On the Existence and Uniqueness of Glosten-Milgrom Price Processes

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Abstract

We study the price-setting problem of market makers under perfect competition in continuous time. Thereby we follow the classic Glosten-Milgrom model [GM85] that defines bid and ask prices as the expectation of a true value of the asset given the market makers partial information that includes the customers trading decisions. The true value is modeled as a Markov process that can be observed by the customers with some noise at Poisson times.

We analyze the price-setting problem by solving a non-standard filtering problem with an endogenous filtration that depends on the bid and ask price process quoted by the market maker. Under some conditions we show existence and uniqueness of the price processes. In a different setting we construct a counterexample to uniqueness. Further, we discuss the behavior of the spread by a convergence result and simulations.

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Chapter 1

Introduction

The aim of this thesis is to discuss the existence and uniqueness of price functions in the Glosten-Milgrom model [GM85]. In this introductory chapter we want to give an overview of the economic context of our topic. This survey of market microstructure theory does not have the aim to be complete, but wants to give an overview for readers that are not familiar with this branch of the literature. For a full synopsis of the topic we recommend the book by O'Hara [O'H07] or the article by Madhavan [Mad00]. Further, we refer to recent contributions to the mathematical literature that applies methods that are similar or related to ours on this topic.

1.1 Liquidity and its price

By liquidity we understand the ability to buy or sell some asset whenever a market participant wishes to do so. The demand for liquidity is filled by market participants that supply liquidity by offering to trade over a positive period of time in the future at prices they specify. Thus, they *set the price* of the asset.

This can happen in many different trading environments. In this thesis we will focus on specialist markets, where one or several market makers (also called specialists) provide liquidity by offering to buy or to sell the respective asset at any time. They quote both a bid price at which they commit themselves to buy and a higher ask price at which they sell. Their counterparts are market participants that use this trading opportunities and that we will call customers in the following.

These two groups have different interests in the market. While we assume that customers have an interest in the long-term value of the asset, the market maker obtains his profit by earning the spread, i.e. the difference between bid and ask price. The actual value is not important to the market maker as long as he can sell assets that he bought earlier at a price at which he makes a profit (or at least no loss).

Before we discuss why this service of the market makers actually costs something, i.e. why he can ask for a charge in form of the spread, we make a short excursion to limit order markets, which are the most common trading environments in todays electronic markets. There are no distinguished groups of traders in this market but there are two types of orders that can be submitted by all traders. In order to submit a limit order the trader has to specify whether he wants to buy or sell, further he has to determine at which price and how much he we wants to trade. The orders are then stored in the limit order book and the trader can cancel it as long it is not executed. The order might be, but does not need to be, executed against a market order (or a matching limit order) in the course of trading. This second type of buy or sell order is always executed immediately against the best limit order in the book.

In limit order markets there is typically also a spread, i.e. a difference between the best bid and the best ask price and it can, as in the specialist market, be interpreted as the price of liquidity. The limit order market is complex to analyze since it contains many interdependencies and allows complicated trading strategies. There is, however, a correspondence between the market maker as price setter and liquidity supplier and the limit order traders on the one side and customers and market order traders on the other side. In this respect the market maker we will speak about in the following can also be seen as an entirety of all limit order traders of such markets.

When the market makers commit themselves to buy or sell at the prices they publish, they face certain risks for which they are compensated by the bid-ask spread. The risk can be decomposed mainly into two components: inventory and information risk.

Inventory risk describes the risk that market makers or other liquidity providers might accumulate large positive or negative inventories in the respective asset and then prices move against them. In a continuous time framework this was studied by Ho and Stoll [HS81] and, more recently, further developed as optimal stochastic control problems by Avellaneda and Stoikov [AS08], Guilbaud and Pham [GP13], Veraart [Ver10] and Cartea and Jaimungal [CJ13] among others.

The second risk market makers take is information risk, i.e. the risk that at least part of the customers have superior (or insider) information about the hidden true value of the asset and trade strategically to their own advantage and therefore to the disadvantage of the market maker. Thus, the market maker faces an adverse selection problem.

Although the nature of the two types of risk is quite different, their effects are somehow similar. Namely, if a customer buys assets, the market maker will most likely raise both his bid and his ask price. On the one hand, because he wants to avoid further buying and stimulate the sell-side to control his inventory, and on the other hand because he believes that the purchase of the customer has conveyed some good news about the true value of the asset. It seems therefore difficult to model both risks simultaneously.

1.2 Glosten-Milgrom prices

In this thesis, we concentrate on information risk, which was first studied by Copeland and Galai [CG83] and more general and in continuous time by Glosten and Milgrom [GM85], who describe the prices as expectations of a hidden true value. This zero expected profit condition can be explained by risk neutrality and perfect competition among market makers.

We develop a continuous time model in Chapter 4 that is very similar to the one in Glosten and Milgrom [GM85]. The focus of this work is to discuss *existence and uniqueness* of the price functions in a mathematically rigorous way. Glosten and Milgrom who assume that these properties hold state in their paper that "General existence of such functions would be difficult to show, since it involves a 'rational expectations' type fixed point condition" (see [GM85, p. 79]).

Already in a static (or one-period) model that we discuss in Chapter 3 showing or disproving the existence or the uniqueness of Glosten-Milgrom prices is a non-trivial issue and there are only a few substantial contributions. Bagnoli, Viswanathan and Holden [BVH01] derive necessary and sufficient conditions for the existence of a so-called linear equilibrium in a one-period model with several strategically behaving insiders. Linearity means that, after observing the size of the arriving market order, the market maker quotes a price per share which is affine linear, but not constant, in the order size. In contrast to our model, the market maker can thus draw conclusions from the size of the order about the type of trader submitting the order. It turns out that linear equilibria only exist in special cases.

Glosten and Milgrom describe the prices as the expectation of a true value that does not change during the trading. As an extension we model the true value as a Markov process with finite state space and again bid and ask prices of the market maker are determined by the zero profit condition given his information about the time-dependent true value of the asset. The introduction of a true value process is to our knowledge new to the Glosten-

Milgrom literature.

However, this information, i.e. the filtration, depends again on the prices the market maker sets, thus he influences the learning environment by setting bid and ask prices and there appears the aforementioned fixed point problem. If, for example, the market maker sets a very large spread, there will be only a small amount of trades on which he can base his estimation of the true value. Mathematically this means that we are faced with a filter problem w.r.t. a filtration that is not exogenously given but that is part of the solution. The filtration depends on the bid and ask price processes which have for their part to be predictable w.r.t. the filtration.

This predictability w.r.t. the filtration of the market makers partial information is central to the idea of Glosten-Milgrom prices (although not formulated in those words in their paper), since it means that prices have to be set by the market maker before a customer can use the opportunity to trade. Since the market maker can differentiate (by the two different prices) between buys and sells he anticipates the information gained by a forthcoming trade in the prices and thus a spread emerges.

The predictability w.r.t. a dependent filtration is an essential difference to other filter problems in market microstructure models with a not directly observable true value of the asset where, however, also point processes are used, see e.g. the article by Zeng [Zen03].

Filtering problems with an endogenous filtration appear in many articles, see Back [Bac92], Back and Baruch [BB04], Lasserre [Las04], Aase, Bjuland, and Øksendal [ABØ12], and Biagini, Hu, Meyer-Brandis, and Øksendal [BHMBØ12], among others. But the inherent fixed point problem which is solved in a Brownian setting is fundamentally different (to the problem we solve) as accumulated purchases and sells are continuous processes and new information arises continuously.

We show that Glosten-Milgrom bid and ask price processes are fixed points of certain functionals acting on the set of stochastic processes and they are given by some deterministic functions of the conditional probabilities of the true value process (under the resulting partial information of the market maker). The conditional probabilities can be obtained as the solution of a system of SDEs.

To our knowledge there have been no previous results on the issue of existence and uniqueness in a continuous time model. Back and Baruch [BB04] derive (in)equalities under which they prove the existence of an equilibrium in the continuous time Gloston-Milgrom model with a strategically behaving insider and two possible states of the true asset value. Then, it is shown numerically that the (in)equalities obtain a solution and an equilibrium is constructed.

1.3 Insiders and the Kyle model

The Glosten-Milgrom prices are the expectation of a hidden true value given the market maker's information of past trades. This only makes sense if the history is somehow connected with the true value, i.e. if customers behave in such way that they transfer information to the market maker. However, there also have to be some customers who behave irrationally, since otherwise the profit of the market maker is strictly negative as was shown by Milgrom and Stokey [MS82].

This decision making of the customers in our model is very similar to Das [Das05, Das08], who also provides methods to simulate the Glosten-Milgrom price process in a discrete time model and examines some statistical properties of the prices in the market model numerically. We explain the mechanism and the corresponding noise parameter ϵ in detail in Chapter 3.

In this thesis we treat two cases, both in the static case and in continuous time. Firstly, if we assume that the information is assigned smoothly to all customers, which means that the noise parameter ϵ in our model has a density and we further assume that this density fulfills certain conditions, we can show in Chapter 5 that existence and uniqueness of Glosten-Milgrom prices hold for this case as we showed in Kühn and Riedel [KR13].

Secondly, another idea concerning decision making is the concept of insiders and noise traders, where the former have superior information of the true value and the latter trade for exogenous reasons. We treat this case in Chapter 6 and show that under certain conditions there are at least two possible Glosten-Milgrom prices by constructing a counterexample to uniqueness.

This might be surprising and to some extent unpleasant since the distinction of insiders and noise traders is a common idea in market microstructure models. Actually, it is central to the second important information risk model which was developed in the same year as the Glosten-Milgrom model by Kyle [Kyl85]. In the one-period case noise traders (randomly) choose a quantity they want to trade, then the insiders can decide how much they want to trade while maximizing their profit from superior information. The risk neutral market maker observes only the accumulated quantity and then sets prices under a zero profit condition. Kyle shows that there is an equilibrium and also introduces a model with continuous trading which was further developed by Back [Bac92].

There are several differences between the Kyle model and the Glosten-Milgrom model. Quantity plays an essential role in the Kyle model and prices are determined after these quantities have been observed by the market maker, while in Glosten-Milgrom models the prices are for a fixed number of assets (most of the time one) and the market maker can revise his prices only after a trade. Also, Kyle models a single price process and can therefore, in contrast to Glosten-Milgrom, not explain the bid-ask spread.

Interestingly, a connection between the Kyle model with continuous trading and the Glosten-Milgrom model was established by Krishnan [Kri92] and more general by Back and Baruch [BB04]. Çetin and Xing [ÇX13] generalize these results and consider the Glosten-Milgrom-type model of Back and Baruch in depth. For a strategically behaving insider who is able to react on the noise trader's action without being seen by the market maker they show the existence of the Glosten-Milgrom equilibrium by constructing a bridge proces.

1.4 Overview

In Chapter 2 we give a short introduction to the methods of the filtering theory applied in this thesis.

In Chapter 3 we introduce the static model and discuss decision making of the customers. We show existence and uniqueness for static Glosten-Milgrom prices in the two cases we consider, namely, 'noise with density' and the insider/noise trader model.

In the subsequent chapter we present a continuous time model of market making which is the one we will work with in the subsequent part of the thesis. We define admissible prices and characterize Glosten-Milgrom pricing strategies in a rigorous way. Further we define the process of conditional probabilities and characterize any solution of the pricing problem as a fixed point of a functional on the stochastic processes.

Chapter 5 proves the existence and uniqueness in the case of 'noise with density' and Chapter 6 gives a counterexample to uniqueness in the insider/noise trader case.

In Chapter 7 we show that the prices converge to a true value, if the true value does not change via a martingale convergence argument. In the final chapter we show in some simulations how the spread depends on model parameters and also via simulations we make a conjecture for the behavior in a model related to ours.

Chapter 2

Filtering

During the proofs of this thesis we will rely on results in stochastic filtering theory. In this chapter we will give a short overview about the innovations approach of stochastic filtering. The chapter can be skipped by readers who are familiar with it.

2.1 The filter equation

When we proof our main results we will heavily depend on filtering techniques and the resulting filtering equations. In the following we want to give a brief summary over this techniques in the context of our model. Thereby we follow [Bré81], although our notation is different. Before we formulate the main result of this chapter precisely in Theorem 2.4 we summarize the idea of filtering.

In filtering problems there is an underlying process Z on a probability space $(\Omega, \mathcal{F}, P)$ that is adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.

However, we do not observe $\mathbb{F}$ but a smaller filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ with $\mathcal{G}_t \subset \mathcal{F}_t$ for all $t \geq 0$. This filtration is generated by a process N, i.e.

$$\mathcal{G}_t = \sigma(N_s, s \leq t).$$

We call N the observation process. As the notation suggests we are here considering observation processes that are point process. There are also models of the Brownian motion-type. For a discussion of these cases we refer to [BC08]. We develop the theory here for one-dimensional N. The extension to the multidimensional case (that we use in the following chapters) complicates notation significantly but can be obtained easily.

We are interested in the estimate of Z_t given the information conveyed by N up to time t, hence we want to calculate

$$\hat{Z}_t = E[Z_t|\mathcal{G}_t].$$

The target is to derive a filter equation that describes how $\hat{Z}$ changes over time and when N changes. Hence the filter equation has the form

$$d\hat{Z}_t = f_t dt + h_t dN_t,$$

where f and h are appropriate functions.

Obviously, this only makes sense if N carries some information about Z. This information is assumed to be disturbed by some measurement noise that also has a point process structure and which introduces more randomness into the model. In other words N is a (quite complex) function of Z and the measurement noise. The aim of filtering is to filter out this noise.

We will now repeat some maybe well-known definitions and then formulate the filtering result precisely in the subsequent theorem.

Definition 2.1. *Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration on $(\Omega, \mathcal{F}, P)$. We say that a real-valued process Y is an $\mathbb{F}$-martingale if Y is adapted to $\mathbb{F}$, if it is integrable and if for all $s, t \geq 0$ with $s < t$*

$$E[Y_t|\mathcal{F}_s] = Y_s \qquad P-a.s.$$

Definition 2.2. *We say that the process Y is $\mathbb{F}$-progressive (or $\mathbb{F}$-progressively measurable) if for all $t \geq 0$ the mapping on $[0,t] \times \Omega$ defined by $(s,\omega) \mapsto Y_s(\omega)$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable.*

Definition 2.3. *Let N be a point process adapted to some filtration $\mathbb{F}$ and let λ be a nonnegative $\mathbb{F}$-progressive process such that for all $t \geq 0$*

$$\int_0^t \lambda_s ds < \infty \qquad P-a.s.$$

If for all nonnegative $\mathbb{F}$-predictable processes C it holds true that

$$E\left[\int_0^\infty C_s dN_s\right] = E\left[\int_0^\infty C_s \lambda_s ds\right]$$

we say that N has $\mathbb{F}$-intensity λ.

Theorem 2.4. *Let there be given a complete probability space $(\Omega, \mathcal{F}, P)$ and on it two filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ such that*

$$\mathcal{G}_t \subset \mathcal{F}_t, \qquad \text{for all } t \geq 0, \qquad \text{where} \qquad \mathcal{G}_t = \sigma(N_s, s \leq t).$$

Thereby, N is a nonexplosive point process with $\mathbb{F}$-intensity λ and $\mathbb{G}$-intensity $\hat{\lambda}$. Let Z be real-valued, adapted to $\mathbb{F}$, bounded and of the form

$$Z_t = Z_0 + \int_0^t f_s ds + m_t,$$

where f_s is a $\mathbb{F}$-progressive process with

$$E\left[\int_0^t |f_s| ds\right] < \infty$$

and m is a $\mathbb{F}$-martingale with mean 0 and with paths of bounded variation on finite intervals. Then, there exists a càdlàg process $\hat{Z}$ with

$$\hat{Z}_t = E[Z_t | \mathcal{G}_t] \qquad P\text{-a.s.}$$

for all t and

$$\hat{Z}_t = E[Z_0] + \int_0^t \hat{f}_s ds + \int_0^t K_s (dN_s - \hat{\lambda}_s ds), \tag{2.1}$$

where $\hat{f}_s$ is a $\mathbb{G}$-progressive process satisfying

$$E\left[\int_0^t C_s f_s ds\right] = E\left[\int_0^t C_s \hat{f}_s ds\right] \tag{2.2}$$

for all nonnegative bounded $\mathbb{G}$-progressive processes C and K is a $\mathbb{G}$-predictable process defined by

$$K_t = \Psi_{1,t} - \Psi_{2,t} + \Psi_{3,t},$$

where $\Psi_{1,t}, \Psi_{2,t}, \Psi_{3,t}$ are $\mathbb{G}$-predictable and satisfy

$$\begin{aligned} E\left[\int_0^t C_s Z_s \lambda_s ds\right] &= E\left[\int_0^t C_s \Psi_{1,s} \hat{\lambda}_s ds\right] \\ E\left[\int_0^t C_s Z_s \hat{\lambda}_s ds\right] &= E\left[\int_0^t C_s \Psi_{2,s} \hat{\lambda}_s ds\right] \\ E\left[\sum_{0<s\le t} C_s \Delta m_s \Delta N_s\right] &= E\left[\int_0^t C_s \Psi_{3,s} \hat{\lambda}_s ds\right] \end{aligned} \tag{2.3}$$

for all nonnegative bounded $\mathbb{G}$-predictable processes C and all $t \ge 0$. It holds that $\Psi_{2,t} = \hat{Z}_{t-}$ and $\Psi_{3,t} = 0$ if the observation N and the process Z never jump at the same time a.s..

(2.1) is called the filter equation. We prove this Theorem in Lemmas 2.5, 2.8 and 2.9.

2.2 Proof of the filter equation

In a first step we consider the projection of Z on the smaller filtration $\mathbb{G}$ up to a fixed time $n \in \mathbb{N}$.

Lemma 2.5. *(comp. [Bré81] IV, T1) Let the filtrations $\mathbb{F}$ and $\mathbb{G}$ be defined as in Theorem 2.4 and let Z be an integrable real-valued process with representation*

$$Z_t = Z_0 + \int_0^t f_s ds + m_t, \tag{2.4}$$

where f_s is a $\mathbb{F}$-progressive process with

$$E\left[\int_0^t |f_s| ds\right] < \infty$$

and m is a $\mathbb{F}$-martingale with mean 0. Let $n \in \mathbb{N}$ be fixed. Then there exists a càdlàg process $\hat{Z} = (\hat{Z}_t)_{t\in[0,n]}$ with $\hat{Z}_t = E[Z_t|\mathcal{G}_t]$ and

$$\hat{Z}_t = E[Z_0] + \int_0^t \hat{f}_s ds + \hat{m}^n_t, \tag{2.5}$$

where $\hat{f}_s$ is a $\mathbb{G}$-progressive process satisfying

$$E\left[\int_0^t C_s f_s ds\right] = E\left[\int_0^t C_s \hat{f}_s ds\right] \tag{2.6}$$

for all nonnegative bounded $\mathbb{G}$-progressive processes C and $\hat{m}^n$ is a càdlàg $\mathbb{G}$-martingale, also with mean 0.

Proof. We define

$$M^n := E[Z_0|\mathcal{G}_n] - E[Z_0] + E\left[\int_0^n f_s ds\middle| \mathcal{G}_n\right] - \int_0^n \hat{f}_s ds + E[m_n|\mathcal{G}_n] \quad (2.7)$$

By [JS87], Theorem 1.42 (b) there exists a uniformly integrable, càdlàg martingale $\hat{m}^n$ such that

$$\hat{m}^n_t = E[M^n|\mathcal{G}_t] \quad (2.8)$$

for $t \in [0, n]$. Note that for this result the completion of the filtration (which is not given here) is not necessary and that the authors include the property to be càdlàg in the definition of martingales (see Definition 1.36 in [JS87]).

We now want to show that $\hat{m}^n_t$ has the form

$$\hat{m}^n_t = E[Z_0|\mathcal{G}_t] - E[Z_0] + E\left[\int_0^t f_s ds\middle| \mathcal{G}_t\right] - \int_0^t \hat{f}_s ds + E[m_t|\mathcal{G}_t], \quad (2.9)$$

since this is equivalent to

$$E\left[Z_0 + \int_0^t f_s ds + m_t\middle| \mathcal{G}_t\right] = E[Z_0] + \int_0^t \hat{f}_s ds + \hat{m}^n_t,$$

which proves (2.5) and hence the lemma. Note that here the version of $t \mapsto E[Z_t|\mathcal{G}_t]$ depends technically on n, thus we denote them by $\hat{Z}^n$. But we can define a right-continuous version of $E[Z_t|\mathcal{G}_t]$ on $\mathbb{R}_+$ by piecing this Versions together via

$$\hat{Z}_t = \sum_{n\in\mathbb{N}} \hat{Z}^n_t 1_{\{n-1\le t<n\}}.$$

Also it is clear that $\hat{m}^n_0 = 0$ and hence we have a mean 0 martingale. To see (2.9) we consider the first, second and last term of M^n that is defined in (2.7) and it is clear that

$$E\left[E[Z_0|\mathcal{G}_n] - E[Z_0] + E[m_n|\mathcal{G}_n]\middle| \mathcal{G}_t\right] = E[Z_0|\mathcal{G}_t] - E[Z_0] + E[m_t|\mathcal{G}_t]$$

Hence, it remains to show that for $t < n$ it holds that

$$E\left[E\left[\int_0^n f_s ds \middle| \mathcal{G}_n\right] - \int_0^n \hat{f}_s ds \middle| \mathcal{G}_t\right] = E\left[\int_0^t f_s ds \middle| \mathcal{G}_t\right] - \int_0^t \hat{f}_s ds,$$

which is equivalent to

$$E\left[\int_0^n f_s ds - \int_0^t f_s ds \middle| \mathcal{G}_t\right] = E\left[\int_0^n \hat{f}_s ds - \int_0^t \hat{f}_s ds \middle| \mathcal{G}_t\right],$$

since $\hat{f}$ is $\mathbb{G}$-progressive. This again holds true iff for all $A \in \mathcal{G}_t$

$$E\left[1_A \int_t^n f_s ds\right] = E\left[1_A \int_t^n \hat{f}_s ds\right],$$

or iff for $C_s(\omega) = 1_A(\omega) 1_{(t,n]}(s)$

$$E\left[\int_0^n C_s f_s ds\right] = E\left[\int_0^n C_s \hat{f}_s ds\right].$$

Since C is a nonnegative bounded $\mathbb{G}$-progressive processes this holds by (2.6) and the proof is completed. □

The existence of $\hat{f}$ can be secured since it can be defined as a Radon-Nikodym derivative. This definition is independent of n in the last lemma. Therefore, consider the progressive σ-Algebra on $\mathbb{R}_+ \times \Omega$ that we denote by $\operatorname{Prog} \mathbb{G}$ and which contains all sets $P \subset \mathbb{R}_+ \times \Omega$ such that the indicator function $1_{\{(t,\omega) \in P\}}$ is progressively measurable in the sense of Definition 2.2. Further we have the two measures

$$\mu_2(dt \times d\omega) = f_t(\omega) dt P(d\omega)$$

and

$$\mu_1(dt \times d\omega) = dt P(d\omega).$$

We then can define $\hat{f}$ as the derivative of μ_2 with respect to μ_1, where we restrict both measures to the progressive σ-Algebra, hence

$$\hat{f} := \frac{d\mu_2|_{\operatorname{Prog} \mathbb{G}}}{d\mu_1|_{\operatorname{Prog} \mathbb{G}}}.$$

For C $\mathbb{G}$-progressive we obtain

$$E\left[\int_0^t C_s\hat{f}_s ds\right] = \int_{[0,t]\times\Omega} C\hat{f}d\mu_1 = \int_{[0,t]\times\Omega} Cd\mu_2 = E\left[\int_0^t C_s f_s ds\right].$$

If there is a version of $s \mapsto E[f_s|\mathcal{G}_s]$ that is $\mathbb{G}$-progressively measurable, then this version can be choosen as $\hat{f}$. For details see [Bré81], remarks $(\alpha) - (\gamma)$.

(2.5) already describes the structure of the filter equation (2.1) in the final form. The remaining task lies in the characterization of $\hat{m}^n$ and the generalization to $\mathbb{R}_+$.

To do this we firstly remark that in Theorem 2.4 we assume that N has an intensity. The following lemma states that this intensity can be assumed to be predictable.

Lemma 2.6. *(comp. [Bré81] II, T13) Let N be a point process with $\mathbb{F}$-intensity λ. Then one can find an $\mathbb{F}$-intensity $\widetilde{\lambda}$ that is predictable.*

Proof. Define $\widetilde{\lambda}$ as the Radon-Nikodym derivative of the restriction on the predictable sigma-field on $\mathbb{F}$ of $P(d\omega)\lambda_t(\omega)dt$ with respect to the restriction on the predictable sigma-field on $\mathbb{F}$ of $P(d\omega)dt$. It follows that $\widetilde{\lambda}$ is an intensity and $\mathbb{F}$-predictable. □

Given this (predictable) intensity we can define

$$\hat{M}_t := N_t - \int_0^t \hat{\lambda}_s ds. \tag{2.10}$$

It can be seen easily that $\hat{M}$ is a martingale with mean 0 that also generates $\mathbb{G}$. We will represent $\hat{m}^n$ by this martingale with the use of a martingale representation theorem. We state a common form (without the complex and technical proof) in the following.

Theorem 2.7. *(comp. [Bré81] III, T9) Let N be a nonexplosive point process and the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ given by $\mathcal{G}_t = \sigma(N_s, s \leq t)$. Suppose that N has the $\mathbb{G}$-predictable intensity $\hat{\lambda}$. Now let M be a right-continuous $\mathbb{G}$-martingale of the form $M_t = E[M_\infty|\mathcal{G}_t]$, where M_∞ is some integrable random variable. Then for each $t \geq 0$*

$$M_t = M_0 + \int_0^t H_s(dN_s - \hat{\lambda}_s ds) \qquad P-a.s., \tag{2.11}$$

where H_s is a $\mathbb{G}$-predictable process such that for all $t \geq 0$

$$\int_0^t |H_s|\hat{\lambda}_s ds < \infty \qquad P - a.s.$$

We will apply this theorem in the following to the process $\hat{m}^n$ from (2.5). Two important requirements are fulfilled for this setting. Firstly, there exists a terminal random variable, which is given by M^n in (2.7). Secondly, the martingale $\hat{m}^n$ is right-continuous by its definition in (2.8).

Both points seem to be difficult to fulfill if one tries to treat the problem on $\mathbb{R}_+$ since the existence of a terminal random variable is hard to establish. As a consequence also the right-continuity turns out to be problematic since $\mathcal{G}_0$ does not contain all P-null sets of $\mathbb{G}$. Hence, the usual assumptions that are subsumed under the "usual conditions" are not fulfilled.

However, in the following Lemma we are able to proof a martingale representation on $\mathbb{R}_+$.

Lemma 2.8. *There exists $\mathbb{G}$-predictable process K such that for all $t \geq 0$*

$$\int_0^t |K_s|\hat{\lambda}_s ds < \infty \qquad P - a.s. \tag{2.12}$$

and

$$\hat{Z}_t = E[Z_0] + \int_0^t \hat{f}_s ds + \int_0^t K_s(dN_s - \hat{\lambda}_s ds) \qquad P - a.s.$$

for all $t \geq 0$.

Proof. Again, we first consider the situation on $I := [0, n]$ for $n \in \mathbb{N}$. By Lemma 2.5 we have that

$$\hat{Z}_t = E[Z_0] + \int_0^t \hat{f}_s ds + \hat{m}_t^n,$$

for all $t \in I$, where $\hat{m}^n$ is a $\mathbb{G}$-martingale. Now, by Theorem 2.7 there exists a $\mathbb{G}$-predictable process H^n on I with

$$\int_0^t |H_s^n|\hat{\lambda}_s ds < \infty \qquad P - \text{a.s.}$$

and

$$\hat{m}_t^n = \hat{m}_0^n + \int_0^t H_s^n(dN_s - \hat{\lambda}_s ds) \qquad P - \text{a.s.},$$

for all $t \in I$. Since $\hat{m}^n$ has mean zero we have that

$$\hat{m}_t^n = \int_0^t H_s^n (dN_s - \hat{\lambda}_s ds) \qquad P-\text{a.s.},$$

again for all $t \in I$. Now define K on $\mathbb{R}_+$ by

$$K_t := \sum_{n \in \mathbb{N}} H_t^n 1_{\{n-1<t\leq n\}}.$$

Then K is $\mathbb{G}$-predictable and it is clear that (2.12) holds.

For $t \geq 0$, let $l := \sup\{n \in \mathbb{N} | n < t\}$. Then

$$\hat{Z}_t = E[Z_0] + \int_0^t \hat{f}_s ds + \hat{m}_t^{l+1},$$

As $\hat{m}_0^0 = 0$ and for $k, n \in \mathbb{N}$ with $k < n$ it holds by (2.9) that $\hat{m}_k^n = \hat{m}_k^k$ P-a.s. we have

$$\begin{aligned}
\hat{m}_t^{l+1} &= \hat{m}_t^{l+1} + \left(\sum_{n=0}^{l} -\hat{m}_n^{n+1} + \hat{m}_n^n \right) \\
&= \hat{m}_t^{l+1} - \hat{m}_l^{l+1} + \left(\sum_{n=1}^{l} \hat{m}_n^n - \hat{m}_{n-1}^n \right) \\
&= \int_0^t H_s^{l+1}(dN_s - \hat{\lambda}_s ds) - \int_0^l H_s^{l+1}(dN_s - \hat{\lambda}_s ds) \\
&\quad + \sum_{n=1}^{l} \int_0^n H_s^n (dN_s - \hat{\lambda}_s ds) - \int_0^{n-1} H_s^n (dN_s - \hat{\lambda}_s ds) \\
&= \int_l^t H_s^{l+1}(dN_s - \hat{\lambda}_s ds) + \sum_{n=1}^{l} \int_{n-1}^n H_s^n (dN_s - \hat{\lambda}_s ds) \\
&= \int_0^t K_s (dN_s - \hat{\lambda}_s ds)
\end{aligned}$$

which proofs the lemma. □

The remaining task is to show that the innovations gain K satisfies (2.3). We do this in the following Lemma and thus conclude the proof of Theorem 2.1.

Lemma 2.9. *The innovations gain K that exists by Theorem 2.8, i.e. the $\mathbb{G}$-predictable process such that $\hat{Z}$ satisfies*

$$\hat{Z}_t = E[Z_0] + \int_0^t \hat{f}_s ds + \int_0^t K_s(dN_s - \hat{\lambda}_s ds), \tag{2.13}$$

is given by

$$K_t = \Psi_{1,t} - \Psi_{2,t} + \Psi_{3,t}, \tag{2.14}$$

where $\Psi_{1,t}, \Psi_{2,t}, \Psi_{3,t}$ satisfy

$$\begin{aligned} E\left[\int_0^t C_s Z_s \lambda_s ds\right] &= E\left[\int_0^t C_s \Psi_{1,s} \hat{\lambda}_s ds\right] \\ E\left[\int_0^t C_s Z_s \hat{\lambda}_s ds\right] &= E\left[\int_0^t C_s \Psi_{2,s} \hat{\lambda}_s ds\right] \\ E\left[\sum_{0<s\leq t} C_s \Delta m_s \Delta N_s\right] &= E\left[\int_0^t C_s \Psi_{3,s} \hat{\lambda}_s ds\right] \end{aligned} \tag{2.15}$$

for all nonnegative bounded $\mathbb{G}$-predictable processes C and all $t \geq 0$.

The functions $\Psi_i, i = 1, 2, 3$ exist since they can be defined as Radon-Nikodym derivatives. With this definition it can be seen that they are also unique in the class of predictable processes.

Proof. Since the existence of K and the Ψ_i is given, it only remains to show that (2.14) holds true. We first consider a process U of the form

$$U_t := \int_0^t H_s d\hat{M}_s$$

where H is a $\mathbb{G}$-predictable process such that U is a bounded $\mathbb{G}$-martingale that satisfies $E[\int_0^t |H_s K_s| \hat{\lambda}_s ds] < \infty$ for all $t \geq 0$. We will later consider a concrete choice for H. It holds that

$$E[Z_t U_t] = E\left[\hat{Z}_t U_t\right], \tag{2.16}$$

since

$$E[Z_t U_t] = E[E[Z_t U_t|\mathcal{G}_t]] = E[E[Z_t|\mathcal{G}_t] U_t] = E\left[\hat{Z}_t U_t\right].$$

We now derive both sides of (2.16). By integration by parts for functions with bounded variation on finite intervals we obtain

$$
\begin{aligned}
Z_tU_t &= \int_0^t Z_{s-}dU_s + \int_0^t U_s dZ_s \\
&= \int_0^t Z_{s-}H_s d\hat{M}_s + \int_0^t U_s(dm_s + f_s ds) \\
&= \int_0^t Z_{s-}H_s(dN_s - \hat{\lambda}_s ds) + \int_0^t U_s f_s ds + \int_0^t U_s dm_s \\
&= \int_0^t Z_{s-}H_s(dN_s - \lambda_s ds) + \int_0^t Z_{s-}H_s(\lambda_s - \hat{\lambda}_s)ds \\
&\quad + \int_0^t U_s f_s ds + \int_0^t U_{s-} dm_s + \sum_{s\le t} H_s \Delta N_s \Delta m_s
\end{aligned}
$$

Since the first and the fourth term are martingales we obtain

$$
E[Z_tU_t] = E\left[\int_0^t U_s f_s ds\right] + E\left[\int_0^t Z_{s-}H_s(\lambda_s - \hat{\lambda}_s)ds + \sum_{s\le t} H_s\Delta N_s \Delta m_s\right],
$$

and by the definition of the Ψ_i in (2.15) we have

$$
E[Z_tU_t] = E\left[\int_0^t U_s f_s ds\right] + E\left[\int_0^t H_s(\Psi_{1,s} - \Psi_{2,s} + \Psi_{3,s})\hat{\lambda}_s ds\right]. \quad (2.17)
$$

In the same way we consider $\hat{Z}U$ and obtain

$$
\begin{aligned}
\hat{Z}_tU_t &= \int_0^t \hat{Z}_{s-}dU_s + \int_0^t U_s d\hat{Z}_s \\
&= \int_0^t \hat{Z}_{s-}H_s d\hat{M}_s + \int_0^t U_s(K_s d\hat{M}_s + \hat{f}_s ds) \\
&= \int_0^t \hat{Z}_{s-}H_s d\hat{M}_s + \int_0^t U_s \hat{f}_s ds + \int_0^t U_s K_s d\hat{M}_s \\
&= \int_0^t \hat{Z}_{s-}H_s d\hat{M}_s + \int_0^t U_s \hat{f}_s ds + \int_0^t U_{s-} K_s d\hat{M}_s + \sum_{s\le t} H_s K_s \Delta N_s
\end{aligned}
$$

Again the first and third term are martingales and we have

$$E\left[\hat{Z}_t U_t\right] = E\left[\int_0^t U_s \hat{f}_s ds + \sum_{s\le t} H_s K_s \Delta N_s\right],$$

and since N has intensity $\hat{\lambda}$ (comp. Definition 2.3) we get

$$E\left[\hat{Z}_t U_t\right] = E\left[\int_0^t U_s \hat{f}_s ds\right] + E\left[\int_0^t H_s K_s \hat{\lambda}_s ds\right].$$

By (2.16), (2.17) and (2.2) we obtain

$$E\left[\int_0^t H_s K_s \hat{\lambda}_s ds\right] = E\left[\int_0^t H_s (\Psi_{1,s} - \Psi_{2,s} + \Psi_{3,s}) \hat{\lambda}_s ds\right] \tag{2.18}$$

for all $t \ge 0$ and all H as described at the brginning of the proof. We now consider H of the form

$$H_t = C_t 1_{\{t \le S_n\}},$$

where C is any nonnegative bounded $\mathbb{G}$-predictable process and S_n is a $\mathbb{G}$-stopping time defined by

$$S_n := \inf\{t \ge 0 | N_t \ge n \text{ or } \int_0^t (1 + |K_s|)\hat{\lambda}_s ds \ge n\}$$

Since this H fullfils the requirements and since (2.18) holds true for all nonnegative bounded $\mathbb{G}$-predictable process C we have that

$$K_t(\omega) 1_{\{t\le S_n(\omega)\}} = (\Psi_{1,t}(\omega) - \Psi_{2,t}(\omega) + \Psi_{3,t}(\omega)) 1_{\{t \le S_n(\omega)\}} \qquad \nu(dt \times d\omega)\text{−a.e.}$$

for all n, where $\nu(dt \times d\omega) = \hat{\lambda}_t(\omega) dt P(d\omega)$ or $dN_t(\omega) P(d\omega)$. For $n \to \infty$ and thus $S_n \to \infty$ P-a.s. we obtain the result of the Lemma. □

Chapter 3

Glosten-Milgrom prices in a static model

The aim of this thesis is to consider Glosten-Milgrom prices in continuous time, as they were introduced in the original paper [GM85]. It is, however, very useful to examine another situation first which we call the static model.

In this model a market maker or specialist is obligated to publish a bid and an ask price, i.e. two real numbers, at which he will buy or sell the asset respectively. His counterpart is a potential customer who can decide whether he accepts one of the offers to trade given his information about a true value of the asset. The customer may decide not to trade at all, but will never accept both offers, since the ask price will be always higher than or equal to the bid price (otherwise the market maker will suffer a secure loss). Note that, even though the order of actions (price-setting of market maker, reaction of customer) is fixed, there is no time dimension in the model, thus, we call it static.

We describe the decision making of both market participants in the first section of this chapter and also discuss the economic motivation and implications in detail, since they are also valid for the continuous time model. The decision making of the customers depends on a random variable ϵ. In a second section we will show that for some choices of ϵ the price setting rationale of the market maker does not have a solution or its result is not unique. In the then following two sections we consider two explicit choices for ϵ, first a model where ϵ has a density that fulfills certain conditions and then the so called insider/noise trader model. In both models we show existence and uniqueness of the market makers prices.

3.1 The static model

Let X be a real-valued random variable which represents the true value of the asset. We assume that X is unknown to all market participants, but the customer has a disturbed valuation given by $X + \epsilon$, where the random variable ϵ is $\mathbb{R} \cup \{\pm\infty\}$-valued and independent of X. The market maker only knows the distribution of X and ϵ.

In the continuous time model that we present in the subsequent chapters we assume that the corresponding process $(X_t)_{t\geq 0}$ has finite state space $\{x_1, \ldots, x_n\} \subset \mathbb{R}$, $n \geq 2$ where $x_{\min} = x_1 < \ldots < x_n = x_{\max}$. Hence this case is of special interest here. In this chapter, unless otherwise stated, we assume X to be a real-valued random variable.

For the rest of this chapter we will only consider ask prices, since a theory for bid prices can be developed completely analogous and the both sides of the market do not interfere with each other. Sometimes we will, however, introduce some notation for the bid side of the market that will be used later.

We assume that a potential customer buys if his valuation is higher than the ask price s, i.e. if $X + \epsilon \geq s$. Thus, the profit of the market maker is given by

$$(s - X)1_{\{X+\epsilon\geq s\}}.$$

The central idea of the Glosten-Milgrom model is, that motivated by risk-neutrality and perfect competition, the price-setting must satisfy a zero-expected-profit condition.

Definition 3.1. *We say that s is a static Glosten-Milgrom ask price if*

$$E[(s - X)1_{\{X+\epsilon\geq s\}}] = 0. \tag{3.1}$$

Before we proceed and think about existence and uniqueness of such prices we want to make some comments on the economic motivation and implications of this model.

Firstly, we remark that risk-neutrality means that the market maker values the possible profit $(s - X)1_{\{X+\epsilon\geq s\}}$ with this number and not with

$$u\left((s - X)1_{\{X+\epsilon\geq s\}}\right),$$

where u is some concave risk-measure.

Further, the zero-expected-profit condition (3.1) is motivated by perfect competition, since market makers are assumed to undercut each others prices as long as they make a profit in expectation. The rationale of this assumption is explained in depth in [O'H07, p. 60]. The concept of perfect competition thus requires a multitude of market makers that compete with each other. Throughout this work we will however only speak of "the" market maker. We can justify this by the following considerations. If there are perfectly competitive market makers, they will all quote the same competitive zero-expected-profit prices. If the customer chooses one of the market makers independently from his valuation of the true value this does not influence the price-setting of the market maker(s) at all. Since we do not allow the market makers to interact in our model we can also just consider one market maker that fills all trades.

Another point we need to comment is the decision making of the customers. We remark that we follow Das [Das05] in defining a buy to occur if $X+\epsilon \geq s$. In the original Glosten-Milgrom paper [GM85] a buy occurs if $\rho E[X|\mathcal{A}] \geq s$, where ρ is an independent random variable which plays the role of ϵ in our model, even though in a multiplicative instead in an additive way. The sigma-algebra $\mathcal{A}$ represents the partial information of the insider. For $\mathcal{A} = \sigma(X)$, the models, including possible interpretation of ϵ and ρ, are quite similar. This interpretation can be given in two ways. ϵ or ρ can be seen as an error in the valuation process of the customers. A large ϵ or ρ then represents an overvaluation because of a mistake. A high variance (if existing) then means that the customers spend not much effort or are not able to estimate the value of the asset correctly. Moreover, a large ϵ or ρ can be seen as the result of the impatience of the customer. Although he might know the value of the asset correctly he overvalues the asset knowingly because he is impatient to buy it at that very moment. In other words, he has a high demand in the buy side liquidity provided by the market maker, for example because he needs to buy a certain amount of assets in a fixed period and he does not know when his next opportunity to trade will come. A high variance then means that there is more impatience in the market.

The definition of the behavior of the customer in a situation where his valuation exactly matches the price of the market maker is a question of taste. We assume that the customer still buys. The whole model would also make sense if we define a buy to occur if $X + \epsilon > s$.

We also note that the behavior of the customer is not rational. A rational exploitation of the given information would be to buy if

$$E[X|X + \epsilon] \geq s.$$

A high realization of $X + \epsilon$ might simply mean that ϵ is large, which the costumer may be well aware of if he knows the distributions of X and ϵ

separately. It was shown by Milgrom and Stokey [MS82] that there has to be some irrational behavior for a price to exist, since otherwise we end up with a no-trade equilibrium.

Finally, note that the volume of each trade is set to one. Hence, we ignore any volume effect.

3.2 Existence, uniqueness and counterexamples

We now turn to the question whether solutions to (3.1) exist and if so, whether they are unique. Generally this question is more difficult to answer than it might look at first sight. It seems that a general statement is hard to formulate, but the numerical simulations that we carried out seemed to support the quite vague statement that a Glosten-Milgrom price exists and is unique if the tails of ϵ are "heavy enough" in comparison to those of X. In other words, there need to be enough badly informed or impatient customers who are prepared to offset the possible losses that the market maker might make due to his imprecise knowledge of the true value X.

We will not further discuss this matter in a general way. In the following we will only consider two choices of ϵ: The model where ϵ has a density and the insider/noise trader model in the subsequent sections. But first we give two simple examples that show cases of nonexistence and non-uniqueness. In the one for nonexistence we assume that the customers are perfectly informed.

Example 3.2. *Let $\epsilon = 0$ and X not essentially bounded from above. Then there exist no $s \in \mathbb{R}_+$ with*

$$E[(s-X)1_{\{X \geq s\}}] = 0,$$

since the integrand is always non-positive and negative with positive probability. For $\epsilon = 0$ and X essentially bounded by $x_{\max}$ only $s = x_{\max}$ is a (trivial) solution.

But also if Glosten-Milgrom prices exist the question of uniqueness is not trivial as the following simple example shows.

Example 3.3. *Let*

$$\epsilon = \begin{cases} 1 & \textit{with probability} \quad \frac{1}{2} \\ -1 & \qquad\qquad\qquad\quad \frac{1}{2} \end{cases}$$

and

$$X = \begin{cases} 1 & \textit{with probability} \quad \frac{3}{4} \\ 3 & \qquad\qquad\qquad\quad \frac{1}{4} \end{cases}$$

Then the static Glosten-Milgrom ask price is not unique.

To see this we check (3.1) for $s = 3$, which is a trivial solution, and $s = \frac{9}{5}$ and obtain

$$E\left[\left(\frac{9}{5} - X\right) 1_{\{X+\epsilon \geq \frac{9}{5}\}}\right] = \left(\frac{9}{5} - 1\right)\frac{3}{4}\frac{1}{2} + \left(\frac{9}{5} - 3\right)\frac{1}{4} = \frac{4}{5}\frac{3}{8} - \frac{6}{5}\frac{1}{4} = 0$$

Before we will discuss the question of existence and uniqueness in the model where ϵ has a density and the insider/noise trader model we want to make some general definitions that help to characterize this problem and will be also central in the subsequent chapters.

Definition 3.4. *We indicate the distribution of X by π. For s such that $P[X + \epsilon \geq s] > 0$ we define*

$$g(s, \pi) := E[X|X + \epsilon \geq s] := \frac{E\left[X 1_{\{X+\epsilon \geq s\}}\right]}{P[X + \epsilon \geq s]},$$

where E is according to π. In the same way we define for later use

$$h(s, \pi) := E[X|X + \epsilon \leq s] := \frac{E\left[X 1_{\{X+\epsilon \leq s\}}\right]}{P[X + \epsilon \leq s]},$$

for all s such that $P[X + \epsilon \leq s] > 0$.

In most situations in this chapter π will be fixed and thus we will sometimes omit it. As mentioned earlier we are most interested in the case where X has a finite state space $\{x_1, \ldots, x_n\}$. We then can think of $\pi = (\pi_i)_{i=1..n}$ simply as a vector of probabilities $P[X = x_i] = \pi_i$.

A central point of proving existence and uniqueness in the static as well as in the continuous time model lies in the fact that the zero-profit condition (3.1) of the static Glosten-Milgrom ask prices translates to a fixed point problem in the following sense.

Lemma 3.5. *If π is given and g is well-defined (i.e. $P[X + \epsilon \geq s] > 0$), then s is a static Glosten-Milgrom ask price (a solution of (3.1)) iff it is a fixed point of g, i.e.*

$$g(s, \pi) = s.$$

Proof. This can be seen by the fact that

$$g(s, \pi) = \frac{E\left[X 1_{\{X+\epsilon \geq s\}}\right]}{P[X + \epsilon \geq s]} = s$$

is equivalent to

$$E\left[X1_{\{X+\epsilon\geq s\}}\right] = E\left[s1_{\{X+\epsilon\geq s\}}\right],$$

which gives (3.1). □

If the Glosten-Milgrom price exists and is unique we can describe them also as a function of the distribution π.

Definition 3.6. *If for a class of distributions Π of X the Glosten-Milgrom ask price exists and is unique for all $\pi \in \Pi$, we define $G : \Pi \mapsto \mathbb{R}$ by*

$$G(\pi) = s$$

where s solves (3.1) or, if g is well-defined,

$$g(s, \pi) = s.$$

In the same way and under analog conditions we define H to be the function that maps the distribution π of X to its corresponding Glosten-Milgrom bid price $H(\pi)$.

3.3 Noise with density

We will now consider the case where the distribution of the noise ϵ has a density that satisfies some conditions. This is a more general version of Das [Das05] who assumes ϵ to be normally distributed. Moreover, we assume that the true value X is bounded in both directions. We formulate the next Lemma that secures that g in Definition 3.4 is well-defined for $s \leq x_{\max}$.

Lemma 3.7. *Let X be bounded between $x_{\min}$ and $x_{\max}$ a.s. and define $C := x_{\max} - x_{\min}$. Let $\Phi(y) := P[\epsilon \geq y]$ be the inverted distribution function of ϵ. If Φ is differentiable (i.e. the distribution of ϵ has density $-\Phi'$) on $[-C, C]$, $\Phi(0) > 0$ and*

$$-\Phi'(y) \leq \frac{K}{C}\Phi(y) \tag{3.2}$$

for all $y \in [-C, C]$ and a constant $K < 1$, it follows that $\Phi(C) > 0$, which implies that $P[X + \epsilon \geq s] > 0$ i.e. the probability that a buy occurs is strictly larger than 0 for all prices $s \leq x_{\max}$.

Proof. Note that Φ is in $[0,1]$ and decreasing by definition. We have

$$\begin{aligned}
\Phi(C) &= \int_0^C \Phi'(t)dt + \Phi(0) \\
&\geq \frac{K}{C}\int_0^C -\Phi(t)dt + \Phi(0) \\
&\geq -\frac{K}{C}C\Phi(0) + \Phi(0) \\
&\geq (1-K)\Phi(0) \\
&> 0
\end{aligned}$$

since $K < 1$ and $\Phi(0) > 0$. Furthermore we have

$$P[X + \epsilon \geq s] = P[\epsilon \geq s - X] \geq P[\epsilon \geq x_{\max} - x_{\min}] = \Phi(C) > 0. \qquad \square$$

Under the same assumptions as of Lemma 3.7 we can now give a first statement of existence and uniqueness.

Theorem 3.8. *Let all assumptions of Lemma 3.7 be fulfilled. Then there exists an unique static Glosten-Milgrom ask price in* $[x_{\min}, x_{\max}]$.

Proof. Since π is fixed we omit it. We consider the derivative of

$$g(s) = E[X|X + \epsilon \geq s] = \frac{E[X\Phi(s-X)]}{E[\Phi(s-X)]}$$

for $x_{\min} \leq s \leq x_{\max}$ which is given by

$$\begin{aligned}
g'(s) &= \frac{E_X[X\Phi'(s-X)]E_Z[\Phi(s-Z)] - E_Z[Z\Phi(s-Z)]E_X[\Phi'(s-X)]}{(E_X[\Phi(s-X)])^2} \\
&= \frac{E_X[E_Z[X\Phi'(s-X)\Phi(s-Z) - Z\Phi(s-Z)\Phi'(s-X)]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&= \frac{E_X[E_Z[-\Phi'(s-X)\Phi(s-Z)(Z-X)]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&\leq \frac{E_X[E_Z[-\Phi'(s-X)\Phi(s-Z)|Z-X|]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&\leq C\frac{E_X[E_Z[-\Phi'(s-X)\Phi(s-Z)]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&\leq C\frac{K}{C}\frac{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]}{E_X[E_Z[\Phi(s-X)\Phi(s-Z)]]} \\
&= K
\end{aligned}$$

Hence, $0 \leq g'(s) \leq K < 1$ for all $s \in [x_{\min}, x_{\max}]$ and therefore

$$|g(s) - g(t)| \leq K|s - t|. \tag{3.3}$$

This implies that g is a contraction which has a unique fixed point by the Banach fixed-point theorem. Together with Lemma 3.5 this yields the Theorem. $\square$

For parametric classes of distributions of ϵ the central condition (3.2) can usually be secured by choosing parameters such that the variance is high. In economic terms this corresponds to informed traders whose information is less precise. Since we need the differentiability only on $[-C, C]$, it is still possible that ϵ takes the values $\pm\infty$.

Note that we make no assumptions on the distribution of X apart from the boundedness but quite explicit assumptions on the distribution of ϵ. The fact that we have existence and uniqueness for a whole class of distributions will be central in the continuous time model. This also implies that the functions G and H from Definition 3.6 are well-defined on all distributions such that X is bounded between $x_{\min}$ and $x_{\max}$.

We have already seen that g is Lipschitz-continuous in s with parameter $K < 1$ in (3.3). If X has finite state space we further obtain Lipschitz-continuity in the distribution π in the following sense.

Lemma 3.9. *Let all assumptions of Lemma 3.7 be fulfilled. In addition assume that X takes only finitely many values, i.e. there exist $x_{\min} = x_1 < ... < x_n = x_{\max}$ and $\pi = (\pi_1, ..., \pi_n)$ such that $P[X = x_i] = \pi_i$ for all i and $\sum_{i=1}^n \pi_i = 1$. Then*

$$|g(s, \pi) - g(\tilde{s}, \tilde{\pi})| \leq K|s - \tilde{s}| + L \sum_{i=1}^{n} |\pi_i - \tilde{\pi}_i|$$

for K from (3.2) and $L = \frac{2x_{\max}}{\Phi(C)^2} < \infty$, all $s, \tilde{s} \in [x_{\min}, x_{\max}]$ and all distributions $\pi, \tilde{\pi}$.

Proof. First, we see that

$$\begin{aligned} |g(s, \pi) - g(\tilde{s}, \tilde{\pi})| &= |g(s, \pi) - g(s, \tilde{\pi}) + g(s, \tilde{\pi}) - g(\tilde{s}, \tilde{\pi})| \\ &\leq |g(s, \pi) - g(s, \tilde{\pi})| + K|s - \tilde{s}| \end{aligned}$$

by (3.3). It remains to show that

$$|g(s, \pi) - g(s, \tilde{\pi})| \leq L \sum_{i=1}^{n} |\pi_i - \tilde{\pi}_i|.$$

To shorten notation we write

$$\alpha(f(X),\pi) := E[f(X)\Phi(s-X)] = \sum_{i=1}^{n} \pi_i f(x_i)\Phi(s-x_i).$$

Hence, we have

$$g(s,\pi) = \frac{\alpha(X,\pi)}{\alpha(1,\pi)}$$

and

$$\begin{aligned}
|g(s,\pi)-g(s,\tilde{\pi})| =& \left|\frac{\alpha(X,\pi)}{\alpha(1,\pi)} - \frac{\alpha(X,\tilde{\pi})}{\alpha(1,\tilde{\pi})}\right| \\
=& \frac{|\alpha(X,\pi)\alpha(1,\tilde{\pi}) - \alpha(X,\tilde{\pi})\alpha(1,\pi)|}{\alpha(1,\pi)\alpha(1,\tilde{\pi})} \\
\leq& \frac{|\alpha(X,\pi)\alpha(1,\tilde{\pi}) - \alpha(X,\pi)\alpha(1,\pi)|}{\alpha(1,\pi)\alpha(1,\tilde{\pi})} \\
&+ \frac{|\alpha(X,\pi)\alpha(1,\pi) - \alpha(X,\tilde{\pi})\alpha(1,\pi)|}{\alpha(1,\pi)\alpha(1,\tilde{\pi})} \\
=& \frac{|\alpha(X,\pi)\sum_{i=1}^{n}(\tilde{\pi}_i - \pi_i)\Phi(s-x_i)|}{\alpha(1,\pi)\alpha(1,\tilde{\pi})} \\
&+ \frac{|\alpha(1,\pi)\sum_{i=1}^{n}(\pi_i - \tilde{\pi}_i)x_i\Phi(s-x_i)|}{\alpha(1,\pi)\alpha(1,\tilde{\pi})} \\
\leq& L\sum_{i=1}^{n}|\pi_i - \tilde{\pi}_i|.
\end{aligned}$$

The last inequality follows from $x_i\Phi(s-x_i) \leq x_{\max} < \infty$ and $\Phi(s-x_i) \leq 1$ for all i and $\alpha(1,\pi) \geq \Phi(s-x_{\min}) \geq \Phi(C) > 0$. This proofs the lemma. □

Lemma 3.10. *Let all assumptions of Lemma 3.9 be fulfilled. Then G is Lipschitz-continuous.*

Proof. Let $s, \tilde{s}$ by such that $G(\phi) = s$, i.e. $g(s,\phi) = s$ and $G(\tilde{\phi}) = \tilde{s}$. We have

$$\begin{aligned}
|s-\tilde{s}| =& \left|g(s,\phi) - g(\tilde{s},\tilde{\phi})\right| \\
\leq& K|s-\tilde{s}| + L\sum_{i=1}^{n}|\phi_i - \tilde{\phi}_i|
\end{aligned}$$

by Lemma 3.9, where $K < 1$ and $L < \infty$. By rearranging we get

$$|G(\phi) - G(\tilde{\phi})| = |s - \tilde{s}| \leq \frac{L}{1-K} \sum_{i=1}^{n} |\phi_i - \tilde{\phi}_i|. \qquad \square$$

It is easy to see, that the restrictions $1 > \Phi(0)$ and

$$-\Phi'(y) \leq \frac{\widetilde{K}}{C}(1 - \Phi(y))$$

for all $y \in [-C, C]$ and a constant $\widetilde{K} < 1$ (together with the differentiability of Φ) are those that we need to obtain unique static Glosten-Milgrom bid prices and similar results to Lemma 3.9 for h and Lemma 3.10 for H. If we merge this with the assumptions of Lemma 3.7 this results in exactly the assumptions of Theorem 5.1 which states existence and uniqueness in the continuous time model.

3.4 Insider and noise trader

Another classical approach in insider information models such as the Kyle-model [Kyl85] is to assume that there are two types of customers. On the one hand there are insiders that know the exact value of the asset, thus $\epsilon = 0$ and there are noise traders that trade for exogenous liquidity reasons and whose behavior does not depend on the quoted prices. This results in $\epsilon = \pm\infty$ since they buy (or sell) at every price. However, it is not known to the market maker which sort of customer he is trading with, but only the distribution of the two (or three) types. Hence, for $\mu, \nu \in (0, 1)$ we consider the distribution

$$\epsilon = \begin{cases} \infty & \text{with probability} \quad \mu \\ 0 & \qquad\qquad\qquad\quad \nu \\ -\infty & \qquad\qquad\qquad\quad 1 - \mu - \nu. \end{cases} \tag{3.4}$$

This model describes a limiting case of the model we considered in the last section where customers have all kinds of noise or preference in their valuation. The advantage of this modelling is that the irrationality is clearly assigned to the noise traders that trade in any case, i.e. for every value of X and more important at any price. The insiders behave rational, since they maximize their profit at every opportunity to trade, which is then (for a buy) given by

$$(X - s)1_{\{X \geq s\}}.$$

This rationality is a fundamental requirement when we think about the optimal behavior of the insiders, which is the aim of all the Kyle-type literature.

However, it is questionable whether the distinction between the two groups is so clear in real markets. Also it seems an unnatural hard assumption that the only source of less demand at higher prices on the buy side are insiders that are offset by these higher prices. It seems plausible that also the liquidity demand of the noise trader is influenced by the prices.

Let us remark that g from Definition 3.4 is well defined, since

$$P[X + \epsilon \geq s] \geq P[\epsilon = \infty] = \mu > 0$$

for all s. However, let X have some atom at x, i.e. $P[X = x] = p > 0$, then

$$g(x, \pi) = \frac{\mu E[X] + \nu p x + \nu E[X 1_{\{X > x\}}]}{\mu + \nu p + \nu P[X > x]} \tag{3.5}$$

and for $\delta > 0$

$$g(x + \delta, \pi) = \frac{\mu E[X] + \nu E[X 1_{\{X \geq x + \delta\}}]}{\mu + \nu P[X \geq x + \delta]}. \tag{3.6}$$

For $\delta \to 0$ the last terms of the numerator and denominator of (3.6) tend to those of (3.5). As the terms resulting from x do not appear this shows that g is not even continuous. Hence it is neither Lipschitz nor a contraction. Thus, we cannot argue as in Theorem 3.8 where we used the Banach fixed-point theorem. It is nevertheless possible to show existence and uniqueness with different methods.

Theorem 3.11. *Let ϵ be as in (3.4) and X integrable. Then there exists an unique static Glosten-Milgrom ask price.*

Proof. If we consider (3.1) for the given ϵ we have

$$E[(s - X)1_{\{X + \epsilon \geq s\}}] = \mu(s - E[X]) + \nu E[(s - X)1_{\{X \geq s\}}] = 0$$

which is equivalent to

$$s = E[X] + \frac{\nu}{\mu} E[(X - s)1_{\{X \geq s\}}]. \tag{3.7}$$

Thus, here the question of existence and uniqueness is equivalent to the question whether

$$f(s) = E[X] + \frac{\nu}{\mu} E[(X - s)1_{\{X \geq s\}}]$$

has a unique fixed point. For $s < t$ we have

$$\begin{aligned} f(s) - f(t) &= \frac{\nu}{\mu} E[(X - s)1_{\{X \geq s\}} - (X - t)1_{\{X \geq t\}}] \\ &= \frac{\nu}{\mu} E[(X - s)1_{\{X \geq s\}} - (X - s)1_{\{X \geq t\}} + (t - s)1_{\{X \geq t\}}] \quad (3.8) \\ &= \frac{\nu}{\mu} E[(X - s)1_{\{0 \leq X - s < t - s\}} + (t - s)1_{\{X - t \geq 0\}}]. \end{aligned}$$

It follows that $f(s) - f(t) \geq 0$ which implies that f is decreasing. Furthermore f is continuous, since

$$\begin{aligned} |f(s) - f(t)| &= \frac{\nu}{\mu} E[(X - s)1_{\{0 \leq X - s < t - s\}} + (t - s)1_{\{X - t \geq 0\}}] \\ &\leq \frac{\nu}{\mu} (t - s) E[1_{\{0 \leq X - s < t - s\}} + 1_{\{X - t \geq 0\}}] \\ &\leq 2\frac{\nu}{\mu} |s - t|. \end{aligned}$$

Hence, there exists a unique solution of (3.7) and an unique static Glosten-Milgrom ask price. □

Note that for any distribution π of X the expression $G(\pi)$ from Definition 3.6 is well-defined and it satisfies

$$G(\pi) = E[X] + \frac{\nu}{\mu} E[(X - G(\pi))1_{\{X \geq G(\pi)\}}]$$

where E is according to π. This equation has the very nice interpretation, that the Glosten-Milgrom ask price is the expectation of X plus some term that contains the ratio of uniformed and insiders and an expectation of the possible losses if the market maker trades with an insider.

Chapter 4

The continuous-time model

In this chapter we will introduce the general continuous time model in the first section. For the rest of this thesis we will discuss this model. We then characterize the Glosten-Milgrom pricing strategies (GMPS) in two equivalent ways in Definition 4.4 and Theorem 4.5 in the second section. In the third section we introduce the process of conditional probabilities and derive the associated filter equations in Lemma 4.10 with the help of the results in Chapter 2. Finally, we describe the GMPS as a fixed point of some functional F that is defined on the admissible pricing strategies (see Theorem 4.14).

4.1 The general framework

In the following we will develop a general model in continuous time for a specialist market, i.e. a market where a market maker or specialist offers to buy or sell at any point in time to the bid and ask prices he quotes.

All random variables that we introduce live on the probability space $(\Omega, \mathcal{F}, P)$ whereas different filtrations are considered. We assume that the càdlàg process $X = (X_t)_{t \geq 0}$, interpreted as the time-dependent true value of the asset, is a time-homogeneous Markov process with finite state space $\{x_1, \ldots, x_n\}$, $n \geq 2$ where $x_{\min} = x_1 < \ldots < x_n = x_{\max}$, and has transition kernel

$$q(i,j) := \lim_{t \to 0} \frac{1}{t} P[X_t = x_j \mid X_0 = x_i] \tag{4.1}$$

for $i \neq j$ and $q(i,i) = -\sum_{j \neq i} q(i,j)$.

The market maker knows the distribution of X but does not know the actual value. The only source of information which is available to the market maker are the trades that take place at the prices he sets.

To model the customer flow, let N be a Poisson process with rate $\lambda > 0$. We denote the ordered jump times of N by $\tau_1 < \tau_2 < \tau_3 \ldots$. We assume that at these times potential customers arrive at the market (unseen by the market maker). As in the static model the customers have some disturbed information about the true value of the asset which is given by $X_{\tau_i} + \epsilon_i$ for the i-th customer where $(\epsilon_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. random variables. We assume that X, N, and $(\epsilon_i)_{i\in\mathbb{N}}$ are independent of each other.

We further assume that the market maker sets a pair of prices according to an $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable mapping $S : \Omega \times [0,\infty) \to \mathbb{R}^2$. We write $S = (\overline{S}, \underline{S})$ to denote ask and bid prices and we only admit prices with $\overline{S}_t(\omega) > \underline{S}_t(\omega)$ for all (ω, t). To be economically meaningful the strategy S has to satisfy some predictability condition that will be given in Definition 4.3.

A potential customer buys one asset if $X_{\tau_i} + \epsilon_i \geq \overline{S}_{\tau_i}$ and sells one asset if $X_{\tau_i} + \epsilon_i \leq \underline{S}_{\tau_i}$. He does nothing if his valuation is within the spread. The motivation of this behavior is the same as in the static case. We studied its interpretation and implications in the first section of Chapter 3.

Definition 4.1. *Let $B_0 = C_0 = T_0 = 0$. We introduce the sequence of random times of actual buys by*

$$B_i := \inf\{\tau_j | \tau_j > B_{i-1}, X_{\tau_j} + \epsilon_j \geq \overline{S}_{\tau_j}\}, \qquad i \geq 1,$$

the sequence of actual sells by

$$C_i := \inf\{\tau_j | \tau_j > C_{i-1}, X_{\tau_j} + \epsilon_j \leq \underline{S}_{\tau_j}\}, \qquad i \geq 1.$$

and the sequence of actual transactions (i.e. buys or sells) by

$$T_i := \inf\{\tau_j | \tau_j > T_{i-1}, X_{\tau_j} + \epsilon_j \geq \overline{S}_{\tau_j} \text{ or } X_{\tau_j} + \epsilon_j \leq \underline{S}_{\tau_j}\}, \qquad i \geq 1.$$

In addition we define the counting processes of actual buys and sells by

$$N_t^B := \sum_{i\geq 1} 1_{\{B_i \leq t\}} \tag{4.2}$$

and N^C and N^T accordingly.

It is a crucial question which information the market maker has when he is setting prices. We assume that he is only observing the history of past trades. The prices at which the trades take place (transaction prices) contain no information since they are set by the market maker himself. The market maker does not observe any direct signal of X such as $X + \epsilon$ but only the reaction of the customers to his prices. This is a crucial point because the market maker thus influences how much information he gets by the setting of his prices.

Definition 4.2. *The filtration of the market maker is given by* $\mathbb{F}^S = (\mathcal{F}_t^S)_{t \geq 0}$, *where*

$$\mathcal{F}_t^S := \sigma(\{B_i \leq s\}, \{C_i \leq s\}, s \leq t, i \in \mathbb{N}) = \sigma\left(N_s^B, N_s^C, s \leq t\right).$$

Since $\mathbb{F}^S$ is generated by counting processes, it is a right-continuous filtration (see Theorem I.25 in [Pro04]). However, it does not in general satisfy the usual conditions, since the null sets are not necessarily included.

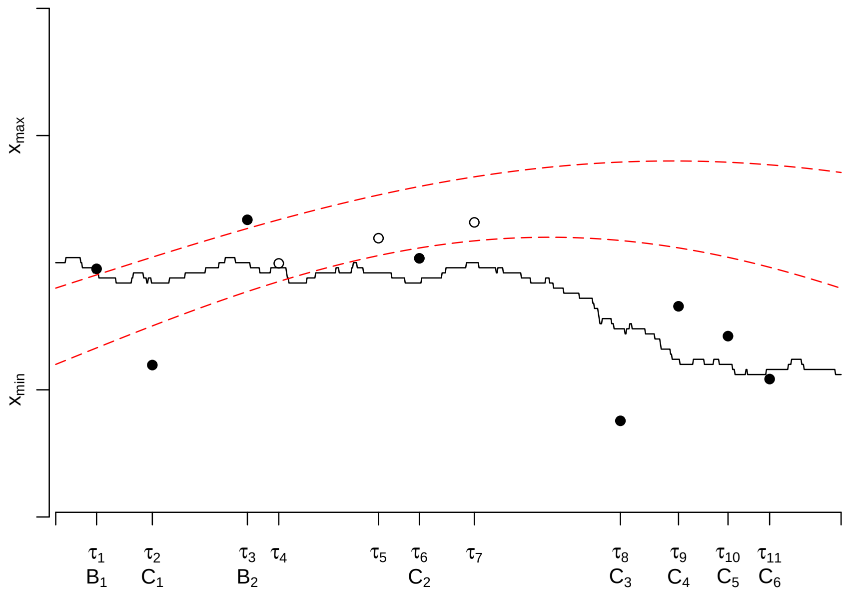

Figure 4.1: The black line represents the true value X and some quoted prices $\overline{S} > \underline{S}$ (here not at equilibrium) are given by the dotted red lines. All potential trades $X_{\tau_i} + \epsilon_i$ are given by the bullets, which are filled if a trade takes place at $\overline{S}$ or $\underline{S}$. Here, X only jumps to the neighboring state, which is not a general restriction. The ϵ_i are normally distributed.

From an economic viewpoint pricing strategies of market makers make sense only if they are $\mathbb{F}^S$-predictable, as $\mathbb{F}^S$ is the information flow of the market maker.

Definition 4.3. *We say that S is an admissible pricing strategy if it is $\mathbb{F}^S$-predictable and $x_{\max} \geq \overline{S}_t(\omega) > \underline{S}_t(\omega) \geq x_{\min}$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$.*

We impose the restriction that the prices lie between $x_{\min}$ and $x_{\max}$ because otherwise there would be either arbitrage opportunities or no trades at all. Note that the definition is quite implicit, since the filtration $\mathbb{F}^S$ depends itself on S.

4.2 Glosten-Milgrom pricing strategies

The model stated above gives a natural, though complex, framework to examine the price-setting of market makers. We now proceed to consider a certain type of price-setting which involves the Glosten-Milgrom idea of risk neutrality and perfect competition between market makers.

Definition 4.4. *We say that an admissible pricing strategy S is a Glosten-Milgrom pricing strategy (GMPS) if*

$$E\left[\sum_{B_i \leq \tau} (\overline{S}_{B_i} - X_{B_i})\right] = 0 \text{ and } E\left[\sum_{C_i \leq \tau} (\underline{S}_{C_i} - X_{C_i})\right] = 0 \tag{4.3}$$

for every bounded $\mathbb{F}^S$-stopping time τ.

Each summand in (4.3) is bounded by $x_{\max} - x_{\min}$ and the sequence of B_i and C_i is included in the Poisson times. This yields integrability of the sums. We assume that in no stochastic time interval it is possible to make a gain in expectation. Note that this definition implies that not only the whole business makes zero profits but both the buy-side and sell-side business separately. This is necessary to comply with the idea of perfect competition. Otherwise the market maker could offset losses on one side with gains on the other side of the market. But this is not possible because other market makers will undercut the price on the side that makes gains.

The Glosten-Milgrom prices can also be characterized by another condition as stated in the following theorem.

Theorem 4.5. *S is a GMPS iff it is admissible and*

$$\overline{S}_{B_i} = E[X_{B_i} | \mathcal{F}^S_{B_i}] \qquad \text{and} \qquad \underline{S}_{C_i} = E[X_{C_i} | \mathcal{F}^S_{C_i}] \qquad P-a.s.$$

for all i.

This means that all trades in a GMPS are executed at a price which is the expectation of X given the information available to market makers at that point of time. The interesting point about this characterization of GMPS is that a trade which occurs at that very moment is included in the filtration but its occurrence and especially its direction is not predictable (but so is S). We also see the connection to the static model. By Lemma 3.5 the Glosten-Milgrom ask price in the static model satisfies

$$s = E[X|X + \epsilon \geq s].$$

Proof of Theorem 4.5. We only consider buys. Let $\overline{S}_{B_i} = E[X_{B_i}|\mathcal{F}^S_{B_i}]$ for all i then

$$\begin{aligned} E\left[\sum_{B_i \leq \tau} (\overline{S}_{B_i} - X_{B_i})\right] &= \sum_{i=1}^{\infty} E\left[1_{\{B_i \leq \tau\}}(\overline{S}_{B_i} - X_{B_i})\right] \\ &= \sum_{i=1}^{\infty} E\left[E\left[1_{\{B_i \leq \tau\}}(\overline{S}_{B_i} - X_{B_i})|\mathcal{F}^S_{B_i}\right]\right] \\ &= \sum_{i=1}^{\infty} E\left[1_{\{B_i \leq \tau\}}(\overline{S}_{B_i} - E\left[X_{B_i}|\mathcal{F}^S_{B_i}\right])\right] \\ &= 0. \end{aligned}$$

Now let S be a GMPS. For fixed $i \in \mathbb{N}$ and $t \in \mathbb{R}_+$ we consider

$$C := \{B_{i-1} \leq t < B_i\} \cap A$$

for $A \in \mathcal{F}^S_t$. For $n \in \mathbb{N}, n > t$, let

$$\kappa_n(\omega) = \begin{cases} B_i(\omega) \wedge n & \text{if} \quad \omega \in C \\ t & \phantom{\text{if}} \quad \omega \notin C, \end{cases} \tag{4.4}$$

hence $t \leq \kappa_n$ and both are bounded $\mathbb{F}^S$-stopping times. Thus, we have

$$\begin{aligned} 0 &= E\left[\sum_{B_j \leq \kappa_n} (\overline{S}_{B_j} - X_{B_j})\right] - E\left[\sum_{B_j \leq t} (\overline{S}_{B_j} - X_{B_j})\right] \\ &= E\left[\sum_{t < B_j \leq \kappa_n} (\overline{S}_{B_j} - X_{B_j})1_C\right] = E\left[(\overline{S}_{B_i} - X_{B_i})1_{C \cap \{B_i \leq n\}}\right] \end{aligned}$$

for all $n > t$ and therefore

$$E\left[(\overline{S}_{B_i} - X_{B_i})1_C\right] = 0.$$

Note that $\mathcal{F}^S_{B_i-}$ is generated by sets of the form $\{t < B_i\} \cap A$, where $A \in \mathcal{F}^S_t$ and $t \in \mathbb{R}_+$which can be written as

$$\{t < B_i\} \cap A = \cup_{t \le t_n \in \mathbb{Q}} \{B_{i-1} \le t_n < B_i\} \cap A.$$

Hence, as $A \in \mathcal{F}^S_t \subset \mathcal{F}^S_{t_n}$ for $t_n \ge t$ the sets of the form like C generate $\mathcal{F}^S_{B_i-}$ and since $\overline{S}$ is predictable it follows that

$$\overline{S}_{B_i} = E\left[X_{B_i} | \mathcal{F}^S_{B_i-}\right].$$

To show that $\mathcal{F}^S_{B_i-} = \mathcal{F}^S_{B_i}$ we consider the marked point process $(T_n, Z_n)_{n\in\mathbb{N}}$, where T are the times of trades defined in Definition 4.1 and

$$Z_n = \begin{cases} 1 & \text{if} \quad T_n = B_i \qquad \text{for some } i \\ -1 & \qquad\;\; T_n = C_i. \end{cases} \tag{4.5}$$

We can now write

$$\mathcal{F}^S_{B_i} = \{A | A = \cup_{n\in\mathbb{N}} A_n \cap \{B_i = T_n\} \text{ for } A_n \in \mathcal{F}^S_{T_n} \text{ for all } n\}$$

and

$$\mathcal{F}^S_{B_i-} = \{A | A = \cup_{n\in\mathbb{N}} A_n \cap \{B_i = T_n\} \text{ for } A_n \in \mathcal{F}^S_{T_n-} \text{ for all } n\}.$$

The first equation can be seen easily. For "$\subset$" of the second it suffices to show that sets of the form $A \cap \{t < B_i\}$, $A \in \mathcal{F}^S_t$ are in the set on the RHS. This can be done by choosing $A_n = A \cap \{t < T_n\}$. For "$\supset$" it again suffices to consider sets of the form $A_n = \widetilde{A}_n \cap \{t < T_n\}$, $\widetilde{A}_n \in \mathcal{F}^S_t$. It then remains to show that $\{B_i = T_n\} \in \mathcal{F}^S_{B_i-}$. However,

$$\{T_n < B_i\} = \cup_{q\in\mathbb{Q}} \{T_n < q\} \cap \{q < B_i\} \in \mathcal{F}^S_{B_i-}$$

and thus

$$\{B_i = T_n\} = \{T_{n-1} < B_i\} \cap \{T_n < B_i\}^c \in \mathcal{F}^S_{B_i-}.$$

Now, by Theorem III, T2 in [Bré81] applied to the marked point process $(T_n, Z_n)_{n\in\mathbb{N}}$ any $A_n \in \mathcal{F}^S_{T_n}$ can be written as

$$A_n = (M_1 \cap \{Z_n = 1\}) \cup (M_2 \cap \{Z_n = -1\})$$

for $M_1, M_2 \in \mathcal{F}^S_{T_n-}$. Since for fixed i $\{B_i = T_n\} = \{Z_n = 1\}$, we have

$$A_n \cap \{B_i = T_n\} = M_1 \cap \{B_i = T_n\}$$

and $\mathcal{F}^S_{B_i-} = \mathcal{F}^S_{B_i}$ follows. □

4.3 The process of conditional probabilities

As we mentioned earlier the filtration of the market maker $\mathbb{F}^S$ does not satisfy the usual conditions, since it does not contain all null sets. We now define the completion $\widetilde{\mathbb{F}}^S$ of $\mathbb{F}^S$.

Definition 4.6. *For any $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable process $S = (\overline{S}, \underline{S})$ let the filtration $\widetilde{\mathbb{F}}^S$ be defined by*

$$\widetilde{\mathcal{F}}^S_t := \mathcal{F}^S_t \vee \mathcal{N},$$

where $\mathcal{N}$ are all P-null sets of $\mathcal{F}$.

$\widetilde{\mathbb{F}}^S$ will be used in the proof of the following lemma, but note that it is not needed to state our results.

Lemma 4.7. *For any $\mathcal{F} \otimes \mathcal{B}([0,\infty))$-measurable process $S = (\overline{S}, \underline{S})$, there exists a unique (up to indistinguishability) $\widetilde{\mathbb{F}}^S$-adapted càdlàg process π^S with*

$$\pi^S_\tau = \left(P\left[X_\tau = x_i | \mathcal{F}^S_\tau\right]\right)_{i=1,\dots,n} \qquad P\text{-a.s.} \tag{4.6}$$

for all finite stopping times τ.

Proof. Since $\widetilde{\mathbb{F}}^S$ satisfies the usual conditions, we can apply Theorems 2.7 and 2.9 of [BC08] to the process $\left(1_{\{X_t = x_i\}}\right)_{i=1,\dots,n}$, which gives us a càdlàg optional projection π^S that is $\widetilde{\mathbb{F}}^S$-adapted and satisfies

$$\pi^S_\tau = \left(P\left[X_\tau = x_i | \widetilde{\mathcal{F}}^S_\tau\right]\right)_{i=1,\dots,n} \qquad P\text{-a.s.}$$

for all finite stopping times τ. Since $E[\,\cdot\,|\mathcal{F}_\tau]$ and $E[\,\cdot\,|\widetilde{\mathcal{F}}_\tau]$ only differ by a P-null set, (4.6) follows. □

We now define a larger filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ by

$$\mathcal{F}_t := \sigma\left(X_s, N_s, \epsilon_i 1_{\{\tau_i \leq s\}}, s \leq t, i \in \mathbb{N}\right) \vee \mathcal{N},$$

which contains all information up to time t.

Definition 4.8. *We define* $\Phi, \Psi : \overline{\mathbb{R}} \mapsto [0,1]$ *by*

$$\Phi(x) = P[\epsilon_1 \geq x] \qquad \textit{and} \qquad \Psi(x) = P[\epsilon_1 \leq x].$$

The following lemma describes the intensity of the jump process N^B that counts actual buys (for a given pricing strategy S) as given in Definition 4.1.

Lemma 4.9. *The* $\mathbb{F}$*-intensity of* N^B *(in the sense of Definition 2.3) is given by* $\lambda\Phi(\overline{S} - X_-)$*.*

Proof. Let C be a nonnegative $\mathbb{F}$-predictable process. Then

$$\begin{aligned}
E\left[\int_0^\infty C_s dN_s^B\right] &= E\left[\sum_{i=1}^\infty C_{\tau_i} 1_{\{X_{\tau_i}+\epsilon_i \geq \overline{S}_{\tau_i}\}}\right] \\
&= \sum_{i=1}^\infty E\left[E\left[C_{\tau_i} 1_{\{X_{\tau_i}+\epsilon_i \geq \overline{S}_{\tau_i}\}} | \mathcal{F}_{\tau_i -}\right]\right] \\
&= \sum_{i=1}^\infty E\left[C_{\tau_i} \Phi(\overline{S}_{\tau_i} - X_{\tau_i -})\right] \\
&= E\left[\int_0^\infty C_s \Phi(\overline{S}_s - X_{s-}) dN_s\right] \\
&= E\left[\int_0^\infty C_s \lambda \Phi(\overline{S}_s - X_{s-}) ds\right]
\end{aligned}$$

where we use $X_{\tau_i} = X_{\tau_i -}$ $P-a.s.$ for the third equation. □

We now derive the filter equation of π^S that is of central importance for the analysis in the following and is based on Theorem 2.4 in Chapter 2.

Lemma 4.10. *The process π^S satisfies the following SDE*

$$\begin{aligned} d\pi_t^{S,i} =& \pi_{t-}^{S,i} \left(\frac{\Phi(\overline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j} \Phi(\overline{S}_t - x_j)} - 1 \right) dN_t^B \\ &+ \pi_{t-}^{S,i} \left(\frac{\Psi(\underline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j} \Psi(\underline{S}_t - x_j)} - 1 \right) dN_t^C \\ &- \left(\lambda \pi_t^{S,i} \left(\Psi(\underline{S}_t - x_i) + \Phi(\overline{S}_t - x_i) \right.\right. \\ &\left. - \sum_j \pi_t^{S,j} \left(\Psi(\underline{S}_t - x_j) + \Phi(\overline{S}_t - x_j) \right) \right) \\ &\left. - \sum_j \pi_t^{S,j} q(j,i) \right) dt. \end{aligned}$$

for all $t \geq 0$, up to indistinguishability, with initial condition $\pi_0^{S,i} = P[X_0 = x_i]$, where Φ and Ψ are given in Definition 4.8.

Proof. We can derive the filter equation for π^S as it is done in Theorem 2.4. The filter equation has the form

$$\pi_t^S = \pi_0^S + \int_0^t K_s^B dN_s^B + \int_0^t K_s^C dN_s^C + \int_0^t \left(-K_s^B \widehat{\lambda}^B - K_s^C \widehat{\lambda}^C + f_s \right) ds,$$

where $\widehat{\lambda}^B$ and $\widehat{\lambda}^C$ are the $\mathbb{F}^S$-intensities of N^B and N^C respectively, f is the $\mathbb{F}^S$-compensator of X, which is given by $\sum_j \pi^{S,j} q(j,\cdot)$ here. The innovations gain K is described in Theorem 2.4 by three summands. The theorem also states that the last one is zero if the observation process and the underlying process do not jump at the same time a.s., which is the case here and that the second one is the left limit of the process that is to be described. Hence, K_s^B can be expressed as $\Psi_s^B - \pi_{s-}^S$ and $K_s^C = \Psi_s^C - \pi_{s-}^S$ respectively, where Ψ_s^B is the unique (up to a $(P \otimes \lambda)$-null set) $\mathbb{F}^S$-predictable process satisfying

$$E\left[\int_0^t C_s 1_{\{X_s = x_i\}} \lambda_s^B ds \right] = E\left[\int_0^t C_s \Psi_s^{B,i} \widehat{\lambda}_s^B ds \right] \tag{4.7}$$

for all $\mathbb{F}^S$-predictable nonnegative bounded processes C, $i = 1, \ldots, n$ and all $t \geq 0$ where $\lambda^B, \widehat{\lambda}^B$ are the $\mathbb{F}, \mathbb{F}^S$-intensities of N^B respectively. A similar equation holds for Ψ_s^C.

From Lemma 4.9 we have that $\lambda^B = \lambda \Phi(\overline{S} - X_-)$ and

$$\begin{aligned} E\left[\lambda^B|\mathcal{F}_s^S\right] &= \sum_{i=1}^n E\left[1_{\{X_{s-}=x_i\}}\lambda\Phi(\overline{S}-x_i)|\mathcal{F}_s^S\right] \\ &= \sum_{i=1}^n \lambda\Phi(\overline{S}-x_i)E\left[1_{\{X_{s-}=x_i\}}|\mathcal{F}_s^S\right] \ \ P\text{-a.s..} \end{aligned}$$

Since, for fixed $s \in \mathbb{R}_+$, $X_s = X_{s-}$ P-a.s., $\lambda\sum_{i=1}^n \pi_{s-}^{S,i}\Phi(\overline{S}_s - x_i)$ is a version of $\widehat{\lambda}_s^B$. From this and as $\pi_-^{S,i}$ is the $\mathbb{F}^S$-compensator of $1_{\{X=x_i\}}$ it follows that

$$\Psi_s^{B,i} = \frac{\pi_{s-}^{S,i}\Phi(\overline{S}_s - x_i)}{\sum_{j=1}^n \pi_{s-}^{S,j}\Phi(\overline{S}_s - x_j)}$$

solves (4.7) which gives us K_s^B and analog K_s^C as stated in the lemma.

For the buy-side part of the dt-term we get

$$-K_s^{B,i}\widehat{\lambda}^B = -\left(\frac{\pi_{s-}^{S,i}\Phi(\overline{S}_s - x_i)}{\sum_{j=1}^n \pi_{s-}^{S,j}\Phi(\overline{S}_s - x_j)} - \pi_s^{S,i}\right)\lambda\sum_{j=1}^n \pi_{s-}^{S,j}\Phi(\overline{S}_s - x_j)$$

which simplifies to

$$-\lambda\pi_{s-}^{S,i}\Phi(\overline{S}_s - x_i) + \lambda\pi_{s-}^{S,i}\sum_{j=1}^n \pi_{s-}^{S,j}\Phi(\overline{S}_s - x_j).$$

Together with f and the similar results for the sell-side we obtain the dt-term as stated in the Lemma which completes the proof. □

4.4 The solution as fixed point

Since π^S is càdlàg , π_{t-}^S is well defined and we can now define the following functional.

Definition 4.11. *For admissible S we define $F(S) : \Omega \times [0, \infty) \to \mathbb{R}^2$ by*

$$F(S)_t := \left(\overline{F(S)}_t, \underline{F(S)}_t\right) := \left(g\left(\overline{S}_t, \pi_{t-}^S\right), h\left(\underline{S}_t, \pi_{t-}^S\right)\right)$$

where g and h are defined in Definition 3.4 and π^S in (4.6).

As a continuous function of $\mathbb{F}^S$-predictable processes $F(S)$ is $\mathbb{F}^S$-predictable. By definition of g and h and the fact that S is admissible it follows that $\overline{F(S)}_t(\omega) \geq \underline{F(S)}_t(\omega)$ for all (ω, t). However, $F(S)$ is not necessarily admissible, since in general $\mathbb{F}^S \neq \mathbb{F}^{F(S)}$.

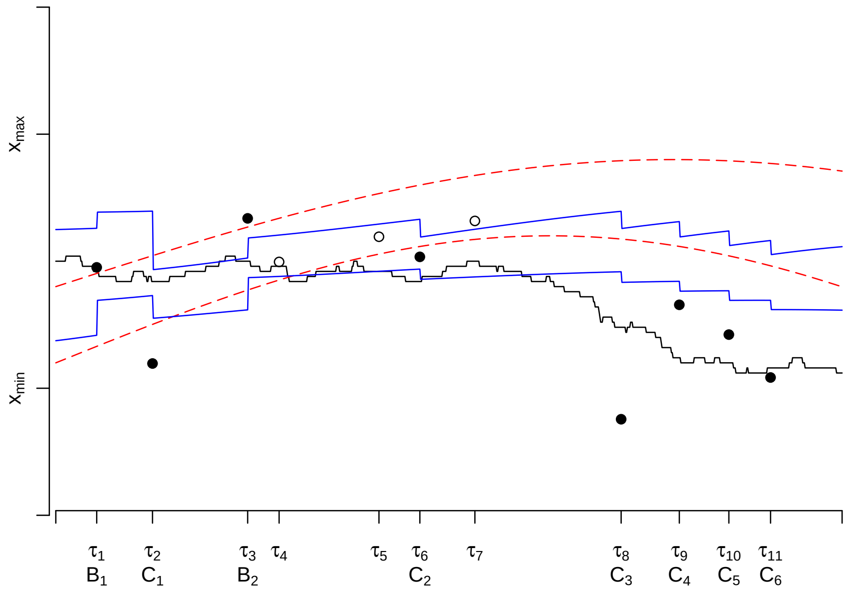

Figure 4.2: We add $F(S)$ to Figure 4.1 which are the fictitious Glosten-Milgrom-prices (i.e. zero-expected-profits) of the market maker if actually prices S are quoted and the market reacts with buys and sells to them.

We now can prove the following Lemma.

Lemma 4.12. *We have*

$$\overline{F(S)}_{B_i} = E[X_{B_i}|\mathcal{F}^S_{B_i}] \qquad \textit{and} \qquad \underline{F(S)}_{C_i} = E[X_{C_i}|\mathcal{F}^S_{C_i}] \qquad P\text{-a.s.}$$

for all i.

Proof. By the filter equation in Lemma 4.10 and the fact that N^B and N^C have no common jumps it follows that

$$\pi^{S,k}_{B_i} = \frac{\pi^{S,k}_{B_i-}\,\Phi(\overline{S}_{B_i} - x_k)}{\sum_{j=1}^n \pi^{S,j}_{B_i-}\,\Phi(\overline{S}_{B_i} - x_j)}, \qquad \text{for all } k. \tag{4.8}$$

By definition and (4.8) we have

$$\begin{aligned}\overline{F(S)}_{B_i} &= g\left(\overline{S}_{B_i}, \pi^S_{B_i-}\right) = \frac{\sum_{j=1}^n x_j \pi^{S,j}_{B_i-} \Phi(\overline{S}_{B_i} - x_j)}{\sum_{j=1}^n \pi^{S,j}_{B_i-} \Phi(\overline{S}_{B_i} - x_j)} \\ &= \sum_{j=1}^n x_j \pi^{S,j}_{B_i} = E[X_{B_i} | \mathcal{F}^S_{B_i}].\end{aligned}$$

The same holds true at the times when a sell occurs. □

The importance of that assertion becomes clear if we compare it with Theorem 4.5, which states that for a GMPS $E[X_{B_i}|\mathcal{F}^S_{B_i}]$ is equal to S_{B_i}. This leads to an intuitive description of F. $F(S)$ are the Glosten-Milgrom-prices (i.e. zero-expected-profits) a market maker would have in mind if actually prices S are quoted and the market reacts with buys and sells to them.

Definition 4.13. *We say that S is a fixed point of F, if $S = F(S)$ $P \otimes \lambda$-a.e. (where λ denotes the Lebesgue-measure on $\mathbb{R}_+$).*

Theorem 4.14. *An admissible strategy S is a solution of the GMPS-problem iff S is a fixed point of F.*

Proof. Identity of predictable sets B up to a $P \otimes \lambda$ null set is equivalent to equality at Poisson times, since

$$P \otimes \lambda(B) = \sum_{j=1}^{\infty} P[(\omega, \tau_j(\omega)) \in B].$$

Let S be a fixed point of F. With Lemma 4.12 we obtain

$$\overline{S}_{B_i} = \overline{F(S)}_{B_i} = E[X_{B_i}|\mathcal{F}^S_{B_i}] \qquad P\text{-a.s.}$$

for all i. Lemma 4.5 now yields that S is a solution.

Now let S be a solution of the GMPS-problem and j be fixed. We consider the set

$$A := \left\{\overline{S}_{\tau_j} \neq \overline{F(S)}_{\tau_j}\right\}$$

which is in $\mathcal{F}_{\tau_j-}$ since S and $F(S)$ are $\mathbb{F}^S$-predictable and hence $\mathbb{F}$-predictable.

Since S is a solution and ϵ_j is independent of $\mathcal{F}_{\tau_j-}$ we have

$$\begin{aligned}0 &= P\left[\overline{S}_{B_i} \neq \overline{F(S)}_{B_i} \text{ for some } i\right] \\ &\geq P[\{\tau_j = B_i \text{ for some } i\} \cap A] = P[\{X_{\tau_j} + \epsilon_j \geq \overline{S}_{\tau_j}\} \cap A] \\ &\geq P[\{\epsilon_j \geq C\} \cap A] = P[A]P[\epsilon_j \geq C].\end{aligned}$$

Since by Lemma 3.7 $P[\epsilon_j \geq C] > 0$ it follows that $P[A] = 0$, in other words, for all τ_j we have

$$\overline{S}_{\tau_j} = \overline{F(S)}_{\tau_j} \qquad P\text{-a.s..}$$ □

Corollary 4.15. *For any GMPS S it holds up to a $P \otimes \lambda$-null set that*

$$S = (G(\pi_-^S), H(\pi_-^S)),$$

where π_-^S is the left-continuous version of the process of the conditional probabilities belonging to S and where G and H are the functions that give the static solutions defined in Definition 3.6.

Proof. Let S be a solution. Then up to a $P \otimes \lambda$-null set it holds that

$$\overline{S}_t = \overline{F(S)}_t = g\left(\overline{S}_t, \pi_{t-}^S\right)$$

by Theorem 4.14 and the definition of F. From this it follows that

$$\overline{S}_t = G(\pi_{t-}^S)$$

by the definition of G in Definition 3.6. □

Chapter 5

Noise with density

In this chapter we show that for a special choice of the ϵ_i the GMPS exists and is unique. This choice corresponds to the assumptions in the static case in Section 3.3, but here we treat buys and sells simultaneously. The result is formulated in the following theorem, which is also the main result of [KR13].

Theorem 5.1. *Let $C := x_{\max} - x_{\min}$ and $\Phi(y) := P[\epsilon_1 \geq y]$ for $y \in \mathbb{R}$ as in Definition 4.8. Assume that Φ is differentiable (i.e. the distribution of ϵ_1 has density $-\Phi'$) on $[-C, C]$, $1 > \Phi(0) > 0$, and*

$$-\Phi'(y) \leq \frac{K}{C} \min\{\Phi(y), 1 - \Phi(y)\}$$

for all $y \in [-C, C]$ and a constant $K < 1$. Then, there exists a Glosten-Milgrom pricing strategy and it is unique up to a $(P \otimes \lambda)$-null set, where λ denotes the Lebesgue measure on $\mathbb{R}_+$.

The rest of the chapter proves this theorem. In the first section we show that F from Definition 4.11 is a contraction (see Lemma 5.4) which can be used to verify uniqueness. In the second section we show by construction that F possesses a fixed point, which is necessary as F does in general not map into the set of admissible strategies and the contraction holds only for admissible sets. Hence, we cannot use a Picard iteration.

5.1 Uniqueness

We first show uniqueness of the solution by proving that F is a contraction. Let S and T be admissible pricing strategies.

Definition 5.2. *For given pricing strategies S, T let*

$$A_s^1 := \{X_{\tau_i} + \epsilon_i \notin [\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}, \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}) \text{ for all } \tau_i \leq s\}.$$

A_s^1 is the event that until s no buy occurred only in one of the two pricing scenarios S and T.

Lemma 5.3. *We have that*

$$P\left[(A_s^1)^c\right] \leq \lambda M E\left[\int_0^s |\overline{S}_u - \overline{T}_u| du\right]$$

where $M := \max\{\Phi'(x) | x \in [-C, C]\}$

Proof. Let Y be the process that counts the number of buys that only occur for one pricing strategy, i.e.

$$Y_t := \sum_{i\in\mathbb{N}} 1_{\left\{\tau_i \leq t, X_{\tau_i}+\epsilon_i \in [\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}, \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\})\right\}}.$$

We now show (essentially with the methods of the proof of Lemma 4.9) that the $\mathbb{F}$-intensity of Y is given by

$$\lambda_t^Y := \lambda(\Phi(\min\{\overline{S}_t, \overline{T}_t\} - X_{t-}) - \Phi(\max\{\overline{S}_t, \overline{T}_t\} - X_{t-})) \leq \lambda M |\overline{S}_t - \overline{T}_t|.$$

Let C be a nonnegative $\mathbb{F}$-predictable process. As $\overline{S}$ and $\overline{T}$ are $\mathbb{F}$-predictable and $P[X_{\tau_i} = X_{\tau_i -}] = 1$, we obtain

$$\begin{aligned}
E\left[\int_0^\infty C_s dY_s\right] &= E\left[\sum_{i=1}^\infty C_{\tau_i} 1_{\{\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\} \leq X_{\tau_i}+\epsilon_i < \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}\}}\right] \\
&= \sum_{i=1}^\infty E\left[E\left[C_{\tau_i} 1_{\{\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\} \leq X_{\tau_i}+\epsilon_i < \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}\}} | \mathcal{F}_{\tau_i -}\right]\right] \\
&= \sum_{i=1}^\infty E\left[C_{\tau_i} \lambda_{\tau_i}^Y\right] = E\left[\int_0^\infty C_s \lambda_s^Y ds\right].
\end{aligned}$$

We define $\tau_Y := \inf\{t \geq 0 \mid Y_t = 1\}$ and get

$$\begin{aligned}
P[(A_s^1)^c] = P[Y_s \neq 0] &= E\left[\int_0^s 1_{\{\tau_Y \geq u\}} dY_u\right] = E\left[\int_0^s 1_{\{\tau_Y \geq u\}} \lambda_u^Y du\right] \\
&\leq E\left[\int_0^s \lambda_u^Y du\right] \leq \lambda M E\left[\int_0^s |\overline{S}_u - \overline{T}_u| du\right]. \qquad \square
\end{aligned}$$

The same holds true for sells. Hence for

$$A_s^2 := \{X_{\tau_i} + \epsilon_i \notin (\min\{\underline{S}_{\tau_i}, \underline{T}_{\tau_i}\}, \max\{\underline{S}_{\tau_i}, \underline{T}_{\tau_i}\}] \text{ for all } \tau_i \leq s\}$$

we obtain a similar estimate and for $A_s := A_s^1 \cap A_s^2$, which is the event that the same buys and sells are observed in the two pricing strategies S and T we have

$$P\left[A_s^c\right] \leq 2\lambda M E\left[\int_0^s \|S_u - T_u\|\, du\right] \qquad \text{where} \tag{5.1}$$

$$\|S_u - T_u\| := \max\left\{|\overline{S}_u - \overline{T}_u|, |\underline{S}_u - \underline{T}_u|\right\}.$$

Lemma 5.4. *There is a constant $K_1 < \infty$ such that*

$$E\left[\int_0^t \|F(S)_s - F(T)_s\|\, ds\right] \leq (K + tK_1) E\left[\int_0^t \|S_s - T_s\|\, ds\right]$$

for all $t \geq 0$ and for K from Theorem 5.1.

Proof. First we estimate the difference of the conditional distributions of the true value resulting from different pricing strategies. We obtain

$$\begin{aligned}
E\left[\left|\pi_s^{S,i} - \pi_s^{T,i}\right|\right] &= E\left[\left|P\left[X_s = x_i|\mathcal{F}_s^S\right] - P\left[X_s = x_i|\mathcal{F}_s^T\right]\right| \left(1_{A_s} + 1_{A_s^c}\right)\right] \\
&\leq E\left[\left|E\left[1_{\{X_s = x_i\}}|\mathcal{F}_s^S\right] - E\left[1_{\{X_s = x_i\}}|\mathcal{F}_s^T\right]\right| 1_{A_s}\right] \\
&\quad + E\left[1_{A_s^c}\right] \\
&= P\left[A_s^c\right] + E[\left|E\left[1_{\{X_s = x_i\}}\left(1_{A_s^c} + 1_{A_s}\right)\right| \mathcal{F}_s^S\right] \\
&\quad - E\left[1_{\{X_s = x_i\}}\left(1_{A_s^c} + 1_{A_s}\right)|\mathcal{F}_s^T\right] |1_{A_s}] \\
&\leq 3P\left[A_s^c\right] + E[|E\left[1_{\{X_s = x_i\}} 1_{A_s}|\mathcal{F}_s^S\right] \\
&\quad - E\left[1_{\{X_s = x_i\}} 1_{A_s}|\mathcal{F}_s^T\right] |1_{A_s}] \\
&= 3P\left[A_s^c\right], \qquad i = 1, \ldots, n.
\end{aligned} \tag{5.2}$$

The last equation holds true since $\mathcal{F}_s^S \cap A_s = \mathcal{F}^T \cap A_s$. The equality of the trace σ-algebras holds due to

$$\mathcal{F}_s^S \cap A_s = \sigma(\{B_i^S \leq u\}, \{C_i^S \leq u\}, u \leq s, i \in \mathbb{N}) \cap A_s,$$

and obviously

$$\{B_i^S \leq u\} \cap A_s = \{B_i^T \leq u\} \cap A_s \text{ and } \{C_i^S \leq u\} \cap A_s = \{C_i^T \leq u\} \cap A_s$$

respectively for all $i \in \mathbb{N}$ and $u \leq s$. Putting (5.1) and (5.2) together we obtain

$$\begin{aligned} E\left[\int_0^t \sum_{i=1}^n |\pi_s^{S,i} - \pi_s^{T,i}| ds\right] &= \int_0^t \sum_{i=1}^n E\left[|\pi_s^{S,i} - \pi_s^{T,i}|\right] ds \\ &\leq \int_0^t \sum_{i=1}^n 3P\left[A_s^c\right] ds \\ &\leq 6n\lambda M \int_0^t E\left[\int_0^s \|S_u - T_u\| \, du\right] ds \\ &\leq 6n\lambda M t E\left[\int_0^t \|S_s - T_s\| \, ds\right]. \end{aligned}$$

Finally, we have

$$\begin{aligned} E\left[\int_0^t \left|\overline{F(S)}_s - \overline{F(T)}_s\right| ds\right] &= E\left[\int_0^t \left|g\left(\overline{S}_s, \pi_{s-}^S\right) - g\left(\overline{T}_s, \pi_{s-}^T\right)\right| ds\right] \\ &\leq E\left[\int_0^t K|\overline{S}_s - \overline{T}_s| + L\sum_{i=1}^n |\pi_s^{S,i} - \pi_s^{T,i}| ds\right] \\ &\leq E\left[\int_0^t K|\overline{S}_s - \overline{T}_s| + 6Ln\lambda Mt \|S_s - T_s\| \, ds\right], \end{aligned}$$

where the first inequality is due to Lemma 3.9 for $K < 1$ defined in Theorem 5.1 and $L = \frac{2x_{\max}}{\Phi(C)^2}$. A similar estimate can be obtained for

$$E\left[\int_0^t \left|\underline{F(S)}_s - \underline{F(T)}_s\right| ds\right].$$

We then get the desired result with $K_1 = 12Ln\lambda M$. □

Proof of uniqueness in Theorem 5.1. Let S, T be two solutions of the GMPS problem. By Theorem 4.14 every solution is a fixed point of F. Applying Lemma 5.4 with some $t > 0$ s.t. $K + tK_1 < 1$ we obtain that S and T coincide $P \otimes \lambda|_{[0,t]}$-a.e.. Note that K and K_1 only depend on $x_{\min}, x_{\max}$, the distribution of the ϵ_i and λ, but it is independent of the probabilities $P[X_0 = x_i]$.

But if $S = T$ $P \otimes \lambda|_{[0,t]}$-a.e. so are the P-completions of $\mathcal{F}_t^S$ and $\mathcal{F}_t^T$. Iteratively it follows that S and T are equal on $[0, \infty)$ as all arguments from above hold true also for a non-trivial $\mathcal{F}_0$. □

5.2 Existence

To show existence we will proceed as follows. We will define an n-dimensional process ϕ as a pathwise solution of a stochastic integral equation, which is what we assume the conditional distribution of the true value under the filtration of a GMPS could look like. We then define prices as the static solutions for every (ω, t), plugging in the conditional distribution of the true value, and construct the corresponding market maker's filtration. Then, we show that, under the constructed filtration, ϕ is adapted and solves the filter equation of the conditional distribution of the true value. This shows with the results in Section 4.4 that we have indeed constructed a GMPS.

We recall Definition 3.6. Let $\phi \in [0,1]^n$ such that $\sum_{i=1}^n \phi_i = 1$ that we interpret as probabilities of a random variable with state space $x_1, \dots, x_n$. With $G(\phi)$ and $H(\phi)$ we denote the unique solutions s of

$$g(s, \phi) = s \text{ and } h(s, \phi) = s$$

respectively where g and h are defined in Definition 3.4. The existence and uniqueness of that solution is secured by Theorem 3.8.

We also recall Definition 4.8 of $\Phi(x) = P[\epsilon_1 \geq x]$ and $\Psi(x) = P[\epsilon_1 \leq x]$.

Proof of existence in Theorem 5.1. Step 1: For $\phi : \Omega \times [0, \infty) \to [0,1]^n$ we consider the SDE

$$\begin{aligned}
\phi_t^i = \phi_0^i &+ \sum_{\tau_k \leq t} \phi^i_{\tau_k -} \left(\frac{\Phi(G(\phi_{\tau_k -}) - x_i)}{\sum_j \phi^j_{\tau_k -} \Phi(G(\phi_{\tau_k -}) - x_j)} - 1 \right) 1_{\{X_{\tau_k} + \epsilon_k \geq G(\phi_{\tau_k -})\}} \\
&+ \sum_{\tau_k \leq t} \phi^i_{\tau_k -} \left(\frac{\Psi(H(\phi_{\tau_k -}) - x_i)}{\sum_j \phi^j_{\tau_k -} \Psi(H(\phi_{\tau_k -}) - x_j)} - 1 \right) 1_{\{X_{\tau_k} + \epsilon_k \leq H(\phi_{\tau_k -})\}} \\
&- \int_0^t \Bigg(\lambda \phi_s^i \Bigg(\Psi(H(\phi_s) - x_i) + \Phi(G(\phi_s) - x_i) \\
&- \sum_j \phi_s^j \left(\Psi(H(\phi_s) - x_j) + \Phi(G(\phi_s) - x_j) \right) \Bigg) \\
&- \sum_j \phi_s^j q(j, i) \Bigg) \, ds
\end{aligned} \tag{5.3}$$

with initial condition $\phi_0^i = P[X_0 = x_i]$ for all $i = 1, \ldots, n$. In a first step we consider this SDE only pathwise and show that it has a unique solution with càdlàg paths (we do not have a filtration yet).

G (and H) are Lipschitz-continuous by Lemma 3.10. Further, the functions Φ and Ψ are differentiable and the derivative is bounded by $\frac{K}{C} < \infty$ on the compact set $[-C, C]$. In addition, Φ and Ψ are bounded by one. By the product rule, it follows that the ds-term in (5.3) considered as a function in ϕ can be modified to a function $f(\phi)$ that is Lipschitz-continuous and f coincides with the original function for all $\phi \in \mathbb{R}^n$ with $\phi^i \geq 0$ and $\sum_{i=1}^n \phi^i = 1$. Then, the system of ordinary differential equations only consisting of the modified ds-terms has a unique solution and, by construction of the ODEs, the solution stays in the set of probabilities. Thus, it also solves the differential equations with the original ds-terms, i.e.

$$\begin{aligned} d\psi_t^i = - \Bigg(\lambda \psi_t^i \Bigg(& \Psi(H(\psi_t) - x_i) + \Phi(G(\psi_t) - x_i) \\ & - \sum_j \psi_t^j \left(\Psi(H(\psi_t) - x_j) + \Phi(G(\psi_t) - x_j) \right) \Bigg) \\ & - \sum_j \psi_t^j q(j,i) \Bigg) \, dt. \end{aligned}$$

We can now construct a candidate for the original problem up to τ_1 by this solution, i.e. $\phi_t := \psi_t$ for all $t < \tau_1$, and

$$\begin{aligned} \phi_{\tau_1}^i = & \psi_{\tau_1-}^i + \psi_{\tau_1-}^i \left(\frac{\Phi(G(\psi_{\tau_1-}) - x_i)}{\sum_j \psi_{\tau_1-}^j \Phi(G(\psi_{\tau_1-}) - x_j)} - 1 \right) 1_{\{X_{\tau_1} + \epsilon_1 \geq G(\psi_{\tau_1-})\}} \\ & + \psi_{\tau_1-}^i \left(\frac{\Psi(H(\psi_{\tau_1-}) - x_i)}{\sum_j \psi_{\tau_1-}^j \Psi(H(\psi_{\tau_1-}) - x_j)} - 1 \right) 1_{\{X_{\tau_1} + \epsilon_1 \leq H(\psi_{\tau_1-})\}}. \end{aligned}$$

We also obtain a solution $\widetilde{\psi}$ of the ordinary differential equation above for every state of $\phi_{\tau_1}^i$ as initial condition. Considered as a parameter-depending differential equation, the solution is continuous in the initial condition. We then define a solution of the original problem on (τ_1, τ_2) by

$$\phi_t = \widetilde{\psi}_{t-\tau_1}$$

and so on. Iteratively we obtain a process that satisfies (5.3) up to all τ_i.

Then, one may define $\phi_t^i(\omega) = 1/n$ for $t \in \mathbb{R}_+$ with $t \geq \sup_{i\in\mathbb{N}} \tau_i(\omega)$. As τ_i are Poisson times, this definition of course only affects a P-null set, but the construction ensures measurability (see Step 2) without needing the usual conditions and without the additional assumption that $\sup_{i\in\mathbb{N}} \tau_i(\omega) = \infty$ for all $\omega \in \Omega$. The process $\phi : \Omega \times \mathbb{R}_+ \mapsto \mathbb{R}^n$ has càdlàg paths at least on $[0, \sup_{i\in\mathbb{N}} \tau_i(\omega))$.

Step 2: We now define

$$S_t := (G(\phi_{t-}), H(\phi_{t-}))$$

on $(0, \sup_{i\in\mathbb{N}} \tau_i(\omega))$ (and maybe $S = (x_{\max}, x_{\min})$ elsewhere) and with it N^B, N^C (with jump times B_k respectively C_k) and $\mathbb{F}^S$ according to Definitions 4.2 and 4.2 respectively. It follows that the jumps in (5.3) only take place at actual buys and sells with prices S. Therefore and by the construction of ϕ (using the continuity in the initial condition), for every $t \in \mathbb{R}$, ϕ_t can be written as a measurable function of $B_k 1_{\{B_k \leq t\}}$ and $C_k 1_{\{C_k \leq t\}}$, $k \in \mathbb{N}$. Thus ϕ_t is $\mathcal{F}_t^S$-measurable, i.e. ϕ is $\mathbb{F}^S$-adapted.

It follows that the process S that is left-continuous on $(0, \sup_{i\in\mathbb{N}} \tau_i(\omega))$ is $\mathbb{F}^S$-predictable and hence admissible in the sense of Definition 4.4. Note that by the pathwise construction we obtain pricing strategies that are $\mathbb{F}^S$-predictable and not only predictable w.r.t. the completed filtration $\widetilde{\mathbb{F}}^S$ that satisfies the usual conditions. By (4.2) we can write (5.3) as

$$\begin{aligned} d\phi_t^i =& \phi_{t-}^i \left(\frac{\Phi(\overline{S}_t - x_i)}{\sum_j \phi_{t-}^j \Phi(\overline{S}_t - x_j)} - 1 \right) dN_t^B \\ &+ \phi_{t-}^i \left(\frac{\Psi(\underline{S}_t - x_i)}{\sum_j \phi_{t-}^j \Psi(\underline{S}_t - x_j)} - 1 \right) dN_t^C \\ &- \Bigg(\lambda \phi_t^i \Bigg(\Psi(\underline{S}_t - x_i) + \Phi(\overline{S}_t - x_i) \\ &- \sum_j \phi_t^j \left(\Psi(\underline{S}_t - x_j) + \Phi(\overline{S}_t - x_j) \right) \Bigg) \\ &- \sum_j \phi_t^j q(j,i) \Bigg) dt. \end{aligned} \tag{5.4}$$

Step 3: We described the filter equation of $\pi_t^{S,i} = P\left[X_t = x_i | \mathcal{F}_\tau^S\right]$ in

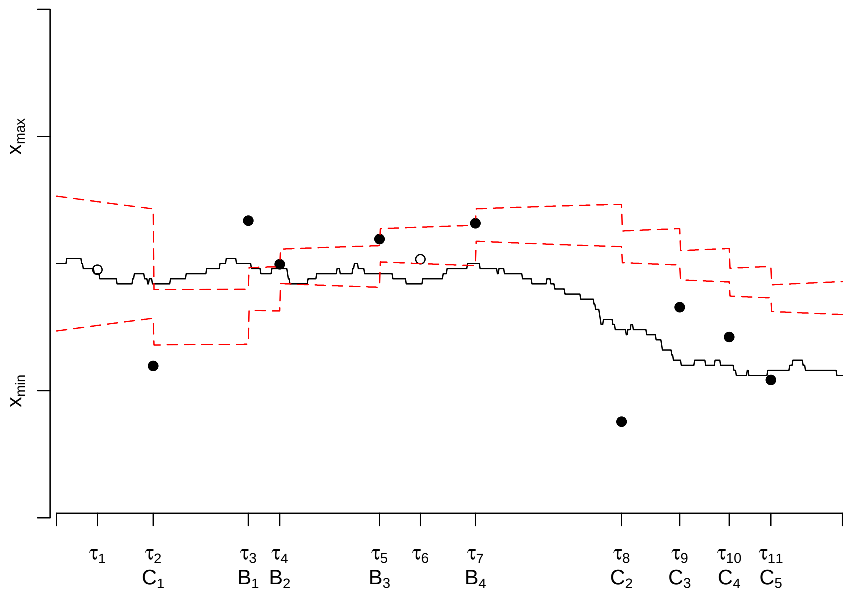

Figure 5.1: Glosten-Milgrom-prices for the same scenario ω as in Figure 4.1.

Lemma 4.10. It is given by

$$\begin{aligned} d\pi_t^{S,i} =& \pi_{t-}^{S,i} \left(\frac{\Phi(\overline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j} \Phi(\overline{S}_t - x_j)} - 1 \right) dN_t^B \\ &+ \pi_{t-}^{S,i} \left(\frac{\Psi(\underline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j} \Psi(\underline{S}_t - x_j)} - 1 \right) dN_t^C \\ &- \Bigg(\lambda \pi_t^{S,i} \Bigg(\Psi(\underline{S}_t - x_i) + \Phi(\overline{S}_t - x_i) \\ &- \sum_j \pi_t^{S,j} \left(\Psi(\underline{S}_t - x_j) + \Phi(\overline{S}_t - x_j) \right) \Bigg) \\ &+ \sum_j \pi_t^{S,j} q(j,i) \Bigg) dt. \end{aligned} \tag{5.5}$$

Note that S depends on ϕ and is fixed in (5.5). In terms of π^S (5.5) has a unique solution and ϕ is obviously a solution of this equation (uniqueness follows as the dt-term considered as a function of π^S is Lipschitz-continuous, thus the arguments are similar but simpler as for (5.3)). Thus, as ϕ and π^S

are both càdlàg , they are indistinguishable. Since S is then given by

$$S_t := (G(\pi_{t-}), H(\pi_{t-}))$$

the result follows from Corollary 4.15. □

Chapter 6

Insider/noise trader model

In the static case the existence and uniqueness of Glosten-Milgrom prices for both choices of ϵ were quite easy to obtain (see Theorem 3.8 and Theorem 3.11, respectively). For the insider/noise trader model, where ϵ obtains only the values 0 or $\pm\infty$, we have this result for the broadest imaginable class of distributions of X. Transferring this to the continuous-time model we have (for fixed t) unique prices given the conditional distribution of X_t up to a null set N_t that depends on t. It is a common observation in stochastic analysis that the union of this sets over all $t > 0$ is not necessarily a null set any more.

In the last chapter we showed that existence and uniqueness hold for the "noise with density"- model. However, the pathwise construction of a solution that we did in *Step 1* in the proof in Section 5.2 does not work in the insider/noise trader model, since the functions Φ and Ψ are by the nature of the distribution of ϵ not Lipschitz-continuous, but have jumps at zero. The behavior at the points where this discontinuity is relevant are the points where the prices hit the states of X, because then a regime change will occur. It might be possible to take this points into account during the construction. We will not do this here, but focus on the uniqueness aspect of the problem.

The uniqueness result in the last chapter depends on the Lipschitz-continuity of g in the first argument in Lemma 5.4. This requirement is not fulfilled here as we showed in (3.5) and (3.6).

Also Lemma 5.3 that is central to the uniqueness argument does not hold as is shown in the following. Remember that

$$A_s^1 := \{X_{\tau_i} + \epsilon_i \notin [\min\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}, \max\{\overline{S}_{\tau_i}, \overline{T}_{\tau_i}\}) \text{ for all } \tau_i \leq s\}$$

is the event that until s no buy occurred only in one of two pricing scenarios S and T. Lemma 5.3 states that the probability of the complement of this event, i.e. there is at least one buy only in one of the two scenarios, is small if the difference of S and T is small. Broadly speaking, if the pricing is about the same, also the same things will happen. However, if $\overline{S}$ is constant and equal to some state x_i of the true value process and $\overline{T} = x_i + \eta$ for some $\eta > 0$ then its clear that

$$E\left[\int_0^s |\overline{S}_u - \overline{T}_u| du\right] = E\left[\int_0^s \eta du\right] = s\eta$$

can be chosen to be arbitrarily small by the choice of η, whereas

$$P\left[(A_s^1)^c\right] \geq P[\tau_1 < s]P[\epsilon = 0]P[X_{\tau_1} = x_i]$$

does not depend on η. Hence, the Lemma does not hold for this choice of ϵ.

Furthermore, in the following we will give an example that the insider/noise trader model implies some surprising results. We construct two pricing scenarios up to the stopping time $\min\{T_1, \widetilde{t}\}$ where T_1 is the first transaction and $\widetilde{t} > 0$ is a constant defined below such that both are GMPS in a certain sense. It is not clear whether we can extend this pricing strategies to GMPS on $\mathbb{R}_+$, but since the two prices are symmetric this is a strong indication that the prices are not generally unique in the insider/noise trader model.

6.1 A counterexample to uniqueness

We choose ϵ to be distributed like

$$\epsilon = \begin{cases} \infty & \text{with probability} \quad \mu = \frac{1}{3} \\ 0 & \qquad\qquad\qquad\quad \nu = \frac{1}{3} \\ -\infty & \qquad\qquad\qquad\quad 1 - \mu - \nu = \frac{1}{3}. \end{cases} \tag{6.1}$$

and we consider the state space

$$\{x_1, x_2, x_3, x_4\} = \{-5, -1, 1, 5\} \tag{6.2}$$

and the vector

$$\pi_0 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \tag{6.3}$$

which we define as the the initial distribution of the process X_t, i.e. $P[X_0 = x_i] = \frac{1}{4}$ for all i. Further we assume that the transition kernel (see (4.1) for the definition) of X is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ q & -q & 0 & 0 \\ 0 & 0 & -q & q \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.4}$$

which means that X jumps with rate q from x_2 to x_1 and from x_3 to x_4 and stays otherwise constant. We assume that

$$q < \frac{\lambda}{6}, \tag{6.5}$$

where λ is the rate of the customer arrival process. Note, that this setting is symmetric around zero. In the following we will construct a solution that is not symmetric. It is clear that this solution reflected at zero is also a solution.

However, before we start to consider how this model can evolve, note that x_3 is the static Glosten-Milgrom ask price in the initial situation. Remember that by $G(\phi)$ we denote the unique solution for given ϕ. $G(\phi)$ must by (3.7) and the definition of μ and ν confine

$$G(\phi) = E[X] + E[(X - G(\phi))1_{\{X \geq G(\phi)\}}]. \tag{6.6}$$

We verify that $G(\pi_0) = 1 = x_3$ by

$$G(\pi_0) = 0 + \left(\pi_0^3(x_3 - x_3) + \phi_0^4(x_4 - x_3)\right) = \frac{1}{4}(5 - 1) = 1.$$

For analog reasons x_2 is the Glosten-Milgrom bid price at the initial situation. Thus, we are faced with the situation that both prices are at a state which are the points where Φ and Ψ are not Lipschitz-continuous.

The aim of the following is not to construct a general GMPS in the sense of Definition 4.4, but rather we construct it for a special stopping time $\tau \leq \min\{T_1, \widetilde{t}\}$ where T_1 is the first transaction and $\widetilde{t} > 0$ is a constant. In this way we avoid the problems that originate from the fact that the distribution functions are not Lipschitz. Hence, we impose the condition that

$$E\left[1_{\{B_1 \leq \tau\}}(\overline{S}_{B_1} - X_{B_1})\right] = 0 \text{ and } E\left[1_{\{C_1 \leq \tau\}}(\underline{S}_{C_1} - X_{C_1})\right] = 0 \tag{6.7}$$

for every $\mathbb{F}^S$-stopping time $\tau \leq \min\{T_1, \widetilde{t}\}$. Of course the filter equation from Lemma 4.10 and the fact that for fixed t any solution is the static solution (Corollary 4.15) still hold true.

We define an ordinary differential equation for $\phi \in \mathbb{R}^4$ which we can think of conditional probabilities of $X = x_i$ up to τ. Let $\phi_0 = \pi_0$ and

$$\begin{aligned} \frac{d\phi_t^1}{dt} &= -\frac{\lambda}{3}\phi_t^3\phi_t^1 + q\phi_t^2 \\ \frac{d\phi_t^2}{dt} &= -\frac{\lambda}{3}\phi_t^3\phi_t^2 - q\phi_t^2 \\ \frac{d\phi_t^3}{dt} &= \frac{\lambda}{3}\phi_t^3(1-\phi_t^3) - q\phi_t^3 \\ \frac{d\phi_t^4}{dt} &= -\frac{\lambda}{3}\phi_t^3\phi_t^4 + q\phi_t^3. \end{aligned} \tag{6.8}$$

This ordinary differential equation with Lipschitz-continuous coefficients has a unique solution that we will also denote by ϕ. We now consider the process

$$\gamma(t) := \Gamma(\phi_t) := \frac{\sum_{i=1}^4 \phi_t^i x_i + \phi_t^4 x_4}{1 + \phi_t^4}, \tag{6.9}$$

which will be the ask price up to time τ later. In the following we want to calculate $\gamma'(0)$. (6.9) can be rewritten as

$$\gamma(t)(1 + \phi_t^4) = \sum_{i=1}^4 \phi_t^i x_i + \phi_t^4 x_4$$

and differentiation yields

$$\gamma'(t)(1+\phi_t^4) + \gamma(t)\frac{d\phi_t^4}{dt} = \frac{d\left(\sum_{i=1}^4 \phi_t^i x_i + \phi_t^4 x_4\right)}{dt}. \tag{6.10}$$

We calculate the RHS and obtain

$$\begin{aligned} \frac{d\left(\sum_{i=1}^4 \phi_t^i x_i + \phi_t^4 x_4\right)}{dt} &= \left(-\frac{\lambda}{3}\phi_t^3\phi_t^1 + q\phi_t^2\right)(-5) + \left(-\frac{\lambda}{3}\phi_t^3\phi_t^2 - q\phi_t^2\right)(-1) \\ &\quad + \left(\frac{\lambda}{3}\phi_t^3(1-\phi_t^3) - q\phi_t^3\right) + 2\left(-\frac{\lambda}{3}\phi_t^3\phi_t^4 + q\phi_t^3\right) 5. \end{aligned}$$

For $t = 0$, i.e. $\phi_0 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ in (6.10) we obtain

$$\begin{aligned}\gamma'(0)\frac{5}{4}+\gamma(0)\left(-\frac{\lambda}{3}\frac{1}{16}+\frac{q}{4}\right)=&\frac{\lambda}{3}\frac{5}{16}-\frac{5q}{4}+\frac{\lambda}{3}\frac{1}{16}+\frac{q}{4}\\&+\frac{\lambda}{3}\frac{3}{16}-\frac{q}{4}-\frac{\lambda}{3}\frac{10}{16}+\frac{10q}{4}\\=&\frac{5q}{4}-\frac{\lambda}{3}\frac{1}{16}.\end{aligned}$$

Since $\gamma(0)=1$ we have

$$\gamma'(0)=\left(\frac{5q}{4}-\frac{\lambda}{3}\frac{1}{16}+\frac{\lambda}{3}\frac{1}{16}-\frac{q}{4}\right)\frac{4}{5}=\frac{4}{5}q.$$

Further we consider the process

$$\delta(t):=\Delta(\phi_t):=\frac{\sum_{i=1}^{4}\phi_t^i x_i+\phi_t^1x_1+\phi_t^2x_2}{1+\phi_t^1+\phi_t^2}, \tag{6.11}$$

which will be the bid price up to time τ later. Again we want to calculate $\delta'(0)$. We write (6.11) as

$$\delta(t)\left(1+\phi_t^1+\phi_t^2\right)=\sum_{i=1}^{4}\phi_t^i x_i+\phi_t^1x_1+\phi_t^2x_2$$

and differentiate to obtain

$$\delta'(t)\left(1+\phi_t^1+\phi_t^2\right)+\delta(t)\left(\frac{d\phi_t^1}{dt}+\frac{d\phi_t^2}{dt}\right)=\frac{d\left(\sum_{i=1}^{4}\phi_t^i x_i+\phi_t^1x_1+\phi_t^2x_2\right)}{dt}. \tag{6.12}$$

The RHS is given by

$$\begin{aligned}\frac{d\left(\sum_{i=1}^{4}\phi_t^i x_i+\phi_t^1x_1+\phi_t^2x_2\right)}{dt}=&2\left(-\frac{\lambda}{3}\phi_t^3\phi_t^1+q\phi_t^2\right)(-5)\\&+2\left(-\frac{\lambda}{3}\phi_t^3\phi_t^2-q\phi_t^2\right)(-1)\\&+\left(\frac{\lambda}{3}\phi_t^3(1-\phi_t^3)-q\phi_t^3\right)\\&+\left(-\frac{\lambda}{3}\phi_t^3\phi_t^4+q\phi_t^3\right)5.\end{aligned}$$

For $t=0$, i.e. $\phi_0=\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ in (6.12) we get

$$\begin{aligned}\delta'(0)\frac{6}{4}+\delta(0)\left(-\frac{\lambda}{3}\frac{1}{16}+\frac{q}{4}-\frac{\lambda}{3}\frac{1}{16}-\frac{q}{4}\right)=&\frac{\lambda}{3}\frac{10}{16}-\frac{10q}{4}+\frac{\lambda}{3}\frac{2}{16}+\frac{2q}{4}\\&+\frac{\lambda}{3}\frac{3}{16}-\frac{q}{4}-\frac{\lambda}{3}\frac{5}{16}+\frac{5q}{4}\\=&\frac{\lambda}{3}\frac{10}{16}-q.\end{aligned}$$

As $\delta(0)=-1$ we obtain

$$\delta'(0)=\left(\frac{\lambda}{3}\frac{10}{16}-q-\frac{\lambda}{3}\frac{2}{16}\right)\frac{4}{6}=\frac{\lambda}{9}-\frac{2}{3}q.$$

It follows that $\gamma'(0)=\frac{4}{5}q>0$ and $\delta'(0)=\frac{\lambda}{9}-\frac{2}{3}q$, which is positive by (6.5). Since the process of probabilities ϕ is continuous and γ and δ are continuous functions of ϕ, it follows that γ and δ are continuous functions in t with positive derivative at zero. Hence, there exists a time $\widetilde{t}>0$ up to which γ and δ are strictly increasing and such that $\gamma(t)\in(1,5)$ and $\delta(t)\in(-1,1)$ for all $t<\widetilde{t}$. Remember that $\gamma(0)=1$ and $\delta(0)=-1$.

Hence for the prices $S_t=(\gamma(t),\delta(t))$ we can filter for the conditional probabilities $\pi_t^{S,i}=P\left[X_t=x_i|\mathcal{F}_t^S\right]$ up to $\tau=\min\{T_1,\widetilde{t}\}$. Since we are before $T_1=\min\{B_1,C_1\}$ we only need to consider the dt-term in Lemma 4.10. We will do this in detail only for π^1 since the other process of conditional probabilities are analog. The differential equation is given by

$$\begin{aligned}\frac{d\pi_t^1}{dt}=&-\lambda\pi_t^1\Bigg(\Psi(\delta(t)-x_1)+\Phi(\gamma(t)-x_1)\\&-\sum_j\pi_t^j\left(\Psi(\delta(t)-x_j)+\Phi(\gamma(t)-x_j)\right)\Bigg)\\&+\sum_j\pi_t^jq(j,i).\end{aligned}$$

Now

$$\Psi(\delta(t)-x_j)+\Phi(\gamma(t)-x_j)=1\qquad\text{for } j\in\{1,2,4\}$$

and

$$\Psi(\delta(t)-x_3)+\Phi(\gamma(t)-x_3)=\frac{2}{3}.$$

By the Definition of the transition matrix in (6.4) it is clear that

$$\sum_j \pi_t^j q(j,i) = \pi_t^2 q.$$

Altogether we get

$$\frac{d\pi_t^1}{dt} = -\lambda\pi_t^1 \left(1 - \pi_t^1 - \pi_t^2 - \frac{2}{3}\pi_t^3 - \pi_t^4\right) + \pi_t^2 q = -\frac{\lambda}{3}\pi_t^1\pi_t^3 + \pi_t^2 q.$$

Thus the process of conditional probabilities is equal to ϕ. Further we see that the functions Γ and Δ describe the solution of the static problem for given conditional distributions and under the condition that $\overline{S} \in (1, 5)$ and $\underline{S} \in (-1, 1)$. It is then clear that if there is a buy or sell before $\widetilde{t}$ which happens with positive probability the conditions for Glosten-Milgrom prices are satisfied. This construction proofs the following theorem.

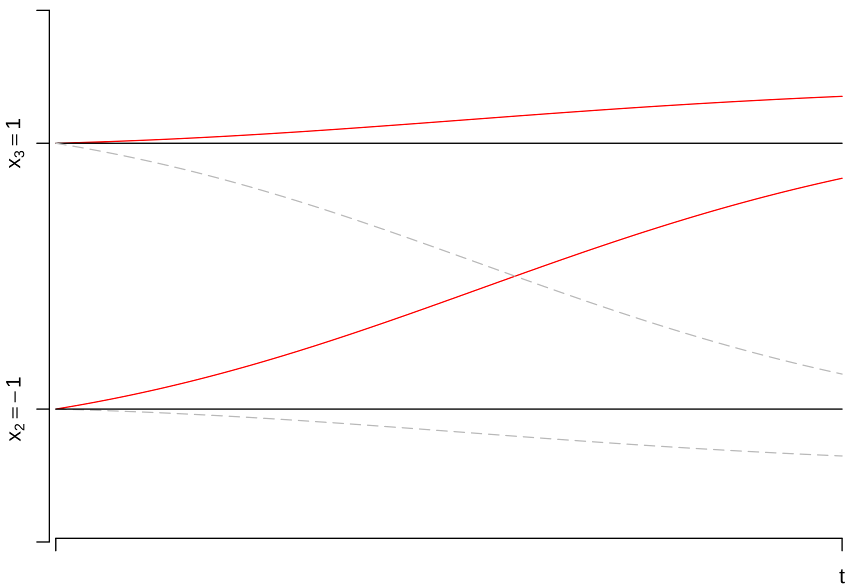

Figure 6.1: The ask and bid prices $\overline{S}$ and $\underline{S}$ of the counterexample that is constructed here are shown in red. No buy or sell has yet occurred. The second solution that is symmetric to the first one is indicated by the dashed grey lines. Both strategies fulfill the GMPS condition at τ and thus are valid strategies. The market maker can choose one of them. Hence, the GMPS is not unique.

Theorem 6.1. *For the setting described by (6.1) - (6.5) the modified GMPS-problem described by (6.7) that is a subproblem of the original problem has at least two solutions. The first solution is described by the differential equation (6.8) and* $\overline{S} = \Gamma(\phi)$ *and* $\underline{S} = \Delta(\phi)$ *(see (6.9) and (6.11) for Definitions). By symmetry a second solution is given by* $\overline{S} = -\Delta(\phi)$ *and* $\underline{S} = -\Gamma(\phi)$.

Chapter 7

Convergence

In this chapter we will present a convergence result for the Glosten-Milgrom prices, if existence and uniqueness hold. The result is not directly connected with the discussion above, but gives a first idea of the characteristics of the prices in our model. It also motivates some of the simulations that we do in the next chapter.

As in most of the literature, we consider the case where $X_t = X_0$ for all t, in other words, in this chapter X stays constant after it was chosen randomly at the beginning. We just write $X_0 = X$ in the following. We further assume that the rate of buys and sells is always larger than a constant that is larger than zero, which implies that the sequence of actual buys does not break off.

Definition 7.1. *Via the times of actual buys B_i (see Definition 4.1) we define the expectation of X at those time $(p_i)_{i\in\mathbb{N}}$ by*

$$p_i = E[X|\mathcal{F}^S_{B_i}],$$

which by Theorem 4.5 coincides with the ask prices $\overline{S}_{B_i}$. Further, we define the ultimate knowledge of the market maker by

$$\mathcal{F}^S_\infty = \sigma\left(\cup_{i\in\mathbb{N}}\mathcal{F}^S_{B_i}\right).$$

We remind the reader of the definition of the process of the conditional probabilities in Lemma 4.7. Since X stays constant in this chapter π satisfies

$$\pi^S_\tau = \left(P\left[X = x_i|\mathcal{F}^S_\tau\right]\right)_{i=1,\dots,n} \qquad P\text{-a.s.}$$

for all finite stopping times τ.

Since there are always buys, F^S_∞ also contains the information about the sells that happen in between buys. For the same reason we also have

$$\mathcal{F}^S_\infty = \sigma\left(\mathcal{F}^S_t, t \geq 0\right).$$

We immediately obtain the following results.

Lemma 7.2. *The process $(p_i)_{i\in\mathbb{N}}$ is a martingale in discrete time and converges a.s. and in L^1 to an $\mathcal{F}^S_\infty$-measurable random variable p_∞ such that*

$$p_i = E[p_\infty | \mathcal{F}^S_{B_i}].$$

Also, $(\pi_t)_{t\geq 0}$ is a martingale in continuous time and converges a.s. and in L^1 to an $\mathcal{F}^S_\infty$-measurable random variables π_∞ such that

$$\pi_t = E[\pi_\infty | \mathcal{F}^S_t].$$

Proof. It is easy to see that

$$E[p_{i+1}|\mathcal{F}^s_{B_i}] = E[E[X|\mathcal{F}^s_{B_{i+1}}]|\mathcal{F}^s_{B_i}] = E[X|\mathcal{F}^s_{B_i}] = p_i,$$

which shows that $(p_i)_{i\in\mathbb{N}}$ is a martingale. As it is bounded by $x_{\max}$ there is a $\mathcal{F}^S_\infty$-measurable random variables π^j_∞ that fulfills the assertion of the lemma by classical martingale convergence results (see for example [Kle08, p. 220f]). The same argument holds true for π and continuous-time martingale theory (see for example [JS87], Theorem 1.42). □

It is, however, not clear that the prices $\overline{S}_{B_i} = p_i$ converge to the true value. As a quite trivial counterexample consider a pure noise trader model such as

$$\epsilon = \begin{cases} \infty & \text{with probability} \quad \frac{1}{2} \\ -\infty & \qquad\qquad\qquad\quad \frac{1}{2} \end{cases} \tag{7.1}$$

and

$$X = \begin{cases} 1 & \text{with probability} \quad \frac{1}{2} \\ 0 & \qquad\qquad\qquad\quad \frac{1}{2}, \end{cases} \tag{7.2}$$

then both Glosten-Milgrom ask and bid price will be $\frac{1}{2} = E[X]$ immediately and for all time.

We need some condition that secures that the market maker is actually learning something.

For this we go back to the static model, where it is easy to derive from the static fixed point equation (3.1) that for any static Glosten-Milgrom ask price s it holds that $s \geq E[X]$ and by analogy for the bid price we obtain that the expectation of X lies always in the spread. If $X = x$ a.s. for some $x \in \mathbb{R}$ then it is trivial that $s = x = E[X]$. Further we have the following Lemma.

Lemma 7.3. *Assume that X has finite state space $x_1 < \ldots < x_n$ and distribution $\pi = (\pi_1, \ldots, \pi_n)$. Further assume that Φ, which is given by $\Phi(x) = P[\epsilon \geq x]$, is strictly decreasing on $[-C, C]$. If the Glosten-Milgrom ask prices $G(\pi)$ satisfies $G(\pi) = E[X] = \sum_{i=1}^n x_i \pi_i$ then $X = G(\pi)$ a.s. and $\pi_j \in \{0, 1\}$ for all j.*

Note that this implies that $G(\pi)$ is one of the x_i and that the bid price is also equal to $G(\pi)$.

Proof. Define q_j, $j = 1 \ldots n$ by

$$q_j = \Phi(G(\pi) - x_j).$$

By the monotonicity of Φ this implies that $q_j - q_k > 0$ for all $j < k$. By Definition 3.6 of G it holds that $G(\pi) = g(G(\pi), \pi)$ (see Lemma 3.5). Again with the Definition 3.4 of g the equation $g(G(\pi), \pi) = E[X]$ can be rewritten as

$$\frac{\sum_{j=1}^n \pi_j x_j q_j}{\sum_{k=1}^n \pi_k q_k} = \sum_{j=1}^n \pi_j x_j$$

which is equivalent to

$$\begin{aligned} 0 &= \sum_{j=1}^n \pi_j x_j q_j - \sum_{i=j}^n \pi_j x_j \sum_{k=1}^n \pi_k q_k \\ &= \sum_{j=1}^n \sum_{k=1}^n \pi_j \pi_k x_j (q_j - q_k) \\ &= \sum_{j<k} \pi_j \pi_k (x_j - x_k)(q_j - q_k). \end{aligned}$$

Since $q_j - q_k > 0$ and $x_j - x_k < 0$ for $j < k$ it follows that $\pi_j \in \{0, 1\}$ for all j. Thus, $X = x_j$ a.s. for some j and $G(\pi) = x_j$. $\square$

We can now proof the following result of convergence.

Theorem 7.4. *Let Φ be strictly decreasing on $[-C, C]$, then the sequence of actual ask prices $\overline{S}_{B_i}$ converges to the true value X, i.e.*

$$\lim_{i \to \infty} \overline{S}_{B_i} = X \qquad a.s..$$

Of course, by analogy also the bid prices converge, i.e.

$$\lim_{i \to \infty} \underline{S}_{C_i} = X \qquad \text{a.s..}$$

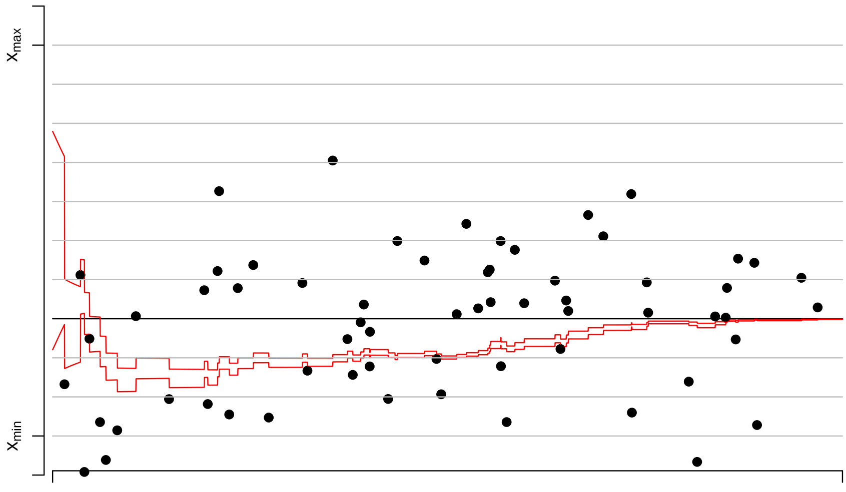

Figure 7.1: The different possible states of X are indicated by the grey lines. The true value X (represented by the black line) is the fourth lowest level, while the initial distribution is centered around the middle value. The bid and ask price (in red) converge to the true value. The bullets mark the customers valuation of the asset.

Proof. By Theorem 4.5 we have that $\overline{S}_{B_i} = E[X|\mathcal{F}^S_{B_i}]$. Hence, $p_i = \overline{S}_{B_i}$ and by Lemma 7.2 it follows that

$$\lim_{i \to \infty} \overline{S}_{B_i} = p_\infty = E[X|\mathcal{F}^S_\infty].$$

From Corollary 4.15 it follows that $\overline{S}_{B_i} = G(\pi_{B_i-})$ a.s. and since G is continuous by Lemma 3.10 it follows that

$$E[X|\mathcal{F}_\infty^S] = \lim_{i\to\infty} G(\pi_{B_i-}) = G(\lim_{i\to\infty} \pi_{B_i-}).$$

Since $B_i \to \infty$ as $i \to \infty$ and π_t is càdlàg we can replace the limit by $\lim_{t\to\infty} \pi_t$. As π_t converges to π_∞ we have

$$G(\pi_\infty) = E[X|\mathcal{F}_\infty^S] = E\left[\sum_{i=1}^n x_i 1_{\{X=x_i\}}|\mathcal{F}_\infty^S\right] = \sum_{i=1}^n x_i \pi_\infty^i,$$

where we again choose the version of π that is càdlàg . The statement of Lemma 7.3 holds then for every $\omega \in \Omega$ and it follows that

$$\pi_\infty^j \in \{0,1\} \qquad \text{a.s. for all } j.$$

This means that with the information $\mathcal{F}_\infty^S$ we know X a.s., more precisely

$$\begin{aligned} P[\{X = x_k\} \cap \{\pi_\infty^j = 1\}] &= E[E[1_{\{X=x_k\}} 1_{\{\pi_\infty^j=1\}}|\mathcal{F}_\infty^S]] \\ &= E[1_{\{\pi_\infty^j=1\}} \pi_\infty^k] = E[\delta_{j,k}\pi_\infty^k] = \delta_{j,k} E[\pi_\infty^k]. \end{aligned}$$

It follows that $\{X = x_k\} \subset \{\pi_\infty^k = 1\}$ P-a.s. and for all k, hence

$$\{X = x_k\} = \{\pi_\infty^k = 1\}.$$

From this it follows that $X = \sum_{k=1}^n x_k \pi_\infty^k = p_\infty$ a.s., hence

$$\lim_{i\to\infty} \overline{S}_{B_i} = X \qquad \text{a.s..}$$

□

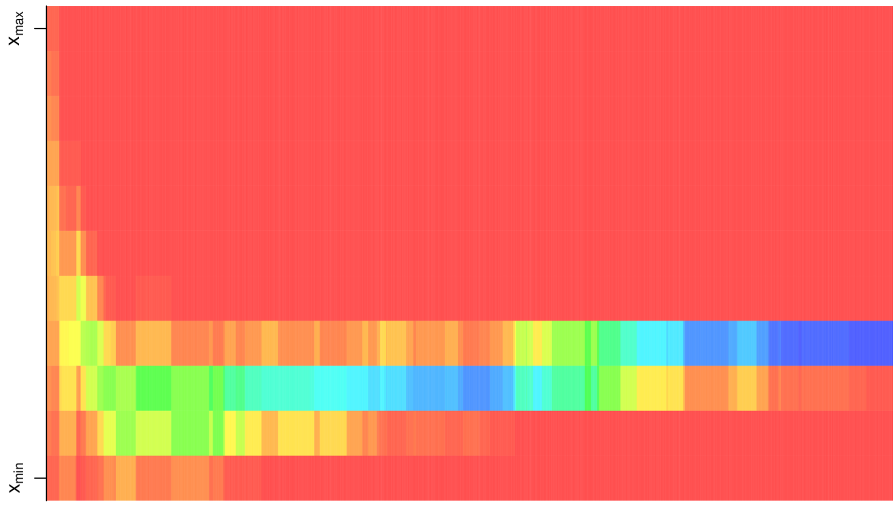

Figure 7.2: For the same scenario as in Figure 7.1 we illustrate the conditional probabilities where red represents a probability equal zero and blue a probability equal one (via yellow and green). Each stripe represents a state of X. Over time the conditional probabilities concentrate at the true value.

Chapter 8

Simulations

In our type of Glosten-Milgrom model many interesting effects and dependencies can be studied by simulations. How does the rate of customer arrivals λ influence the spread, its size, the actual trading intensities or the information content of the trades? How does the distribution of the ϵ_i influence those outcomes? How does the initial distribution of X? Can anything be said about how the transition kernel q affects the results? To which extent do the prices reflect the true value? How long does it take after a large jump in X (probably caused by some sort of shock) until the prices reflect again the true value reasonably good?

Obviously, we can not answer all these questions. Or, to put it differently, we can not check whether simulation results fit to the plenty of already existing economic literature to all of this points. Hence, we will consider only one setting, in which we assume that ϵ_i is normally distributed with mean 0 and standard deviation σ and X is an approximation of Brownian motion in discrete space. This means that X is only allowed to jump to the neighboring states with a rate such that

$$Var[X_t] = Var[W_t]$$

for all $t \geq 0$ if W is a Brownian motion. Since the state space is bounded we assume that X is reflected at the boundaries $x_{\min}$ and $x_{\max}$.

In the first section of this chapter we concentrate on the parameters λ and σ and how they influence the size of the spread. Our model here can also be seen as a discretized version of a model with continuous state space with X being Brownian motion. We discuss this in the second section and conclude the thesis with a conjecture for this model.

8.1 The size of the spread

Our aim in this section is to study the dependence of the size of the spread on the parameters λ, which is the rate of customer arrivals, and σ, the standard deviation of the error terms ϵ_i. But before we present our simulation setting and the results we want to make some general remarks on the size of the spread and the information flow.

It is clear that to some extent the spread reflects the information the market maker has. If he has full knowledge of the true value X_t at time t the spread will be zero. If his information is imprecise the spread will typically be large. There are two effects on the spread:

Firstly, at the trades the market maker learns something about the true value, his information about X increases and the spread typically decreases. We showed in the last chapter that if the true value stays constant, then, under some conditions, both prices converge to the true value and the spread tends to zero. Thus, the market maker collects more and more information and since X does not change, he can identify the true value.

Secondly, in the times when nothing (new) can be observed by the market maker the true value will change as described in the transition kernel q of the process X. Since the market maker corrects for this effect, typically a widening of the spread will be the result. But this is not necessarily so. If the spread is large the market maker can also learn that its quite likely that customers leave the market without trading because the true value is inside the spread. This may lead to a narrowing of the spread. Thus, the effect of no trades over a certain amount of time is not clear. It depends on the transition kernel q and on the effects a change in the conditional probabilities has on the prices, thus, on the functions G and H. However, this relation is hard to capture.

When X is constant the learning effect dominates. In the situation where no trades happen at all (which has zero probability), it is clear that the conditional probabilities will tend to the stationary distribution of the process X and the according prices and spread will be obtained. It is an interesting question whether one of the two effects dominates or whether there might be some sort of equilibrium where the two effects counterbalance each other.

However, it seems clear that it is very hard to make such assertions in a rigorous and general way because the behavior depends heavily on the nature of q. Thus we confine ourselves with the following simulation.

We choose the state space to be $[0, 10]$ divided by steps of 0.1, i.e. we have 101 different states. The time runs from 0 to 5 in steps of size 0.001. We choose X_0 to be distributed like a discretized version of the Normal distribution with mean 5 and standard deviation 1. The standard deviation is chosen freely here.

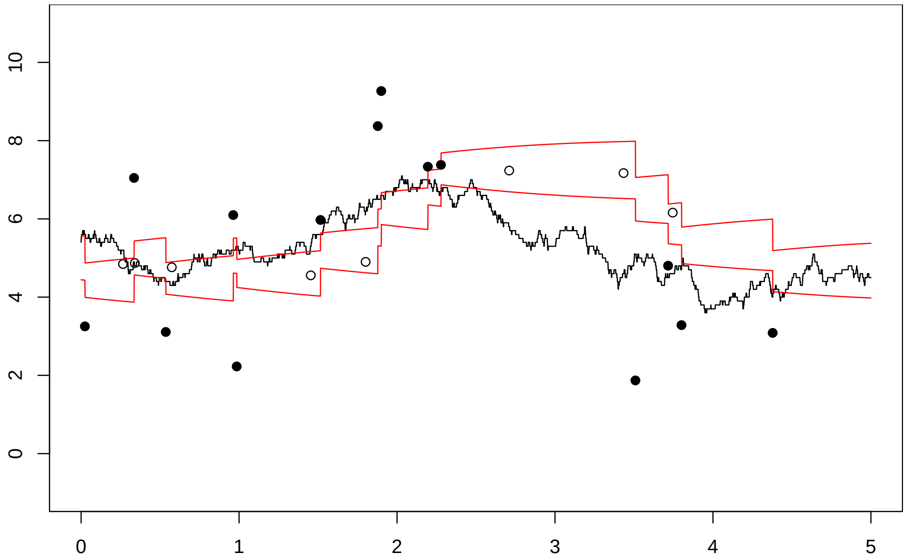

Figure 8.1: The rate of customer arrivals is $\lambda = 5$, the standard deviation of ϵ_i is $\sigma = 1.5$.

Our interest lies in the size of the spread. As described above there are two effect and as it turns out in simulations they more or less balance each other. Thus, we consider the spread only after some time of balancing. For our very simple statics in the following we only consider outcomes after $t = 1$.

We start with a setting where the rate of customer arrivals is $\lambda = 5$ and the standard deviation of ϵ_i is $\sigma = 1.5$. In the scenario shown in Figure 8.1 the average spread for $t \in [1, 5]$ is 1.12850 with an empirical variance of 0.03047.

In a first step we increase the rate of customer arrival to $\lambda = 20$ as shown in Figure 8.2. In the otherwise same scenario as in Figure 8.1 the average spread after $t = 1$ decreases to 0.46888 and the empirical variance to 0.00717.

If we further increase λ to 100, we see that the spread becomes even smaller (see Figure 8.3) and is in average 0.20411 with an empirical variance of 0.00073. This is no surprise since X is piecewise constant and in that case the convergence result of Theorem 7.4 dominates. The information gathering effect more and more dominates the disturbance effect of q. However, it seems difficult to formulate this mathematically rigorous, since we can not describe the rate of convergence in Theorem 7.4 explicitly. This is due to the fact that this theorem uses the martingale convergence theorem for which little is known in terms of the rate of convergence.

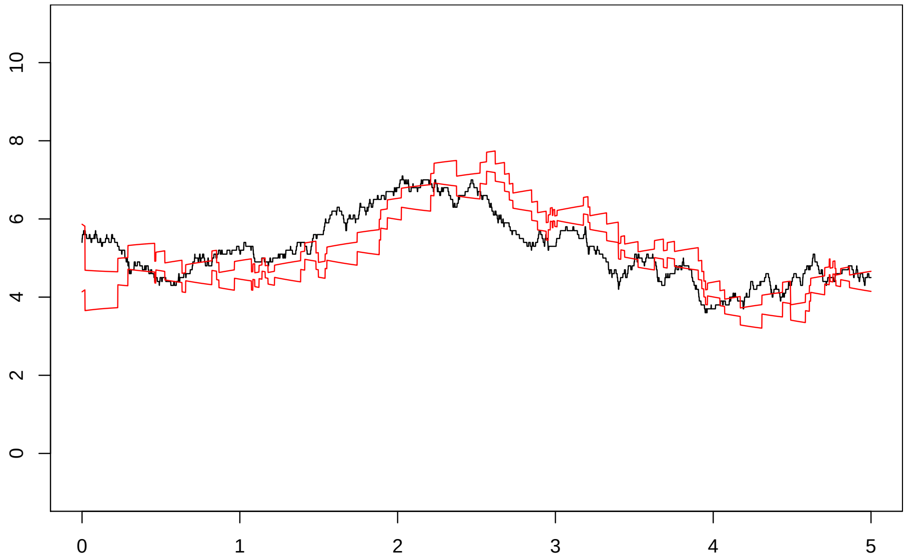

Figure 8.2: The rate of customer arrivals is $\lambda = 20$, the standard deviation of ϵ_i is still $\sigma = 1.5$. We omit the bullets that show the customers valuations for reasons of clarity.

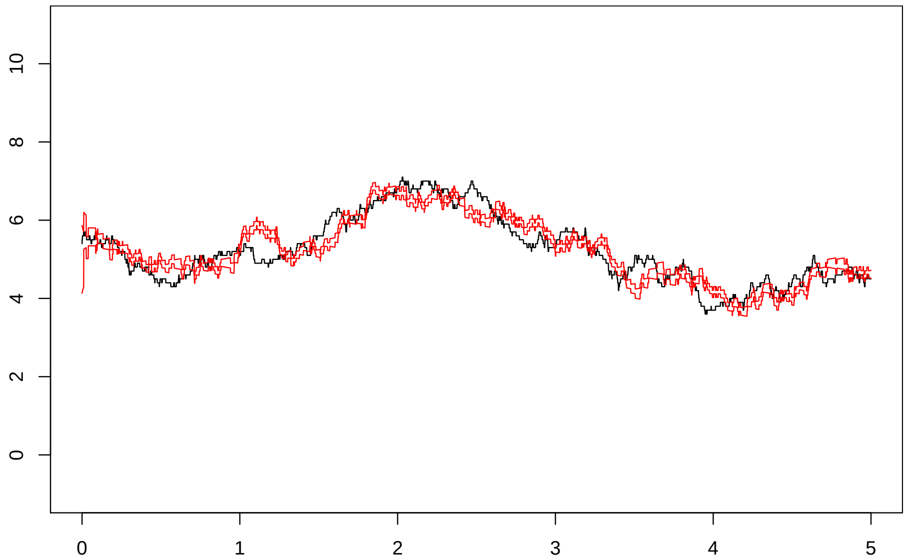

Figure 8.3: The rate of customer arrivals is $\lambda = 100$, the standard deviation of ϵ_i is again $\sigma = 1.5$. For large λ the prices seem to converge against the true value.

We now go back to the case where λ is 5 as in Figure 8.1 but increase the standard deviation from 1.5 to 3. The outcome can be seen in Figure 8.4. The average spread in this scenario obtains a value of 0.9028 and an empirical variation of 0.01327. Hence, it is smaller than in the case with $\sigma = 1.5$.

If we increase σ further as in Figure 8.5 we have an average spread of 0.7922 and an empirical variation of 0.01581. The effect that with increasing σ the spread decreases can be explained by the following considerations. If σ is large customers are more of a noise-type, i.e. their behavior is only connected marginally with the state of X and hence the information content for the market maker is smaller, the business of the market maker is less risky and he thus quotes smaller spreads.

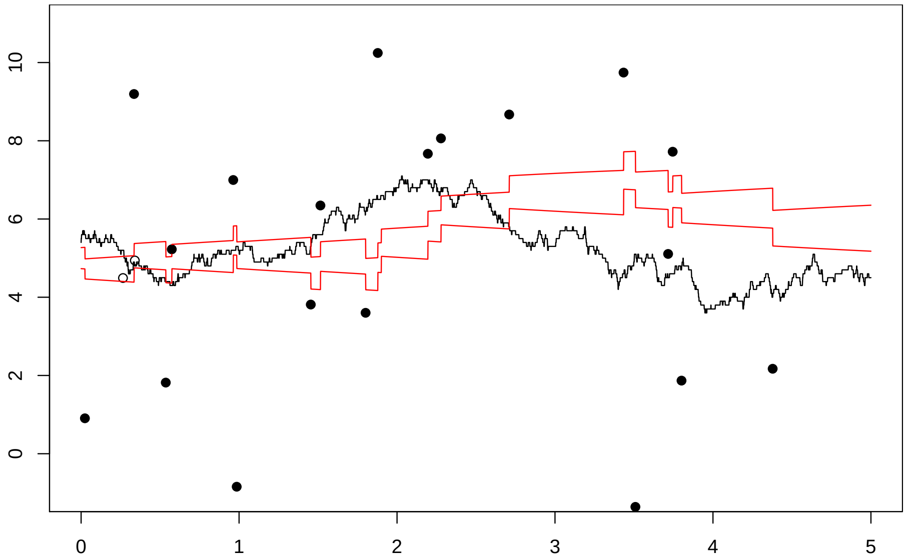

Figure 8.4: The rate of customer arrivals is $\lambda = 5$. We increased the standard deviation of ϵ_i to $\sigma = 3$.

However, this does not mean that the market makers prices reflect the true value as our last concrete example shows. In Figure 8.6 we now increase both, the rate of the customers λ to 100 and the standard deviation σ to 5. The result are price processes that have a small spread but little to do with the true value process. Actually, it is clear that the prices are mostly a random walk. They go up at every buy and down at the sells, but this behavior is triggered mostly by the outcome of the respective ϵ_i, which are iid, and has little to do with the current value of X.

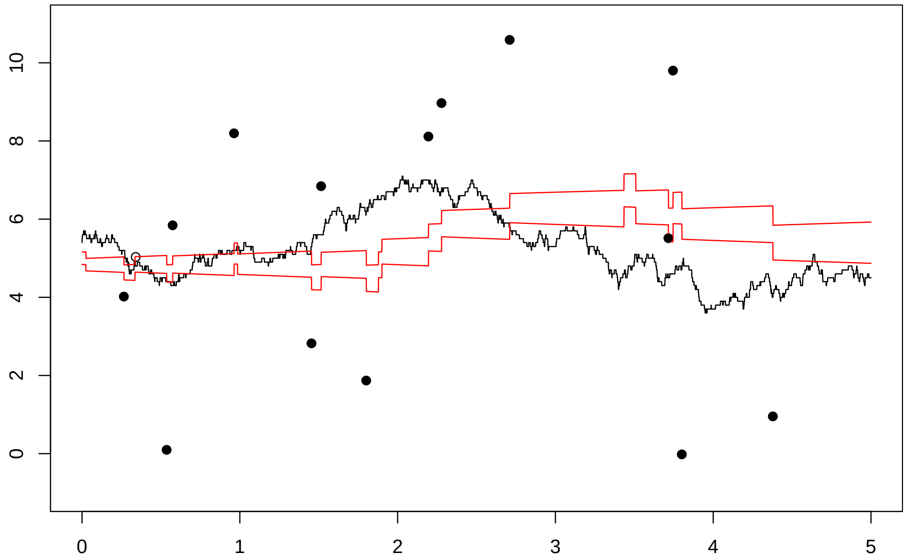

Figure 8.5: For $\lambda = 5$ and $\sigma = 5$ the spread further decreases.

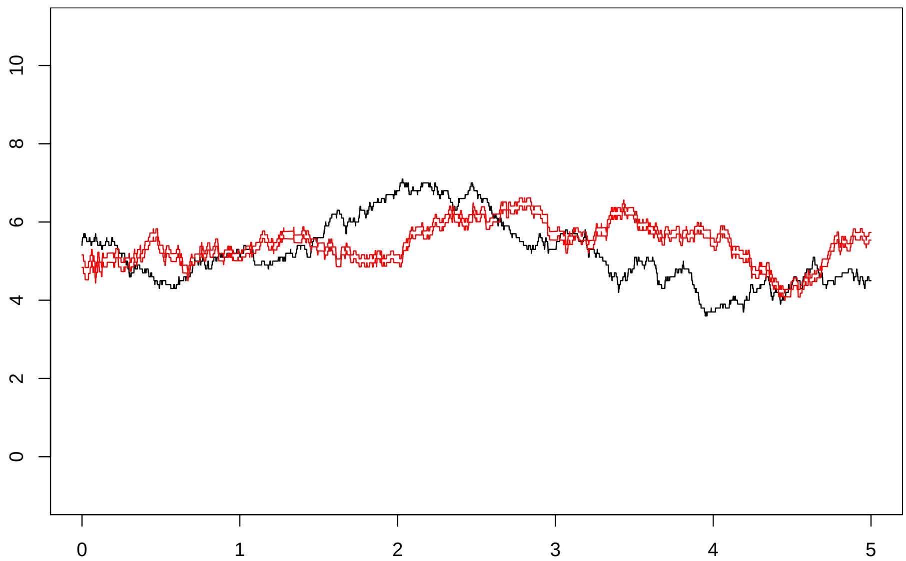

Figure 8.6: For $\lambda = 100$ and $\sigma = 5$ the spread is small, but the market makers prices do not reflect the prices at all.

8.2 A conjecture for Brownian motion

Throughout this thesis we assumed that X has a finite state space $x_1 < \ldots < x_n$. Given the assumptions of the last section it is natural to ask, whether we can formulate our model if X is a process defined on $\mathbb{R}$ instead. If we reconsider Sections 4.1 and 4.2 that describe the definition of the model this seems to be possible. However, all methods in the subsequent work that are aimed to prove existence and uniqueness are based on the consideration of the process of conditional probabilities π (see Lemma 4.7). An extension to a real-valued process X faces several problems:

- It is essential that there is an existence and uniqueness result in the static case (such as Theorem 3.8). This clearly requires some restrictions on the class of conditional distributions that occur, but in advance, i.e. before we know that they exist, little is known how they look like.
- The results that we present in Chapter 5 are much harder to achieve in this setting. There are still general results in filter theory, such as that for a fixed pricing strategy the conditional distributions exist (see [BC08], Theorem 2.1). However, a pathwise construction as in Section 5.2 seems very difficult, since we can not work with results on multidimensional differential equations.
- Also, it is essential for our arguments on uniqueness in Section 5.1 that there is a Lipschitz-continuity of G in π (see Lemma 3.10), i.e. similar conditional distributions lead to similar ask (and bid) prices. It is not clear how to formulate this adequately in a continuous state space.

However, let us look at some simulations in which we pretend to be already in the case of real Brownian motion (actually, one could question whether Brownian motion is a good process to represent the true value process since it does not exhibit any jumps).

We stick to the model described in the last section with the setting that we considered initially, i.e. $\lambda = 5, \sigma = 1.5$ and the initial distribution is normal with mean 5 and standard deviation 1. In Figure 8.7 we see the conditional probabilities represented by color and in Figure 8.8 at times $t = 3, 4, 5$. The pictures heavily suggest that the conditional distributions are (at least approximately) normal.

This also seems to hold true (in the limit) if the initial distribution is not normal. In Figures 8.9 and 8.10 we see the same pictures of conditional distributions as described above for another scenario but with an uniform initial distribution. Again the pictures suggest that the distributions are

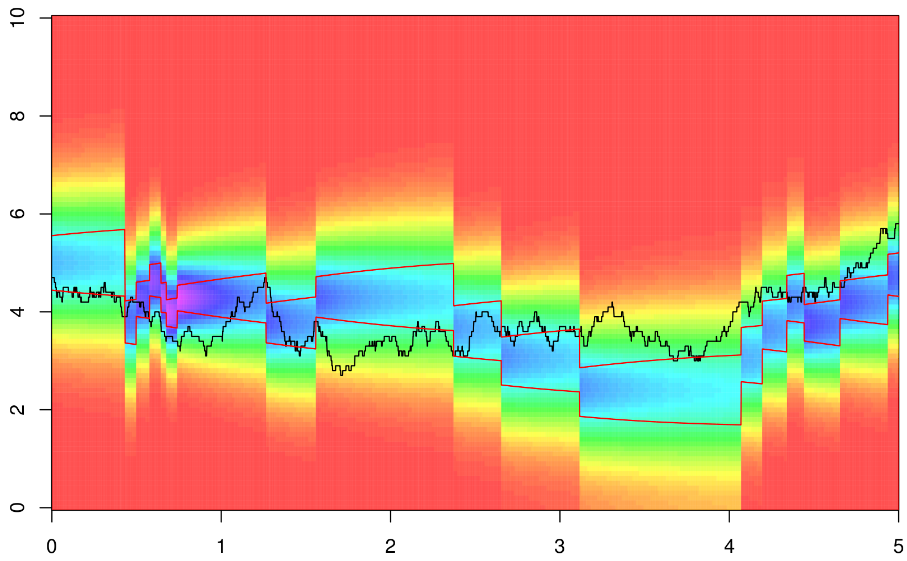

Figure 8.7: The true value, which is an approximation to Brownian motion, is given by the black line, the Glosten-Milgrom prices by the red lines. In the background the conditional probabilities are shown, where red indicates a probability equal zero, blue and purple indicate a high probability.

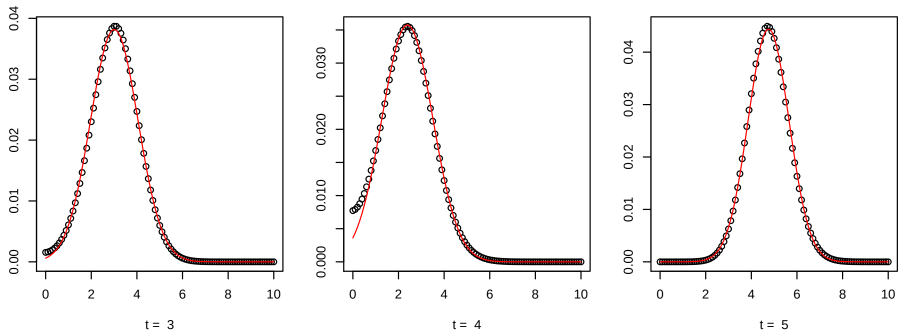

Figure 8.8: The conditional probabilties at all 101 states at different times $t = 3, 4, 5$ are given by the bullets. The red line is the density of a mean-variance fitted normal distribution.

normal. If all conditional distributions are approximately normal at least an existence and uniqueness result in the static case might be obtainable. The second and third points seems still to be difficult to solve.

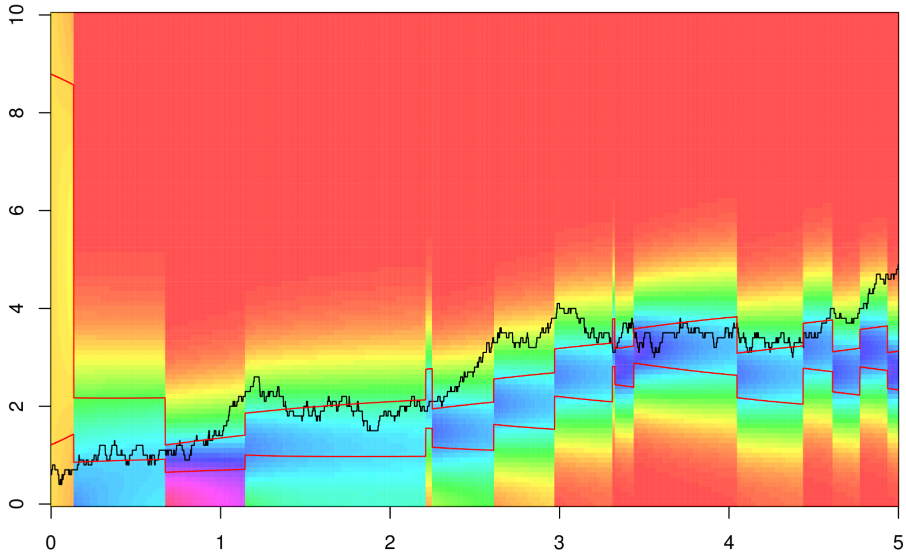

Figure 8.9: A similar simulation to Figure 8.7 but with an uniform initial distribution.

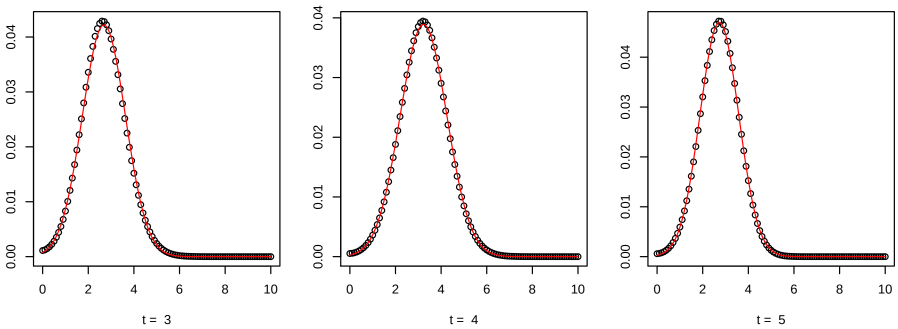

Figure 8.10: The conditional probabilities at all 101 states at different times $t = 3, 4, 5$ are given by the bullets. The red line is the density of a mean-variance fitted normal distribution. After some time of balancing the conditional distributions seem to be normal.

Deutsche Zusammenfassung

Ziel dieser Doktorarbeit ist es, die Existenz und Eindeutigkeit von Glosten-Milgrom Preisstrategien zu untersuchen. Diese Strategien kennzeichnen im klassischen Modell von Glosten und Milgrom [GM85] das Preissetzungsverhalten eines Market Makers. Dieser tritt als alleiniger Liquiditätsanbieter für das zu untersuchende Wertpapier auf, das heißt alle anderen Marktteilnehmer, die im Folgenden als Kunden bezeichnet werden, können nur mit ihm handeln.

Die ökonomische Kernfrage ist nun, wie der Market Maker seine Preise wählt und warum und in welcher Höhe er für seine Leistung entschädigt wird. Der Market Maker trägt im Wesentlichen zwei Risiken, nämlich das Inventar- und das Informationsrisiko, wobei wir uns hier auf letzteres konzentrieren. Eine Übersicht über die verschiedenen Ansätze zu diesen Problemen und aktuelle Forschungsergebnisse sind in Kapitel 1 dargestellt.

Das Informationsrisiko besteht darin, dass die Kunden mit denen der Market Maker handelt besser über den wahren Wert X informiert sind als er selbst. Sie entscheiden anhand der vom Market Maker gesetzten Preise und einer eigenen, privaten Information über den Wert des Papieres darüber, ob sie kaufen oder verkaufen. Die private Information setzt sich aus besagtem wahren Wert X und Störtermen ϵ_i zusammen. Dabei modellieren wir X nicht wie in der bisherigen Literatur als über die Zeit konstante Zufallsvariable sondern als Markov-Prozess.

Der Market Maker kann allerdings aus dem Verhalten der Kunden, nämlich aus deren Käufen und Verkäufen, etwas über den wahren Wert X des Papieres lernen. Die zentrale Glosten-Milgrom Bedingung ist nun, dass die Preise $S = (\overline{S}, \underline{S})$ des Market Makers dem Erwartungswert des wahren Wertes X unter seiner Information $\mathbb{F}^S$, also den vergangenen Käufen und Verkäufen, entsprechen. Es handelt sich also um einen risikoneutralen Market Maker bei vollständigem Wettbewerb. Dabei antizipiert der Ask Preis $\overline{S}$ des Market Makers weiterhin, dass die nächste Transaktion ein Kauf eines Kunden ist, bzw. der Bid Preis $\underline{S}$, dass es sich bei der nächsten Transaktion um einen Verkauf durch einen Kunden handelt (siehe auch Theorem 3 unten).

Hieraus ergibt sich ein Endogenitätsproblem, da die gewonnene Information $\mathbb{F}^S$ von den Preisen S abhängt, die Preise sich aber wiederum aus der Information erklären. Ziel der Arbeit ist es, das daraus resultierende Fixpunktproblem zu untersuchen, was damit nach unserem Wissensstand zum ersten Mal auf mathematisch rigorose Weise geschieht. Diese Art von Endogenitätsproblem scheint in Modellen mit Poisson-Prozessen eine Neuerung darzustellen.

Ein wichtiges Hilfsmittel in dieser Arbeit stellt die Theorie des stochastischen Filterns dar. Diese Theorie wird in Kapitel 2 zunächst für Punktprozesse in Anlehnung an [Bré81] dargestellt. Hauptergebnis ist die Herleitung der Filtergleichung für diese Art von Problemen.

Das zeitstetige Modell, das wir auch im Folgenden näher betrachten wollen, wird in Kapitel 4 ausführlich beschrieben. In der Arbeit wird zunächst in Kapitel 3 der statische Fall, d.h. ein Einperiodenmodell, behandelt, welches die Situation an festen Zeitpunkten beschreibt. Für dieses Modell werden, für verschiedene Vorraussetzungen an ϵ, Existenz- und Eindeutigkeitsaussagen bewiesen (siehe Theorem 3.8 und 3.11), die später im stetigen Fall benutzt werden.

Das zeitstetige Modell und allgemeine Resultate

Auf dem Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, P)$ betrachten wir zunächst den càdlàg Prozess $X = (X_t)_{t\geq 0}$, der den wahren Wert des Wertpapieres beschreibt. Wir nehmen an, dass dieser ein zeithomogener Markov-Prozess mit endlichem Zustandsraum $\{x_1, \dots, x_n\}$, $n \geq 2$ ist, wobei

$$x_{\min} = x_1 < \ldots < x_n = x_{\max}.$$

Der Übergangskern wird mit q bezeichnet.

Der Market Maker kennt X nicht, sondern beobachtet nur die aus X und seinen Preisen resultierenden Käufe und Verkäufe. Diese werden wie folgt modelliert: Sei N ein Poisson-Prozess mit konstanter Rate $\lambda > 0$. Die Sprungzeiten von N bezeichnen wir mit $\tau_1 < \tau_2 < \tau_3 \ldots$. Zu diesen Zeiten erreichen Kunden den Markt und erhalten eine gestörte Information $X_{\tau_i} + \epsilon_i$ über den wahren Wert des Papiers. Dabei bezeichnet die Folge $(\epsilon_i)_{i\in\mathbb{N}}$ den Störterm, der als unabhängig identisch verteilte Folge von Zufallsvariablen modelliert wird. Außerdem wird angenommen, dass X, N, und $(\epsilon_i)_{i\in\mathbb{N}}$ wechselseitig unabhängig sind.

Sei $S : \Omega \times [0, \infty) \to \mathbb{R}^2$ eine gegebene, $\mathcal{F} \otimes \mathcal{B}([0, \infty))$-messbare Preisfunktion des Market Makers. Wir bezeichnen mit $S = (\overline{S}, \underline{S})$ jeweils Ask und Bid Preis und nehmen an, dass $\overline{S}_t(\omega) > \underline{S}_t(\omega)$ für alle (ω, t). Ein Kunde kauft ein Wertpapier, falls $X_{\tau_i} + \epsilon_i \geq \overline{S}_{\tau_i}$ und verkauft falls $X_{\tau_i} + \epsilon_i \leq \underline{S}_{\tau_i}$. Er handelt nicht, falls seine Bewertung im Spread liegt.

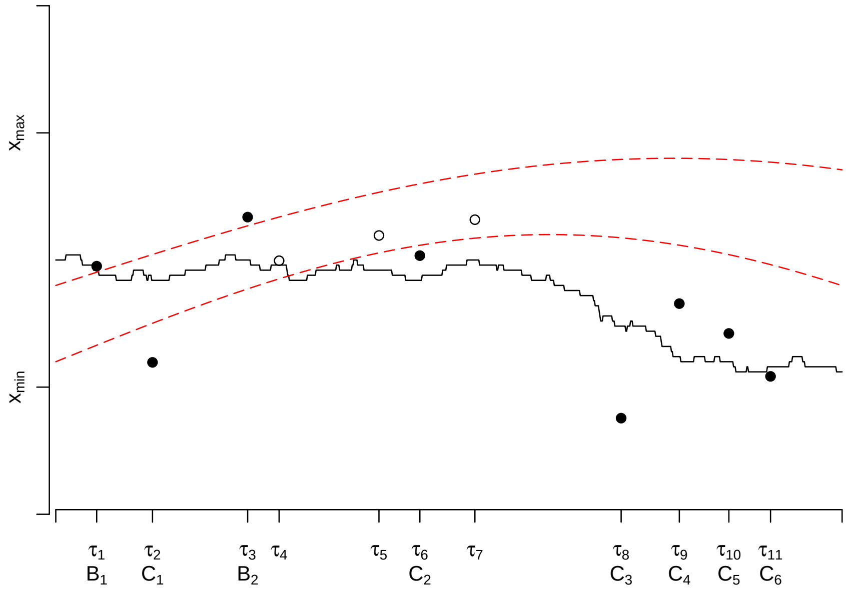

Figure 1: Die schwarze Linie bezeichnet X, die roten gestrichelten Linien die beiden Preise $\overline{S} > \underline{S}$. Die Bewertungen der Kunden $X_{\tau_i} + \epsilon_i$ sind durch die Punkte markiert, die schwarz sind, falls ein Kauf oder Verkauf stattgefunden hat.

Wir erhalten so die Folge der tatsächlich stattgefundenen Käufe und Verkäufe $(B_i)_{i \in \mathbb{N}}$ und $(C_i)_{i \in \mathbb{N}}$ als Teilfolge von $(\tau_i)_{i \in \mathbb{N}}$, die zugehörigen Zählprozesse N_B und N_C (vergl. Definition 4.1 und 4.2) und die Filtration des Market Makers $\mathbb{F}^S$, die von diesen Ereignissen erzeugt wird (siehe Definition 4.2).

Definition 1. *Wir bezeichnen S als zulässige Preisstrategie, falls sie $\mathbb{F}^S$-vorhersehbar ist und $x_{\max} \geq \overline{S}_t(\omega) > \underline{S}_t(\omega) \geq x_{\min}$ für alle $(\omega, t) \in \Omega \times \mathbb{R}_+$.*

Dabei ist die (im ökonomischen Sinne) wesentliche Eigenschaft die Vorhersehbarkeit bezüglich $\mathbb{F}^S$. Wir betrachten dann Glosten-Milgrom Preise, die sich wie folgt beschreiben lassen.

Definition 2. *Eine zulässige Preisstrategie S ist eine Glosten-Milgrom Preisstrategie (GMPS), falls*

$$E\left[\sum_{B_i\le\tau}(\overline{S}_{B_i}-X_{B_i})\right]=0 \textit{ und } E\left[\sum_{C_i\le\tau}(\underline{S}_{C_i}-X_{C_i})\right]=0$$

für jede beschränkte $\mathbb{F}^S$-Stoppzeit τ.

Eine GMPS zeichnet sich also dadurch aus, dass der Erwartungswert der Summen der Profite des Market Makers, jeweils aus Käufen und Verkäufen, bewertet am stattfindenden Zeitpunkt, gleich 0 ist. Wir zeigen zunächst folgende Äquivalenz.

Theorem 3. *S ist eine GMPS genau dann, wenn sie zulässig ist und*

$$\overline{S}_{B_i}=E[X_{B_i}|\mathcal{F}^S_{B_i}] \qquad \textit{und} \qquad \underline{S}_{C_i}=E[X_{C_i}|\mathcal{F}^S_{C_i}] \qquad P-f.s.$$

für alle i.

Die Ask Preise $\overline{S}$ zu den Kaufzeitpunkten B_i entsprechen also dem Erwartungswert von X zum gleichen Zeitpunkt, gegeben die Information $\mathcal{F}^S_{B_i}$. Diese enthält alle vergangenen Transaktionszeitpunkte und die Art der Transaktion, insbesondere auch die Information, dass gerade ein Kauf stattgefunden hat. Gleiches gilt für den Bid Preis $\underline{S}$ und Verkaufszeitpunkte C_i.

Zentraler Teil des Kapitels 4 ist die Konstruktion eines Funktionals F auf dem Raum der (nicht notwendigerweise zulässigen) Preisstrategien, für das wir folgendes Theorem beweisen.

Theorem 4. *Eine zulässige Preisstrategie S ist eine GMPS genau dann, wenn S ein Fixpunkt von F ist.*

Dabei sagen wir, dass S ein Fixpunkt von F ist, falls $S=F(S)$ $P\otimes\lambda$-f.ü. (λ bezeichnet das Lebesgue-Maß auf $\mathbb{R}_+$). Wir benötigen hierzu zum einen eine Existenz- und Eindeutigkeitsaussage für den statischen Fall, wie sie bereits in Kapitel 3 gezeigt wird, und außerdem den Prozess der bedingten Wahrscheinlichkeiten π^S, der durch folgendes Lemma beschrieben wird.

Lemma 5. *Für jeden $\mathcal{F}\otimes\mathcal{B}([0,\infty))$-messbaren Prozess $S=(\overline{S},\underline{S})$ existiert ein (bis auf Ununterscheidbarkeit) eindeutiger $\widetilde{\mathbb{F}}^S$-adaptierter càdlàg Prozess π^S mit*

$$\pi^S_\tau=\left(P\left[X_\tau=x_i|\mathcal{F}^S_\tau\right]\right)_{i=1,\dots,n} \qquad P\text{-}f.s.$$

für alle endlichen Stoppzeiten τ.

Hier bezeichnet $\widetilde{\mathbb{F}}^S$ die Vervollständigung von $\mathbb{F}^S$. Für π^S können wir mit den Resultaten aus Kapitel 2 bei gegebenem S explizit die Filtergleichung berechnen, für welche die Vervollständigung nicht nötig ist.

Lemma 6. *Sei $\Phi(x) = P[\epsilon_1 \geq x]$ und $\Psi(x) = P[\epsilon_1 \leq x]$. Der Prozess π^S genügt folgender Stochastischer Differentialgleichung:*

$$\begin{aligned} d\pi_t^{S,i} =& \pi_{t-}^{S,i} \left(\frac{\Phi(\overline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j} \Phi(\overline{S}_t - x_j)} - 1 \right) dN_t^B \\ &+ \pi_{t-}^{S,i} \left(\frac{\Psi(\underline{S}_t - x_i)}{\sum_j \pi_{t-}^{S,j} \Psi(\underline{S}_t - x_j)} - 1 \right) dN_t^C \\ &- \Bigg(\lambda \pi_t^{S,i} \Bigg(\Psi(\underline{S}_t - x_i) + \Phi(\overline{S}_t - x_i) \\ &- \sum_j \pi_t^{S,j} \left(\Psi(\underline{S}_t - x_j) + \Phi(\overline{S}_t - x_j) \right) \Bigg) \\ &- \sum_j \pi_t^{S,j} q(j,i) \Bigg) dt. \end{aligned}$$

für alle $t \geq 0$, bis auf Ununterscheidbarkeit, mit Anfangsbedingung $\pi_0^{S,i} = P[X_0 = x_i]$.

Betrachtung für verschiedene Verteilungen von ϵ_1

Wir betrachten das Modell für verschiedene Verteilungen der Störterme, die unabhängig und identisch zu ϵ_1 verteilt sind. Zunächst zeigen wir in Kapitel 5 für den Fall, dass ϵ_1 eine Dichte hat und weitere Annahmen erfüllt sind, folgende Aussage.

Theorem 7. *Sei $C := x_{\max} - x_{\min}$ und $\Phi(y) := P[\epsilon_1 \geq y]$, $y \in \mathbb{R}$. Sei Φ differenzierbar (d.h. die Verteilung von ϵ_1 hat die Dichte $-\Phi'$) auf $[-C, C]$, $1 > \Phi(0) > 0$, und*

$$-\Phi'(y) \leq \frac{K}{C} \min\{\Phi(y), 1 - \Phi(y)\}$$

für alle $y \in [-C, C]$ und eine Konstante $K < 1$. Dann existiert eine Glosten-Milgrom Preisstrategie und diese ist eindeutig bis auf eine $(P \otimes \lambda)$-Nullmenge.

Die wesentliche Idee des Eindeutigkeitsteils des Beweises in Abschnitt 5.1 besteht darin zu zeigen, dass F unter den Vorraussetzungen dieses Satzes eine Kontraktion ist, was für hinreichend kleines t durch folgendes Lemma gegeben ist.

Lemma 8. *Es existiert eine Konstante $K_1 < \infty$ so, dass*

$$E\left[\int_0^t \|F(S)_s - F(T)_s\|\, ds\right] \leq (K + tK_1) E\left[\int_0^t \|S_s - T_s\|\, ds\right]$$

für alle $t \geq 0$ und für K aus Theorem 7.

Daraus lässt sich dann die Eindeutigkeit herleiten. Allerdings liefert dies nicht die Existenz, da der Raum der Preisstrategien unter F nicht abgeschlossen ist. Eine Lösung kann aber in Abschnitt 5.2 pfadweise konstruiert und damit die Existenz gezeigt werden. Diese Rückführung auf den deterministischen Fall macht Gebrauch von der Tatsache, dass für eine Differentialgleichung mit Lipschitz-stetigen Koeffizienten eine eindeutige Lösung existiert. Die allgemeinen Resultate und die Resultate für diese Wahl von ϵ sind Inhalt von [KR13].

In Kapitel 6 betrachten wir eine andere Verteilung von ϵ_1, das sogenannte Insider/Noise Trader Modell. Hierbei beobachten die Kunden entweder den Wert X ungestört, also $\epsilon_i = 0$ (wir sprechen in diesem Fall von einem Insider) oder die Kunden kaufen rein zufällig. Im zweiten Fall entspricht dies $\epsilon_i = \pm\infty$ und wir bezeichnen diese Kunden als Noise Trader. Wir betrachten also für $\mu, \nu \in (0,1)$

$$\epsilon_1 = \begin{cases} \infty & \text{mit Wahrscheinlichkeit} \quad \mu \\ 0 & \qquad\qquad\qquad\qquad\quad \nu \\ -\infty & \qquad\qquad\qquad\qquad\quad 1 - \mu - \nu. \end{cases}$$

Eine Existenzaussage ist für dieses Modell schwierig, da die pfadweise Konstruktion fehlschlägt. Grund dafür ist die Unstetigkeit der Verteilungsfunktion von ϵ_1, die sich in der Weise fortsetzt, dass das zugehörige deterministische Problem bei pfadweiser Betrachtung keine Lipschitzkoeffizienten mehr aufweist.

Wir betrachten dann das GMPS-Problem bis zu einer Stoppzeit $\tau = \min\{T_1, \widetilde{t}\}$, wobei T_1 der erste Transaktionszeitpunkt ist, also entweder B_1 oder C_1 und $\widetilde{t}$ eine deterministische Konstante echt größer 0. Für diese Situation, in der alle vorherigen Resultate über die Gestalt der Preise gültig sind (insbesondere Theorem 4), können wir zwei gültige und symmetrische GMPSen konstruieren. Dies stellt also ein Gegenbeispiel zur Eindeutigkeit dar.

Konvergenz und Simulationen

Die letzten beiden Kapitel beschäftigen sich nicht mit der Existenz- und Eindeutigkeitsfrage, sondern betrachten das Modell und dessen Eigenschaften. Eine Kernfrage der ökonomischen Literatur ist die Größe des Spreads und seine Abhängigkeit von bestimmten Parametern. Zunächst betrachten wir in Kapitel 7 den Fall, dass X zwar zufällig, aber über die Zeit konstant ist. In diesem Fall lernt der Market Maker (unter gewissen Vorraussetzungen) immer mehr über den wahren Wert und seine Preise konvergieren gegen X.

Theorem 9. *Sei Φ streng monoton fallend auf $[-C, C]$, dann konvergiert die Folge der tatsächlichen Kaufspreise $\overline{S}_{B_i}$ gegen den wahren Wert X, d.h.*

$$\lim_{i\to\infty} \overline{S}_{B_i} = X \qquad \textit{f.s.}.$$

Gleiches gilt natürlich auch für die Verkaufspreise $\underline{S}_{C_i}$.

In Kapitel 8 untersuchen wir die Abhängigkeit des Spreads von den Parametern für ein Modell, dass für X eine Näherung an die Brownsche Bewegung annimmt und die ϵ_i normalverteilt sind.

Dabei zeigt sich in den Simulationen, dass je größer die Ankunftsrate der Kunden beim Market Maker λ, desto kleiner der Spread ist. Außerdem entsprechen die Preise bei höherem λ eher dem wahren Wert. Dies kann mit der Konvergenz aus dem vorherigen Kapitel erklärt werden. Abschnittsweise ist X konstant, wenn nun das Lernen des Market Makers durch viele Transaktionen schnell verläuft, nähern sich die Preise dem wahren Wert.

In einem zweiten Teil betrachten wir die Abhängigkeit des Spreads von der Varianz der Fehlerterme ϵ_i. Bei hoher Varianz ist der Spread klein. Dies kann so erklärt werden, dass einerseits der Market Maker zwar weniger aus den Transaktionen lernen kann, da die Kunden willkürlicher handeln, er aber andererseits weniger Risiko eingeht, da die Kunden eher auch zu für sie ungünstigen Preisen zu handeln bereit sind. Die Verringerung des Spreads geht dabei aber nicht mit einer besseren Schätzung des wahren Wertes einher. In der Tat können beide erheblich differieren.

Im letzten Abschnitt wird untersucht, wie die bedingten Verteilungen in einem völlig "normalen" Modell, das heißt falls X eine Brownsche Bewegung ist, aussehen könnten. Dieses passt zwar nicht in unserer theoretisches Modell, da wir in dieser Arbeit einen endlichen Zustandsraum des Prozesses X vorraussetzen (statt $\mathbb{R}$), dennoch ergeben sich interessante Resultate. Die bedingten Verteilungen scheinen auf lange Sicht näherungsweise normalverteilt zu sein.

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