem 2.21), it is widely open, except for the linear case, where it is trivial since in this case every U_α^A equals $\mathbb{C}^A$. The third problem was marked on top position among "open problems on amoebas" by Rullgård in his thesis ([**77**, p. 60]; here formally introduced as Problem 2.20).

We solve the first problem for all multivariate polynomials with a simplex Newton polytope with one additional monomial whose exponent is contained in the interior of the Newton polytope. Polynomials in this class can also be regarded as *supported on a circuit* (see e.g., [**4, 68**]) and are also investigated in other contexts (see e.g., [**1, 73**]). For polynomials in this class we give upper and lower bounds for the existence of a bounded complement component in the amoeba and prove furthermore that the upper bound is optimal in the sense that it becomes sharp under extremal conditions (Theorems 4.8, 4.10 and 4.13).

For the special case that the interior lattice point y is the barycenter of the simplex, which is the Newton polytope, we can even give a complete description of the set U_y^A by showing that its complement is locally (in $\mathbb{C}^A$) exactly the region bounded by the trajectory of an (eventually rotated) *hypocycloid curve* (Theorem 4.20). This solves not only the first problem for this class, but also allows us to show connectivity of the sets U_y^A, i.e., answers the second question (Corollary 4.25) and generalizes broadly the one example by Passare and Rullgård mentioned above. All these results can also be found in the article [**87**].

Furthermore we are able to answer the second problem affirmatively for all univariate polynomials where the set of exponents A equals $\text{conv}(A) \cap \mathbb{Z}$ (Theorem 3.12; we call such polynomials *minimally sparse* – see Chapter 3, Sections 2 and 3).

Besides these results we investigate the univariate case, i.e., *trinomials* of the form $z^s + p + qz^{-t}$ with $p, q \in \mathbb{C}$. The question how to choose coefficients of trinomials such that the absolute values of the roots show a specific effect are classical questions firstly discussed in late 19th resp. early 20th century (see e.g., [**8, 33, 45**]). A typical question, which we investigate here, is how to choose the coefficients such that a certain number of roots have at most a certain absolute value. Algebraically, this question was solved by Bohl in 1908 ([**8**]), but the geometrical and topological structures behind it are, despite the fact that more than a hundred years have passed, unknown.

We reinterpret these questions in terms of amoeba theory to discover a beautiful geometry and topology hidden behind the algebraic properties. We show that a trinomial has a root of modulus $|z^*| \in \mathbb{R}_{>0}$ if and only if the coefficient p is located on the trajectory of an explicitly computable (eventually rotated) *hypotrochoid curve* (Theorem 4.32). Furthermore, we show that a trinomial has two roots of the same modulus if and only if p is located on an explicitly computable 1-fan corresponding to nodes of hypotrochoids (Theorem 4.40).

On the amoeba side these results imply that for trinomials every U_α^A with $\alpha \neq 0$ can, roughly spoken, be deformation retracted to an $(s+t)$-sheeted covering of an S^1 (Theorem 4.51). This does not only allow us to solve the problems (1) and (2) also for trinomials, but additionally yields the fundamental group for the particular U_α^A and proves in particular that they are not simply connected, which was done for no other class of polynomials before.

Furthermore, the results on trinomials have an impact on question (3). There exists a theorem by Rullgård giving some necessary and some (different) sufficient conditions for $U_\alpha^A \neq \emptyset$ for $\alpha \in (\text{conv}(A) \cap \mathbb{Z}^n) \setminus A$ (Theorem 2.19; see also [**77**, Theorem 11]). Unfortunately, we discover a gap in the proof about the sufficient conditions (see Chapter 2, Section 4). Fortunately, we can use our results on trinomials to close it (Theorem 4.43) and provide, to the best of my knowledge, the first explicit example of an amoeba containing a complement component with an order α, which is not contained in the support set A (Example 4.44; see also Figure 4.10).

2.2. Combinatorial Aspects and Sparsity. A further related topic, which we discuss, is motivated by the fact that on the one hand, in configuration spaces $\mathbb{C}^A$ by definition all coefficients of polynomials are contained in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. But, on the other hand, for every sequence $(c_r)_{r\in\mathbb{N}} \in \mathbb{C}^*$ of a coefficient with $\lim_{r\to\infty} c_r = 0$ no (new) complement component may appear in the corresponding amoeba in the limit if it does not already exist for coefficients in the sequence. Thus, it makes sense to investigate an *augmented configuration space* $\mathbb{C}_\Diamond^A$ where we allow coefficients c_α (and hence also the corresponding monomial) to vanish unless α is a vertex in $\text{conv}(A)$. We require the latter since we want to preserve the Newton polytopes of polynomials in $\mathbb{C}_\Diamond^A$ with respect to those in $\mathbb{C}^A$ and correspondingly the logarithmic limit set, i.e., the direction of the tentacles of their amoebas.

We show that for a given lattice polytope P the set of all configuration spaces $\mathbb{C}^A$ with $\text{conv}(A) = P$ forms a boolean lattice $L(P)$ with respect to a relation $\sqsubseteq$ induced by set-theoretic inclusion on the sets $A \subset \mathbb{Z}^n$ (Theorem 3.2). The lattice structure solves the upper conflict nicely and naturally, since we can show that the augmented configuration space $\mathbb{C}_\Diamond^A$ is nothing else than $\bigcup_{\mathbb{C}^B \sqsubseteq \mathbb{C}^A} \mathbb{C}^B$, i.e., the union of all elements in the *order ideal* $\mathcal{O}(\{\mathbb{C}^A\})$ of $\mathbb{C}^A$ with respect to $L(P)$ (Corollary 3.3).

Furthermore, if a set $U_\alpha^A \subseteq \mathbb{C}^A$ is empty, then the lattice preserves this property for the whole order ideal, since then for every $\mathbb{C}^B \sqsubseteq \mathbb{C}^A$ the corresponding set $U_\alpha^B \subseteq \mathbb{C}^B$ is also empty (Theorem 3.6). This result is in particular an independent motivation for a prominent problem (see Problem 3.4 here) posed by Passare and Rullgård ([**62**]; see also [**66**]) asking

(4) Do maximally sparse polynomials have solid amoebas?

"Solid amoeba" means that the existing complement components of the amoeba corresponds to vertices in the Newton polytope (via the order map) and "maximally sparse" means that every monomial in the initial polynomial has an exponent, which is a vertex in the Newton polytope. The configuration space, which contains maximally sparse polynomials is exactly the minimal element of every lattice of configuration spaces. Hence, this question is indeed motivated by our Theorem 3.6, since it, roughly spoken, implies that if a U_α^A is empty in some configuration space, then it is also empty in the "corresponding" maximally sparse case.

Note that Problem (3) was treated and announced to be solved by Nisse in [**52**] using a coamoeba approach. In this thesis we provide independent, rather elementary proofs for rich classes of polynomials (Theorems 3.9 and 3.10).

2.3. The Boundary of Amoebas. Since amoebas are closed sets, another evident problem is to describe their boundary. Obviously, a point may only be contained in the boundary $\partial\mathcal{A}(f)$ of an amoeba $\mathcal{A}(f)$, if it is contained in the *contour* of $\mathcal{A}(f)$, i.e., the image of all critical points of $\text{Log}_{|\mathcal{V}(f)}$. Mikhalkin proved (see [**41, 43**]) that the points of $\mathcal{V}(f)$, which are critical under the Log-map, coincide with the set $S(f)$ of all points with real image under the *logarithmic Gauss map*

$$\gamma : \mathcal{V}(f) \to \mathbb{P}_{\mathbb{C}}^{n-1}, \quad (z_1, \ldots, z_n) \mapsto \left(z_1 \cdot \frac{\partial f}{\partial z_1}(\mathbf{z}) : \cdots : z_n \cdot \frac{\partial f}{\partial z_n}(\mathbf{z})\right).$$

which is a composition of a branch of the holomorphic logarithm and the usual Gauss map, which maps every point of a (non-singular) variety on the (projective) normal vector of its corresponding tangent space. In other words, a point $\mathbf{w} \in \mathbb{R}^n$ may only belong to the boundary $\partial\mathcal{A}(f)$ if in the intersection of its fiber $\mathbb{F}_{\mathbf{w}}$ with respect to the Log map and the variety $\mathcal{V}(f)$ exists a point $\mathbf{z} \in (\mathbb{C}^*)^n$, which belongs to the set $S(f)$ of critical points of the logarithmic Gauss map (see Corollary 3.14).

Unfortunately, in general the contour is a strict superset of the boundary of an amoeba (see e.g., [**66**]), but it is completely unknown, what separates these two sets. We provide such a distinction here by proving that a point $\mathbf{w} \in \mathbb{R}^n$ may only be contained in the boundary $\partial\mathcal{A}(f)$ of an amoeba $\mathcal{A}(f)$ if for *every* point $\mathbf{z}$ in the intersection $\mathcal{V}(f) \cap \mathbb{F}_{\mathbf{w}}$ holds that $\mathbf{z} \in S(f)$ (Theorem 3.15).

2.4. Approximation of Amoebas. For the computation and approximation of amoebas and coamoebas for us the starting point is Purbhoo's outstanding article [**70**]. Although this topic was initiated by Theobald in [**85**] by computation of the contour of amoebas, which we just defined above, nowadays the canonical approach to approximate amoebas is to find a way to solve the following *membership problem* (see Problem 2.26)

(5) Let f be a multivariate Laurent polynomial and $\mathbf{w} \in \mathbb{R}^n$. Decide, whether $\mathbf{w}$ is contained in the amoeba $\mathcal{A}(f)$ of f.

Note that an analogue problem may be formulated for coamoebas (Problem 2.33). Purbhoo gave a first solution for Problem (5). He presented a certificate, which he denotes as "*lopsidedness*", for a point to be in the amoebas complement (and even to figure out, in which complement component with respect to its order it is contained). With a relaxation based on iterated resultants it is possible to use this certificate to approximate the amoeba up to an ε-neighbourhood of its boundary (see Theorem 2.28; see also [**70**]). The degree is hereby growing exponentially in the number of steps the approximation process takes.

A fact that makes this result even stronger is that the lopsidedness-condition turned out to be a genuine instrument to prove structural results in amoeba theory. We use it e.g., to construct a path in configuration spaces (Theorem 4.24) or to prove a statement about the location of real roots in real trinomials (Theorem 4.39).

Anyhow, some issues remain, which keep this Problem (5) very worthy to investigate. Firstly, Purbhoo's result only works for amoebas, not for coamoebas. Secondly, his certificate is not an *algebraic certificate* in the strict sense, i.e., it is not based on a polynomial inequality, which can be computed out of the initial polynomial and the investigated point $\mathbf{w} \in \mathbb{R}^n$. Thirdly, there is no canonical way to implement his result. Of course, there is no obstruction against an implementation, but his algorithm does not connect canonically to existing computational approaches or software.

We solve all these issues here with an alternative, certificate based approach via *semidefinite programming* (SDP) and *sums of squares* (SOS). *Semidefinite programming* is a generalization of linear programming with the difference that one optimizes over the cone of positive semidefinite matrices instead of the positive orthant and the constraints are given by linear matrix inequalities instead of linear inequalities (see e.g., [**6, 35, 39**]). A real polynomial f (of degree $2d$) is a *sum of squares* if it can be written as sum $f = \sum_{j=1}^{r} s_j^2$ of real polynomials s_j (of degree d).

Our idea is to transform a given n-variate complex polynomial f into two $2n$-variate real polynomials f^{re} and f^{im} by rewriting every variable z_j as $x_j + i \cdot y_j$ and then take the real and imaginary part of f. With that it is possible to reformulate the membership problem as a system of real polynomial equations. The Real Nullstellensatz (Theorem 5.3; see e.g., [**7**]) guarantees that this system has no solution if and only if there is a certificate of the form $G + H + 1 = 0$ where G is a polynomial in the ideal defined by the constraints of the problem and H is a sum of squares, which yields an algebraic certificate for the membership problem as well for the amoeba as for the coamoeba case (Corollary 5.4 and Theorem 5.8).

Since it is well known that a polynomial is a sum of squares if and only if a particular positive semidefinite matrix exists, a certificate for the membership problem can be found by proving infeasibility of a certain semidefinite optimization problem. And since there is software available as well for the translation from SOS to SDP as for the solving of SDPs, our approach can be implemented straightforwardly. We use the software packages SOSTOOLS and SeDuMi here and present a couple of examples as well for the approximation of amoebas as for related problems, which can be tackled with our approach (see Chapter 5, Section 4 for further details).

Since the three remaining issues are solved, the final, important question is, how good our approach (in terms of complexity) is compared to Purbhoo's. Here, we show our main theoretical result of this chapter (Corollary 5.17), stating that our SDP-based approach has the same complexity as Purbhoo's lopsided-based approach. More precise, we show that for every certificate of degree at most d for a point $\mathbf{w} \in \mathbb{R}^n$ not to be in a certain amoeba, which can be found by lopsidedness and iterated resultants, we find a certificate for the same point and the same amoeba of degree at most $2d$.

3. Structure of the Thesis

In Chapter 2, the preliminaries, we present all known results needed for this thesis and introduce most of the open problems, which we solve or partially solve.

Specifically, we begin with some basic properties about amoebas, e.g., regarding closedness, convexity and the logarithmic limit set. As a next step we introduce the already mentioned order map. We describe how it associates every complement component of an amoeba uniquely to an integral point in the Newton polytope (of the defining polynomial). We discuss elaborately the different connections between amoebas and tropical geometry. Firstly, we give an introduction into tropical geometry, and afterwards we explain how amoebas build a bridge between classical and tropical world via Maslov dequantization and how the homotopy of an amoeba is encoded in its spine, a tropical hypersurface.

In the following we define the configuration space of amoebas and recall the most important results on it. Since the configuration space is a central object of investigation for us, we also formally provide most of the key problems here. Afterwards, we formally introduce the membership Problem (5) and recall Purbhoo's lopsidedness condition and his main result to use this condition and a relaxation on the initial polynomial to solve the membership problem. Finally, we give a short overview about coamoebas, present some core results of the past years and formulate a membership problem for coamoebas.

We begin Chapter 3 with discussing the fiber structure (of amoebas) provided by the Log-map. We show in particular that the naturally given fiber bundle of the Log-map induces a fiber function, which will be the crucial puzzle piece connecting varieties and configuration spaces.

In the following section we show that the set of all configurations spaces $\mathbb{C}^A$ with $A \subset \mathbb{Z}^n$ and $\text{conv}(A) = P$, where P is an arbitrary given lattice polytope, has a genuine boolean lattice structure with a relation yielded by a set-theoretical inclusion. We show that this lattice structure not only generalizes notation nicely, but also harmonizes with the structure of amoebas and motivates Passare's and Rullgård's Problem (3). We have a closer look at this problem by investigating maximally and minimally sparse polynomials.

We answer Question (3) affirmatively for a large class of polynomials (Theorems 3.9 and 3.10) and prove Problem (2) for univariate minimally sparse polynomials (Theorem 3.12).

In the final section of this chapter we discuss the boundary of amoebas. We introduce the Gauss- as well as the logarithmic Gauss-map and present Mikhalkin's result on the boundary of amoebas resp. its contour (Theorem 3.13). Afterwards, we improve this result by showing that a point $\mathbf{w} \in \mathbb{R}^n$ in the contour may only belong to the boundary, if *every* point in the intersection of the variety and the fiber $\mathbb{F}_{\mathbf{w}}$ is critical under the logarithmic Gauss-map (Theorem 3.15).

In Chapter 4 we focus on geometrical or topological questions of amoebas and present all results of this thesis related to problems of this kind. The main class $\mathcal{P}_\Delta^y$ of investigation contains polynomials with $n+2$ monomials and a simplex Newton polytope such that the lattice point given by the $(n+2)$nd monomial is contained in the interior. We show for $n \geq 2$ that the corresponding amoebas have at most one bounded complement component. We provide a bundle of minor results, but in particular, we formulate and prove our main Theorems 4.8, 4.10 and 4.13 providing bounds on the coefficients for the existence of a bounded complement component in amoebas of polynomials in this class and thus solving Problem (1).

In the following we have a close look at the special case of polynomials in $\mathcal{P}_\Delta^y$ where the inner lattice point is the barycenter of the Newton polytope. For this class we give a full, local, geometrical description of the configuration space by proving that the set $(U_y^A)^c$ of all polynomials, where the amoeba has no bounded complement component, coincides exactly with the region bounded by a certain hypocycloid curve (Theorem 4.20). Furthermore, we solve Problem (2) by showing that U_y^A is connected (Corollary 4.25).

The univariate elements of the class $\mathcal{P}_\Delta^y$ are trinomials. We show that many classical 19th resp. early 20th century problems on trinomials can be reinterpreted in terms of amoeba theory, which allows us to derive an amazing rich geometrical and topological structure hidden in the corresponding configuration space. As main theorems we show that the existence of a root with a certain modulus is equivalent to the containedness of a certain coefficient in the trajectory of an explicitly computable hypotrochoid curve (Theorem 4.32) and the existence of a certain complement component of the amoeba is equivalent to the non-containedness of a certain coefficient in a particular, explicitly computable 1-fan (Theorem 4.40). This result allows us to close the gap in one of Rullgård's proofs, which we discovered earlier (Theorem 4.43). Furthermore, we show that, in the case of trinomials, at least for all but one α, the sets U_α^A can be deformation retracted to an $(s+t)$-sheeted cover of an S^1 (Theorem 4.51). This result also solves en passant the Problems (1) and (2) for trinomials. Finally, we use trinomials to disprove that complement components of amoebas are always monotonically growing in the absolute value of its (via the order map) corresponding coefficient (Theorem 4.53).

In Chapter 5 we present a new approach to approximate amoebas with the use of semidefinite programming and sums of squares. In particular, we show that this approach can solve the membership Problem (5) for amoebas and coamoebas via an algebraic certificate, which can in the amoeba case be computed as efficiently as Purbhoo's lopsided-based certificate, which is state of the art. In detail, we proceed as follows.

We begin with a short introduction into semidefinite programming and sums of squares. In particular, we present the Real Nullstellensatz as main theoretical backbone of our method. Afterwards, we show how the membership problem for amoebas and coamoebas can be solved via usage of the Real Nullstellensatz (Corollary 5.4 and Theorem 5.8). More precisely, we show how the Real Nullstellensatz can certify that a point $\mathbf{w} \in \mathbb{R}^n$ is contained in the complement of a given amoeba (analogously for coamoebas). Furthermore, we prove that (for $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ and $\mathbf{w} \in \mathbb{R}^n$) if there is a lopsidedness-based certificate of degree d, then our approach yields an SOS-based certificate of degree at most $2d$ and thus our method is as efficient as Purbhoo's (Corollary 5.17).Finally, we present examples solved with the implementation of our approach and some possible further applications.

In Chapter 6 we give a resume and an overview about the in my opinion most interesting problems in amoeba and coamoeba theory, which are resp. remain contemporarily open.

Parts of this thesis were already published, accepted for publication or are part of ongoing projects. The content of Chapter 4, Sections 1 and 2 is based on joint work with Thorsten Theobald and is contained in [**87**]. The content of Chapter 5 is based on joint work with Thorsten Theobald and is contained in [**86**]. The content of Chapter 3, Section 4 is part of an ongoing project joint with Franziska Schröter.

CHAPTER 2

Preliminaries

In this section we introduce *amoebas* as projection of varieties in the complex torus (given by a Laurent polynomial) on its componentwise logarithmized absolute values. We present all relevant known properties as well as the open problems on amoeba theory, which we investigate in this thesis. Furthermore, we present some background information about related topics as e.g., tropical geometry. For general background on amoebas see e.g., [**42, 66, 77**].

In Section 1 we define amoebas, fix the basic notation we use and present some first, classical, elemental results – e.g., concerning closedness of amoebas, convexity and bijective correspondence between complement components and Laurent series expansions. Furthermore, we discuss the logarithmic limit set.

In Section 2 we explain how complement components of an amoeba can be uniquely related to lattice points in the Newton polytope of its corresponding polynomial via the *order map*.

In Section 3 we point out the connection between amoebas and tropical geometry. Next to an introduction into tropical geometry itself we explain how amoebas connect the "classical world" with the "tropical world". Furthermore, we show that crucial information about an amoeba, specifically its homotopy, is encoded in an associated tropical hypersurface – the *spine*.

In Section 4 we introduce the *configuration space*, which contains all polynomials formed by a sum of monomials with a particular fixed set of exponents. In this section we additionally provide a major part of the problems we tackle in this thesis (see Section 4) – in particular the key Problems 2.22 and 2.25 – since most contemporary problems concerning the topology or geometry of can be formulated in terms of subsets of the configuration space. In order to motivate these problems and to give an overview we recall the most important amoeba related results on the configuration space (mostly proven by Rullgård who initiated a systematic investigation of this space as part of his thesis; see [**77**]).

Next to the geometrical and topological structure of amoebas, the second part of this thesis takes charge of the approximation of amoebas. It culminates in the *membership problem* (see Problems 2.26 and 2.27), which we present in Section 5. It asks to decide efficiently whether a given point in $\mathbb{R}^n$ is contained in a given amoeba. Next to the problem itself, we recall the main known results on this topic based on Purbhoo's *lopsidedness condition* and iterated resultants (see also [**70**]).

In Section 6 we have a look on *coamoebas*, which can be understood as dual objects to amoebas since they are given by the projection of a variety in the complex torus on its arguments. We provide a selection of main results and open questions on coamoebas. In particular, we introduce the membership problem (Problem 2.33) for coamoebas, for which we present a solution in Chapter 5.

1. General Aspects of Amoebas

In this section we introduce *amoebas* – the key objects of this dissertation. We introduce the notation for *coefficients*, *support* and *varieties* of *Laurent polynomials* and define their *amoebas* and *unlog-amoebas* afterwards. We mention some very basic properties of amoebas with respect to closedness, convexity, continuous behavior of the complement under changing of coefficients (Theorem 2.3) and their structure at infinity given by the *logarithmic limit set.*

Let $\mathbb{C}[\mathbf{z}^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ denote the *ring of complex Laurent polynomials.* Every of its elements is of the form

$$f \quad = \quad \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha,$$

where the *support set* $A \subset \mathbb{Z}^n$ is finite and the *coefficients* satisfy $b_\alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Note that we will omit to write down the assumption of finiteness of support sets A for the rest of the thesis.

To every Laurent polynomial f of the upper form we associate a *Newton polytope* $\text{New}(f)$, which is the convex hull of the support set A

$$\text{New}(f) \quad = \quad \text{conv}(A),$$

see e.g., Figure 2.2.

For a complex Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ the *variety* $\mathcal{V}(f)$ is given by the subset of the algebraic torus $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \{0\}$ where f vanishes, i.e.,

$$\mathcal{V}(f) \quad = \quad \{\mathbf{z} \in (\mathbb{C}^*)^n \ : \ f(\mathbf{z}) = 0\}.$$

The *amoeba* $\mathcal{A}(f)$ of a Laurent polynomial f (introduced by Gelfand, Kapranov, and Zelevinsky; [**23**]) is the image of its variety $\mathcal{V}(f)$ under the Log-map

$$\text{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n, \ (z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|),$$

where $|z|$ denotes the complex absolute value of a complex number z. In Figure 2.2 we show some examples for amoebas in $\mathbb{R}^2$.

The definition of amoebas can be generalized straightforwardly from hypersurfaces to varieties of ideals $I \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$. Let the variety of an ideal $I \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ be given by

$$\mathcal{V}(I) \quad = \quad \{\mathbf{z} \in (\mathbb{C}^*)^n \ : \ f(\mathbf{z}) = 0 \text{ for all } f \in I\}.$$

Then, we define the amoeba of I by $\mathcal{A}(I) = \text{Log}(\mathcal{V}(I))$. In most parts of the thesis we focus on the hypersurface case of amoebas; only some of our results about the approximation of amoebas in Chapter 5 will deal with amoebas of ideals.

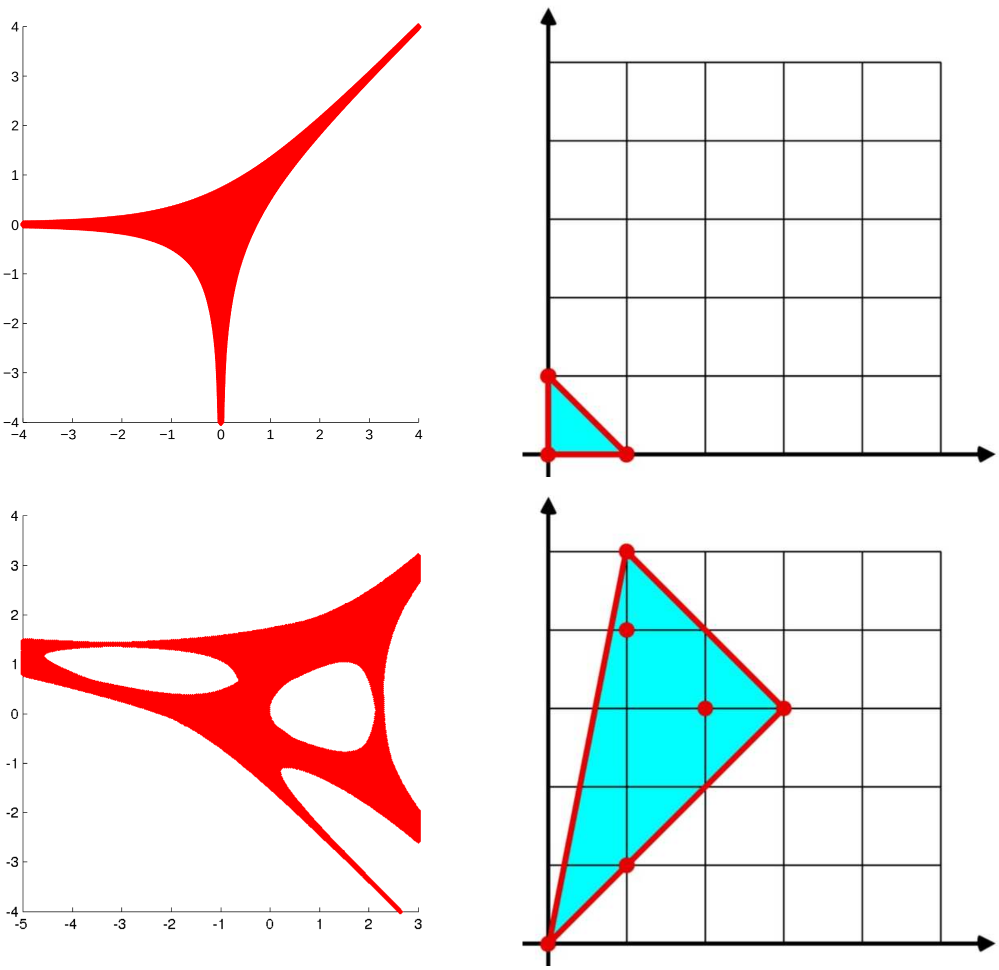


FIGURE 2.2. The amoebas of the polynomials $f = z_1 + z_2 + 1$ and $f = z_1^3 z_2^3 - 9z_1^2 z_2^3 + z_1 z_2^5 - 4z_1 z_2^4 - 4z_1 z_2 + 1$ and their corresponding Newton polytopes.

In some special contexts it is more convenient to investigate the *unlog amoeba* $\mathcal{U}(I)$ of an ideal I (or equivalently of a single polynomial f). The unlog amoeba is given as the image of the variety $\mathcal{V}(I)$ under the componentwise absolute value map

$$|\cdot| : (\mathbb{C}^*)^n \to \mathbb{R}^n_{>0}, \ (z_1, \ldots, z_n) \mapsto (|z_1|, \ldots, |z_n|).$$

We present some general, basic facts about amoebas.

Theorem 2.1 (Gelfand, Kapranov, Zelevinsky [**23**]). *For $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ the amoeba $\mathcal{A}(f)$ is a closed set with non-empty complement component.*

Let $(\mathcal{A}(f))^c$ denote the complement of an amoeba $\mathcal{A}(f)$. We say $E \subseteq (\mathcal{A}(f))^c$ with $E \neq \emptyset$ is a *complement component* of $\mathcal{A}(f)$ if E and $(\mathcal{A}(f))^c \setminus E$ are not connected.

Theorem 2.2 (Gelfand, Kapranov, Zelevinsky [**23**])**.** *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$. Every complement component of the amoeba $\mathcal{A}(f)$ is convex. The set of all complement components of $\mathcal{A}(f)$ corresponds bijectively to the set of all Laurent expansions of $1/f$ centered at the origin.*

The upper theorem gives not only a strong motivation to investigate amoebas, but it allows us furthermore to distinguish between different complement components via different Laurent series expansions of $1/f$. Thus, an expression like "the" or "a certain complement component" is well defined. In the following Section 2 we will be able to simplify the language of complement components even more with the introduction of the *order map*.

For a Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with fixed support set $A \subset \mathbb{Z}^n$ the size and even the existence of a certain complement component of its amoeba $\mathcal{A}(f)$ depends on the choice of the coefficients of f. We discuss this fact and the bundle of problems implied by it in detail in Section 4, when we introduce the *configuration space* of amoebas. For the moment, we only point out that complement components behave nicely with respect to the fact that they do not vanish due to a small change of coefficients.

Theorem 2.3 (Forsberg, Passare, Tsikh [**20**])**.** *Let $A \subset \mathbb{Z}^n$ and $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha$ with $b_\alpha \in \mathbb{C}^*$ and $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. Then the number of complement components of $\mathcal{A}(f)$ is lower semicontinuous under changing of the coefficients of f.*

This theorem will be generalized in Theorem 2.17 such that it becomes in particular obvious that "the same" complement components of $\mathcal{A}(f)$ (in the sense of the *order map*, which we introduce in Section 2) are preserved under a slight changing of the coefficients.

An amoeba has finitely many "*tentacles*" with different directions, which were the reason for the term "amoeba" (see, again, e.g., Figure 2.2). The tentacles direct to a set of points at infinity called the *logarithmic limit set*, which was defined by Bergman [**2**] in the following way. For a given ideal I with amoeba $\mathcal{A}(I)$ and an arbitrary positive real number r one defines a sequence $(\mathcal{A}_r(I))_{r \in \mathbb{R}}$ given by

$$\mathcal{A}_r(I) \quad = \quad 1/r \cdot \mathcal{A}(I) \cap S^n,$$

where $1/r \cdot \mathcal{A}(I) = \{1/r \cdot \mathbf{w} : \mathbf{w} \in \mathcal{A}(I)\}$ and S^n denotes the n-dimensional unit sphere $S^n = \{\mathbf{w} \in \mathbb{R}^n \ : \ ||\mathbf{w}||_2 = 1\}$.

The logarithmic limit set $\mathcal{A}_\infty(I)$ is given by $\mathcal{A}_\infty(I) = \lim_{r \to \infty} \mathcal{A}_r(I)$. It was shown by Bieri and Groves (see [**3**]) that the logarithmic limit set is a rational, polyhedral fan on the unit sphere (see also [**38**]). In Section 3 we will see that, in case of a hypersurface $\mathcal{V}(f)$ with $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$, the logarithmic limit set $\mathcal{A}_\infty(f)$ is induced by the elements of the normal fan of the Newton polytope $\text{New}(f)$ with codimension at least one.

2. The Order Map

Many questions about amoebas are, more precisely, questions about the existence and behavior of their complement components with respect to the support and the coefficients of their corresponding Laurent polynomial. Since the standard setting is to fix the support set and allow the coefficients to vary (i.e., fix a *configuration space*; see Section 4), it makes sense to ask for a map from the set of complement components into a structure determined by the support. This is done by the *order map*, which maps injectively into $\text{conv}(A) \cap \mathbb{Z}^n$. In this section we introduce the order map and discuss some of its properties and consequences yielded by its existence.

Let $A \subset \mathbb{Z}^n$ and $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. We already advised that we want to investigate the behavior of the complement components of $\mathcal{A}(f)$ with respect to changing the coefficient of f but keeping the support set A fixed. Thus, it is heuristically convincing that it would be convenient if one was able to establish a correspondence between the set of complement components of the amoeba $\mathcal{A}(f)$ and the elements of A. It turns out that A is not sufficient but that we have to take all integral points in the Newton polytope $\text{New}(f)$ of f, which is the convex hull of A.

It was shown by Forsberg, Passare and Tsikh (see [**20**]) that such a desired correspondence indeed exists via the *order map*, which is given by

$$\text{ord} : \mathbb{R}^n \setminus \mathcal{A}(f) \to \mathbb{R}^n, \quad \mathbf{w} \mapsto \frac{1}{(2\pi i)^n} \int_{\text{Log}^{-1}(\mathbf{w})} \frac{z_j \partial_j f(z)}{f(z)} \frac{dz_1 \cdots dz_n}{z_1 \cdots z_n}, \quad 1 \leq j \leq n\,.$$

The following theorem states that the order map has all the requested properties mentioned above.

Theorem 2.4 (Forsberg, Passare, Tsikh [**20**])**.** *The image of the order map is contained in* $\text{New}(f) \cap \mathbb{Z}^n$. *Let* $\mathbf{w}, \mathbf{w}' \in (\mathcal{A}(f))^c$. *Then* $\mathbf{w}$ *and* $\mathbf{w}'$ *belong to the same complement component of* $\mathcal{A}(f)$ *if and only if* $\text{ord}(\mathbf{w}) = \text{ord}(\mathbf{w}')$.

The theorem shows that the order map yields an injective map from the the set of complement components of $\mathcal{A}(f)$ to $\text{New}(f) \cap \mathbb{Z}^n$, i.e., $\text{conv}(A) \cap \mathbb{Z}^n$. Hence, we use the following notation for complement components

$$E_\alpha(f) \quad = \quad \{\mathbf{w} \in (\mathcal{A}(f))^c \ : \ \text{ord}(\mathbf{w}) = \alpha\} \quad \text{ for every } \alpha \in \text{New}(f) \cap \mathbb{Z}^n\,.$$

Unfortunately, the set of complement components neither corresponds bijectively to $\text{New}(f) \cap \mathbb{Z}^n$ nor to $A \subset \text{New}(f) \cap \mathbb{Z}^n$ as the following example shows.

Example 2.5. Let $f = z_1^2 z_2 + z_1 z_2^2 + 0.5 \cdot z_1 z_2 + 1$. This polynomial has support set $A = \{(0,0),(1,2),(2,1),(1,1)\}$ but the complement component $E_{(1,1)}(f)$ is empty as Figure 2.3 shows. A formal proof for that fact follows later from Theorem 4.8.

Assume one knows that a point $\mathbf{w} \in \mathbb{R}^n$ is contained in a complement component of $\mathcal{A}(f)$. In general, it is not clear a priori how to compute its order easily and efficiently. The following theorem yields that the computation is trivial in some special cases.

Theorem 2.6 (Forsberg, Passare, Tsikh [**20**])**.** *Let* $A \subset \mathbb{Z}^n$ *and* $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ *with* $b_\alpha \in \mathbb{C}^*$ *and* $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. *Assume there exists an* $\alpha' \in A$ *and a* $\mathbf{w} \in (\mathcal{A}(f))^c$ *such that for all* $\mathbf{z} \in (\mathbb{C}^*)^n$ *with* $\text{Log}(\mathbf{z}) = \mathbf{w}$ *holds* $|b_{\alpha'} \mathbf{z}^{\alpha'}| > |\sum_{\alpha \in A \setminus \{\alpha'\}} b_\alpha \mathbf{z}^\alpha|$. *Then* $\text{ord}(\mathbf{w}) = \alpha'$, *i.e.,* $\mathbf{w} \in E_{\alpha'}(f)$.

With the Theorems 2.4 and 2.6 one obtains some first upper and lower bounds for the number of complement components of an amoeba $\mathcal{A}(f)$.

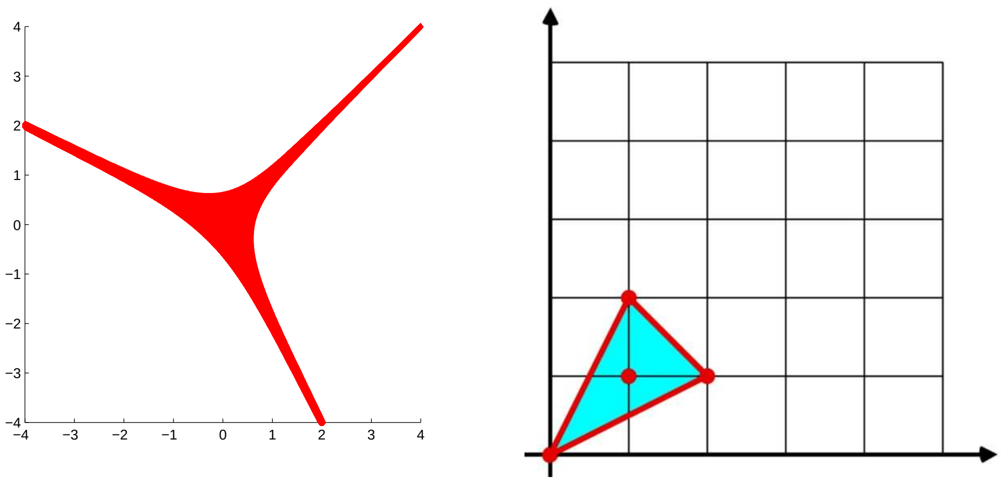


FIGURE 2.3. The amoeba of the polynomial $f = z_1^2 z_2 + z_1 z_2^2 + 0.5 \cdot z_1 z_2 + 1$ and its corresponding Newton polytopes.

Corollary 2.7 (Forsberg, Passare, Tsikh [**20**]). *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with support set $A \subset \mathbb{Z}^n$. If $\alpha \in A$ is a vertex of $\mathrm{New}(f)$, then $E_\alpha(f) \neq \emptyset$. In particular, the number of complement components of $\mathcal{A}(f)$ is bounded from below by the number of vertices of $\mathrm{New}(f)$ and bounded from above by $\#(\mathrm{New}(f) \cap \mathbb{Z}^n)$.*

The first part of the statement basically follows from the fact that for a point $\mathbf{w} \in E_{\alpha'}(f)$ and a vector $\mathbf{v}$ contained in the dual cone of α' in the normal fan of $\mathrm{New}(f)$ holds $\lim_{\lambda \to \infty} |f(\mathbf{w} + \lambda \mathbf{v})| - |(\mathbf{w} + \lambda \mathbf{v})^{\alpha'}| = 0$. The rest follows from the Theorems 2.4 and 2.6.

A consequence of this corollary is that the order map is bijective for linear polynomials. In fact, the linear case is even completely understood due to the following result of Forsberg, Passare and Tsikh.

Theorem 2.8 (Forsberg, Passare, Tsikh [**20**]). *For a linear polynomial $f = b_0 + \sum_{i=1}^n b_i z_i \in \mathbb{C}[\mathbf{z}]$ and a point $\mathbf{z} \in (\mathbb{C}^*)^n$, $\mathrm{Log}(\mathbf{z})$ is contained in a complement component if and only if $|b_0| > \sum_{j=1}^n |b_i z_i|$ or $|b_i z_i| > |b_0| + \sum_{j \neq i} |b_j z_j|$ for some $i \in \{1, \ldots, n\}$.*

Unfortunately, it is known for no other class of polynomials (except some special examples) how to choose the coefficients such that a specific complement component exists. This question can be regarded as *the* classical resp. *the* first prominent problem on amoebas, since it was already marked as an open problem by Gelfand, Kapranov and Zelevinsky (vaguer formulated in the original since the order map was not known at this time; see [**23**, Chapter 6, Remark 1.10])

Problem 2.9. *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with support set $A \subset \mathbb{Z}^n$ and $\alpha \in \mathrm{conv}(A)$. Figure out, how $E_\alpha(f)$ depends on the the coefficients of f. In particular, give sufficient and necessary conditions on the coefficients such that $E_\alpha(f) \neq \emptyset$.*

This very initial and general problem was later splitted resp. followed up by more specific problems on the configuration space (see Section 4), the membership of points in the complement components of amoebas (see Section 5) and the boundary of amoebas (see Chapter 3, Section 4).

3. Connections to Tropical Geometry

Tropical geometry has been an emerging topic in mathematics within roundabout the last ten years. It investigates the geometrical properties of the *tropical semi-ring* $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, which is given by the operations $a \oplus b = \max(a, b)$ and $a \odot b = a + b$ (where some expositions prefer the minimum instead of the maximum). Thus, the neutral elements for tropical addition is $-\infty$ and the neutral element for tropical multiplication is 0. For a general introduction to tropical geometry see e.g., [**10, 22, 38, 74**].

Analogously to classical polynomials one can define a *tropical polynomial* in the following way. A *tropical monomial* is a function

$$\mathbb{R}^n \to \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto b_\alpha \odot \mathbf{x}^\alpha = b_\alpha \odot x_1^{\alpha_1} \odot \cdots \odot x_n^{\alpha_n}$$

with $b_\alpha \in \mathbb{R}$ and $\alpha \in \mathbb{N}^n$. Note that a tropical monomial in terms of classical operations is the linear form $b_\alpha + \langle \mathbf{x}, \alpha \rangle$. A *tropical polynomial* is a tropical sum of tropical monomials, i.e., it is a function

$$\mathbb{R}^n \to \mathbb{R}, \quad (x_1, \ldots, x_n) \mapsto \bigoplus_{\alpha \in A}^{d} b_\alpha \odot \mathbf{x}^\alpha,$$

where $A \subset \mathbb{N}^n$ is a support set as in the classical case and $b_\alpha \in \mathbb{R}$ (note that a tropical monomial $b_\alpha \odot \mathbf{x}^\alpha$ does not vanish if $b_\alpha = 0$).

For a tropical polynomial h, the *tropical hypersurface* (or *tropical variety*) $\mathcal{T}(h)$ is defined as the set of points where the maximum is attained at least twice. Tropical hypersurfaces are polyhedral complexes, which are geometrically dual to a subdivision of the Newton polytope of h. This subdivision is induced by lifting every lattice point $\alpha \in A$ to (α, b_α) and then projecting down the upper hull of the lifted Newton polytope to $\mathbb{R}^n$ (i.e., it is a regular subdivision; see Figure 2.4).

Indeed, amoeba theory forms a cornerstone of tropical geometry. On the one hand, the *Maslov dequantization*, which is the canonical way to transform the variety of a classical polynomial into the variety of a tropical one, leads via the amoeba of the classical polynomial. On the other hand, for every given amoeba there exists a tropical hypersurface – the *spine* – which is a deformation retract of the amoeba. We give a brief overview about both of these connections.

For a given semi-ring R_0 a *quantization* is a family of semi-rings R_h, $h \geq 0$ such that R_s and R_t are isomorphic for every $s, t > 0$ but no R_s with $s > 0$ is isomorphic to R_0; one calls R_s with $s > 0$ a *quantized version* of R_0 (see e.g., [**42**]). Maslov observed ([**40**]) that the standard semi-ring $\mathbb{R}_{>0}$ is a quantized version of the tropical $(\max, +)$ semi-ring. This can e.g., be done by defining $R_h = (\mathbb{R}, \oplus_h, \odot_h)$ with

$$x \oplus_h y \;=\; (x^{\frac{1}{h}} + y^{\frac{1}{h}})^h, \quad x \odot_h y \;=\; x + y \quad \text{for } x, y \in \mathbb{R}. \tag{2.1}$$

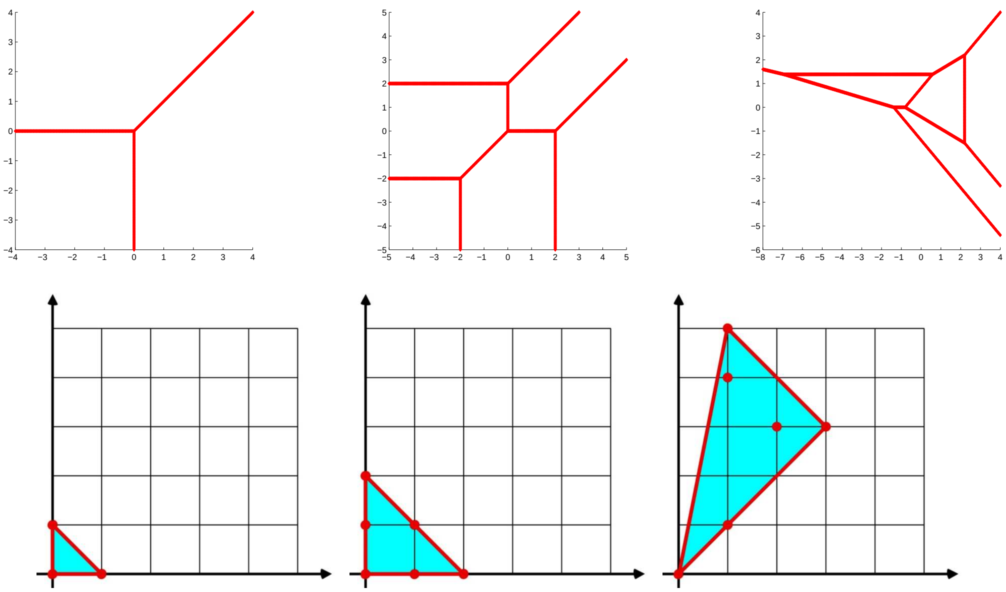


FIGURE 2.4. The tropical hypersurfaces of the tropical polynomials $f = 1z_1 \oplus 1z_2 \oplus 1$, $f = 1z_1^2 \oplus 3z_1 \oplus 1z_2^2 \oplus 3z_2 \oplus 3z_1z_2 \oplus 1$ and $f = 0z_1^3z_2^3 \oplus \log(9)z_1^2z_2^3 \oplus 0z_1z_2^5 \oplus \log(4)z_1z_2^4 \oplus \log(4)z_1z_2 \oplus 0$ and their corresponding Newton polytopes.

R_1 is isomorphic to the classical semi-ring on $\mathbb{R}_{>0}$ (i.e., with classical operations "+" and "·"; see [**40**]) and the isomorphism between R_1 and R_h is given by $x \mapsto x^h$. R_0 is the tropical semi-ring, which is not isomorphic to R_1 since $x \oplus x = x$ (see [**42**]).

Here, we use an isomorphic approach defining $R_t = (\mathbb{R}, \oplus_t, \odot_t)$ with

$$x \oplus_t y \;=\; \log_t(t^x + t^y), \quad x \odot_t y \;=\; \log_t(t^{x+y}) \quad \text{for } x, y \in \mathbb{R}$$

for every $t \geq e$. For $t = e$ we obtain a ring isomorphic to the classical operations "+" and "·" on $\mathbb{R}_{>0}$ given by the quantization (2.1) and the isomorphism $[0,1] \to \mathbb{R}_{\geq e}$, $h \mapsto e^{1/h}$. For $t \to \infty$ we obtain the tropical "$\oplus$" and "$\odot$".

Let $\text{Log}_t : (\mathbb{C}^*)^n \to \mathbb{R}$, $\mathbf{z} \mapsto (\log_t |z_1|, \ldots, \log_t |z_n|) = (x_1, \ldots, x_n)$ and $A \subset \mathbb{Z}^n$ a support set. Maslov and Viro (see [**40, 88**]; see also [**42**]) showed that for every $t \geq e$ and every polynomial

$$g_t \;=\; \bigoplus_{\alpha \in A}{}_t \; b_\alpha \odot_t \langle \alpha, \mathbf{x} \rangle,$$

mapping from $(R_t)^n$ to R_t, the function $f_t = (\log_t)^{-1} \circ g_t \circ \text{Log}_t$ is a "classical polynomial mapping from $(\mathbb{C}^*)^n$ to $\mathbb{C}$ given by $f_t = \sum_{\alpha \in A} t^{b_\alpha} \mathbf{z}^\alpha$ with standard "+" and "·".

Thus, for every $t \geq e$ the polynomial f_t has a variety $\mathcal{V}(f_t) \in (\mathbb{C}^*)^n$ and a corresponding amoeba $\mathcal{A}(f_t) \subset \mathbb{R}^n$ given by $\text{Log}_t(\mathcal{V}(f_t))$. Since $\mathcal{A}(f_e)$ is a "classical" amoeba and $\lim_{t \to \infty}(\mathcal{A}(f_t))$ is a tropical hypersurface, Maslov dequantization indeed yields tropical polynomials with tropical hypersurfaces out of classical polynomials with classical hypersurfaces by taking their amoebas and deforming them due to changing of the log-basis.

A more elegant way to perform this process is via Puiseux-series. First, we define valuations (see [**9**, p. 386]).

Definition 2.10. For a ring $(R, \oplus, \odot)$ and a totally ordered commutative group $(G, +)$ a *valuation* on R with values in G is a map $\nu : R \to G \cup \{\infty\}$, which satisfies the following axioms:

$$\begin{aligned} \nu(x \odot y) &= \nu(x) + \nu(y) \text{ for every } x, y \in R, \\ \nu(x \oplus y) &\geq \inf\{\nu(x), \nu(y)\} \text{ for every } x, y \in R, \\ \nu(1) &= 0 \text{ and } \nu(0) \ = \ \infty. \end{aligned}$$

In the case $G = \mathbb{R}$ we say that R is *real valuated*.

On real valuated fields F the valuation map $\nu : F \to \mathbb{R} \cup \{\infty\}$ induces a norm on F (see [**9**, p. 428 et seq.]; see also, e.g., [**38**, p. 64]), which is given by

$$|\cdot|_\nu : F \to \mathbb{R}, \quad z \mapsto e^{-\nu(z)}.$$

The norm $|\cdot|_\nu$ is *non-Archimedean*, i.e., it satisfies $|x+y|_\nu \leq \max\{|x|_\nu, |y|_\nu\}$. For more details on valuations see, e.g., [**9, 69**], see also [**3**].

The *field of Puiseux-series* $\mathbb{K}$ (see, e.g., [**15, 38, 42, 44**]) is given by all formal power sums $\sum_q b_q t^q$ with $b_q \in \mathbb{C}^*$, and the support set of all q is a well ordered subset of the rational numbers, such that all q share a common denominator. On the field of Puiseux-series $\mathbb{K}$ there exists a real valuation map $\mathrm{val} : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$, which is given by

$$\mathrm{val}\left(\sum_q b_q t^q\right) \ = \ \min\{q \, : \, b_q \neq 0\}.$$

Note that the minimum always exists due to the requirement that the support set of every element in $\mathbb{K}$ is well ordered. Hence, $\mathbb{K}$ is a real valuated field.

Kapranov (see [**15, 31**]; see also, e.g., [**42, 44**]) defined for a given algebraic variety $\mathcal{V}(f) \subset (\mathbb{K}^*)^n$ (i.e., here $f \in \mathbb{K}[\mathbf{z}]$ is a polynomial over the field of Puiseux-series) its *non-Archimedean amoeba* $\mathcal{A}_\mathbb{K}(f)$ by $\mathcal{A}_\mathbb{K}(f) = \overline{\mathrm{Log}_\mathbb{K}(\mathcal{V}(f))}$, where $\mathrm{Log}_\mathbb{K}$ is given by

$$\mathrm{Log}_\mathbb{K} : (\mathbb{K}^*)^n \to \mathbb{R}^n, \ (z_1, \ldots, z_n) \mapsto (\log|z_1|_{\mathrm{val}}, \ldots, \log|z_n|_{\mathrm{val}}),$$

where $|\cdot|_{\mathrm{val}}$ denotes the norm on $\mathbb{K}$ induced by the valuation val, i.e., $\log|z_j|_{\mathrm{val}} = -\mathrm{val}(z_j)$ for every $1 \leq j \leq n$. Note that $\mathrm{Log}_\mathbb{K}$ is well defined here, since we assumed that $\mathcal{V}(f) \subset (\mathbb{K}^*)^n$. In fact, this non-Archimedean amoeba is nothing else than a tropical hypersurface.

Theorem 2.11 (Kapranov [**31**]; see also [**42, 74**]). *Let $A \subset \mathbb{Z}^n$, $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \in \mathbb{K}[\mathbf{z}]$ with $\mathcal{V}(f) \subset (\mathbb{K}^*)^n$. Then the non-Archimedean amoeba $\mathcal{A}_\mathbb{K}(f)$ equals the tropical hypersurface $\mathcal{T}(h)$ for $h = \bigoplus_{\alpha \in A} -\mathrm{val}(b_\alpha) \odot \mathbf{w}^\alpha$.*

Indeed, the non-Archimedian amoeba is not only a tropical hypersurface but it furthermore coincides with the result of the Maslov dequantization in the following way. Let d be the Euclidean metric in $\mathbb{R}^n$. The *Hausdorff metric* for two closed sets $A, B \subset \mathbb{R}^n$ is given by $d_{\mathrm{Hausd}}(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}$.

Theorem 2.12 (Mikhalkin [**43**], Rullgård [**77**]; see also [**42**]). *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n \subset (\mathbb{K}^*)^n$ and non-Archimedean amoeba $\mathcal{A}_{\mathbb{K}}(f) = \overline{\mathrm{Log}_{\mathbb{K}}(\mathcal{V}(f))}$. Then the amoebas $\mathcal{A}(f_t)$ given by Maslov dequantization converge against $\mathcal{A}_{\mathbb{K}}(f)$ in Hausdorff metric.*

For more details on Maslov dequantization, Puiseux-series and non-Archimedean amoebas see e.g., [**10, 15, 22, 38, 42, 44, 74**]. Note that the notation differs slightly from source to source. E.g., Mikhalkin avoids taking closures but uses a generalized version of Puiseux-series allowing real exponents (see [**42, 43, 44**]). Furthermore, Mikhalkin denotes $\mathrm{Log}_{\mathbb{K}}$ ("val" in his notation) as "non-Archimedean valuation" (see [**42**, p. 26]).

Notice that not every tropical hypersurface can be realized out of a classical one in the ways just described. Conditions and obstructions for such a realization were e.g., recently given in [**11**].

Maslov dequantization describes in general how to construct tropical hypersurfaces out of classical polynomials resp. their amoebas. But it is not useful in a practical sense if we start with a certain given amoeba and want to associate a certain tropical hypersurface, which preserves structural properties of the amoeba. Hence, we discuss some special choices for tropical coefficients.

The easiest way to construct a non-trivial tropical hypersurface out of a given amoeba, is to use the valuation coinciding with the usual complex absolute value $|\cdot|$. Thus, for a given polynomial $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha$ we define its *tropicalization* as

$$\mathrm{trop}(f) \quad = \quad \bigoplus_{\alpha \in A} \log|b_\alpha| \odot \mathbf{w}^\alpha,$$

where $\mathbf{w} = \mathrm{Log}(\mathbf{z})$. Similarly, we define the *complement-induced tropicalization* $\mathcal{C}(f) = \mathcal{T}(\mathrm{trop}(f_{|C}))$. For the amoeba $\mathcal{A}(f)$ of f let

$$C \quad = \quad \{\alpha \in \mathrm{conv}(A) \cap \mathbb{Z}^n \ : \ E_\alpha(f) \neq \emptyset\}$$

and

$$\mathrm{trop}(f_{|C}) \quad = \quad \bigoplus_{\alpha \in C} \log|b_\alpha| \odot \mathbf{w}^\alpha.$$

For both $\mathcal{T}(\mathrm{trop}(f))$ and $\mathcal{C}(f) = \mathcal{T}(\mathrm{trop}(f_{|C}))$ see, e.g., [**66, 77**].

Unfortunately, for a given f both $\mathcal{T}(\mathrm{trop}(f))$ and $\mathcal{C}(f)$ do not need to be homotopy equivalent to $\mathcal{A}(f)$ in general (see Figure 2.5 for a counterexample; see also e.g., [**66**]). But this property holds for the *spine* of an amoeba introduced by Passare and Rullgård (see [**63, 77**]), which we provide in the following. Although the spine is more complicated to define, it is naturally related to the order map.

The Ronkin function of a polynomial f is given by

$$N_f : \mathbb{R}^n \to \mathbb{R}, \qquad \mathbf{w} \ \mapsto \ \frac{1}{(2\pi i)^n} \int_{\mathrm{Log}^{-1}(\mathbf{w})} \frac{\log|f(z_1,\ldots,z_n)|}{z_1 \cdots z_n} \, dz_1 \cdots dz_n.$$

Theorem 2.13 (Ronkin [**75**])**.** *For $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$, N_f is a convex function, which is affine linear on the complement components of $\mathcal{A}(f)$. Furthermore, for $\alpha \in \operatorname{conv}(A) \cap \mathbb{Z}^n$ and $\mathbf{w} \in E_\alpha(f)$ we have* $\operatorname{grad}(N_f)(\mathbf{w}) = \operatorname{ord}(\mathbf{w})$.

Observe that the fiber of a point $\mathbf{w} \in \mathbb{R}^n$ under the Log–map is a real n–torus (see Chapter 3, Section 1 for a detailed description). The value of the Ronkin function at a point $\mathbf{w} \in \mathbb{R}^n$ can be interpreted as a computation of the average value of the f restricted to the fiber $\mathbb{F}_{\mathbf{w}} = \operatorname{Log}^{-1}(\mathbf{w})$ of $\mathbf{w}$. This can be seen very nicely e.g., by Purbhoo's approach using iterated resultants of f to approximate $\mathcal{A}(f)$ and its spine (see Section 5).

By the affine linearity of $N_f(\mathbf{w})$ on every $E_\alpha(f)$, we have for all $\mathbf{w} \in E_\alpha(f)$ that $N_f(\mathbf{w}) = \beta_\alpha + \langle \alpha, \mathbf{w} \rangle$ with Ronkin coefficient

$$\beta_\alpha \quad = \quad \log|b_\alpha| + \operatorname{Re}\left[\frac{1}{(2\pi i)^n} \int_{\operatorname{Log}^{-1}(0)} \log\left(\frac{f(\mathbf{z})}{b_\alpha \cdot \mathbf{z}^\alpha}\right) \frac{dz_1 \wedge \ldots \wedge dz_n}{z_1 \cdots z_n}\right], \tag{2.2}$$

see e.g., [**66**].

Note that if $\alpha \in A$ is a vertex of $\operatorname{conv}(A)$, then $\beta_\alpha = \log|b_\alpha|$. The *spine* $\mathcal{S}(f)$ of $\mathcal{A}(f)$ is defined as the tropical hypersurface of the tropical polynomial

$$\bigoplus_{\alpha \in C} \beta_\alpha \odot \mathbf{w}^\alpha$$

and is therefore dual to an integral, regular subdivision of $\operatorname{New}(f)$ (see [**63, 66, 77**]). The reason for investigating the spine, which is with respect to the complement induced tropical hypersurface much more complicated, is the following theorem.

Theorem 2.14 (Passare, Rullgård [**63**])**.** *For $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ the spine $\mathcal{S}(f)$ is a deformation retract of $\mathcal{A}(f)$.*

For the complement induced tropical hypersurface the same property is only known for certain special cases. For our purposes we need the following theorem.

Theorem 2.15 (Rullgård [**77**])**.** *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with at most $2n$ monomials such that for all $k \in \{1, \ldots, n-1\}$ no $k+2$ of its exponent vectors lie in an affine k-dimensional subspace. Then $\mathcal{C}(f)$ is a deformation retract of $\mathcal{A}(f)$.*

Although one would not expect this property to be true in general for $\mathcal{C}(f)$, it has – to the best of my knowledge – not been disproven yet. In particular no explicit counterexamples are given in the literature. Hence, we introduce this as an open problem here.

Problem 2.16. *Is there any $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ such that $\mathcal{A}(f)$ and $\mathcal{C}(f)$ are not homotopy equivalent?*

We provide such a counterexample in this thesis in Corollary 4.45.

Note that Theorem 2.14 yields that the logarithmic limit set $\mathcal{A}_\infty(f)$ of an amoeba (of a hypersurface) is given by the elements in the normal fan of $\operatorname{conv}(A) = \operatorname{New}(f)$ with codimension at least one (see Section 1). Note furthermore that by the fundamental theorem of tropical geometry (see e.g., [**38**]) the direction $\mathbf{w}$ lies in the logarithmic limit if and only if the initial ideal given by the initial term $\text{in}_{\mathbf{w}}(f)$ with weight $\mathbf{w}$ does not equal $\langle 1 \rangle$. For background about initial ideals see e.g., [**12**].

Despite the Theorems 2.14 and 2.15, the computation of the homotopy of amoebas (i.e., the question, which (inner) complement components exist) with respect to the choice of coefficients remains an open key question. The reason is that for both the computation of the spine $\mathcal{S}(f)$ and the computation of the complement induced tropical hypersurface $\mathcal{C}(f)$ knowledge of the set $C = \{\alpha \in \text{conv}(A) \cap \mathbb{Z}^n : E_\alpha(f) \neq \emptyset\}$ of the orders of *existing* complement components in $\mathcal{A}(f)$ is required. Thus, for a given $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$, *if* one knows, which complement components of $\mathcal{A}(f)$ exist, *then* $\mathcal{A}(f)$ can be retracted into a tropical hypersurface, i.e., piecewise linear object, $\mathcal{S}(f)$. But it is a widely open problem, *how* to compute C (see Figure 2.5; see also e.g., [**77**]).

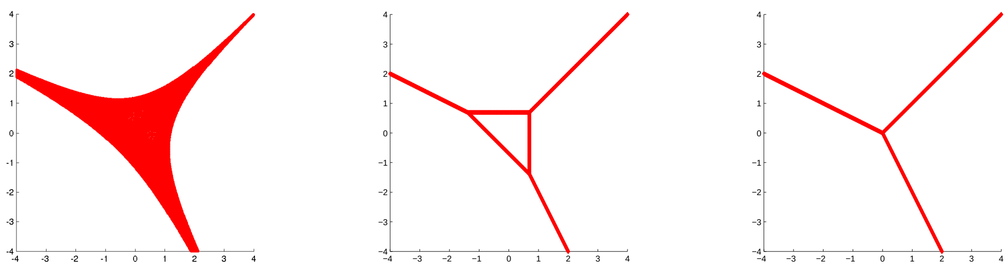


FIGURE 2.5. Let $f = 1 + z_1^2 z_2 + z_1 z_2^2 - 2z_1 z_2$. The left figure shows the amoeba $\mathcal{A}(f)$, the middle one the tropical hypersurface $\mathcal{T}(\text{trop}(f))$ and the left one the complement induced tropical hypersurface $\mathcal{T}(\text{trop}(f_{|C}))$. Obviously, $\mathcal{A}(f)$ and $\mathcal{T}(\text{trop}(f_{|C}))$ are homotopy equivalent, but $\mathcal{T}(\text{trop}(f))$ and $\mathcal{T}(\text{trop}(f_{|C}))$ are not.

4. The Configuration Space of Amoebas

As already mentioned in Section 2 the usual proceeding for the investigation of amoebas is to fix a support set $A \subset \mathbb{Z}^n$ and allow coefficients to vary. This turns into a *configuration space* $\mathbb{C}^A$ of all polynomials with support set A. This space was already defined and used by Gelfand, Kapranov and Zelevinsky (see [**23**, Chapter 5, p. 165]) and for the first time systematically investigated (with respect to amoebas) by Rullgård (see in particular [**76, 77**]; see also [**42, 63**]). A lot of open questions about amoebas are questions about this particular space and a huge part of this thesis concerns about problems on it.

We introduce the configuration space together with Rullgård's related main results in this section and present a bunch of open problems on it.

For a fixed, support set $A \subset \mathbb{Z}^n$ we define the *configuration space* $\mathbb{C}^A$ as the set of all Laurent polynomials with non-vanishing complex coefficients and support set A, i.e.,

$$\mathbb{C}^A \quad = \quad \left\{ f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \in \mathbb{C}\left[\mathbf{z}^{\pm 1}\right] \ : \ b_\alpha \in \mathbb{C}^* \right\}.$$

Since for a fixed support set every polynomial $f \in \mathbb{C}^A$ is uniquely determined by its coefficients, we can identify $\mathbb{C}^A$ with $(\mathbb{C}^*)^d$ where $d = \#A$ (see e.g., [**23, 42, 63, 77**]).

In some situations it will be more convenient to investigate the closure $\overline{\mathbb{C}^A}$ or the *augmented configuration space* $\mathbb{C}^A_\Diamond$ given by

$$\mathbb{C}^A_\Diamond \quad = \quad \left\{ f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \in \mathbb{C}[\mathbf{z}^{\pm 1}] \ : \ b_\alpha \in \mathbb{C}, \, \mathrm{New}(f) = \mathrm{conv}(A) \right\}.$$

I.e., in $\mathbb{C}^A_\Diamond$ we allow coefficients to be zero as long as the exponent of the corresponding monomial is no vertex of $\mathrm{conv}(A)$ and thus the Newton polytope is identical for all polynomials in $\mathbb{C}^A_\Diamond$.

The space $\mathbb{C}^A_\Diamond$ turns out to be a natural augmentation of $\mathbb{C}^A$ since, for a given lattice polytope P, the set

$$L(P) \quad = \quad \left\{ \mathbb{C}^A \ : \ A \subset \mathbb{Z}^n, \, \mathrm{conv}(A) = P \right\}$$

forms a boolean lattice with a relation $\sqsubseteq$ induced by the inclusion relation of sets $A, B \subset \mathbb{Z}^n$ as we show in Theorem 3.2. Under this viewpoint $\mathbb{C}^A_\Diamond$ is the union of all elements $\mathbb{C}^B \in L(P)$ with $\mathbb{C}^B \sqsubseteq \mathbb{C}^A$, i.e., the union of all elements in the *order ideal* of $\mathbb{C}^A$ (Corollary 3.3).

With respect to $\mathbb{C}^A$ we are mainly interested in the question how to choose the coefficients of a polynomial such that a certain complement component of an amoeba exists, i.e., we are interested in understanding the sets

$$U^A_\alpha \quad = \quad \left\{ f \in \mathbb{C}^A \ : \ E_\alpha(f) \neq \emptyset \right\} \qquad \text{for } \alpha \in \mathrm{conv}(A) \cap \mathbb{Z}^n.$$

Note that $U^A_\alpha \neq \emptyset$ implies $\alpha \in \mathrm{conv}(A) \cap \mathbb{Z}^n$ by Theorem 2.4. The systematic investigation of the sets U^A_α was initialized by Rullgård in his PhD-thesis. We recall some of his main results.

Theorem 2.17 (Rullgård [**77**])**.** *Let $A \subset \mathbb{Z}^n$ and $\alpha \in \mathrm{conv}(A) \cap \mathbb{Z}^n$. The set $U^A_\alpha \subseteq \mathbb{C}^A$ is open and semi-algebraic.*

Theorem 2.18 (Rullgård [**76, 77**])**.** *Let $A \subset \mathbb{Z}^n$, $f \in \mathbb{C}^A$ and $B, C \subseteq A$ with $B \cap C = \emptyset$. Then there exists a $g \in \mathbb{C}^A$ such that*

$$f + g \quad \in \quad \bigcap_{\alpha \in B} U^A_\alpha \cap \bigcap_{\alpha \in C} (U^A_\alpha)^c.$$

Unfortunately, this theorem gives no hint about sets U^A_α for $\alpha \in \mathrm{conv}(A) \cap \mathbb{Z}^n$ but $\alpha \notin A$. To the best of my knowledge the only known result regarding this case is the following theorem by Rullgård (see [**77**, Theorem 11]). Let for a finite set $A \subset \mathbb{Z}^n$ the lattice generated by A be denoted by $\mathcal{L}_A$, i.e.,

$$\mathcal{L}_A \quad = \quad \left\{ \sum_{\alpha \in A} \lambda_\alpha \cdot \alpha \ : \ \lambda_\alpha \in \mathbb{Z} \right\} \subseteq \mathbb{Z}^n.$$

Theorem 2.19 (Rullgård [**77**])**.** *Let $A \subset \mathbb{Z}^n$ be a support set and l a line in $\mathbb{R}^n$.*

(1) *If $\alpha \in \mathbb{Z}^n$ and $U_\alpha^A \neq \emptyset$, then $\alpha \in \operatorname{conv}(A) \cap \mathcal{L}_A$.*
(2) *If $\alpha \in \operatorname{conv}(A \cap l) \cap \mathcal{L}_{A \cap l}$, then $U_\alpha^A \neq \emptyset$.*

Unfortunately, the proof of the theorem is a bit scanty and, in particular, Part (2) is incomplete. Part (1) uses a rather abstract argument (see [**77**, Theorems 7 and 11]). The issues of Part (2) arise since it relies on a genericity argument. Specifically, the statement used here is the following. Assume w.l.o.g. that $\mathcal{L}_{A \cap l} = \mathbb{Z}$ and $\{0, N\} \subset A \cap l \subseteq \{0, 1, \ldots, N\}$. Then all roots of the polynomial $z^N - 1 + \sum_{s \in A \setminus \{0,N\}} b_s z^s$ have different absolute values for generic choices of $b_s \in \mathbb{C}^*$. But although this argument is convincing, it is absolutely non-trivial, because in a worst case $A \setminus \{0, N\} = \{s\}$. Hence, a necessary condition is that for a generic choice of $p \in \mathbb{C}^*$ and every $N \in \mathbb{N}_{>s}$ the trinomial $z^N + pz^s - 1$ has no two roots of the same absolute value.

In Chapter 4, Section 3 we fix this proof (see Theorem 4.43). Furthermore, we give a construction method and an example for an amoeba of a multivariate polynomial with a complement component whose order is $\alpha \in (\operatorname{conv}(A) \cap \mathbb{Z}^n) \setminus A$ (Example 4.44).

Note that, although Theorem 2.19 can be fixed, it yields no complete solution to the initial problem it was motivated by.

Problem 2.20. *Let $A \subset \mathbb{Z}^n$ and $\alpha \in (\operatorname{conv}(A) \cap \mathbb{Z}^n) \setminus A$. What is a sufficient and necessary condition such that $U_\alpha^A \neq \emptyset$?*

This problem is the first one mentioned by Rullgård in his section describing a couple of open problems on amoebas (see [**77**, Problem 1, p. 60]). In fact, it is a much stronger version of the prominent Problem 3.4 about maximally sparse polynomials, which we introduce and discuss in Chapter 3, Section 2 and partially solve in Chapter 3, Section 3.

In many cases it is convenient to investigate $\mathbb{P}_\mathbb{C}^A$ instead of $\mathbb{C}^A$ since varieties of polynomials remain invariant under scalar multiplication. To keep notation simple we usually just write $\mathbb{C}^A$ with slight abuse of notation.

Theorem 2.21 (Rullgård [**76, 77**])**.** *Let $A \subset \mathbb{Z}^n$ and $\alpha \in \operatorname{conv}(A) \cap \mathbb{Z}^n$ sucht that $U_\alpha^A \neq \emptyset \neq (U_\alpha^A)^c$ in $\mathbb{P}_\mathbb{C}^A$. Then the intersection of $(U_\alpha^A)^c \subset \mathbb{P}_\mathbb{C}^A$ with any complex projective line in $\mathbb{P}_\mathbb{C}^A$ is non-empty and connected.*

Interestingly, the question whether the sets U_α^A are connected is widely open and turns out to be a or maybe even the most difficult key problem in understanding the structure of the configuration space of amoebas. It was already marked as an open problem by Rullgård in his thesis.

Problem 2.22 (Rullgård [**77**])**.** *Let $A \subset \mathbb{Z}^n$ and $\alpha \in \operatorname{conv}(A) \cap \mathbb{Z}^n$. Is U_α^A connected?*

We will discuss this problem broadly in Chapter 4 and solve it for rich classes of polynomials (Theorem 3.12, Corollary 4.25 and Corollary 4.50).

With respect to Theorem 2.21 it is manifest to ask the same question for the sets U_α^A itself.

Problem 2.23. *Let $A \subset \mathbb{Z}^n$ and $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$ sucht that $U_\alpha^A \neq \emptyset$. Is the intersection of $U_\alpha^A \subset \mathbb{P}_\mathbb{C}^A$ with a generic complex projective line in $\mathbb{P}_\mathbb{C}^A$ non-empty and connected?*

The term "generic" comes in here, since one class of counterexamples was found by Rullgård. It is the set U_0^A for polynomials $g(z + z^{-1}) + c$ with $c \in \mathbb{C}^*$ and $g \in \mathbb{C}[z^{\pm 1}]$ (see [**77**, Example 5, p 58]). But this counterexample is very special since it only works for very specific complex lines and only for U_0^A in this specific class making use of the symmetry of the given Laurent polynomial.

We prove that, although the answer on the question in Problem 2.23 is affirmative in certain special cases (see Lemma 4.22), it can be, roughly spoken, regarded as negative in general (Theorem 4.40 yields that the answer is negative for arbitrary trinomials for every up to one special U_α^A).

Since we will show that for certain sets $A \subset \mathbb{Z}^n$ certain sets U_α^A are connected, it is self-evident to try to understand the topology of these sets even better. A natural question following up is whether connected sets U_α^A are always *simply connected*. Recall that a topological space is simply connected if it is path-connected and its fundamental group is trivial (see e.g., [**28**]). We call a set simply connected if it is homotopy equivalent to a simply connected topological space.

Problem 2.24. *Let $A \subset \mathbb{Z}^n$ and $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$ such that U_α^A is connected. Is U_α^A then also simply connected? If not, what is the corresponding fundamental group?*

Besides all open questions about the topological structure of the sets U_α^A formulated in the different problems above, one has to clearly mark out that the geometrical and algebraic structure of the sets U_α^A are, of course, also of interest. Its investigation is nothing else than an abstracter approach on the initial Problem 2.9 by Gelfand, Kapranov and Zelevinsky, since every path γ through the space $\mathbb{R}^n$ containing an amoeba $\mathcal{A}(f)$ can be embedded in $\mathbb{C}^A$ by fixing the initial point $\mathbf{w} \in \mathbb{R}^n$ of γ and adjusting the coefficients of f instead.

Unfortunately, concerning these questions the situation is quite devastating so far. Besides the linear case, which we already mentioned as completely understood (see Theorem 2.8), nothing is known about the geometrical and algebraic structure of the sets U_α^A for any class of polynomials except the very special case of a standard-simplex with edge length $n+1$ and an inner lattice point $(1, \ldots, 1)$ in the support. I.e., here $A = \{(0, \ldots, 0), (n+1) \cdot e_1, \ldots, (n+1) \cdot e_n, (1, \ldots, 1)\}$ where e_j denotes the vectors of the standard basis. For this particular class, Passare and Rullgård showed – again via making use of the symmetry of the corresponding polynomials – that the set $(U_{(1,\ldots,1)}^A)^c$ is contained in an area bounded by the *Steiner curve* (see [**63**] and [**77**, Example 6, p.59], see also [**66**]). In his PhD-thesis Rullgård does not particularly deal with these sort of problems except mentioning the particular example.

Problem 2.25. *Let $A \subset \mathbb{Z}^n$ and $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$. Find an algebraic or geometrical description of U_α^A. Alternatively, find (at least) bounds to approximate U_α^A.*

Since Passare's and Rullgård's example in 2003/04 no progress was made in this direction. But, large parts of this thesis deal with this problem and we provide a bunch on results on it (Theorems 4.8, 4.10, 4.13, 4.19 and 4.40) for different classes of polynomials, which we do not want to discuss in detail at this point, except remarking that Theorem 4.19 generalizes and strengthens the example of Passare and Rullgård distinctly.

5. The Membership Problem and the Lopsidedness Condition

The study of computational questions about amoebas has been initiated by Theobald in [**85**], where primary certain special classes of amoebas (e.g., two-dimensional amoebas, amoebas of Grassmannians) were studied. In the particular paper an approximation of amoebas is achieved by computing its *contour* as a superset of its boundary (see Chapter 3, Section 4). In the following years a second approach for approximating amoebas arose by investigating the following natural and fundamental computational *membership problem.*

Problem 2.26. *Let $f \in [\mathbf{z}^{\pm 1}]$ and $\mathbf{w} \in \mathbb{R}^n$. Decide efficiently, whether $\mathbf{w} \in \mathcal{A}(f)$.*

Analogously, the membership problem can be extended to amoebas of ideals.

Problem 2.27. *Let $I \subseteq \mathbb{C}[\mathbf{z}^{\pm 1}]$ and $\mathbf{w} \in \mathbb{R}^n$. Decide efficiently, whether $\mathbf{w} \in \mathcal{A}(I)$.*

The term "decide efficiently" means that we are interested in an algorithm, which does not depend on f resp. I or $\mathbf{w}$ to make a decision. A solution of these problems is interesting since it would in particular allow to approximate amoebas in an efficient way.

State of the art of Problem 2.26 is an approximation process by Purbhoo ([**70**]) based on iterated resultants and a condition called *lopsidedness*. We recall his main results.

Let $A = \{\alpha(1), \ldots, \alpha(d)\} \subset \mathbb{Z}^n$ and $f = \sum_{j=1}^d b_j \mathbf{z}^{\alpha(j)} = \sum_{j=1}^d m_j(\mathbf{z}) \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ be a Laurent polynomial with coefficients $b_j \in \mathbb{C}$ and with monomials $m_1, \ldots, m_d$. For a given $\mathbf{w} \in \mathbb{R}^n$ we define $f\{\mathbf{w}\}$ to be the following sequence of numbers in $\mathbb{R}_{\geq 0}$.

$$f\{\mathbf{w}\} \quad = \quad \left(|m_1(\text{Log}^{-1}(\mathbf{w}))|, \ldots, |m_d(\text{Log}^{-1}(\mathbf{w}))|\right).$$

A sequence of positive real numbers is called *lopsided* if one of the numbers is greater than the sum of all the others (for convenience, we also say "f is lopsided at $\mathbf{w}$"). Defining

$$\mathcal{LA}(f) \quad = \quad \{\mathbf{w} \in \mathbb{R}^n \ : \ f\{\mathbf{w}\} \text{ is not lopsided}\},$$

it is easy to see that $\mathcal{A}(f) \subseteq \mathcal{LA}(f)$. If $\mathbf{w} \in \mathcal{A}(f)$, then the evaluated monomials $m_j(\text{Log}^{-1}(\mathbf{w}))$, interpreted as scalars in $\mathbb{C}$, would have to sum up to zero for an appropriate argument $\phi \in (S^1)^n$. This can obviously not happen if f is lopsided at $\mathbf{w}$ (see Figure 2.6).

In order to establish a converging hierarchy of approximations of $\mathcal{A}(f)$, set

$$\begin{aligned}\tilde{f}_r(\mathbf{z}) \quad &= \quad \prod_{k_1=0}^{r-1} \cdots \prod_{k_n=0}^{r-1} f\left(e^{2\pi i k_1/r} z_1, \ldots, e^{2\pi i k_d/r} z_n\right) \\ &= \quad \text{res}\left(\text{res}\left(\ldots \text{res}(f(u_1 z_1, \ldots, u_n z_n), u_1^r - 1), \ldots, u_{n-1}^r - 1\right), u_n^r - 1\right),\end{aligned}$$

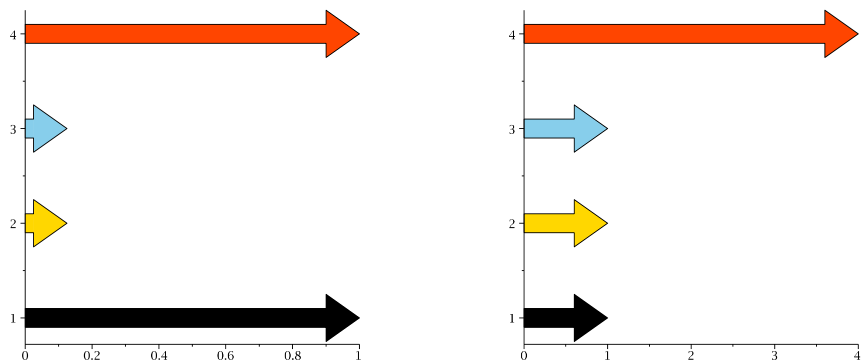


FIGURE 2.6. The absolute value of the monomials of the polynomial $f = z_1^2 z_2 + z_1 z_2^2 - 4 \cdot z_1 z_2 + 1$ evaluated at $p_1 = (0.5 \cdot e^{i \cdot 0.2\pi}, 0.5 \cdot e^{i \cdot 1.05 \cdot \pi})$ and $p_2 = (e^{i \cdot 0.1\pi}, e^{i \cdot 0.6\pi})$. Obviously f is lopsided at p_2 but not at p_1.

where $\text{res}(f, x)$ denotes the resultant with respect to x. Note that $\mathcal{A}(f) = \mathcal{A}(\tilde{f}_r)$. Indeed, $\tilde{f}_r$ is given by a product of polynomials f_j where each f_j is obtained from f and also $\mathcal{V}(f_j)$ from $\mathcal{V}(f)$ by a group action $\mathbf{1} \mapsto (\phi_1, \ldots, \phi_n)$ on $(S^1)^n \subset (\mathbb{C}^*)^n$ (where $\mathbf{1}$ denotes the trivial element in the real torus, i.e., the origin in its universal covering). Thus, $\text{Log}(\mathcal{V}(f_j)) = \text{Log}(\mathcal{V}(f))$ for all j, since Log is invariant on $(S^1)^n$.

As a main result, the following theorem holds.

Theorem 2.28 (Purbhoo [**70**])**.** *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$. For $r \to \infty$ the family $\mathcal{LA}(\tilde{f}_r)$ converges uniformly to $\mathcal{A}(f)$. For every $\varepsilon > 0$ there exists an integer N_ε such that for any $r \geq N_\varepsilon$ holds: If $\mathbf{w} \in \mathbb{R}^n$ and a ball with radius ε around $\mathbf{w}$ is completely contained in the complement of $\mathcal{A}(f)$, then $\mathbf{w} \in \mathcal{LA}(\tilde{f}_r)$.*

The theorem can be transferred to ideals as follows.

Theorem 2.29 (Purbhoo [**70**])**.** *Let $I \subset \mathbb{C}[\mathbf{z}^{\pm 1}]$ be an ideal. Then $\mathbf{w} \in \mathbb{R}^n$ is in $\mathcal{A}(I)$ if and only if $f\{\mathbf{w}\}$ is not lopsided for every $f \in I$.*

Furthermore, if a point $\mathbf{w}$ is lopsided, then it is possible to recover the order of the complement component $\mathbf{w}$ is located in. This is a consequence of Theorem 2.6 (although Purbhoo gives an own proof).

Theorem 2.30 (Purbhoo [**70**])**.** *Let $r \in \mathbb{N}$, $\mathbf{w} \in \mathbb{R}^n$ with $\mathbf{w} \notin \mathcal{LA}(\tilde{f}_r)$ and $\mathbf{w} \in E_\alpha(f)$. Then the dominating term in $\tilde{f}_r$ has the exponent vector $r^n \cdot \alpha$.*

A further, very nice fact about Purbhoo's approach is that his relaxation process can also be used to approximate the spine. The reason is that $\lim_{r \to \infty} r^n \cdot \log|\tilde{f}_r(\mathbf{z})|$ is a Riemann sum for the Ronkin function N_f (see [**70**, pp. 21] for further details; see also Section 3). Thus, Purbhoo's relaxation process gives a nice, easy, discrete explanation about what the Ronkin function does. The relaxation colorfully puts a lattice on a fiber torus (see also Chapter 3, Section 1), evaluates f at every lattice point and computes the intermediate value. For $r \to \infty$ the lattice is constantly refined and hence in the limit every point of the fiber torus is evaluated. Evaluation at the limit coincides with evaluating N_f.

6. Coamoebas

In this section we give a brief introduction to *coamoebas*, which were to the best of our knowledge first defined by Passare in 2004 (see [**54**]) and can be regarded as dual objects for amoebas.

Let $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ be a Laurent polynomial with support $A \subset \mathbb{Z}^n$ and variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. Similarly as the amoeba one defines the *coamoeba* co$\mathcal{A}(f)$ of f as the image of $\mathcal{V}(f)$ under the Arg-map given by

$$\text{Arg} : (\mathbb{C}^*)^n \to (S^1)^n, \ (z_1, \ldots, z_n) \mapsto (\arg(z_1), \ldots, \arg(z_n)),$$

where $\arg(z)$ denotes the argument of a complex number z. For convenience the coamoeba is often not investigated in the real n-torus $(S^1)^n$ but in its universal covering $\mathbb{R}^n$. To keep notation simple we do not distinguish between these two target spaces with slight abuse of notation. Exactly in the same way as for amoebas the definition of coamoebas can be generalized from hypersurfaces to varieties of ideals (see Section 1).

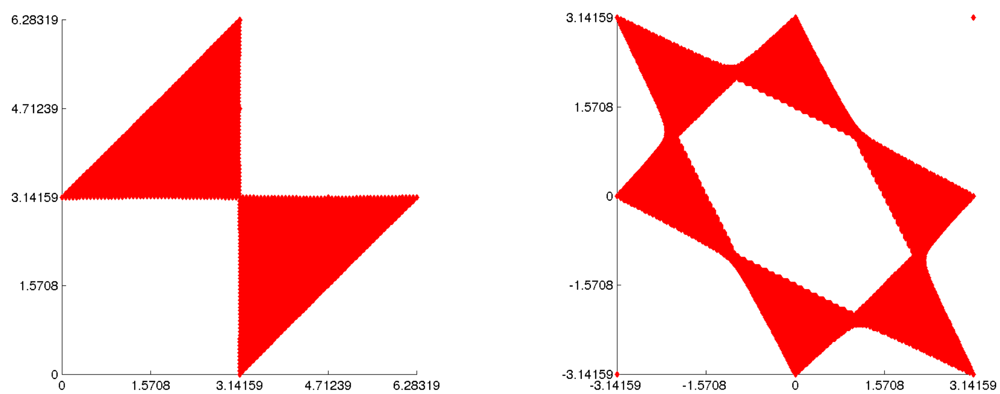


FIGURE 2.7. The coamoebas of $f = z_1 + z_2 + 1$ and $f = z_1^2 z_2 + z_1 z_2^2 + 0.5 \cdot z_1 z_2 + 1$.

Coamoebas do not preserve all of the nice properties of amoebas. For example, in general coamoebas are neither open nor closed sets. But it was shown but Nisse and Sottile ([**54**]) that the boundary of the closure is described by an object they denote as *phase limit set*, which can be considered as the counterpart of the logarithmic limit set introduced in Section 1.

Recall that for an $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ we figured out in Section 3 that the logarithmic limit $\mathcal{A}_\infty(f)$ set consists precisely of every $\mathbf{w}$ with $\text{in}_\mathbf{w}(f) \neq 1$, which is by Hilbert's Nullstellensatz (see Theorem 5.2) equivalent to $\mathcal{V}(\text{in}_\mathbf{w}(f)) \neq \emptyset$. Now the *phase limit set* co$\mathcal{A}_\infty(f)$ is defined by Nisse and Sottile as

$$\text{co}\mathcal{A}_\infty(f) \quad = \bigcup_{\mathbf{w} \in (\mathbb{R}^*)^n} \text{co}\mathcal{A}(\text{in}_\mathbf{w}(\mathcal{V}(f))).$$

With the phase limit set they show the following theorem.

Theorem 2.31 (Nisse, Sottile [**54**])**.** *For $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ the closure of co$\mathcal{A}(f)$ equals co$\mathcal{A}(f) \cup$ co$\mathcal{A}_\infty(f)$.*

This result can as well as the result about the description of logarithmic limit sets be generalized to ideals (see [**54**]). Furthermore, they use phase limit sets to describe *non-Archimedean coamoebas*, which are "the image of a subvariety of a torus over a non-Archimedean field $\mathbb{K}$ with complex residue field under an argument map" (see [**53**, Abstract]).

Unfortunately, neither an order map nor a useful approximation algorithm (like Purbhoo's lopsidedness certificate joint with his approximation process based on iterated resultants; see Section 5) are known for coamoebas so far. Indeed, these two problems can be considered as key problems on contemporary coamoeba theory.

Problem 2.32. *Is there an order map for coamoebas? I.e., a suitable map from the set of complement components of a coamoeba co$\mathcal{A}(f)$, $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$, into $\mathcal{L} \cap P$, where $\mathcal{L} \subset \mathbb{R}^n$ is a lattice and P is a polytope both determined by f.*

The second problem culminates in the membership problem for coamoebas.

Problem 2.33. *Let $I \subset \mathbb{C}[\mathbf{z}^{\pm 1}]$ and $\phi \in (S^1)^n$. Decide efficiently whether $\phi \in$ co$\mathcal{A}(I)$.*

A very fruitful first step to tackle both of these problems was recently done by Forsgård and Johansson ([**21**]). They introduce a lopsidedness certificate for coamoebas, which we call *colopsidedness* here, in the following way. Let $A \subset \mathbb{Z}^n$, $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \subset \mathbb{C}[\mathbf{z}^{\pm 1}]$ and $\phi \in [0, 2\pi)^n$. Then a sequence of complex numbers on the complex unit circle

$$f\{\phi\} \quad = \quad (\arg(b_\alpha) \cdot e^{i \cdot \pi \langle \phi, \alpha \rangle})_{\alpha \in A}$$

is *colopsided* if there exists a hyperplane $H \subset \mathbb{C}$ with $0 \in H$ and positive halfspace H^+ such that $f\{\phi\} \subset H^+$ but $f\{\phi\} \not\subset H$ (for convenience we say "f is colopsided at ϕ"). Similarly as for amoebas and lopsidedness one defines

$$\text{co}\mathcal{L}\mathcal{A}(f) \quad = \quad \{\phi \in [0, 2\pi)^n \ : \ f\{\phi\} \text{ is not colopsided}\}.$$

It is easy to see that co$\mathcal{A}(f) \subset$ co$\mathcal{L}\mathcal{A}(f)$. If $\phi \in$ co$\mathcal{A}(f)$ then the evaluated monomials of f, interpreted as complex numbers, have to sum up to zero. But this can obviously never happen, if f is colopsided at ϕ, meaning that the sum of evaluated monomials is a complex number located in the strict positive halfspace of a hyperplane H containing the origin (see Figure 2.8).

Forsgård and Johansson use this property to give an order map for the complement components of a coamoeba, which contain colopsided points. Let $A \subset \mathbb{Z}^n$ be a support set and M_A be a matrix with columns $\alpha \in A$. Then we denote the matrix M_B as the *Gale dual* matrix (see [**21**] for further details; see also e.g., [**23**]).

Theorem 2.34 (Forsgård, Johansson [**21**])**.** *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with support set $A \subset \mathbb{Z}^n$. Let S be the set of complement components of the closure of co$\mathcal{A}(f)$, which contain at least one point, which is colopsided. Then there exists an explicitly describable, surjective (order) map from S to $\mathcal{L}_B \cap Z_B$ where $\mathcal{L}_B$ is a lattice and Z_B a zonotope both depending on the Gale dual M_B of M_A.*

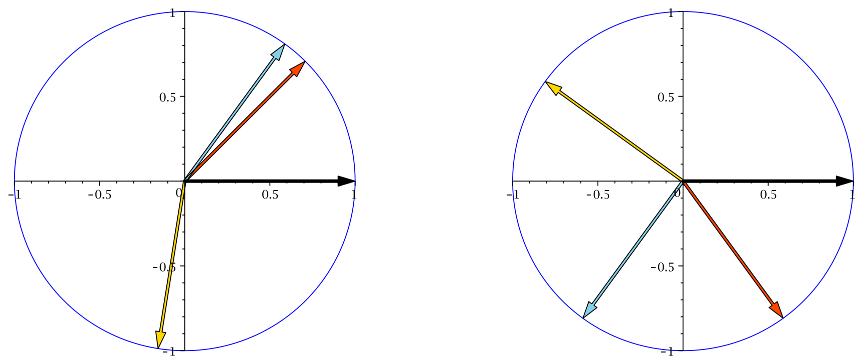


FIGURE 2.8. The arguments of the monomials of the polynomial $f = z_1^2 z_2 + z_1 z_2^2 - 4 \cdot z_1 z_2 + 1$ evaluated at $p_1 = (0.5 \cdot e^{i \cdot 0.2\pi}, 0.5 \cdot e^{i \cdot 1.05 \cdot \pi})$ and $p_2 = (e^{i \cdot 0.1\pi}, e^{i \cdot 0.6\pi})$. Obviously f is colopsided at p_1 but not on p_2.

For the special case of coamoebas of A-discriminants in two variables, Nilsson and Passare proved that the zonotope Z_B mentioned in the upper theorem exactly covers $(S^1)^2 \setminus \text{co}\mathcal{A}(f)$ ([**49**], see also [**48**]). This result was later generalized to arbitrary dimension by Passare and Sottile ([**65**]). A-discriminants were already discussed by Gelfand, Kapranov and Zelevinsky in [**23**] and also have many connections to amoebas. We define and treat A-discriminants in Chapter 4, Section 1.3. We show in particular that for amoebas of genus at most one an upper bound on the coefficients to obtain an amoeba without bounded (i.e., "inner") complement component becomes sharp if and only if the corresponding polynomial is located on the A-discriminant (Theorem 4.17).

In this thesis we present a way to solve Problem 2.33 with the use of semidefinite programming (see Chapter 5, Section 2, Theorem 5.8).

CHAPTER 3

Fibers, Lattices, Sparsity and Boundaries

In this chapter we investigate different aspects of amoebas, namely *fibers, the lattice of configurations spaces, maximally resp. minimally sparse amoebas* and the *boundary of amoebas*. The investigation of these topics has two aims. On the one hand, it will lead to a more general, abstract structure, where amoeba theory can be understood in (lattice of configuration spaces) and to crucial proof techniques for the main topics of this thesis (via the fiber structure of the Log-map, which we point out here). On the other hand, we provide a couple of stand alone results. In particular, we partially solve Passare's and Rullgård's problem asking whether maximally sparse polynomials have solid amoebas (Problem 3.4 and Theorems 3.9 and 3.10) and we give a strengthening of Mikhalkin's result about the boundary of amoebas (Theorem 3.15).

In Section 1 we describe the fiber bundle induced by the Log-map. In particular, we introduce the *fiber function* of a Laurent polynomial f given by its restriction to a certain fiber. In the latter part of the thesis we use this fiber function as a crucial instrument to discover the structure of the configuration space $\mathbb{C}^A$ (e.g., Theorems 4.13, 4.19 and 4.32).

In Section 2 we show that for a given integer polytope P the set of all configurations spaces $\mathbb{C}^A$ with $\text{conv}(A) = P$ forms a boolean lattice (Theorem 3.2), which we call *lattice of configuration spaces*. This lattice nicely embeds augmented configuration spaces (Theorem 3.3) and furthermore motivates Passare's and Rullgård's question about maximally sparse polynomials (Problem 3.4) since its defining relation respects emptiness of sets U_α^A (Theorem 3.6).

In Section 3 we have a closer look at the maximal and minimal element of a lattice of configuration spaces, which happen to be configuration spaces of polynomials whose Newton polytope is minimally resp. maximally sparse. We provide a proof of Passare's and Rullgård's Problem 3.4 for special classes of Newton polytopes. Furthermore, we prove that Rullgård's question about the connectivity of sets U_α^A (Problem 2.22) has a positive answer for univariate, minimally sparse polynomials.

In Section 4 we investigate the boundary of amoebas. We recall Mikhalkin's result stating that the *contour* of an amoeba, a superset of its boundary, is given by the image of the critical values of the *logarithmic Gauss-map* under the Log-map (Theorem 3.13). We show that this statement can be strengthened in the way that a contour point may only be a boundary point if *every* point in the intersection of variety and fiber is critical under the logarithmic Gauss-map (Theorem 3.15).

1. Fibers of the Log-map

One key difficulty, which makes the investigation of configuration spaces $\mathbb{C}^A$ resp. their structure complicated, is the fact that, in general, it is absolutely unclear, how $\mathbb{C}^A$ is related to varieties (and thus, their amoebas). More precisely, for a general $A \subset \mathbb{Z}^n$ it is unclear how a changing of coefficients changes the variety of the corresponding polynomial.

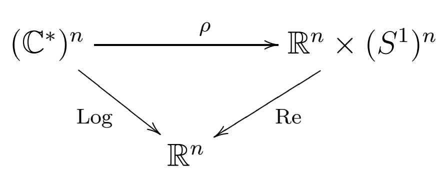
In the univariate case, this correspondence is given by Newton identities (see e.g., [**12, 82**]), which are combinatorially already hard to handle for a sufficient large degree. In the multivariate case, no such correspondence is given at all for a general $A \subset \mathbb{Z}^n$.

Roughly, our main idea to face this problem is to make use of the natural fiber bundle, which comes with the Log-map. It induces a fiber function given by a restriction of the original polynomial f to a certain fiber. The question, whether a certain point is in the amoeba is equivalent to the question whether the corresponding fiber function has non-empty variety. This fact yields the desired correspondence in many cases, since the variety of the fiber function depends linearly on the coefficients of f.

Let $A \subset \mathbb{Z}^n$ be a support set and $f = \sum_{\alpha \in A} b_\alpha \cdot \mathbf{z}^\alpha \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ be an arbitrary Laurent polynomial. We denote the fiber of a point $\mathbf{w} \in \mathbb{R}^n$ under the Log-map as

$$\mathbb{F}_{\mathbf{w}} \quad = \quad \operatorname{Log}^{-1}(\mathbf{w}) = \{\mathbf{z} \in (\mathbb{C}^*)^n \ : \ |\mathbf{z}| = |\operatorname{Log}^{-1}(\mathbf{w})|\}.$$

Observe that the Log-map comes with a fiber bundle $\mathbb{F}_{\mathbf{w}} \to (\mathbb{C}^*)^n \to \mathbb{R}^n$ given by the homeomorphism $\rho : (\mathbb{C}^*)^n \to \mathbb{R}^n \times (S^1)^n,\ (z_1, \ldots, z_n) \mapsto (\log(z_1), \ldots, \log(z_n))$ for some chosen local branch of the holomorphic log. Then the following diagram commutes



where Re denotes the projection on the real part. Thus, $\mathbb{F}_{\mathbf{w}}$ is homeomorphic to a real n-torus $(S^1)^n$.

f and a point $\operatorname{Log}(\mathbf{z}) = \mathbf{w} \in \mathbb{R}^n$ induce a function

$$f^{|\mathbf{z}|} : [0, 2\pi)^n \to \mathbb{C}, \quad \phi \mapsto \sum_{\alpha \in A} b_i \cdot |\mathbf{z}|^\alpha \cdot e^{i \cdot \langle \phi, \alpha \rangle}$$

on the fiber $\mathbb{F}_{\mathbf{w}}$ by $f^{|\mathbf{z}|}(\phi) = f\left(|\mathbf{z}| \cdot e^{i \cdot \phi}\right)$. Note that the fiber $\mathbb{F}_{\mathbf{w}}$ is identified with $[0, 2\pi)^n$ here, which is possible due to the fibration described above. We call $f^{|\mathbf{z}|}$ the *fiber function* of f at $\mathbf{w}$ resp. at $|\mathbf{z}|$. Log yields a map

$$\varphi : (\mathbb{C}^*)^n \to \{\pi_{\mathbf{w}} : (\mathbb{C}^*)^n \to \mathbb{F}_{\mathbf{w}} : \mathbf{w} \in \mathbb{R}^n\}, \quad \mathbf{z} \mapsto \pi_{\operatorname{Log}(\mathbf{z})}$$

mapping every point $\mathbf{z}$ to the projection $\pi_{\operatorname{Log}(\mathbf{z})}$ of $(\mathbb{C}^*)^n$ to the fiber $\mathbb{F}_{\operatorname{Log}(\mathbf{z})} = \mathbb{F}_{\mathbf{w}}$. For every $\operatorname{Log}(\mathbf{z}) = \mathbf{w} \in \mathbb{R}^n$ the fiber function $f^{|\mathbf{z}|}$ is the pushforward $\varphi(\mathbf{z})_*(f)$ of f and hence the zero set of $f^{|\mathbf{z}|}$ is

$$\{\phi \in [0, 2\pi)^n \ : \ \varphi(\mathbf{z})_*(f)(\phi) = 0\} \ = \ \pi_{\mathbf{w}}(\mathcal{V}(f)) \ = \ \mathcal{V}(f) \cap \mathbb{F}_{\mathbf{w}}.$$

Since $f^{|\mathbf{z}|}$ is a regular function on $\mathbb{F}_{\mathbf{w}}$ (resp. on $[0,2\pi)^n$), we denote its zero set as $\mathcal{V}(f^{|\mathbf{z}|})$. By the definition of the amoeba it follows for all $\mathbf{z} \in (\mathbb{C}^*)^n$ that

$$\mathrm{Log}(\mathbf{z}) \in \mathcal{A}(f) \quad \Leftrightarrow \quad \mathcal{V}(f^{|\mathbf{z}|}) \neq \emptyset.$$

For a general background about fibrations and regular functions see e.g., [**28, 80**]. We can subsume this section in the following way. The set of points mapped to a specific point $\mathbf{w} \in \mathbb{R}^n$ by Log is a real n-torus $\mathbb{F}_{\mathbf{w}} \subset (\mathbb{C}^*)^n$. For a given Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ the point $\mathbf{w}$ belongs to the amoeba $\mathcal{A}(f)$ if and only if f vanishes at a point $e^{\mathbf{w}+i\cdot\phi} \in (\mathbb{C}^*)^n$, i.e., $\phi \in \mathbb{F}_{\mathbf{w}}$ (see Figure 3.2). This is in particular the case if the fiber function $f^{|\mathbf{z}|}$ obtained by restriction of the preimage of f to $\mathbb{F}_{\mathbf{w}}$ has a non-empty variety. Although this observation is not very deep at a first glance, a careful investigation of fiber functions $f^{|\mathbf{z}|}$ will be crucial to obtain several main results of this thesis.

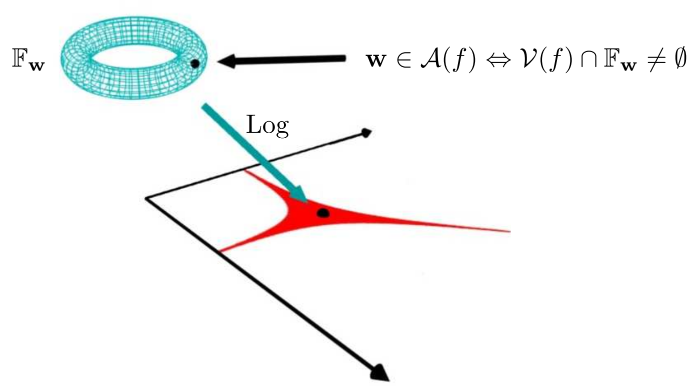


FIGURE 3.2. An amoeba $\mathcal{A}(f)$ with a fiber (with respect to the Log-map) of a point $\mathbf{w} \in \mathbb{R}^n$.

2. The Lattice of Configuration Spaces

Recall that for a given lattice polytope P (i.e., all vertices of P are elements of $\mathbb{Z}^n$) the set $L(P)$, as defined in Chapter 2, Section 4 denotes the set of all configuration spaces $\mathbb{C}^A$ of amoebas with $\mathrm{conv}(A) = P$. We show that $L(P)$ is a boolean lattice with respect to an inclusion relation $\sqsubseteq$, which naturally embeds the augmented configuration space, which we introduced in Chapter 2, Section 4, into its lattice structure. Furthermore, we have a closer look on the maximal and minimal elements of the lattice $(L(P), \sqsubseteq)$, which turn out to correspond to *maximally* and *minimally sparse amoebas*. We present a famous problem on maximally sparse amoebas given by Passare and Rullgård (Problem 3.4; see also [**62**]) and a conjecture on minimally sparse amoebas (Conjecture 3.8), which we both partially solve later in Section 3.

We begin with a proof for the lattice structure on $L(P)$. Recall (see e.g., [**83, 89**]) that a *lattice* is a partially ordered set (poset) S with respect to some order relation $\sqsubseteq$

such that for all $x, y \in S$ there is an $\inf\{x,y\} \in S$ and a $\sup\{x,y\} \in S$. A lattice $(S, \sqsubseteq)$ is *distributive* if for all $x, y, z \in S$ holds

$$\begin{aligned} \sup\{\inf\{x,y\}, \inf\{x,z\}\} &= \inf\{x, \sup\{y,z\}\} \text{ and} \\ \inf\{\sup\{x,y\}, \sup\{x,z\}\} &= \sup\{x, \inf\{y,z\}\}. \end{aligned}$$

A lattice $(S, \sqsubseteq)$ has a *maximum* $1 \in S$ if $\sup\{1, x\} = 1$ for all $x \in S$; it has a *minimum* $0 \in S$ if $\inf\{0, x\} = 0$ for all $x \in S$. A lattice with maximum and minimum is called *bounded*. A bounded lattice $(S, \sqsubseteq)$ is called *complemented* if for every $x \in S$ there is an x^c such that $\sup\{x, x^c\} = 1$ and $\inf\{x, x^c\} = 0$. A distributive, complemented lattice is called a *boolean lattice* or a *boolean algebra*.

Let $(S, \sqsubseteq)$ be a poset and $M \subset S$. The *order ideal* $\mathcal{O}(M)$ of M (with respect to $(S, \sqsubseteq)$) is given by

$$\mathcal{O}(M) = \{x \in S \;:\; x \sqsubseteq y \text{ for some } y \in M\};$$

analogously the *dual order ideal* or *filter* $\mathcal{F}(M)$ of M (with respect to $(S, \sqsubseteq)$) is given by

$$\mathcal{F}(M) = \{x \in S \;:\; y \sqsubseteq x \text{ for some } y \in M\}.$$

To prove the lattice structure on $L(P)$ we use the following well known facts about lattices (see e.g., [**83, 89**]).

Lemma 3.1. *The power set $\mathcal{P}(N)$ of a finite set N is a boolean algebra with respect to the inclusion $\subseteq$. If $(S, \sqsubseteq)$ is a boolean algebra, then $(\mathcal{O}(\{x\}), \sqsubseteq)$ and $(\mathcal{F}(\{x\}), \sqsubseteq)$ are boolean algebras as well for every $x \in S$.*

With this lemma we are able to prove the boolean lattice structure on $L(P)$.

Theorem 3.2. *For every lattice polytope P the set*

$$L(P) = \left\{\mathbb{C}^A \;:\; A \subset \mathbb{Z}^n, \operatorname{conv}(A) = P\right\}$$

is a boolean lattice with respect to the order relation $\sqsubseteq$ given by $\mathbb{C}^A \sqsubseteq \mathbb{C}^B :\Leftrightarrow A \subseteq B$ for all $\mathbb{C}^A, \mathbb{C}^B \in L(P)$.

Proof. By definition of $\sqsubseteq$ it suffices to show that the set $M(P) = \{A \subset \mathbb{Z}^n : \operatorname{conv}(A) = P\}$ is a boolean lattice with respect to inclusion. Let $V(P)$ denote the vertex set of P. Then $M(P)$ is the filter $\mathcal{F}(\{V(P)\})$ in the power set $\mathcal{P}(P \cap \mathbb{Z}^n)$ and the theorem follows with Lemma 3.1. □

We call $(L(P), \sqsubseteq)$ the *lattice of configuration spaces* with respect to a lattice polytope P. Recall that we defined for a given $A \subset \mathbb{Z}^n$ the *augmented configuration space* as $\mathbb{C}^A_\Diamond = \{f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \;:\; b_\alpha \in \mathbb{C}, \operatorname{New}(f) = \operatorname{conv}(A)\}$ (see Chapter 2, Section 4). Theorem 3.2 yields that this is indeed a natural extension of $\mathbb{C}^A$.

Corollary 3.3. *Let P be a lattice polytope, $(L(P), \sqsubseteq)$ its lattice of configuration spaces and $\mathbb{C}^A \in (L(P), \sqsubseteq)$. Then $\mathbb{C}^A_\Diamond = \bigcup_{B \sqsubseteq A} \mathbb{C}^B$, i.e., $\mathbb{C}^A_\Diamond$ is the union of all elements in the order ideal $\mathcal{O}(\{\mathbb{C}^A\})$.*

Proof. Follows immediately from Theorem 3.2 and the definitions of augmented configuration spaces and order ideals. □

Note that $\inf(L(P)) = \mathbb{C}^{V(P)}$ (where $V(P)$ is the vertex set of P) and $\sup(L(P)) = \mathbb{C}^{P \cap \mathbb{Z}^n}$. We call $\inf(L(P))$ the *maximally sparse element* of $(L(P), \sqsubseteq)$ and $\sup(L(P))$ the *minimally sparse element* of $(L(P), \sqsubseteq)$. In case of $A = V(P)$, we also say A resp. some $f \in \mathbb{C}^A$ or its amoeba $\mathcal{A}(f)$ is *maximally / minimally sparse* with slight abuse of notation.

Maximally sparse polynomials are well known entities in amoeba theory, basically due to the following problem (resp. conjecture) by Passare and Rullgård.

Problem 3.4 (Passare, Rullgård [**62, 66**])**.** *Do maximally sparse polynomials have solid amoebas? I.e., let $\mathbb{C}^A = \inf(L(P))$ for some lattice polytope P. Is $U_\alpha^A = \emptyset$ for every $\alpha \in (P \cap \mathbb{Z}^n) \setminus A$?*

In Section 3 we discuss maximally and minimally sparse polynomials on its own and present in Theorem 3.9 and Theorem 3.10 proofs of the conjecture for rich classes of Newton polytopes. Our treatment of polynomials with amoebas of genus at most one (Chapter 4, Section 1) will yield an independent proof for the case that $\operatorname{conv}(A)$ is a simplex (Corollary 4.9).

Surprisingly, so far minimally sparse polynomials, the counterpart of maximally sparse polynomials, have not been discussed with respect to their amoebas at all. So, the first question is whether there is an intrinsic counterpart to Problem 3.4. But the solution for this problem is an immediate consequence of one of Rullgård's theorems.

Remark 3.5. *Let $\mathbb{C}^A = \sup(L(P))$ for some lattice polytope P. Then $U_\alpha^A \neq \emptyset$ for every $\alpha \in P \cap \mathbb{Z}^n$ by Theorem 2.18.*

Problem 3.4 is motivated and Remark 3.5 is reflected by the following theorem, which shows that emptiness of a set U_α^A is respected by the lattice structure.

Theorem 3.6. *Let $(L(P), \sqsubseteq)$ be a lattice of configuration spaces, $\mathbb{C}^A \in (L(P), \sqsubseteq)$ and $\alpha \in P \cap \mathbb{Z}^n$. If $U_\alpha^A = \emptyset$, then $U_\alpha^B = \emptyset$ for every $\mathbb{C}^B \in \mathcal{O}(\{\mathbb{C}^A\}) \subseteq (L(P), \sqsubseteq)$.*

Proof. Let $U_\alpha^A = \emptyset$ and $\mathbb{C}^B \in (L(P), \sqsubseteq)$ with $\mathbb{C}^B \sqsubseteq \mathbb{C}^A$. Assume, $U_\alpha^B \neq \emptyset$. Then, there exists an $f \in \mathbb{C}^B$ with $E_\alpha(f) \neq \emptyset$. Let $\mathbf{w} \in E_\alpha(f) \subset \mathbb{R}^n$. Hence $\min_{\mathbf{z} \in \mathbb{F}_\mathbf{w}} |f(\mathbf{z})| > 0$. Note that the minimum exists since the fiber $\mathbb{F}_\mathbf{w}$ is a compact set.

$\mathbb{C}^B \sqsubseteq \mathbb{C}^A$ yields $B \subseteq A$. Let $A \setminus B = \{\alpha(1), \ldots, \alpha(k)\}$. Let

$$\varepsilon = \frac{\min_{\mathbf{z} \in \mathbb{F}_\mathbf{w}} |f(\mathbf{z})|}{2 \cdot \sum_{j=1}^k e^{\langle \mathbf{w}, \alpha(j) \rangle}} \ > \ 0$$

and $g = f + \varepsilon \cdot \sum_{j=1}^k \mathbf{z}^{\alpha(j)}$. Thus, $g \in \mathbb{C}^A$ and, by construction, $g(\mathbf{z}) \neq 0$ for every $\mathbf{z} \in \mathbb{F}_\mathbf{w}$. Thus, $\mathbf{w} \in E_\alpha(g)$ and therefore $U_\alpha^A \neq \emptyset$, which is a contradiction. □

Corollary 3.7. *The sets U_α^A are open in $\mathbb{C}_\Diamond^A$ for every $A \subset \mathbb{Z}^n$ and $\alpha \in \operatorname{conv}(A) \cap \mathbb{Z}^n$.*

Note that this corollary implies in particular that the number of complement components is also lower semicontinuous under changing coefficients in the augmented configuration space $\mathbb{C}_\Diamond^A$ (as an extension to Theorem 2.3).

Proof. Follows immediately from Theorem 2.17 and the argument in the proof of Theorem 3.6. □

A natural question motivated by Remark 3.5 is if in the minimally sparse case all U_α^A are connected, which is Rullgård's Problem 2.22 for this special instance. Although we do not want to give a conjecture if Problem 2.22 has an affirmative answer for all configuration spaces, we conjecture it to be true in the minimally sparse case.

Conjecture 3.8. *Let $(L(P), \sqsubseteq)$ be a lattice of configuration spaces and $\mathbb{C}^A = \sup(L(P))$ its minimally sparse element. Then every $U_\alpha^A \subseteq \mathbb{C}^A$ with $\alpha \in A \cap \mathbb{Z}^n$ is connected.*

In the following Section 3 we give two possible heuristics how the sets U_α^A might behave in the minimally sparse case. One of them supports this conjecture. We motivate the particular one and hence this conjecture by proving it for the univariate case (Theorem 3.12).

3. Amoebas of Minimally and Maximally Sparse Polynomials

In the last section we discussed the lattice of configuration spaces and the famous Problem 3.4 about maximally sparse polynomials. Moreover, we conjectured (Conjecture 3.8) the connectivity of sets U_α^A in configuration spaces of minimally sparse polynomials. In this section we have a closer look at this problem and this conjecture. Here, we make (only) use of Rullgård's tropical Theorem 2.15 and a result of Forsberg, Passare and Tsikh about *directional orders* of complement components to give an affirmative answer of Problem 3.4 for a huge set of Newton polytopes (Theorems 3.9 and Theorem 3.10).

Afterwards, we show in Theorem 3.12 that the Conjecture 3.8 can be proved straightforwardly in the univariate case.

Passare and Rullgård asked in [**62**] whether every maximally sparse polynomial has a solid amoeba. In Chapter 4, Section 1 it will follow from Theorem 4.8 that this is true if the Newton polytope is a simplex (Corollary 4.9). One key ingredient to prove this statement is Rullgard's Theorem 2.15 stating that the complement induced tropical hypersurface $\mathcal{C}(f)$ of a polynomial f is a deformation retract of the amoeba $\mathcal{A}(f)$, if the support A of f contains at most $2n$ points such that no $k+2$ of them are contained in a k-dimensional affine subspace of $\mathbb{R}^n$. But, when focussing on maximally sparse polynomials themselves, Rullgård's result even allows to solve Problem 3.4 for this whole particular class described in the theorem.

Theorem 3.9. *Let $A = \{\alpha(1), \ldots, \alpha(d)\} \subset \mathbb{Z}^n$ maximally sparse such that $d \leq 2n$ and no $k+2$ elements of A are contained in a k-dimensional subspace of $\mathbb{R}^n$. Then $\mathcal{A}(f)$ is solid.*

Proof. Let $f \in \mathbb{C}^A$. The assumptions guarantee that we can apply Rullgård's Theorem 2.15. Thus, the complement induced tropical hypersurface $\mathcal{C}(f)$ is a deformation retract of the amoeba $\mathcal{A}(f)$. Since every element of A is a vertex of $\mathrm{conv}(A)$ and since $\mathcal{C}(f)$ is dual to a regular subdivision of $\mathrm{conv}(A)$, which is induced by the elements of A lifted by their valuations, every component of $\mathbb{R}^n \setminus \mathcal{C}(f)$ has to be unbounded. □

Problem 3.4 has obviously a positive answer in the univariate case since every polynomial of the form $z^d + s$ with $s \in \mathbb{C}^*$ has only roots of the same absolute value (see also e.g., [**66**]). The following theorem shows that this observation can be generalized to multivariate polynomials f, whose Newton polytope is of a form that lets f behave "like a maximally sparse univariate polynomial" in a certain direction in space.

Theorem 3.10. *Let $A = \{\alpha(1), \ldots, \alpha(d)\} \subset \mathbb{Z}^n$ maximally sparse. If there is a 1-dimensional $\mathbb{Q}$-vectorspace V such that the orthogonal projection of A on V has cardinality two, then $\mathcal{A}(f)$ has no bounded complement components for every $f \in \mathbb{C}^A$.*

The projection property can also be described by assuming that the vertex set of the polytope is contained in exactly two parallel hyperplanes. Observe that a rich class of polytopes has this property. For example parallelepipeds, pyramids, prisms, crosspolytopes (see e.g., [**91**]) or 2-level polytopes (including e.g., stable set polytopes of perfect graphs and weakly Hannar polytopes; see [**25, 27, 78**]). Note that e.g., zonotopes, centrally symmetric polytopes or direct sums of polytopes do not in general have this property.

The theorem can be deduced immediately by Part (1) of Rullgård's Theorem 2.19. But since the approach leading to this Theorem 2.19 is rather abstract, we give an alternative, longer, but more elemental proof here.

To prove the theorem we introduce the *directional order* defined by Forsberg, Passare and Tsikh. Let $A \subset \mathbb{Z}^n$, $f \in \mathbb{C}^A$ whose amoeba $\mathcal{A}(f)$ has a complement component $E_\alpha(f) \neq \emptyset$ (with order α). Let $\eta \in (\mathbb{Z}^*)^n$. Then the *directional order* of $E_\alpha(f)$ with respect to η is defined as $\langle \eta, \alpha \rangle \in \mathbb{Z}$ (see [**20**]). In particular, we will use the following lemma.

Lemma 3.11 (Forsberg, Passare, Tsikh [**20**])**.** *Let $A \subset \mathbb{Z}^n$ and $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha$, $\mathbf{w} \in (\mathcal{A}(f))^c$ and $\eta = (\eta_1, \ldots, \eta_n) \in (\mathbb{Z}^*)^n$. Then the directional order $\langle \eta, \mathrm{ord}(\mathbf{w}) \rangle$ is given by the number of roots minus the number of poles of the univariate Laurent polynomial*

$$y \mapsto f(c_1 y^{\eta_1}, \ldots, c_n y^{\eta_n})$$

inside the unit circle $|y| = 1$, where $\mathbf{c} \in (\mathbb{C}^*)^n$ with $\mathrm{Log}(\mathbf{c}) = \mathbf{w}$.

Proof. (Theorem 3.10) Let the conditions of the theorem be satisfied and let π denote the projection of $\mathbb{R}^n$ on the subspace V defined in the theorem. Let $\pi(A) = \{\zeta(0), \zeta(1)\}$. Since all $A \subset \mathbb{Z}^n$ and $V \subset \mathbb{Q}^n$ we have $\pi(A) \subset \mathbb{Q}^n$. Since the homotopy of amoebas is invariant under translation of the Newton polytope, we can assume that $\zeta(0)$ is the origin. An outline of the proof is that the directional order of a complement component with respect to a vector $\eta \in V$ with $||\eta||_2 = \kappa \in \mathbb{R}$ needs to be zero or $||\zeta(1)||_2$ under π and that this cannot happen with lattice points in the interior of $\mathrm{conv}(A)$.

Assume there exists a Laurent polynomial $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \in \mathbb{C}^A$ such that $\mathbf{w} \in E_{\alpha'}(f) \neq \emptyset$ for some $\alpha' \in \mathbb{Z}^n$ in the interior of $\mathrm{conv}(A)$. Let $\eta \in V \cap \mathbb{Z}^n$. Since $\pi(A) = \{\zeta(0), \zeta(1)\}$, the hyperplanes $H_1 = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \eta \rangle = 0\}$ and $H_2 = \{\mathbf{x} \in \mathbb{R}^n : \langle \mathbf{x}, \eta \rangle = \langle \zeta(1), \eta \rangle\}$ are supporting hyperplanes of $\mathrm{conv}(A)$ with $A \subset H_1 \cup H_2$. Since α' is in the interior of $\mathrm{conv}(A)$, we have $\alpha' \notin H_1 \cup H_2$. Thus, for $\zeta = \pi(\alpha')$ holds that $\zeta \in \mathrm{conv}(\{0, \zeta(1)\})$, but $\zeta \notin \{0, \zeta(1)\}$. In particular, since α' is the order of $E_{\alpha'}(f)$, the directional order of $E_{\alpha'}(f)$ with respect to η is $\langle \zeta, \eta \rangle \notin \{0, \langle \zeta(1), \eta \rangle\}$.

On the other hand, by Lemma 3.11 the directional order $\langle \alpha', \eta \rangle$ is given by the number of roots minus the number of poles of the univariate polynomial

$$h : y \mapsto f(c_1 y^{\eta_1}, \ldots, c_n y^{\eta_n}) = \sum_{\alpha \in A} b_\alpha \mathbf{c}^\alpha \cdot \mathbf{y}^{\langle \alpha, \eta \rangle},$$

where $\mathrm{Log}(\mathbf{c}) = \mathbf{w} \in E_{\alpha'}(f)$. But since $A \subset H_1 \cup H_2$ we have $\langle \alpha, \eta \rangle \in \{0, \langle \zeta(1), \eta \rangle\}$ for every $\alpha \in A$ and hence

$$h \quad = \quad y^{\langle \zeta(1), \eta \rangle} + s \quad \text{ for some } s \in \mathbb{C}^*.$$

Note that $\langle \zeta(1), \eta \rangle \in \mathbb{Z}$ since all α as well as η are in $\mathbb{Z}^n$. Therefore, in particular all roots of h have the same modulus and thus the directional order $\langle \alpha', \eta \rangle$ is zero or $\langle \zeta(1), \eta \rangle$, which is a contradiction. Hence, $E_{\alpha'}(f) = \emptyset$. □

We now head over to the minimally sparse case. Let $A \subset \mathbb{Z}^n$. The lesser sparse $\mathrm{conv}(A) \cap \mathbb{Z}^n$ is, the higher the dimension of $\mathbb{C}^A$ is and hence the more degrees of freedom are given in $\mathbb{C}^A$. This leads to two possible but unfortunately contrary heuristics concerning Rullgård's Problem 2.22.

(1) The more degrees of freedom in $\mathbb{C}^A$ are given, the more possibilities exist to construct a path between two points in a given set U_α^A. Thus, in particular in the minimally sparse case we should expect U_α^A to be connected (if there exists a configuration space where they are connected at all).

(2) With more degrees of freedom in $\mathbb{C}^A$, more non-connected components of a set U_α^A may arise, which might vanish in a sparser configuration space (i.e., a subspace of $\mathbb{C}^{\mathrm{conv}(A) \cup \mathbb{Z}^n}$).

In Section 2 we followed the first heuristic by formulating Conjecture 3.8. We now motivate this conjecture by proving it for the univariate case.

Theorem 3.12. *Let $A = \{0, 1, \ldots, d\} \subseteq \mathbb{Z}$ such that $\mathbb{C}^A$ is minimally sparse. Then every $\bigcap_{j \in B} U_j^A$ with $B \subseteq A$ is path-connected in $\mathbb{C}_\Diamond^A$.*

Proof. Let $B \subseteq A$ such that $\bigcap_{j \in B} U_j^A \neq \emptyset$. Let $f, g \in \bigcap_{j \in B} U_j^A \subseteq \mathbb{C}^A$. Due to the fundamental theorem of algebra both f and g are reducible to linear factors, i.e., there are $a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{C}^*$ with $f = (z - a_1) \cdots (z - a_d)$, $g = (z - b_1) \cdots (z - b_d)$. Let w.l.o.g. $|a_1| \leq \cdots \leq |a_d|$ and $|b_1| \leq \cdots \leq |b_d|$. Note that $|a_j| < |a_{j+1}|$ if and only if $|b_j| < |b_{j+1}|$. We construct a path γ_1 in $\mathbb{C}_\Diamond^A$ from f to $h = (z - |b_1| \cdot \arg(a_1)) \cdots (z - |b_d| \cdot \arg(a_d))$, which is completely contained in $\bigcap_{j \in B} U_j^A$. I.e., we have to shift every $|a_j|$ to $|b_j|$ successively. If two roots share the same modulus in initial configuration, then we may shift them simultaneously. Hence, w.l.o.g. let $|a_j| < |a_{j+1}|$ for all j.

Therefore, and by Newton identities (see e.g., [**82**]) the construction of a path γ_1 is reduced to finding a homeomorphism, mapping $\{|a_1|, \ldots, |a_d|\} \subset \mathbb{R}_{>0}$ to $\{|b_1|, \ldots, |b_d|\} \subset \mathbb{R}_{>0}$, which, clearly, always exists.

We construct a path γ_2 on $\mathbb{C}^{\mathrm{conv}(A) \cap \mathbb{Z}}$ from h to g by successively shifting $|b_j| \cdot \arg(a_j)$ to b_j via successively adjusting the arguments and using Newton identities. Since $g, h \in \bigcap_{j \in B} U_j^A$ and no root changes its modulus during γ_2, we have $\gamma_2 \subseteq \bigcap_{j \in B} U_j^A$. Hence, we constructed a path $\gamma_2 \circ \gamma_1$ from f to g in $\bigcap_{j \in B} U_j^A$. □

4. The Boundary of Amoebas

Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a complex, non-singular Laurent polynomial with variety $\mathcal{V}(f) \in (\mathbb{C}^*)^n$. Besides the membership problem and the characterization of its configuration space, a third, central problem on amoebas is to characterize and to compute its boundary. Obviously, a boundary point of $\mathcal{A}(f)$ is a critical value (i.e., the image of a critical point) of the Log-map restricted of $\mathcal{V}(f)$. We call the set of critical values of $\text{Log}_{|\mathcal{V}(f)}$ the *contour* of $\mathcal{A}(f)$ (see e.g., [**66**]).

It was proven by G. Mikhalkin in [**41**] and [**43**] that the critical points of $\text{Log}_{|\mathcal{V}(f)}$ (and thus also the contour) are given by the points with a real image under the *logarithmic Gauss-map*.

For a non-singular variety $\mathcal{V}(f)$ (interpreted as a complex, smooth n-manifold) the *Gauss-map* is given by

$$G : \mathcal{V}(f) \to \mathbb{P}_{\mathbb{C}}^{n-1}, \quad (z_1, \ldots, z_n) \mapsto \left(\frac{\partial f}{\partial z_1}(\mathbf{z}) : \ldots : \frac{\partial f}{\partial z_n}(\mathbf{z}) \right).$$

Geometrically, the Gauss-map maps every point $\mathbf{z}$ on the linear space that is parallel to the tangent space $T_{\mathcal{V}(f)}(\mathbf{z})$ of $\mathcal{V}(f)$ at $\mathbf{z}$, where the complex projective image vector $G(\mathbf{z})$ is exactly the projective normal vector of $T_{\mathcal{V}(f)}(\mathbf{z})$. Hence, the image of the Gauss-map is in bijection with the normal field resp. the tangent bundle of $\mathcal{V}(f)$.

The *logarithmic Gauss-map*, introduced by Kapranov ([**30**]), is a composition of a branch of a holomorphic logarithm of each coordinate with the conventional Gauss-map. It is given by

$$\gamma : \mathcal{V}(f) \to \mathbb{P}_{\mathbb{C}}^{n-1}, \quad (z_1, \ldots, z_n) \mapsto \left(z_1 \cdot \frac{\partial f}{\partial z_1}(\mathbf{z}) : \cdots : z_n \cdot \frac{\partial f}{\partial z_n}(\mathbf{z}) \right).$$

For a given variety $\mathcal{V}(f)$ we define the set $S(f)$ by

$$S(f) \quad = \quad \{\mathbf{z} \in \mathcal{V}(f) \ : \ \gamma(\mathbf{z}) \in \mathbb{P}_{\mathbb{R}}^{n-1} \subset \mathbb{P}_{\mathbb{C}}^{n-1}\}.$$

Theorem 3.13 (Mikhalkin [**41, 43**])**.** *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with $\mathcal{V}(f) \in (\mathbb{C}^*)^n$. Then the critical values of the* Log *map equal $S(f)$ (and thus the contour of $\mathcal{A}(f)$ equals* $\text{Log}(S(f))$*).*

With respect to the description of the boundary of an amoeba, the theorem yields the following corollary (recall the notations regarding fibers from Section 1), stating that a point $\mathbf{w} \in \mathbb{R}^n$ may only be a boundary point of an amoeba $\mathcal{A}(f)$, if there exists a point in the intersection of its fiber $\mathbb{F}_{\mathbf{w}}$ with resp. to the Log-map and the variety $\mathcal{V}(f)$, which belongs to the set $S(f)$.

Corollary 3.14. *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with $\mathcal{V}(f) \in (\mathbb{C}^*)^n$ and let $\mathbf{w} \in \mathbb{R}^n$. Then*

$$\mathbf{w} \in \partial\mathcal{A}(f) \ \textit{ implies that } \ \mathbb{F}_{\mathbf{w}} \cap \mathcal{V}(f) \cap S(f) \neq \emptyset.$$

The aim of this section is to prove the following Theorem 3.15, which is a strengthening of this statement. It states that a point $\mathbf{w} \in \mathbb{R}^n$ may only be a boundary point of an amoeba $\mathcal{A}(f)$, if *every* point in the (non-empty) intersection of its fiber $\mathbb{F}_{\mathbf{w}}$ and the variety $\mathcal{V}(f)$ belongs to the set $S(f)$. This result is part of ongoing work joint with Franziska Schröter.

Theorem 3.15. *Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ non-singular and let $\mathbf{w} \in \mathbb{R}^n$. Then*

$$\mathbf{w} \in \partial\mathcal{A}(f) \text{ implies that there exists no } \mathbf{z} \in \mathbb{F}_{\mathbf{w}} \cap \mathcal{V}(f) \text{ with } \mathbf{z} \notin S(f).$$

First, we have to give a clean definition of the *boundary* $\partial\mathcal{A}(f)$ of a given amoeba $\mathcal{A}(f)$. Let $A \subset \mathbb{Z}^n$ and w.l.o.g. $0 \in A$. We introduce a *configuration metric* $d^A : \mathbb{C}^A \times \mathbb{C}^A \to \mathbb{R}_{>0}$ in the following way. Let $f = \sum_{\alpha \in A} b_\alpha \cdot \mathbf{z}^\alpha$ and $g = \sum_{\alpha \in A} c_\alpha \cdot \mathbf{z}^\alpha$. Then we define

$$d^A(f,g) \;=\; \left(\sum_{\alpha \in A} \left|\frac{b_\alpha}{b_0} - \frac{c_\alpha}{c_0}\right|^2\right)^{1/2}.$$

Note that in a strict sense d^A is defined on $\mathbb{P}^A_{\mathbb{C}}$ without its points at infinity instead of $\mathbb{C}^A$. The reason for the upper definition is that varieties and, hence, also amoebas of polynomials are invariant under scaling of the coefficients. Hence, we want those polynomials to have distance zero in a metric. By the axioms of a metric this means that such polynomials need to coincide, such that we have to work on $\mathbb{P}^A_{\mathbb{C}}$. We omit a distinction between $\mathbb{C}^A$ and $\mathbb{P}^A_{\mathbb{C}}$ with slight abuse of notation (see also the related comment in Chapter 2, Section 4).

With the configuration metric we define for $f \in \mathbb{C}^A$ a point $\mathbf{w} \in \mathcal{A}(f)$ as a *boundary point* of $\mathcal{A}(f)$ if there exists an $\alpha \in \operatorname{conv}(A) \cap \mathbb{Z}^n$ such that for every $\varepsilon > 0$ there exists an $g \in \mathcal{B}^A_\varepsilon(f) \subset \mathbb{C}^A$ such that $\mathbf{w} \in E_\alpha(g)$. Here, $\mathcal{B}^A_\varepsilon(f)$ denotes the ball with radius ε with respect to the configuration metric on $\mathbb{C}^A$ around the polynomial f.

Unfortunately, the definition has to be that complicated since bounded complement components can appear somewhere inside an amoeba for an arbitrary small changing of the coefficients and we want to consider the points where a complement component appears in such a case also as a boundary point (see e.g., Chapter 4, Section 1, where such effects appear).

For every polynomial $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ we denote its real and imaginary part as $f^{\text{re}}, f^{\text{im}} \in \mathbb{R}[\mathbf{x}, \mathbf{y}] = \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, which are given by

$$f(\mathbf{z}) \;=\; f(\mathbf{x} + i\mathbf{y}) \;=\; f^{\text{re}}(\mathbf{x}, \mathbf{y}) + i \cdot f^{\text{im}}(\mathbf{x}, \mathbf{y}).$$

We have to make some comments about this approach. With this notation we have obviously $\mathcal{V}(f) = \mathcal{V}(f^{\text{re}}) \cap \mathcal{V}(f^{\text{im}})$. Note that after the upper embedding the variety $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ obviously coincides with the intersection of the *real* varieties of f^{re} and f^{im} when these are investigated in $\mathbb{R}^{2n}$. To keep notation as simple as possible, we omit this here with slight abuse of notation and only mark out that we investigate varieties, which are *real* manifolds.

Note furthermore that if we assume that $\mathcal{V}(f)$ is non-singular, then we also assume that $\mathcal{V}(f^{\text{re}})$ and $\mathcal{V}(f^{\text{im}})$ are non-singular after the embedding of $\mathcal{V}(f)$ into $\mathbb{R}^{2n}$ with slight abuse of notation. I mention this here, since I see no reason why these two assumptions should be equivalent (but I do not know it).

Let $\mathbf{z} \in (\mathbb{C}^*)^n$ with $\text{Log}(\mathbf{z}) = \mathbf{w}$. Note that in particular (in $\mathbb{R}^{2n}$) also for the real part holds $\mathcal{V}(f^{\text{re}}) \cap \mathbb{F}_{\mathbf{w}} = \mathcal{V}(f^{|\mathbf{z}|,\text{re}})$ (analogously for the imaginary part), where $f^{|\mathbf{z}|,\text{re}} = f^{\text{re}}_{|\mathbb{F}_{\mathbf{w}}}$ (see Section 1). The following lemma describes the structure of $\mathcal{V}(f^{|\mathbf{z}|,\text{re}})$.

Lemma 3.16. *Let $f \in \mathbb{C}[\mathbf{z}]$ with $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ non-singular and $\text{Log}(\mathbf{z}) = \mathbf{w} \in \mathbb{R}^n$ with $\mathbb{F}_{\mathbf{w}} \cap \mathcal{V}(f) \neq \emptyset$. Then $\mathcal{V}(f^{|\mathbf{z}|,\text{re}})$ and $\mathcal{V}(f^{|\mathbf{z}|,\text{im}})$ generically are real, smooth $(n-1)$-manifolds.*

Note that $\mathcal{V}(f^{|\mathbf{z}|,\text{re}})$ and $\mathcal{V}(f^{|\mathbf{z}|,\text{im}})$ in general neither are connected nor non-singular (and thus the term "generically" appears in the lemma).

Proof. We only show the real case. $\mathcal{V}(f^{\text{re}})$ is a real, smooth, non-singular $(2n-1)$-manifold in $\mathbb{R}^{2n}$. The fiber $\mathbb{F}_{\mathbf{w}}$ is given by the n real, smooth, non-singular $(2n-1)$-hypersurfaces $x_j^2 + y_j^2 = w_j^2$. Since $\mathbb{F}_{\mathbf{w}} \cap \mathcal{V}(f^{\text{re}}) \neq \emptyset$ by assumption, it is the intersection of $n+1$ real, smooth, non-singular $(2n-1)$-manifolds, which is generically a real, smooth, $(n-1)$-manifold. □

Lemma 3.17. *Let $f \in \mathbb{C}[\mathbf{z}]$ with $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$. A point $\mathbf{z} \in \mathcal{V}(f)$ is critical under the* Log*-map if and only if it is critical under the* Arg*-map.*

This statement follows already (at least) implicitly from Mikhalkin's argumentation on the Log-Gauss map (see e.g., [**41**]) and was also observed by Nisse and Passare before (see [**51**]). For convenience, we give an own proof here.

Proof. We choose a local branch of the holomorphic log, interpret $(\mathbb{C}^*)^n$ as $\mathbb{R}^{2n}$ and Log- and Arg-map as linear maps from $\mathbb{R}^{2n}$ to $\mathbb{R}^n$ with $\text{Log}(\mathbb{R}^{2n}) \perp \text{Arg}(\mathbb{R}^{2n})$. Let $T_{\mathcal{V}(f)}(\mathbf{z})$ denote the tangent space of $\mathbf{z} \in \mathcal{V}(f)$. Note that $\dim(T_{\mathcal{V}(f)}(\mathbf{z})) = 2n-2$.

Observe that $\dim(\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z})^c)) = \dim(\text{Arg}(T_{\mathcal{V}(f)}(\mathbf{z})^c)) = 1$. The reason is that $T_{\mathcal{V}(f)}(\mathbf{z})^c$ is spanned by the (complex) normal vector $\mathbf{t}$ of $T_{\mathcal{V}(f)}(\mathbf{z})$ and hence we know $\dim(\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z})^c)) + \dim(\text{Arg}(T_{\mathcal{V}(f)}(\mathbf{z})^c)) = 2$. And since $\mathbf{t}$ has an absolute value and an argument and both Log and Arg are linear here, $T_{\mathcal{V}(f)}(\mathbf{z})^c$ may not vanish under one of these maps.

Let $\mathbf{z} \in \mathcal{V}(f)$ be critical under the Log-map. This is the case if and only if the Jacobian of Log does not have full rank at $\mathbf{z}$, i.e., $\dim(\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z}))) \leq n-1$. Since furthermore $\dim(\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z})^c)) = 1$ we have $\dim((\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z})))) = n-1$ and hence $\dim(\text{Ker}(\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z})))) = n-1$. This means, we can choose an orthogonal basis $B = \{b_1, \ldots, b_{2n-2}\} \subset \mathbb{R}^{2n}$ of $T_{\mathcal{V}(f)}(\mathbf{z})$ with $b_1, \ldots, b_{n-1} \in \text{Ker}(\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z})))$.

Since $\text{Log}(\mathbb{R}^{2n}) \perp \text{Arg}(\mathbb{R}^{2n})$ we have $\text{Ker}(\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z}))) \subseteq \text{Arg}(T_{\mathcal{V}(f)}(\mathbf{z}))$, i.e., in particular, $\text{Arg}(\{b_1, \ldots, b_{n-1}\})$ is an immersion. But since $\dim(\text{Log}(T_{\mathcal{V}(f)}(\mathbf{z}))) = n-1$, then $\text{Log}(\{b_n, \ldots, b_{2n-2}\})$ is also an immersion. Thus, with $\text{Log}(\mathbb{R}^{2n}) \perp \text{Arg}(\mathbb{R}^{2n})$, this yields $\{b_n, \ldots, b_{2n-2}\} \subset \text{Ker}(\text{Arg}(T_{\mathcal{V}(f)}(\mathbf{z})))$, i.e., $\dim(\text{Ker}(\text{Arg}(T_{\mathcal{V}(f)}(\mathbf{z})))) = n-1$ and therefore $\dim(\text{Arg}(T_{\mathcal{V}(f)}(\mathbf{z}))) = n-1$. Hence, $\mathbf{z}$ is critical under the Arg-map. Vice versa the argument works analogously. □

Lemma 3.18. *Let $f \in \mathbb{C}[\mathbf{z}]$ with $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ non-singular and $\mathbf{z} \in \mathcal{V}(f)$ with $\text{Arg}(\mathbf{z}) = \phi \in [0, 2\pi)^n$ be a non-critical point under the* Arg*-map. Then neither $T_{\mathcal{V}(f^{|\mathbf{z}|,\text{re}})}(\phi) = T_{\mathcal{V}(f^{|\mathbf{z}|,\text{im}})}(\phi)$ nor $\mathcal{V}(f^{|\mathbf{z}|,\text{re}})$ or $\mathcal{V}(f^{|\mathbf{z}|,\text{im}})$ are singular at ϕ.*

Note that $T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})}(\phi)$ is the tangent space of the variety $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ of the real part of the fiber function $f^{|\mathbf{z}|}$ at ϕ (analoguesly for the imaginary part). I.e., $T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})}(\phi)$ is a subset of the fiber $\mathbb{F}_{\mathbf{w}}$ with real dimension $n-1$, if $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ is smooth at ϕ.

Proof. Assume first $T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})}(\phi) = T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})}(\phi)$ and $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}}), \mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})$ non-singular at ϕ. Since $T_{\mathcal{V}(f^{|\mathbf{z}|})}(\phi) = T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})}(\phi) \cap T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})}(\phi)$ and $\dim(\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})) = \dim(\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})) = n-1$ by Lemma 3.16, we have $\dim(T_{\mathcal{V}(f^{|\mathbf{z}|})}(\phi)) = n-1$. Since $\mathcal{V}(f^{|\mathbf{z}|}) = \mathcal{V}(f) \cap \mathbb{F}_{\mathbf{w}}$ for $\mathbf{w} = \mathrm{Log}(\mathbf{z})$ and, hence, $T_{\mathcal{V}(f^{|\mathbf{z}|})}(\phi) = T_{\mathcal{V}(f)}(\phi) \cap \mathbb{F}_{\mathbf{w}}$, there is an immersion of an $(n-1)$-dimensional subspace of $T_{\mathcal{V}(f)}(\mathbf{z})$ into $\mathrm{Arg}(T_{\mathcal{V}(f)}(\mathbf{z}))$, which yields that $\mathbf{z}$ is critical with the argumentation from Lemma 3.17.

Assume now $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ is singular at ϕ. Then $T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})}(\phi) = [0, 2\pi)^n$ and hence, $T_{\mathcal{V}(f^{|\mathbf{z}|})}(\phi) = T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})}(\phi)$. Since $\mathcal{V}(f)$ is non-singular, $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})$ may not be singular at ϕ either. Hence, $\dim(T_{\mathcal{V}(f^{|\mathbf{z}|})}(\phi)) = n-1$. The rest works the same way as above. □

With these lemmata we can prove our main theorem in this section. An outline of the proof is that if there exists a point $\mathbf{z} \in \mathcal{V}(f) \cap \mathbb{F}_{\mathbf{w}}$ with $\mathbf{z} \notin S(f)$, then the manifolds $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ and $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})$ intersect regularly in $\mathrm{Arg}(\mathbf{z})$. But this means that they also intersect for every small perturbation of the coefficients of f. This is a contradiction with the assumption that $\mathbf{w} \in \partial\mathcal{A}(f)$, which means that there is a small perturbation of the coefficients yielding $\mathcal{V}(f) \cap F_{\mathbf{w}} = \emptyset$, i.e., $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}}) \cap \mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}}) = \emptyset$.

Proof. (Theorem 3.15) Since $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ we can assume w.l.o.g. that $f \in \mathbb{C}[\mathbf{z}]$. Let $\mathbf{w} \in \partial\mathcal{A}(f)$ and assume that there is a $\mathbf{z} \in \mathcal{V}(f) \cap \mathbb{F}_{\mathbf{w}}$ with $\mathbf{z} \notin S(f)$. By Theorem 3.13 this means that $\mathbf{z}$ is not a critical point under the Log-map and hence, by Lemma 3.17, $\mathbf{z}$ is not a critical point under the Arg-map, too. Thus, by Lemma 3.18 both $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ and $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})$ are regular at ϕ and $T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})}(\phi) \neq T_{\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})}(\phi)$. Therefore, $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ and $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})$ intersect regularly in ϕ. Hence, there exists a $\delta > 0$ such that the intersection of every perturbation of $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ and $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})$ of at most δ is not empty.

Since $\mathbf{w} \in \partial\mathcal{A}(f)$ we find a $g \in \mathcal{B}_{\varepsilon}^{A}(f) \subset \mathbb{C}^A$ for every arbitrary small $\varepsilon > 0$ such that $\mathbf{w} \notin \mathcal{A}(g)$, i.e., $\mathcal{V}(g) \cap \mathbb{F}_{\mathbf{w}} = \emptyset$, and thus in particular $\mathcal{V}(g^{|\mathbf{z}|,\mathrm{re}}) \cap \mathcal{V}(g^{|\mathbf{z}|,\mathrm{im}}) = \emptyset$. But $f^{\mathrm{re}}, f^{\mathrm{im}}$ and hence also $\mathcal{V}(f^{\mathrm{re}}), \mathcal{V}(f^{\mathrm{im}})$ are continuous under changing of coefficients of f. Therefore, by definition of the fiber function, $\mathcal{V}(g^{|\mathbf{z}|,\mathrm{re}})$ and $\mathcal{V}(g^{|\mathbf{z}|,\mathrm{im}})$ are arbitrary small perturbations of $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ and $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})$. Thus, $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{re}})$ and $\mathcal{V}(f^{|\mathbf{z}|,\mathrm{im}})$ may not intersect regularly in ϕ. This is a contradiction. □

We close the section with an example. Let f be a Laurent polynomial given by

$$f \quad = \quad -2z_1^2 - 2z_1z_2^2 + 1,5e^{i\pi\cdot 0.5}z_1^{-1}z_2^{-1} + c.$$

We investigate the fiber $\mathbb{F}_{(0,0)}$ of the point $\mathrm{Log}((1,1))$ for $c = -1.2, -2.7, -4.6$ and -4.9 depicted in Figure 3.3. In all pictures the red curve marks $\mathcal{V}(f^{|(1,1)|,\mathrm{re}})$ and the green curve marks $\mathcal{V}(f^{|(1,1)|,\mathrm{im}})$. Hence, the points in the intersection of the red and the green curve are the points where the real and the imaginary part of $f^{|(1,1)|}$ vanishes, i.e., these are the intersection points of the fiber $\mathbb{F}_{(0,0)}$ with the variety $\mathcal{V}(f)$ of f.

The blue curve marks the arguments of points on the complex unit sphere, which are critical points under the Log-Gauss map (i.e., the critical points of γ on the fiber $\mathbb{F}_{(0,0)}$).

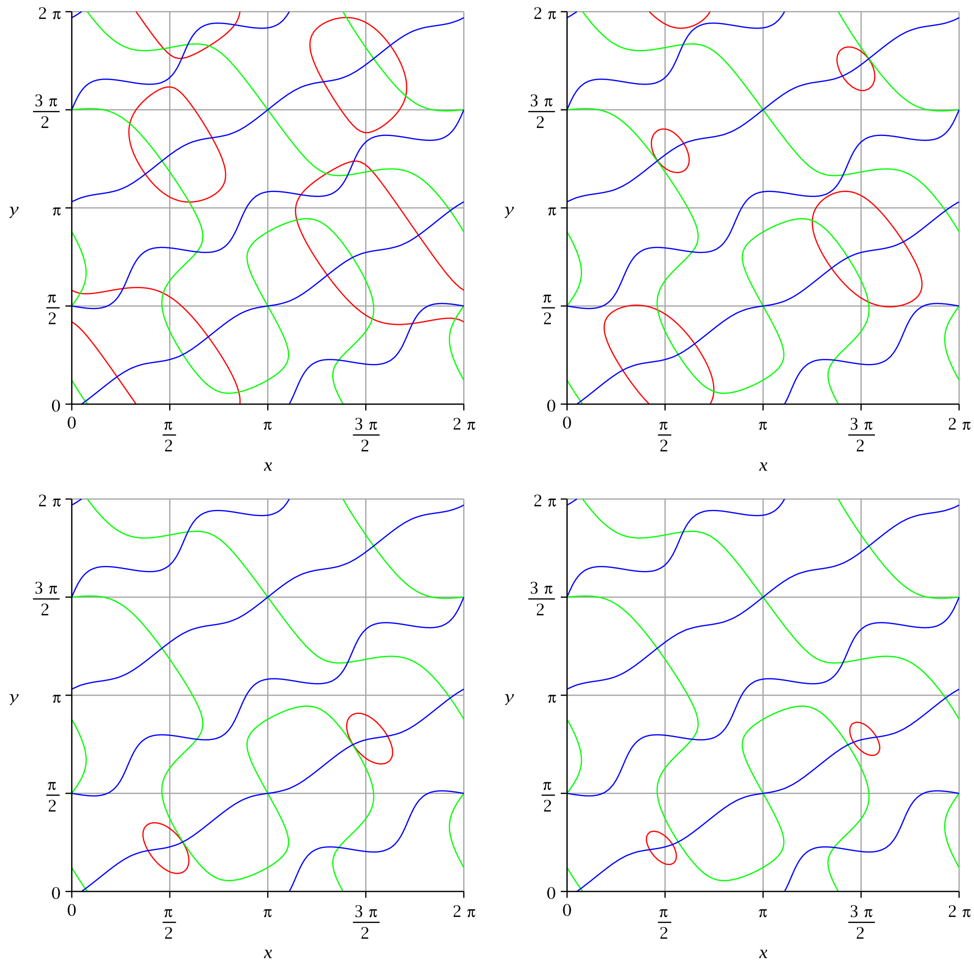


FIGURE 3.3. The behavior of $f = -2z_1^2 - 2z_1z_2^2 + 1,5e^{i\pi\cdot 0,5}z_1^{-1}z_2^{-1} + c$ on the fiber $\mathbb{F}_{(0,0)}$ for $c \in \{-1.2, -2.7, -4.6, -4.9\}$.

Thus, $(0,0)$ is part of the contour if there is a point where the red, green and blue curve intersect (Corollary 3.14) and $(0,0)$ may only be a boundary point if all intersection points of the red and the green curve also intersect the blue one (Theorem 3.15). Note that in this example a changing of the coefficient c along the real axis only changes the red curve.

Observe that in the upper left picture , i.e., $c = -1.2$, of Figure 3.3 the red and green curve intersect regularly in several points and hence in this case $(0,0) \in \mathcal{A}(f)$. In the upper right picture , i.e., $c = -2.7$, there are two intersection points where all three curves intersect. But there are other points where (only) the red and the green curve intersect regularly. Thus, in this case $(0,0)$ is part of the contour but still $(0,0) \in \mathcal{A}(f) \setminus \partial\mathcal{A}(f)$. In the lower left picture, i.e., $c = -4.6$, the only two intersection points of the red and the green curve also intersect the blue one. Hence, in this case $(0,0)$ might be part of the boundary. In the lower right picture, i.e., $c = -4.9$, the red and the green curve do not intersect in any point anymore. Therefore, we have $(0,0) \notin \mathcal{A}(f)$.

Note that the values $c = -2.7$ and $c = -4.6$ are not the exact values of c for $(0, 0)$ to be in the contour resp. in the boundary of $\mathcal{A}(f)$. They are guessed approximations of the exact values to show the described behavior exemplarily in Figure 3.3.

CHAPTER 4

The Geometry and Topology of Amoebas

The focus of this chapter is the geometrical and topological structure of configuration spaces of amoebas. More precisely, we are interested in the structure of the sets U_α^A and its related problems introduced in Chapter 2, Section 4, in particular the Problems 2.22 and 2.25.

In earlier investigations of configuration spaces the authors usually concentrated on gaining general results about configuration spaces, which led in particular to the theorems, which we discussed in Chapter 2, Section 4. As a special instance only linear polynomials have been discussed carefully so far (see Theorem 2.8; see also [**20**]). But since the problems mentioned above resp. in Chapter 2, Section 4 are open since round about ten years without any strong progress, it is a convincing strategy to restrict to special classes of polynomials.

This is what we do in this Chapter. In Section 1 we investigate polynomials with a simplex Newton polytope in dimension $n \geq 2$ and a support set $A \subset \mathbb{Z}^n$ with exactly $n+2$ elements where the non-vertex element of A is contained in the interior of $\text{conv}(A)$. We call amoebas in the set $\mathcal{P}_\Delta^y$ of such polynomials "*of genus at most one*" since they have at most one bounded complement component (Theorem 4.1). We provide upper and lower bounds of the coefficients for the existence of this complement component (Theorems 4.8, 4.10 and 4.13), where the upper bound gets sharp under extremal conditions. Furthermore, we show that the upper bound is connected to A-*discriminants* and Purbhoo's *lopsidedness condition* (Theorems 4.15 and 4.17).

If one restricts to the special case that the lattice point in the interior is the barycenter of the simplex, then one can even give a complete description of the set U_y^A where $y \in A$ is the inner lattice point. We discuss this case in Section 2. We show that the complement of the set U_y^A locally is given by the region bounded by the trajectory of a *hypocycloid curve* (Theorem 4.20). With this result we can furthermore conclude that Problem 2.22 has an affirmative answer for this class, i.e., U_y^A is connected.

In the last Section 3, we investigate the univariate polynomials of the class $\mathcal{P}_\Delta^y$, i.e., trinomials. In the univariate case amoebas of trinomials *can* have more than one bounded complement component. It turns out that in the univariate case our investigated problems are very closely related to reformulations of classical problems on trinomials, which were investigated in the late 19th resp. the early 20th century, in terms of amoeba theory. We introduce some of these problems in Section 3. It turns out that they are algebraically well understood, but neither geometrically nor topologically. We provide such explanations resp. solutions. Specifically, we show that a trinomial has a root of a certain modulus if an only if a certain coefficient is located on the trajectory of a *hypotrochoid curve*

(Theorem 4.32). Furthermore, it has multiple roots of the same modulus if and only if the coefficient is located on a specific 1-fan (Theorem 4.40). This result allows us not only to close the gap in Rullgård's Theorem 2.19 (Theorem 4.43). It allows us furthermore to describe the topological structure of sets U_α^A, since this theorem allows us to show that the sets U_α^A can (for all but one α) be deformation retracted to an $(s+t)$-sheeted covering of an S^1 (Theorem 4.51). Therefore, their fundamental group is $\mathbb{Z}$ and hence they are in particular path-connected but not simply connected. This solves the Problems 2.22, 2.24 and 2.25 (for this class).

Finally, we provide a counterexample for the question if complement components of amoebas are monotonically growing in the absolute value of the coefficient corresponding to this component via the order map (Theorem 4.53). This result also follows from the initial Theorems 4.32 and 4.40 about trinomials.

1. Amoebas of Genus at Most One

Despite the various general, structural results on configuration spaces $\mathbb{C}^A$ with $A \subset \mathbb{Z}^n$, which we presented in Chapter 2, Section 4, almost nothing is known concerning the existence and explicit characterization of the complement components $E_\alpha(f) \subset \mathbb{R}^n$ with orders $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$ in terms of the coefficients of a given Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ (Problem 2.25). Hence, understanding the geometrical and topological structure of the sets U_α^A in the configuration space of amoebas is a widely open problem. For amoebas of linear polynomials an explicit characterization exists, given by Forsberg, Passare and Tsikh (Theorem 2.8; [**20**]). In this case we have seen that there exist no complement components except the "trivial" ones whose orders correspond to the vertices of the standard simplex. Thus, those amoebas are particular instances of amoebas of genus zero.

As a step towards better understanding the structure of amoebas of general, nonlinear varieties, in the next two Sections 1 and 2 we study a class of polynomials whose amoebas can have at most one bounded complement component. The results presented in these two sections were recently published in [**87**].

We assume $n \geq 2$ and choose a support set $A = \{\alpha(0), \ldots, \alpha(n), y\}$, such that $\text{conv}(A)$ is a full-dimensional lattice simplex $\Delta \subset \mathbb{R}^n$, such that $\alpha(0), \ldots, \alpha(n) \in \mathbb{Z}^n$ are the vertices of Δ and $y \in \mathbb{Z}^n$ is contained in the interior of Δ.

Let $\mathcal{P}_\Delta$ denote the class of all Laurent polynomials with Newton polytope Δ and let $\mathcal{P}_\Delta^y \subset \mathcal{P}_\Delta$ denote the class of Laurent polynomials of the form

$$f \;=\; b_0 \cdot \mathbf{z}^{\alpha(0)} + b_1 \cdot \mathbf{z}^{\alpha(1)} + \cdots + b_n \cdot \mathbf{z}^{\alpha(n)} + c \cdot \mathbf{z}^y\,, \quad b_i \in \mathbb{C}^*, c \in \mathbb{C}. \tag{4.1}$$

Since $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$, we can assume that $\alpha(0)$ is the origin and $b_0 = 1$. Polynomials in $\mathcal{P}_\Delta$ have exactly $n+2$ monomials. Note that we do not require that $\#(\Delta \cap \mathbb{Z}^n) = n+2$, since the simplex Δ may contain further lattice points as long as the corresponding coefficients are zero. Observe that for this special instance of the support set A we have $\mathcal{P}_\Delta^y = \mathbb{C}_\Diamond^A$ and $\mathcal{P}_\Delta = \bigcup_{\mathbb{C}^A \in L(\Delta)} \mathbb{C}^A$ (see Chapter 3, Section 2). For general background on lattice point simplices (with one inner lattice point) see e.g., [**1, 73**]), and we remark that f can be regarded as *supported on a circuit* (a set of points that is affinely dependent, but with all proper subsets affinely independent; see [**23**] and also e.g., [**4, 68**]).

In order to study the amoebas of polynomials in $\mathcal{P}_\Delta^y$, we investigate the parametric family of polynomials

$$(4.2)\quad f_\kappa \;=\; \left[|c|\cdot e^{i\cdot\arg(c)}\cdot \mathbf{z}^y + \sum_{i=0}^n b_i\cdot \mathbf{z}^{\alpha(i)}\right]_{|c|=\kappa} \;=\; \kappa\cdot e^{i\cdot\arg(c)}\cdot \mathbf{z}^y + \sum_{i=0}^n b_i\cdot \mathbf{z}^{\alpha(i)}.$$

As a first key fact, we show that amoebas $\mathcal{A}(f)$ of polynomials $f \in \mathcal{P}_\Delta^y$ can have at most one bounded complement component. Thus, there exist only two possible homotopy types for $\mathcal{A}(f)$ in this case.

Theorem 4.1. *Let $n \geq 2$ and $f \in \mathcal{P}_\Delta^y$ of the form* (4.1). *Then $\mathcal{A}(f)$ has at most one bounded complement component.*

Proof. Since $n \geq 2$ we have $\#A \leq 2n$ and by construction no $k+2$ points in A are contained in an affine k-subspace of $\mathbb{R}^n$. Thus, by Theorem 2.15 in this case the complement induced tropical hypersurface $\mathcal{C}(f)$ is a deformation retract of $\mathcal{A}(f)$. Since $\mathcal{C}(f)$ is dual to a subdivision of $\operatorname{conv}(A)$ induced by a lifting of A, $\mathcal{C}(f)$ can have at most one bounded component. This follows e.g., since A is a circuit and circuits come with only two possible triangulations (see e.g., [**23**, Chapter 7, p. 217]). □

When we study polynomials $f \in \mathcal{P}_\Delta^y$ of the Form (4.1), we call the monomials $b_i\mathbf{z}^{\alpha(i)}$ the *outer monomials* and $c\cdot\mathbf{z}^y$ the *inner monomial*.

Apparently, these polynomials form a "simplest" class of polynomials where the characterization of the corresponding amoebas becomes "difficult". Since an exact description of the complement components (and, in particular, the homotopy) is not available, one of our main goals is to provide bounds on the coefficients to determine the homotopy type of $\mathcal{A}(f)$ (i.e., solve Problem 2.25 for this class).

As a starting point, recall that the complement components of amoebas of linear polynomials are well understood (Theorem 2.8). The following statement captures a slight generalization of this result to Newton polytopes that might contain interior lattice points.

Theorem 4.2. *Let $f = \sum_{i=0}^n b_i\mathbf{z}^{\alpha(i)}$ such that* $\operatorname{New}(f)$ *is an n-simplex. For $\mathbf{z} \in (\mathbb{C}^*)^n$ we have* $\operatorname{Log}(\mathbf{z}) \in E_{\alpha(i)}(f)$ *if and only if* $\left|b_i\mathbf{z}^{\alpha(i)}\right| > \sum_{j\neq i}\left|b_j\mathbf{z}^{\alpha(j)}\right|$.

Note that this theorem refers to the maximally sparse case. Recall that an arbitrary polynomial f is called *maximally sparse* if for all non-vertices α of $\operatorname{New}(f)$ we have $b_i = 0$ (see Chapter 3, Section 2).

For the convenience of the reader we provide a proof of Theorem 4.2, which is analogous to the proof of statement [**20**, Proposition 4.2].

Proof. The direction "$\Leftarrow$" is obvious. For the converse direction let $\mathbf{z} \in (\mathbb{C}^*)^n$ with $|b_i\mathbf{z}^{\alpha(i)}| \leq \sum_{j\neq i}|b_j\mathbf{z}^{\alpha(j)}|$ for all $i \in \{1,\ldots,n\}$. Since the case $n = 1$ is trivial, assume $n \geq 2$. We normalize such that $\alpha(0) = 0 \in \mathbb{Z}^n$ and $\arg(b_0) = 0 \in [0, 2\pi)$.

Order the monomials by norm such that $|b_j\mathbf{z}^{\alpha(j)}| \leq |b_{j+1}\mathbf{z}^{\alpha(j+1)}|$ for $j \in \{0,\ldots,n\}$ and let m denote the largest integer such that $\sum_{j=0}^{m-1} |b_j\mathbf{z}^{\alpha(j)}| < \sum_{j=m}^{n} |b_j\mathbf{z}^{\alpha(j)}|$. By choice of $\mathbf{z}$ we have $m < n$. We denote $t_1 = \sum_{j=0}^{m-1} |b_j\mathbf{z}^{\alpha(j)}|$, $t_2 = |b_m\mathbf{z}^{\alpha(m)}|$ and $t_3 = \sum_{j=m+1}^{n} |b_j\mathbf{z}^{\alpha(j)}|$. By the choice of m we have $t_1 + t_2 \geq t_3$, $t_1 + t_3 \geq t_2$ and $t_2 + t_3 \geq t_1$. Hence, t_1, t_2, t_3 form the edge lengths of a triangle and thus there are $\psi_1, \psi_2 \in [0, 2\pi)$ with

$$\sum_{j=0}^{m-1} |b_j\mathbf{z}^{\alpha(j)}| + |b_m\mathbf{z}^{\alpha(m)}| \cdot e^{i\cdot\psi_1} + \sum_{j=m+1}^{n} |b_j\mathbf{z}^{\alpha(j)}| \cdot e^{i\cdot\psi_2} \;\; = \;\; 0.$$

Since the integer vectors $\alpha(1),\ldots,\alpha(n)$ are linearly independent, we can find $\phi \in [0,2\pi)^n$ such that $\sum_{j=0}^{n} b_j|\mathbf{z}|^{\alpha(j)} \cdot e^{i\cdot\langle\alpha(i),\phi\rangle} = 0$ and thus $\operatorname{Log}|\mathbf{z}| \in \mathcal{A}(f)$.

Finally, one can show that all extreme points of the closure of $\mathcal{A}(f)$ satisfy the required inequalities, which we omit here. □

Thus, the class $\mathcal{P}_\Delta^y$ is a natural generalization of maximally sparse polynomials with simplex Newton polytope. Note that the above proof technique does not extend to the case of simplices with interior integer points since then the set of all exponent vectors is not affinely independent.

1.1. The Equilibrium and First Bounds. In this section we provide a lower and a rough upper bound on the coefficients for the existence of the inner complement component of the amoeba of a polynomial $f \in \mathcal{P}_\Delta^y$. These bounds – which are stated in Theorem 4.8 – are based on investigating the *equilibrium points* (as defined in Definition 4.3). We remark that, as a special case, the lower bound in Theorem 4.8 implies immediately that maximally sparse polynomials with simplex Newton polytope have solid amoebas (Corollary 4.9).

We introduce the *(modular) equilibrium* $\mathcal{E}(f)$ of a given Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$

$$\mathcal{E}(f) \;\; = \;\; \left\{\mathbf{w} \in \mathbb{R}^n \,:\, |m_i(\operatorname{Log}^{-1}(\mathbf{w}))| = |m_j(\operatorname{Log}^{-1}(\mathbf{w}))| \text{ for some } 1 \leq i \neq j \leq d\right\}.$$

I.e., the modular equilibrium is the piecewise linear subset of $\mathbb{R}^n$ containing all points $\mathbf{w}$ such that at least two monomials of f attain the same modular value on the fiber torus $\mathbb{F}_{\mathbf{w}} = \operatorname{Log}^{-1}(\mathbf{w})$. From this definition of the modular equilibrium it follows immediately that both the tropical hypersurface $\mathcal{T}(\operatorname{trop}(f))$ and the complement induced tropical hypersurface $\mathcal{C}(f) = \mathcal{T}(\operatorname{trop}(f_{|C})$ are subsets of the equilibrium $\mathcal{E}(f)$ (see Chapter 2, Section 3).

Definition 4.3. Let $A = \{\alpha(1),\ldots,\alpha(d)\} \subset \mathbb{Z}^n$ and $f \in \mathbb{C}^A$. We define the *(modular) equilibrium point* $\operatorname{eq}(\alpha(j_1),\ldots,\alpha(j_{n+1}))$ as the point of the modular equilibrium $\mathcal{E}(f)$ where at least the monomials $b_{\alpha(j_1)}\mathbf{z}^{\alpha(j_1)},\ldots,b_{\alpha(j_n)}\mathbf{z}^{\alpha(j_n)}$ (with $\alpha(j_1),\ldots,\alpha(j_{n+1}) \in A$) have the same modular value.

For $f \in \mathcal{P}_\Delta^y$ of the form (4.3) we abbreviate the notation in the following way. We denote $\operatorname{eq}(y)$ as the point of the modular equilibrium $\mathcal{E}(f)$ where at least all monomials but $b_y\mathbf{z}^y$ have the same modular value. Similarly, for $0 \leq j \leq n$ we denote $\operatorname{eq}(j)$ as the point in $\mathcal{E}(f)$ where at least all monomials but $b_j\mathbf{z}^{\alpha(j)}$ have the same modular value.

It is not completely obvious that equilibrium points are always well defined. The following Lemma yields that this is the case and how they can be computed.

Lemma 4.4. *Let* $A = \{\alpha(1), \ldots, \alpha(d)\} \subset \mathbb{Z}^n$, $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha$ *with* $b_\alpha \in \mathbb{C}^*$ *and let* $B \in \mathbb{Z}^{n \times n}$ *be the matrix with columns* $\alpha(1) - \alpha(n+1), \ldots, \alpha(n) - \alpha(n+1)$. *Then the equilibrium point* $\text{eq}(\alpha(1), \ldots, \alpha(n+1)) \in \mathbb{R}^n$ *is the unique solution* $\mathbf{x} \in \mathbb{R}^n$ *of the system of linear equations* $B^t \cdot \mathbf{x} = (\text{Log}(b_{\alpha(n+1)}, \ldots, b_{\alpha(n+1)}))^t - (\text{Log}(b_{\alpha(1)}, \ldots, b_{\alpha(n)}))^t$.

PROOF. The equilibrium point $\text{eq}(\alpha(1), \ldots, \alpha(n+1))$ is the point where all monomials $b_{\alpha(1)}\mathbf{z}^{\alpha(1)}, \ldots, b_{\alpha(n+1)}\mathbf{z}^{\alpha(n+1)}$ share the same modular value. Hence $\text{eq}(\alpha(1), \ldots, \alpha(n+1))$ satisfies the n linear equations

$$\begin{aligned} \log|b_{\alpha(i)}| + \langle \mathbf{w}, \alpha(i) \rangle &= \log|b_{\alpha(n+1)}| + \langle \mathbf{w}, \alpha(n+1) \rangle \quad \Leftrightarrow \\ \langle \mathbf{w}, \alpha(i) - \alpha(n+1) \rangle &= \log|b_{\alpha(n+1)}| - \log|b_{\alpha(i)}|. \end{aligned}$$

Each of these equations coincides with one row of the linear system

$$B^t \cdot \mathbf{x} = (\text{Log}(b_{\alpha(n+1)}, \ldots, b_{\alpha(n+1)}))^t - (\text{Log}(b_{\alpha(1)}, \ldots, b_{\alpha(n)}))^t.$$

$\square$

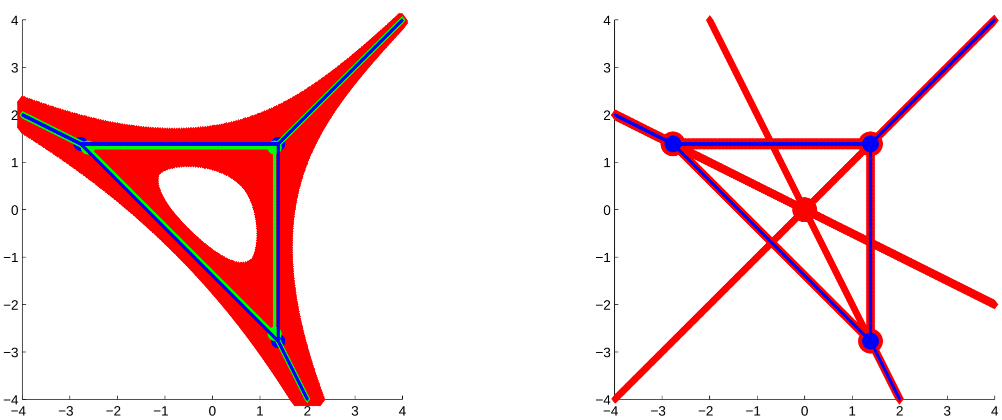


FIGURE 4.2. Let $f = 1 + z_1^2 z_2 + z_1 z_2^2 - 4z_1 z_2$. Left picture: the amoeba $\mathcal{A}(f)$ (red) with the spine $\mathcal{S}(f)$ (green, light) and the complement-induced tropical hypersurface $\mathcal{C}(f)$ (blue, dark). Note that $\mathcal{S}(f)$ and $\mathcal{C}(f)$ coincide on the outer tentacles of $\mathcal{A}(f)$. Right picture: the equilibrium $\mathcal{E}(f)$ (red) together with $\mathcal{C}(f)$ (blue, dark). Note that $\mathcal{C}(f) \subset \mathcal{E}(f)$. The equilibrium points introduced in Definition 4.3 are marked by big red points.

From now on we concentrate on $f \in \mathcal{P}_\Delta^y$ of the form (4.1). For the rest of this section we assume $\alpha(0) = 0$ and $b_0 = 1$. We denote M_Δ as the integral $n \times n$ matrix with columns $\alpha(1), \ldots, \alpha(n)$.

The following lemma states how the spine $\mathcal{S}(f)$ of the amoeba $\mathcal{A}(f)$ is related to $\mathcal{C}(f)$ for polynomials $f \in \mathcal{P}_\Delta^y$.

Lemma 4.5. *Let $f \in \mathcal{P}_\Delta^y$.*

(a) *If $\mathcal{A}(f)$ is solid then the inner vertex of $\mathcal{S}(f)$ is the equilibrium point* $\mathrm{eq}(y)$ *and $\mathcal{S}(f)$ coincides with the complement-induced tropicalization $\mathcal{C}(f)$.*

(b) *If $\mathcal{A}(f)$ has genus one, then $\mathcal{S}(f)$ and $\mathcal{C}(f)$ are homotopy equivalent, their inner simplices $\Sigma_{\mathcal{S}(f)}$ and $\Sigma_{\mathcal{C}(f)}$ are similar and all faces not belonging to the inner simplices coincide in all points lying outside of both inner simplices.*

Proof. (a) If $\mathcal{A}(f)$ is solid then the order of any complement component of $\mathcal{A}(f)$ is a vertex of $\mathrm{New}(f)$ and hence for every Ronkin coefficient $\beta_{\alpha(i)}$ we have $\beta_{\alpha(i)} = \log|b_i|$ and therefore $\mathcal{S}(f) = \mathcal{C}(f)$.

(b) Let $\mathcal{A}(f)$ have genus one. $\mathcal{S}(f)$ and $\mathcal{C}(f)$ coincide in all points lying outside of both inner simplices since for any vertex $\alpha(i)$ of $\mathrm{New}(f)$ we have $\beta_{\alpha(i)} = \log|b_i|$. As $n \geq 2$, homotopy equivalence follows from Theorem 2.15. Since $\mathcal{S}(f)$ and $\mathcal{C}(f)$ are tropical hypersurfaces dual to the same triangulation of $\mathrm{New}(f)$, $\Sigma_{\mathcal{S}(f)}$ and $\Sigma_{\mathcal{C}(f)}$ are similar. □

This fact was already depicted in Figure 4.2. In the following we often write $f \in \mathcal{P}_\Delta^y$ as a sum of monomials

$$f(\mathbf{z}) \;=\; m_0(\mathbf{z}) + m_1(\mathbf{z}) + \cdots + m_n(\mathbf{z}) + m_y(\mathbf{z}), \tag{4.3}$$

with each $m_i(\mathbf{z})$ representing the according monomial of f in the notation of (4.1).

Lemma 4.6. *Let $n \geq 2$, $\alpha(0) = 0$, $b_0 = 1$ and $f \in \mathcal{P}_\Delta^y$ such that $\mathcal{A}(f)$ has genus one.*

(a) *If $\mathbf{z} \in (\mathbb{C}^*)^n$ with $\mathrm{Log}(\mathbf{z}) = \mathrm{eq}(y)$ then $|m_y(\mathbf{z})| > 1$.*

(b) *The equilibrium point $\mathrm{eq}(y)$ is contained in the interior of the simplex with vertices $\mathrm{eq}(0), \ldots, \mathrm{eq}(n)$.*

Proof. (a) Assume that $|m_y(\mathbf{z})| \leq 1$. Due to definition of $\mathrm{eq}(y)$ and $\mathrm{Log}(\mathbf{z}) = \mathrm{eq}(y)$ we know $|m_i(\mathbf{z})| = 1$ for all $i \in \{0, \ldots, n\}$. Hence, we have $\mathrm{eq}(y) \in \mathcal{C}(f)$. By Lemma 4.4 $\mathrm{eq}(y)$ is the unique point where the infinite cells of $\mathcal{C}(f)$ intersect. Thus, $\mathcal{C}(f)$ has genus zero. This yields a contradiction since $\mathcal{A}(f)$ has genus one and $\mathcal{C}(f)$ is a deformation retract of $\mathcal{A}(f)$ for $n \geq 2$ by Theorem 2.15.

(b) Let Σ' be the simplex with vertices $\mathrm{eq}(0), \ldots, \mathrm{eq}(n)$. By definition of $\mathcal{C}(f)$ we have for all $\mathbf{z} \in (\mathbb{C}^*)^n$: If $|m_y(\mathbf{z})| > |m_i(\mathbf{z})|$ for all $i \in \{0, \ldots, n\}$, then $\mathrm{Log}(\mathbf{z})$ is contained in the interior of Σ'. With (a) the assertion follows. □

Let $f \in \mathcal{P}_\Delta^y$, and consider f with a varying angle $\arg(c)$. An angle $\arg(c)$ is called in *extreme opposition* if there exists some $\mathbf{z} \in (\mathbb{C}^*)^n$ with

$$\arg(m_y(\mathbf{z})) \;=\; \arg(m_i(\mathbf{z})) + \pi \mod 2\pi, \qquad \text{for all } i \in \{0, \ldots, n\}\,. \tag{4.4}$$

Since condition (4.4) is actually independent of the modulus of $\mathbf{z}$ (and also of the modulus of the coefficients), we call $\arg(\mathbf{z})$ an *extremal phase*.

Lemma 4.7. *Let f be in $\mathcal{P}_\Delta^y$, where we consider* $\arg(c)$ *as a parameter. Then there always exists some choice of* $\arg(c)$ *such that* $\arg(c)$ *is in extreme opposition.*

Proof. By multiplying f with a Laurent monomial, we can assume $\alpha(0) = 0$ and $b_0 = 1$.

Setting $\phi = \arg(\mathbf{z})$, the condition (4.4) is a linear condition in ϕ. Using the non-singular integral matrix M_Δ introduced above, the image of $[0, 2\pi)^n$ under the mapping $\phi \mapsto M_\Delta \phi$ is a D-fold covering of $[0, 2\pi)^n$ where $D = \det(M_\Delta)$. Hence, there exists $\phi \in [0, 2\pi)^n$ with

$$M_\Delta^t \cdot \phi \;=\; -(\arg(b_1), \ldots, \arg(b_n))^t \mod 2\pi,$$

and indeed the number of distinct solutions for ϕ in $[0, 2\pi)^n$ is D. Setting $\arg(c) = \pi - \langle \phi, y \rangle$ we obtain the result. □

Recall the parametric family f_κ of polynomials in $\mathcal{P}_\Delta^y$, which we introduced in (4.2). For such a parametric family f_κ we are interested in those parameters κ where the genus of $\mathcal{A}(f_\kappa)$ changes. We say that $\mathcal{A}(f_\kappa)$ *switches* from genus zero to one at κ_0, if $E_y(f_{\kappa_0}) = \emptyset$ and for every (sufficiently small) $\varepsilon > 0$ we have $E_y(f_{\kappa_0+\varepsilon}) \neq \emptyset$. Note that, if κ is sufficiently large, then $\mathcal{A}(f_\kappa)$ is always of genus one (e.g., by the lopsidedness criterion; see Chapter 2, Section 5).

For a parameter value $\kappa_1 \in \mathbb{R}$ with $E_y(f_{\kappa_1}) \neq \emptyset$ we are furthermore interested in characterizing the point where the complement component E_y appears first (with respect to values $\kappa < \kappa_1$ in the parametric family). Formally, we say that the inner complement component $E_y(f_{\kappa_1})$ *appears first* at $\mathbf{w} \in \mathrm{Log}\left((\mathbb{C}^*)^n\right)$ if the following conditions hold:

(a) $\mathbf{w} \in E_y(f_{\kappa_1})$, and
(b) there exists a $\kappa_0 < \kappa_1$ such that $E_y(f_{\kappa_0}) = \emptyset$ and for every $\kappa \in [\kappa_0, \kappa_1]$ we have $E_y(f_\kappa) = \emptyset$ or $\mathbf{w} \in E_y(f_\kappa)$.

For every such κ_1 this point is unique due to convexity of complement components and will be denoted by $\mathrm{app}(f_{\kappa_1})$.

Let $K \subset \mathbb{R}_{\geq 0}$ for some given parametric family f_κ denote a set of parameters where $\mathcal{A}(f_\kappa)$ switches from genus zero to one. Then we say f_κ *switches the last time* from genus zero to one at $\kappa^* = \max K$. In the following we are in particular interested in the corresponding point $\mathrm{app}(f_{\kappa^*})$ where the inner complement component *finally appears* and which we denote as $\mathbf{a}(f_\kappa)$. Note that this maximum always exists since K is bounded from above (e.g., due to lopsidedness condition).

Let M_Δ^j be the matrix obtained by replacing the j-th column of M_Δ by y. For convenience of notation we define

$$\Theta \;=\; \prod_{i=1}^{n} b_i^{\det\left(M_\Delta^i\right)/\det(M_\Delta)}. \tag{4.5}$$

With the results of the lemmata we are able to establish the main theorem of this section.

Theorem 4.8. *Let $n \geq 2$, let f_κ be a parametric family of the form (4.2) in $\mathcal{P}_\Delta^y$ with $\alpha(0) = 0$, $b_0 = 1$, and let Θ be defined by (4.5).*

(a) *For $\kappa = |\Theta|$ we have $\mathrm{eq}(y) = \mathrm{eq}(0) = \cdots = \mathrm{eq}(n)$. Hence, in particular, $\mathcal{A}(f_\kappa)$ is solid for all choices of $\arg(c)$ whenever $\kappa \leq |\Theta|$.*

(b) *For* $\kappa > (n+1) \cdot |\Theta|$ *we have* $\text{eq}(y) \notin \mathcal{A}(f_\kappa)$ *and hence* $\mathcal{A}(f_\kappa)$ *has genus one. If additionally* $\arg(c)$ *is in extreme opposition and the inner complement component* $E_y(f_\kappa)$ *appears finally at the point* $\text{eq}(y)$ *then this bound is sharp, i.e.,* $\text{eq}(y) \in \mathcal{A}(f_{(n+1)\cdot|\Theta|})$.

Note that the question *if* the inner complement component appears finally at $\text{eq}(y)$ will be discussed in the next section.

PROOF. As initial preparation, we note that for $f \in \mathcal{P}_\Delta^y$ and any $\mathbf{z} \in (\mathbb{C}^*)^n$ with $\text{Log}(\mathbf{z}) = \text{eq}(y)$ we have $|m_y(\mathbf{z})| = |c|/|\Theta|$. Namely, by Lemma 4.4 we have

$$|m_y(\mathbf{z})| \;=\; |c| \cdot e^{\langle \text{eq}(y), y\rangle} \;=\; |c| \cdot \exp\left(-\left\langle \left(M_\Delta^t\right)^{-1} \cdot \text{Log}(b), y\right\rangle\right)$$

and the claim follows with Cramer's rule.

(a) Let $\mathbf{z} \in (\mathbb{C}^*)^n$ with $\text{Log}(\mathbf{z}) = \text{eq}(y)$. By Lemma 4.4 we have $|m_i(\mathbf{z})| = 1$ for all $i \in \{0, \ldots, n\}$. If $\kappa = |\Theta|$ we have $|m_y(\mathbf{z})| = 1$ as well due to initial calculation. Hence, by definition of $\text{eq}(y)$ and of the $\text{eq}(k)$, all equilibrium points coincide. The solidness of $\mathcal{A}(f_\kappa)$ for such κ follows from Lemma 4.6.

(b) Assume $\text{eq}(y) \in \mathcal{A}(f_\kappa)$ for some $\kappa > (n+1) \cdot |\Theta|$. Then there exists a point $\mathbf{z} \in (\mathbb{C}^*)^n$ with $\text{Log}(\mathbf{z}) = \text{eq}(y)$ and $f_\kappa(\mathbf{z}) = 0$. By the definition of $\text{eq}(y)$ and our initial calculation, we have $|m_y(\mathbf{z})| = \kappa/|\Theta|$ and $|m_i(\mathbf{z})| = 1$, and thus

$$\frac{\kappa}{|\Theta|} \cdot e^{i\cdot(\arg(c) + \langle\phi, y\rangle)} + 1 + \sum_{j=1}^{n} e^{i\cdot(\arg(b_j) + \langle\phi, \alpha(j)\rangle)} \;=\; 0. \tag{4.6}$$

But since each exponential term has modulus 1, this implies $\kappa \leq |\Theta| \cdot (n+1)$, contradicting the precondition.

Since $\text{eq}(y) \in \text{conv}\{\text{eq}(0), \ldots, \text{eq}(n)\}$ (Lemma 4.6 (b)), the fact $\text{eq}(y) \notin \mathcal{A}(f)$ already implies $\text{eq}(y) \in E_y(f)$, and thus $E_y(f) \neq \emptyset$.

Recall the definition of the fiber function $f^{|\mathbf{z}|}$ from Chapter 3, Section 1. Assume now that the inner complement component $E_y(f_\kappa)$ appears finally at $\text{eq}(y)$. It suffices to show that $\text{eq}(y) \in \mathcal{A}(f_{(n+1)|\Theta|})$. If $\arg(c)$ is in extreme opposition then (by definition of an extremal phase) there exists a $\phi \in [0, 2\pi)^n$ satisfying (4.6) with $\arg(c) + \langle\phi, y\rangle = \pi + \arg(b_j) + \langle\phi, \alpha(j)\rangle$. Hence, on the fiber $\mathbb{F}_{\text{eq}(y)}$ of $\text{eq}(y)$ at ϕ the (κ depending) fiber function attains $f^{|\text{Log}^{-1}(\text{eq}(y))|}(\phi) = -\kappa + (n+1)|\Theta|$ and we have $\text{eq}(y) \in \mathcal{A}(f_{(n+1)|\Theta|})$ for our choice of κ. □

Theorem 4.8 yields the following corollary, which solves the Problem 3.4 for the special class $\mathcal{P}_\Delta^y$. Note that this is a special case of the (already proven) Theorem 3.9.

Corollary 4.9. *Maximally sparse polynomials with simplex Newton polytope have solid amoebas.*

PROOF. For $n = 1$, the amoeba $\mathcal{A}(f)$ of a maximally sparse polynomial f is a single point. For $n \geq 2$ and f_κ of the form (4.2), Theorem 4.8 (a) yields that $\mathcal{A}(f_\kappa)$ is solid for all $\kappa \leq |\Theta|$. Since $|\Theta| > 0$, $\mathcal{A}(f_\kappa)$ is in particular solid for $\kappa = 0$, i.e., if f is maximally sparse. □

1.2. Points of Appearance of the Inner Complement Component and Sharp Bounds. In the previous section we gave a lower and an upper bound for $\mathcal{A}(f)$ having genus zero respectively one via investigating the fiber function $f^{|\operatorname{Log}^{-1}(\operatorname{eq}(y))|}$. We have seen that if the inner complement component appears finally at $\operatorname{eq}(y)$ (and $\arg(c)$ is in extreme opposition), then the upper bound gets sharp. In this section we investigate in general where the complement component appears finally and how this point is related to $\operatorname{eq}(y)$. Based on this investigation, we provide lower and upper bounds partially improving Theorem 4.8 (see a comparison at the end of the section). We show that, under some extremal condition, the upper bound is tight and the inner complement component appears finally at a unique, explicitly computable minimum $\mathbf{a}(f_\kappa)$, which happens to coincide with $\operatorname{eq}(y)$ if and only if the inner lattice point is the barycenter of the Newton polytope (Theorems 4.10, 4.13 and Corollary 4.12).

As before, let Δ be a lattice n-simplex and y be in the interior of Δ. Again, we consider the parametric family f_κ as introduced in (4.2). In the first statement we assume that $y = 0$.

Theorem 4.10. *Let $n \geq 2$ and f_κ be a parametric family of polynomials in $\mathcal{P}_\Delta^0$ with $f_\kappa = \kappa \cdot e^{i\cdot\arg(c)} + \sum_{i=0}^n m_i(\mathbf{z}) = \kappa \cdot e^{i\cdot\arg(c)} + \sum_{i=0}^n b_i \cdot \mathbf{z}^{\alpha(i)}$. Let $\operatorname{Log}(\mathbf{z}) = \mathbf{w} \in \mathbb{R}^n$ and assume that $|m_0\left(\operatorname{Log}^{-1}(\mathbf{w})\right)| \leq \cdots \leq |m_n\left(\operatorname{Log}^{-1}(\mathbf{w})\right)|$. Then there exists a $\kappa \in \mathbb{R}_{>0}$ such that*

$$\kappa \;\geq\; \sum_{j=2}^n |m_j\left(\operatorname{Log}^{-1}(\mathbf{w})\right)| \quad \textit{and} \quad \mathbf{w} \notin E_y(f_\kappa).$$

Proof. Since $\alpha(0), \ldots, \alpha(n)$ form a simplex, there is a dual basis $\alpha(1)^*, \ldots, \alpha(n)^* \in \mathbb{Q}^n$ with $\langle \alpha(j)^*, \alpha(k)\rangle = 0$ for all $k \notin \{j, 0\}$. An outline of the proof is that we choose $\lambda_1, \ldots, \lambda_n \in [0, 2\pi)$ such that for $\phi = \sum_{j=1}^n \lambda_j \alpha(j)^*$ and $\operatorname{Log}(\mathbf{z}) = \mathbf{w}$ the κ depending fiber functions satisfy $f_\kappa^{|\mathbf{z}|}(\phi) = 0$ for some $\kappa \in \mathbb{R}_{>0}$ sufficiently large.

We can choose $\lambda_2, \ldots, \lambda_n \in [0, 2\pi)$ with

$$e^{i\cdot(\arg(b_j) + \langle \lambda_j \alpha(j)^*, \alpha(j)\rangle)} \;=\; \arg(c) + \pi \text{ for all } j \in \{2, \ldots, n\}.$$

We may finally choose $\lambda_1 \in [0, 2\pi)$ such that the sum of the two shortest monomials

$$|m_0\left(\mathbf{z}\right)| \cdot e^{i\cdot\left(\arg(b_0) + \sum_{j=1}^n \langle \lambda_j \alpha(j)^*, \alpha(0)\rangle\right)} + |m_1\left(\mathbf{z}\right)| \cdot e^{i\cdot(\arg(b_1) + \langle \lambda_1 \alpha(1)^*, \alpha(1)\rangle)}$$

is either zero or a complex number with argument $\arg(c) + \pi$, due to the following Rouché-type principle from complex analysis. Recall that the winding number of a closed curve γ in the complex plane around a point z is given by $\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}$ (see e.g., [**32**]).

Claim. For $A, B \in \mathbb{C}$ with $A > B$ and $r, s \geq 1$ the function $g(\phi) = A \cdot e^{i\cdot r\phi} + B \cdot e^{i\cdot s\phi}$ with $\phi \in [0, 2\pi)$ has a non-zero winding number with respect to the origin.

Clearly, the function $A \cdot e^{i\cdot r\phi}$ has a non-zero winding number. Now, assuming that g has a winding number of zero, there would exist some $t \in (0, 1)$ such that $h(\phi) = A \cdot e^{i\cdot r\phi} + t \cdot B \cdot e^{i\cdot s\phi}$ has a zero ϕ outside the origin. This is a contradiction.

Altogether, for $\phi = \sum_{j=1}^{n} \lambda_i \cdot \alpha(j)^*$, we get $f_\kappa^{|\mathbf{z}|}(\phi) = (\kappa - \sum_{j=2}^{n} |m_j(\mathbf{z})| + \xi) \cdot e^{i \cdot \arg(c)}$ with $\xi \in \mathbb{R}_{<0}$ for $|m_1(\mathbf{z})| > |m_0(\mathbf{z})|$ and hence $\xi \in \mathbb{R}_{\leq 0}$ for $|m_1(|\mathbf{z}|)| = |m_0(\mathbf{z})|$. Thus, we have $f_\kappa^{|\mathbf{z}|}(\phi) = 0$ for $\kappa = |\xi| + \sum_{j=2}^{n} |m_j(\mathbf{z})|$. This yields $\operatorname{Log}(\mathbf{z}) \notin E_y(f_\kappa)$ for such choice of κ. □

Our goal is to characterize the κ, for which the amoeba $\mathcal{A}(f_\kappa)$ switches the last time from genus zero to genus one. First, we consider the case of $\arg(c)$ in extreme opposition and then we use this case to provide a bound for the general case.

Let $\arg(c)$ be in extreme opposition for f_κ (note that this property is independent of the choice of κ). For a point $\mathbf{z} \in (\mathbb{C}^*)^n$ with $\operatorname{Log}(\mathbf{z}) = \mathbf{w}$, the κ depending fiber function $f_\kappa^{|\mathbf{z}|}$ on the fiber $\mathbb{F}_\mathbf{w}$ of $\mathbf{w}$ evaluates for an extremal phase ϕ to

$$f_\kappa^{|\mathbf{z}|}(\phi) \quad = \quad \left(\kappa \cdot e^{\langle \mathbf{w}, y\rangle} - 1 - \sum_{j=1}^{n} |b_j| \cdot e^{\langle \mathbf{w}, \alpha(j)\rangle}\right) \cdot e^{i \cdot \psi}$$

for some angle $\psi \in [0, 2\pi)$. Since we are only interested in the zeros of $f_\kappa^{|\mathbf{z}|}$, we can always assume $\psi = 0$. Clearly, $\mathbf{w} \in E_y(f_\kappa)$ whenever $\kappa \cdot e^{\langle \mathbf{w}, y\rangle} > 1 + \sum_{j=1}^{n} |b_j| \cdot e^{\langle \mathbf{w}, \alpha(j)\rangle}$.

Since an extremal phase ϕ yields the minimal real value of f on a fiber $\mathbb{F}_\mathbf{w}$ and since $\mathcal{A}(f_\kappa)$ has genus one if $E_y(f_\kappa) \neq \emptyset$, the κ^* where $\mathcal{A}(f_\kappa)$ switches its genus the last time is given by

$$\min_{\mathbf{w} \in \mathbb{R}^n} \left(e^{-\langle \mathbf{w}, y\rangle} + \sum_{j=1}^{n} |b_j| \cdot e^{\langle \mathbf{w}, \alpha(j) - y\rangle}\right) \in \mathbb{R}_{>0}. \tag{4.7}$$

The minimizer $\mathbf{w}^*$ then has to be the point $\mathbf{a}(f_\kappa)$ where the inner complement component finally appears for $\arg(c)$ in extreme opposition, since $\mathbf{w}^* \notin E_y(f_{\kappa^*})$, $\mathbf{w}^* \in E_y(f_\kappa)$ for all $\kappa > \kappa^*$ and for all $\mathbf{w} \neq \mathbf{w}^*$ there is a $\kappa > \kappa^*$ such that $\mathbf{w} \notin E_y(f_\kappa)$. Notice that the location of the point $\mathbf{a}(f_\kappa)$ does not depend on κ (the notation might be sort of misleading here).

In the following we set $\widehat{M_\Delta} = (\alpha(j)_i - y_i)_{1 \leq i,j \leq n}$ and $\widehat{M_\Delta^j}$ as the matrix obtained by replacing the j-th column of $\widehat{M_\Delta}$ by y.

Lemma 4.11. *Let $\alpha(0) = 0$, $b_0 = 1$, and $\arg(c)$ be in extreme opposition for f_κ. The point $\mathbf{a}(f_\kappa)$ where the inner complement finally appears is given by $\operatorname{eq}(y) + \mathbf{s}^*$, where $\mathbf{s}^*$ is the solution of the system of linear equations*

$$M_\Delta^t \cdot \mathbf{s} \quad = \quad (\gamma_1, \ldots, \gamma_n)^t \tag{4.8}$$

with $\gamma_j = \log\left(\det(\widehat{M_\Delta^j}) / \det(\widehat{M_\Delta})\right)$ for $j \in \{1, \ldots, n\}$.

Proof. It suffices to show that $\operatorname{eq}(y) + \mathbf{s}^*$ solves the Problem (4.7). Substituting $\mathbf{w} = \operatorname{eq}(y) + \mathbf{s}$ into (4.7) and applying Lemma 4.4 and Theorem 4.8 simplifies the problem to

$$|\Theta| \cdot \min_{\mathbf{s} \in \mathbb{R}^n} \left(e^{-\langle \mathbf{s}, y\rangle} + \sum_{j=1}^{n} e^{\langle \mathbf{s}, \alpha(j) - y\rangle}\right).$$

To compute the global minimum of $e^{-\langle \mathbf{s},y\rangle} + \sum_{j=1}^n e^{\langle \mathbf{s},\alpha(j)-y\rangle}$ we observe that the partial derivatives

$$\frac{\partial f_\kappa}{\partial s_i} = -y_i \cdot e^{-\langle \mathbf{s},y\rangle} + \sum_{j=1}^n (\alpha(j)_i - y_i) \cdot e^{\langle \mathbf{s},\alpha(j)-y\rangle}$$

vanish if and only if $\sum_{j=1}^n (\alpha(j)_i - y_i) \cdot e^{\langle \mathbf{s},\alpha(j)\rangle} = y_i$ for all $i \in \{1,\ldots,n\}$. We obtain $\widehat{M_\Delta} \cdot \left(e^{\langle \mathbf{s},\alpha(1)\rangle},\ldots,e^{\langle \mathbf{s},\alpha(n)\rangle}\right)^t = y$, and hence $e^{\langle \mathbf{s},\alpha(j)\rangle} = \det(\widehat{M^j_\Delta})/\det(\widehat{M_\Delta})$ for $j \in \{1,\ldots,n\}$. Setting $\gamma_j = \log\det(\widehat{M^j_\Delta}) - \log\det(\widehat{M_\Delta}) > 0$ yields $\langle \mathbf{s},\alpha(j)\rangle = \gamma_j$. Thus, we obtain a system of linear equations (4.8). Since its solution is unique and $\lim_{|\mathbf{s}|\to\infty} f(\mathbf{s}) = \infty$ this critical point has to be a minimum. □

Note that, by Lemma 4.4 and 4.11, the point $\mathbf{a}(f_\kappa)$ is the solution of the system of linear equations given by

$$M^t_\Delta \cdot \mathbf{x} = (\gamma_1 - \log|b_1|,\ldots,\gamma_n - \log|b_n|)^t. \tag{4.9}$$

Hence, $\mathbf{a}(f_\kappa)$ can be computed explicitly in terms of the coefficients and exponents of f.

Corollary 4.12. *Let* $\arg(c)$ *be in extreme opposition for* f_κ*. The point* $\mathbf{a}(f_\kappa)$*, where the inner complement component appears finally, coincides with the equilibrium point* $\mathrm{eq}(0)$ *if and only if* y *is the barycenter of* Δ*, i.e., if and only if*

$$\sum_{j=1}^n \alpha(j) = (n+1)\cdot y.$$

Proof. Since $b_0 \in \mathbb{C}^*$ and $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$ we may assume $\alpha(0) = 0$, $b_0 = 1$ (otherwise divide f by $b_0 \cdot \mathbf{z}^{\alpha(0)}$). Then the result follows from $\sum_{j=1}^n (\alpha(j)_i - y_i) \cdot e^{\langle \mathbf{s},\alpha(j)\rangle} = y_i$ for all $i \in \{1,\ldots,n\}$. □

With these statements we can prove the main theorem of this section.

Theorem 4.13. *Let* f_κ *be a parametric family of polynomials in* $\mathcal{P}^y_\Delta$ *of the Form* (4.2) *with* $\alpha(0) = 0$, $b_0 = 1$*, let* $\arg(c)$ *be in extreme opposition and set*

$$\widehat{\Theta} = \prod_{j=1}^n \left(\frac{\det(\widehat{M_\Delta})\cdot b_j}{\det\left(\widehat{M^j_\Delta}\right)}\right)^{\det\left(M^j_\Delta\right)/\det(M_\Delta)}. \tag{4.10}$$

$\mathcal{A}(f_\kappa)$ *switches the last time from genus zero to one at*

$$\kappa = |\widehat{\Theta}| \cdot \left(1 + \sum_{j=1}^n \frac{\det\left(\widehat{M^j_\Delta}\right)}{\det(\widehat{M_\Delta})}\right). \tag{4.11}$$

For all other choices of $\arg(c)$ *we have: If* $\mathcal{A}(f_\kappa)$ *is solid, then* κ *is strictly bounded from above by the right hand side of* (4.11).

Proof. Let $\arg(c)$ be in extreme opposition. By Lemma 4.11 it is easy to verify that for an extremal phase $\phi' \in [0, 2\pi)^n$ we have $e^{-\langle \mathbf{a}(f_\kappa), y\rangle} \cdot e^{i \cdot \langle \phi', y\rangle} = \widehat{\Theta}$. We know that $\mathcal{A}(f_\kappa)$ switches the last time from genus zero to one at

$$\kappa^* = \min_{s \in \mathrm{Log}((\mathbb{C}^*)^n)} \left(e^{-\langle \mathrm{eq}(y)+s, y\rangle} + \sum_{j=1}^{n} |b_j| \cdot e^{\langle \mathrm{eq}(y)+s, \alpha(j)-y\rangle} \right).$$

Due to above calculation of $\widehat{\Theta}$ and (4.9) this is equivalent to (4.11).

Let $\arg(c)$ be not in extreme opposition. By the upper argumentation we have $E_y(f_\kappa) = \emptyset$ if and only if $\mathcal{V}(f_\kappa^{|\mathrm{Log}^{-1}(\mathbf{a}(f_\kappa))|}) \neq \emptyset$. Let $\phi \in [0, 2\pi)^n$ be a zero of $f_\kappa^{|\mathrm{Log}^{-1}(\mathbf{a}(f_\kappa))|}$. Since $\arg(c)$ is not in extreme opposition, not all outer monomial have the same argument at $f_\kappa^{|\mathrm{Log}^{-1}(\mathbf{a}(f_\kappa))|}(\phi)$ and therefore $|f_\kappa^{|\mathrm{Log}^{-1}(\mathbf{a}(f_\kappa))|}(\phi)| < |\widehat{\Theta}| \cdot \left(1 + \sum_{j=1}^{n} \frac{\det(\widehat{M}_\Delta^j)}{\det(\widehat{M}_\Delta)}\right)$. □

It follows from the argumentation above that the upper bound, which we computed in Theorem 4.13, which guarantees amoebas of polynomials in $\mathcal{P}_\Delta^y$ to be solid, improves the upper bound from Theorem 4.8 (b) in all cases but the one in Corollary 4.12. The reason is basically that we minimize over all $\mathbb{R}^n$ instead of just choosing the equilibrium point.

For the lower bound computed in Theorem 4.10 notice that it holds for all κ, and hence improves the lower bound from Theorem 4.8 (a), if there exists only one κ such that $f_\kappa \in \partial U_y^A$ (i.e., if the genus switches only once from zero to one for κ running from 0 to ∞). The question if this is the case is closely related to the question whether the set U_y^A is connected, which was (for general polynomials) introduced as Problem 2.22 in Chapter 2, Section 4.

1.3. Connection to Lopsidedness and A-discriminants. In the following section we investigate the configuration space $\mathcal{P}_\Delta^y$ for polynomials in our class of interest from two other points of view, namely, lopsidedness and A-discriminants.

Recall the definition of lopsidedness and the main results about it, which we introduced in Chapter 2, Section 5. As before, let $A \subset \mathbb{Z}^n$ be an arbitrary support set, let $\mathbb{C}_\Diamond^A$ be the corresponding augmented configuration space (see Chapter 2 Section 4 and Chapter 3, Section 2), and let U_α^A be the set of polynomials $f \in \mathbb{C}_\Diamond^A$ whose amoeba has a complement component of order α.

For $f \in \mathbb{C}_\Diamond^A$ let furthermore $\mathbb{T}(f)$ denote the real d-torus of polynomials in $\mathbb{C}_\Diamond^A$, whose coefficients have the same absolute values as the coefficients of f, i.e., for $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha$ we have

$$\mathbb{T}(f) = \left\{ \sum_{\alpha \in A} e^{i \cdot \psi_\alpha} \cdot b_\alpha \mathbf{z}^\alpha \ : \ \psi_\alpha \in [0, 2\pi) \text{ for all } \alpha \in A \right\}.$$

It is an easy consequence of the definition of lopsidedness that the following proposition holds (which is, to the best of our knowledge, surprisingly nowhere mentioned in the literature).

Proposition 4.14. *Let $A \subset \mathbb{Z}^n$ and $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha$. Assume that $E_{\alpha'}(f)$ is non-empty for some $\alpha' \in \operatorname{conv}(A) \cap \mathbb{Z}^n$ and that there exists some $\mathbf{w} \in E_{\alpha'}(f)$ such that $f\{\mathbf{w}\}$ is lopsided. Then $g\{\mathbf{w}\}$ is lopsided for every $g \in \mathbb{T}(f)$. In particular $\mathbb{T}(f) \subset U_{\alpha'}^A$.*

Proof. Since $g\{\mathbf{w}\} = f\{\mathbf{w}\}$ for every $g \in \mathbb{T}(f)$, for every $\mathbf{w} \in E_{\alpha'}(f)$ with $f\{\mathbf{w}\}$ lopsided we have $g\{\mathbf{w}\}$ lopsided as well. Thus $E_{\alpha'}(g) \neq \emptyset$ and hence $g \in U_{\alpha'}^A$. □

The proposition basically says that, if one has a polynomial f whose amoeba has a complement component $E_\alpha(f)$ and f is lopsided at some point $\mathbf{w} \in E_\alpha(f)$, then for *every* polynomial g in the torus $\mathbb{T}(f)$ g is lopsided at $\mathbf{w}$ and hence in particular, $E_\alpha(g) \neq \emptyset$. Since the lopsidedness property is preserved for an increasement of $|b_{\alpha'}|$, the proposition yields that every U_α^A, although it might be not connected, contains a special, say, "main" connectivity component.

The following Theorem 4.15 shows that for polynomials in $\mathcal{P}_\Delta^y$ the converse of Proposition 4.14 is also true. In this statement it is convenient to have the origin as the interior lattice point. Hence, we set $A = \{\alpha(0), \ldots, \alpha(n), 0\}$. We may always assume this since we can divide f by $\mathbf{z}^y$ due to $\mathcal{V}(f) \subset (\mathbb{C}^*)^n$.

Theorem 4.15. *Let $f_c = c + \sum_{j=0}^n b_j \mathbf{z}^{\alpha(j)} = c + \sum_{j=0}^n m_j(\mathbf{z})$ be a parametric family in $\mathcal{P}_\Delta^0$ with complex parameter $c \in \mathbb{C}$, and let $\mathbf{a} = \mathbf{a}(f_{|c|})$ be the point where the inner complement component appears finally for positive real parameter values $|c|$ and $\arg(c)$ in extreme opposition. If there exists some $d \in \mathbb{C}^*$ such that $\mathbb{T}(f_d) \subset U_0^A$, then $f_d\{\mathbf{a}\}$ is lopsided with $|d|$ as the maximal term.*

Proof. Let $d \in \mathbb{C}^*$ with $\mathbb{T}(f_d) \subset U_0^A$. First we show that for every $c \in \mathbb{C}$ with $|c| \geq |d|$ the amoeba $\mathcal{A}(f_c)$ is of genus one.

The parametric family f_c forms a complex line in $\mathcal{P}_\Delta^0$, which can be interpreted as a real plane H. By Theorem 2.21 the intersection of $(U_\alpha^A)^c$ with an arbitrary projective line in $\mathbb{C}_\Diamond^A$ (viewed as a projective space) is non-empty and connected (even for arbitrary A). For the parameter value $c = 0$ we are in the maximally sparse case, and thus Corollary 4.9 implies $f_0 \in (U_0^A)^c$. By the precondition $\mathbb{T}(f_d) \subset U_0^A$, the set $C = \{f_c : c = |d| \cdot e^{i \cdot \phi}, \phi \in [0, 2\pi)\} \subset \mathbb{T}(f_d)$ is contained in U_0^A. Considered in the plane H, the set C is a circle around the origin. Now the connectedness result implies that for $|c| \geq |d|$ the amoeba $\mathcal{A}(f_c)$ is of genus one (see Figure 4.3 for an illustration).

For $\arg(c)$ in extreme opposition, let $\kappa^* \in \mathbb{R}$ be the value where $\mathcal{A}(f_{|c|})$ switches the last time from genus zero to one. By Theorem 4.13, the upper bound is attained at some point $\mathbf{z} \in (\mathbb{C}^*)^n$ with $\operatorname{Log}(\mathbf{z}) = \mathbf{a}$ and extremal phase ϕ. Hence, by evaluating the fiber function $f^{|\mathbf{z}|}$ on the fiber $\mathbb{F}_\mathbf{a}$ at ϕ we obtain $\kappa^* = \sum_{j=0}^n |b_j| \cdot e^{\langle \mathbf{a}, \alpha(j) \rangle}$. The auxiliary statement yields that $\kappa^* < |d|$, and thus $|d| > \sum_{j=0}^n |b_j| \cdot e^{\langle \mathbf{a}, \alpha(j) \rangle} = \sum_{j=0}^n \left| m_j \left(\operatorname{Log}^{-1}(\mathbf{a}) \right) \right|$. □

The same argumentation yields some nice local, structural consequences on $\mathbb{C}^A$ for general support sets.

Corollary 4.16. *Let $A \subset \mathbb{Z}^n$ with $0 \in A$ and let $f_c = c + \sum_{\alpha \in A \setminus \{0\}} b_\alpha \mathbf{z}^\alpha$ be a parametric family. Let $c_1, c_2 \in \mathbb{C}$ be parameters with $|c_1| < |c_2|$ such that $f_{c_1} \notin U_0^A$. If there exists an $\kappa \in (|c_1|, |c_2|) \subset \mathbb{R}$ such that for $C = \{f_c \in \mathbb{C}^A \ : \ |c| = \kappa\}$ holds $C \subset U_0^A$, then $f_{c_2} \in U_0^A$.*

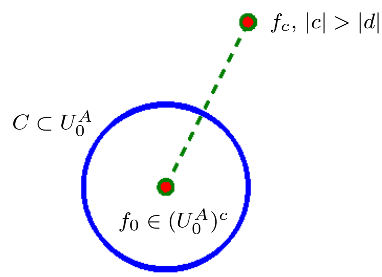


FIGURE 4.3. The real plane H in the proof of Theorem 4.15.

The Corollary can be regarded as a local refinement of Proposition 4.14 on $\mathbb{C}^A$, avoiding the explict use of the lopsidedness condition.

PROOF. Follows immediately from the argument in the first part of the proof of Theorem 4.15 (see Figure 4.3). □

We recall some of the terminology for A-*discriminants*. For a support set $A \subset \mathbb{Z}^n$ let $\nabla_0 \subset \mathbb{C}^A$ denote the set of all polynomials f such that there exists a point $\mathbf{z}^* \in (\mathbb{C}^*)^n$ with

$$f(\mathbf{z}^*) \;=\; \frac{\partial f}{\partial z_1}(\mathbf{z}^*) \;=\; \cdots \;=\; \frac{\partial f}{\partial z_n}(\mathbf{z}^*) \;=\; 0$$

and let ∇_A denote the Zariski closure of ∇_0. If the variety ∇_A is of codimension one, then the A-*discriminant* Δ_A is defined as the irreducible, integral polynomial in the coefficients $b_1, \ldots, b_d$ of $f \in \mathbb{C}^A$ as variables, which vanishes on ∇_A. The A-discriminant is unique up to sign (see [**23**, Chapter 9, p. 271]).

The following theorem shows that, for polynomials in $\mathcal{P}_\Delta^y$, there is a strong connection between their A-discriminants and the topology of their amoebas. Here, $\overline{U_y^A}$ denotes the topological closure of the set U_y^A. Set $A = \{\alpha(0), \ldots, \alpha(n), y\}$.

Theorem 4.17. *Let $\alpha(0) = 0$, $b_0 = 1$. A polynomial $f = c \cdot \mathbf{z}^y + 1 + \sum_{i=1}^n b_i \cdot \mathbf{z}^{\alpha(i)} \in \mathcal{P}_\Delta^y$ is contained in ∇_A if and only if the expression*

$$c + \widehat{\Theta} \cdot \left(1 + \sum_{j=1}^{n} \frac{\det\left(\widehat{M}_\Delta^j\right)}{\det(\widehat{M}_\Delta)}\right) \tag{4.12}$$

in the variables $b_1, \ldots, b_n, c$ vanishes. Here, $\widehat{\Theta}$ is defined as in (4.10).

Note that a power of the summands of (4.12) is an irreducible binomial with rational coefficients. Up to normalizing the coefficients, this is the A-discriminant.

Corollary 4.18. *Let $\alpha(0) = 0$ and $b_0 = 1$. The A-discriminant Δ_A is a binomial whose variety coincides with the set of projective points* $(1 : b_1 : \ldots : b_n : c)$ *where* $\arg(c)$ *is in* *extreme opposition and* $\mathcal{A}(f_{|c|})$ *switches the last time from genus zero to genus one exactly at the value* $|c| = |\widehat{\Theta}| \cdot (1 + \sum_{j=1}^{n} \det(\widehat{M_{\Delta}^{j}}) / \det(\widehat{M_{\Delta}}))$.

PROOF. (Theorem 4.17) For the given polynomial $f \in \mathcal{P}_{\Delta}^{y}$ we have

$$\frac{\partial f}{\partial z_j} \;=\; y_j \cdot c \cdot \mathbf{z}^{y-e_j} + \sum_{i=1}^{n} b_i \cdot \alpha(i)_j \cdot \mathbf{z}^{\alpha(i)-e_j}\,, \quad 1 \leq j \leq n, \tag{4.13}$$

where e_j denotes the j-th unit vector. Assume that arbitrary $b_1, \ldots, b_n \in \mathbb{C}^*$ are fixed. Substituting f into $\mathbf{z}^{e_j}$ times the expression (4.13) yields a regular system of linear equations in $(\mathbf{z}^{\alpha(1)}, \ldots, \mathbf{z}^{\alpha(n)})$. The regularity comes from the fact that $\alpha(1), \ldots, \alpha(n)$ are the vertices of a simplex. Hence, there are only finitely many solutions $\mathbf{z}^* \in (\mathbb{C}^*)^n$ such that all partial derivatives vanish, and all of these solutions have the same modulus. For any such solution $\mathbf{z}^*$, solving $f = 0$ for c, yields a unique and non-zero c such that the polynomial f corresponding to the coefficients $b_1, \ldots, b_n, c$ is in ∇_A. This argumentation shows furthermore that ∇_A is a subvariety of codimension one and hence Δ_A exists. Observe that $\mathbf{z}^*$ does not depend on c.

Let now $\phi' \in [0, 2\pi)^n$ be an extremal phase. Then $(\mathbf{a}(f_\kappa), \phi') = \mathbf{z}^*$ since we know $\frac{\partial f}{\partial z_j}(\mathbf{a}(f_\kappa), \phi') = 0$ for all $j \in \{1, \ldots, n\}$ from the last section (see the proof of Lemma 4.11). But since further $f_\kappa^{|\operatorname{Log}^{-1}(\mathbf{a}(f_\kappa))|}(\phi') = 0$ if and only if c is in extreme opposition and its modulus equals the bound from Theorem 4.13, we have $f \in \nabla_A$ if and only if (4.12) vanishes. ☐

PROOF. (Corollary 4.18) Expression (4.12) is a Laurent binomial in the variables $b_1, \ldots, b_n, c$ with rational coefficients and monomials in distinct variables. Now the statement follows from Theorem 4.17 via Theorem 4.13. ☐

We remark that a different connection between A-discriminants and amoebas was investigated by Passare, Sadykov and Tsikh in [**64**] who studied the amoebas of A-discriminantal hypersurfaces. For further connections between A-discriminants and polynomials in the class $\mathcal{P}_{\Delta}^{y}$, see also [**4, 5**].

1.4. An Example. Since all bounds and formulas are quite overwhelming we apply the results of Section 1 to an example defined in the following, which we discuss in this Section 1.4. Let $A = \{\alpha(0), \alpha(1), \alpha(2), y\} = \{(0,0), (7,1), (1,2), (1,1)\}$ and $\Delta = \operatorname{conv}(A)$ as usual. We investigate $f \in \mathcal{P}_{\Delta}^{y}$ given by

$$f \;=\; 1 + c \cdot z_1 z_2 + z_1^7 z_2 + 2 \cdot i \cdot z_1 z_2^2.$$

We start with the calculation of the equilibrium point $\operatorname{eq}(y)$ (Definition 4.3) and the rough bounds shown in Theorem 4.8. With $M_{\Delta} = (\alpha(j)_i)_{1 \leq i,j \leq n}$ and M_{Δ}^{j} given by replacing the j-th column in M_{Δ} by y we have

$$M_{\Delta} \;=\; \begin{pmatrix} 7 & 1 \\ 1 & 2 \end{pmatrix}, \quad M_{\Delta}^{1} \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad M_{\Delta}^{2} \;=\; \begin{pmatrix} 7 & 1 \\ 1 & 1 \end{pmatrix}$$

$\operatorname{eq}(y)$ is the point in $\mathbb{R}^n$ where all outer monomials are in equilibrium, i.e., attain norm 1 due to the constant term here. Hence, the following system of linear equations needs to be satisfied:

$$\begin{aligned} e^{\langle(7,1),\mathrm{eq}(y)\rangle} &= 1 \\ e^{\langle(1,2),\mathrm{eq}(y)\rangle} &= 1/2 \\ \Leftrightarrow\; 7\,\mathrm{eq}(y)_1 + \mathrm{eq}(y)_2 &= 0 \\ \mathrm{eq}(y)_1 + \mathrm{eq}(y)_2 &= -\log 2 \\ \Rightarrow \mathrm{eq}(y) &= 1/13 \cdot (\log 2, -7\log 2). \end{aligned}$$

Thus, to ensure that the inner monomial is the dominant term on the fiber $f^{|\,\mathrm{Log}^{-1}(\mathrm{eq}(y))|}$ of $\mathrm{eq}(y)$, we need to have

$$\begin{aligned} |c| \cdot \mathrm{eq}(y)_1 \cdot \mathrm{eq}(y)_2 &> 1 \\ \Leftrightarrow\; |c| \cdot e^{1/13 \log 2} \cdot e^{-7/13 \log 2} &> 1 \\ \Leftrightarrow\; |c| &> 2^{6/13} \approx 1.377. \end{aligned}$$

This lower bound coincides, as claimed in Theorem 4.8, with the lower bound given by $|\Theta|$ (see (4.5)) since

$$\Theta = \sqrt[\det(M_\Delta)]{b_1^{\det(M_\Delta^1)} \cdot b_2^{\det(M_\Delta^2)}} = \sqrt[13]{1^1 \cdot 2^6}.$$

The same theorem states that we have an upper bound for $\mathcal{A}(f)$ to be solid, which is given by

$$|c| \le (n+1) \cdot |\Theta| \approx 4.131.$$

Now, we compute the improved upper bound via $\widehat{\Theta}$. With $\widehat{M_\Delta} = (\alpha(j)_i - y_i)_{1 \le i,j \le n}$ and $\widehat{M_\Delta^j}$ given by replacing the j-th column in $\widehat{M_\Delta}$ by y we have we have

$$\widehat{M_\Delta} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}, \quad \widehat{M_\Delta^1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \widehat{M_\Delta^2} = \begin{pmatrix} 6 & 0 \\ 1 & 1 \end{pmatrix}$$

By Lemma 4.11 the point $\mathbf{a}(f_\kappa)$, where the inner complement component finally appears, is given by

$$\min_{\mathbf{w} \in \mathrm{Log}((\mathbb{C}^*)^2)} \left(e^{\langle -y, \mathbf{w}\rangle} + e^{\langle \alpha(1)-y, \mathbf{w}\rangle} + 2 \cdot e^{\langle \alpha(2)-y, \mathbf{w}\rangle} \right).$$

Since we define $\mathbf{s} = \mathbf{w} - \mathrm{eq}(y)$ and since $e^{\langle -y, \mathrm{eq}(y)\rangle} = \Theta$, we obtain

$$|\Theta| \cdot \widehat{f} = |\Theta| \cdot \left(e^{\langle(-1,-1),\mathbf{s}\rangle} + e^{\langle(6,0),\mathbf{s}\rangle} + 2 \cdot e^{\langle(0,1),\mathbf{s}\rangle} \right).$$

We compute

$$\begin{aligned} \frac{\partial \widehat{f}}{\partial s_1} &= -1 \cdot e^{\langle(-1,-1),\mathbf{s}\rangle} + 6 \cdot e^{\langle(6,0),\mathbf{s}\rangle}, \\ \frac{\partial \widehat{f}}{\partial s_2} &= -1 \cdot e^{\langle(-1,-1),\mathbf{s}\rangle} + 1 \cdot e^{\langle(0,1),\mathbf{s}\rangle}. \end{aligned}$$

Thus, for $\mathbf{s}^*$ to be a global minimum, i.e., both partial derivatives to vanish, we need

$$6 \cdot e^{\langle(7,1),\mathbf{s}^*\rangle} = 1, \qquad 1 \cdot e^{\langle(1,2),\mathbf{s}^*\rangle} = 1.$$

This yields

$$\begin{aligned} 7s_1^* + s_2^* &= -\log 6 \\ s_1^* + 2s_2^* &= 0 \\ \Rightarrow \mathbf{s}^* &= 1/13 \cdot (-2\log 6, \log 6). \end{aligned}$$

Hence, the global minimum is attained at

$$\mathbf{a}(f_\kappa) \;=\; \mathbf{s}^* + \mathrm{eq}(y) \;=\; 1/13 \cdot (-2\log 6 + \log 2, \log 6 - 7\log 2).$$

We compute $|\widehat{\Theta}|$:

$$|\widehat{\Theta}| \;=\; e^{-\langle \mathbf{a}(f_\kappa), y\rangle} = e^{1/13\cdot(1\cdot(2\log 6 - \log 2) + 1\cdot(-\log 6 + 7\log 2))} \;=\; \sqrt[13]{6\cdot 2^6} \;\approx\; 1.5805.$$

The upper bound is given by the assumption that there exists an argument in the fiber $f^{|\operatorname{Log}^{-1}(\mathbf{a}(f_\kappa))|}$ of the global minimum (i.e., the point, where the inner complement component finally appears) $\mathbf{a}(f_\kappa)$ under the Log-map such that the inner monomial is in extreme opposition (see Section 1.2). If this is the case, then we have

$$\begin{aligned} |c| &= |\widehat{\Theta}| \cdot \left(1 + e^{\langle \alpha(1), \mathbf{s}^*\rangle} + e^{\langle \alpha(2), \mathbf{s}^*\rangle}\right) \\ &= \sqrt[13]{6\cdot 2^6} \cdot \left(1 + e^{\langle (7,1), 1/13\cdot(-2\log 6, \log 6)\rangle} + e^{\langle (1,2), 1/13\cdot(-2\log 6, \log 6)\rangle}\right) \\ &= \sqrt[13]{6\cdot 2^6} \cdot (1 + 1/6 + 1) \;\approx\; 3.4244. \end{aligned}$$

Furthermore, the final change of the genus (i.e., the appearance of an inner complement that does not vanish for increasing $|c|$) may due to Theorem 4.10 not happen before

$$|c| \;=\; \sqrt[13]{6\cdot 2^6} \cdot (1 - 1/6 + 1) \;\approx\; 2.8976.$$

Notice that also in this case we obtain the correct upper bound by using the formula given in Theorem 4.13

$$\widehat{\Theta} \;=\; \prod_{j=1}^{2} \left(\frac{\det(\widehat{M_\Delta}) \cdot b_j}{\det\left(\widehat{M_\Delta^j}\right)} \right)^{\det\left(M_\Delta^j\right)/\det(M_\Delta)} \;=\; \left(\frac{1\cdot 6}{1}\right)^1 \cdot \left(\frac{2\cdot i\cdot 6}{6}\right)^{6/13} \;=\; \sqrt[13]{6\cdot(2\cdot i)^6}$$

and (the bound):

$$|c| \;>\; |\widehat{\Theta}| \cdot \left(1 + \sum_{j=1}^{2} \frac{\det\left(\widehat{M_\Delta^j}\right)}{\det(\widehat{M_\Delta})} \right) \;=\; \sqrt[13]{6\cdot 2^6} \cdot (1 + 1/6 + 1).$$

Finally, we compute the corresponding A-discriminant for the case $1 + c\cdot z_1 z_2 + b_1 z_1^7 z_2 + b_2 \cdot z_1 z_2^2$. Due to Theorem 4.17 it is given by the variety of the polynomial $c + \widehat{\Theta} \cdot \left(1 + \sum_{j=1}^{2} \frac{\det\left(\widehat{M_\Delta^j}\right)}{\det(\widehat{M_\Delta})}\right)$ and therefore the A-discriminant is given by

$$\Delta_A(f) \;=\; c^{13} + (13/6)^{13} \cdot 6\, b_1 b_2^6.$$

2. Amoebas of Genus at Most One with Barycentric Simplex Newton Polytope

In this section we treat polynomials in $\mathcal{P}_\Delta^y$ where the exponent of the inner monomial is the barycenter of the simplex spanned by the exponents of the outer monomials. We call such a pair (Δ, y) *barycentric*. For this class we provide a complete classification of the configuration space of amoebas, i.e., the set U_y^A and its complement $(U_y^A)^c$. In particular, we are able to answer Rullgård's question for this barycentric case by showing that set U_y^A is path-connected (Corollary 4.25).

In [**63**, Proposition 2] Passare and Rullgård showed that the amoeba of $f(\mathbf{z}) = 1 + c \cdot z_1 \cdots z_n + \sum_{i=1}^n z_i^{n+1} \in \mathbb{C}[\mathbf{z}]$ has a complement component of order $(1, \ldots, 1)$ if and only if $0 \notin \mathcal{A}(f)$. Moreover, this component exists if and only if $c \notin \{-t_1 - \cdots - t_n : t_i \in \mathbb{C}, |t_i| = 1, t_1 \cdots t_n = 1\}$. We generalize this result as well as our Corollary 4.12 to the following theorem. From now on let $n \geq 2$, $A = \{\alpha(0), \ldots, \alpha(n), y\}$ and $\text{conv}(A) = \Delta$.

Theorem 4.19. *Let (Δ, y) be barycentric, and let f_c be a family of parametric polynomials in $\mathcal{P}_\Delta^y$ with parameter $c \in \mathbb{C}$ (i.e., $|c|$ and $\arg(c)$). Then for every parameter value $c \in \mathbb{C}$ the following statements are equivalent:*

(a) $f_c \in U_y^A$ *(i.e., $\mathcal{A}(f_c)$ has genus one),*

(b) $\text{eq}(y) \in E_y(f_c)$,

(c) $c \notin \left\{ -|\Theta| \cdot \sum_{j=0}^{n} e^{i \cdot (\arg(b_j) + \langle \alpha(j) - y, \phi \rangle)} : \phi \in [0, 2\pi)^n \right\}$.

Proof. Since the inner lattice point y is the barycenter, we have $f_c = c \cdot \mathbf{z}^y + \sum_{j=0}^n b_j \cdot \mathbf{z}^{\alpha(j)}$ and $\sum_{j=0}^n \alpha(j) = (n+1) \cdot y$. As usual, we may assume $b_0 = 1$ and $\alpha(0) = 0$.

(b) $\Leftrightarrow$ (c): Since $\alpha(0), \ldots, \alpha(n)$ form a simplex, the equilibrium point $\text{eq}(y)$ is unique. At $\text{eq}(y)$ we have for the outer monomials $|b_i| \cdot e^{\langle \alpha(i), \text{eq}(y) \rangle} = 1$ (Definition 4.3) and furthermore $e^{\langle y, \text{eq}(y) \rangle} = 1/|\Theta|$ (proof of Theorem 4.8). Hence, at $\text{eq}(y)$ the fiber function is given by

$$f_c^{|\operatorname{Log}^{-1}(\text{eq}(y))|}(\phi) \quad = \quad c \cdot e^{i \cdot \langle y, \phi \rangle} + |\Theta| \cdot \sum_{j=0}^{n} e^{i \cdot (\arg(b_j) + \langle \alpha(j), \phi \rangle)}.$$

Thus, if and only if the condition (c) is satisfied, the variety $\mathcal{V}(f_c^{|\operatorname{Log}^{-1}(\text{eq}(y))|})$ of the fiber function $f_c^{|\operatorname{Log}^{-1}(\text{eq}(y))|}$ is empty and therefore $\text{eq}(y) \notin \mathcal{A}(f_c)$. Since by Theorem 4.8 (a) $\text{eq}(y)$ may be contained in the complement of $\mathcal{A}(f_c)$ only if c is the dominating term, we have with Theorem 2.15 that $\text{eq}(y) \notin \mathcal{A}(f_c)$ if and only if $\text{eq}(y) \in E_y(f_c)$.

(b) $\Rightarrow$ (a) is trivial. (a) $\Rightarrow$ (b): Since we are only interested in $\mathcal{V}(f_c)$ we may normalize such that $y = 0$ and hence $\sum_{i=1}^n \alpha(i) = -\alpha(0)$. We show that $\mathcal{A}(f)$ is symmetric around $\text{eq}(y)$: Assume that $\text{eq}(y) + \mathbf{w} \in E_y(f_c)$ for an arbitrary $\mathbf{w} \in \mathbb{R}^n$. Setting $\lambda_j = \langle \alpha(j), \text{eq}(y) + \mathbf{w} \rangle$ for $j \in \{1, \ldots, n\}$ we obtain

$$\langle \alpha(0), \text{eq}(y) + \mathbf{w} \rangle \;=\; -\sum_{j=1}^{n} \langle \alpha(j), \text{eq}(y) + \mathbf{w} \rangle \;=\; -\sum_{j=1}^{n} \lambda_j. \tag{4.14}$$

Then, for any permutation of the λ_j, there exists a $\mathbf{w}'$ with $\langle \alpha(j), \mathrm{eq}(y) + \mathbf{w}' \rangle = \lambda_j$ for $j \in \{0, \ldots, n\} \setminus \{k, l\}$ and $\langle \alpha(k), \mathrm{eq}(y) + \mathbf{w}' \rangle = \lambda_l$, $\langle \alpha(l), \mathrm{eq}(y) + \mathbf{w}' \rangle = \lambda_k$. This is obvious for $k, l \in \{1, \ldots, n\}$. Thus, let $k = 0, l = 1$ and $\langle \alpha(0), \mathrm{eq}(y) + \mathbf{w}' \rangle = \lambda_1$. Then, by (4.14), we have

$$\langle \alpha(1), \mathrm{eq}(y) + \mathbf{w}' \rangle \ = \ -\langle \alpha(0), \mathrm{eq}(y) + \mathbf{w}' \rangle - \sum_{j=2}^{n} \langle \alpha(j), \mathrm{eq}(y) + \mathbf{w}' \rangle \ = \ -\sum_{j=1}^{n} \lambda_j,$$

i.e., every permutation of the lengths of the monomials at $\mathrm{eq}(y) + \mathbf{w}$ is realized at some point $\mathrm{eq}(y) + \mathbf{w}'$. Similarly, let $\phi \in [0, 2\pi)^n$ with $\exp(i \cdot \langle \alpha(j), \phi \rangle) = \psi_j \in [0, 2\pi)$. With the same argumentation, there exists a ϕ' realizing every given permutation of the ψ_j. Altogether, such a permutation is realized by some $\mathbb{C}^*$-basis transformation on $(\mathbb{C}^*)^n$. Thus, if $\mathbf{w}'$ realizes some permutation of the λ_j, then there exists a isomorphism $\pi : \mathbb{F}_{\mathrm{eq}(y)+\mathbf{w}} \to \mathbb{F}_{\mathrm{eq}(y)+\mathbf{w}'}$ between the fibers $\mathbb{F}_{\mathrm{eq}(y)+\mathbf{w}}$ and $\mathbb{F}_{\mathrm{eq}(y)+\mathbf{w}'}$. Thus, we have for all $\psi \in [0, 2\pi)^n$

$$f^{|\operatorname{Log}^{-1}(\mathrm{eq}(y)+\mathbf{w})|}(\psi) = f^{|\operatorname{Log}^{-1}(\mathrm{eq}(y)+\mathbf{w}')|}(\pi(\psi)),$$

and hence for all such $\mathbf{w}'$

$$\mathrm{eq}(y) + \mathbf{w} \in E_y(f_c) \ \Rightarrow \ \mathrm{eq}(y) + \mathbf{w}' \in E_y(f_c). \tag{4.15}$$

Now investigate the complement-induced tropical hypersurface $\mathcal{C}(f_c - c)$ (see Chapter 2, Section 3) with $\mathrm{eq}(y)$ as unique vertex. Let $A_0, \ldots, A_n$ denote the cells given by the decomposition $\mathbb{R}^n \setminus \mathcal{C}(f_c - c)$. Since $E_y(f_c)$ is an open set and $\mathcal{C}(f_c - c)$ has codimension one in $\mathbb{R}^n$, we can assume that $\mathbf{w}$ is contained in the interior of some A_i. The fact that every permutation of the λ_i is realized at some point $\mathrm{eq}(y) + \mathbf{w}'$ together with (4.15) yields: If $\mathrm{eq}(y) + \mathbf{w} \in A_i$, then there exists some $\mathrm{eq}(y) + \mathbf{w}' \in E_y(f_c)$ for every $A_j \neq A_i$. Since $\mathrm{eq}(y)$ is the unique vertex of $\mathcal{C}(f_c - c)$ and due to convexity of $E_y(f_c)$, this implies for every $\mathbf{w} \neq 0$

$$\mathrm{eq}(y) + \mathbf{w} \in E_y(f_c) \ \Rightarrow \ \mathrm{eq}(y) \in E_y(f_c).$$

□

Theorem 4.19 yields that understanding U_y^A and its complement can be reduced to understanding the c depeding fiber function $f_c^{|\operatorname{Log}^{-1}(\mathrm{eq}(y))|}$ and its variety. With this approach we will be able to provide a geometrical description of U_y^A and $(U_y^A)^c$.

For $R > r$, a *hypocycloid* with parameters R, r is the parametric curve in $\mathbb{R}^2 \cong \mathbb{C}$ given by

$$[0, 2\pi) \to \mathbb{C}, \quad \phi \mapsto (R - r) \cdot e^{i \cdot \phi} + r \cdot e^{i \cdot \left(\frac{r-R}{r}\right) \cdot \phi}. \tag{4.16}$$

Geometrically, it is the trajectory of some fixed point on a circle with radius r rolling (from the interior) on a circle with radius R. Hypocycloids are special instances of *hypotrochoids*, which we introduce in Section 3. See Figure 4.6 where the central figure shows the trajectory of a hypocycloid.

The main part of this section is attended to proving the following nice and explicit characterization of $\partial(U_y^A)^c$.

Theorem 4.20. *Let (Δ, y) be barycentric. For given $b_0, \ldots, b_n \in \mathbb{C}^*$ the intersection of the set $\partial(U_y^A)^c$ with the complex line $\{(b_0, \ldots, b_n, c) : c \in \mathbb{C}\}$ is given by the (eventually rotated) hypocycloid with parameters $R = (n+1) \cdot |\Theta|$, $r = |\Theta|$ and with cusps at*

$$\arg(c) \quad = \quad \pi \cdot \left(1 + \frac{2k - \sum_{i=1}^n \arg(b_i)}{n+1}\right), \qquad k \in \{0, \ldots, n\}.$$

We have already seen that it suffices to treat the case $y = 0$. Let $f_c \in \mathcal{P}_\Delta^0$ be a parametric family with $\sum_{i=0}^n \alpha(i) = 0$ and fixed $b_0, \ldots, b_n \in \mathbb{C}^*$, $b_0 = 1$. For f_c consider the set

$$S \quad = \quad \left\{c \in \mathbb{C} \ : \ \mathcal{V}(f_c^{|\operatorname{Log}^{-1}(\operatorname{eq}(y))|}) \neq \emptyset\right\} \tag{4.17}$$

as a subset of $\mathbb{R}^2 \cong \mathbb{C}$. Theorem 4.19 shows that S is exactly the set of all $c \in \mathbb{C}$ such that the inner complement component of $\mathcal{A}(f)$ exists. Hence, $S \subseteq \mathbb{R}^2$ is located in the space $\mathcal{P}_\Delta^0$ intersected with the complex line $\{(b_0, \ldots, b_n, c) : c \in \mathbb{C}\}$ induced by the family f_c. It contains all coefficient vectors of polynomials not belonging to U_y^A. As a first step towards the proof of Theorem 4.20 we show a technical result on the set S.

Lemma 4.21. *Let $k = -n + 1 + (-1)^{n+1}$ and*

$$F : [k, n] \times [0, 2\pi) \to \mathbb{C}, \quad (\mu, \psi) \mapsto |\Theta| \cdot \mu \cdot e^{i \cdot \psi} + |\Theta| \cdot e^{i \cdot (-n \cdot \psi + \sum_{j=1}^n \arg(b_j))}. \tag{4.18}$$

Then

(1) *The image of F is contained in the set S defined in* (4.17).
(2) *Up to a rotation, the curve parameterized by $\phi \mapsto F(n, \phi)$ for $\phi \in [0, 2\pi)$ is a hypocycloid* (4.16) *with $R = (n+1) \cdot |\Theta|$, $r = |\Theta|$.*

Proof. By (4.17) and the definition of the fiber function (see Chapter 3, Section 1) the set S is given by the image of the function $g : [0, 2\pi)^n \to \mathbb{C}, \phi \mapsto -|\Theta| \cdot \sum_{j=0}^n e^{i \cdot (\arg(b_j) + \langle \alpha(j), \phi \rangle)}$ (Theorem 4.19). The idea of the proof is that the image of g restricted to some particular subset of $[0, 2\pi)^n$ is exactly the image of F.

Again, let $\alpha(1)^*, \ldots, \alpha(n)^* \in \mathbb{Q}^n$ denote the dual basis of $\alpha(1), \ldots, \alpha(n)$, and set

$$h(\phi) \quad = \quad g(\phi) + |\Theta| \cdot e^{i \cdot \langle \alpha(0), \phi \rangle} \quad = \quad -|\Theta| \cdot \sum_{j=1}^n e^{i \cdot (\arg(b_j) + \langle \alpha(j), \phi \rangle)}.$$

Further let $\psi \in [0, 2\pi)$ and σ_ψ denote the segment $[-k \cdot |\Theta| \cdot e^{i \cdot \psi}, n \cdot |\Theta| \cdot e^{i \cdot \psi}] \subset \mathbb{C}$.

First, we discuss the case of even n. For fixed ψ, let $M = \{\phi_\xi : \xi \in [0, \pi]\}$ with

$$\phi_\xi \quad = \quad \sum_{j=1}^{n/2} (\psi - \arg(b_j) + \xi) \cdot \alpha(j)^* + \sum_{j=n/2+1}^{n} (\psi - \arg(b_j) - \xi) \cdot \alpha(j)^*.$$

Since $\arg(b_j) + \langle\alpha(j), \phi_\xi\rangle = \psi + \xi$ for $j \leq n/2$ (resp. $\psi - \xi$ for $j > n/2$) and since all summands have norm $|\Theta|$, we see $e^{i\cdot(\arg(b_j)+\langle\alpha(j),\phi_\xi\rangle)} + e^{i\cdot(\arg(b_j)-\langle\alpha(j+n/2),\phi_\xi\rangle)} \in \sigma_\psi$. Thus, $h(\phi_\xi) \in \sigma_\psi$ for all $\phi_\xi \in M$.

Since furthermore the real part of $h(\phi_\xi)\cdot e^{-i\cdot\psi}$ is given by $n\cdot\cos(\xi)$, the image of $h(M)$ is σ_ψ, i.e., $\{|\Theta|\cdot\mu\cdot e^{i\cdot\psi} : \mu \in [k,n]\}$. Finally, we have for every $\phi_\xi \in M$

$$\begin{aligned}\langle\alpha(0), \phi_\xi\rangle &= \Big\langle -\sum_{j=1}^{n}\alpha(j), \phi_\xi\Big\rangle = \sum_{j=1}^{n/2}\arg(b_j) - \psi + \xi + \sum_{j=n/2+1}^{n}\arg(b_j) - \psi - \xi \\ &= \sum_{j=1}^{n}\arg(b_j) - \psi.\end{aligned}$$

Hence, the set $g(M) = \{h(\phi_\xi) + |\Theta|\cdot e^{i\cdot\langle\alpha(0),\phi_\xi\rangle} : \phi_\xi \in M\}$ coincides with the set $\{|\Theta|\cdot(\mu\cdot e^{i\cdot\psi} + e^{i\cdot(\sum_{j=1}^{n}(\arg(b_j)-\psi))}) : \mu \in [k,n]\}$, i.e., $g(M) = F([k,n],\psi)$.

If n is odd, the argumentation is analogous up to the fact that we redefine $M = \{\phi_\xi : \xi \in [0,\pi]\}$ by

$$\begin{aligned}\phi_\xi &= (\psi - \arg(b_1))\cdot\alpha(1)^* + \sum_{j=2}^{\lceil n/2\rceil}(\psi - \arg(b_j) + \xi)\cdot\alpha(j)^* + \\ &\quad \sum_{j=\lceil n/2\rceil+1}^{n}(\psi - \arg(b_j) - \xi)\cdot\alpha(j)^*.\end{aligned}$$

This proves the first statement.

For the choice of R and r we obtain the hypocycloid curve $\{|\Theta|\cdot n\cdot e^{i\cdot\phi} + |\Theta|\cdot e^{-i\cdot n|\Theta|\phi} : \phi \in [0,2\pi)\}$, which coincides with the image of $F(n,\psi), \psi \in [0,2\pi)$ up to a coordinate change given by $\psi \mapsto \left(\frac{\sum_{i=1}^{n}\arg(b_i)}{n+1}\right) + \phi$. This is the second statement. □

Indeed, the next lemma states that the set S defined in (4.17) *exactly* coincides with the region defined by the hypocycloid curve. See Section 2.1 for a detailed calculation.

Lemma 4.22. *The set S equals the region T whose boundary is (up to rotation) the hypocycloid with parameter $R = (n+1)\cdot|\Theta|$, $r = |\Theta|$ given by $\phi \mapsto F(n,\phi)$ for $\phi \in [0,2\pi)$. In particular, S is simply connected.*

Note that this lemma shows that Problem 2.23 has an affirmative answer for the sets U_y^A in the case of polynomials with barycentric simplex Newton polytope, which we treat in this section for all complex lines in $\mathbb{C}^A$, which are given by fixing the "outer" coefficients and taking the "inner" one as variable. Here and in the following we sometimes identify $f_b = \sum_{i=1}^{d} b_i\cdot\mathbf{z}^{\alpha(i)} \in \mathbb{C}_\Diamond^A$ with its coefficient vector $b = (b_1,\ldots,b_d)$ and use an abreviate notation $b \in \mathbb{C}_\Diamond^A$. With the results obtained up to this point we are now able to prove Theorem 4.20:

Proof. (Theorem 4.20) Again, we may assume that y is the origin. For $b_0,\ldots,b_n \in \mathbb{C}^*$ we investigate the parametric family $f_c = c + \sum_{j=0}^{n} b_i\cdot\mathbf{z}^{\alpha(i)} \in \mathcal{P}_\Delta^0$ with a parameter $c \in \mathbb{C}$. On this complex line in the space of amoebas we want to describe $\partial(U_y^A)^c$.

By Theorem 4.19 (c), $\mathcal{A}(f_c)$ has genus one if and only if $c \notin \{|\Theta| \cdot \sum_{j=0}^{n} e^{i\cdot(\arg(b_j)+\langle\alpha(j)-y,\phi\rangle)} : \phi \in [0,2\pi)^n\}$ (recall that Θ depends on the choice of the b_i), which is the complement of S by definition. Therefore

$$\partial\left((U_y^A)^c\right) \cap \{(b_0,\ldots,b_n,c) : c \in \mathbb{C}\} \;=\; \partial S.$$

By Lemma 4.22, ∂S is up to a rotation a hypocycloid with parameters $R = (n+1)\cdot|\Theta|$, $r = |\Theta|$ around the origin. The location of the cusps follows from the definition of the ∂S-describing function F in (4.18) solving $i \cdot \lambda = -i \cdot (n \cdot \lambda + \sum_{j=1}^{n} \arg(b_j)) \mod 2\pi$. □

Example 4.23. For the parametric family of polynomials $f_c = 1 + 2.4 \cdot z_1^2 z_2 + c \cdot z_1 z_2^3 + (1 + 1.3i) \cdot z_1 z_2^8$, the set $\mathcal{P}_\Delta^y \cap \{(1 : 2.4 : 1 + 1.3 \cdot i : c) : c \in \mathbb{C}\}$ (with $\Delta = \text{conv}\{(0,0),(1,2),(2,1),(1,1)\}$) is illustrated in Figure 4.4. The non-real choice of one of the "outer" coefficients causes a rotation of the set as described in Theorem 4.20.

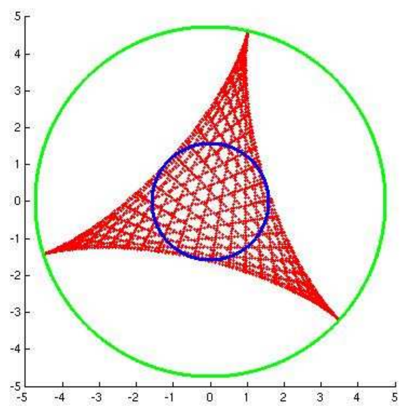


FIGURE 4.4. A meshplot of S. The green (light) circle has radius $3 \cdot |\Theta|$ and the blue (dark) circle has radius $|\Theta|$ with $|\Theta| \approx 1.5789$.

Finally, we show path-connectivity of the set U_y^A and therefore answer Rullgård's question for all spaces of amoebas of polynomials with barycentric simplex Newton polytopes with one inner lattice point (see Corollary 4.25). As a cornerstone, we show the following general result about configuration spaces of amoebas.

Theorem 4.24. *Let $A = \{\alpha(1),\ldots,\alpha(d)\} \subset \mathbb{Z}^n$ and $j \in \{1,\ldots,d\}$. If for every $b \in \mathbb{C}_\Diamond^A$ the set $\{(b_1,\ldots,b_d) : b_j \in \mathbb{C}\} \cap \left(U_{\alpha(j)}^A\right)^c$ is simply connected, then $U_{\alpha(j)}^A$ is path-connected.*

Note that $\{(b_1,\ldots,b_d) : b_j \in \mathbb{C}^*\}$ is a complex line in $\mathbb{C}_\Diamond^A$, where all parameters $b_1,\ldots,b_{j-1},b_{j+1},\ldots,b_d$ are arbitrary, fixed complex numbers with $b_i \neq 0$ if $\alpha(i)$ is a vertex of $\text{conv}(A)$.

PROOF. We identify $b \in \mathbb{C}_\Diamond^A$ with $f_b = \sum_{i=1}^{d} b_i \cdot \mathbf{z}^{\alpha(i)} \in \mathbb{C}_\Diamond^A$. Since no assumptions are made about the $\alpha(j)$ here we may choose $j = 1$ to abbreviate notation. Let $a, b \in$

$U^A_{\alpha(1)} \subseteq \mathbb{C}^A_\Diamond$. We construct an explicit path γ between a and b such that $\gamma \in U^A_{\alpha(1)}$. Let $[a,b]$ denote the line segment $a + \mu \cdot (b-a) \subset \mathbb{C}^A_\Diamond, \mu \in [0,1]$. For the construction of the path we need a value $\kappa \in \mathbb{R}_{>0}$ for the norm of the first coordinate of points in $\mathbb{C}^A_\Diamond$ such that every point on $[a,b]$ is lopsided. This is guaranteed by

$$\kappa \;=\; 1 + \max_{c \in [a,b]} \min_{\mathbf{w} \in \mathbb{R}^n} \left\{ \sum_{i=2}^{d} |c_i| \cdot e^{\langle \mathbf{w}, \alpha(i) - \alpha(1) \rangle} \right\} \in \mathbb{R}_{>0}. \tag{4.19}$$

Define the points $a', b' \in \mathbb{C}^A_\Diamond$ by

$$a' \;=\; (\kappa \cdot \arg(a_1), a_2, \ldots, a_d), \quad b' \;=\; (\kappa \cdot \arg(b_1), b_2, \ldots, b_d).$$

The choice of κ guarantees that the polynomials $f_{a'}$ and $f_{b'}$ are lopsided at some point with the monomial with exponent $\alpha(1)$ as the dominant term and therefore $a', b' \in U^A_{\alpha(1)}$. Since for every $b \in \mathbb{C}^A_\Diamond$ the set $\{(b_1, \ldots, b_d) : b_1 \in \mathbb{C}\} \cap (U^A_{\alpha(1)})^c$ is simply connected and since $a, a', b, b' \in U^A_{\alpha(1)}$, there exists a path γ_1 from a to a' and a path γ_2 from b' to b with $\gamma_1 \subset \{(a_1, a_2, \ldots, a_d) : a_1 \in \mathbb{C}\} \cap U^A_{\alpha(1)}$ and $\gamma_2 \subset \{(b_1, b_2, \ldots, b_d) : b_1 \in \mathbb{C}\} \cap U^A_{\alpha(1)}$. Let

$$d \;=\; (\kappa \cdot \arg(b_1), \arg(b_2) \cdot |a_2|, \ldots, \arg(b_d) \cdot |a_d|).$$

Since there is a $\mathbf{w} \in \mathbb{R}^n$ with $\mathbf{w} \in E_{\alpha(1)}(f_{a'})$ and $f_{a'}\{\mathbf{w}\}$ lopsided we have

$$\mathbb{T}(f_{a'}) \;=\; \left\{ f' = \kappa \cdot e^{i \cdot \psi_1} \cdot \mathbf{z}^{\alpha(1)} + \sum_{j=2}^{d} e^{i \cdot \psi_j} \cdot a_j \cdot \mathbf{z}^{\alpha(j)} \;:\; \psi_j \in [0, 2\pi) \text{ for all } j \right\} \;\subset\; U^A_{\alpha(1)}$$

by Proposition 4.14. Since furthermore $d \in \mathbb{T}(f_{a'})$, there exists a path $\gamma_3 \subset \mathbb{T}(f_{a'}) \subset U^A_{\alpha(1)}$ from a' to d.

Let γ_4 denote the line segment

$$\gamma_4 \;=\; \left\{ d + \lambda \cdot (0, \arg(b_2) \cdot (|b_2| - |a_2|), \ldots, \arg(b_d) \cdot (|b_d| - |a_d|)), \quad \lambda \in [0,1] \right\}.$$

By construction $\gamma_4(\lambda) \in \mathbb{T}(f_{a+\lambda(b-a)})$ for all $\lambda \in [0,1]$. Since for every $\lambda \in [0,1]$ the first coordinate of $\gamma_4(\lambda)$ has norm κ, it follows from (4.19) and Proposition 4.14 that there is a $\mathbf{w} \in \mathbb{R}^n$ such that $f_{\gamma_4(\lambda)}\{\mathbf{w}\}$ is lopsided and in $E_{\alpha(1)}(f_{\gamma_4(\lambda)})$. Hence, $\gamma_4 \subset U^A_{\alpha(1)}$. Therefore, $\gamma = \gamma_2 \circ \gamma_4 \circ \gamma_3 \circ \gamma_1$ is a path from a to b with $\gamma \in U^A_{\alpha(1)}$. □

Now, we have everything we need to prove that the answer on Rullgård's Problem 2.22 is affirmative for amoebas of polynomials with barycentric simplex Newton polytope.

Corollary 4.25. *If (Δ, y) is barycentric then U^A_y is path-connected.*

Proof. All S (see (4.17)) are simply connected (Lemma 4.22) and contain the origin (Theorem 4.8). Thus, U^A_y is path-connected by Theorem 4.24. □

2.1. Proof of Technical Lemmata. We provide the calculations for the proof of Lemma 4.22.

Lemma 4.26. *Let T denote the region whose boundary is the (rotated) hypocycloid given by $\phi \mapsto F(n,\phi)$ for $\phi \in [0,2\pi)$. Then $S \subseteq T$ and $\partial T \subseteq \partial S$.*

PROOF. By Theorem 4.19, S is given by the image of the function $g : [0,2\pi)^n \to \mathbb{C}, \phi \mapsto -|\Theta| \cdot \sum_{j=0}^{n} e^{i\cdot(\arg(b_j)+\langle \alpha(j),\phi\rangle)}$ with $\alpha(0) = -\sum_{j=1}^{n} \alpha(j)$. In order to show $S \subseteq T$, it suffices to show that every critical point of g is mapped in T, because every boundary point of S is image of a critical point of g.

Once more, we use the dual basis $\alpha(1)^*, \ldots, \alpha(n)^*$ of $\alpha(1), \ldots, \alpha(n))$, i.e., $\phi = \sum_{j=1}^{n} \phi_j \cdot \alpha(j)^*$. Furthermore, we can assume $\arg(b_1) = \cdots = \arg(b_n) = 0$ since we can replace ϕ_j by $-\arg(b_j) + \phi_j$. We have

$$\frac{\partial g}{\partial \phi_j}(\phi) \;=\; -|\Theta| \cdot i \cdot \left(e^{i\cdot\phi_j} - e^{-i\cdot(\sum_{l=1}^{n}\phi_l - \arg(b_0))}\right),$$

and thus,

$$\begin{aligned}\mathrm{Re}\left(\frac{\partial g}{\partial \phi_j}(\phi)\right) &= |\Theta| \cdot \left(\sin(\phi_j) - \sin\left(-\sum_{l=1}^{n}\phi_l + \arg(b_0)\right)\right)\\ \text{and }\ \mathrm{Im}\left(\frac{\partial g}{\partial \phi_j}(\phi)\right) &= |\Theta| \cdot \left(-\cos(\phi_j) + \cos\left(-\sum_{l=1}^{n}\phi_l + \arg(b_0)\right)\right).\end{aligned}$$

ϕ is a critical point of g if and only if $\mathrm{Re}(\nabla g(\phi)) = \lambda_\phi \cdot \mathrm{Im}(\nabla g(\phi))$ with $\lambda_\phi \in \mathbb{R}$, i.e., if and only if for all $j \in \{1,\ldots,n\}$:

$$\lambda_\phi \cdot \cos(\phi_j) + \sin(\phi_j) \;=\; \lambda_\phi \cdot \cos\left(-\sum_{l=1}^{n}\phi_l + \arg(b_0)\right) + \sin\left(-\sum_{l=1}^{n}\phi_l + \arg(b_0)\right).$$

Since the right hand term is independent of j, this implies

$$\lambda_\phi \cdot \cos(\phi_j) + \sin(\phi_j) \;=\; \lambda_\phi \cdot \cos(\phi_k) + \sin(\phi_k)$$

for all $j,k \in \{1,\ldots,n\}$. This is in particular true if $\cos(\phi_j) = \cos(\phi_k)$ and $\sin(\phi_j) = \sin(\phi_k)$, i.e., if all $e^{i\cdot\phi_j}$ have the same argument, that is, $g(\phi)$ is located on the (rotated) hypocycloid given by $F(n,\psi), \psi \in [0,2\pi)$ (see (4.18), Lemma 4.21).

The function $h(\phi_j) = \lambda_\phi \cdot \cos(\phi_j) + \sin(\phi_j)$ is a periodic function in the interval $[0,2\pi)$, which has a vanishing derivative exactly at the points ϕ_j with $\tan(\phi_j) = 1/\lambda_\phi$. tan is π-periodic and strictly increasing on the interval $(-\pi/2,\pi/2)$. Therefore, for a fixed solution ϕ_n of $h(\phi_n)$, for every $j \in \{1,\ldots,n-1\}$ there are exactly two possibilities: either $\phi_j = \phi_n$ or ϕ_j is the unique solution distinct from ϕ_n with $h(\phi_j) = h(\phi_n)$, and that one coincides with $-\sum_{l=1}^{n}\phi_l + \arg(b_0)$.

Thus, if ϕ is a critical point with $g(\phi) \notin F(n,[0,2\pi))$, then there are ϕ_j (we choose here $j = 1,\ldots,s$ for some $1 \leq s < n-1$ since every outer monomial has the same properties) satisfying $\lambda_\phi \cdot \cos(\phi_j) = \cos\left(-\sum_{l=1}^{n}\phi_l + \arg(b_0)\right)$ and $\sin(\phi_j) = \sin\left(-\sum_{l=1}^{n}\phi_l + \arg(b_0)\right)$, which means that $\arg(e^{i\cdot\phi_1}) = \cdots = \arg(e^{i\cdot\phi_s}) = \arg(e^{-i\cdot\sum_{l=1}^n \phi_l + \arg(b_0)})$ and $\arg(e^{i\cdot\phi_{s+1}}) = \cdots = \arg(e^{i\cdot\phi_n})$. Hence, $\phi_1 = \cdots = \phi_s$, $\phi_{s+1} = \cdots = \phi_n$ and

$$\begin{aligned}\phi_1 &= -\sum_{l=1}^n \phi_l + \arg(b_0) = -s\cdot\phi_1 - (n-s)\cdot\phi_n + \arg(b_0)\\ &= -\frac{n-s}{s+1}\cdot\phi_n + \arg(b_0).\end{aligned}$$

Thus, $g(\phi)$ is located on the curve given by the hypocycloid with parameters $R = (n+1)|\Theta|$ and $r' = (s+1)|\Theta|$ rotated by $\arg(b_0)$ (see (4.16)).

Since S is a subset of the closed ball $\mathcal{B}_{(n+1)\cdot|\Theta|}(0)$ with radius $(n+1)\cdot|\Theta|$ around the origin (Theorem 4.8), it is bounded and since $(U_y^A)^c$ is a closed set, we have $\partial S \subset S$. Since the trajectory of every hypocycloid with parameters $R = n+1$ and $r \in \{2,\ldots,n-1\}$ is a subset of T (coinciding with $F(n,\psi), \psi \in [0,2\pi)$ at the cusps), we have $S \subseteq T$.

Since the sets S and T are closed, the statement $\partial T \subseteq \partial S$ follows from $S \subseteq T$ and $\partial T \subseteq S$. The first of these conditions has just been shown and the second one is Lemma 4.21 in connection with the definition of T. □

PROOF. (Lemma 4.22) By Lemma 4.26 we know that $S \subseteq T$ with $\partial T \subseteq \partial S$. Furthermore, by (4.18) the image of F is contained in S. Hence, the lemma is proven if we can show that the image of F equals T (which is simply connected by definition).

Let $k = n - 1 + (-1)^n$. We may assume $\arg(b_1),\ldots,\arg(b_n) = 0$ again (otherwise we transform the basis of $\phi_1,\ldots,\phi_n$ as in other proofs before). F satisfies an $(n+1)$-quasi-periodicity condition $F(\mu, j\cdot\psi) = e^{i\cdot(2\pi j)/(n+1)}\cdot F(\mu,\psi)$ with $\mu \in [k,n], \psi \in [0, 2\pi/(n+1)], j \in \{0,\ldots,n\}$. In particular,

$$\{F(\mu, j\cdot 2\pi/(n+1)) : \mu \in [k,n]\} = \left\{e^{i\cdot 2\pi\cdot j/(n+1)}\cdot\mu : \mu \in [-k,n]\right\} \tag{4.20}$$

for $j \in \{0,\ldots,n\}$.

We know that the path $\gamma_n(\psi) = F(n,\psi)$ with $\psi \in [0,2\pi)$ is a hypocycloid (Lemma 4.21). Let $T = T_1 \cup \cdots \cup T_{n+1}$ where

$$T_j = T \cap \{x \in \mathbb{C} : \arg(x) \in [(j-1)\cdot 2\pi/(n+1), j\cdot 2\pi/(n+1)]\}.$$

We show that the image of F equals T and thus is in particular simply connected. This follows from the quasi-periodicity, if the image of $F(\mu,\psi)$ with $\psi \in [0, 2\pi/(n+1)]$ covers T_1.

The path-segment $\gamma_n(\psi)$ with $\psi \in [0, 2\pi/(n+1)]$ is a loop-free path, which is injective in the argument. The path-segment $\gamma_0(\psi)$ with $\psi \in [0, 2\pi/(n+1)]$ is a segment of a circle in $(T\setminus T_1)\cup\partial T_1$, which is also injective in the argument. Thus, for every $\psi \in (0, 2\pi/(n+1))$ the segment $\sigma_\psi = [F(0,\psi), F(n,\psi)]$ intersects $\{x \in [0,n)\} \cup \{x \in e^{i\cdot 2\pi/(n+1)}\cdot[0,n)\}$ at some point t_ψ. This implies with (4.20) that F covers the homotopy $H : [0, 2\pi/(n+1)] \to \{[x_1,x_2] \subset \mathbb{C}\}, \psi \mapsto [t_\psi, \gamma_n(\psi)]$ of line segments with $H(0) = [n,n], H(2\pi/(n+1)) = e^{i\cdot 2\pi/(n+1)}\cdot[n,n]$. The image of H is T_1. Hence, the image of $F(\mu,\psi)$ with $\psi \in [0, 2\pi/(n+1)]$ covers T_1 and therefore the image of F is T. Since $\mathrm{im}(F) \subseteq S$ and $S \subseteq T$ we have $S = T$ and thus, S is simply connected. □

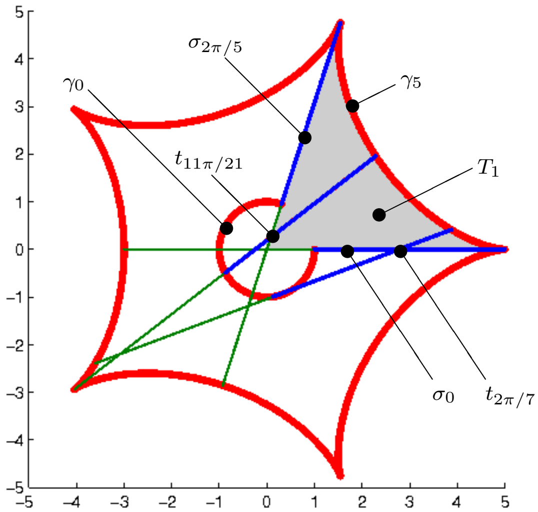


FIGURE 4.5. Illustration of the covering of the set S by the function F.

Example 4.27. Figure 4.5 illustrates the proof of Lemma 4.22 for the case of $f = c + \sum_{j=1}^{5} \mathbf{z}^{\alpha(j)} \in \mathbb{C}[z_1, \ldots, z_4]$ with $\sum_{j=1}^{5} \alpha(j) = 0$. Here, $\Theta = 1$ hence $R = 5$ and $r = 1$. Due to the quasi-periodicity it suffices to cover the grey region T_1. Of course, γ_5 is the hypocycloid with the upper values of R and r, and γ_0 is the circle of radius one around the origin. In the figure one can see the path-segments $\sigma(0)$ and $\sigma(\frac{2}{5}\pi)$ yielding the start- and endpoint of the homotopy H (the two cusps intersecting T_1) and the path-segments $\sigma(\frac{2}{7}\pi)$ and $\sigma(\frac{11}{21}\pi)$, which yield $H(\frac{2}{7}\pi)$ and $H(\frac{11}{21}\pi)$ given by the subsegments from the point on γ_5 to $t_{2\pi/7}$ resp. $t_{11\pi/21}$. One can see how the complete region T_1 is covered by these subsegments given by H, since, figuratively, the segments sweep over the whole grey region T_1.

3. Amoebas of Trinomials

The investigation of *univariate trinomials* i.e., (Laurent) polynomials of the form

$$z^s + p + qz^{-t} \in \mathbb{C}[z^{\pm 1}]$$

with $p, q \in \mathbb{C}$ is a truly classical nineteenth and early twentieth century problem (see e.g., [**8, 33, 45**]). At this time mathematicians started to ask how the variety (which is a finite set of complex points here) depends on the choice of the coefficients p, q. For example, how the roots can be arranged geometrically, how many of them do not exceed a certain, given absolute value or how coefficients can be chosen, such that two roots share the same absolute value.

Algebraically, these questions are well understood due to a result of P. Bohl from 1908 (see Theorems 4.28, 4.29; see also [**8**]). But, although the investigation of trinomials went on in modern times (e.g., [**14, 18**]), the space of coefficients and in particular its geometrical and topological properties are still not understood.

In this section we recall some classical problems and results. Afterwards we reinterpret the problems in terms of amoeba theory and show that with its tools – in particular the fiber structure of the Log-map (see Chapter 3, Section 1) – we gain the capability to solve these problems and uncover a beautiful geometrical and topological structure hidden in the configuration space of trinomials.

Our main results are that a trinomial has a root of a given modulus $|z^*| \in \mathbb{R}_{>0}$ if and only if one of the coefficients p is located on the (eventually rotated) trajectory of a *hypotrochoid curve* depending on the choice of the second coefficient q, the support set and, of course, $|z^*|$ itself (Theorem 4.32). Hypotrochoids are generalizations of hypocycloids discussed in Section 2 and have a lot of nice geometrical properties.

Furthermore, a second root shares the same modulus $|z^*|$ if and only if p is located on a node of the particular hypotrochoid, which means in general that there are two roots with the same modulus if and only if p is located in a particular 1-fan determined by the support set and q (Theorem 4.40).

This result has a couple of consequences. It allows us in Theorem 4.43 to close the gap we found in the proof of Rullgård's Theorem 2.19 (see Chapter 2, Section 4) and yields a way to construct an amoeba with a complement component, whose order is not contained in its support set (Example 4.44). Theorem 4.43 resp. Example 4.44 show furthermore that there exist amoebas, which are not homotopy equivalent to its complement induced tropical hypersurface (Corollary 4.45). This solves Problem 2.16.

But Theorem 4.40 discovers additionally the topology of the configuration space of all trinomials. It turns out that each set $U_j^A \subset \mathbb{C}^A$ with $j \in \{-t+1, \ldots, s-1\} \setminus \{0\}$ can be deformation retracted to a closed path, which is an $(s+t)$-sheeted covering of an S^1 (Theorem 4.51). This result yields strong consequences for questions concerning amoebas. It implies that the sets U_j^A are always connected for trinomials and thus solves Rullgård's problem 2.22 for this class. Furthermore, it allows to compute the fundamental group of the set U_j^A and in particular disproves that these sets are simply connected (Problem 2.24).

Besides that, although not precisely discussed in the literature yet, it is a quite obvious question whether complement components grow monotonically in the absolute value of their "corresponding" coefficient (via the order map). We both, motivate and formally introduce, this problem in Section 3.4 and in particular give a trinomial counterexample for the suggested monotonic behavior of complement components (Theorem 4.53).

3.1. Classical problems and classical results. In this section we investigate trinomials of the form

$$f \quad = \quad z^s + p + qz^{-t} \in \mathbb{C}[z^{\pm 1}],$$

with $(p,q) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $s,t \in \mathbb{N}^*$. We can always assume that s,t are coprime, since otherwise we might remap $z^{\gcd(s,t)} \mapsto z$ without changing the amoeba of f. Let $f = (z-a_1)\cdots(z-a_{s+t})$ with $|a_1| \leq \cdots \leq |a_{s+t}|$. Note that for $A = \{-t, 0, s\}$ and $B = \{0, s\}$ the set of trinomials with $(p,q) \in \mathbb{C}^2 \setminus \{(0,0)\}$ is the set of all trinomials in $\mathbb{C}_\Diamond^A \cup \mathbb{C}^B$, where $\mathbb{C}_\Diamond^A$ is the augmented configuration space of $\mathbb{C}^A$ (see Section 2). To keep notation consistent, from Section 3.2 on we restrict ourselves w.l.o.g. to the case $\mathbb{C}^A_\Diamond$.

In the late 19th century mathematicians started to investigate the connection between the roots of trinomials (often, in particular their modulis) and the choice of their coefficients. E.g., in a very early work Nekrassoff describes in 1887 how roots of trinomials are located in certain regions ("Contouren") of the complex plane ([**45**]). In 1907, Landau gave an upper bound for the smallest absolute value of the roots of a trinomial ([**33**]), which was generalized to polynomials by Fejér one year later ([**17**]). The inverse of this question, i.e., the number of roots $k \in \mathbb{N}$ with modulus lower than a given $|z^*| \in \mathbb{R}_{>0}$ can be computed by a result of Bohl, also from 1908 (see [**8**]). Specifically, he showed the following two theorems

Theorem 4.28. *(Bohl 1908)*

$$\begin{array}{rcll} \text{If } |q| & > & |z^*|^{s+t} + |p| \cdot |z^*|^t, & \text{then} \quad k = 0. \\ \text{If } |z^*|^{s+t} & > & |q| + |p| \cdot |z^*|^t, & \text{then} \quad k = s+t. \\ \text{If } |p| \cdot |z^*|^t & > & |q| + |z^*|^{s+t}, & \text{then} \quad k = t. \end{array}$$

Note that, from the viewpoint of amoeba theory, this theorem is obvious since it means that f is lopsided at $|z^*|$ (see Chapter 2, Section 5).

If none of the upper inequalities holds, we define a triangle Δ with edges of length $|z^*|^{s+t}$, $|p| \cdot |z^*|^t$ and $|q|$. Let α be the argument opposing $|z^*|^{s+t}$ and β be the argument opposing $|p| \cdot |z^*|^t$.

Theorem 4.29. *(Bohl 1908) If Δ is well defined, then the number of roots k with modulus lower than $|z^*| \in \mathbb{R}_{>0}$ is given by the number of integers located in the open interval given by*

$$\frac{(s+t)(\pi + \arg(p) - \arg(q)) - t(\pi - \arg(q))}{2\pi} - \frac{(s+t)\alpha + t\beta}{2\pi} \tag{4.21}$$

and

$$\frac{(s+t)(\pi + \arg(p) - \arg(q)) - t(\pi - \arg(q))}{2\pi} + \frac{(s+t)\alpha + t\beta}{2\pi}. \tag{4.22}$$

Since the theorems are quite abstract, we give an example.

Example 4.30. Let $f = z^3 + z + \sqrt{2}$ and $|z^*| = 1$. Then $\alpha = \beta = \pi/8$. Thus, k is the number of integers between

$$\begin{array}{rcl} \dfrac{3(\pi + 0 - 0 - \pi)}{2\pi} - \dfrac{3/8\pi + 1/8\pi}{2\pi} & = & -\dfrac{1}{4} \quad \text{and} \\ \dfrac{3(\pi + 0 - 0 - \pi)}{2\pi} + \dfrac{3/8\pi + 1/8\pi}{2\pi} & = & +\dfrac{1}{4}. \end{array}$$

Since this is only the origin, we have $k = 1$. A double check with Maple yields that the roots of f have approximately modulus

$$0.83403883, 1.30216004 \text{ and } 1.30216004.$$

Unfortunately, these theorems give – again spoken in the language of amoebas – no explanation for the geometrical or topological structure of the configuration space $\mathbb{C}^A$ resp. the sets U_α^A in it. Amazingly, despite the fact that these theorems were proven over one hundred years ago and people kept on investigating trinomials until nowadays (see e.g., [**14, 18**]), no evident progress was made w.r.t. the geometrical and topological structure.

To make things more explicitly, here we concentrate on the following problem, whose *algebraic* counterpart can be regarded as solved by Bohl's theorems.

Problem 4.31. *Let $f = z^{s+t} + pz^s + q \in \mathbb{C}[z]$, with $(p, q) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and $|z^*| \in \mathbb{R}_{>0}$.*

1. *How can one choose p, q such that f has a root with modulus $|z^*| \in \mathbb{R}_{>0}$? What is, for given q the geometrical structure of the set of all p such that f has a root with modulus $|z^*| \in \mathbb{R}_{>0}$?*
2. *How can one choose p, q such that f has roots of the same modulus? What is, for given q the geometrical structure of the set of all p such that this is the case?*
3. *Which geometrical and topological properties does the space $\{(p, q) \in \mathbb{C}^2 \setminus \{(0, 0)\}\}$ with respect to the absolute values of roots of the corresponding polynomials have?.*

3.2. Modulis of Trinomials. Let $A = \{-t, 0, s\}$ and let $f = z^s + p + qz^{-t}$ be a trinomial. Our general goal is to describe the topology of the sets U_j^A in $\mathbb{C}^A$, i.e., the topology of the sets of all polynomials with roots $a_1, \ldots, a_{s+t}$ (with $|a_1| \leq \cdots \leq |a_{s+t}|$) such that $|a_j| \neq |a_{j+1}|$. In this section we investigate the special case of a fixed q and describe the structure of U_j^A restricted to the corresponding complex linear subspaces of $\mathbb{C}^A$. Note that, with respect to U_j^A, we can switch between $\mathbb{C}^A$ and $\mathbb{C}_\Diamond^A$ for trinomials. If there is a trinomial f with $p = 0$ or $q = 0$, then all roots have the same modulus. Thus, the amoeba is always solid in such a case.

The description of these subsets of U_j^A is mainly based on two key observations. Firstly, we show that f has a root with modulus $|z^*|$ for some $z^* \in \mathbb{C}^*$ (i.e., the fiber $\mathbb{F}_{\log|z^*|}$ of a point $z^* \in \mathbb{C}^*$ with respect to the Log-map intersects the variety $\mathcal{V}(f)$ of f) if and only if the coefficient p of f is located on the trajectory of a certain *hypotrochoid curve* depending on s, t, q and $|z^*|$ in a $\mathbb{C}$-subspace of the configuration space $\mathbb{C}_\Diamond^A$ (Theorem 4.32). Secondly, we show that the existence of two roots with identical modulus corresponds to a node on the trajectory of one of these hypotrochoids. Moreover, we re-prove a classical result by Sommerville (Proposition 4.36) showing that all these nodes are located on a 1-fan in $\mathbb{C} \cong \mathbb{R}^2$, which yields the desired local description of the sets U_j^A for all $j \neq 0$ (Theorem 4.40).

A *hypotrochoid* with parameters $R, r, d \in \mathbb{R}_{>0}$, $R \geq r$ is the parametric curve γ in $\mathbb{R}^2 \cong \mathbb{C}$ given by

$$\gamma : [0, 2\pi) \to \mathbb{C}, \quad \phi \mapsto (R - r) \cdot e^{i \cdot \phi} + d \cdot e^{i \cdot \left(\frac{r-R}{r}\right) \cdot \phi}. \tag{4.23}$$

We say that a curve γ is a *hypotrochoid up to a rotation* if there exists some reparametrization $\rho_k : [0, 2\pi) \to [0, 2\pi), \phi \mapsto k + \phi \mod 2\pi$ with $k \in [0, 2\pi)$, such that $\gamma \circ \rho_k^{-1}$ is a hypotrochoid.

See Figure 4.6 for some examples of hypotrochoids. Note that hypocycloids (introduced in Section 2) are a special instance of hypotrochoids given by choosing $d = r$. One can show that further, special instances are ellipses ($R = 2r$) and rhodonea curves ($R - r = d$). Additional information on hypotrochoids can, e.g., be found in [**19**].

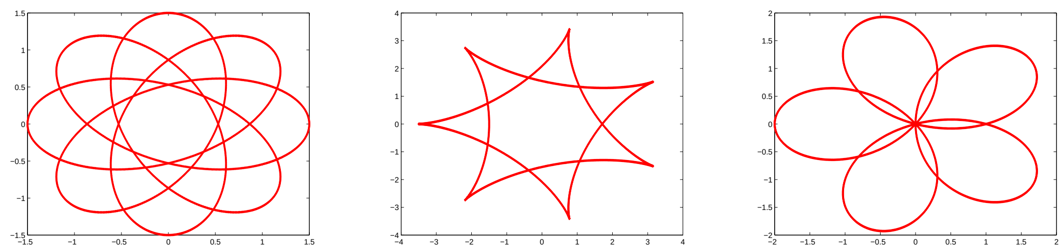


FIGURE 4.6. Hypotrochoids for $(R, r, d) = (8/3, 5/3, 1/2), (7/2, 5/2, 5/2)$ and $(5, 4, 1)$. The second curve is a hypocycloid and the third one is a rhodonea curve, which are both special instances of hypotrochoids.

Geometrically, a hypotrochoid is the trajectory of some fixed point with distance d from the center of a circle with radius r rolling in the interior of a circle with radius $R > r$ (see Figure 4.7).

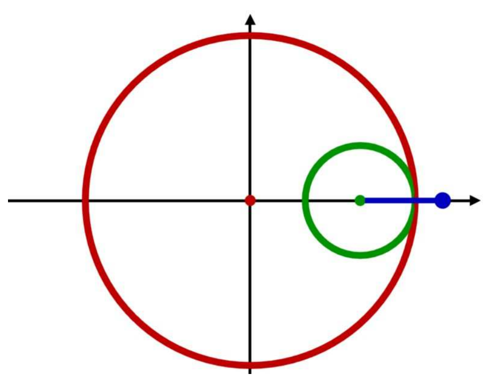


FIGURE 4.7. A geometrical explanation of a hypotrochoid. The green circle with radius r rolls inside the red circle of radius R. The hypotrochoid describes the trajectory of the blue point with distance d to the center of the green circle. The trajectory has finite length if $R/r \in \mathbb{Q}$.

With hypotrochoids defined and the fiber functions $f^{|\mathbf{z}|}$ of a polynomial f (see Chapter 3, Section 1) we have already all tools to solve Part (1) of Problem 4.31.

Theorem 4.32. *Let $f = z^s + p + qz^{-t}$, $(p, q) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ be a trinomial and $z^* \in \mathbb{C}^*$. f has a root of modulus $|z^*|$ if and only if p is located on the trajectory of a hypotrochoid up to a rotation with parameters $R = |z^*|^s/t \cdot (t + s)$, $r = |z^*|^s/t \cdot s$ and $d = |q| \cdot |z^*|^{-(s+t)}$.*

Note that, with respect to the proof of the theorem, it is a well known fact that the varieties of general trigonometric trinomials (which the fiber functions $f^{|z^*|}$ of trinomials happen to be) are the trajectory of a hypotrochoid (see e.g., [**46**]). For convenience, we prove this fact as part of the regular proof of the above theorem.

Proof. Let $f = z^s + p + qz^{-t}$ be a (Laurent-) trinomial. By definition of the fiber function $f^{|\mathbf{z}|}$ (see Chapter 3, Section 1), f has a root with modulus $|z^*| \in \mathbb{R}_{>0}$ if and only if $\mathcal{V}(f) \cap \mathbb{F}_{\log|z^*|} \neq \emptyset$, i.e., if and only if $\mathcal{V}(f^{|z^*|}) \neq \emptyset$, i.e., if and only if

$$(4.24) \qquad p + |z^*|^s \cdot e^{i \cdot s \cdot \phi} + |q| \cdot |z^*|^{-t} \cdot e^{i \cdot (\arg(q) - t \cdot \phi)} \; = \; 0 \quad \text{for some } \phi \in [0, 2\pi).$$

With the given parameters we obtain $(R - r) = |z^*|^s/t \cdot (t + s) - |z^*|^s/t \cdot s = |z^*|^s$ and $(r - R)/r = -|z^*|^s/(|z^*|^s/t \cdot s) = -t/s$. We insert the parameters in the fiber function $f^{|z^*|}$ and set $\phi' = \phi \cdot s$. With (4.24) we obtain

$$\mathcal{V}(f^{|z^*|}) \neq \emptyset \;\Leftrightarrow\; -p \in S_{|z^*|} \; = \; \left\{(R - r) \cdot e^{i \cdot \phi'} + d \cdot e^{i \cdot \left(\arg(q) + \frac{(r-R)}{r} \cdot \phi'\right)} : \phi' \in [0, 2\pi)\right\}.$$

By (4.23) $S_{|z^*|}$ is the trajectory of a hypotrochoid up to a rotation. □

Example 4.33. Let $f = z^5 + p + \frac{1}{2}z^{-3}$, $g = z^5 + p + \frac{5}{2}z^{-2}$, $h = z^4 + p + z^{-1}$. Then f, g resp. h has a root of modulus one if and only if $p \in \mathbb{C}$ is located on the trajectory of the hypotrochoids with parameters $(R, r, d) = (8/3, 5/3, 1/2), (7/2, 5/2, 5/2)$ resp. $(5, 4, 1)$ (see Figure 4.6).

The set $S_{|z|}$ defined in the proof of Theorem 4.32 is the image of the map $f^{|z|} - p$. In order to achieve a local description of the configuration space of trinomials, we need to show certain properties of $f^{|z|} - p$ and its trajectory first.

Lemma 4.34. *Let* $\gamma : [0, 2\pi) \to \mathbb{C}, \phi \mapsto e^{i \cdot s\phi} + e^{i \cdot (\arg(q) - t\phi)}$. *Then for every* $\phi \in [0, 2\pi)$ *and every* $k \in \{1, \ldots, s + t\}$ *we have*

$$\gamma(\phi + 2\pi k/(s + t)) \; = \; e^{i \cdot 2\pi k s/(s+t)} \cdot \gamma(\phi).$$

Due to this lemma and since s generates the complete group $\mathbb{Z}_{s+t}$ (recall that $\gcd(s, t) = 1$), we say $f^{|z|} - p$ is $(2\pi/(s + t))$-*quasi-periodic*. Geometrically, this means that the trajectory of $f^{|z|} - p$ is invariant under rotation of $2\pi k/(s + t)$ around the origin.

Proof. Let $\phi \in [0, 2\pi)$ and let $k \in \{1, \ldots, s + t\}$. Then we have

$$\begin{aligned}
\gamma(\phi + 2\pi k/(s + t)) \; &= \; e^{i \cdot s(\phi + 2\pi k/(s+t))} + e^{i \cdot (\arg(q) - t(\phi + 2\pi k/(s+t))} \\
&= \; e^{i \cdot 2\pi k s/(s+t)} \cdot \left(e^{is\phi} + e^{i \cdot (\arg(q) - t\phi)} \cdot e^{-i \cdot (2\pi k(s+t)/(s+t))}\right) \\
&= \; e^{i \cdot 2\pi k s/(s+t)} \cdot \gamma(\phi).
\end{aligned}$$

□

For parameters $s, t \in \mathbb{N}^*$ and $q \in \mathbb{C}^*$, e.g., given by a trinomial $f = z^s + p + qz^{-t}$, we define a 1-fan

$$(4.25) \; F(s, t, q) \; = \; \{\lambda \cdot e^{i \cdot (s \arg(q) + \pi \cdot k)/(s+t)} : \lambda \in \mathbb{R}_{\geq 0}, k \in \{0, 1, \ldots, 2(s + t) - 1\}\}.$$

Note that $F(s,t,q) \subset \mathbb{C}$, where $\mathbb{C}$ can be regarded as a subset of the augmented configuration space $\mathbb{C}^A_\Diamond$. We write $F(s,t,q) = F^{\text{odd}}(s,t,q) \cup F^{\text{even}}(s,t,q)$, where $F^{\text{odd}}(s,t,q)$ resp. $F^{\text{even}}(s,t,q)$ consists of the halfrays with odd k resp. even k (see Figure 4.9).

Lemma 4.35. *Let $f = z^s + p + qz^{-t}$, with $p \in \mathbb{C}$, $q \in \mathbb{C}^*$ and $|z^*| \in \mathbb{R}_{>0}$. Then the trajectory of $f^{|z^*|} - p$ is axially symmetric along every ray of $F(s,t,q)$. In particular,* $\arg(e^{i\cdot\phi s}) = \arg(e^{i\cdot(\arg(q) - \phi t)}) \mod \pi$ *if and only if $f^{|z^*|}(\phi) - p \in F(s,t,q)$.*

Proof. Let $\arg(q) = 0$. Then $f^{|z^*|} - p$ is a real curve and thus $f^{|z^*|}(\phi) = f^{|z^*|}(-\phi)$, i.e., $f^{|z^*|} - p$ is axially symmetric along the real line. The symmetry along the other rays of $F(s,t,q)$ follows from quasi-periodicity (see Lemma 4.34) of $f^{|z^*|} - p$ and from the fact that $f^{|z^*|}(\phi) = f^{|z^*|}(\phi + \pi)$ for all $\phi \in [0,\pi)$ if both s and t are odd. The general assumption follows since the trajectory of $f^{|z^*|} - p$ for $\arg(q) = \psi \in [0, 2\pi)$ is the image of the one for $\arg(q) = 0$ under the bijection $z \mapsto e^{i\cdot s\psi/(s+t)} z$. □

Proposition 4.36. *Let $\gamma : [0,2\pi) \to \mathbb{C}, \quad \phi \mapsto |z^*|^s e^{i\cdot s\phi} + |q||z^*|^{-t} e^{i\cdot(\arg(q) - t\phi)}$ be a hypotrochoid. Then all singularities of the trajectory of γ are located on $F(s,t,q)$.*

This fact was already observed by Sommerville in 1920 ([**81**]). Since this classical article is written very compactly, we present our own proof here.

Proof. W.l.o.g. we may assume $|z^*| = 1$ since $F(s,t,q)$ is invariant under scaling. We need to show the proposition for nodes and cusps. Since the trajectory for $\arg(q) = \psi \in [0,2\pi)$ is the image of the one for $\arg(q) = 0$ under the bijection $z \mapsto e^{i\cdot s\psi/(s+t)} z$, for nodes we need to show that for all $\phi, \psi \in [0,2\pi)$ with $\phi \neq \psi$ one has

$$\gamma(\phi) = \gamma(\psi) \;\; \Rightarrow \;\; \gamma\left(\phi - \frac{\arg(q)}{s+t}\right) \; = \; \lambda \cdot e^{i\cdot(ks\pi/(s+t))} \mod 2\pi,$$

with $\lambda \in \mathbb{R}_{>0}$ and $k \in \{0, \ldots, 2(s+t) - 1\}$. Note that for arbitrary $x, y \in [0, 2\pi)$

$$\begin{aligned} e^{i\cdot x} - e^{i\cdot y} &= \cos(x) + i\cdot\sin(x) - \cos(y) - i\cdot\sin(y) \\ &= \cos(x) - \cos(y) + i\cdot(\sin(x) - \sin(y)) \\ &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) + i\left(2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\right) \\ &= 2\sin\left(\frac{x-y}{2}\right)\cdot i e^{i\cdot\frac{x+y}{2}}. \end{aligned}$$

Hence

$$\begin{aligned}
& \gamma(\phi) \;=\; \gamma(\psi) \\
\Leftrightarrow\;& e^{i\cdot s\phi} + |q|\cdot e^{i\cdot(\arg(q)-t\phi)} \;=\; e^{i\cdot s\psi} + |q|\cdot e^{i\cdot(\arg(q)-t\psi)} \\
\Leftrightarrow\;& e^{i\cdot s\phi} - e^{i\cdot s\psi} = |q|\cdot\left(e^{i\cdot(\arg(q)-t\psi)} - e^{i\cdot(\arg(q)-t\phi)}\right) \\
\Leftrightarrow\;& (\phi+\psi)s \;=\; 2\arg(q) - (\phi+\psi)t + 2k\pi \quad \text{and} \\
& \sin\left(\frac{(\phi-\psi)s}{2}\right) \;=\; \pm|q|\cdot\sin\left(\frac{(\phi-\psi)t}{2}\right) \\
\Leftrightarrow\;& \sin\left(\frac{(\phi-(\frac{2\arg(q)+2k\pi}{s+t}-\phi))s}{2}\right) \;=\; \pm|q|\cdot\sin\left(\frac{(\phi-(\frac{2\arg(q)+2k\pi}{s+t}-\phi))t}{2}\right) \\
\Leftrightarrow\;& \sin\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot s\right)\cos\left(\frac{k\pi s}{s+t}\right) - \cos\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot s\right)\sin\left(\frac{k\pi s}{s+t}\right) = \\
& \pm|q|\cdot\sin\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot t\right)\cos\left(\frac{k\pi t}{s+t}\right) - \cos\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot t\right)\sin\left(\frac{k\pi t}{s+t}\right).
\end{aligned}$$

Observe $\cos(k\pi t/(s+t)) = \cos((-2k\pi(s+t)+k\pi t)/(s+t)) = \cos(k\pi - k\pi s/(s+t)) = (-1)^k\cos(k\pi s/(s+t))$ and analogously $\sin(k\pi t/(s+t)) = (-1)^{k+1}\sin(k\pi s/(s+t))$. Thus,

$$\begin{aligned}
& \gamma(\phi) \;=\; \gamma(\psi) \\
\Leftrightarrow\;& \left(\sin\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot s\right) \pm |q|\cdot(-1)^k\sin\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot t\right)\right)\cdot\cos\left(\frac{k\pi s}{s+t}\right) = \\
& \left(\cos\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot s\right) \pm |q|\cdot(-1)^{k+1}\cos\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot t\right)\right)\cdot\sin\left(\frac{k\pi s}{s+t}\right) \\
\Leftrightarrow\;& \frac{\left(\sin\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot s\right) \pm |q|\cdot(-1)^k\sin\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot t\right)\right)}{\left(\cos\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot s\right) \pm |q|\cdot(-1)^{k+1}\cos\left(\left(\phi-\frac{\arg(q)}{s+t}\right)\cdot t\right)\right)} \;=\; \tan\left(\frac{k\pi s}{s+t}\right) \\
\Leftrightarrow\;& \gamma\left(\tan\left(\phi-\frac{\arg(q)}{s+t}\right)\right) \;=\; \tan\left(\frac{k\pi s}{s+t}\right).
\end{aligned}$$

For cusps the proof is way easier since cusps may only occur at $\phi \in [0, 2\pi)$ with

$$e^{i\cdot\phi s} \;=\; e^{i\cdot(\arg(q)-\phi t)} \;\Leftrightarrow\; \phi \;=\; \arg(q)/(s+t).$$

Since $\gamma(\arg(q)/s+t) = \lambda\cdot e^{i\cdot\arg(q)s/(s+t)}$ the statement follow by quasi-periodicity provided by Lemma 4.34. □

We can now use the location of the nodes on the trajectory of hypotrochoids to show that for every fixed $q \in \mathbb{C}^*$ and every j the complement of U_j^A is always contained in the 1-fan $F(s,t,q)$.

Theorem 4.37. *Let $f = z^s + p + qz^{-t}$ with $p \in \mathbb{C}, q \in \mathbb{C}^*$. If $f \notin U_j^A$ for some $j \in \{-t+1,\ldots,s-1\}$, then $p \in F(s,t,q)$.*

Proof. Let $f = (z - a_1) \cdots (z - a_{s+t})$ with $|a_1| \leq \cdots \leq |a_{s+t}|$. Assume $f \notin U_j^A$, for some $j \in \{-t, -t+1, \ldots, s-1\}$. Hence, there is a z^* with $|a_j| = |a_{j+1}| = |z^*|$. Therefore, there exist $\phi, \psi \in [0, 2\pi)$ with $\phi \neq \psi$ and $f^{|z^*|}(\phi) = f^{|z^*|}(\psi) = 0$ resp. $f^{|z^*|}(\phi) - p = f^{|z^*|}(\psi) - p = -p$. $f^{|z^*|} - p$ is a hypotrochoid (by Theorem 4.32), i.e., a closed curve. Thus, $f^{|z^*|} - p$ has a node at $p \in \mathbb{C}$. Lemma 4.36 implies $p \in F(s,t,q)$. Hence, for a fixed $q \in \mathbb{C}^*$, i.e., $f \in \{(1,p,q) : p \in \mathbb{C}\}$ we have $f \in F(s,t,q)$, if $f \notin U_j^A$. If f has a multiple root a_j, then the derivative $\partial f / \partial z$ of f vanishes at a_j and thus in particular its derivative of $\partial f^{|a_j|} / \partial \phi$ vanishes at $\arg(a_j)$. This means due to $\partial f^{|a_j|}/\partial \phi = \partial (f^{|a_j|} - p)/\partial \phi$ that the hypotrochoid $f^{|a_j|} - p$ is singular at $\arg(a_j)$ and thus Lemma 4.36 yields $p \in F(s,t,q)$. □

Now, we want to describe, which subsets of $F(s,t,q)$ belong to a set U_j^A. This requires the following lemma.

Lemma 4.38. *Let $f = z^s + p + qz^{-t}$ with $p, q \in \mathbb{C}^*$ and $w \in \mathbb{R}_{>0}$. Then there are at most two roots $a_j, a_k \in \mathcal{V}(f)$ with $|a_j| = |a_k| = w$.*

Proof. The lemma follows from Bohl's Theorem 4.29. Indeed, assume there were three roots a_1, a_2, a_3 such that $|a_1| = |a_2| = |a_3| = w$. Let k be the number of roots in the open circle $\mathcal{B}_w(0) \subset \mathbb{C}$ with radius w around the origin. Hence, for $\varepsilon > 0$ we have $\#(\mathcal{B}_{w+\varepsilon}(0) \cap \mathcal{V}(f)) = k + 3$. Since there are roots with modulus w, the triangle Δ with edges of length w^s, $|p|$ and $|q|w^{-t}$ is either well defined or degenerated to a line and k is the number of integers in the open interval I bounded by (4.21) and (4.22). Changing the radius of the circle from w to $w + \varepsilon$ changes I only by a $\delta_\varepsilon > 0$. Thus, k increases at most by two, which is a contradiction as long as Δ is well defined for $w + \varepsilon$. If the latter is not the case, i.e., Δ is a line, then the interval I contains 0 resp. at least $t - 2$ resp. $(s + t) - 2$ integers. This also yields a contradiction with Theorem 4.28. □

To describe, which subsets of $F(s,t,q)$ belong to a set U_j^A, we make use of the following surprising observation about real trinomials, which is also a nice stand-alone statement.

Theorem 4.39. *Let $f = z^s + p + qz^{-t}$ with $p, q \in \mathbb{R}^*$ and $\mathcal{V}(f) = \{a_1 \ldots, a_{s+t}\}$, such that $|a_1| \leq \cdots \leq |a_{s+t}|$. Assume a_j is real. Then $j \in \{1, t, t+1, s+t\}$.*

Proof. Let a_j be a real root of f. Then all three monomials $(a_j)^s, p$ and qa^{-t} are real. Thus, one of the monomials equals the sum of the two others. Hence, if we continuously increase the modulus of the dominating monomial by $\varepsilon > 0$, then the resulting polynomial g is lopsided at $\mathrm{Log}(a_j)$ (see Chapter 2, Section 5) and the ordering of the zeros is preserved (under the right labeling for the case that f has a multiple real root). By Theorem 2.30 the complement component $E_\alpha(g)$ of $\mathcal{A}(g)$, which contains $\log|a_j|$ has order $s, 0$ or $-t$. Therefore, $\log|a_j|$ is in the closure of the complement components $E_{-t}(f), E_0(f)$ or $E_s(f)$ (where $E_0(f)$ is degenerated to the empty set in the case that a_j is a multiple real root). □

With this theorem we obtained the final tool needed to solve Part (2) of Problem 4.31.

Theorem 4.40. *Let $f_p = z^{s+t} + pz^t + q$ with $q \in \mathbb{C}^*$ be a parametric family with parameter $p \in \mathbb{C}$. For $j \in \{1, \ldots, s+t-1\} \setminus \{t\}$ it holds that*

$$U_j^A \cap \{(1,p,q) : p \in \mathbb{C}\} \;=\; \begin{cases} \{(1,p,q) : p \in \mathbb{C}\} \setminus F^{\text{odd}}(s,t,q) & \text{if } s \cdot j \text{ is odd,} \\ \{(1,p,q) : p \in \mathbb{C}\} \setminus F^{\text{even}}(s,t,q) & \text{if } s \cdot j \text{ is even.} \end{cases}$$

For U_t^A holds the same result with the exception that $f_p \in U_t^A$, also if there exists a $z \in \mathbb{C}^*$ such that f_p is lopsided with dominating term pz^t.

Note that as a consequence of this theorem $U_j^A \cap \{(1,p,q) : p \in \mathbb{C}\}$ is not connected. Thus, it is a partial solution of Problem 2.23.

Proof. Let $a_1, \ldots, a_{s+t} \in \mathbb{C}^*$ denote the roots of f_p (depending on p) with $|a_1| \leq \cdots \leq |a_{s+t}|$. Theorem 4.37 yields that we only need to investigate the case $p \in F(s,t,q)$. An outline of the proof is that we can reduce the whole problem to the case of real trinomials. For real trinomials the membership of a polynomial in a set U_α^A is determined by the location of its real roots. This location can be done via the lopsidedness condition and Theorem 4.39.

Since $F(s,t,q)$ is invariant under changing $|q|$, we may assume $|q| = 1$. And since we might remap $z \mapsto e^{i \cdot \pi \cdot \arg(q)/(s+t)} z$ yielding $f_p \mapsto e^{i \cdot \pi \arg(q)} \cdot (z^{s+t} + p' z^t + |q|)$ with $p' = e^{i \cdot \pi \cdot \arg(q)(t-1)/(s+t)}$, we can even assume $q = 1$.

For $p = 0$ every root has modulus 1 and thus $f_0 \notin U_\alpha^A$ for every $\alpha \in \{1, 2, \ldots, s+t-1\}$.

For the special case $p \in \mathbb{R}$ the real roots of f_p then are given by $x^s + p = -x^{-t}$, where $x \in \mathbb{R}$. Due to monotonicity of x^s and x^{-t} this leads, depending on signs and the parity of s and t, to the four situations depicted in Figure 4.8.

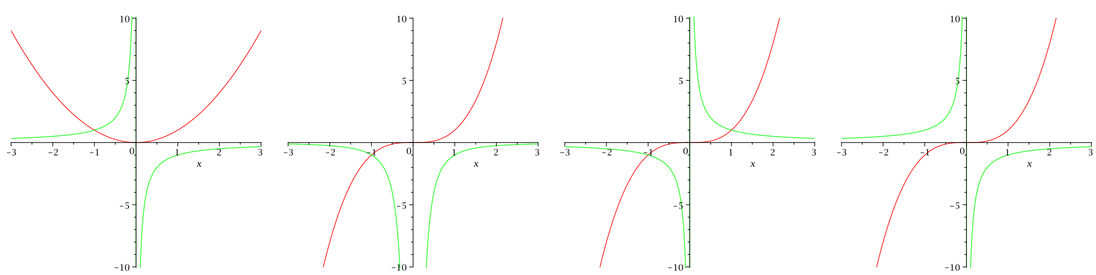


Figure 4.8. Real root situation for $x^s + p = -x^{-t}$ with $p = 0$ here.

In particular, for every $p \in \mathbb{R}$ all roots of f_p are either real or appear in complex conjugated pairs, i.e., in pairs of two roots with the same modulus. By Lemma 4.38 it suffices in the real case to determine, which a_j are real in order to determine, which U_j^A intersect $F^{\text{odd}}(s,t,1)$ resp. $F^{\text{even}}(s,t,1)$. Recall that by Theorem 4.39 the only possible real roots are a_1, a_t, a_{t+1} and a_{s+t}.

Let s be odd and t be even. Since by Theorem 4.37 for every j holds that $f_p \notin U_j^A$ only if $f_p \in F(s,t,q)$, it suffices due to quasi-periodicity (Lemma 4.34) to investigate the case $p \in \mathbb{R}$. Due to monotonicity of $x^s + p$ and x^{-t} the polynomial f_p has either one or three real roots and thus $s+t-1$ resp. $s+t-3$ complex conjugated roots.

There is a unique choice of p such that f_p switches from one to three real roots. This can be seen e.g., by the fact that for such a p the discriminant of f_p has to vanish and for fixed q the discriminant of a trinomial has a unique root in p (see e.g., [**26**]).

Let $p^* \in \mathbb{R}_{<0}$ be the point where f_p switches from one to three real roots, i.e., lopsidedness is attained at some point w (see the proof of Theorem 4.39). Hence, we know that for every $p < p^*$, we have $w \in U_0^A$ (Theorem 2.28; recall that $p^* < 0$). Thus, if three real roots exist, then a_t and a_{t+1} need to be two of them (Theorem 2.30). Since s is odd and t even Theorem 4.39 yields that the third real root is a_1 for every $p < 0$. Therefore, we have $|a_r| = |a_{r+1}|$ for every $r \in \{2, 4, \ldots, t+s-1\}$ if $p < 0$. Since s is odd and t is even $p \in \mathbb{R}_{<0}$ is equivalent to $p \in F(s,t,q)^{\text{even}}$.

Let now $p > 0$, i.e., f_p has in particular only one real root x. We have $x < 0$ since $(-a_1)\cdots(-a_{s+t}) = q = 1$ and $z \cdot \overline{z} \in \mathbb{R}_{>0}$ for every $z \in \mathbb{C}^*$. Thus $x^s < 0$ and $x^{-t} > 0$ and hence $x^s = x^{-t} + p$ and therefore the norm of the real root tends to infinity for $p \to \infty$. Since in this case x^s is the dominating term, Theorem 4.39 yields that $x \in \partial \overline{E_{s+t}(f)}$ and thus $x = a_{s+t}$. Hence, we have $|a_r| = |a_{r+1}|$ for every $r \in \{1, 3, \ldots, t+s-2\}$ if $p > 0$, which is equivalent to $p \in F(s,t,q)^{\text{odd}}$.

The case s even and t odd follows directly from the same argumentation.

Let as final case s and t both be odd. Since $f_p \in F(s,t,q)$ we can, due to quasi-periodicity (Lemma 4.34), assume that $p \in \mathbb{R}$ or $p = \lambda \cdot e^{i \cdot \pi/(2(s+t))}$.

Assume first $p \in \mathbb{R}$, i.e., $p \in F(s,t,q)^{\text{even}}$. Then, since s, t odd, f_p is symmetric in $\pm p$ and the real roots of f_p are given by $x^s + p = -x^{-t}$, which obviously only exist if $|p|$ is large enough. Furthermore, there exists a unique choice of $|p|$ such that f_p switches from zero to two real roots (see also [**26**]). With Theorem 4.39 it follows that the real roots are a_t and a_{t+1} and thus we have $|a_r| = |a_{r+1}|$ for every $r \in \{1, 3, \ldots, t+s-2\} \setminus \{t\}$.

Assume now $p = \lambda \cdot e^{i \cdot \pi/(2(s+t))}$, i.e., $p \in F(s,t,q)^{\text{even}}$. We map $z \mapsto -z$ and obtain

$$\begin{aligned} & (-z)^s + e^{i \cdot \pi/(2(s+t))} \cdot (-z)^{-t} + 1 \\ = \;& -z^s + e^{i \cdot \pi/(2(s+t))} \cdot (e^{i \cdot \pi(2(s+t))/(2(s+t))} \cdot z)^{-t} + 1 \\ = \;& -z^s + e^{i \cdot \pi(t+1)} \cdot z^{-t} + 1 \\ = \;& -z^s + z^{-t} + 1. \end{aligned}$$

Then, since s and t are odd, f_p is symmetric in $\pm p$ and the real roots of f_p are given by $x^s + p = x^{-t}$. Obviously, f_p has two real roots for all choices of $p \in \mathbb{R}$ (see also [**26**]) and for these roots the dominating term is either x^s or x^{-t}. Thus, the real roots are a_1 and a_{s+t} (see the proof of Theorem 4.39) and we have $|a_r| = |a_{r+1}|$ for every $r \in \{2, 4, \ldots, t+s-3\}$. □

Example 4.41. We compute the absolute values of the roots of some real trinomials to depict exemplarily the different situations in the upper theorem.

(1) Let $f = x^5 + 6x^2 + 1$, i.e., s is odd, t is even and $p \in F(s,t,q)^{\text{even}}$. The absolute values of $\mathcal{V}(f)$ are approximately

$$0.4082, 0.4082, 1.8030, 1.8030, 1.8462,$$

i.e., $a_5 = a_{s+t}$ is the unique real root and thus $f \in U_0^A \cap U_2^A \cap U_4^A \cap U_5^A$.

(2) Let $f = x^5 - 6x^2 + 1$, i.e., s is odd, t is even and $p \in F(s,t,q)^{\text{odd}}$. The absolute values of $\mathcal{V}(f)$ are approximately

$$0.4060, 0.4106, 1.7849, 1.8332, 1.8332,$$

i.e., $a_1, a_2 = a_t$ and $a_3 = a_{t+1}$ are the real roots and thus $f \in U_0^A \cap U_1^A \cap U_2^A \cap U_3^A \cap U_5^A$.

(3) $f = x^5 + 6x^3 + 1$, i.e., s is even, t is odd and $p \in F(s,t,q)^{\text{even}}$. The absolute values of $\mathcal{V}(f)$ are approximately

$$0.5416, 0.5546, 0.5546, 2.4498, 2.4498,$$

i.e., a_1 is the unique real root and thus $f \in U_0^A \cap U_1^A \cap U_3^A \cap U_5^A$.

(4) Let $f = x^4 + 0.5x^1 + 1$, i.e., s is odd, t is odd and $p \in F(s,t,q)^{\text{even}}$. The absolute values of $\mathcal{V}(f)$ are approximately

$$0.916, 0.916, 1.091, 1.091,$$

i.e., there exist no real roots und thus $f \in U_0^A \cap U_2^A \cap U_4^A$.

Example 4.42. As in Example 4.33, let $f = z^5 + p + 0.5z^{-3}$, $g = z^5 + p + 2.5z^{-2}$ and $h = z^4 + p + z^{-1}$. Then f, g, h have two roots with the same modulus if and only if p is located on the rays of the blue 1-fan in Figure 4.9.

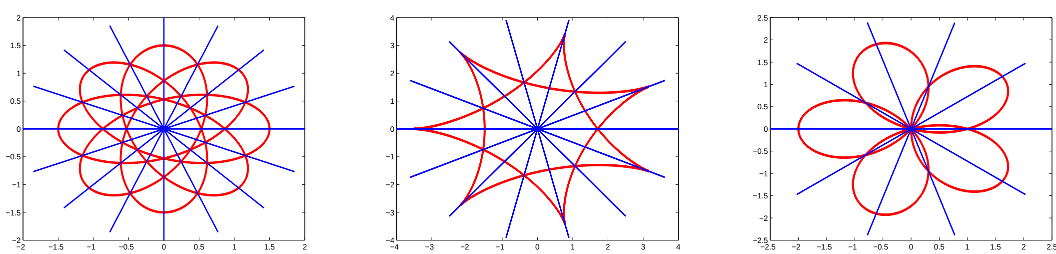


FIGURE 4.9. Three hypotrochoids: The trajectory of $f^{|1|} - p$ for $f = z^5 + p + 0.5z^{-3}$, $f = z^5 + p + 2.5z^{-2}$ and $f = z^4 + p + z^{-1}$ with their corresponding 1-fan $F(s,t,q)$ (the blue rays).

Using Theorem 4.40 we are able to close the gap in the proof of Part (2) of Rullgård's Theorem 2.19, where he gave, for $A \subset \mathbb{Z}^n$, a sufficient condition for an $U_\alpha^A \neq \emptyset$ if $\alpha \notin A$. His statement is implied by the following theorem. Recall that for a set $A \subset \mathbb{Z}^n$ the lattice generated by A is denoted by $\mathcal{L}_A$ (see Chapter 2, Section 4).

Theorem 4.43. *Let $A \subset \mathbb{Z}^n$ and $l \in \mathbb{R}^n$ a line with $\mathcal{L}_{A \cap l} = \mathbb{Z}^n \cap l$. Then $U_\alpha^A \neq \emptyset$ for all $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n \cap l$. In particular, this is also true for $\alpha \notin A$.*

PROOF. We can assume that l intersects $\mathbb{Z}^n$ in at least 3 points since otherwise the theorem is trivial. Since it is a well known fact that for a translation or rotation of A by a unimodular matrix M (i.e., a map $\sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha \mapsto \sum_{M(\alpha) \in M(A)} b_\alpha \mathbf{z}^{M(\alpha)}$) holds

$U_{M(\alpha)}^{M(A)} = U_\alpha^A$ (see e.g., [**20**, Remark 2.3] and [**77**, Theorem 7]), we may assume w.l.o.g. $l = \{(z_1, 0, \ldots, 0) : z_1 \in \mathbb{Z}\}$ and $A \cap l = \{(0, \ldots, 0), (s, 0, \ldots, 0), (s+t, 0, \ldots, 0)\}$ with s, t coprime.

Let $\alpha = (\alpha_1, 0, \ldots, 0) \in (\text{conv}(A) \cap l) \setminus A$. Note that it suffices to investigate the case $\alpha \notin A$ since otherwise we can construct a polynomial in U_α^A due to Theorem 2.6 easily. Let $f = z_1^{s+t} + e^{i \cdot \frac{\pi}{2(s+t)}} z_1^s + 1$ be a trinomial in $\mathbf{z}$, which is invariant on $z_2, \ldots, z_n$. Since, by construction, $f \notin F(s, t, 1)$, Theorem 4.40 yields $E_\alpha(f) \neq \emptyset$ for all $\alpha \in (\text{conv}(A) \cap \mathbb{Z}) \setminus \{s\}$.

Let $\text{Log}(\mathbf{z}) = \mathbf{w} \in E_\alpha(f)$. We define $\delta = \min_{\phi \in [0,2\pi)^n} \left| f^{|\mathbf{z}|}(\phi) \right| > 0$ as the minimal value attained by f at the fiber over $\mathbf{w}$ (note that the minimum is greater than 0 since $\mathbf{w} \notin \mathcal{A}(f)$ and the minimum exists since the fiber $\mathbb{F}_\mathbf{w}$ is compact). We define $\kappa = \max_{\alpha \in A \setminus (A \cap l)} |\mathbf{z}^\alpha|$ as the maximal modular value a monomial with exponent in A but not belonging to f attains at $\text{Log}^{-1}(\mathbf{w})$. Let furthermore $d = \#A - 3$.

Now we construct a polynomial $g = f + \sum_{\alpha \in A \setminus l} \varepsilon\, \mathbf{z}^\alpha \in \mathbb{C}^A$ with $\varepsilon = \delta/(2d\kappa)$. If we evaluate g at an arbitrary point $\mathbf{v} \in \mathbb{F}_\mathbf{w}$, we obtain

$$|g(\mathbf{v})| \;\geq\; |f(\mathbf{v})| - \left| \sum_{\alpha \in A \setminus l} \varepsilon\, \mathbf{v}^\alpha \right| \;\geq\; |f(\mathbf{v})| - \sum_{\alpha \in A \setminus l} \varepsilon\, |\mathbf{v}^\alpha| \;\geq\; \delta - d \cdot \frac{\delta}{2d\kappa} \cdot \kappa \;=\; \frac{\delta}{2} \;>\; 0.$$

Thus, $\mathbf{w} \notin \mathcal{A}(g)$.

We compute the order of $\mathbf{w}$ with respect to g. Let, for each $j \in \{1, \ldots, n\}$, f_j and g_j be the univariate polynomials in z_j obtained from f and g by setting $z_i = v_i$ for all $i \neq j$. Since $|f_1(z_1)| = |f(z_1, v_2, \ldots, v_n)| \geq \delta$ for all z_1 with $|z_1| = |v_1|$ and $|g_1(z_1)|$ differs from $|f_1(z_1)|$ by at most $\delta/2$, we have $w_1 \in E_{\alpha_1}(f_1) \Rightarrow w_1 \in E_{\alpha_1}(g_1)$. For every other j we know that $f_j(z_j)$ equals the constant given by $f_1(v_1)$. Thus, $|f(\mathbf{v})| = \delta$ belongs to the constant term of g_j and $\sum_{\alpha \in A \setminus l} \varepsilon |\mathbf{v}^\alpha| \leq \delta/2$ for all $v_j \in \mathbb{F}_{w_j}$. Thus, g_j is lopsided in v_j with the constant term being the dominating term. Hence, $w_j \in E_0(g_j)$ for all $1 < j \leq n$ by Theorem 2.6. Therefore, we have in total $\text{ord}(\mathbf{w}) = \alpha$ with respect to g and thus $g \in U_\alpha^A$, i.e., $U_\alpha^A \neq \emptyset$. □

We use the construction of the upper theorem to provide an example for an amoeba of a multivariate polynomial with a complement component of order α, although α is not contained in the support set $A \subset \mathbb{Z}^n$ of the corresponding polynomial. For practical construction we obviously do not need to compute minima of fiber functions or maxima attained by monomials but may just choose $\varepsilon > 0$ sufficiently small.

Example 4.44. Let $f = z_2 + e^{i \cdot 0.2\pi} \cdot z_1 z_2 + z_1 z_3 + \varepsilon \cdot (z_1 + z_1^2 + z_1 z_2^2 + z_1^2 z_2^2)$ with $\varepsilon \in \mathbb{R}_{>0}$. If ε is sufficiently small, then $E_{(1,2)}(f) \neq \emptyset$ by Theorem 4.43 although $(1, 2)$ is not contained in the support set A (see Figure 4.10).

A nice as well as sad consequence of this example resp. Theorem 4.43 is that the complement induced tropical hypersurface $\mathcal{C}(f)$ of a polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ in general is not homotopy equivalent to its amoeba $\mathcal{A}(f)$, which solves Problem 2.16.

Corollary 4.45. *For general $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ $\mathcal{A}(f)$ and $\mathcal{C}(f)$ are not homotopy equivalent.*

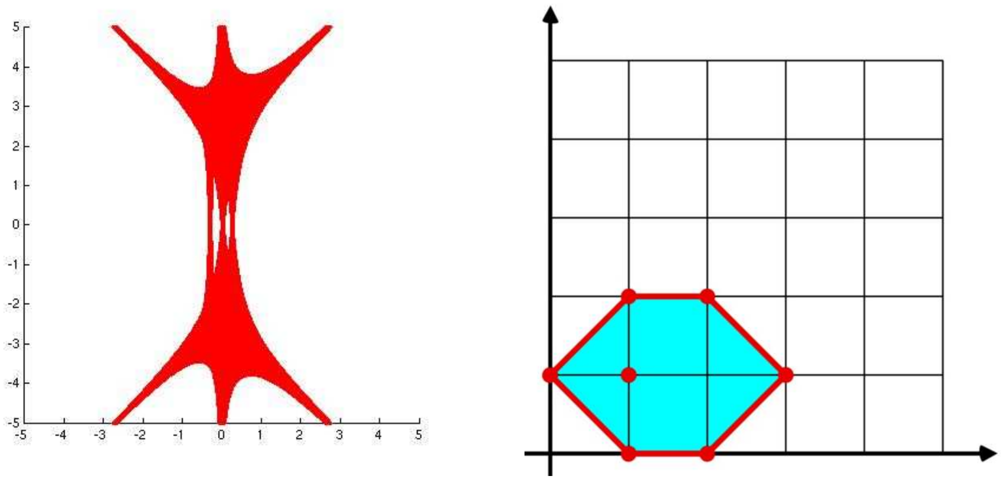


FIGURE 4.10. Left picture: The amoeba of $f = z_2 + e^{i \cdot 0.2\pi} \cdot z_1 z_2 + z_1 z_2^3 + \varepsilon \cdot (z_1 + z_1^2 + z_1 z_2^2 + z_1^2 z_2^2)$ for $\varepsilon = 0.1$; right picture: The Newton polytope of f.

PROOF. Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ with support set $A \subset \mathbb{Z}^n$ such that $\mathcal{A}(f)$ has a complement component with order $\alpha \notin A$. Such polynomials exist due to Theorem 4.43 (and an explicit one is given in Example 4.44). Since f does not have a term $b_\alpha \mathbf{z}^\alpha$, we can formally add such a monomial with $b_\alpha = 0$. We investigate the corresponding tropical polynomial $\text{trop}(f_{|C})$. Since $\log|b_\alpha| = -\infty$ in the tropical semi-ring, there exists no $\mathbf{w} \in \mathbb{R}^n$ such that $\log|b_\alpha|\mathbf{w}^\alpha$ attains the maximum in $\text{trop}(f_{|C})$. Hence, the homotopy of the corresponding tropical hypersurface $\mathcal{T}(\text{trop}(f_{|C}))$, which is exactly the complement induced tropical hypersurface $\mathcal{C}(f)$, needs to have another homotopy than $\mathcal{A}(f)$. $\square$

3.3. The Topological Structure of the Configuration Space of Trinomials. The aim of this section is to determine the topological structure of the complete configuration space of trinomials with respect to their amoebas. We do this in the following sense. We say two polynomials $f, g \in \mathbb{C}^A$ are *equivalent* if their amoebas are homotopy equivalent, i.e.,

$$f \sim g \quad :\Leftrightarrow \quad f \in U_j^A \text{ if and only if } g \in U_j^A \text{ for all } j \in \{-t, \ldots, s\}.$$

As a main result we show that for every $j \in \{-t+1, \ldots, s-1\} \setminus \{0\}$ both the set $U_j^A \subseteq \mathbb{C}^A$ and its complement $(U_j^A)^c \subset \mathbb{C}^A$ can be deformation retracted to an $(s+t)$-sheeted covering of an S^1. This answers not only Part (3) of Problem 4.31 but allows us furthermore to answer our initial question concerning the topology of the sets U_α^A in the configuration space $\mathbb{C}^A$ (Problems 2.22 and 2.24) since such coverings are connected but their fundamental group is non-trivial and thus they are not simply connected (see e.g., [**28**]).

As a motivation we provide an example showing that a set $U_j^A \subseteq \mathbb{C}^A$ can be connected although none of the sets $U_j^A \cap \{(1, p, q) \in \mathbb{C}^A : p \in \mathbb{C}\}$ for $q \in \mathbb{C}^*$ fixed is.

Example 4.46. Let $f = z^2 + 1.5 \cdot e^{i \cdot \arg(p)} + e^{i \cdot \arg(q)} z^{-1}$ with $f = (z - a_1)(z - a_2)(z - a_3)$ and $|a_1| \leq |a_2| \leq |a_3|$. We want to construct a path γ in $\mathbb{C}^A$ from $(p_1, q_1) = (1.5 \cdot e^{i \cdot \pi/2}, 1)$ to $(p_2, q_2) = (1.5 \cdot e^{-i \cdot \pi/6}, 1)$ such that $\gamma \in U_1^A$, i.e., $|a_2| \neq |a_3|$ for every point on γ. Theorem 4.40 implies that this is impossible if $\arg(q)$ remains constant for every point on γ. Analogously, there are e.g., points on the paths $\eta_1 : [0,1] \to \mathbb{C}^A, k \mapsto (1.5 \cdot e^{i \cdot \pi/2}, e^{i \cdot 2k\pi})$ and $\eta_2 : [0,1] \to \mathbb{C}^A, k \mapsto (1.5 \cdot e^{i(1/4+k) \cdot \pi/2}, e^{i \cdot 2k\pi})$ with $|a_2| = |a_3|$. But there is a path given by $\eta_3 : [0,1] \to \mathbb{C}^A, k \mapsto (1.5 \cdot e^{i(1/4+2k/3) \cdot \pi/2}, e^{i \cdot 2k\pi})$ from (p_1, q_1) to (p_2, q_2) that is completely contained in U_1^A (see Figure 4.11).

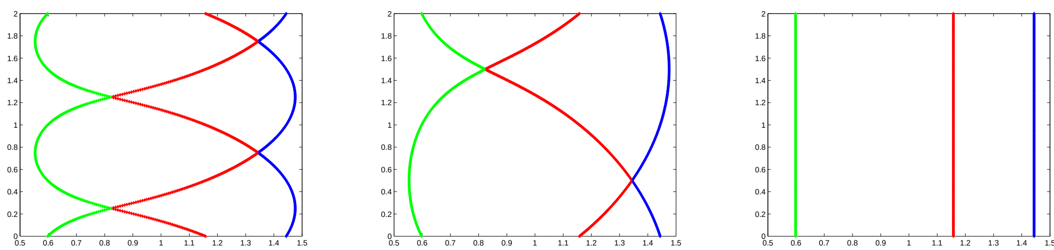


FIGURE 4.11. The modulis of the roots of f along the paths η_1, η_2 and η_3 (from left to right).

As a first step to prove connectivity of the sets $U_j^A \subseteq \mathbb{C}^A$ compute its fundamental group, we show that it suffices to investigate the situation of fixed $|p|$ and $|q|$.

Lemma 4.47. *Every set $U_j^A \subseteq \mathbb{C}^A$ with $j \in \{-t+1, \ldots, s-1\} \setminus \{0\}$ and its complement can be deformation retracted to a subset $\widehat{U}_j^A$ of the real torus $T = \{\left(e^{i \cdot \arg(p)}, e^{i \cdot \arg(q)}\right) : p, q \in \mathbb{C}^*\}$.*

PROOF. We can identify $\mathbb{C}^A$ with $(\mathbb{C}^*)^2$. $(\mathbb{C}^*)^2$ comes with a natural fibration $\mathbb{R}_{>0} \to (\mathbb{C}^*)^2 \to (S^1)^2$ given by the Arg map, which can be regarded as the canonical counterpart of the fibration of the Log-map described in Chapter 3, Section 1. I.e., the following diagram commutes

$$\begin{array}{ccc} \mathbb{C}^A & \xrightarrow{\quad h \quad} & T \times \mathbb{R}_{>0} \\ & {\scriptstyle \mathrm{Arg}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & T & \end{array}$$

where h is a homeomorphism and π is the natural projection on the first component. Now, we investigate the homotopy

$$F : (\mathbb{C}^*)^2 \times [0,1] \to (\mathbb{C}^*)^2, \quad ((p,q), l) \mapsto \left(\frac{p}{(1-l) + l \cdot |p|}, \frac{q}{(1-l) + l \cdot |q|} \right).$$

Obviously, we have the identity for $l = 0$ and the projection on T for $l = 1$. Recall that by Theorem 4.40 U_j^A is invariant under changing $|q|$ and, for $q \in \mathbb{C}^*$ fixed holds that $(p,q) \in U_j^A$ implies $(\lambda p, q) \in U_j^A$ for every $\lambda \in \mathbb{R}_{>0}$. Hence, $F_{|U_j^A}$ is indeed a deformation retraction of U_j^A to a subset of T. For the complement the argumentation works the same way. □

Lemma 4.48. *Let $f = z^s + p + qz^{-t}$ with $p, q \in \mathbb{C}^*$. Then $f \sim g$ for every g on the path $\gamma_{(p,q)} : [0,1] \to \mathbb{C}^A, \quad \phi \mapsto (p \cdot e^{-i \cdot 2\pi t\phi/(s+t)}, q \cdot e^{i \cdot 2\pi\phi})$. In particular, for $p, q \in \mathbb{C}^*$ on the real torus*

$$T_{(p,q)} \quad = \quad \{(|p|e^{i\pi \arg(p)}, |q|e^{i\pi \arg(q)}) \ : \ \arg(p), \arg(q) \in [0, 2\pi)\} \subset \mathbb{C}^A \tag{4.26}$$

for polynomials f and g with coefficients $(\arg(p), \arg(q))$ and $(-2\pi t/(s+t) \arg(p), \arg(q))$ holds $f \sim g$.

Proof. We have $f \sim g$ if and only if f and g are contained in the same sets U_j^A. For every fixed $q \in \mathbb{C}^*$ we have by Theorem 4.40 that $f \in U_j^A$ if and only if $f \notin F(s,t,q)^{\text{odd}}$ and $s \cdot j$ is odd resp. $f \notin F(s,t,q)^{\text{even}}$ and $s \cdot j$ is even. We also have $f \in U_0^A$ if f is lopsided at some point $z \in \mathbb{C}$ with dominating term p. Since lopsidedness is either given for every point on a torus $T_{(p,q)}$ or for none (Proposition 4.14), we can omit this special case.

By Definition (4.25) of $F(s,t,q)$ each of its rays is given by $\lambda \cdot e^{(i \cdot s \arg(q) + k\pi)/(s+t)}$. Thus, the path every point p on such a ray takes for continuously moving $\arg(q)$ to $2\pi + \arg(q)$ is given by $\gamma_{(p,q)}$. Since $F(s,t,q)$ is invariant under changing $|q|$, the lemma follows. □

Figure 4.12 depicts how the structure of $\mathbb{C}^A$ locally changes under changing $\arg(q)$.

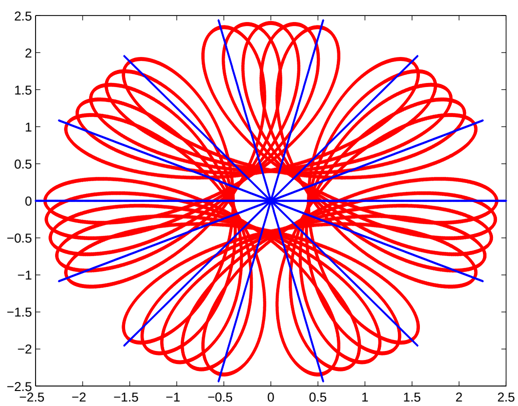


Figure 4.12. The trajectory of $f^{|1|} - p$ for $f = z^5 + p + 1.4e^{i \arg(q)} z^{-2}$ and $\arg(q) \in \{0, \pi/20, \pi/10, 3/20\pi, \pi/5\}$.

Note that since $\gcd(s,t) = 1$ we have $2\pi \cdot ks/(s+t) \equiv 0 \mod 2\pi$ if and only if $k \in (s+t)\mathbb{Z}$. And since $(p,q) \sim (p \cdot e^{i \cdot 2\pi s/(s+t)}, q)$, every U_j^A and $(U_j^A)^c$ with $j \in \{-t+1, \ldots, s-1\} \setminus \{0\}$ is invariant under the group $\mathbb{Z}_{s+t}$ acting on $T_{(p,q)}$ by

$$\mathbb{Z}_{s+t} \times T_{(p,q)} \to T_{(p,q)}, \quad (k, (p,q)) \mapsto (p \cdot e^{i \cdot 2\pi ks/(s+t)}, q). \tag{4.27}$$

For the real standard torus $T = \{(e^{i \cdot \arg(p)}, e^{i \cdot \arg(q)}) \ : \ p, q \in \mathbb{C}^*\}$ let $\rho_{(\arg(p), \arg(q))}$ denote the path given by $\gamma_{(s+t-1)(p,q)} \circ \gamma_{(s+t-2)(p,q)} \circ \cdots \circ \gamma_{1(p,q)} \circ \gamma_{0(p,q)}$ for some $p, q \in T$, where $k(p,q)$ denotes the image under the upper group action of $\mathbb{Z}_{s+t}$ on T. Observe that $\rho_{(\arg(p),\arg(q))}$ is closed and not contractable on T by construction.

With this construction we can describe the sets $\widehat{U}_j^A$ and its complements on the real standard torus T.

Lemma 4.49. *Let $j \in \{+1, \ldots, s+t-1\} \setminus \{t\}$. Then*

$$\begin{aligned} \rho_{(0,0)} &= (\widehat{U}_j^A)^c \text{ and } \rho_{(\pi/(s+t),0)} \text{ is a deformation retract of } \widehat{U}_j^A \text{ for } s \cdot j \text{ even}, \\ \rho_{(\pi/(s+t),0)} &= (\widehat{U}_j^A)^c \text{ and } \rho_{(0,0)} \text{ is a deformation retract of } \widehat{U}_j^A \text{ for } s \cdot j \text{ odd}. \end{aligned}$$

Proof. Recall that by Lemma 4.47 $\widehat{U}_j^A$ is the deformation retract of U_j^A to a subset of the standard torus T. Let $s \cdot j$ be even and $j \neq t$. By Theorem 4.40 $f = z^{s+t} + pz^t + q$ does not belong to U_j^A if and only if $p \in F(s,t,q)^{\text{even}}$. By (4.25) for $|p| = |q| = 1$ this is the case if and only if $\arg(p) \neq (\arg(q)s + 2\pi k)/(s+t)$ for $k \in \{1, \ldots, s+t\}$. By definition of $\rho(p,q)$ these are exactly the points on $\rho(0,0)$.

Now, we investigate $\widehat{U}_j^A = T \setminus (\widehat{U}_j^A)^c = T \setminus \rho(0,0)$. Since $\rho_{(\pi/(s+t),0)}$ is obtained from $\rho_{(0,0)}$ by the translation $(\arg(p), \arg(q)) \mapsto (\arg(p) + \pi/(s+t), \arg(q))$, we have $\rho_{(\pi/(s+t),0)} \subset U_j^A$. We investigate the homotopy

$$\begin{aligned} \widehat{F} : \widehat{U}_j^A \times [0,1] &\to \widehat{U}_j^A, \\ ((\arg(p), \arg(q)), l) &\mapsto \left(\arg(p) + l \cdot \left(\frac{\arg(q)s + \pi}{s+t} - \left(\arg(p) \mod \frac{2\pi}{s+t} \right) \right), \arg(q) \right). \end{aligned}$$

Obviously, we have $\widehat{F}(U_j^A, 0) = \widehat{U}_j^A$ and since $(\arg(p), \arg(q)) \in \rho_{(\pi/(s+t),0)} \Leftrightarrow \arg(p) = (\arg(q)s + (1+2k)\pi)/(s+t)$ for $k \in \{1, \ldots, s+t\}$ we have $\widehat{F}(U_j^A, 1) = \rho_{(\pi/(s+t),0)}$ (see Figure 4.13).

Since $\widehat{F}$ is continuous in $\arg(q)$ and the second coordinate of the image is independent of l, it suffices to prove the homotopy for the first image coordinate for an arbitrary, fixed $\arg(q)$. For a fixed $\arg(q)$ the set $\widehat{U}_j^A$ is given by all $\arg(p) \neq (\arg(q)s + 2\pi k)/(s+t)$ for $k \in \{1, \ldots, s+t\}$. Thus, it consists of $s+t$ separated, open segments with middlepoints $(\arg(q)s + (1+2k)\pi)/(s+t)$, where $k \in \{0, \ldots, s+t-1\}$. Each segment is contracted to its middlepoint by $\widehat{F}$ and hence $\widehat{F}$ indeed is a deformation retraction of $\widehat{U}_j^A$ to $\rho_{(\pi/(s+t),0)}$. For $s \cdot j$ odd with $j \neq t$ the proof works analogously. □

Corollary 4.50. *For each $j \in \{-t+1, \ldots, s-1\} \setminus \{0\}$ the sets U_j^A and $(U_j^A)^c$ are path-connected in $\mathbb{C}^A$.*

Proof. By Lemma 4.47 and 4.49 the sets U_j^A and $(U_j^A)^c$ can be deformation retracted to one closed path. □

Now we have all tools to prove the main theorem of this section, which describes the topology of the sets U_j^A, their complements and, which yields answers to Problem 2.22 for trinomials and to Problem 2.24.

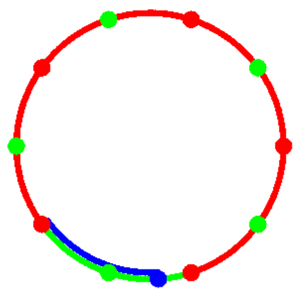


FIGURE 4.13. Situation for $s+t=5$ and $\arg(q)=0$. For a fixed $\arg(q)$, the set $\widehat{U}_j^A$ is the union of $s+t$ open segments between the red points (one is exemplarily depicted in green colour here). Each of the segments is retracted to their green middlepoint under $\widehat{F}$. For a point $\arg(p)$ (the blue point here), the corresponding value $\arg(p) \mod \frac{2\pi}{s+t}$ is the length of the blue segment. Thus, indeed, $\widehat{F}(U_j^A, 1) = \rho_{(\pi/(s+t),0)}$.

Theorem 4.51. *For $A = \{s, 0, -t\}$ each $j \in \{-t+1, \ldots, s-1\} \setminus \{0\}$ both $U_j^A \subseteq \mathbb{C}^A$ and $(U_j^A)^c \subset \mathbb{C}^A$ are homotopic to an $(s+t)$-sheeted covering of an S^1. In particular, we have $\pi_1(U_j^A) = \pi_1((U_j^A)^c) = \mathbb{Z}$ and thus the sets U_j^A are not simply connected.*

PROOF. By Lemma 4.47 and 4.49 U_j^A and $(U_j^A)^c$ can be deformation retracted to the closed paths $\rho_{(0,0)}$ and $\rho_{(\pi/(s+t),0)}$. Since both pathes are homeomorphic the second part of the theorem follows and it suffices to investigate $\rho_{(0,0)}$.

As described in (4.27) $\mathbb{Z}_{s+t}$ acts on the standard torus T, mapping both paths to themselves, i.e., $\mathbb{Z}_{s+t}$ also acts on $\rho_{(0,0)}$. Since for each $\arg(q)$ there are exactly $s+t$ possible $\arg(p)$ such that $(\arg(p), \arg(q)) \in \rho_{(0,0)}$, $\rho_{(0,0)}$ is mapped to an S^1 under the quotient map of T to the orbit space $T/\mathbb{Z}_{s+t}$. Furthermore, $\rho_{(0,0)}$ is path-connected (Corollary 4.50). Obviously, $\rho_{(0,0)}$ is locally path-connected and for every open neighbourhood U of a point $(\arg(p), \arg(q)) \in \rho_{(0,0)}$ and two arbitrary $g_1, g_2 \in \mathbb{Z}_{s+t}$ holds $g_1(U) \cap g_2(U) = \emptyset$. Thus, $\rho_{(0,0)}$ with $h : \rho_{(0,0)} \to S^1$ is a normal covering space with deck transformation group $\mathbb{Z}_{s+t}$, which is isomorphic to $\pi_1(S^1)/h_*(\pi_1(\rho_{(0,0)}))$, where h_* denotes the map of fundamental groups induced by h (see e.g., [**28**, p. 72]). Therefore, $h_*(\pi_1(\rho_{(0,0)})) = (s+t)\mathbb{Z}$ and, since h maps $z \mapsto z^{s+t}$, we have $\pi_1(\rho_{(0,0)}) = \mathbb{Z}$. □

3.4. Monotonicity of Complement Components of Amoebas. Let $A \subset \mathbb{Z}^n$ and $f = \sum_{\alpha \in A} b_\alpha \mathbf{z}^\alpha$. Assume that the corresponding amoeba $\mathcal{A}(f)$ has a complement component $E_\alpha(f) \neq \emptyset$ of order $\alpha \in A$. Then it is a typical observation that the complement component grows if one increases the modulus of the "corresponding" coefficient b_α. See Figure 4.15 for an example.

Precisely, we say that a complement component $E_\alpha(f)$ is *monotonically growing* in $|b_\alpha|$ if for every $\mathbf{w} \in \mathbb{R}^n$ holds that

$$\mathbf{w} \in E_\alpha(f) \quad \Rightarrow \quad \mathbf{w} \in E_\alpha(f + \lambda \cdot b_\alpha \cdot \mathbf{z}^\alpha)$$

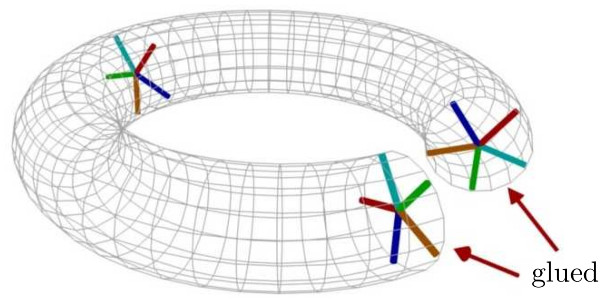


FIGURE 4.14. The set $\{f = z^2 + p + e^{i \cdot \arg(q)} z^{-3} : p \in \mathbb{C}, |p| \leq 1, \arg(q) \in [0, 2\pi)\}$ in the corresponding subset (a real full torus) of its configuration space $\mathbb{C}^A_\Diamond$. Note that we need restrict to $|p| > 0$ if we want to investigate sets in $\mathbb{C}^A$.

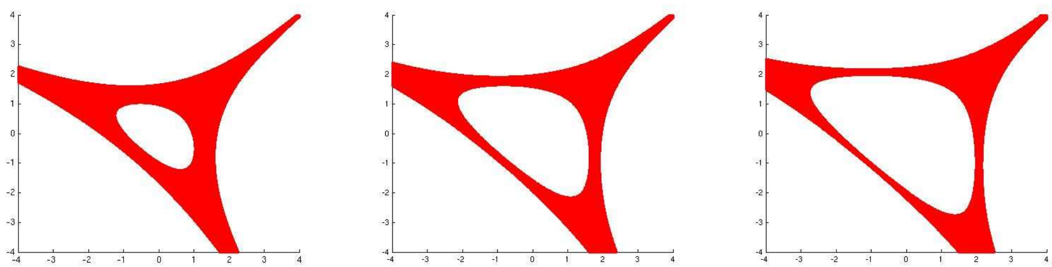


FIGURE 4.15. The amoebas of $f = z_1^2 z_2 + z_1 z_2^2 + c z_1 z_2 + 1$ for $c = -4, -6$ and -8.

for every $\lambda \in \mathbb{R}_{>0}$.

This yields the following problem.

Problem 4.52. *Does for every $A \subset \mathbb{Z}^n$ and $f \in \mathbb{C}^A$ every complement component $E_\alpha(f)$, $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$, grow monotonically in $|b_\alpha|$?*

Indeed, it is not clear a priori whether one would expect this question to have a positive or a negative answer. On the one hand, it is clear that complement components behave in a way, which could be described as "growing monotonically around the center", since for every fixed $\mathbf{w} \in \mathbb{R}^n$ f will be lopsided with $b_\alpha \mathbf{z}^\alpha$ for large $|b_\alpha|$ and will maintain this property for increasing $|b_\alpha|$. Hence, one may only expect complement components to "flutter" on its boundary while growing in total in $|b_\alpha|$. On the other hand, it would be surprising if every complement component had this property since this would imply that the structure of the sets $U_\alpha^A \subseteq \mathbb{C}^A$ is rather simple. For example one could always apply Theorem 4.24 and thus prove connectivity of all U_α^A, i.e., a solution to Rullgård's Problem 2.22, would follow immediately.

Now we show that Problem in general 4.52 has a negative answer by giving a counterexample. More precisely, we show the following theorem.

Theorem 4.53. *For trinomials $f = z^s + p + q \cdot z^{-t}$ with $p \in \mathbb{C}$, $q \in \mathbb{C}^*$ the complement component $E_0(f)$ is not monotonically growing in general.*

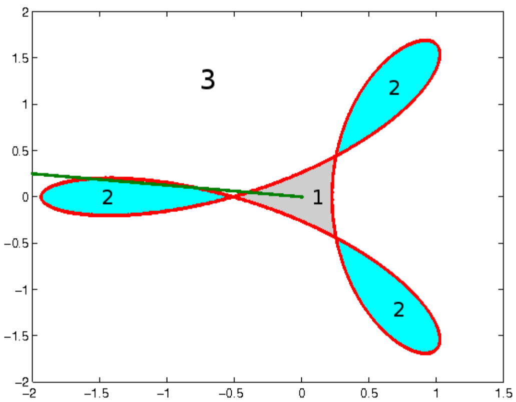


FIGURE 4.16. The trajectory of $\mathcal{V}(f_p^{|z|} - p)$ for $f_p = z^2 + p \cdot e^{i \cdot \varepsilon\pi} + z^{-1}$, $z = 0.925$ and $\varepsilon > 0$ sufficiently small. The green line marks the values attained by p for increasing $|p|$.

PROOF. To prove the theorem, we provide a counterexample to Problem 4.52. Let $f_p = z^2 - p \cdot e^{i \cdot \varepsilon\pi} + z^{-1}$ with $p \in \mathbb{R}_{>0}$ be a parametric family with $\varepsilon > 0$ sufficiently small. We investigate the variety of the fiber function $f^{|z|}$ for the point $z = 0.925$ (see Figure 4.16).

For growing p three roots pass the circle around the origin with radius $|0.925|$ for certain, different values v_1, v_2, v_3 of p. See Figure 4.16; the green line depicts the (growing) values of p, the three intersection points with the trajectory of the hypotrochoid curve are v_1, v_2 and v_3. Since by Proposition 4.36 no roots are conjugated, every complement component $U_{-1}^A, U_0^A, U_1^A, U_2^A$ exists. Thus, for every $p \in \mathbb{R}_{>0} \setminus \{v_1, v_2, v_3\}$, $w = \log|z|$ is contained in exactly one $E_i(f_p)$ and $\overline{E_i}(f_p) \cap \overline{E_j}(f_p) = v_r \Rightarrow |i - j| = 1$ (i.e., a v_r passing $|0.925|$ changes the order of the complement component w is contained in by ± 1).

For $p = 0$ all roots of f are located on the circle $|z| = 1$. Hence for all p in the grey region 1 in Figure 4.16, $w \in E_{-1}(f_p)$. The white region 3 in Figure 4.16 contains p with $|p| > |0.925|^2 + |0.925|$, i.e., f_p is lopsided at w with the monomial p as dominating term. By lopsidedness condition (Theorem 2.30) this implies $w \in E_0(f_p)$. The closure of each turquoise region 2 in Figure 4.16 is connected with region 1 by a node. Hence, by passing from 1 to 2 along the node changes the order by 2. Therefore, for all p in region 2, $w \in E_1(f_p)$. But that means in total for p growing from 0 to ∞, that w is located in $E_{-1}(f_p), E_0(f_p), E_1(f_p), E_0(f_p)$.

Thus, $E_0(f_p)$ is not monotonically growing in p, i.e., Problem 4.52 does not have a positive answer in general. ☐

CHAPTER 5

Approximation of Amoebas and Coamoebas by Sums of Squares

In the Sections 5 and 6 of Chapter 2 we introduced the membership problems for amoebas and coamoebas (Problems 2.26 and 2.33) as a natural question on providing a proper solution for the problem of computing resp. approximating amoebas and coamoebas.

While for coamoebas no algorithm to solve the membership problem is known at all so far, Purbhoo provided in [**70**] a characterization for the points in the complement of a hypersurface amoeba, which can be used to numerically approximate the amoeba (see Chapter 2, Section 5). His *lopsidedness criterion* provides an inequality-based certificate for non-containedness of a point in an amoeba. But it does not provide an *algebraic* certificate in the sense of a polynomial identity certifying the non-containedness. The certificates are given by iterated resultants. With this technique the amoeba can be approximated by a limit process. The computational efforts of computing the resultants are growing quite fast, and the convergence is slow.

A different approach to tackle computational problems on amoebas is to apply suitable Nullstellen- or Positivstellensätze from real algebraic geometry or complex geometry. For some natural problems a direct approach via the Nullstellensatz (applied on a realification of the problem) is possible. Using a degree truncation approach, this allows to find sum-of-squares-based polynomial identities, which certify that a certain point is located outside of an amoeba or coamoeba. In particular, it is well known from recent lines of research in computational semialgebraic geometry (see, e.g., [**34, 35, 56**]) that these certificates can be computed via semidefinite programming (SDP).

In this chapter, we discuss theoretical foundations as well as some practical issues of such an approach, thus establishing new connections between amoebas, semialgebraic and convex algebraic geometry and semidefinite programming. Firstly, in Section 1 we recover central facts about *semidefinite optimization problems* (SDPs) (as a natural extension of *linear optimization problems* (LPs)) and *sums of squares* (SOS) – in particular the Real Nullstellensatz (Theorem 5.3).

In Section 2 we present various Nullstellensatz-type formulations (Statements 5.4 and 5.9) based on the Real Nullstellensatz and compare their properties to a recent toric Nullstellensatz of Niculescu and Putinar ([**47**]). Using a degree truncation approach this yields effective approximation hierarchies for both the amoeba and the coamoeba membership problems (Theorem 5.13).

The main theoretical contribution is contained in Section 3. For one of our approaches, we can provide degree bounds for the certificates (Corollary 5.17). It is remarkable and even somewhat surprising that these degree bounds are derived from Purbhoo's lopsidedness criterion (which is not at all sum-of-squares-based). We also show that in certain cases (such as for the Grassmannian of lines) the degree bounds can be reduced to simpler amoebas (Theorem 5.19).

In Section 4 we provide some actual computations on this symbolic-numerical approach. Besides providing results on the membership problem itself, we will also consider more sophisticated versions (such as bounding the diameter of a complement component for certain classes).

We remark that all results of this chapter can be found in the Paper [**86**].

1. Semidefinite Optimization and the Real Nullstellensatz

We give a brief overview about the theory of semidefinite programming, sums of squares and its relation to the feasibility of real systems of polynomial equations. For a detailed introduction into the topic see e.g., [**6, 35, 39**].

Recall that for a given matrix $A \in \mathbb{R}^{m \times n}$ and vectors $b \in \mathbb{R}^m, c \in \mathbb{R}^n$ a *linear optimization problem* (LP) is given by

$$\begin{aligned} &\min\langle c, x\rangle \text{ such that} \\ &Ax \leq b. \end{aligned}$$

Geometrically, the region of feasibility of an LP is a polyhedron P, located in the positive orthant of $\mathbb{R}^n$. It is given as an intersection of half-spaces, where each half-space is yielded by one row of the matrix A interpreted as linear form. The vector c gives the direction in which to minimize. If the LP has a unique solution, then this solution will be a vertex of the polyhedron P. Note that LPs can be solved in polynomial time. For more details about linear programming see e.g., [**79**].

A *semidefinite optimization problem* (SDP) is a generalization of a linear programming problem. Let $C, A_1, \ldots, A_m$ be real, symmetric $n \times n$ matrices and $b_1, \ldots, b_m \in \mathbb{R}^m$. A *semidefinite optimization problem* is given by

$$\begin{aligned} &\inf\langle X, C\rangle \text{ such that} \\ &\langle A_i, X\rangle = b_i \text{ for } 1 \leq i \leq m \text{ and} \\ &X \succeq 0, \end{aligned}$$

where $X \succeq 0$ means that X is *positive semidefinite* and $\langle \cdot, \cdot \rangle$ denotes the inner product on real, symmetric $n \times n$ matrices, which is for matrices $A = (a_{ij}), B = (b_{ij})$ given by $\langle A, B\rangle = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \cdot b_{ij}$.

SDPs are generalizations of LPs since every LP can be written as an SDP via use of diagonal matrices. SDPs are both a convenient and convincing generalization of LPs since the cone of positive semidefinite matrices is, as well as the positive orthant, self-dual. This allows to express a dual problem to every (primal) SDP and many of the properties of LPs also hold for SDPs (e.g., weak duality, strong duality if the SDP is strictly feasible; see e.g., [**6, 35**]). Furthermore, SDPs are solvable up to an ε-error in polynomial time.

Geometrically, the region of feasibility of an SDP is a *spectrahedron* – a convex subset of the cone of positive semidefinite matrices, which is given by linear matrix inequalities.

Since SDPs are generalizations of LPs, obviously spectrahedra are generalizations of polyhedra. See e.g., [**24, 67**] for some basic properties of spectrahedra.

Positive semidefinite matrices and semidefinite optimization problems are closely related to real algebraic geometry since they can be used to relax solutions of real systems of polynomial optimization problems in the following way.

A real polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of total degree $2d$ is called a *sum of squares* (SOS) if there are real polynomials $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most d such that $f = \sum_{j=1}^{r} g_j^2$. Obviously, every SOS polynomial is in particular non-negative. The fact that a real polynomial is SOS can be expressed in terms of positive semidefinite matrices.

Proposition 5.1. *Let $g \in \mathbb{R}[x_1, \ldots, x_n], \operatorname{tdeg}(g) = 2d$ and Y the vector of all monomials in $x_1, \ldots, x_n$ with degree $\leq d$. g is a sum of squares if and only if there exists a matrix Q with $Q \succeq 0$ and*

$$g = Y^T Q Y.$$

Now one can rewrite a real system of polynomial optimization given by

$$\begin{aligned} &\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) \text{ such that} \\ &g_i(\mathbf{x}) \geq 0 \text{ for } 1 \leq i \leq m \end{aligned}$$

to

$$\begin{aligned} &\max p \in \mathbb{R} \text{ such that} \\ &f(\mathbf{x}) - p \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n \text{ with} \\ &g_i(\mathbf{x}) \geq 0 \text{ for } 1 \leq i \leq m. \end{aligned}$$

Since testing non-negativity of real polynomials is NP-hard, one is interested in relaxing this problem. This can be done by testing whether a polynomial $f - p$ is a sum of squares instead. We omit, how this transformation can be written exactly under preservation of the constraints (Schmüdgen's and Putinar's theorem, see e.g., [**35**]).

Here, we will be only interested if a certain real polynomial system of equations has a real solution. For this problem it suffices to investigate the *Real Nullstellensatz*, which can be seen as a real counterpart of Hilbert's Nullstellensatz, which we want to recall first (see e.g., [**7, 13**]).

Theorem 5.2 (Hilbert's Nullstellensatz)**.** *For polynomials $g_1, \ldots, g_r, f \in \mathbb{C}[z_1, \ldots, z_n]$ and $I = \langle g_1, \ldots, g_r \rangle \subset \mathbb{C}[z_1, \ldots, z_n]$ the following statements are equivalent:*

- $\mathcal{V}(I) \subseteq \mathcal{V}(f)$
- $f^s \in I$ *for some* $s \in \mathbb{N}^*$.

Note that this in particular implies that $\mathcal{V}(I) = \emptyset$ if and only if $1 \in I$. The real Nullstellensatz states that basically, up to some sums of squares, the same holds for the real variety $\mathcal{V}_{\mathbb{R}}(I) = \mathcal{V}(I) \cap \mathbb{R}^n$ of an ideal I.

Theorem 5.3 (Real Nullstellensatz)**.** *For polynomials* $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$ *and* $I = \langle g_1, \ldots, g_r \rangle \subset \mathbb{R}[x_1, \ldots, x_n]$ *the following statements are equivalent:*

- *The real variety* $\mathcal{V}_{\mathbb{R}}(I)$ *is empty.*
- *There exist a polynomial* $G \in I$ *and a sum of squares polynomial* H *with*

$$G + H + 1 \ = \ 0\,.$$

2. The Solution of Membership Problems Via the Real Nullstellensatz

For technical reasons (as explained below) it will often be convenient to consider in the definition of a coamoeba also those points $z \in \mathcal{V}(I)$, which have a zero-component. Namely, if a zero z of I has a zero-component $z_j = 0$, then we associate this component to *any* phase. Call this modified version of a coamoeba $\text{co}\mathcal{A}'(I)$. Note that for principal ideals $I = \langle f \rangle$ the difference between $\text{co}\mathcal{A}(I)$ and $\text{co}\mathcal{A}'(I)$ solely may occur at points, which are contained in the closure of $\text{co}\mathcal{A}(I)$. The set-theoretic difference of $\text{co}\mathcal{A}(I)$ and $\text{co}\mathcal{A}'(I)$ is a lower-dimensional subset of $\mathbb{R}^n$ (since in each environment of a point in $\text{co}\mathcal{A}'(I) \setminus \text{co}\mathcal{A}(I)$ we have a coamoeba point).

Given $\lambda \in (0, \infty)^n$, the question if λ is contained in the unlog-amoeba $\mathcal{U}(I)$ can be phrased as the real solvability of a real system of polynomial equations. For a polynomial $f \in \mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$ let, again, $f^{\text{re}}, f^{\text{im}} \in \mathbb{R}[\mathbf{x}, \mathbf{y}] = \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be its real and imaginary parts (see Chapter 3, Section 4), i.e.,

$$f(\mathbf{z}) \ = \ f(\mathbf{x} + i\mathbf{y}) \ = \ f^{\text{re}}(\mathbf{x}, \mathbf{y}) + i \cdot f^{\text{im}}(\mathbf{x}, \mathbf{y})\,.$$

We consider the ideal $I' \subset \mathbb{R}[\mathbf{x}, \mathbf{y}]$ generated by the polynomials

$$\{f_j^{\text{re}}, f_j^{\text{im}} \,:\, 1 \le j \le r\} \cup \left\{x_k^2 + y_k^2 - \lambda_k^2 \,:\, 1 \le k \le n\right\}. \tag{5.1}$$

Corollary 5.4. *Let* $I = \langle f_1, \ldots, f_r \rangle$, *and* $\lambda \in (0, \infty)^n$. *Either the point* λ *is contained in* $\mathcal{U}(I)$, *or there exist a polynomial* $G \in I' \subset \mathbb{R}[\mathbf{x}, \mathbf{y}]$ *and a sum of squares polynomial* $H \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ *with*

$$G + H + 1 \ = \ 0\,. \tag{5.2}$$

Proof. For any polynomial $f \in \mathbb{C}[\mathbf{z}]$ it suffices to observe that a point $z = x + iy$ is contained in $\mathcal{V}(f)$ if and only if $(x, y) \in \mathcal{V}_{\mathbb{R}}(f^{\text{re}}) \cap \mathcal{V}_{\mathbb{R}}(f^{\text{im}})$, and that $|z_k| = \lambda_k$ if and only $(x, y) \in \mathcal{V}_{\mathbb{R}}(x_k^2 + y_k^2 - \lambda_k^2)$. Then the statement follows from Theorem 5.3. □

Corollary 5.4 states that for any point $\lambda \notin \mathcal{U}(I)$ there exists a certificate

$$\sum_{j=1}^{r} p_j f_j^{\text{re}} + \sum_{j=1}^{r} p_j' f_j^{\text{im}} + \sum_{k=1}^{n} q_k (x_k^2 + y_k^2 - \lambda_k^2) + H + 1 \ = \ 0$$

with polynomials p_j, p_j', q_k and a sum of squares H.

Remark 5.5. *By the following lemma (which is easy to check), the sum of squares condition 5.2 can also be stated shortly as*

$$-1 \text{ is a sum of squares in the quotient ring } \mathbb{R}[\mathbf{x}, \mathbf{y}]/I' .$$

Lemma 5.6 (Parrilo [**57**]). *Let $I = \langle g_1, \ldots, g_r \rangle \subset \mathbb{R}[\mathbf{x}]$ and $f \in \mathbb{R}[\mathbf{x}]$. There exist $p_1, \ldots, p_k \in \mathbb{R}[\mathbf{x}]$ such that*

$$f + \sum_i p_i g_i \text{ is a sum of squares in } \mathbb{R}[\mathbf{x}]$$

if and only if f is a sum of squares in $\mathbb{R}[\mathbf{x}]/I$.

Any two of these equivalent conditions in Lemma 5.6 is a certificate for the nonnegativity of f on the variety I.

Before stating a coamoeba version, we note the following normalization properties. Whenever it is needed for amoebas, we can assume that λ is the all-1-vector $\mathbf{1}$. Similarly, for coamoebas we can assume that all components of the point under investigation have argument 0.

Lemma 5.7. *Let $I = \langle f_1, \ldots, f_r \rangle$.*

(1) *A point $(\lambda_1, \ldots, \lambda_n) \in (0, \infty)^n$ is contained in $\mathcal{U}(I)$ if and only if $\mathbf{1}$ is contained in $\mathcal{U}(\langle g_1, \ldots, g_r \rangle)$, where*

$$g_j(z_1, \ldots, z_n) \;=\; f_j(\lambda_1 z_1, \ldots, \lambda_n z_n) , \quad 1 \le j \le r .$$

(2) *A point $(z_1, \ldots, z_n)$ is contained in $\mathcal{V}(I)$ with $\arg z_j = \mu_j$ if and only if the (nonnegative) real vector y with $y_j = z_j e^{-i\mu_j}$ is contained in $\mathcal{V}(g_1, \ldots, g_r)$ where*

$$g_j(z_1, \ldots, z_n) \;=\; f_j(z_1 e^{i\mu_1}, \ldots, z_n e^{i\mu_n}) , \quad 1 \le j \le r .$$

Proof. A point $(z_1, \ldots, z_n)$ is contained in $\mathcal{V}(I)$ with $|z_j| = \lambda_j$ if and only if the vector y defined by $y_j = z_j/\lambda_j$ is contained in $\mathcal{V}(g_1, \ldots, g_r)$ with $|y_j| = 1$. The second statement follows analogously. □

Theorem 5.8. *Let $I = \langle f_1, \ldots, f_r \rangle$. The point $(0, \ldots, 0)$ is contained in the complement of the coamoeba co$\mathcal{A}'(I)$ if and only if there exists a polynomial identity*

$$\sum_{i=1}^{r} c_i \cdot f_i(\mathbf{x}^2, \mathbf{y})^{\mathrm{re}} + \sum_{i=1}^{r} c_i' \cdot f_i(\mathbf{x}^2, \mathbf{y})^{\mathrm{im}} + \sum_{j=1}^{n} d_j \cdot y_j + H + 1 \;=\; 0 \tag{5.3}$$

with polynomials $c_i, c_i', d_j \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ and a sum of squares H. Here, $f_i(\mathbf{x}^2, \mathbf{y})$ abbreviates $f_i(x_1^2, \ldots, x_n^2, y_1, \ldots, y_n)$.

Proof. Note that the statement $0 \in \mathrm{co}\mathcal{A}'(I)$ is equivalent to $\{z = x + iy \in \mathbb{C}^n : z \in \mathcal{V}(I)$ and $x_i \ge 0,\, y_i = 0,\, 1 \le i \le n\} \ne \emptyset$. Moreover, observe that the condition $x_i \ge 0$ can be replaced by considering x_i^2 in the arguments of $f_1, \ldots, f_r$. Hence, by Theorem 5.3 the statement $0 \notin \mathrm{co}\mathcal{A}'(I)$ is equivalent to the existence of a polynomial identity of the form (5.3). □

Observe that in the proof the use of co$\mathcal{A}'(I)$ (rather than co$\mathcal{A}(I)$) allowed to use the basic Nullstellensatz (rather than a Positivstellensatz, which would have introduced several sum of squares polynomials).

The following variant of the Nullstellensatz approach will allow to obtain degree bounds (see Section 3). For vectors $\alpha(1), \ldots, \alpha(d) \in \mathbb{N}_0^n$ and coefficients $b_1, \ldots, b_d \in \mathbb{C}^*$ let $f = \sum_{j=1}^{d} b_j \cdot z^{\alpha(j)} \in \mathbb{C}[\mathbf{z}]$. For any given values of $\lambda_1, \ldots, \lambda_n$ set

$$\mu_{\alpha(j)} \;=\; \lambda^{\alpha(j)} \;=\; \lambda_1^{\alpha(j)_1} \cdots \lambda_n^{\alpha(j)_n}\,, \quad 1 \le j \le d\,.$$

If the rank of the matrix with columns $\alpha(1), \ldots, \alpha(d)$ is n (i.e., the vectors $\alpha(1), \ldots, \alpha(d)$ span $\mathbb{R}^n$) then the λ-values can be reconstructed uniquely from the μ-values. We come up with the following variant of a Nullstellensatz.

Let $m_{\alpha(j)}$ be the monomial $m_{\alpha(j)} = \mathbf{z}^{\alpha(j)} = z_1^{\alpha(j)_1} \cdots z_n^{\alpha(j)_n}$. For every $f_j \in \mathbb{C}[\mathbf{z}]$ we denote the corresponding support set as $A_j \subset \mathbb{Z}^n$. We consider the ideal $I^* \subset \mathbb{R}[\mathbf{x}, \mathbf{y}]$ generated by the polynomials

$$\{f_j^{\mathrm{re}}, f_j^{\mathrm{im}} \,:\, 1 \le j \le r\} \cup \left\{ (m_{\alpha(k)}^{\mathrm{re}})^2 + (m_{\alpha(k)}^{\mathrm{im}})^2 - \mu_{\alpha(k)}^2 \,:\, \alpha(k) \in \bigcup_{j=1}^{r} A_j \right\}. \tag{5.4}$$

Corollary 5.9. *Let $f_1, \ldots, f_r \in \mathbb{C}[\mathbf{z}]$ with support sets $A_1, \ldots, A_r \in \mathbb{Z}^n$. Let $I = \langle f_1, \ldots, f_r \rangle$ and assume that $\bigcup_{j=1}^{r} A_j$ spans $\mathbb{R}^n$. Further, let $\lambda \in (0, \infty)^n$. Either a point $(\lambda_1, \ldots, \lambda_n)$ is contained in $\mathcal{U}(I)$, or there exist polynomials $G \in I^* \subset \mathbb{R}[\mathbf{x}, \mathbf{y}]$ and a sum of squares polynomial $H \in \mathbb{R}[\mathbf{x}, \mathbf{y}]$ with*

$$G + H + 1 \;=\; 0\,. \tag{5.5}$$

For hypersurface amoebas of real polynomials, the membership problem relates to the following statement of Niculescu and Putinar [**47**]. Let $p = p(\mathbf{x}, \mathbf{y}) \in \mathbb{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be a real polynomial. Then p can be written as a complex polynomial $p(\mathbf{x}, \mathbf{y}) = P(\mathbf{z}, \overline{\mathbf{z}})$ with $P \in \mathbb{C}[z_1 \ldots, z_n, \overline{z}_1, \ldots, \overline{z}_n]$ and $\overline{P(\mathbf{z}, \overline{\mathbf{z}})} = P(\mathbf{z}, \overline{\mathbf{z}})$. Note that there exists a polynomial $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with

$$p(\mathbf{x}, \mathbf{y})^2 \;=\; |P(\mathbf{z}, \bar{\mathbf{z}})|^2 \;=\; |Q(\mathbf{z})|^2 \quad \text{for } \mathbf{z} \in T^n\,,$$

where $T = \{z \in \mathbb{C} \,:\, |z| = 1\}$.

The following statement can be obtained by applying the Nullstellensatz on the set $\{\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in (\mathbb{C}^*)^n \,:\, |q(\mathbf{z})|^2 = 1, |z_1|^2 = 1, \ldots, |z_n|^2 = 1\}$, then applying Putinar's Theorem [**71**] on the multiplier polynomial of $|q(\mathbf{z})|^2$ (see [**47**]).

Proposition 5.10. *Let $q \in \mathbb{C}[z_1, \ldots, z_n]$. Then $q(\mathbf{z}) \ne 0$ for all $\mathbf{z} \in T^n$ if and only if there are complex polynomials $p_1, \ldots, p_k, r_1, \ldots, r_l \in \mathbb{C}[z_1, \ldots, z_n]$ with*

$$1 + |p_1(\mathbf{z})|^2 + \cdots + |p_k(\mathbf{z})|^2 \;=\; |q(\mathbf{z})|^2(|r_1(\mathbf{z})|^2 + \cdots + |r_l(\mathbf{z})|^2)\,, \quad \textit{for } \mathbf{z} \in T^n\,. \tag{5.6}$$

Note that the statement is not an identity of polynomials, but an identity for all $\mathbf{z}$ in the real n-torus T^n.

While Proposition 5.10 provides a nice structural result, due to the following reasons we prefer Corollary 5.4 for actual computations. In representation (5.6), *two* sums of squares polynomials (rather than just one as in (5.2)) are needed in the representation, and the degree is increased (by the squaring process). Moreover, the theorem is not really a representation theorem (in terms of an identity of polynomials), but an identity over T^n; therefore in order to express this computationally, the polynomials hidden in this equivalence (i.e., the polynomials $1 - |z_1|^2, \ldots, 1 - |z_n|^2$) have to be additionally used.

2.1. SOS-based Approximations. By putting degree truncations on the certificates, we can transform the theoretic statements into effective algorithmic procedures for constructing certificates. The idea of degree truncations in polynomial identities follows the same principles of the degree truncations with various types of Nullstellen- and Positivstellensätze in [**37, 34, 56**]. It is instructive to have a look at two simple examples first.

Example 5.11. Let f be the polynomial $f = z + z_0$ with a complex constant $z_0 = x_0 + iy_0$. The ideal I of interest is defined by

$$\begin{aligned} h_1 &= f^{\mathrm{re}} = x + x_0\,, \\ h_2 &= f^{\mathrm{im}} = y + y_0\,, \\ h_3 &= x^2 + y^2 - \lambda^2\,. \end{aligned}$$

For values of $\lambda \geq 0$, which correspond to points outside the amoeba (i.e., $\lambda^2 \neq x_0^2 + y_0^2$), we have $\mathcal{V}_{\mathbb{C}}(I) = \emptyset$ and thus the Gröbner basis G of $\langle h_1, h_2, h_3 \rangle$ is $G = \{1\}$. The corresponding multiplier polynomials p_i to represent 1 as a linear combination $\sum_i p_i h_i$ are

$$p_1 = \frac{-x + x_0}{x_0^2 + y_0^2 - \lambda^2}\,, \quad p_2 = \frac{-y + y_0}{x_0^2 + y_0^2 - \lambda^2}\,, \quad p_3 = \frac{1}{x_0^2 + y_0^2 - \lambda^2}\,.$$

Hence, in particular, -1 can be written as a sum of squares in the quotient ring $\mathbb{R}[x]/I$. The necessary degree with regard to equation (5.2) is just 2.

For $\lambda^2 = a^2 + b^2$, the Gröbner basis (w.r.t. a lexicographic variable ordering with $x \succ y$) is

$$x + a,\ y + b\,.$$

The point $(-a, -b)$ is contained in $\mathcal{V}_{\mathbb{R}}(I)$; thus in this case there does not exist a Nullstellensatz-type certificate.

Example 5.12. Consider the polynomial $f = z_1 + z_2 + 5$ with $z_j = x_j + iy_j$. The ideal I of interest is defined by

$$\begin{aligned} h_1 &= x_1 + x_2 + 5\,, \\ h_2 &= y_1 + y_2\,, \\ h_3 &= x_1^2 + y_1^2 - \lambda_1^2\,, \\ h_4 &= x_2^2 + y_2^2 - \lambda_2^2\,. \end{aligned}$$

Consider $\lambda_1 = 2$, $\lambda_2 = 3$. Using a lexicographic ordering with $x_1 \succ x_2 \succ y_1 \succ y_2$, a Gröbner basis is

$$y_2^2\,,\ y_1 + y_2\,,\ x_2 + 3\,,\ x_1 + 2\,.$$

The standard monomials are 1 and y_2. It is easy to see that -1 is not a sum of squares in the quotient ring, which reflects the fact that $(2,3) \in \mathcal{U}_I$.

Using the degree truncation approach for sums of squares we can define the set C_k as the set of points in $\mathbb{R}^n$ such that there exists a certificate, say, in the standard approach, of degree at most $2k$. And similarly for coamoebas, where we denote the sequence by D_k. These sequences are the basis of the effective implementation (see Section 4).

Theorem 5.13. *Let $I = \langle f_1, \ldots, f_r\rangle$ and $k_0 = \max_i \lceil \deg f_i/2 \rceil$. The sequence $(C_k)_{k \geq k_0}$ converges pointwise to the complement of the unlog amoeba $\mathcal{U}(I)$, and it is monotone increasing in the set-theoretic sense, i.e., $C_k \subset C_{k+1}$ for $k \geq k_0$.*

Similarly, the sequence $(D_k)_{k \geq k_0}$ converges pointwise to the complement of the coamoeba $co\mathcal{A}(I)$, and it is monotone increasing in the set-theoretic sense, i.e., $D_k \subset D_{k+1}$ for $k \geq k_0$.

Proof. For any given point $\mathbf{w} \in \mathbb{R}^n$ in the complement of the amoeba there exists a certificate of minimal degree, say d. For $k < \lceil d/2 \rceil$ the point z is not contained in C_k and for $k \geq \lceil d/2 \rceil$ the point $\mathbf{w}$ is contained in C_k. In particular, the relaxation process is monotone increasing. And analogously for coamoebas. □

Recall from Section 1 that it is well-known (and at the heart of current developments in optimization of polynomial functions, see [**34, 56**] or e.g., the survey [**35**]; see e.g., [**90**] for a further comprehensive treatment) that SOS conditions of bounded degree can be phrased as semidefinite programs. Semidefinite programs can be solved efficiently both in theory and in practice.

Recall in particular that any sum-of-squares polynomial H can be expressed as YQY^T, where Q is a symmetric positive semidefinite matrix (abbreviated $Q \succeq 0$) and Y is a vector of monomials (Proposition 5.1).

Similarly, by the degree restriction the linear combination in (5.2) or (5.5) can be integrated into the semidefinite formulation by a comparison of coefficients.

3. Special Certificates and a Proof of Effectivity

For a certain class of amoebas, we can provide some explicit classes of Nullstellensatz-type certificates. As a first warmup-example, we illustrate some ideas for constructing special certificates systematically for linear amoebas in the standard approach. Then we show how to construct special certificates for the monomial-based approach. In this section we concentrate on the case of hypersurface amoebas.

3.1. Linear Amoebas in the Standard Approach. Let $f = az_1 + bz_2 + c$ be a general linear polynomial in two variables with real coefficients $a, b, c \in \mathbb{R}$. We consider certificates of the form (5.2) based on the third binomial formula, where we use the sums of squares $(x_1 - x_2)^2$ and $(y_1 - y_2)^2$. For simplicity assume $a, b > 0$. The representation

$$(ax_1 + bx_2 - c)(ax_1 + bx_2 + c) + (ay_1 + by_2)(ay_1 + by_2)$$
$$-(a^2 + ab)(x_1^2 + y_1^2 - \lambda_1^2) - (b^2 + ab)(x_2^2 + y_2^2 - \lambda_2^2) + ab(x_1 - x_2)^2 + ab(y_1 - y_2)^2$$

simplifies to

$$(a^2 + ab)\lambda_1^2 + (b^2 + ab)\lambda_2^2 - c^2 \,. \tag{5.7}$$

Assume that the point (λ_1, λ_2) is not contained in the unlog amoeba $\mathcal{U}_f$. In order to obtain the desired polynomial identity (5.2) certifying containedness in the complement of $\mathcal{U}(f)$, we need this term to be smaller than zero. Then, by scaling, we can bring this to -1, so that by adding of $+1$ (i.e., replacing '0' in the formula by '1'), we obtain the polynomial identity (5.2).

Example 5.14. Let $a = 1$, $b = 2$, $c = 5$. The curve (in λ_1, λ_2) given by $(a^2 + ab)\lambda_1^2 + (b^2 + ab)\lambda_2^2 - c^2$ has a logarithmic image that is shown in Figure 5.2. By projective symmetry, analogous special certificates can be obtained within the two other complement components.

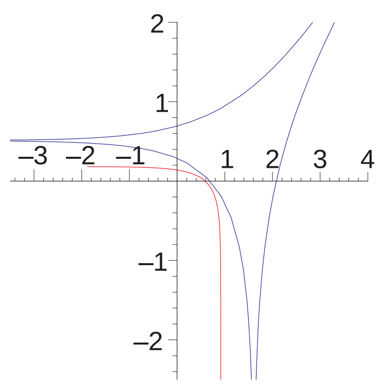


FIGURE 5.2. The boundary of an amoeba of a linear polynomial (blue) and the boundary of the outer region, for which the special certificates of degree 2 exist (red).

Since (by homogenizing the polynomial) there is a symmetry, we obtain similarly an approximation of the other two complement components. Hence we have:

Lemma 5.15. *For the points in the regions defined by the outer regions of the curves* (5.7) *there exist certificates of degree at most 2.*

3.2. The Monomial-based Approach. For the monomial-based approach based on Corollary 5.9 we can provide special certificates for a much more general class. Our point of departure is Purbhoo's *lopsidedness* criterion introduced in Chapter 2, Section 5 (see also [**70**]), which guarantees that a point belongs to the complement of an amoeba $\mathcal{A}(f)$. In particular, we can provide degree bounds for these certificates.

In the following let $\alpha(1), \ldots, \alpha(d) \in \mathbb{N}_0^n$ span $\mathbb{R}^n$ and $f = \sum_{j=1}^d b_j \mathbf{z}^{\alpha(j)} \in \mathbb{C}[\mathbf{z}]$ with monomials $m_{\alpha(j)} = \mathbf{z}^{\alpha(j)} = z_1^{\alpha(j)_1} \cdots z_n^{\alpha(j)_n}$.

Based on Theorem 2.28 one can devise a converging sequence of approximations for the amoeba. Note, however, that the lopsidedness criterion is not a Nullstellensatz in a strict sense since it does not provide a *polynomial* identity certifying membership in the complement of the amoeba.

The aim of this section is to determine how our SOS approximation is related to lopsidedness (recall the notations and results presented in Chapter 2, Section 5) and transform the lopsidedness certificate into a certificate for the Nullstellensätze presented in Section 2.

By Lemma 5.7 we can assume that the point λ under investigation is the all-1-vector $\mathbf{1}$. In this situation lopsidedness means that there is an index $j \in \{1, \ldots, d\}$ with $|b_j| > \sum_{i \neq j} |b_i|$. If the lopsidedness condition is satisfied in λ, then the following statement provides a certificate of the form $G + H + 1$ with bounded degree.

Corresponding to the definition of I^* in (5.4), let the polynomials $s_1, \ldots, s_{d+2}$ be defined by

$$s_i \;=\; \left(\frac{b_i^{\mathrm{re}}}{|b_i|} \cdot \left(\mathbf{z}^{\alpha(i)}\right)^{\mathrm{re}}\right)^2 + \left(\frac{b_i^{\mathrm{im}}}{|b_i|} \cdot \left(\mathbf{z}^{\alpha(i)}\right)^{\mathrm{im}}\right)^2 - 1\,, \quad 1 \leq i \leq d\,,$$

and $s_{d+1} = f^{\mathrm{re}}$, $s_{d+2} = f^{\mathrm{im}}$.

Theorem 5.16. *If the point $\lambda = \mathbf{1}$ is contained in the complement of $\mathcal{U}(f)$ with $f\{0\}$ being lopsided and dominating element $|m_{\alpha(1)}(\mathbf{1})|$, then there exists a certificate of total degree $2 \cdot \mathrm{tdeg}(f)$, which is given by*

$$\sum_{i=1}^{d+2} s_i g_i + H + 1 \;=\; 0\,, \tag{5.8}$$

where

$$g_1 \;=\; |b_1|^2\,, \quad g_i \;=\; -|b_i| \cdot \sum_{k=2}^{d} |b_k|\,, \quad 2 \leq i \leq d\,,$$

$$g_{d+1} \;=\; \left(-b_1 \cdot \mathbf{z}^{\alpha(1)} + \sum_{i=2}^{d} b_i \cdot \mathbf{z}^{\alpha(i)}\right)^{\mathrm{re}}, \quad g_{d+2} \;=\; \left(-b_1 \cdot \mathbf{z}^{\alpha(1)} + \sum_{i=2}^{d} b_i \cdot \mathbf{z}^{\alpha(i)}\right)^{\mathrm{im}},$$

$$H \;=\; \sum_{2 \leq i < j \leq d} |b_i| \cdot |b_j| \cdot \left(\frac{b_i^{\mathrm{re}}}{|b_i|} \cdot \left(\mathbf{z}^{\alpha(i)}\right)^{\mathrm{re}} - \frac{b_j^{\mathrm{re}}}{|b_j|} \cdot \left(\mathbf{z}^{\alpha(j)}\right)^{\mathrm{re}}\right)^2$$
$$+ |b_i| \cdot |b_j| \cdot \left(\frac{b_i^{\mathrm{im}}}{|b_i|} \cdot \left(\mathbf{z}^{\alpha(i)}\right)^{\mathrm{im}} - \frac{b_j^{\mathrm{im}}}{|b_j|} \cdot \left(\mathbf{z}^{\alpha(j)}\right)^{\mathrm{im}}\right)^2.$$

Proof. By the binomial theorem $(a + b) \cdot (a - b) = a^2 - b^2$, we substitute the polynomials s_i and g_j into $s_{d+1} g_{d+1} + s_{d+2} g_{d+2}$ yields

$$-\left(b_1^{\mathrm{re}} \cdot \left(\mathbf{z}^{\alpha(1)}\right)^{\mathrm{re}}\right)^2 + \left(\sum_{i=2}^{d} (b_i \cdot \mathbf{z}^{\alpha(i)})^{\mathrm{re}}\right)^2 - \left(b_1^{\mathrm{im}} \cdot \left(\mathbf{z}^{\alpha(1)}\right)^{\mathrm{im}}\right)^2 + \left(\sum_{i=2}^{d} (b_i \cdot \mathbf{z}^{\alpha(i)})^{\mathrm{im}}\right)^2.$$

Adding $g_1 s_1$ and the SOS term H yields

$$-|b_1|^2 + \left(\sum_{j=2}^{d} \left(\frac{b_j^{\text{re}}}{|b_j|} \cdot \left(\mathbf{z}^{\alpha(j)}\right)^{\text{re}} \right)^2 + \left(\frac{b_j^{\text{im}}}{|b_j|} \cdot \left(\mathbf{z}^{\alpha(j)}\right)^{\text{im}} \right)^2 \right) \cdot \left(|b_j| \cdot \sum_{k=2}^{d} |b_k| \right).$$

Hence, the expression $\sum_{i=1}^{d+2} s_i g_i + H$ in (5.8) in total results in

$$-|b_1|^2 + \left(\sum_{i=2}^{d} |b_i| \right)^2 ,$$

which, since all $|b_i| \geq 0$, is the certificate we wanted to obtain since we assumed lopsidedness with dominating term $|m_{\alpha(1)}(\mathbf{1})|$. By rescaling, we can bring the constant to -1. □

We say that there exists a certificate for a point $\mathbf{w}$ in the complement of the amoeba $\mathcal{A}(f)$ if there exists a certificate for the point $\mathbf{1}$ in the complement of the unlog amoeba $\mathcal{U}(g)$ in the sense of Theorem 5.16, where g is defined as in Lemma 5.7 and $\lambda_i = |\log^{-1}(w_i)|$.

Corollary 5.17. *Let $r \in \mathbb{N}$.*

1. *For any $\mathbf{w} \in \mathbb{R}^n \setminus \mathcal{LA}(\tilde{f}_r) \subset \mathbb{R}^n \setminus \mathcal{A}(f)$ there exists a certificate of degree at most $2 \cdot r^n \cdot \deg(f)$, which can be computed explicitly.*
2. *The certificate determines the order of the complement component $\mathbf{w}$ belongs to.*

Proof. By definition of g, we have $\mathbf{w} \in \mathcal{A}(f)$ if and only if $\mathbf{1} \in \mathcal{U}(g)$. Further $\mathbf{1}$ belongs to $\mathcal{LA}(\tilde{g}_r)$ if and only if $\tilde{g}_r\{0\}$ is not lopsided. Applying Theorem 5.16 on the function $\tilde{g}_r$ yields a certificate for $\mathbf{w}$ in the log amoeba $\mathcal{A}(f)$. Since we have $\text{tdeg}(\tilde{g}_r) = \text{tdeg}(g) \cdot r^n = \text{tdeg}(f) \cdot r^n$ due to the definition of $\tilde{g}_r$ and of g the result follows.

For the second statement, note that passing over from f to g does not change the order of any point in the complement of the amoebas. Now it suffices to show that the dominating term (which occurs in a distinguished way in the certificate) determines the order of the complement component. The latter statement follows from Purbhoo's Theorem 2.30 that if $\mathbf{w} \notin \mathcal{LA}(\hat{f}_r)$ and the order of the complement component $\mathbf{w}$ belongs to is $\alpha(i)$ then the dominant term in $\tilde{f}_r$ has the exponent $r^n \cdot \alpha(i)$ (see also [**70**, Proposition 4.1]). □

Theorem 5.18. *For linear hyperplane amoebas in $\mathbb{R}^n$, any point in the complement of the amoeba has a certificate whose sum of squares is a sum of squares of affine functions.*

Proof. By the explicit characterization of linear hyperplane amoebas in Theorem 2.8 (see also [**20**, Corollary 4.3]), any point in the complement is lopsided. Hence, the statement follows from Theorem 5.16. □

3.3. Simplified Expressions. From a slightly more general point of view, monomial-based certificates can be seen as a special case of the following construction. Whenever the defining polynomials of a variety origin from simpler polynomials with algebraically independent monomials, then the approximation of the amoeba can be simplified.

For an ideal I let $V = \mathcal{V}(I) \subset (\mathbb{C}^*)^n$ be its subvariety in $(\mathbb{C}^*)^n$. Let $\gamma_1, \ldots, \gamma_k$ be k monomials in n variables, say, $\gamma_i = \mathbf{y}^{\alpha(i)} = y_1^{\alpha(i)_1} y_2^{\alpha(i)_2} \cdots y_n^{\alpha(i)_n}$, where $\alpha(i) = (\alpha(i)_1, \ldots, \alpha(i)_n) \in \mathbb{Z}^n$. They define a homomorphism γ of algebraic groups from $(\mathbb{C}^*)^n$ to $(\mathbb{C}^*)^k$. For any subvariety W of $(\mathbb{C}^*)^k$, the inverse image $\gamma^{-1}(W)$ is a subvariety of $(\mathbb{C}^*)^n$. Note that the map γ is onto if and only if the vectors $\alpha(1), \ldots, \alpha(k)$ are linearly independent (see [**85**, Lemma 4.1]).

Let J be an ideal with $\mathcal{V}(J) = \gamma^{-1}(V)$. If the map γ is onto, then computing the amoeba of J can be reduced to the computation of the amoeba of I. Let γ' denote the restriction of γ to the multiplicative subgroup $(0, \infty)^n$. Then the following diagram is a commutative diagram of multiplicative abelian groups:

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \xrightarrow{\gamma} & (\mathbb{C}^*)^k \\ \downarrow & & \downarrow \\ (0,\infty)^n & \xrightarrow{\gamma'} & (0,\infty)^k \end{array}$$

where the vertical maps are taking coordinate-wise absolute value. For vectors $p = (p_1, \ldots, p_n)$ in $(\mathbb{C}^*)^n$ we write $|p| = (|p_1|, \ldots, |p_n|) \in (0, \infty)^n$, and similarly for vectors of length k. Further, for $V \subset (\mathbb{C}^*)^n$ let $|V| = \{|p| \, : \, p \in V\}$. If the map γ is onto then $|\gamma^{-1}(V)| = \gamma'^{-1}(|V|)$ (see [**85**]).

Theorem 5.19. *If a point outside of an unlog amoeba $\mathcal{U}(I)$ has a certificate of total degree d then a point outside of the unlog $\mathcal{U}(J)$ has a certificate of degree $d \cdot D$, where D is the maximal total degree of the monomials $\gamma_1, \ldots, \gamma_k$.*

In particular, this statement applies to the certificates from Statements 5.4 and 5.9.

PROOF. Let p be a point outside of the unlog amoeba of $V \subset (\mathbb{C}^*)^n$, which has a certificate of total degree d. By Corollary (5.4), the certificate consists of a polynomial $G(\mathbf{x}, \mathbf{y})$ in the real ideal $I' \subset \mathbb{R}[\mathbf{x}, \mathbf{y}]$ from (5.1) and by real sums of squares of polynomials in $\mathbb{R}[\mathbf{x}, \mathbf{y}]$. For the polynomials in the ideals, we observe that the realification process carries over to the substitution process. W.l.o.g. we can assume that γ_i is a product of just two factors. Then, with $\mathbf{z} = z_1 + iz_2$, $\mathbf{z} = pq$ we have $z_1 + iz_2 = (p_1 + ip_2)(q_1 + iq_2)$ and use the real substitutions $z_1 \equiv p_1q_1 - p_2q_2$, $z_2 \equiv p_1q_2 + p_2q_1$. And in the same way the real sum of squares remain real sums of squares (of the polynomials in p_i, q_j) after substituting. □

Example 5.20. Let $\mathbb{G}_{1,3}$ denote the Grassmannian of lines in 3-space. It is the variety in $\mathbb{P}^5_{\mathbb{C}}$ defined by

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0 \, ,$$

which we consider as a subvariety of $(\mathbb{C}^*)^6$. The three terms in this quadratic equation involve distinct variables and hence correspond to linearly independent exponent vectors. Note that $\mathbb{G}_{1,3}$ equals $\gamma^{-1}(V)$ where

$$\gamma : (\mathbb{C}^*)^6 \to (\mathbb{C}^*)^3 \, , \; (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}) \mapsto (p_{01}p_{23}, \, p_{02}p_{13}, \, p_{03}p_{12})$$

and V denotes the plane in 3-space defined by the linear equation $x - y + z = 0$. Since by Theorem 5.18 any point in the complement has a certificate of degree 2, Theorem 5.19

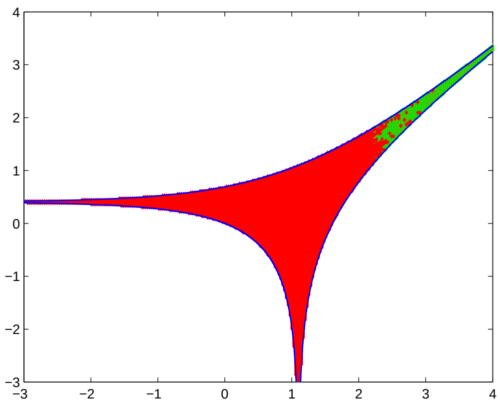


Figure 5.3. SOS certificates of linear amoebas restricted to degree 2. The red (dark) points represent infeasible SDPs, and in the green (light) points numerical instabilities were reported in the computations.

implies that every point in the complement of the Grassmannian amoeba has a certificate of degree 4.

4. Examples and Applications

We close this chapter by providing some computational results in order to confirm the validity of our approach. The subsequent computations have been performed on top of SOSTools [**55**], which is a Matlab package for computing sums of squares based computations. The SDP package underlying SOSTools is SeDuMi [**84**].

Example 5.21. For the test case of a linear polynomial $f = z_1 + 2z_2 + 3$, the boundary contour of the amoeba $\mathcal{A}(f)$ can be explicitly described, and it is given by the curves

$$\begin{aligned} \exp(z_1) &= 2 \cdot \exp(z_2) + 3\,, \\ 2 \cdot \exp(z_2) &= \exp(z_1) + 3\,, \\ 3 &= \exp(z_1) + 2 \cdot \exp(z_2)\,, \end{aligned}$$

see [**20**]. We compute the amoeba of f with our SDP via SOSTools on a grid of size 250×250 lattice points in the area $[-3, 4]^2$. In the SDP we restrict to polynomials of degree 2. By Theorem 5.16, the approximation is exact in that case (up to numerical issues). Figure 5.3 visualizes the SDP-based computation of the SOS certificates. In the figure, at the white outer regions, certificates are found. At the red points the SDP is infeasible; at the green points feasibility cannot be proven by the solver, but numerical instabilities are reported by the SDP solver. The issue of numerical stability of our SOS-based amoeba computations and of general SOS computations is an important issue in convex algebraic geometry, which deserves further study.

Example 5.22. As in Figure 4.2 we consider the class of polynomials $f = z_1^2 z_2 + z_1 z_2^2 + c \cdot z_1 z_2 + 1$ with some constant $c \in \mathbb{R}$. We use the monomial-based approach from

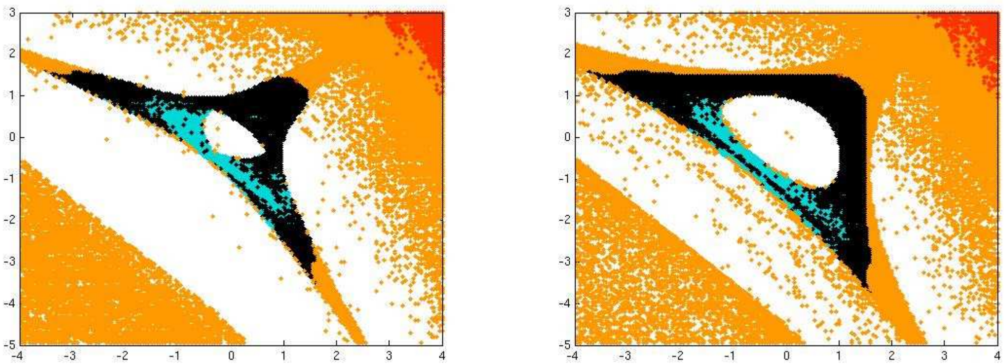


FIGURE 5.4. The amoeba of $f = 1 + z_1^2 z_2 + z_1 z_2^2 + c \cdot z_1 z_2$ approximated with SOSTOOLS for $c = 2$ and $c = -4$.

Corollary 5.9. In order to compute whether a given point $(\mu_1, \mu_2) \in \mathbb{R}_{>0}$ is contained in the unlog amoeba $\mathcal{U}(f)$, we have to consider the polynomials

$$\begin{aligned}
h_1 &= (x_1^2 x_2 - 2x_1 y_1 y_2 - y_1^2 \cdot x_2) + (x_1 x_2^2 - x_1 y_2^2 - 2y_1 x_2 y_2) \\
&\quad + c \cdot (x_1 x_2 - y_1 y_2) + 1\,, \\
h_2 &= (x_1^2 y_2 + 2x_1 y_1 x_2 - y_1^2 y_2) + (2x_1 x_2 y_2 + y_1 x_2^2 - y_1 y_2^2) + c \cdot (x_1 y_2 + x_2 y_1)\,, \\
h_3 &= (x_1^2 x_2 - 2x_1 y_1 y_2 - y_1^2 x_2)^2 + (x_1^2 y_2 + 2x_1 y_1 x_2 - y_1^2 y_2)^2 - (\mu_1^2 \cdot \mu_2)^2\,, \\
h_4 &= (x_1 x_2^2 - x_1 y_2^2 - 2y_1 x_2 y_2)^2 + (2x_1 x_2 y_2 + y_1 x_2^2 - y_1 y_2^2)^2 - (\mu_1 \cdot \mu_2^2)^2\,, \\
h_5 &= (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2 - (\mu_1 \cdot \mu_2)^2\,.
\end{aligned}$$

For the case $c = 2$ and $c = -4$ we investigate 160×160 points in the area $[-4, 4] \times [-5, 3]$ and restrict the polynomials multiplied with the constraints to degree 3. The resulting log-amoeba $\mathcal{A}(f)$ is depicted in Figure 5.4. At the white points the SDP is feasible and thus these points belong to the complement component. At the orange points the SDP is recognized as feasible with numerical issues (within a pre-defined range). At the black points the SDP was infeasible without and at the turquoise points with numerical issues reported. At the red points the program stopped due to exceeding numerical problems. The union of the black, the turquoise and part of the orange points provides the (degree bounded) approximation of the amoeba.

4.1. The Diameter of Inner Complement Components. We briefly discuss that the SOS-based certificates can also be used for more sophisticated questions rather than the pure membership problem. For this, we investigate the the class $\mathcal{P}_\Delta^y$ of polynomials $f \in \mathbb{C}[z_1, \ldots, z_n]$ introduced in Chapter 4, Section 1 again. Recall that the Newton polytope of polynomials in $\mathcal{P}_\Delta^y$ is a simplex and that these polynomials have $n + 2$ monomials such that exactly one of their exponents is located in the interior of the simplex Δ (see (4.1)). Recall furthermore that amoebas of polynomials in $\mathcal{P}_\Delta^y$ have at most one inner complement component (Theorem 4.1).

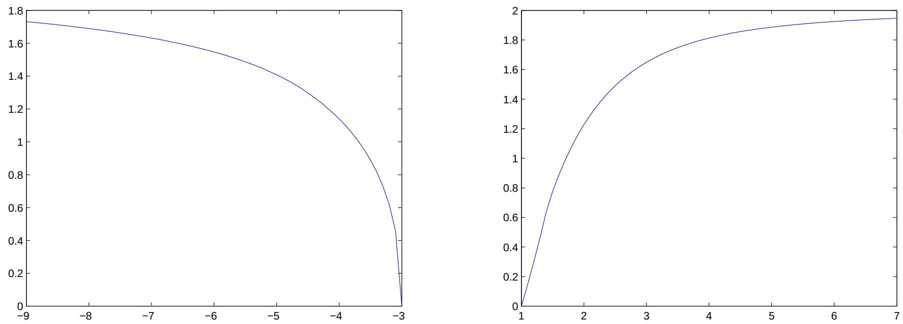


FIGURE 5.5. Lower bound for the diameter of the inner complement component of $f = 1 + z_1^2 z_2 + z_1 z_2^2 + c \cdot z_1 z_2$ and $c \in [-9, -3]$ resp. $c \in [1, 7]$.

Let $f = \sum_{i=0}^{n} b_i \cdot \mathbf{z}^{\alpha(i)} + c \cdot \mathbf{z}^{y}$, and let c denote the coefficient of the inner monomial. By Lemma 4.11 and Theorem 4.13 for $|c| \to \infty$ the inner complement component appears at the image under the Log–map of the uniquely determined minimum δ of the function $\hat{f} = \frac{f}{\arg(b_0) \cdot \mathbf{z}^{\alpha(0)}}$. δ is explicitly computable. This yields the opportunity to certify that a complement component of the unlog amoeba has a certain diameter d under the scaling $|\mathbf{z}| \mapsto |\mathbf{z}|^2$ of the (unlog) amoebas basis space by solving the SDP corresponding to

$$\sum_{j=1}^{3} s_i g_i + H + 1 \quad = \quad 0 \tag{5.9}$$

with polynomials $s_1 = \sum_{i=1}^{n} (|\delta_i|^2 - |z_i|^2)^2 - d^2/4$, $s_2 = \sum_{i=0}^{n} \left(b_i \cdot \mathbf{z}^{\alpha(i)}\right)^{\text{re}} + (c \cdot \mathbf{z}^{y})^{\text{re}}$ and $s_3 = \sum_{i=0}^{n} \left(b_i \cdot \mathbf{z}^{\alpha(i)}\right)^{\text{im}} + (c \cdot \mathbf{z}^{y})^{\text{im}}$, where $g_i \in \mathbb{C}[\mathbf{z}]$ (restricted to some total degree) and H is an SOS polynomial.

Feasibility of the SDP certifies that there exists no point $\mathbf{v} \in \mathcal{V}(f) \cap \partial \mathcal{B}_{d/2}(\delta)$ (where $\mathcal{B}_{d/2}(\delta)$ denotes the ball with radius $d/2$ centered in δ) in the rescaled amoebas basis space. Hence, the corresponding inner complement component of the unlog amoeba has at least a diameter d in that space. We have to investigate the rescaled basis space of the unlog amoeba here in order to transform the generic condition $(|\delta_i| - |z_i|)^2 - d^2/4$ on the basis space of $\mathcal{U}(f)$ into a polynomial condition, which is given by s_1 here.

Note that this works not only for polynomials in the class under investigation, but for every polynomial as long as one knows, where a complement component appears.

Example 5.23. Let $f = 1 + z_1^2 z_2 + z_1 z_2^2 + c \cdot z_1 z_2$ with a real parameter c. For this class, the inner lattice point is the barycenter of the simplex (see Chapter 4, Section 2) and the inner complement component appears at the point $(1, 1)$, i.e., under the Log-map at the origin of $\text{Log}(\mathbb{R}^2)$. The inner complement component exists for $c > 1$ and $c < -3$ (see Theorem 4.20). We compute a bound for the diameter of the inner complement component using the upper SDP (5.9) for the intervals $[-9, -3]$ and $[1, 7]$ with step length 0.1. For any of these points we compute 14 SDPs in order to estimate the radius (based on binary search). We obtain the bounds shown in Figure 5.5.

Observe that these bounds are lower bounds since feasibility of the SDP certifies membership in the complement of the amoeba but infeasibility only certifies that no certificate with polynomials of degree at most k (i.e., 3 in our case) exists.

This approach also yields lower bound for the diameter of the inner complement component of the (log) amoeba, since, e.g., the image of the circle $\sum_{i=1}^{n}(|\delta_i|^2 - |z_i|^2)^2 - r^2$ under the Log-map, i.e., $\sum_{i=1}^{n}(\log|\delta_i| - \log|z_i|)^2 - r^2$, contains the triangle

$$\operatorname{conv}\left(\left\{\left(|\delta_1|\cdot e^{r/2}, \ldots, |\delta_n|\right), \ldots, \left(|\delta_1|, \ldots, |\delta_n|\cdot e^{r/2}\right), \left(|\delta_1|\cdot e^{\frac{r}{2\sqrt{n}}}, \ldots, |\delta_n|\cdot e^{\frac{r}{2\sqrt{n}}}\right)\right\}\right).$$

Hence the double radius of the incircle of that triangle is a lower bound for the diameter of the inner complement component of the amoeba.

CHAPTER 6

Resume and Open Problems

In this final chapter we recapitulate the problems we have investigated, the advances we made on them and provide questions that remain open.

The Configuration Space of Amoebas

A major part of this thesis concentrates on understanding the geometrical and topological structure of the configuration space of amoebas. Specifically, we dealt with the following problems and questions. Let $A \subset \mathbb{Z}^n$.

(1) What are necessary and sufficient conditions such that $U_\alpha^A \neq \emptyset$ for $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n \setminus A$ (Problem 2.20)?
(2) Is every U_α^A connected for $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$ (Problem 2.22)?
(3) Is the intersection of U_α^A with a generic complex projective line non-empty and connected for $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$ (Problem 2.23)?
(4) Is for $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$ the set U_α^A simply connected, if U_α^A is (path-)connected? If not, what is its fundamental group (Problem 2.24)?
(5) Let $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$. Give an exact algebraic or geometrical description of U_α^A or (at least) bounds to approximate U_α^A (Problem 2.25; see also Problem 2.9).

With respect to (1) state of the art were Rullgård's Theorems 2.15 and 2.19 providing some necessary and some sufficient conditions on A and α. Although (1) remains open, since we do not provide strengthenings on these conditions, we figured out that the proof of Theorem 2.19 had a gap, which we were able to close (Theorem 4.43). Furthermore, we give (for appropriate $A \subset \mathbb{Z}^n$) an explicit construction method yielding (to the best of my knowledge) the first non-univariate example for a polynomial $f \in U_\alpha^A$ with $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n \setminus A$ (Example 4.44).

We put very much effort on Question (2) leading finally to an affirmative answer for polynomials with barycentric simplex Newton polytope (Corollary 4.25), for trinomials (Corollary 4.50) and for minimally sparse univariate polynomials (3.12). These are the first classes of polynomials where this question is solved for at all (up to linear polynomials, where the question is trivial).

On the one hand, I would consider this question, which, of course, remains open in general, as one of the most interesting problems in contemporary amoeba theory. On the other hand, I had to learn that it is seemingly extremely complicated in general and thus I recommend to "approach with caution". I cannot even give a good guess, whether the answer is "yes" or "no" in general; I conjecture it to be affirmative in the case that $A = \text{conv}(A) \cap \mathbb{Z}^n$ (Conjecture 3.8), which is the multivariate version of Theorem 3.12 (see Chapter 3, Section 3).

We solve Question (3) – the general answer is negative since in general U_α^A of trinomials do not have this property (see Chapter 4, Section 3). Anyhow, there might be special choices of A such that the property holds, which remains unclear. For example, the property holds for all intersections of U_y^A of polynomials with barycentric simplex Newton polytopes with the complex line given by fixing all but the inner coefficient (Lemma 4.22; see Chapter 4, Section 2 for further details). A similar statement holds for the U_α^A corresponding to the exponent of the "middle" term of a trinomial (see Chapter 4, Section 3)

We also solve the first question of (4), where the general answer is negative again. If $A = \{0, s, s+t\}$ with s, t coprime and $\alpha \in (\text{conv}(A) \cap \mathbb{Z})) \setminus \{0, s, s+t\}$, then U_α^A can be deformation retracted to an $(s+t)$-sheeted covering of an S^1 (Theorem 4.51), which is not simply connected. The same statement answers the second question of (4) for trinomials, since in this case the fundamental group of the particular U_α^A is $\mathbb{Z}$. The problem remains open for other classes of polynomials. But, since already the rather easy class we investigate here yield (next to a lot of complications) a rather complicated structure of the configuration space, one has to face the fact that there is at least no genuine approach for other classes in sight.

Problem (5), again, is way too general for a complete solution. We deliver the desired exact description for polynomials with barycentric simplex Newton polytope (Theorem 4.19 and for trinomials Theorems 4.40 and the results of Section 3.3).

Furthermore, we provide explicitly computable, exact upper and lower bounds for polynomials with simplex Newton polytope and one additional inner lattice point ("amoebas of genus at most one" resp. "amoebas supported on a circuit"), where the upper bound becomes sharp under some extremal conditions (Theorems 4.8, 4.10 and 4.13). Next to the linear case investigated by Forsberg, Passare and Tsikh (Theorem 2.8) and an example by Passare and Rullgård (see Chapter 2, Section 4), these are the first classes this question is solved for at all.

Maybe the most interesting consequence of these results is that the structure of the configuration space, which is discovered here, is very rich and beautiful – geometrically related to such classical, well known objects as hypotrochoid curves and the possibility to deformation retract whole sets U_α^A to one specific, nice, closed path, which topologically can be tackled properly.

In my opinion the questions (1), (5) and (2) with the restriction mentioned above are worthiest to be investigated in future. In particular, I would suggest Question (5) to be solvable for some appropriate classes of polynomials.

The Lattice of Configuration Spaces and Sparsity

In the past the usual approach for the investigation of amoebas was to fix the support set $A \subset \mathbb{Z}^n$ and thus the corresponding configuration space. To the best of my knowledge there are almost no results explaining how configuration spaces are related to each other with respect to existing combinatorial relations between the defining support sets of this configuration spaces (of course, only with respect to amoeba theory – I am not aware of known structures for similar objects in other fields of mathematics, which I would strongly conjecture to be investigated and understood better). The only result I am aware of is a theorem by Rullgård (roughly) stating that homeomorphism between two configuration spaces given by a regular linear map between their support sets induces a homotopy between the corresponding amoebas (and also respects orders and Ronkin coefficients; see [**77**, Theorem 7]).

It is a surprising fact that this relation between spaces is so rarely investigated because one can switch from one configuration space $\mathbb{C}^A$ to another one $\mathbb{C}^B$ with $B \subset A$ easily by investigating sequences of polynomials in $\mathbb{C}^A$ where certain coefficients vanish in the limit. Since varieties of polynomials and thus also amoebas are continuous along such sequences, a fruitful structure can be expected behind this relationship.

We show that this is indeed the case. If P is an integral n-polytope, then the set $L(P)$ of all configuration spaces $\mathbb{C}^A$ with $\text{conv}(A) = P$ forms a boolean lattice with respect to a relation induced by set theoretical inclusion (Theorem 3.2). In this lattice every augmented configuration space $\mathbb{C}^A_\Diamond$ (see Chapter 2, Section 4) is the union of all elements in the the order ideal $\mathcal{O}(\{\mathbb{C}^A\})$ of the configuration space $\mathbb{C}^A$ (Corollary 3.3). Indeed, the structure of a lattice of configuration spaces has an impact on the amoebas in the configuration spaces, which are its elements. We show this exemplarily by proving that if a set $U_\alpha^A = \emptyset$, then $U_\alpha^B = \emptyset$ for every $\mathbb{C}^B$ contained in the order ideal $\mathcal{O}(\{\mathbb{C}^A\})$ (Theorem 3.6). As a consequence the sets U_α^A are also open on augmented configuration spaces (Corollary 3.7) and we obtain an independent motivation that "*maximally sparse polynomials have solid amoebas*" (Problem 3.4) as conjectured by Passare and Rullgård (see [**62, 66**]), since the configuration space of maximally sparse polynomials with Newton polytope P is the minimal element in $L(P)$ (see Chapter 3, Section 2 for further details). Furthermore, we give easy proofs (independent of Nisse's approach in [**52**]) of this conjecture for certain, rich classes of Newton polytopes (Theorems 3.9 and 3.10).

For the future it would surely be nice to find an elemental proof for the maximally sparse conjecture (Problem 3.4) for *all* Newton polytopes. Independent of that I suggest it as very worthy to go on to investigate the connection between (boolean) lattices and (configuration spaces of) amoebas. The theory of lattices is very rich and well developed (see e.g., [**83, 89**]) and I truly expect more pearls to be hidden here, which are waiting to be discovered.

The Approximation of Amoebas

Since an exact description of amoebas and coamoebas is not known basically except for the linear case, one genuine task is to find algorithms to approximate amoebas and coamoebas. In the last years the usual approach, which was used to tackle this problem, was to try to solve the *membership problem* (see Problems 2.26 and 2.33). While for coamoebas no algorithm solving this problem is known so far, for amoebas state of the art is Purbhoo's lopsidedness criterion and his corresponding relaxation process based on iterated resultants (see Chapter 2, Section 5), which yields a certificate-based solution of the Problem 2.26 for amoebas. But, besides the fact that there is no canonical implementation, the strongest issue arising with Purbhoo's solution is that it does not provide a certificate in a strict sense, i.e., it does not provide an *algebraic certificate* based on polynomial inequalities.

We provide a new approach to the membership problem via using semidefinite programming and sums of squares. Specifically, we show that the membership problem both for amoebas and for coamoebas can be expressed in terms of real polynomial inequalities and therefore an (algebraic) certificate is given in terms of sums of squares via the Real Nullstellensatz (Corollaries 5.4, 5.9 and Theorem 5.8).

This new approach is very attractive – not only, because it can be implemented straightforwardly by using the well established software for semidefinite programming (we use SeDuMi and SOSTools here), but in particular since we can additionally prove that (on the amoeba side) our approach is as good as Purbhoo's in terms of complexity (Corollary 5.17).

Furthermore, we have shown that the approach works from a practical point of view. The implementation was done and a couple of approximations were made (see e.g., Figures 5.3 and 5.4) and also related problems can be tackled in practice (see Chapter 5, Section 4 for further details).

Anyhow, although the problem can be regarded as solved from a theoretical point of view (except, of course, that one might look for further applications of this method), from the practical point of view the method is far away from being optimally implemented. In particular, we had to learn that – even if the related SDPs have small degrees – numerical issues appear in an terrifying amount (see Example 5.22 and Figure 5.4).

In discussions e.g., with Pablo Parrilo and Frank Valentin we found out that similar phenomena were observed for other SDP-based algorithms and that they are conjectured to be caused by a "bad" choice of polynomial bases (at least as one reason). Anyhow, there exists almost no literature about these effects (see e.g., [**36**] as one of the rare exceptions). Thus, it seems as a very worthy future project to collect data about numerical issues for solving SDP problems with different monomial bases with the aim to be able to make suggestions of good choices for such bases.

The Boundary of Amoebas

Similarly as the membership problem for complement components of amoebas it is an open problem how to determine the points in $\mathbb{R}^n$, which belong to the boundary of amoebas (recall that amoebas are closed sets; see Theorem 2.1). Obviously, for a Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ a boundary point $\mathbf{w} \in \partial\mathcal{A}(f)$ has to be the image of a critical point of the Log-map, i.e., it has to be part of the *contour* of the amoeba. The contour is a nice structure to work with, since it can be computed properly due to a result of Mikhalkin (Theorem 3.13) stating that the critical points of the Log-map are exactly the points $S(f)$, which are mapped to a real projective vector under the *logarithmic Gauss map* (see Chapter 3, Section 4 and in particular [**43, 41, 85**] for further details).

On the other hand, in general the contour is a strict superset of the boundary (see e.g., [**66**]). So, the only fact known about a boundary point is that in the intersection of its fiber and the variety of the defining polynomial there *exist* a point in $S(f)$ (Corollary 3.14). But it is unknown how to distinguish between boundary and non-boundary points in the contour.

We provide a necessary condition for such a distinction by a strengthening of the existing statement. We show that if a point $\mathbf{w}$ of the contour belongs to the boundary $\partial\mathcal{A}(f)$, then *every* point in the intersection of its fiber $\mathbb{F}_{\mathbf{w}}$ and the variety $\mathcal{V}(f)$ has to belong to $S(f)$ (Theorem 3.15). We show this by investigating the submanifolds of $\mathbb{F}_{\mathbf{w}}$ where the real- and imaginary part of f vanish. In particular, we study how they may intersect. These submanifolds can be nicely visualized – at least for $n = 2$ where they are curves (e.g., with MAPLE; see Figure 3.3).

Unfortunately, it is not clear if this necessary condition is also sufficient or at least generically sufficient (I believe the latter to be the case). Furthermore, there is no algorithm so far to check the criteria of our theorem efficiently. I consider both questions as very interesting and solvable. They are, as well as the presented result, part of an ongoing project joint with Franziska Schröter.

Miscellaneous Results

The thesis contains some further, more special results, which do not fit properly in one of the upper sections. The most important are:

- Let $f \in \mathcal{P}_{\Delta}^{y}$ be a multivariate polynomial with amoeba of genus at most one (see Chapter 4, Section 1). $\mathcal{A}(f)$ is solid and the upper bound computed in Theorem 4.13 is attained with equality if and only if f belongs to the A-discriminant ∇_A of $\mathbb{C}^A$ (Theorem 4.17). Furthermore, if the upper bound is is exceeded, then there exists an explicitly computable point in the bounded complement component, where f is lopsided (Theorem 4.15).
- Let $A \subset \mathbb{Z}^n$ and $\alpha \in A$. If for every complex line $l_\alpha \subset \mathbb{C}^A$ given by fixing all coefficients but b_α corresponding to $\mathbf{z}^\alpha$ holds that $l_\alpha \cap (U_\alpha^A)^c$ is simply connected, then U_α^A is connected (Theorem 4.24).
- For univariate trinomials $f = z^{s+t} + pz^t + q \in \mathbb{C}[z]$ since the late 19th century the correspondence between location of the roots and the choice of the coefficients is investigated (see [**8, 33, 45**]). Although algebraically well understood due to theorems by Bohl from 1908 (Theorems 4.28 and 4.29), it is since over years not understood, what is, for fixed q, the geometrical structure of all p such that f has a root of a certain absolute value or two roots of the same absolute value (see Problem 4.31; see also Chapter 4, Section 3 for further details). We prove via usage of amoeba theory that f has a root of a certain, given absolute value $|z^*| \in \mathbb{R}_{>0}$ if and only if p is located on the trajectory of a specific hypotrochoid curve depending on s, t, q and $|z^*|$ (Theorem 4.32). For $p, q \neq 0$ we show that pairs of at most two roots have the same absolute value (Lemma 4.38) and such pairs exist if and only if (for q fixed) p is located on a specific 1-fan $F(s, t, q)$ (Theorem 4.40; see Figure 4.9 for trajectories of hypotrochoids and the 1-fan $F(s, t, q)$). Furthermore, we prove that if $p, q \in \mathbb{R}^*$ and $\mathcal{V}(f) = \{a_1, \ldots, a_{s+t}\}$

with $|a_1| \leq \cdots \leq |a_{s+t}|$, then the only roots, which may be real are a_1, a_t, a_{t+1} and a_{s+t} (Theorem 4.39).

- Let $A \subset \mathbb{Z}^n$ and $\alpha \in \text{conv}(A) \cap \mathbb{Z}^n$ but $\alpha \notin A$. Rullgård gave a sufficient condition on α depending on the lattice $\mathcal{L}_A$ generated by A for $U_\alpha^A \neq \emptyset$ (Theorem 2.19, Part (2); see also [**77**, Theorem 11]). Unfortunately, his proof has a gap. We close this gap (Theorem 4.43) and provide the (to the best of my knowledge) first known example of an amoeba with a complement component of order α with $\alpha \notin A$ (see Example 4.44 and Figure 4.10).
- It was unclear whether the complement induced tropical hypersurface $\mathcal{C}(f)$ given by a Laurent polynomial $f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ is always homotopy equivalent with its amoeba $\mathcal{A}(f)$ (Problem 2.16). We prove that this is not the case in general (Corollary 4.45).
- It was for $A \subset \mathbb{Z}^n$ an open question whether an existing complement component of order $\alpha \in A$ increases monotonically with increasing absolute value of the coefficient b_α (see Problem 4.52). We prove that this is not the case in general (Theorem 4.53).

Bibliography

1. G. Averkov, *On the size of lattice simplices with a single interior lattice point*, SIAM J. Discrete Math. **26** (2012), no. 4, 515–526.
2. G.M. Bergman, *The logarithmic limit-set of an algebraic variety*, Trans. Amer. Math. Soc. **157** (1971), 459–469.
3. R. Bieri and J.R.J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Angew. Math. **347** (1984), 168–195.
4. F. Bihan, *Polynomial systems supported on circuits and dessins d'enfants*, J. Lond. Math. Soc. **75** (2007), no. 1, 116–132.
5. F. Bihan, J.M. Rojas, and C.E. Stella, *Faster real feasibility via circuit discriminants*, ISSAC 2009—Proceedings of the 2009 International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2009, pp. 39–46.
6. G. Blekherman, P. Parrilo, and R. Thomas, *Semidefinite optimization and convex algebraic geometry*, 2012.
7. J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 36, Springer-Verlag, Berlin, 1998.
8. P. Bohl, *Zur Theorie der trinomischen Gleichungen*, Math. Ann. **65** (1908), no. 4, 556–566.
9. N. Bourbaki, *Elements of mathematics. Commutative algebra*, Hermann, Paris, 1972.
10. E. Brugallé, *Un peu de géométrie tropicale*, Quadrature **74** (2009), 10–22, Versions in English, Swedish and Portuguese aviable; see arXiv:0911.2203.
11. E. Brugallé and K.M. Shaw, *Obstructions to approximating tropical curves in surfaces via intersection theory*, Preprint, arXiv:1110.0533, 2011.
12. D. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms*, third ed., Undergraduate Texts in Mathematics, Springer, New York, 2007.
13. C.N. Delzell and A. Prestel, *Positive Polynomials*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001.
14. K. Dilcher, J.D. Nulton, and K.B. Stolarsky, *The zeros of a certain family of trinomials*, Glasgow Math. J. **34** (1992), no. 1, 55–74.
15. M. Einsiedler, M.M. Kapranov, and D. Lind, *Non-Archimedean amoebas and tropical varieties*, J. Reine Angew. Math. **601** (2006), 139–157.
16. M. Einsiedler, D. Lind, R. Miles, and T. Ward, *Expansive subdynamics for algebraic $\mathbb{Z}^d$-actions*, Ergodic Theory Dynam. Systems **21** (2001), no. 6, 1695–1729.
17. L. Fejér, *Über die Wurzel vom kleinsten absoluten Betrage einer algebraischen Gleichung*, Math. Ann. **65** (1908), no. 3, 413–423.
18. H. Fell, *The geometry of zeros of trinomial equations*, Rend. Circ. Mat. Palermo (2) **29** (1980), no. 2, 303–336.
19. K. Fladt, *Analytische Geometrie spezieller ebener Kurven*, Akademische Verlagsgesellschaft, Frankfurt am Main, 1962.
20. M. Forsberg, M. Passare, and A. Tsikh, *Laurent determinants and arrangements of hyperplane amoebas*, Adv. Math. **151** (2000), 45–70.
21. J. Forsgård and P. Johansson, *An order map for lopsided coamoebas*, Preprint, arXiv:1205.2014, 2012.

22. A. Gathmann, *Tropical algebraic geometry*, Jahresber. Deutsch. Math.-Verein. **108** (2006), no. 1, 3–32.
23. I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*, Birkhäuser, Boston, 1994.
24. A.J. Goldman and M. Ramana, *Some geometric results in semidefinite programming*, J. Global Optim. **7** (1995), no. 1, 33–50.
25. J. Gouveia, R.Z. Robinson, and R.R. Thomas, *Polytopes of minimum positive semidefinite rank*, Preprint, , arXiv:1205.5306, 2012.
26. G. Greenfield and D. Drucker, *On the discriminant of a trinomial*, Linear Algebra Appl. **62** (1984), 105–112.
27. A.B. Hansen, *A certain class of polytopes associated with independent systems*, Mathematica Scandinavica **41** (1977), 225–241.
28. A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2001.
29. A. Henriques, *An analogue of convexity for complements of amoebas of varieties of higher codimension, an answer to a question asked by B. Sturmfels*, Adv. Geom. **4** (2004), no. 1, 61–73.
30. M.M. Kapranov, *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. **290** (1991), no. 2, 277–285.
31. M.M. Kapranov, *Amoebas over non-archimedian fields*, preprint, 2000.
32. K. Königsberger, *Analysis 2*, Springer-Verlag, Berlin, 1993.
33. E. Landau, *Sur quelques généralisations du théorème de M.Picard*, Ann. Sci. École Norm. Sup. (3) **24** (1907), 179–201.
34. J.B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim. **11** (2001), no. 3, 796–817.
35. M. Laurent, *Sums of squares, moment matrices and optimization over polynomials*, Emerging applications of algebraic geometry, IMA Vol. Math. Appl., vol. 149, Springer, New York, 2009, pp. 157–270.
36. J. Löberg and P. Parrilo, *From coefficients to samples: a new approach to SOS optimization*, IEEE Conference on Decision and Control, vol. 3, 2004, pp. 3154–3159.
37. J.A. De Loera, J. Lee, P.N. Malkin, and S. Margulies, *Hilbert's Nullstellensatz and an algorithm for proving combinatorial infeasibility*, Proc. ISSAC 2008, ACM, New York, 2008, pp. 197–206.
38. D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, Book manuscript, 2012.
39. M. Marshall, *Positive polynomials and sums of squares*, Mathematical Surveys and Monographs, vol. 146, American Mathematical Society, Providence, RI, 2008.
40. V.P. Maslov, *On a new superposition principle for optimization problem*, Séminaire sur les équations aux dérivées partielles, 1985–1986, École Polytech., Palaiseau, 1986, pp. Exp. No. XXIV, 14.
41. G. Mikhalkin, *Real algebraic curves, the moment map and amoebas*, Ann. of Math. **151** (2000), no. 1, 309–326.
42. ———, *Amoebas of algebraic varieties and tropical geometry*, Different Faces of Geometry (S. K. Donaldson, Y. Eliashberg, and M. Gromov, eds.), Kluwer, New York, 2004, pp. 257–300.
43. ———, *Decomposition into pairs-of-pants for complex algebraic hypersurfaces*, Topology **43** (2004), no. 5, 1035–1065.
44. ———, *Enumerative tropical algebraic geometry in* $\mathbb{R}^2$, J. Amer. Math. Soc. **18** (2005), no. 2, 313–377.
45. P. Nekrassoff, *Ueber trinomische Gleichungen*, Math. Ann. **29** (1887), no. 3, 413–430.
46. S. Neuwirth, *The maximum modulus of a trigonometric trinomial*, J. Anal. Math. **104** (2008), 371–396.
47. S.-I. Niculescu and M. Putinar, *A toric Positivstellensatz with applications to delay systems*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 5-6, 327–329.
48. L. Nilsson, *Amoebas, Discriminants, and Hypergeometric Functions*, Ph.D. thesis, Stockholm University, 2009.
49. L. Nilsson and M. Passare, *Discriminant coamoebas in dimension two*, J. Commut. Algebra **2** (2010), 447–471.

50. L. Nilsson and M. Passare, *Mellin Transforms of Multivariate Rational Functions*, J. Geom. Anal. **23** (2013), no. 1, 24–46.
51. M. Nisse, *Coamoebas of complex algebraic plane curves and the logarithmic gauss map*, Preprint, arXiv:0805.2872, 2008.
52. ———, *Maximally sparse polynomials have solid amoebas*, To appear in Proceedings of the 2011 Bellairs Workshop in Number Theory: Non-Archimedean and Tropical Geometry, arXiv:0704.2216, 2011.
53. M. Nisse and F. Sottile, *Non-archimedean coamoebae*, Preprint, arxiv:1110.1033, 2011.
54. ———, *The phase limit set of a variety*, To appear in Algebra Number Theory, arXiv:1106.0096, 2011.
55. A. Papachristodoulou, P.A. Parrilo, S. Prajna, and P. Seiler, *SOSTOOLS and its control applications*, Positive polynomials in control, Lecture Notes in Control and Inform. Sci., vol. 312, Springer, Berlin, 2005, pp. 273–292.
56. P.A. Parrilo, *Semidefinite programming relaxations for semialgebraic problems*, Math. Program. **96** (2003), no. 2, Ser. B, 293–320.
57. ———, *Exploiting algebraic structure in sum of squares programs*, Positive Polynomials in Control, Lecture Notes in Control and Inform. Sci., vol. 312, Springer, Berlin, 2005, pp. 181–194.
58. M. Passare, *How to compute $\sum 1/n^2$ by solving triangles*, Amer. Math. Monthly **115** (2008), no. 8, 745–752.
59. M. Passare, D. Pochekutov, and A. Tskih, *Amoebas of complex hypersurfaces in statistical thermodynamics*, Math. Phys. Anal. Geom. **15** (2012).
60. M. Passare and J.-J. Risler, *On the curvature of the real amoeba*, Proceedings of the Gökova Geometry-Topology Conference 2010, Int. Press, Somerville, MA, 2011, pp. 129–134.
61. M. Passare, J.M. Rojas, and B. Shapiro, *New multiplier sequences via discriminant amoebae*, Mosc. Math. J. **11** (2011), no. 3, 547–560, 631.
62. M. Passare and H. Rullgård, *Multiple Laurent series and polynomial amoebas*, Actes des Rencontres d'Analyse Complexe (Poitiers-Futuroscope, 1999), Atlantique, Poitiers, 2002, pp. 123–129.
63. ———, *Amoebas, Monge-Ampére measures and triangulations of the Newton polytope*, Duke Math. J. **121** (2004), no. 3, 481–507.
64. M. Passare, T. Sadykov, and A. Tsikh, *Singularities of hypergeometric functions in several variables*, Compositio Math. **141** (2005), 787–810.
65. M. Passare and F. Sottile, *Discriminant coamoebas through homology*, To appear in J. Commutative Algebra, arXiv:1201.6649, 2012.
66. M. Passare and A. Tsikh, *Amoebas: their spines and their contours*, Idempotent Mathematics and Mathematical Physics, Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 275–288.
67. G. Pataki, *On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues*, Math. Oper. Res. **23** (1998), no. 2, 339–358.
68. P. Pébay, M.J. Rojas, and D.C. Thompson, *Optimizing n-variate $(n+k)$-nomials for small k*, Theoret. Comput. Sci. **412** (2011), no. 16, 1457–1469.
69. P.Samuel and O. Zariski, *Commutative algebra. Vol. II*, Springer-Verlag, New York, 1975.
70. K. Purbhoo, *A Nullstellensatz for amoebas*, Duke Math. J. **14** (2008), no. 3, 407–445.
71. M. Putinar, *Positive polynomials on compact semi-algebraic sets*, Indiana Univ. Math. J. **42** (1993), no. 3, 969–984.
72. A. Rashkovskii, *A remark on amoebas in higher codimensions*, Analysis and mathematical physics, Trends Math., Birkhäuser, Basel, 2009, pp. 465–471.
73. B. Reznik, *Lattice point simplices*, Discrete Math. **60** (1986), 219–242.
74. J. Richter-Gebert, B. Sturmfels, and T. Theobald, *First steps in tropical geometry*, Idempotent Mathematics and Mathematical Physics, Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 289–317.

75. L.I. Ronkin, *On zeros of almost periodic functions generated by functions holomorphic in a multicircular domain*, Complex Analysis in Modern Mathematics (Russian), FAZIS, Moscow, 2001, pp. 239–251.
76. H. Rullgård, *Stratification des espaces de polynômes de Laurent et la structure de leurs amibes.*, C. R. Acad. Sci. Paris, Sèrie I **331** (2000), 355–358.
77. ______, *Topics in geometry, analysis and inverse problems*, Ph.D. thesis, Stockholm University, 2003.
78. R. Sanyal, A. Werner, and G. Ziegler, *On Kalai's conjecture concerning centrally symmetric polytopes*, Discrete & Computational Geometry **41** (2009), no. 2, 183–198.
79. A. Schrijver, *Theory of linear and integer programming*, Wiley-Interscience Series in Discrete Mathematics, John Wiley & Sons Ltd., Chichester, 1986.
80. I.R. Shafarevich, *Basic algebraic geometry 1*, Springer-Verlag, Berlin, 1994, Varieties in projective space.
81. D.M.Y. Sommerville, *The Singularities of the Algebraic Trochoids*, Proc. London Math. Soc. **s2-18** (1920), no. 1, 385–392.
82. R.P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.
83. ______, *Enumerative combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 2012.
84. J.F. Sturm, *Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones*, Optim. Methods Softw. **11/12** (1999), no. 1-4, 625–653.
85. T. Theobald, *Computing amoebas*, Experiment. Math. **11** (2002), no. 4, 513–526 (2003).
86. T. Theobald and T. de Wolff, *Approximating amoebas and coamoebas by sums of squares*, 2011, To appear in Math. Comp., arXiv:1101.4114.
87. T. Theobald and T. de Wolff, *Amoebas of genus at most one*, Adv. Math. **239** (2013), 190–213.
88. O. Viro, *Dequantization of real algebraic geometry on logarithmic paper*, European Congress of Mathematics, Vol. I (Barcelona, 2000), Progr. Math., vol. 201, Birkhäuser, Basel, 2001, pp. 135–146.
89. M.L. Wachs, *Poset topology: tools and applications*, Geometric combinatorics, IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., pp. 497–615.
90. H. Wolkowicz, R. Saigal, and L. Vandenberghe (eds.), *Handbook of semidefinite programming*, International Series in Operations Research & Management Science, 27, Kluwer Academic Publishers, Boston, MA, 2000.
91. G.M. Ziegler, *Lectures on polytopes*, Springer Verlag, 2007.

APPENDIX A

Curriculum Vitae of Timo de Wolff

Name: Timo de Wolff.
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Place of birth: Hamburg (Germany).

Education.

02/2009 – 04/2013: PhD in Mathematics at the Goethe University in Frankfurt / Main.
- **PhD-thesis:** "On the Geometry, Topology and Approximation of Amoebas".
- **Advisor:** Prof. Dr. Thorsten Theobald.
- **Referees:**
  - Prof. Dr. Thorsten Theobald (Goethe Univ., Frankfurt / Main),
  - Prof. Dr. Hannah Markwig (Univ. d. Saarlandes, Saarbrücken),
  - Prof. Dr. Ilia Itenberg (Univ. Pierre et Marie Curie (Paris 6), Paris).
- **Total grade:** Summa cum laude (with highest honors).

10/2004 – 01/2009: Studies of Mathematics at the Goethe University in Frankfurt / Main.
- **Master thesis (Diplomarbeit):** "Polytope mit speziellen Simplizes".
- **Advisor:** Prof. Dr. Thorsten Theobald.
- **Total grade:** Mit Auszeichnung (with highest honors).

10/2002 – 01/2011: Studies of Philosophy at the Goethe University in Frankfurt / Main.
- **Master thesis (Magisterarbeit):** "Goodmans "New Riddle of Induction" – Eine Analyse auf mathematischer Grundlage".
- **Advisor:** Prof. Dr. Wilhelm K. Essler.
- **Total grade:** Mit Auszeichnung (with highest honors).

My main academical teachers were *Thorsten Theobald* (discrete mathematics, algebraic and tropical geometry) and *Wolfgang Metzler* (topology). I would also name *Gerhard Burde* (topology) whose joint seminar with Wolfgang Metzler I visited a couple of semesters. For the topics in mathematics I did not focus on in my diploma or PhD-thesis I would in particular name *Ralph Neininger* (stochastics) and *Markus Pflaum* (analysis) from whom I heared multiple lectures.

Employement.

02/2009 – today: Scientific research assistant at the Institute of Mathematics at the Goethe University in Frankfurt / Main.

01/2010 – 12/2012: Member of the DFG-project "**Computational Tropical Geometry**" at the Institute of Mathematics at the Goethe Universtiy in Frankfurt / Main.

Publications.

- **Articles:**
  - "*Low Dimensional Test Sets for Nonnegativity of Even Symmetric Forms*" (joint with S. Iliman), in proceedings, see `http://arxiv.org/abs/1303.4241`.
  - "*Separating Inequalities for Nonnegative Polynomials that Are not Sums of Squares*" (joint with S. Iliman), in proceedings, see `http://arxiv.org/abs/1201.4061`.
  - "*Polytopes with Special Simplices*", in proceedings, see `http://arxiv.org/abs/1009.6158v1`.
  - "*Approximating Amoebas and Coamoebas by Sums of Squares*" (joint with T. Theobald), to appear in Math. Comp.; see also `http://arxiv.org/abs/1101.4114`.
  - "*Amoebas of Genus at Most One*" (joint with T. Theobald), Adv. Math. **239** (2013), 190-213; see also `http://arxiv.org/abs/1108.2456`.
- **Thesis:**
  - "*On the Geometry, Topology and Approximation of Amoebas*", PhD-thesis (Dissertation) for mathematics, 2013.
  - "*Goodman's "New Riddle of Induction"* – Eine Analyse auf mathematischer Grundlage", Master thesis (Magisterarbeit) for philosophy (in German), 2010.
  - "*Polytope mit speziellen Simplizes*", Master thesis (Diplomarbeit) for mathematics (in German), 2008.
- **In Preparation:**
  - "*On the Geometry and Topology of Modulis of Roots of Trinomials*" (working title; joint with T. Theobald).

Talks.

- "*An Introduction to Amoeba Theory*"
  - PhD-students seminar, Goethe University, Frankfurt am Main, October 28th, 2012.
- "*Tropical Approaches to Amoebas Supported on a Circuit*"
  - DMV-Jahrestagung, Univ. des Saarlandes, Saarbrücken, September 18th, 2012.
- "*Separating Inequalities for Nonnegative Polynomials that Are not Sums of Squares*"
  - CWI, Amsterdam, June 8th, 2012
- "*Roots of Trinomials from the Viewpoint of Amoeba Theory*"
  - Goethe University, Frankfurt am Main, June 27th, 2012,
  - Université Pierre et Marie Curie (Paris 6), Paris, June 6th, 2012,
  - Philipps-Universität, Marburg, May 23rd, 2012.

- "*The Configuration Space of Amoebas*" (poster presentation)
  - Workshop "Tropical Geometry", ICMS, Edinburgh, April 2nd, 2012.
- "*Contemporary Key Problems in Amoeba Theory*"
  - Université de Genève, February 1st, 2012.
- "*The Configuration Space of Amoebas with Barycentric Simplex Newton Polytope*"
  - "4th PhD Students Conference on Tropical Geometry", Technical University, Kaiserslautern, July 30th, 2011,
  - Goethe University, Frankfurt am Main, July 5th, 2011.
- "*Approximation of Amoebas and Coamoebas by Sums of Squares*" (poster presentation)
  - "MEGA 2011", Stockholm University, May 31st, 2011.
- "*(Co)Amoebas: Different Aspects of Approximation, Boundary and Homotopy*"
  - Courant Research Centre, Göttingen, November 27th, 2010.
- "*Amoebas of Genus at Most One*"
  - Goethe University, Frankfurt am Main, December 18th, 2009,
  - "1st PhD-students Conference on Tropical Geometry", TU-Berlin, December 11th, 2009.
- "*Polytopes with Special Simplices*"
  - Kolkom 2010, Max Planck Institut Informatik, Saarbrücken, November 12th, 2010,
  - DMV-Studierendenkonferenz, Ruhr Universität, Bochum, October 2nd, 2009.

Further Conferences and Workshops (selected).

- "*3rd PhD Students Conference on Tropical Geometry*", CRCG in Göttingen, January 28th - 30th, 2011.
- "*Algebraic Optimization and Semidefinite Programming*, EIDMA Minicourse at the CWI in Amsterdam, May 31th - June 4th, 2010.
- "*Symposium "Diskrete Mathematik"*", Erwin Schrödinger Institute, Vienna, May 14th - 15th, 2010.
- "*New Trends in Algorithms for Real Algebraic Geometry*, Oberwolfach, November 22nd - 28th, 2009.
- "*Introductory Workshop: Tropical Geometry*", MSRI in Berkeley, August 24th - 28th, 2009.

Teaching.

WiTe 12/13: Seminar "*Optimization*" (Institute of Mathematics, joint with Thorsten Theobald and Christian Trabandt).

SuTe 11: Seminar "*Geometry and combinatorics of spectrahedra*" (Institute of Mathematics, lecturer: Thorsten Theobald).

SuTe 10: Tutorial for the lecture "*Tropical geometry*" (Institute of Mathematics, lecturer: Thorsten Theobald).

WiTe 09/10: Seminar: "*Basic problems of induction and probability*" (Institute of Philosophy, joint with Wilhelm K. Essler).

WiTe 09/10: Tutorial for the lecture "*Symbolic computation and Gröbner bases*" (Institute of Mathematics, lecturer: Thorsten Theobald).
SuTe 09: Tutorial for the lecture "*Lattices and cryptography*" (Institute of Mathematics, lecturer: Claus-Peter Schnorr).
SuTe 08: Seminar: "*Selected topics of the Philosophy of Mathematics*" (Institute of Philosphy, joint with Wilhelm K. Essler and Cordian Riener).
SuTe 07: Seminar: "*Mathematics for philosophers*" (Institute of Philosophy).
SuTe 07: Tutorial for the lecture "*Discrete mathematics*" (Institute of Mathematics, lecturer: Thorsten Theobald).
WiTe 06/07: Tutorial for the lecture "*Logic I*" (Institute of Philosophy, lecturer: André Fuhrmann).
SuTe 06: Tutorial for the lecture "*Logic II*" (Institute of Philosophy, lecturer: Wilhelm K. Essler).
WiTe 05/06: Tutorials for the lecture "*Logic I*" (Institute of Philosophy, lecturer: Wilhelm K. Essler).
WiTe 04/05: Tutorial for the lecture "*Logic I*" (Institute of Philosophy, lecturer: Wilhelm K. Essler).

Scholarships and Honors.

- Minor price of the DMV–Studierendenkonferenz 2009 for the Diploma thesis "*Polytope mit speziellen Simplizes*" ("*Polytopes with special simplices*").
- Honors for the best graduation in mathematics at the Goethe University in winter term 08/09.
- 10/2007 – 12/2009: Scholarship of the "*Studienstiftung des deutschen Volkes*" ("*German National Academic Foundation*").

Professional Activities.

- Referee for "*Arkiv for Matematik*", "*Experimental Mathematics*" and "*Mathematical Proceedings*".
- Coorganizer of the conference "*MEGA 2013*" (June 3rd - 7th, 2013) (`http://www.math.uni-frankfurt.de/mega2013/`).
- Organizer of the workshop "*Math across the Main*" (March 7th, 2013).
- Coorganizer of the workshop "*Discrete, Tropical and Algebraic Geometry*" (May 5th – 7th, 2011) (`http://www.math.uni-frankfurt.de/geometry2011/`).
- Organizer of the "*2nd PhD students conference on tropical geometry*" (June 16th / 17th, 2010) (`http://www.math.uni-frankfurt.de/phd-tropical/`).
- Member of three professoral recruitement committees at the Goethe University in Frankfurt (two as studentical representative at the Institute of Philosophy and one as PhD students / PostDocs representative at the Institute of Mathematics).
- Administrative organizer of the "AG & Oberseminar Diskrete Mathematik" since 2009.

Students.

- Co-advisor for *Florian Landsgesell* (Bachelor thesis, 2011).
- Informal co-supervisor for *Sadik Iliman* (Master thesis, 2011), *Jan Hofmann* (Diploma thesis, 2011) and *Sandra Kiefer* (Master thesis, 2013).