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Consumption-Investment Problems with Stochastic Mortality Risk

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Non-Technical Summary

Life cycle consumption-investment models often assume a deterministic time of death or, if at all, include deterministic mortality risk given by mortality tables. Only a few recent papers allow for stochastic mortality risk driven by a diffusive component. In reality, cancer and other critical illnesses suggest that there is a significant jump component in the individual hazard rate of death that is not captured by the life cycle consumption-investment models. This raises the question of the importance of mortality risk in life cycle consumption-investment models, and what impact a jump component in the hazard rate of death has. My paper aims to close this gap in the literature.

The main feature of my model is the uncertain time of death due to mortality risk. Mortality risk is rarely considered in continuous-time life cycle models, whereas it is often analyzed in the insurance literature. The main difference between the life cycle and the insurance perception of mortality risk is that the insurance literature considers aggregate mortality rates. In contrast, in the individual consumption-investment decision, I focus on an agent's perspective and consider the individual mortality risk of an agent. This difference affects both interpretation and modeling. If the actuarial literature considers jumps, it focuses mostly on negative jumps since these yield decreased profits of annuities. Furthermore, it considers jumps in aggregate mortality rates. In my model, individual positive shocks that occur with higher intensity are more important. These can be interpreted as health problems with a permanent impact, e.g. medical disasters like a cancer detection or an accident.

I analyze the impact of mortality risk in a life cycle consumption-investment model. Further model features are unspanned labor income risk, short-sale and liquidity constraints and a simple insurance. The mortality process is calibrated to mortality data for Germany and allows uncertainty driven by a diffusive and jump component. I compare results with deterministic time of death and stochastic time of death. Furthermore, I provide sensitivity analyses with respect to the agent's characteristics and the financial market. I also analyze the impact of the insurance and of a bequest motive.

The mortality calibration shows that allowing for jumps in the hazard rate of death significantly increases the fit to the data. Considering the results, the model with deterministic time of death and the model with stochastic time of death produce nearly identical optimal consumption and portfolio holdings as long as mortality risk is very low. This highlights that early in the lifetime mortality risk can be neglected for simplicity. The endogenous insurance is used by the agents to allocate optimal bequest such that all bequest is intended. In a model without insurance, agents leave more than twice as much bequest on average if they face mortality risk. Most bequest is accidental then. Without insurance, the consumption profile over the life cycle shows a hump-shaped pattern. A diffusive component in the hazard rate of death has almost no influence, whereas a jump component has a significant impact. The agent is better off since a mortality jump acts as a death indicator. A jump in mortality risk also crucially affects the consumption and investment decision. Together with the importance in the mortality calibration, this highlights that mortality risk should be modeled with a jump component in life cycle consumption-investment models. In contrast, a diffusive component can be left out. In general, mortality risk should not be neglected when considering the retirement phase of the life or health related insurance products.

To summarize, I find that mortality risk is negligible early in the life cycle. Furthermore a diffusive component in the hazard rate of death has no significant effect on the optimal decisions and the welfare level of the agent. In contrast, a jump component is strongly desired by the agent and also significantly increases the fit to mortality data.

Consumption-Investment Problems with Stochastic Mortality Risk

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ABSTRACT: I numerically solve realistically calibrated life cycle consumption-investment problems in continuous time featuring stochastic mortality risk driven by jumps, unspanned labor income as well as short-sale and liquidity constraints and a simple insurance. I compare models with deterministic and stochastic hazard rate of death to a model without mortality risk. Mortality risk has only minor effects on the optimal controls early in the life cycle but it becomes crucial in later years. A diffusive component in the hazard rate of death has no significant impact, whereas a jump component is desired by the agent and influences optimal controls and wealth evolution. The insurance is used to ensure optimal bequest such that there is no accidental bequest. In the absence of the insurance, the biggest part of bequest is accidental.

KEYWORDS: Stochastic mortality risk, Health jumps, Labor income risk, Portfolio choice, Insurance

JEL-CLASSIFICATION: D91, G11

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1 Introduction

Life cycle consumption-investment models often assume a deterministic time of death or, if at all, include deterministic mortality risk given by mortality tables. Only a few recent papers allow for stochastic mortality risk driven by a diffusive component. In reality, cancer and other critical illnesses suggest that there is a significant jump component in the individual hazard rate of death that is not captured by the life cycle consumption-investment models. This raises the question of the importance of mortality risk in life cycle consumption-investment models, and what impact a jump component in the hazard rate of death has. My paper aims to close this gap in the literature.

I analyze the impact of mortality risk in a life cycle consumption-investment model. Further model features are unspanned labor income risk, short-sale and liquidity constraints and a simple insurance. The mortality process is calibrated to mortality data for Germany and allows uncertainty driven by a diffusive and jump component. I compare results with deterministic time of death and stochastic time of death. Furthermore, I provide sensitivity analyses with respect to the agent's characteristics and the financial market. I also analyze the impact of the insurance and of a bequest motive.

The mortality calibration shows that allowing for jumps in the hazard rate of death significantly increases the fit to the data. Considering the results, the model with deterministic time of death and the model with stochastic time of death produce nearly identical optimal consumption and portfolio holdings as long as mortality risk is very low. This highlights that early in the lifetime mortality risk can be neglected for simplicity. The endogenous insurance is used by the agents to allocate optimal bequest such that all bequest is intended. In a model without insurance, agents leave more than twice as much bequest on average if they face mortality risk. Most bequest is accidental then. Without insurance, the consumption profile over the life cycle shows a hump-shaped pattern. A diffusive component in the hazard rate of death has almost no influence, whereas a jump component has a significant impact. The agent is better off since a mortality jump acts as a death indicator. A jump in mortality risk also crucially affects the consumption and investment decision. Together with the importance in the mortality calibration, this highlights that mortality risk should be modeled with a jump component in life cycle consumption-investment models. In contrast, a diffusive component can be left out. In general, mortality risk should not be neglected when considering the retirement phase of the life or health related insurance products.

The remaining paper is organized as follows. In Section 2, I give an overview of the related literature considering life cycle consumption-investment models especially with a focus on mortality risk and unspanned labor income. Section 3 introduces the general model setup of a life cycle consumption-investment problem with stochastic mortality risk.

In Section 4, I provide analytical results for a complete market case with deterministic labor income. I compare the results with deterministic mortality risk to results without mortality risk. Section 5 provides the calibration of the models with a special focus on the mortality risk calibration using German mortality data. In Section 6, I illustrate numerical results with unspanned labor income and short-sale and liquidity constraints for the model with deterministic hazard rate of death. I provide sensitivity analyses regarding the preference, labor income and asset parameters. Section 7 compares the numerical results with deterministic hazard rate of death with the results of an equivalent model without mortality risk. Especially, I consider the impact of the insurance and the bequest motive. In Section 8, I allow for a stochastic hazard rate of death driven by a diffusive and jump component. I comment on the importance of the insurance in the different models and consider the impact of the diffusive and jump component. Finally, Section 9 concludes and gives an outline for further research.

2 Literature

The main feature of my model is the uncertain time of death due to mortality risk. Mortality risk is rarely considered in continuous-time life cycle models, whereas it is often analyzed in the insurance literature. Cairns, Blake, and Dowd (2008) review several approaches for modeling mortality risk, both in discrete time and in continuous time. They present and compare different models and interpret mortality patterns from an insurance point of view. Especially, they stress the importance of a diffusive component since mortality rates fluctuate from year to year with a high volatility. They interpret these fluctuations as weather dependent, e.g. a hot summer or a cold winter increases mortality risk significantly especially for old agents. The models that they review do not include jumps. The main difference between the life cycle and the insurance perception of mortality risk is that the insurance literature considers aggregate mortality rates. In contrast, in the individual consumption-investment decision, I focus on an agent's perspective and consider the individual mortality risk of an agent. This difference affects both interpretation and modeling. If the actuarial literature considers jumps, it focuses mostly on negative jumps since these yield decreased profits of annuities. Furthermore, it considers jumps in aggregate mortality rates. Negative jumps in aggregate mortality rates are mainly interpreted as medical progress. Since there are little explanations for the interesting case of negative jumps, most models neglect the jump component. Positive aggregate jumps can occur due to catastrophic events like war, earthquake, tsunami, pandemic or epidemic. These jumps are mainly transitory and affect mortality rates for few years only. After the catastrophic event, the old mortality pattern returns for survivors. These positive jumps have a higher magnitude but also a low intensity and are mainly considered in life insurance as well as property or casualty insurance. In my model, individual positive shocks that occur with higher intensity are more important. These can be interpreted as health problems with a permanent impact, e.g. medical disasters like a cancer detection or an accident.

There are several papers that study the effect of mortality risk analytically. The first paper that analyzes the uncertain lifetime analytically in a continuous-time framework is Yaari (1965). Closely related to my model is the setup of Richard (1975) who solves the portfolio problem with several risky assets in continuous time with mortality risk, deterministic labor income and an endogenous insurance in closed form. He deduces that efficient portfolios are identical compared to a setup without mortality risk if the time of death is independent of other sources of risk. Ye (2006) and Pliska and Ye (2007) provide closely related extensions to the model. Amongst other things, they overcome problems with the terminal condition and present the resulting life insurance rules. Other papers studying the effect of mortality risk are Blanchard (1985), Blanchet-Scalliet et al. (2008) and Blanchet-Scalliet, Karoui, and Martellini (2005). Blanchet-Scalliet et al. (2008) extend the model setup of Merton (1971) with a stochastic time horizon. They show that the optimal portfolio decisions are influenced by the stochastic time of death if randomness in stocks and time of death are related. They derive their main results using a martingale approach. Kraft and Steffensen (2008) extend the setup with a state switch between "alive" and "dead" to a Markov chain with more states, interpreted as "unemployed" or "disabled". They focus on the optimal consumption pattern and neglect the asset allocation dimension. Bruhn and Steffensen (2011) also use the Markov chain approach and focus on a family with several mortality processes. They calculate the corresponding optimal consumption, investment and insurance decision. Kwak, Shin, and Choi (2011) consider the life insurance purchase of a family as well. They use a HARA utility as a weighted average from parents' and children's utility. The parents can buy life insurance to protect their children from an income loss that occurs if the parents suffer an early death. Huang, Milevsky, and Salisbury (2012) use the basic model from Yaari (1965) for a comparison of optimal consumption in two frameworks: one with deterministic mortality rates and another one with stochastic mortality rates modeled with a diffusive component. In the simple model with deterministic mortality rates, their results coincide with the results from Yaari (1965). Their analysis focuses on the effect of mortality risk on consumption for retired agents. They do not include an asset allocation dimension and do not model income. However, all papers mentioned in this paragraph analyze the uncertainty with respect to the time of death analytically and none of them provides results for a calibrated life cycle in continuous time or includes jumps in the hazard rate of death.

Huang, Milevsky, and Wang (2008) solve a continuous-time life cycle model with deterministic mortality risk numerically. They include a life insurance and focus especially on a correlation between labor income and the stock market. To the best of my knowledge, Kraft, Schendel, and Steffensen (2014) is the only continuous-time paper with a realistically calibrated life cycle and stochastic mortality risk. They include health jumps which increase the mortality risk significantly. Their focus lies on the analysis of the optimal term life insurance demand of a family that faces the risk of an unexpected early death of the wage earner.

In contrast to continuous-time models, there is a vast literature about discrete-time models that analyze life cycle models with deterministic mortality risk using numerical methods. The seminal paper of Cocco, Gomes, and Maenhout (2005) provides numerical results for a realistically calibrated lifetime optimization problem which also features mortality risk, unspanned labor income as well as short-sale and borrowing constraints. Especially, they analyze how unspanned labor income can be calibrated to fit real data. In a similar way, Cocco (2005) analyzes the effect of housing on asset allocation in a calibrated model. Recent research like Horneff et al. (2008), Horneff et al. (2009), Horneff, Maurer, and Stamos (2008), Maurer et al. (2013) and Chai et al. (2011) mostly focuses on different types of annuities in realistic setups considering the impact on retirement planning and the possibility to hedge longevity risk. Horneff et al. (2008) compare different strategies for retirees to invest their wealth with focus on annuities and phased withdrawal plans. Additionally, they provide a comprehensive literature overview regarding the retirement payout research. Horneff, Maurer, and Stamos (2008) numerically solve a life cycle model for an agent with Epstein-Zin preferences who can invest in illiquid life annuities with non-zero initial cost. This implies that insurance decisions cannot be revised and the insurance decision is not independent from the asset allocation decision. Furthermore, they explore the welfare impact of an incomplete insurance market that allows for gradual investment. Horneff et al. (2009) consider a different type of annuities, called survival-contingent investment-linked annuities. These products have the advantage of participating in the stock market and simultaneously pooling longevity risk. They give the surviving agents the extra return from dying agents (survival credit) at the cost of illiquidity. In contrast to classical annuities that can be considered as an implicit investment in a bond, these investment-linked annuities deliver an implicit investment in a specific portfolio. They also use Epstein-Zin preferences and allow for differences in believes considering mortality risk. Chai et al. (2011) make the model more realistic by allowing the agent to endogenously choose if he wants to work, how much hours he wants to work and when he wants to retire. They use power utility and a modified Cobb-Douglas function for the trade-off between leisure and consumption. They also allow for illiquid life annuities (not investment-linked) and differences in believes with regard to survival probabilities. Love (2010) considers the modeling of a family together with demographic shocks that allow for a varying family size. The model also features mortality risk and a simple term life insurance. Hubener, Maurer, and Rogalla (2013) focus on a couple with two mortality processes in the retirement state. They do not consider a working stage with labor income and especially analyze the optimal life insurance and annuity demand.

Recent discrete-time life cycle models also include stochastic mortality risk, however there are only few published papers. Cocco and Gomes (2012) model mortality rates as a random walk with drift. They consider the effect of stochastic mortality rates by allowing to choose the date of retirement endogenously and they analyze longevity bonds as investment products. Maurer et al. (2013) allow for systematic longevity risk, which they model with stochastic mortality tables. They make the annuity product more complex and analyze the impact of variable investment-linked deferred annuities. These products allow the payout period to begin at some predetermined specific age. As far as I know, there is no discrete-time life cycle portfolio choice paper that allows for jumps in the mortality rate.

Besides mortality risk, another important feature of my model is unspanned labor income. Merton (1971) analyzes the effect of deterministic labor income that can be summarized as an implicit investment in the risk-free asset. Bodie, Merton, and Samuelson (1992) focus on spanned labor income in a life cycle model in which agents can decide about their work effort. However, the importance of unspanned labor income is well known in the literature and outlined e.g. by Viceira (2001) as well as Cocco, Gomes, and Maenhout (2005). In continuous time, Munk and Sørensen (2010) provide results for a realistically calibrated life cycle with unspanned labor income risk, but with deterministic time of death. They analyze the optimal consumption and investment decisions in a setup with power utility, unspanned labor income and a stochastic risk-free rate.

3 Model Setup

I introduce a general model and the corresponding life cycle consumption-investment problem. I focus on four specifications of the general model that differ with respect to the mortality structure.

Financial Assets I consider a continuous-time model. The economy consists of two assets. The risk-free rate is constant and denoted by r. The agent can invest in the risky stock S with dynamics

$$\begin{aligned} dS_t &= S_t \left[\mu_S \, dt + \sigma_S \, dW_t^S \right] \\ &= S_t \left[(r + \sigma_S \lambda) \, dt + \sigma_S \, dW_t^S \right] \end{aligned}$$

with constant parameters μ_S, σ_S and a standard Brownian motion $W^S = (W_t^S)$. The parameter $\lambda = \frac{\mu_S - r}{\sigma_S}$ denotes the market price of risk. The second asset is a riskless bond B which yields the risk-free rate. The price dynamics are given by

$$dB_t = B_t r\, dt.$$

Labor Income The agent receives a stream of income Y until his time of death. Before retirement, Y is interpreted as labor income. After retirement, the payment stream can be interpreted as pension that is payed by the government and is related to earnings before retirement. Y is modeled with dynamics

$$dY_t = Y_t \left[\mu_Y(t)\, dt + \sigma_Y(t) \left(\rho(t)\, dW_t^S + \sqrt{(1-\rho(t)^2)}\, dW_t^Y \right) \right], \tag{1}$$

for $t \in [0, \tau)$ with volatility $\sigma_Y(t)$ and a standard Brownian motion $W^Y = (W_t^Y)$ that is independent of W^S. The correlation between Y and the stock S is captured by $\rho(t)$. The drift $\mu_Y(t)$ allocates the expected labor income over the life cycle.

Mortality In the general model with stochastic time of death, the agent faces mortality risk. He has an uncertain lifetime which is modeled by a doubly stochastic stopping time τ. The time of death is the first jump of the jump process $N^D = (N_t^D)$ with time-dependent and stochastic intensity π_t. The jump process N^D is independent of all other sources of risk, i.e. the Brownian motions W^S and W^Y. The hazard rate of death π_t follows

$$d\pi_t = \mu_\pi(t)\pi_t\, dt + \sigma_\pi(t)\pi_t\, dW_t^\pi + \beta(t)\, dN_t^\pi \tag{2}$$

with another Brownian motion $W^\pi = (W_t^\pi)$ and a jump process $N^\pi = (N_t^\pi)$ with time-dependent intensity $\kappa(t)$ and magnitude $\beta(t)$. These sources of risk are independent from the other Brownian motions and the death jump process. I use the drift parameter to model that the risk of dying increases when the agent gets older ($\mu_\pi(t) > 0$). The diffusion parameter captures small fluctuations in the hazard rate of death, e.g. given by a common cold. The jump term accounts for catastrophic events, like a critical illness ($\beta(t) > 0$).

Insurance The agent has the possibility to contract a simple insurance as in Richard (1975). The insurance can be interpreted as a life insurance. However, the results show that agents optimally sell the life insurance to the insurance company and therefore, I construct the insurance in the inverse way. For a notional of 1, the insurance pays $\pi_t\, dt$ if the insured person stays alive. If the agent dies, the insurance receives the notional. The insurance is continuously traded without transaction costs such that the agent can adjust the notional continuously. The agent chooses η_t which is the fraction of his financial wealth he wants to insure. The financial wealth of the agent is denoted by X_t. Thus, as long as the agent lives, he continuously receives $\eta_t \pi_t X_t \, dt$ from the insurance and has to pay $\eta_{\tau^-} X_{\tau^-}$ to the insurance when he dies. The insurance is actuarially fair by construction.

The assumption of such an insurance can be justified as follows: If one considers a world with a large amount of identical agents facing the same mortality risk that are all equally insured, the insurance breaks even almost surely. Thus, assuming a competitive insurance market with risk neutral insurance companies would lead to such an insurance contract in the absence of administrative fees.[1] Due to the unrealistic features of the insurance and the non-availability of such an insurance in the real world, I also consider a model setup where agents are not allowed to contract the insurance.

Preferences of the Agent The agent lives from time 0 to τ. I consider an agent with utility of the constant relative risk aversion (CRRA) type with risk aversion parameter γ. The agent wants to maximize lifetime utility of consumption and terminal wealth at every point in time $t \in [0, \tau)$. The expected utility at time t is given by

$$\mathrm{E}_{t,x,y,\pi}\left[\int_t^\tau \mathrm{e}^{-\delta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma} \, ds + \epsilon(t) \mathrm{e}^{-\delta(\tau-t)} \frac{X_\tau^{1-\gamma}}{1-\gamma}\right], \tag{3}$$

where δ is the time preference rate and ϵ captures the weight of the bequest motive. X_τ is the financial wealth that the agent leaves as bequest. Due to the insurance choice, the financial wealth for bequest is calculated as

$$X_\tau = (1 - \eta_{\tau^-}) X_{\tau^-},$$

since the remaining financial wealth has to be payed to the insurance.

To gain intuition, I convert the model with uncertain but finite horizon to a model with infinite horizon. For a moment, I drop the bequest motive and assume a deterministic hazard rate of death. Then, I rewrite the expected utility as

$$\mathrm{E}_{t,x,y,\pi}\left[\int_t^\infty \mathrm{e}^{-\int_t^v \delta + \pi(s) ds} u(c_v) \, dv\right]. \tag{4}$$

Considering (4) in detail, we see that the setup is like an infinite horizon model but with a new time preference rate $\delta + \pi(t)$. Furthermore, the time preference rate is not constant, which might yield severe problems considering model solving since this may lead to

[1] Risk averse insurance companies or insurance companies facing administrative costs would demand a higher payment in the case of death or pay a lower rate if the agent survives.

time-inconsistent behavior according to Marín-Solano and Navas (2009, 2010).[2] However, Bommier (2006b) disentangles uncertainty with respect to death and time by assuming that the agent knows that he cannot die in a certain interval. Then, he analyzes conditions under which the agent has time-consistent preferences. According to his calculations, the preferences I use in this paper are time-consistent.

Financial Wealth Dynamics and the Optimization Problem The financial wealth of the agent is denoted by X. The agent chooses optimal consumption c and optimal portfolio holdings θ, where θ denotes the fraction of financial wealth invested into the stock. The remaining financial wealth $(1-\theta)X$ is invested into the bond. Furthermore, the agent decides about the fraction of financial wealth that is insured η. Given the consumption, portfolio holdings and insurance decision, the financial wealth evolution for $t \in [0,\tau]$ has the dynamics

$$dX_t = X_{t^-}\left[\left(r+\pi_{t^-}\eta_{t^-}+\theta_{t^-}\lambda\sigma_S\right)dt+\theta_{t^-}\sigma_S\,dW_t^S\right]+(Y_{t^-}-c_{t^-})\,dt-\eta_{t^-}X_{t^-}\,dN_t^D. \quad (5)$$

The objective of the agent is maximizing utility from intermediate consumption and terminal wealth. Hence, the optimization problem of the agent is given by

$$\begin{aligned}\max_{c,\theta,\eta}\quad & \mathrm{E}_{t,x,y,\pi}\left[\int_t^\tau \mathrm{e}^{-\delta(u-t)}\frac{c_u^{1-\gamma}}{1-\gamma}du+\epsilon(t)\mathrm{e}^{-\delta(\tau-t)}\frac{X_\tau^{1-\gamma}}{1-\gamma}\right]\\ \text{s.t.}\quad & dX_t = X_{t^-}\left[\left(r+\pi_{t^-}\eta_{t^-}+\theta_{t^-}\lambda\sigma_S\right)dt+\theta_{t^-}\sigma_S\,dW_t^S\right]+(Y_{t^-}-c_{t^-})\,dt-\eta_{t^-}X_{t^-}\,dN_t^D. \quad (6)\end{aligned}$$

I define the value function (indirect utility) J as

$$J(t,x,y,\pi)=\sup_{\{c_s,\theta_s,\eta_s\}_{s\in[t,\tau)}}\mathrm{E}_{t,x,y,\pi}\left[\int_t^\tau \mathrm{e}^{-\delta(u-t)}\frac{c_u^{1-\gamma}}{1-\gamma}du+\epsilon(t)\mathrm{e}^{-\delta(\tau-t)}\frac{X_\tau^{1-\gamma}}{1-\gamma}\right]. \quad (7)$$

Calculating the corresponding Hamilton-Jacobi-Bellman equation (HJB) yields

$$\begin{aligned}\delta J = \sup_{c,\theta,\eta}\quad & \left\{\frac{c^{1-\gamma}}{1-\gamma}+J_t+J_x\left[x\left(r+\eta\pi+\theta\lambda\sigma_S\right)+y-c\right]+\frac{1}{2}J_{xx}x^2\theta^2\sigma_S^2\right.\\ & +J_y y\mu_Y+\frac{1}{2}J_{yy}y^2\sigma_Y^2+J_{xy}xy\sigma_S\sigma_Y\rho\theta\\ & +J_\pi\pi\mu_\pi+\frac{1}{2}J_{\pi\pi}\pi^2\sigma_\pi^2\\ & +\pi\left[J(\tau,(1-\eta)x,0,0)-J(t,x,y,\pi)\right]\end{aligned}$$

[2] Marín-Solano and Navas (2009) comment on the problems occurring with time-dependent time preference rates and illustrate how to derive Hamilton-Jacobi-Bellman equations for pre-commited, naive and sophisticated agents with time-inconsistent preferences. Marín-Solano and Navas (2010) apply the approach in a Merton (1969, 1971) setup to derive optimal consumption and portfolio rules for these agents.

$$+\kappa\left[J(t,x,y,\pi+\beta)-J(t,x,y,\pi)\right]\Big\}, \tag{8}$$

where I use subscripts for partial derivatives of J, for example: $J_x = \frac{\partial J}{\partial x}$.

Special Cases In the following sections, I consider four model specifications with different mortality structure. Increasing in uncertainty with respect to mortality risk, the four specifications are derived from the general model as follows:

In model D, the time of death is deterministic and known to the agent. I get this model from the above setup by setting τ equal to a constant which implicates $\pi = \kappa = 0$. This case leads to a classical finite horizon model. The insurance has no effect, π is not needed as state variable and the HJB (8) gets the terminal condition $J(\tau,x,y) = \epsilon(\tau)\frac{X_\tau^{1-\gamma}}{1-\gamma}$.

The model S has a stochastic time of death, but the hazard rate of death itself is not stochastic. I get the setup by setting $\sigma_\pi = \kappa = 0$. Furthermore, π as state variable is redundant in this case, since there is no additional uncertainty captured by π due to the time state variable t.

In the model SD the agent faces stochastic mortality risk with a diffusion component. I set $\kappa = 0$.

The SDJ model with most sources of risk additionally includes the jump component in the hazard rate of death. This setup is exactly as presented above.

4 Analytical Results

This section provides analytical results for a special case with deterministic labor income for the models D and S. I simplify the problem by setting $\sigma_Y = 0$. In this case, the market is complete and the labor income can be replicated using the riskless bond. Alternatively, I could set $\sigma_Y \neq 0$ but $\rho = \pm 1$ instead. Then, labor income is spanned (it can be replicated using the stock and the bond), the market is complete and analytical results can be derived. However, a perfect correlation of the stock and labor income is highly unrealistic, which is why I stick to the assumption of $\sigma_Y = 0$ here.

First, I present the results for the model with deterministic time of death D in the special case of deterministic labor income. The results are summarized in the following proposition.

Proposition 1. *For a complete market case with deterministic labor income, $\sigma_Y = 0$, the optimization problem (6) of the model D can be solved analytically. The indirect utility (7) can be expressed as*

$$J^D(t,x,y) = \frac{1}{1-\gamma}\left(x + yf^D(t)\right)^{1-\gamma} g^D(t)^\gamma,$$

for $t \in [0,\tau]$ *with*

$$
\begin{aligned}
f^D(t) &= \int_t^\tau e^{\int_t^s \mu_Y(u) - r\, du}\, ds, \\
g^D(t) &= \int_t^\tau e^{\left(\frac{1-\gamma}{\gamma} r - \frac{\delta}{\gamma} + \frac{1}{2}\frac{(1-\gamma)\lambda^2}{\gamma^2}\right)(s-t)}\, ds + \epsilon(\tau)^{\frac{1}{\gamma}} e^{\left(\frac{1-\gamma}{\gamma} r - \frac{\delta}{\gamma} + \frac{1}{2}\frac{(1-\gamma)\lambda^2}{\gamma^2}\right)(\tau-t)} \\
&= \frac{1}{\frac{1-\gamma}{\gamma} r - \frac{\delta}{\gamma} + \frac{1}{2}\frac{(1-\gamma)\lambda^2}{\gamma^2}} \left(e^{\left(\frac{1-\gamma}{\gamma} r - \frac{\delta}{\gamma} + \frac{1}{2}\frac{(1-\gamma)\lambda^2}{\gamma^2}\right)(\tau-t)} - 1 \right) \\
&\quad + \epsilon(\tau)^{\frac{1}{\gamma}} e^{\left(\frac{1-\gamma}{\gamma} r - \frac{\delta}{\gamma} + \frac{1}{2}\frac{(1-\gamma)\lambda^2}{\gamma^2}\right)(\tau-t)}.
\end{aligned}
$$

For $t \in [0,\tau)$*, the optimal controls for consumption and portfolio holdings are*

$$
\begin{aligned}
c^D(t,x,y) &= \frac{x + y f^D(t)}{g^D(t)}, \\
\theta^D(t,x,y) &= \frac{1}{\gamma}\frac{x + y f^D(t)}{x}\frac{\lambda}{\sigma_S}.
\end{aligned}
$$

Proof. The formulas can be verified by substituting the results for J, c, θ into the HJB (8). The derivation follows the lines of the proof of Proposition 2 and is therefore skipped here. □

Next, I consider the model S with stochastic time of death. In the case of deterministic labor income ($\sigma_Y = 0$), the optimization problem (6) can be solved analytically using a similar approach as above. The results are summarized below.

Proposition 2. *With deterministic labor income,* $\sigma_Y = 0$*, the optimization problem (6) of the model S can be solved analytically and the indirect utility (7) is given by*

$$
J^S(t,x,y) = \mathbb{1}_{\{t<\tau\}} \left(\frac{1}{1-\gamma} \left(x + y f^S(t) \right)^{1-\gamma} g^S(t)^\gamma \right) + \mathbb{1}_{\{t=\tau\}} \left(\epsilon(t) \frac{x^{1-\gamma}}{1-\gamma} \right),
$$

for $t \in [0,\tau]$ *where*

$$
\begin{aligned}
f^S(t) &= \int_t^\infty e^{\int_t^s \mu_Y(u) - r - \pi(u)\, du}\, ds, \\
g^S(t) &= \int_t^\infty e^{\int_t^s \frac{1-\gamma}{\gamma} r - \frac{1}{\gamma}\delta - \pi(u) + \frac{1}{2}\frac{1-\gamma}{\gamma^2}\lambda^2\, du} \left(1 + \pi(s)\epsilon(s)^{\frac{1}{\gamma}} \right) ds.
\end{aligned}
$$

The optimal controls for $t \in [0,\tau)$ *are*

$$
\begin{aligned}
c^S(t,x,y) &= \frac{x + y f^S(t)}{g^S(t)}, \\
\theta^S(t,x,y) &= \frac{1}{\gamma}\frac{x + y f^S(t)}{x}\frac{\lambda}{\sigma_S},
\end{aligned}
$$

$$\eta^S(t,x,y) = 1 - \frac{x + y f^S(t)}{g^S(t) x} \epsilon(t)^{\frac{1}{\gamma}}.$$

Proof. The formulas can be verified by substituting the results for J, c, θ, η into the HJB (8). A derivation is given in Appendix A. □

Insurance Demand Note that η can be rewritten as

$$\eta^S(t,x,y) = 1 - \frac{c^S(t,x,y)}{x} \epsilon(t)^{\frac{1}{\gamma}}. \tag{9}$$

The optimal fraction of insured wealth depends only on the actual consumption-to-wealth ratio, the weighting of the bequest motive and the risk aversion parameter. But, it is independent of the hazard rate of death. This is worth noting since the mortality risk is the intensity that triggers the event of death and is therefore crucial for insurance related payoffs. However, due to the insurance being actuarially fair and the agent being risk neutral with respect to the time of death, the fraction insured is independent of the mortality risk. Substituting (9) into the bequest if death occurs yields

$$\left(1 - \eta^S(t,x,y)\right) x = \left(1 - 1 + \frac{c^S(t,x,y)}{x} \epsilon(t)^{\frac{1}{\gamma}}\right) x = c^S(t,x,y) \epsilon(t)^{\frac{1}{\gamma}}.$$

The amount of bequest is expressed as the actual consumption level weighted with the bequest importance parameter to the power of the intertemporal elasticity of substitution. This indicates that in the complete market the insurance is only used to ensure optimal bequest if death occurs. Furthermore, the bequest is intended and not accidental due to the insurance.

Comparing the Complete Market Formulas I start the comparison with a simpler model without bequest motive. It follows immediately that it is optimal for the agent to insure his whole wealth: If his wealth is positive and the agent survives, he receives an insurance premium which is a benefit for the agent. If the agent dies, he loses all his wealth. However, the agent is indifferent between leaving money on the table and paying the notional to the insurance since he has no bequest motive. Furthermore, one can prove the intuitive statement that the agent wants to fully insure his financial wealth if there is no bequest motive via substituting $\epsilon = 0$ in the results from Proposition 2. This yields $\eta(t,x,y) = 1$ and hence, the whole financial wealth is insured.

I consider the analytical results for the complete market of the Propositions 1 and 2 and substitute $\epsilon = 0$. The structure of the indirect utility and optimal controls is identical in both setups. The effect of mortality risk is embedded in the functions g and f. Writing the results from Proposition 1 and 2 in a slightly different manner (using $\epsilon = 0$), I get

$$
\begin{aligned}
f^D(t) &= \int_t^\tau e^{\int_t^s \mu_Y(u) - r\,du}\, ds, \\
f^S(t) &= \int_t^\infty e^{\int_t^s \mu_Y(u) - r\,du}\, e^{-\int_t^s \pi(u)\,du}\, ds, \\
g^D(t) &= \int_t^\tau e^{\left(\frac{1-\gamma}{\gamma} r - \frac{\delta}{\gamma} + \frac{1}{2}\frac{(1-\gamma)\lambda^2}{\gamma^2}\right)(s-t)}\, ds, \\
g^S(t) &= \int_t^\infty e^{\left(\frac{1-\gamma}{\gamma} r - \frac{\delta}{\gamma} + \frac{1}{2}\frac{(1-\gamma)\lambda^2}{\gamma^2}\right)(s-t)}\, e^{-\int_t^s \pi(u)\,du}\, ds.
\end{aligned}
$$

The only difference is that the integrals of the model D run to infinity instead to τ and the term $e^{-\int_t^s \pi(u)\,du}$, which equals the survival probability from time t to s, is added. Thus, the effect of added mortality risk can be interpreted as a different discounting here. Nothing else changes.

Now adding the bequest motive, the formulas for f stay the same and the g formulas get the additional bequest term as in the Propositions. The bequest term enters weighted with the hazard rate of death in the S model for each possible time of death to capture the uncertainty with respect to death. In contrast, the bequest occurs at the fixed time τ in the D model and is discounted. Hence, the different weighting of the bequest can also be interpreted as a different discounting.

Whereas the formulas are useful to provide a rough intuition, analytical solutions are not possible for realistically calibrated setups in which labor income uncertainty is an important source of risk.

5 Calibration

For a numerical analysis of the effects of mortality risk, I calibrate the models. The calibration and the parameters that I choose for assets, labor income and preferences are closely related to the calibration used by Munk and Sørensen (2010) and Cocco, Gomes, and Maenhout (2005). For calibrating the mortality risk I use German mortality data.

Assets I choose the parameters for the assets as

$$
\begin{aligned}
\mu_S &= 0.06, \\
\sigma_S &= 0.2, \\
r &= 0.02,
\end{aligned}
$$

which are similar to the values of Munk and Sørensen (2010) and Cocco, Gomes, and Maenhout (2005).

Labor Income Calibrating the labor income process, I mainly follow Munk and Sørensen (2010). They adapt the calibration from Cocco, Gomes, and Maenhout (2005) to a continuous time model. Cocco, Gomes, and Maenhout (2005) model the drift of the labor income process before retirement as a polynomial of the age of the agent for different education groups (no high school, high school, college) using data from the Panel Study of Income Dynamics (PSID). After retirement, the drift is set to zero and the agent receives a fixed fraction of the income before retirement as pension. The continuous-time adaption from Munk and Sørensen (2010) yields for the labor income drift

$$\mu_Y(t) = \begin{cases} \xi_0 + b + 2ct + 3dt^2 & \text{for } t < T_{ret}, \\ -(1-P) & \text{for } T_{ret} \le t \le T_{ret}+1, \\ 0 & \text{for } t > T_{ret}+1, \end{cases} \tag{10}$$

where b, c, d are constant parameters dependent on the education level, originally estimated by Cocco, Gomes, and Maenhout (2005). ξ_0 captures an real wage increase that is independent of the age and the education. P determines the income reduction when going into retirement. T_{ret} denotes the age at which the agent retires. I take the parameter calibration from Munk and Sørensen (2010) (Table 4) for college graduates

$$\begin{aligned} T_{ret} &= 65, & b &= 0.3194, \\ \xi_0 &= 0.02, & c &= -0.00577, \\ P &= 0.93887, & d &= 0.000033, \\ Y_0 &= 13\,912. \end{aligned}$$

Y_0 denotes the starting income at the age of 20 which depends on the education. The volatility function that I use allows different values for the working period and the retirement period with linear interpolation when the status changes, hence

$$\sigma_Y(t) = \begin{cases} \sigma_Y^w & \text{for } t < T_{ret}, \\ \sigma_Y^w - \left(\sigma_Y^w - \sigma_Y^r\right)(t - T_{ret}) & \text{for } T_{ret} \le t \le T_{ret}+1, \\ \sigma_Y^r & \text{for } t > T_{ret}+1. \end{cases} \tag{11}$$

I assume the same structure for the correlation with the risky asset

$$\rho(t) = \begin{cases} \rho^w & \text{for } t < T_{ret}, \\ \rho^w - \left(\rho^w - \rho^r\right)(t - T_{ret}) & \text{for } T_{ret} \le t \le T_{ret}+1, \\ \rho^r & \text{for } t > T_{ret}+1. \end{cases} \tag{12}$$

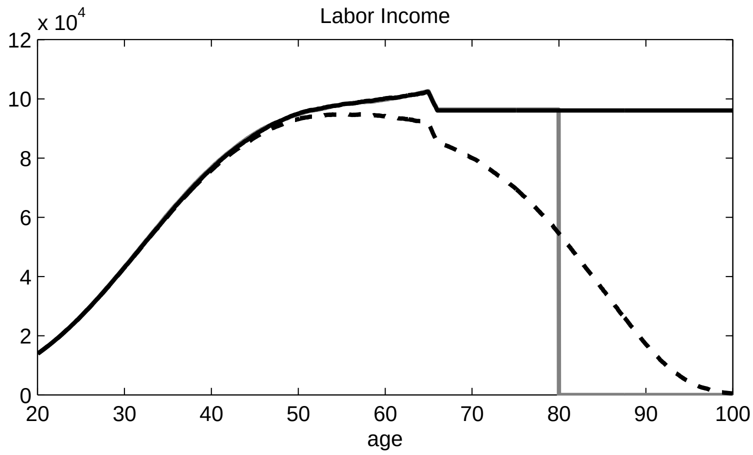

Figure 1: Average Labor Income Profile over the Life Cycle. The black solid line depicts the average labor income of living agents in a model with stochastic time of death (S). The dashed line shows the expected labor income with stochastic time of death. The grey line corresponds to the model with deterministic time of death (D). The labor income process is calibrated with the values from Munk and Sørensen (2010) for college graduates. The parameter calibration is stated in Section 5.

In the benchmark calibration, I stick to the values used by Munk and Sørensen (2010) for the volatility and correlation:

$$\sigma_Y^w = 0.2, \qquad \rho^w = 0,$$
$$\sigma_Y^r = 0, \qquad \rho^r = 0.$$

The resulting pattern for expected labor income is shown in Figure 1.

Mortality Risk To calibrate the mortality process, I use mortality data for Germany. I calibrate the models D, S, SD and SDJ separately. In the deterministic time of death model D, I set $\tau = 80$, which is the average life expectancy in Germany for newborns in 2010.[3] For calibrating the stochastic time of death models, I use age-dependent mortality data.[4] The mortality data is given for males and females separately. I weight both genders equally and calculate the corresponding average values. Considering the drift component of the hazard rate of death, I rewrite the initial value and drift parameter to the classical

[3] The number is estimated using life expectancy at birth, total (years). "Life expectancy at birth indicates the number of years a newborn infant would live if prevailing patterns of mortality at the time of its birth were to stay the same throughout its life." Source: WDI and GDF 2010. The estimate for a child in Germany at 2010 is given by 79.988. The source for this information, from which I directly cite here, is The World Bank Group. The data is available online at: `http://search.worldbank.org/data?qterm=SP.DYN.LE00.IN`, last access: January 21, 2014.

[4] Mortality data is taken of a Life table for Germany "Sterbetafel 2009/11, Statistisches Bundesamt, 2013", available online at: `https://www.destatis.de/DE/ZahlenFakten/GesellschaftStaat/Bevoelkerung/Sterbefaelle/Tabellen/SterbetafelDeutschland.html`, last access: January 21, 2014.

	b	m	σ_π
S	8.8	84.56	0.00
SD	8.3	83.68	0.05
SDJ	4.7	87.55	0.10

Table 1: Mortality Process Parameters. The table gives the drift and diffusion parameter calibration of the mortality process that I use in Section 8.

Gompertz form. The initial value and drift are then expressed as

$$\mu_\pi = \frac{1}{b},$$
$$\pi_0 = \frac{1}{b} e^{\frac{x-m}{b}}$$

with constant parameters x, m, b. The interpretation of x is the age at $t = 0$. m sets the x-axis displacement and the growth rate b influences the steepness of the curve. I set the starting age $x = 20$ in all models. In Section 6 and 7, I show results for the model S and compare them to the deterministic time of death model D. In this case, when I have no diffusive and jump component, I can directly express the hazard rate of death and get the standard Gompertz form

$$\pi(t) = \frac{1}{b} e^{\left(\frac{x+t-m}{b}\right)}.$$

In the model S in Section 6 and 7, I set $b = 8.9$ and $m = 85.1$, which leads to an average age of death of $E[\tau] = 80$. Considering the comparison of the S, SD and SDJ model in Section 8, I have to stick to the stochastic differential equation definition (2) of the hazard rate of death. I calibrate the models such that the average age of death equals $E[\tau] = 80$ in all models. Furthermore, the parameters are set such that the time of death distribution fits the empirical values of Germany. The calibration of the parameters m, b and the diffusive component σ_π are given in Table 1. Note that the model S is calibrated differently in Section 8 compared to Section 6 and 7.[5] Comparing the S, SD and SDJ model calibrations, we can see that the model S has no uncertainty about future mortality patterns, whereas the model SD has little uncertainty about future mortality rates, and the model SDJ has plenty of uncertainty considering the future mortality situation. Next, I give the results for the calibration of the jump term of the hazard rate of death in the SDJ model. I interpret a jump in the hazard rate of death as a critical illness and use gender-averaged

[5] The different calibration results from the differences in the calculation method. The first method directly expresses π, whereas the second method expresses the increase of π via a differential equation. Numerically small deviations are not avoidable although both expressions are analytically identical. The difference between both techniques decreases with a decreasing numerical choice of the length of dt. Although the effect is very small, I use slightly different calibrations here to capture the impact.

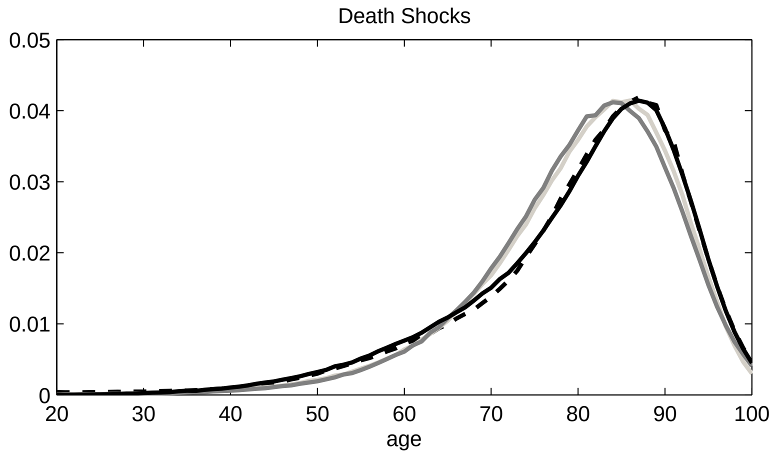

Figure 2: Biometric Risk Calibration Results. The graph compares the gender-averaged German mortality data (dashed line) with my model calibrations (solid lines). The figure depicts the death distribution for a normalized population of size 1. My simulated death shocks are averaged after simulating 1 000 000 times of death. The light line represents the model S, the grey line the model SD and the black line the model SDJ. The calibration can be found in Section 5.

cancer data for Germany for the calibration.[6] The health jump calibration is taken from Kraft, Schendel, and Steffensen (2014) where it is described in detail. Thus, I set the health jump intensity to

$$\kappa(t) = 0.02489\,\mathrm{e}^{-\left(\frac{\min(t,65)-66.96}{29.42}\right)^2},$$

and the corresponding jump magnitude to

$$\beta(t) = 0.048 + 0.0008t.$$

Figure 2 depicts the empirical mortality pattern, compared to the resulting mortality patterns of the S, SD and SDJ calibration. We directly see that the calibration of the model SDJ fits the empirical mortality pattern pretty well, whereas the S and SD calibration do significantly worse. In early years for the age of 20 until 35, none of the three models is able to capture the relatively high empirical number of deaths. Then up to the age of 63, the models S and SD underestimate the number of deaths significantly, whereas the model SDJ only slightly overestimates the empirical values. Afterwards roughly until the age of 85, the models S and SD consequently overestimate death rates. Especially, the peak appears too early. After the age of 85, the models S and SD underestimate the empirical values again. The model SDJ and the data show a strong co-movement, especially after the age of 75. The calibration results highlight that the jump component is essential to explain the empirical mortality pattern.

[6] The German cancer data is taken from "Cancer in Germany 2007/2008, German Centre for Cancer Registry Data & Robert Koch Institute, 8th Edition 2012", available online at: `http://www.rki.de/EN/Content/Health_Monitoring/Cancer_Registry/cancer_registry_node.html`, last access: January 21, 2014.

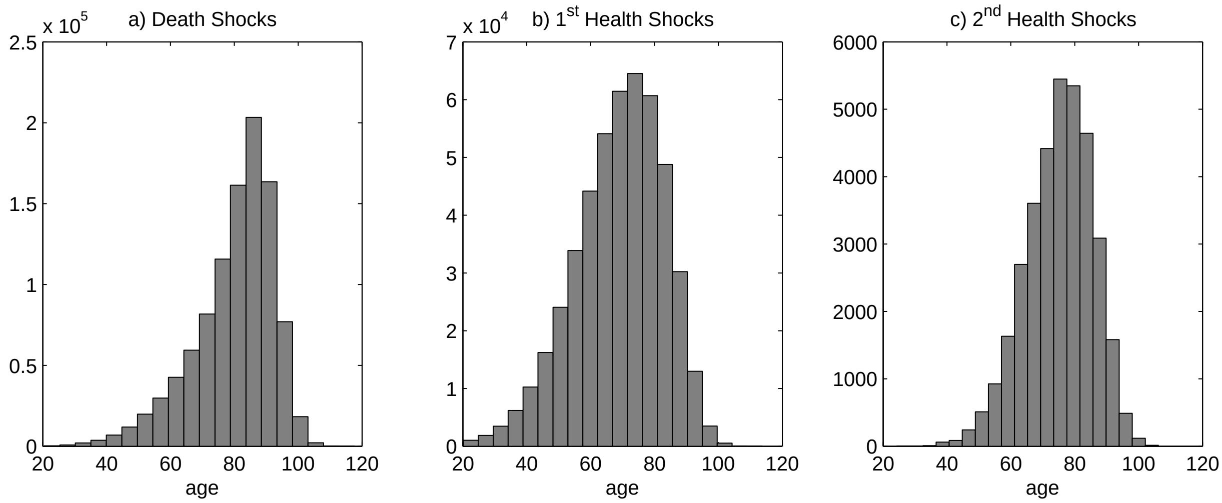

Figure 3: Histogram of Death and Health Shocks in the SDJ Model. The graphs depict the histogram of death and health jumps after 1 000 000 simulations in the model SDJ with the calibration of Section 5. a) shows the death jump distribution, b) presents the histogram of the first health shock and c) depicts the second health shocks.

Figure 3 depicts histograms of death and health shocks for the SDJ model after 1 000 000 simulations. In the sample the average time of death is 80. The average time of the first health shock is 68.9, where 47.8% of the population face health shocks during their lifetime. 3.5% face at least two health shocks with the second one at the average age of 75.1. Only 0.2% of the population face three health shocks or more.

Preferences I choose the risk aversion, the time preference rate and the weight of the bequest motive according to

$$\delta = 0.03,$$
$$\gamma = 4,$$
$$\epsilon = 3.$$

These values are taken from Munk and Sørensen (2010). I set the initial financial wealth equal to the first labor income $X_0 = Y_0 = 13\,912$.

Unfortunately, there is no clear evidence about the impact of a bequest motive in the literature.[7] Cocco, Gomes, and Maenhout (2005) model a bequest motive between 0 and 5. Munk and Sørensen (2010) use values of 1 or 3 for the weight of the bequest motive. In my model, agents face mortality risk throughout their life which raises the question how a bequest motive changes over time. On the one hand, young people normally do not think about a potential death and bequest, whereas the topic is more present to older people. This might indicate an increasing bequest motive over the lifetime. On the other hand, young people contract term life insurance to protect their partner and children in the case of death. In later years, the children are grown up and there is no substantial need for a bequest. These thoughts indicate a decreasing bequest motive. Since there is no clear evidence how the bequest motive should behave over the life cycle, I stick to a constant bequest motive here although the model setup allows a time-dependent bequest motive.

6 Results with Stochastic Time of Death

This section provides numerical results and interpretation for the model S with stochastic time of death but a deterministic hazard rate of death. I show policy functions, the optimal controls and the wealth evolution over the life cyle and provide sensitivity analyses.

Starting from now, I present numerical results for models with unspanned labor income, short-sale and liquidity constraints. This means, I omit the restriction of $\sigma_Y = 0$ from Section 4 to make labor income stochastic and unspanned. Furthermore, agents are restricted to portfolio holdings $\theta_t \in [0,1], \forall t$, which avoids short selling. Additionally, agents have to choose optimal controls such that $X_t > 0, \forall t$. Thus, for low levels of financial wealth, agents have to consume less than their income and are not allowed to invest in the risky asset if financial wealth could go negative. I also restrict the insurance decision to $\eta_t \in [0,1], \forall t$ such that only available financial wealth can be insured and negative insurance holdings are forbidden.

In the following, I show results that are based on the numerical procedure described in Appendix B for different calibrations. All models include the constraints introduced here. The benchmark calibration is as presented in Section 5 for the model S. The figures depict averaged results after 100 000 simulations. Especially, I analyze the effect of the bequest

[7] Abel (1985) argues that agents leave unintended bequest due to mortality risk although agents do not have a bequest motive. Bernheim, Shleifer, and Summers (1985) consider a strategic bequest motive. Parents use bequest to influence their children's behavior. They claim that agents have a strategic bequest motive and provide empirical evidence for their results. Hurd (1989) analyzes empirically whether agents have a bequest motive. According to his results, the motive for intended bequest is nearly zero and observed bequest is mostly accidental. Laitner (2002) provides a comparison of different models with bequest using calibrated simulations. In order to match the data, the models need a positive bequest motive and he gets best results if agents care roughly the same about themselves and their children. Lockwood (2012) argues that the bequest motive is crucial for agents not to insure wealth. He uses the bequest motive as a possible explanation for the annuity puzzle. In his simulations, he gets best results (realistic amounts of insured wealth) if he assumes a bequest motive.

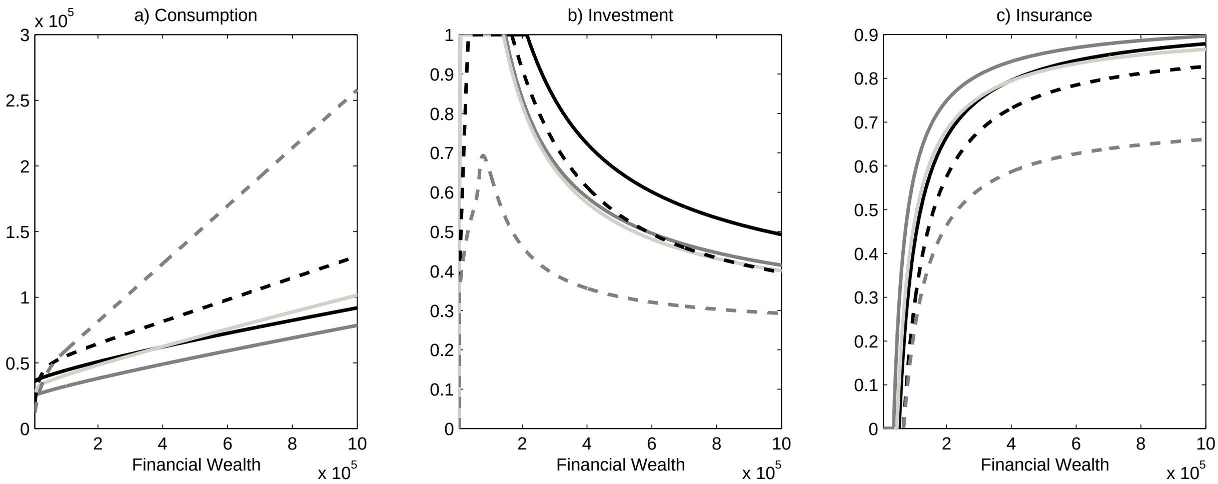

Figure 4: Policy Functions for the S Model. The graphs depict policy functions with varying financial wealth for differently aged agents for fixed labor income and age. The labor income is fixed to 50 000 for all lines. The solid lines are for an agent in the working phase of the age of 25 (dark line), age 40 (grey line) and age 60 (light line). The dashed lines represent a retired agent of the age of 70 (dark line) and age 90 (grey line). a) plots consumption for varying financial wealth, b) shows the corresponding portfolio holdings and c) depicts the insurance decision. The model S is calibrated with the parameters of Section 5.

motive, the driver of the optimal insurance decision and the impact of the labor income parameters. Furthermore, I provide sensitivity analyses with respect to preference and asset parameters.

Policy Functions First, I present policy functions for different ages and a fixed income of 50 000 in Figure 4. On the x-axis is financial wealth. The graphs illustrate the optimal portfolio holdings, fraction of insured wealth and consumption for the working phase (age 25, 40 and 60, solid lines) and for the retirement phase (age 70 and 90, dashed lines). Lemma 3 in Appendix B shows that the optimal controls only depend on the fraction $\frac{x}{y}$. Therefore, it is sufficient to restrict the consideration on the policy functions for financial wealth. Policy functions for labor income would deliver the inverse results.

The consumption graph increases in financial wealth, as usual. Consumption steepness is increasing in age, which can be interpreted as an increase in the time preference rate due to the increased mortality risk. Older agents with more financial wealth want to use their excess wealth before they die and, thus, consume more. Poor old agents consume less than younger ones since they want to accumulate at least a bit financial wealth for bequest.

The amount riskily invested is generally decreasing in financial wealth and decreasing in age. The effect of the secure labor income in retirement is an increase of risky investments. Since labor income changes from being risky to being risk-free, the implicit risky investment due to labor income is gone and, hence, the optimal explicit risky investment increases to offset this effect. With more financial wealth, labor income becomes less important. For diversification purposes, the agent reduces the risky investment and the

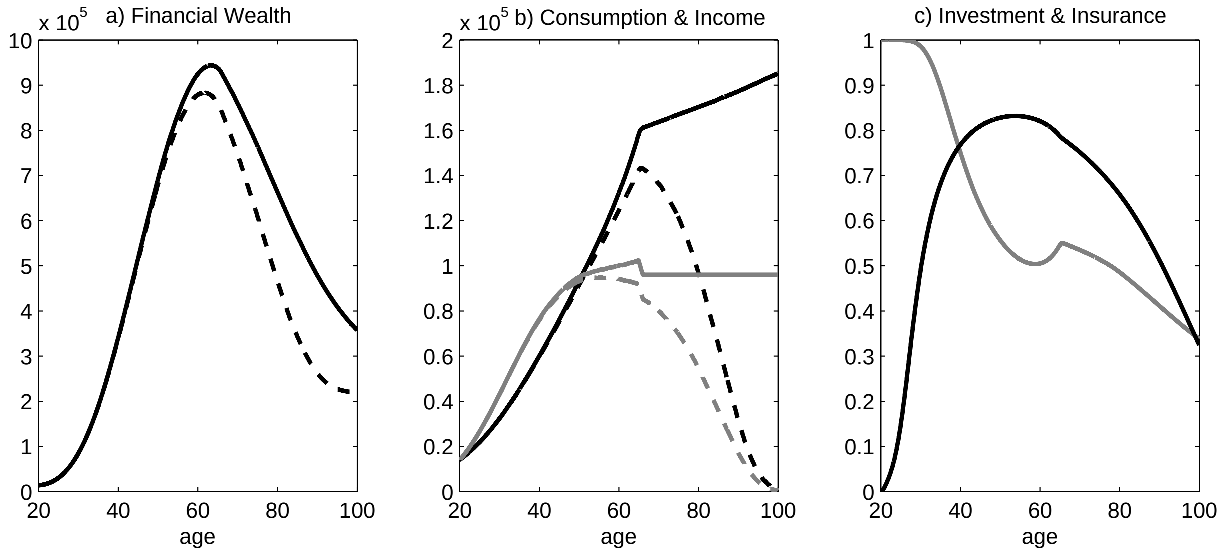

Figure 5: Optimal Wealth and Controls over the Life Cycle in the Model S. a) depicts the average optimal financial wealth over the life cycle. The solid line represent the average financial wealth of living agents. The dashed line shows the average financial wealth of all agents (dead agents are counted with their amount of bequest). b) depicts average optimal consumption (dark lines) and labor income (grey lines) over the life cycle. Solid lines indicate averages for living agents only, whereas dashed lines also include dead agents. c) presents the average optimal fraction riskily invested (grey line) and the average optimal fraction insured (dark line). The model calibration is given in Section 5.

effect of the change in volatility is also less pronounced. For very low levels of financial wealth, the risky investment goes to zero due to the introduced liquidity constraint.

The insurance offers mainly a choice between actual wealth and bequest. Contracting the insurance means more wealth while alive but less bequest. The insurance holdings are increasing in financial wealth for all ages. More financial wealth means more wealth for bequest. This allows the agent to increase the fraction insured since the bequest motive is already fulfilled. One can observe that the insurance holdings for the ages 25 and 60 intersect at the same level of financial wealth as the corresponding consumption lines. This indicates a relation between the insurance choice and the consumption level, as in the complete market in Section 4.

Benchmark Results Figure 5 depicts the average optimal controls and average wealth evolution over the life cycle for the benchmark calibration.

The average financial wealth of living agents and the expected financial wealth of all agents are increasing until shortly before retirement. This is due to excess income over consumption in early years, a positive average return of the invested financial wealth and the insurance premium for living agents. Afterwards, the financial wealth decreases due to excess consumption over income. However, the positive return from investment and the increasing insurance premium mitigate the decrease. The expected financial wealth of all agents approaches the average amount of bequest.

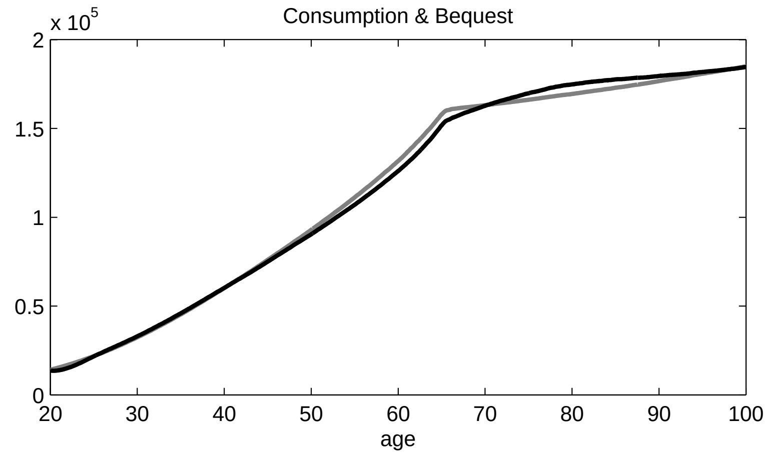

Figure 6: Consumption and Expected Bequest over the Life Cycle. The grey line depicts the average optimal consumption of living agents in the model S with a bequest motive of $\epsilon = 1$. The dark line corresponds to the average optimal bequest that would occur if the agent died (calculated as $(1-\eta_t)X_t$). The bequest motive is changed to $\epsilon = 1$, the remaining calibration is as presented in Section 5 for the model S.

The consumption of the living agents is increasing throughout the lifetime with a jump in consumption growth at retirement. The expected consumption of all agents (where dead agents consume zero, i.e. $c_t = 0$ for $t \geq \tau$) is increasing until retirement and decreases to zero afterwards due to mortality. Comparing consumption with labor income, we can see that labor income exceeds consumption until the age of 50 in order to accumulate financial wealth. Afterwards, consumption exceeds labor income to ensure an increasing consumption path over the life cycle and dissave excess financial wealth.

The fraction riskily invested is decreasing over the lifetime, as usual. With mortality risk, the risky investment decreases with increasing risk of dying. At retirement, the labor income volatility decreases which affects the implicit risky investment. Therefore, we observe the slight increase in risky investment at retirement to offset this effect.

The insurance holdings show a hump-shaped pattern. The next paragraph explains that the agent chooses the insurance holding at every point in time such that if death occurs, he realizes the desired amount of financial wealth for bequest. With the optimal insurance decision the agent avoids accidental bequest. The insurance is the best way for the agent to allocate wealth to bequest since other controls are not affected and the financial wealth evolution only slightly depends on the insurance holdings as long as mortality risk is low. Hence, the hump-shaped insurance choice over the life cycle is explained by the hump-shaped financial wealth graph.

The Impact of the Insurance To shed light on how the agent uses the insurance and to identify the main driver of the insurance decision, I consider the optimal insurance decision together with the amount of bequest in detail here. Figure 6 depicts the optimal average consumption (grey line) in a model where the bequest motive is set to $\epsilon = 1$. The black line represents the amount of bequest that the agent would leave if he would die

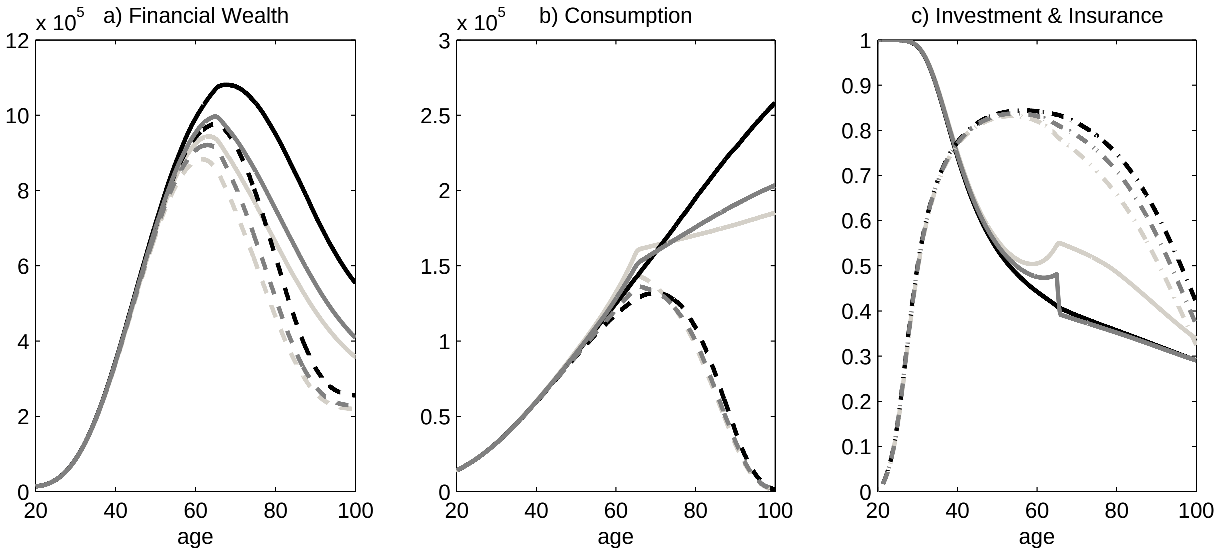

Figure 7: Sensitivity Analysis for Retirement Income. The graphs compare the average optimal controls and financial wealth evolution for different calibrations of the retirement income in the S model: $\sigma_Y^r = 0$ and $\rho^r = 0$ (light lines), $\sigma_Y^r = 0.1$ and $\rho^r = 0.2$ (grey lines), $\sigma_Y^r = 0.2$ and $\rho^r = 0$ (dark lines). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) shows the average financial wealth evolution, b) the average optimal consumption and c) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

at the age specified on the x-axis. The bequest is determined by the insurance decision and the actual financial wealth, calculated as $(1-\eta_t)X_t$. The utility function (3) indicates that in a model with a bequest motive of $\epsilon = 1$, the bequest is valued equal to one year consumption. The figure depicts a strong co-movement of the consumption and the bequest line. Thus, in the model the insurance is used by the agents to allocate optimal bequest and not to capture other investment or hedging motives. Furthermore, the graph shows that there should be approximately no accidental bequest in the presence of the insurance. Hence, the agents use the insurance as outlined in the complete market setup in Section 4 where I show this result analytically.

The Impact of Income The most important source of wealth in this setup is income. Here, I consider the impact of varying the parameters that drive the income process.

Figure 7 depicts the effects of changing the retirement income. The income drift remains unchanged, i.e. equals zero during the retirement phase, but I allow for a risky retirement income. I compare the results with the benchmark calibration with a secure retirement income, given by the light lines.

The grey lines depict results for a risky retirement income ($\sigma_Y^r = 0.1$) and a positive correlation with the stock ($\rho^r = 0.2$). This parametrization can be justified by the assumption that the retirement income depends on the financial situation of the country which is positively linked to a stock market index. Alternatively, one could argue that the agent

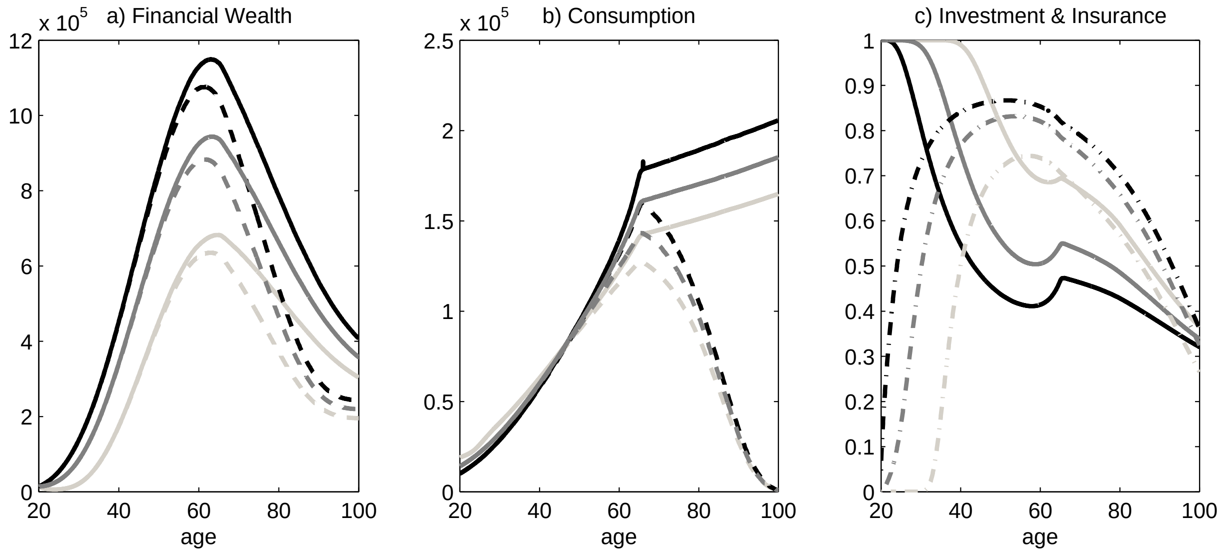

Figure 8: Sensitivity Analysis for Labor Income Volatility. The graphs compare the average optimal controls and financial wealth evolution for different values of labor income volatility in the working phase of the S model: $\sigma_Y^w = 0.15$ (light lines), $\sigma_Y^w = 0.2$ (grey lines) and $\sigma_Y^w = 0.25$ (dark lines). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) shows the average financial wealth evolution, b) the average optimal consumption and c) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

contracted retirement products with a stock-related return. The main change appears at the portfolio holdings when going into retirement. Portfolio holdings decrease since retirement income now has an implicit investment in the risky asset due to the positive correlation with the stock. To incorporate this effect, the explicit investment decreases for diversification purposes. Thus, retirement portfolio holdings are lower with risky retirement income and a positive correlation with the stock. Furthermore, retirement consumption growth is larger and the agent accumulates more financial wealth to capture retirement income risk.

The dark lines show a setup where I assume that labor income and pension payments have the same patterns of risk. Hence, I use $\sigma_Y^r = 0.2$ and $\rho^r = 0$ during retirement. This pattern might be more realistic if there is no social security system that accounts for a certain retirement income. The graphs for consumption and portfolio holdings have no significant change in steepness when retirement occurs. Thus, the changing consumption and portfolio holdings during retirement in the benchmark model occur due to a change in volatility and correlation and not due to the changing drift at retirement. Compared with the benchmark calibration, retirement portfolio holdings are lower due to the implicit risky investment of retirement income. Retirement consumption growth and the amount of accumulated financial wealth increases further.

Up to now, I examined changes in retirement income. In the following, I vary the income volatility during the working phase $\sigma_Y^w \in \{0.15, 0.2, 0.25\}$ in Figure 8. The less risky the income, the more similar are the working and the retirement phase. For a lower income volatility, the consumption increases in early years since the agent needs less financial wealth as buffer for negative income shocks. Due to more consumption in early years, the agent accumulates less financial wealth and has less bequest on average as well as less consumption in later years. With a less risky income, there is more investment in the risky asset since the implicit risky investment due to labor income is lower. Furthermore, the wealth effect increases the risky investment as well. The fraction insured decreases due to the wealth effects. After retirement, the setups are identical. Hence, the observed differences in consumption, portfolio holdings and insurance holdings after retirement are all due to the wealth effect.

So far, I restricted my analysis to a change in the volatility and the correlation of income. Figure 9 analyzes the effect of different expected earnings profiles over the life cycle. In addition to the college degree calibration, I use the parameter calibration from Munk and Sørensen (2010) for agents with high school degree and no high school education. The income graph shows that the college graduate starts with less initial labor income but has a huge expected increase in income, whereas the loss due to retirement is rather small. Agents with high school education start with the highest initial labor supply but have a huge decrease in labor income when getting retired. The agents without high school education are on average worse off throughout their lifetime but income does not decrease as much when going into retirement. The consumption graph shows that the college graduate consumes on average more throughout his lifetime although he starts with less initial labor supply. This is due to the high expected labor income in the future and his willingness to smooth consumption. Therefore, financial wealth is lower for college graduates in early years. At the age of 40, college graduates have the most wealth, due to the labor income increase. The average bequest is highest for college graduates and lowest for agents with no high school education. The portfolio holdings graph shows that a higher education translates into more risky investment. However, at the age of 60, the agent with high school degree invests less riskily compared to the agent without high school education. The intuition is taken from Cocco, Gomes, and Maenhout (2005). They state that due to the large income decrease when going into retirement, the high school agent shifts the portfolio to be less risky since labor income is an implicit riskless investment during retirement. Considering the insurance, the agent with college degree always has the smallest demand for insurance. In early years, this is due to little wealth that yields low insurance demand. In later years, the college graduate has a huge amount of labor income. Since insurance holdings decrease with increasing labor income, I deduce that the labor income effect dominates the financial wealth effect here. The high school and no high school graphs again intersect around the age of 60. Since the agent with high school degree has a more reduced labor income in retirement, he increases insurance holdings to

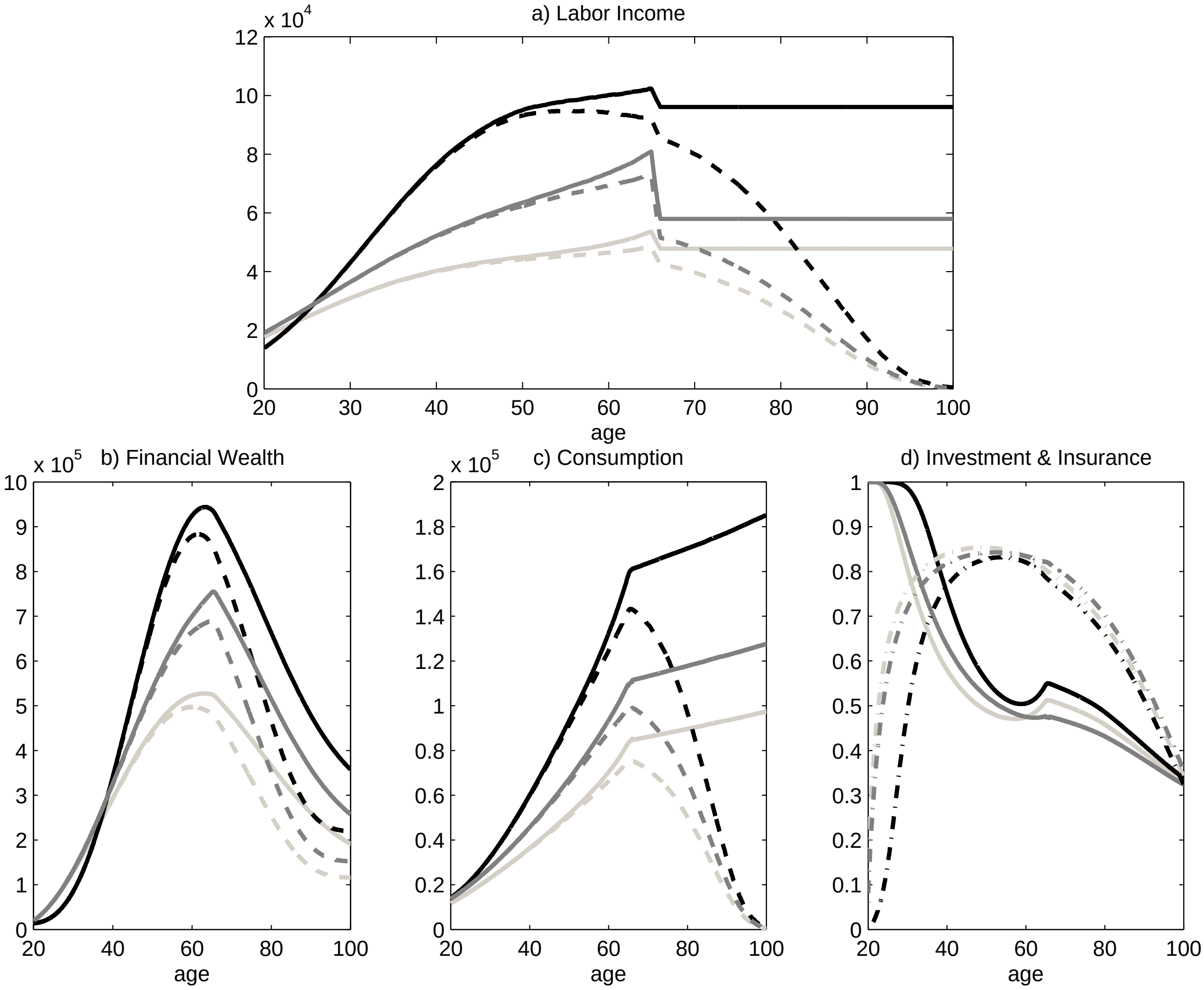

Figure 9: Sensitivity Analysis for the Earnings Profile over the Life Cycle. The graphs compare the average optimal controls, financial wealth evolution and labor income for different calibrations of the earnings profile over the life cycle in the S model: The black lines depict the benchmark results with a college graduate calibration. The grey lines correspond to the calibration for agents with a high school degree: $b = 0.1682$, $c = -0.00323$, $d = 0.000020$, $P = 0.68212$, $Y_0 = 19\,107$. The light lines represent results for agents without high school degree: $b = 0.1684$, $c = -0.00353$, $d = 0.000023$, $P = 0.88983$, $Y_0 = 17\,763$. These values are taken from Munk and Sørensen (2010). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) depicts the average earnings over the life cycle, b) shows the average financial wealth evolution, c) the average optimal consumption and d) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

anticipate the effect and to be able to leave the desired amount of bequest.

The Impact of a Bequest Motive Previously, I highlighted the strong relation between the optimal insurance choice and the bequest motive. Here, I examine the effect of the bequest motive in detail. I illustrate the results for different importance of the bequest motive $\epsilon \in \{1,3,5\}$ in Figure 10. The consumption graph depicts that optimal consumption is nearly unaffected. The lower the bequest motive, the higher is the consumption since agents want less financial wealth as bequest and have more to consume. However, the

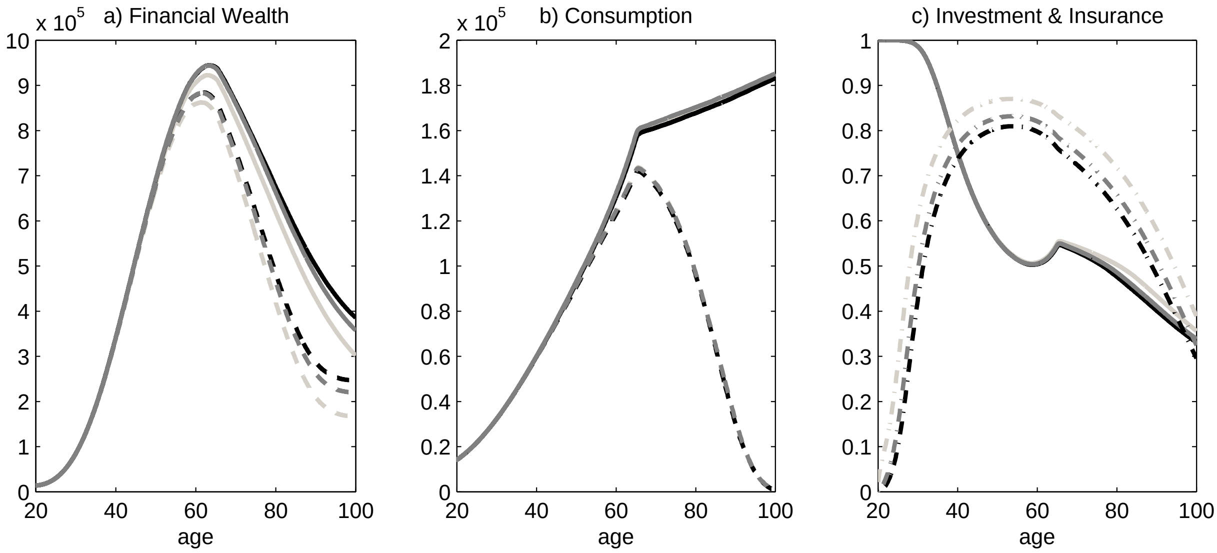

Figure 10: Sensitivity Analysis for the Bequest Motive. The graphs compare the average optimal controls and financial wealth evolution for different values of the importance of the bequest motive in the S model: $\epsilon = 1$ (light lines), $\epsilon = 3$ (grey lines) and $\epsilon = 5$ (dark lines). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) shows the average financial wealth evolution, b) the average optimal consumption and c) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

magnitude of this effect is very low. The amount riskily invested is almost unchanged as well. A small decrease in later years for an increasing bequest motive can be observed due to the wealth effect. As expected, the higher the bequest motive is the higher is the average bequest. Furthermore, living agents accumulate more financial wealth if they face a higher bequest motive. Considering the insurance holdings, we observe that a higher bequest motive goes along with less insurance. Less insurance means more bequest if death occurs, which gives the result an intuition. Note that the change in optimal consumption and portfolio holdings is very small, whereas there is a larger effect on optimal insurance holdings. These results also support the previous statement that mainly the insurance is used to realize the optimal financial wealth for bequest.

Sensitivity Analyses for Preference Parameters I provide further sensitivity analyses and investigate the impact of the preference parameters.

I start with the sensitivity with respect to the relative risk aversion and provide results for $\gamma \in \{1.5, 4, 7\}$ in Figure 11. Intuitively, the portfolio holdings are lower throughout the lifetime for a higher level of relative risk aversion. Furthermore, the amount of accumulated financial wealth is higher for a high value of relative risk aversion. These changes reflect the willingness of the investor to decrease risk. Agents that are less risk averse invest more in the risky asset which has a higher expected return. Therefore, average consumption is higher for agents that are less risk averse. For the same reason,

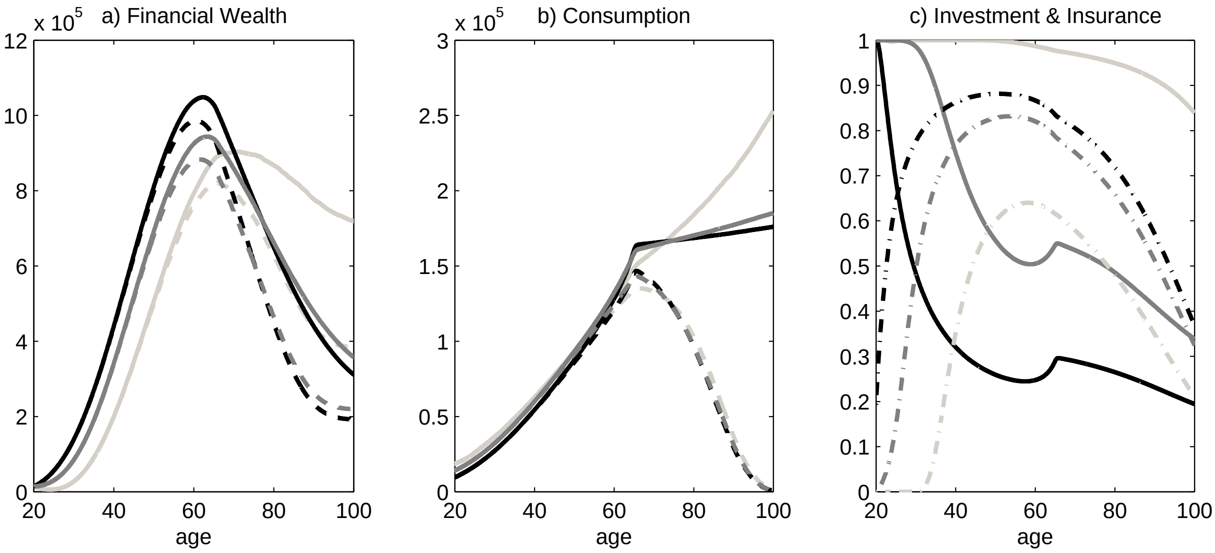

Figure 11: Sensitivity Analysis for Risk Aversion. The graphs compare the average optimal controls and financial wealth evolution for different values of the relative risk aversion coefficient of the S model: $\gamma = 1.5$ (light lines), $\gamma = 4$ (grey lines) and $\gamma = 7$ (dark lines). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) shows the average financial wealth evolution, b) the average optimal consumption and c) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

less risk averse investors have on average more wealth in later years and also more bequest. This comes at the cost of a higher variation in financial wealth evolution and optimal consumption compared to agents that are more risk averse (not shown in the figure). In early years, more risk averse agents accumulate more financial wealth due to less consumption and their willingness to get a buffer in order to handle future shocks in labor income. After retirement when labor income is certain, the agent does not need that buffer anymore and dissaves. The consumption graph highlights that the risk aversion crucially influences the valuation of labor income risk by the agents. For the less risk averse agent the consumption steepness is nearly unaffected by the different income volatility in the working and retirement stage, whereas the more risk averse agent has a very pronounced change in steepness. The insurance holdings are adjusted to ensure optimal bequest at each point in time. This yields less insurance for the less risk averse investor and more insurance for the more risk averse investor.

Now, I consider the effect of changing the time preference rate $\delta \in \{0.01, 0.03, 0.1\}$. The results are depicted in Figure 12. Intuitively, I expect more consumption today and less tomorrow with a higher time preference rate. This is exactly what the consumption graph shows. Especially after retirement, there is a huge effect with regard to consumption growth. Actually, consumption is decreasing when agents have a high time preference rate. For high values of δ, the fraction insured is lower. Insuring the financial wealth means getting a payoff today but having to pay something later when death occurs. Since future

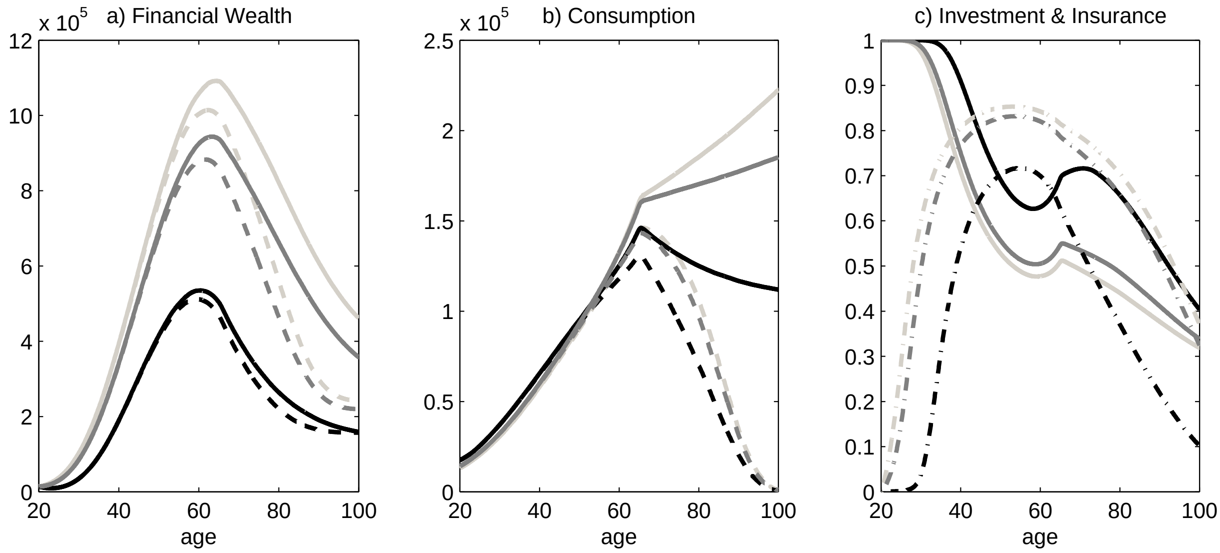

Figure 12: Sensitivity Analysis for the Time Preference Rate. The graphs compare the average optimal controls and financial wealth evolution for different values of the time preference rate of the S model: $\delta = 0.01$ (light lines), $\delta = 0.03$ (grey lines) and $\delta = 0.1$ (dark lines). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) shows the average financial wealth evolution, b) the average optimal consumption and c) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

payoffs decrease in value and present payments increase for a higher time preference rate, I would expect a higher fraction of insured wealth. However, the financial wealth of the agent decreases compared to a lower δ which is due to the high consumption in early years. In order to leave the desired bequest, the agent has to decrease his insurance holdings. This effect dominates the payment time effect and overall, the insurance holdings are lower for a higher δ. Although the insurance holdings are lower, the agent with high δ leaves less bequest on average. With a high time preference rate, agents accumulate less financial wealth since they consume more. This explains the low bequest on average, which is intuitive since bequest occurs in the future compared to actual consumption. The fraction riskily invested increases with increasing time preference rate due to the wealth effect.

Sensitivity Analyses for Asset Parameters I analyze the effect of a changing investment opportunity set. In particular, I vary μ_S, σ_S and r and show the corresponding graphs.

I investigate the stock drift and vary $\mu_S \in \{0.04, 0.06, 0.08\}$. The results are depicted in Figure 13. Intuitively, there is more risky investment with a higher stock drift. Furthermore, average wealth and bequest increase. The same holds true for consumption. These effects are due to a better investment opportunity set since agents are strictly better off with a higher stock drift. In later years, the fraction insured is higher due to more

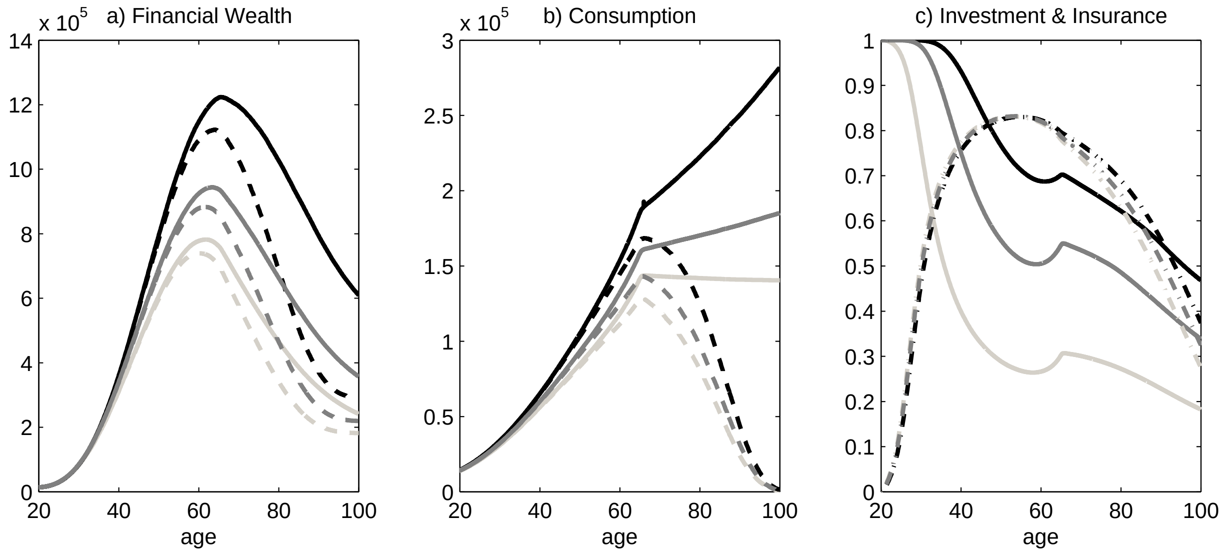

Figure 13: Sensitivity Analysis for the Stock Drift. The graphs compare the average optimal controls and financial wealth evolution for different values of the stock drift in the S model: $\mu_S = 0.04$ (light lines), $\mu_S = 0.06$ (grey lines) and $\mu_S = 0.08$ (dark lines). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) shows the average financial wealth evolution, b) the average optimal consumption and c) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

available financial wealth. In early years, the fraction insured is slightly lower. This can be explained by actually relatively low financial wealth but the willingness to leave more bequest in the case of death, due to the anticipated future wealth and consumption evolution. However, these thoughts can be summarized by stating that the insurance decision is corrected to ensure the optimal amount of bequest throughout the life cycle.

In Figure 14, I vary $\sigma_S \in \{0.05, 0.2, 0.4\}$. A higher volatility yields less risky investment due to the increased risk. Furthermore, a higher stock volatility yields less consumption, less financial wealth and less bequest on average since the investment opportunity set worsens. Due to less risky investment, the agent has less financial wealth since the risky asset has a higher expected return compared to the risk-free asset. For a higher volatility, the fraction insured increases in early years and decreases in later years. The interpretation is identical to the previous passage. Changing the stock volatility has a similar effect like varying the stock drift. It changes the investment opportunity set in a way that the agent is strictly better off or strictly worse off.

I vary the risk-free rate $r \in \{0.008, 0.02, 0.04\}$ and present the results in Figure 15. Increasing r yields decreasing portfolio holdings since the attractiveness of the risk-free asset increased in comparison to the risky asset. However, average financial wealth and average consumption are lower for a high r since the expected return of the risky asset is higher than the return of the risk-free asset. Although the investment opportunity set improved, the agent consumes less, accumulates less financial wealth and has less bequest

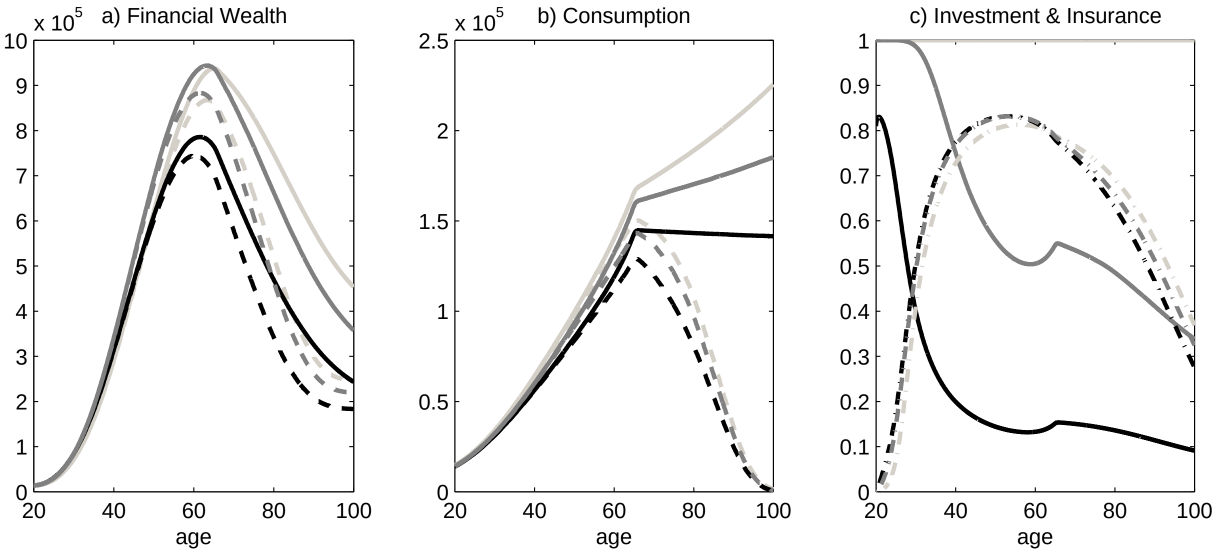

Figure 14: Sensitivity Analysis for the Stock Volatility. The graphs compare the average optimal controls and financial wealth evolution for different values of the stock volatility of the S model: $\sigma_S = 0.05$ (light lines), $\sigma_S = 0.2$ (grey lines) and $\sigma_S = 0.4$ (dark lines). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) shows the average financial wealth evolution, b) the average optimal consumption and c) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

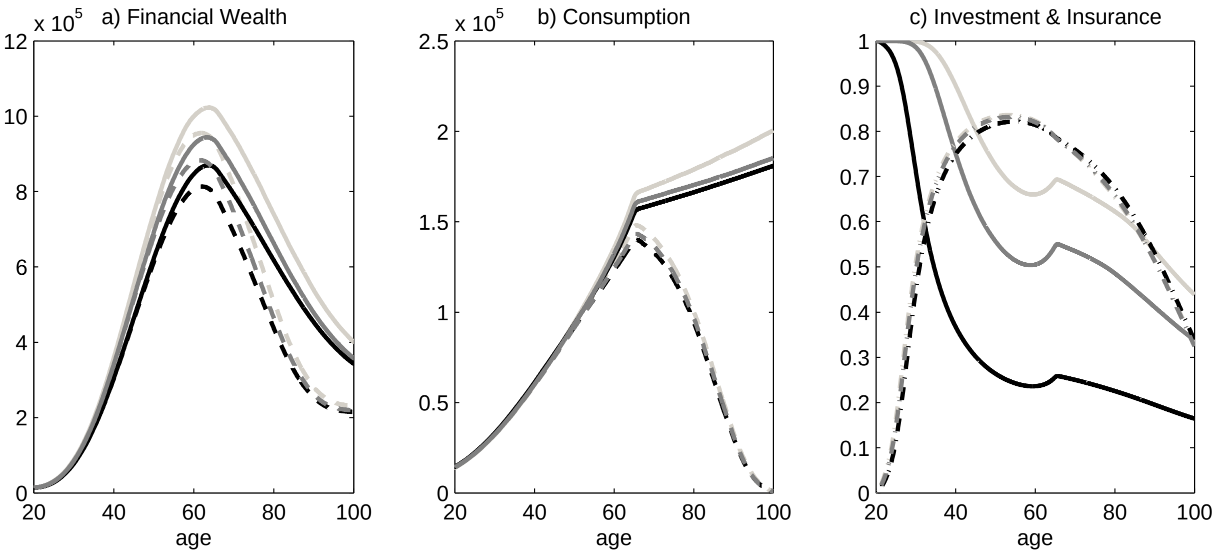

Figure 15: Sensitivity Analysis for the Risk-Free Rate. The graphs compare the average optimal controls and financial wealth evolution for different values of the risk-free rate in the S model: $r = 0.008$ (light lines), $r = 0.02$ (grey lines) and $r = 0.04$ (dark lines). The solid lines represent results for living agents, whereas the dashed lines are average results including dead agents. a) shows the average financial wealth evolution, b) the average optimal consumption and c) the average optimal risky investment (solid lines) and the average optimal insurance decision (dash-dotted lines). The other parameters are calibrated as described in Section 5.

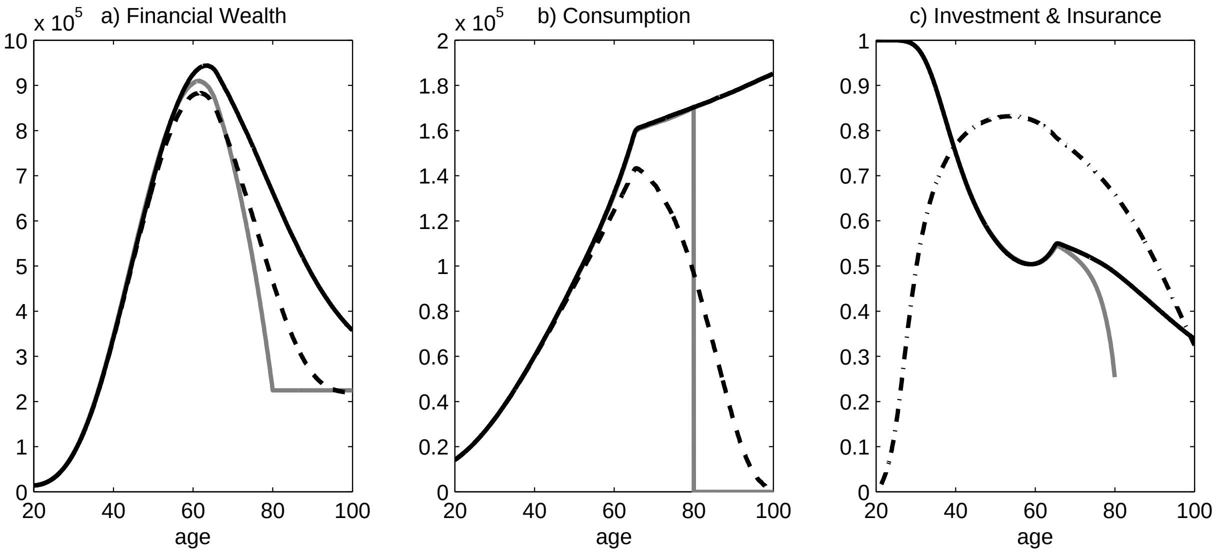

Figure 16: Model Comparison (D,S). The graphs compare the deterministic time of death model (grey lines) with the stochastic time of death model (dark lines). a) depicts the optimal average financial wealth over the life cycle. The solid lines include living agents only, whereas the dashed line includes all agents. b) shows the average optimal consumption for living agents only (solid lines) and for all agents (dashed line). c) presents the optimal fraction riskily invested (solid lines) and for the S model the average optimal fraction insured (dash-dotted line). The model calibrations are given in Section 5.

on average. However, the variation of optimal consumption, financial wealth evolution and bequest is smaller (not shown in the figure) due to the less risky investment. This is more important for the agent than the small reduction of expected consumption and bequest.

7 Deterministic vs Stochastic Time of Death

This section compares the stochastic time of death model S, presented in detail in the previous section, with the standard model D with deterministic time of death. The comparison highlights the effects of considering deterministic mortality risk instead of a fixed time of death. Especially, I comment on the role of the insurance and the bequest motive. The benchmark calibration of the models can be found in Section 5 and details on the numerical approach are given in the Appendix B. All figures depict means calculated from 100 000 simulations.

Benchmark Comparison Figure 16 depicts the corresponding graphs for a comparison of the models D and S with the benchmark calibration.

The expected financial wealth looks similar in both models and mainly co-moves until the age of 55. Surprisingly, the average bequest is identical in both models. In the deterministic time of death setup the bequest motive only matters at the time of death and the amount of bequest can be planned, whereas the bequest motive is important throughout the lifetime in the setup with mortality risk. With a deterministic time of death, all bequest is intended. In the setup with mortality risk, one would expect that total bequest splits into an intended part (due to the bequest motive) and an accidental part (due to death occurring while having more or less wealth as the agent wants to allocate as bequest). However, the average bequest is almost identical in both setups. The previous section shows that the endogenous insurance partially explains this issue: The agent chooses the insurance holding at every point in time such that if death occurs, he realizes the desired financial wealth for bequest. In this way, the agent avoids accidental bequest. The same level of bequest is nevertheless surprising. Although both models have the same expected time of death, the distribution of deaths is completely different. Hence, the identical average bequest shows that agents are risk neutral with respect to the time of death as they only care about the expected time of death and not about higher moments.

Considering the graph for consumption, there is a nearly identical pattern for living agents in both models. Consumption is increasing throughout the lifetime with a change in consumption growth at retirement. Hence, the insurance premium in the S model is not used to consume more but for accumulating more financial wealth.

The fraction riskily invested is decreasing over the lifetime in both models and almost identical until retirement. Then, the portfolio holdings decrease faster in the deterministic time of death model. Since the agent wants to reach the desired amount of bequest, he reduces risk. With certain time of death, this can be realized via a low risky investment shortly before the time of death. With mortality risk, the risky investment decreases as the hazard rate of death increases.

Details about the insurance holdings in the S model are given in the previous section. In the model D, there is no insurance decision due to the certain time of death.

The results highlight that mortality risk together with an actuarially fair insurance has nearly no impact until the age of 60. As long as the hazard rate of death is low, the differences between both models are negligible. The resulting financial wealth evolution and optimal controls over the life cycle are similar considering living agents. The differences between both models become important if one considers the retirement stage in detail, or considers mortality risk related assets or insurance products. In contrast, the assumption of a deterministic time of death can be used to simplify computations if the focus is the working period of the life cycle.

The Impact of the Insurance Here, I examine the impact of the absence of the insurance. The previous sections show that the insurance is used to ensure the optimal financial wealth for bequest. Thus, the insurance takes out a crucial implication of mortality risk, namely accidental bequest. Furthermore, the structure of the insurance is rather unrealistic. No insurance company allows to contract such an insurance in the real

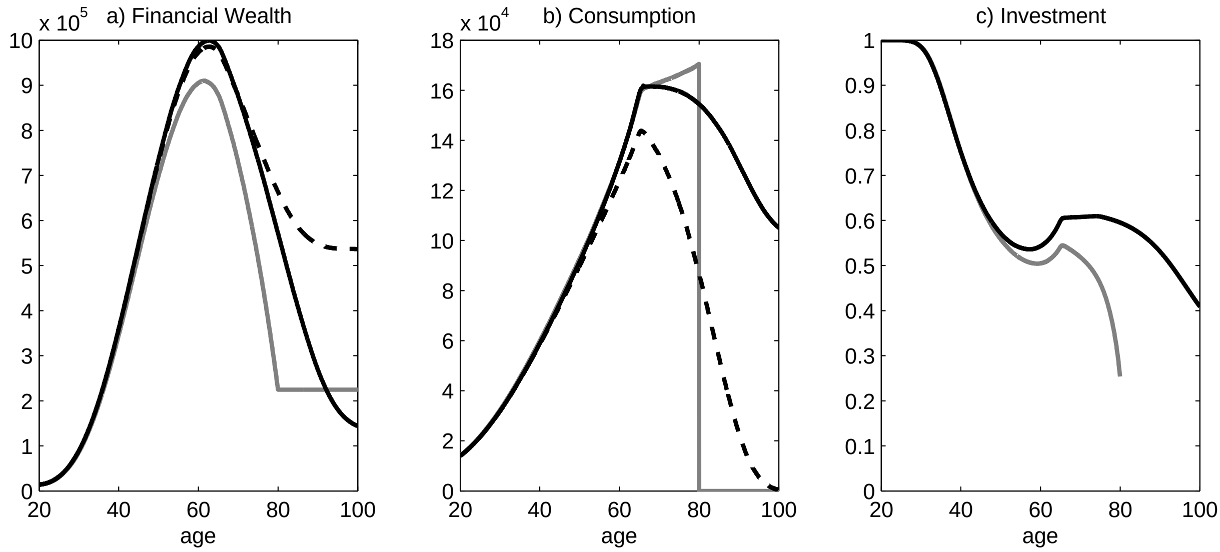

Figure 17: Model Comparison (D,S) without Insurance. The graphs compare the deterministic time of death model (grey lines) to the stochastic time of death model without insurance (dark lines). a) depicts the optimal average financial wealth over the life cycle. The solid lines include living agents only, whereas the dashed line includes all agents. b) shows the average optimal consumption for living agents only (solid lines) and for all agents (dashed line). c) presents the optimal fraction riskily invested. The models are calibrated as presented in Section 5.

world. Therefore, I consider the implications of a setup where the agents are not allowed to insure their financial wealth and compare it to the D model with deterministic time of death. Figure 17 depicts the results.

Considering the bequest first, the intuition presented before gets verified now. Without insurance, the average bequest is more than two times higher in the model with mortality risk compared to the model without mortality risk or the S model with insurance. Hence, the major part of bequest is accidental now. The financial wealth graph of living agents shows that agents save more in early years. In particular, the peak is higher. The absence of the insurance amplifies the risk of outliving available financial wealth. Longevity risk becomes important here. The older the agent gets, the higher is the sure payout that he receives from the insurance in the benchmark model. In the absence of the insurance, the agent has to accumulate more wealth to encounter longevity risk. However, this increases accidental bequest in the case of an early death. In later years, the financial wealth of living agents is significantly lower without insurance. On the one hand, this is due to the increasing mortality risk. The agent wants to use his wealth before he dies. On the other hand, the financial wealth is lower due to the missing insurance premium. Agents that are older than 95 leave on average less bequest compared to agents in the deterministic time of death model.

The consumption graph for the model S now has a hump-shaped pattern. Due to the missing insurance premium and the fear of outliving available wealth, the agent is not able to afford that much consumption when getting older. With insurance, he expects a

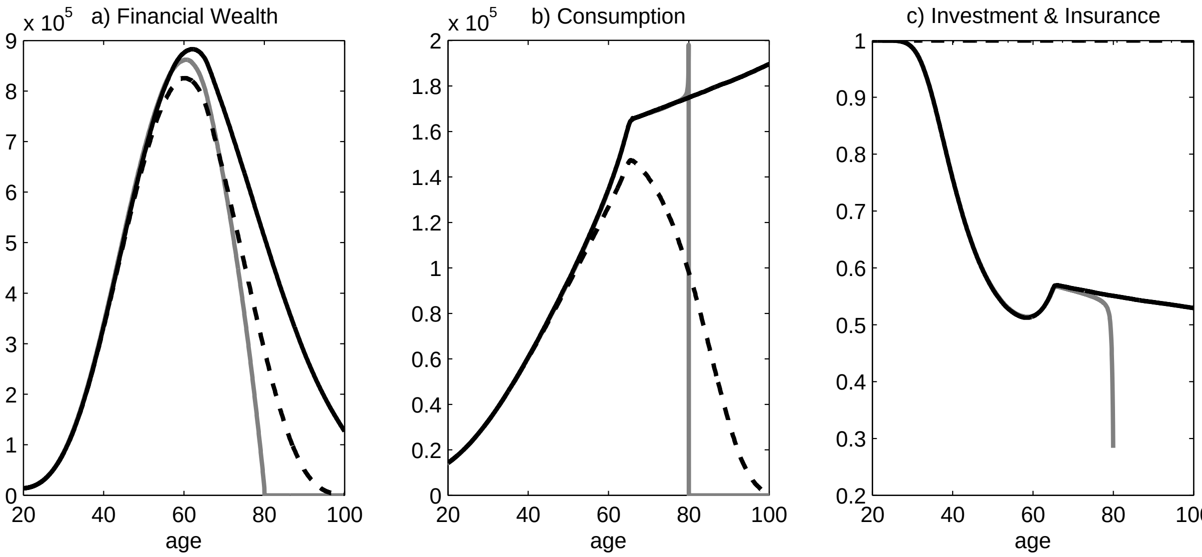

Figure 18: Model Comparison (D,S) without Bequest Motive. The graphs compare the deterministic time of death model (grey lines) to the stochastic time of death model (dark lines) with $\epsilon = 0$ in both models. a) depicts the optimal average financial wealth over the life cycle. The solid lines include living agents only, whereas the dashed line includes all agents. b) shows the average optimal consumption for living agents only (solid lines) and for all agents (dashed line). c) presents the optimal fraction riskily invested (solid lines) and for the S model the average optimal fraction insured (dash-dotted line). The remaining parameters are calibrated as described in Section 5 for the models S and D.

high sure cash flow if he survives. Without insurance, he faces longevity risk as he has to use his own accumulated financial wealth to finance consumption. Therefore, consumption decreases in the absence of the insurance. However, consumption is on average still above the retirement income. Actually, an agent at the age of 100 dissaves. Comparing the models S and D, the optimal consumption is again almost identical until retirement. Afterwards, the differences are crucial with a different consumption growth sign.

Considering portfolio holdings, we see again more risky investment in the model S in later years. Without insurance, the agent has less financial wealth in later years which results in more risky investment. This effect increases with increasing mortality rates when the agent gets older as the insurance premium increases.

The results show more realistic patterns in the absence of an insurance. The agents face accidental bequest and the consumption graph depicts a hump-shaped pattern. These two effects are not observable in the presence of the insurance. Furthermore, the annuity puzzle and the lack of existence of an insurance like the one assumed here indicate that the simple insurance should be omitted in order to get realistic life cycle results. Without insurance, there are more differences between the models S and D after the age of 50. For younger agents, mortality risk is negligible again.

The Impact of a Bequest Motive I consider the models S and D without bequest motive in Figure 18. In the S model, the insurance holdings are trivial: It is always

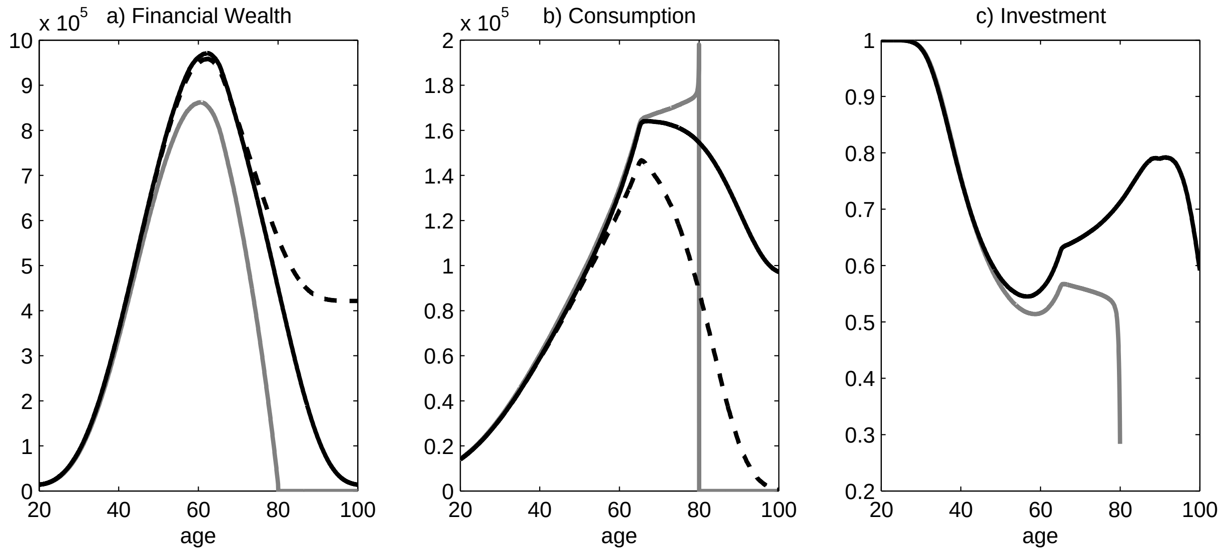

Figure 19: Model Comparison (D,S) without Insurance and without Bequest Motive. The graphs compare the deterministic time of death model (grey lines) with the stochastic time of death model (dark lines) where both models are without insurance and without bequest motive ($\epsilon = 0$). a) depicts the optimal average financial wealth over the life cycle. The solid lines include living agents only, whereas the dashed line includes all agents. b) shows the average optimal consumption for living agents only (solid lines) and for all agents (dashed line). c) presents the optimal fraction riskily invested. The remaining calibration of the models S and D is given in Section 5.

optimal for all agents to insure all the financial wealth. In both models, agents leave no bequest: in the deterministic time of death model due to the certainty of death whereas in the model with mortality risk due to the insurance holdings. The financial wealth of living agents is again almost identical in the S and D model in early years. In later years, financial wealth is reduced to capture the absence of the bequest motive. The optimal consumption is nearly identical in both models until short before the time of death in the model without mortality risk. Then, the agent increases consumption in order to get a very low level of financial wealth since there is no benefit from leaving money on the table. The portfolio holdings are nearly identical until retirement as well. In the model D, the risky investment is lower during retirement and decreases extremely shortly before the certain time of death in order to avoid risk and reach the desired amount of bequest which equals zero. The absence of the bequest motive yields a higher risky investment in later years in the S model due to less financial wealth. Comparing the models S and D, the previous results with insurance remain valid. There are no significant differences until retirement and mortality risk can be neglected in the working phase of the life cycle.

Figure 19 depicts a comparison of the models S and D without bequest motive and without insurance. The financial wealth graph highlights again the effect of accidental bequest in the absence of the insurance. Although the financial wealth of living agents approaches zero when the agents get close to the age of 100, the average expected financial wealth, and thus the bequest, is more than four times of last year's consumption. This effect is driven by agents that die during a phase of the life cycle in which they have a huge amount of financial wealth in order to save for the later years and to capture the impact of longevity risk. The consumption graph shows again a hump-shaped pattern without insurance. Until retirement, the consumption mainly co-moves in both models however, afterwards consumption increases in the model with certain time of death and decreases in the model with mortality risk. The portfolio holdings co-move in early years. Without insurance and bequest motive, the portfolio holdings increase in the S model between the age of 60 and 90 due to the fast decreasing financial wealth. Afterwards, the average portfolio holdings decrease due to the constraint that ensures positive wealth. Risky investment is forbidden for very low levels of financial wealth, which reduces average risky investment. Comparing both models without bequest motive and without insurance, the previous intuition of the comparison without insurance remains unaffected. Until the age of 50, mortality risk is negligible, whereas it has a crucial impact afterwards, especially during retirement.

8 Deterministic vs Stochastic Hazard Rate of Death

In this section, I allow the hazard rate of death to have a diffusive and a jump component. I analyze the impact of the stochastic hazard rate of death and compare the results with the deterministic hazard rate model. In order to avoid any impact of the numerical calculation technique, I use the same approach for all models. The solution method differs from the algorithm used before due to the hazard rate of death as additional state variable. Details considering the numerical approach are given in Appendix B. In the following, I present results from the S, SD and SDJ model with the calibration of Section 5. The figures depict averaged results after 100 000 simulations.

Benchmark Comparison Figure 20 compares the optimal controls and the wealth evolution of the three models. The S and SD results are nearly identical for all optimal controls and for the financial wealth evolution. This is not surprising since the SD model has only little uncertainty with respect to future mortality rates. In detail, we observe that in the SD model the agent has a little more financial wealth, more consumption and more risky investment over the life cycle compared to the S model.

Comparing the S and SD model with the SDJ model shows more differences. The SDJ model delivers more financial wealth, more bequest, a higher amount of consumption and more risky investment. These results indicate that the agent is better off when he faces jump risk in the hazard rate of death. To verify the intuition and to evaluate the importance of the stochastic hazard rate of death, I calculate a certainty equivalent

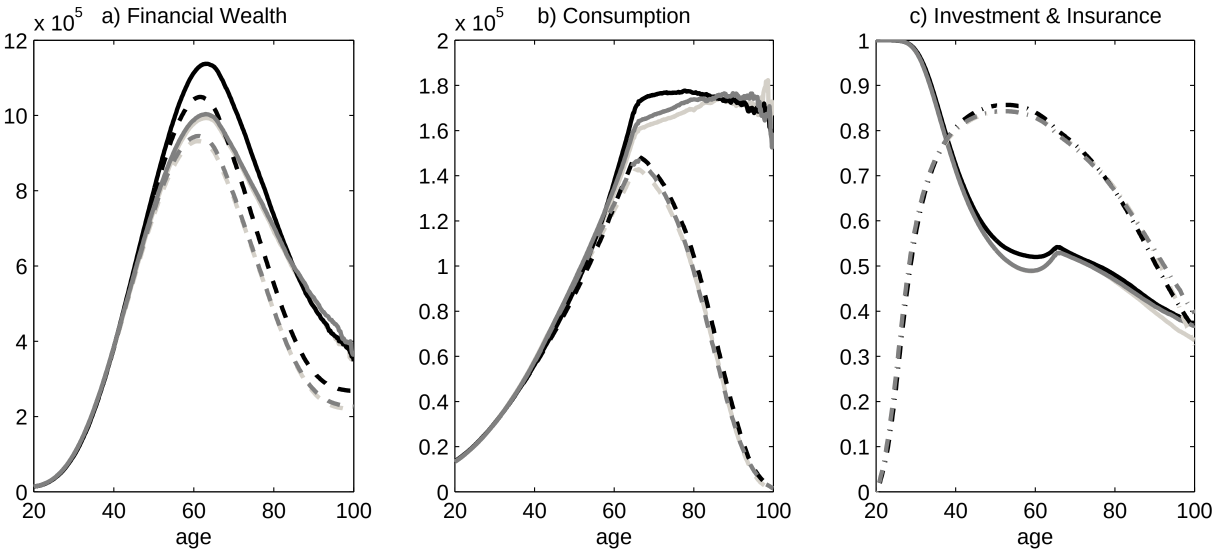

Figure 20: Model Comparison (S,SD,SDJ). This figure compares the model S with deterministic hazard rate of death (light lines) with stochastic hazard rate of death models. The SD model (grey lines) has little uncertainty with respect to future mortality risk modeled with a diffusive component in the hazard rate. The SDJ model (dark lines) additionally has a jump component and more uncertainty with respect to future mortality risk. a) depicts the optimal average financial wealth over the life cycle. The solid lines include living agents only, whereas the dashed lines include all agents. b) shows the average optimal consumption for living agents only (solid lines) and for all agents (dashed lines). c) presents the optimal fraction riskily invested (solid lines) and the optimal insurance decision (dash-dotted lines). The parameters are calibrated as stated in Section 5 for the models S, SD and SDJ.

defined by the inverse value function

$$CE(t,x,y,\pi) = \left[(1-\gamma)J(t,x,y,\pi)\right]^{\frac{1}{1-\gamma}}.$$

Table 2 provides the percentage gain of the SD and SDJ model compared to the S model calculated by the ratio of certainty equivalents. Adding a diffusion has only a minor effect with a gain smaller than 1.5%. However, the agent prefers the stochastic hazard rate of death. Adding the jump component has a significant impact over the life cycle.

	age 20	age 50	age 80
SD with Insurance	0.02	1.38	0.82
SD without Insurance	0.03	1.38	0.88
SDJ with Insurance	8.45	28.82	16.33
SDJ without Insurance	8.41	28.61	16.80

Table 2: Welfare Impact of the Stochastic Mortality Risk. The table compares the certainty equivalent of the S model with the SD and SDJ model with insurance and without insurance. The table gives the percentage gain in the certainty equivalent induced by the stochastic hazard rate of death. For the agent with age 20, I use the values $t = 0$, $x = 13912$, $y = 13912$, $\pi = 0.000074$. For the agent aged 50, the values are $t = 30$, $x = 750000$, $y = 92500$, $\pi = 0.0022$. For the agent at the age of 80, I have $t = 60$, $x = 690000$, $y = 91000$, $\pi = 0.0635$. I do the calculations with the benchmark calibration of the models S, SD and SDJ given in Section 5.

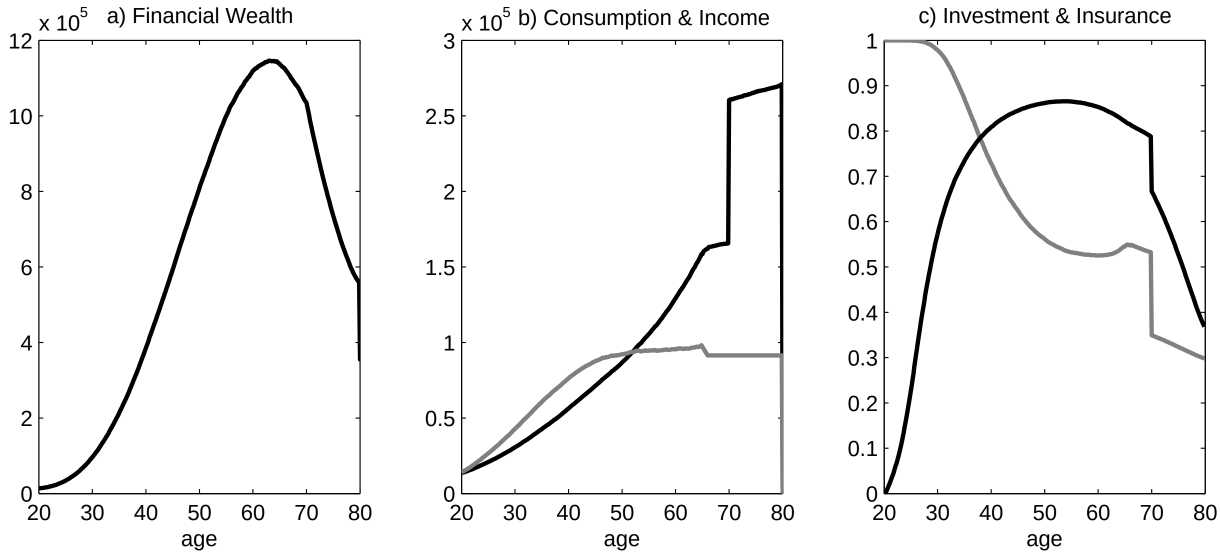

Figure 21: Sample Mortality Jump in the SDJ Model. The graphs depict the optimal wealth and controls in the SDJ model when agents face a jump in the hazard rate of death at the age of 70 and the death jump occurs at the age of 80. a) depicts the financial wealth evolution. b) shows the optimal consumption (dark line) and the income (grey line). c) presents the optimal fraction insured (dark line) and riskily invested (grey line). The model is calibrated with the SDJ parameters given in Section 5.

Middle-aged agents are more than 25% better off if they face stochastic mortality risk with jumps. This also highlights the importance of a jump component in the hazard rate of death. I extend the results of Huang, Milevsky, and Salisbury (2012). They state that in an analytically tractable model of retirement consumption, the stochastic mortality risk is relatively unimportant from an individual's perspective. Here, I approve this result for a realistically calibrated life cycle setup with unspanned labor income risk if the stochastic mortality risk is driven by a pure diffusive component. Moreover, I show that if the stochastic mortality risk is also driven by a jump component, the statement does not hold and the jump component becomes crucial from an individual's perspective. These results are valid in the models with and without the insurance.

In order to gain more intuition about the impact of a jump in the hazard rate of death, I consider graphs where all agents face a shock at the same time. Figure 21 depicts the results for agents with a health shock at the age of 70 and death occurs at the age of 80. When the health shock occurs, mortality risk increases, the expected remaining lifetime decreases and the insurance premium increases. The optimal reaction to a health shock is an increase in consumption and a decrease in risky investment to capture the decreased expected remaining lifetime. The fraction insured is also effected in order to ensure optimal bequest. Due to the co-movement of expected bequest and optimal consumption, it is clear that the fraction insured decreases to reach a higher amount of desired bequest. Financial wealth growth reduces, i.e. the higher insurance premium due to the increased mortality risk does not offset the change in optimal controls. The reaction to a critical illness shock is more pronounced, the earlier the shock occurs. A second or third health

	age 20	age 50	age 80
S	0.14	8.15	13.43
SD	0.13	8.15	13.36
SDJ	0.17	8.33	12.96

Table 3: Welfare Impact of the Insurance. The table depicts the certainty equivalent increase in percentage when agents have access to the insurance market for differently aged agents. For the agent with age 20, I use the values $t = 0$, $x = 13\,912$, $y = 13\,912$, $\pi = 0.000074$. For the agent aged 50, the values are $t = 30$, $x = 750\,000$, $y = 92\,500$, $\pi = 0.0022$. For the agent at the age of 80, I have $t = 60$, $x = 690\,000$, $y = 91\,000$, $\pi = 0.0635$. I do the calculations with the benchmark calibration of the models S, SD and SDJ given in Section 5.

shock further amplifies the effect.

The reason why the agent prefers a setup with jump risk in the hazard rate of death is that a jump gives an indication prior to a high death probability and the agent has the possibility to react in an optimal way. The agent likes extreme probabilities and dislikes probabilities in between. If the risk of death is very low, the agent faces the risk of a sudden death, whereas he especially faces longevity risk if mortality risk is very high. However, the agent can use the insurance to mitigate these extreme cases and can react in an optimal way to the predominant state. If there is a 50-50 chance of surviving the next few years, optimal decisions are mainly a tradeoff between both possible cases and semi-optimal whatever happens. In the S and SD model, the agent faces this situation especially in middle and older years. In the SDJ model, the agent faces a low probability of death as long as no jump has occurred and a high probability of death afterwards. This more extreme distribution explains the importance of the jump component for the individual agent.

A more accurate calibration, impact on the optimal controls and financial wealth evolution as well as a significant importance for the individual agent highlight that jumps in the hazard rate of death are important in life cycle consumption-investment problems with mortality risk. On the contrary, a diffusive component has only a minor effect.

The Impact of the Insurance In this paragraph, I consider the importance of the insurance in the three models and present results in the absence of the insurance. Table 3 gives the increase in the certainty equivalent of having access to the insurance market for the S, SD and SDJ model. At the age of 20, the agent is not significantly better off when having access to the insurance market. This is not surprising since the agent has little financial wealth to insurance and mortality risk is extremely low. At the age of 20, agents ideally invest nothing in the insurance at all. Due to the relatively high importance of the bequest motive and the low wealth early in the life cycle, the agent would even prefer to sell the insurance in order to get more bequest in the case of death. The middle-aged

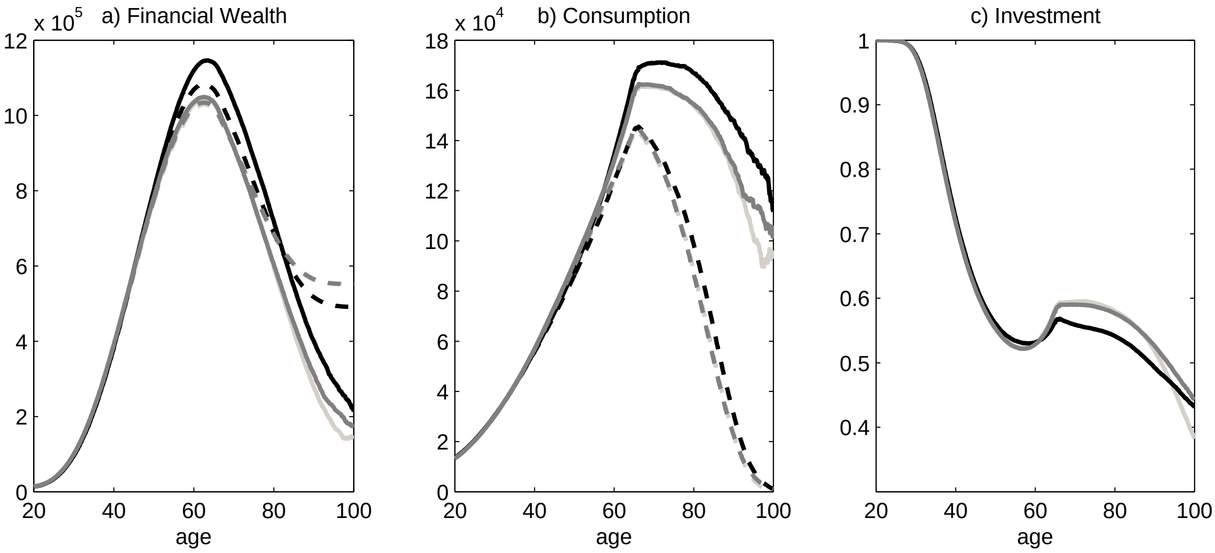

Figure 22: Model Comparison (S,SD,SDJ) without Insurance. The figure compares the deterministic hazard rate of death model S (light lines) with stochastic hazard rate of death models. The SD model (grey lines) has little uncertainty with respect to future mortality risk modeled with a diffusive component in the hazard rate. The SDJ model (dark lines) additionally has a jump component and more uncertainty with respect to future mortality risk. a) depicts the optimal average financial wealth over the life cycle. The solid lines include living agents only, whereas the dashed lines include all agents. b) shows the average optimal consumption for living agents only (solid lines) and for all agents (dashed lines). c) presents the optimal fraction riskily invested. The models S, SD and SDJ are calibrated as stated in Section 5.

agent is significantly better off when having access to the insurance market in all models. Now, the agent has accumulated more financial wealth than he wants to hold for bequest. Therefore, he uses the insurance to ensure optimal bequest and he gets the insurance premium as additional source of income. For old agents, the importance of the insurance further increases although less financial wealth is available. Due to a higher mortality risk, the insurance becomes more important. Comparing the three models, a stochastic hazard rate of death has no significant impact on the importance of the insurance.

Figure 22 compares the three models in the absence of the insurance. Again, the S and SD model produce nearly identical results. As in Huang, Milevsky, and Salisbury (2012), the withdrawal rate at retirement, i.e. at the age of 65, is slightly higher in the model with a diffusive component. Comparing the models S and SD to the SDJ model, we see more financial wealth and more consumption in the model with jump risk. However, the average bequest is lower although consumption and financial wealth are higher in the SDJ model. Hence, the SDJ model delivers significantly less accidental bequest. For an explanation, I consider the effect of mortality jumps in the model without insurance.

Figure 23 depicts the effect of a health shock at the age of 70 and a death shock at the age of 80 in the SDJ model without insurance. Consumption increases and financial wealth decreases when the health jump occurs. Since it is likely that death occurs a few years after a health jump, the agent has time to reduce financial wealth to prepare the bequest. This explains the lower fraction of bequest being accidental and also highlights

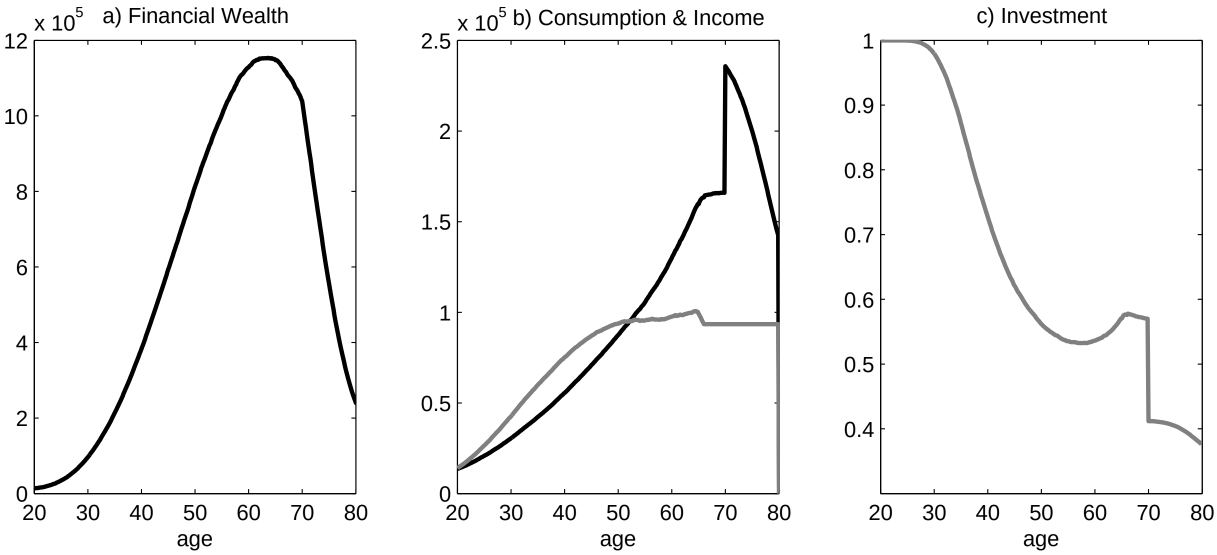

Figure 23: Sample Mortality Jump in the SDJ Model without Insurance. The figure shows the optimal wealth and controls in the SDJ model without insurance if agents face a mortality jump at the age of 70 and a death jump at the age of 80. a) shows the financial wealth evolution. b) presents the optimal consumption (dark line) and the income (grey line). c) depicts the optimal fraction riskily invested. The model is calibrated with the SDJ parameters given in Section 5.

why agents are better off in the SDJ model. Compared to the model with insurance, we see that financial wealth decreases even faster. This is necessary since the insurance is not available as a possibility to reduce financial wealth at death. Therefore, the financial wealth is reduced in advance. Consumption jumps after the mortality shock but decreases rapidly afterwards. Without an insurance premium, the agent cannot afford the high level of consumption over plenty of years without facing crucial longevity risk. Hence, consumption growth decreases after the jump. The risky investment reduces when the jump occurs as a reaction to the reduced expected remaining lifetime. Again, the earlier the shock occurs, the more pronounced is the effect and additional shocks further amplify the effect.

Without insurance, the inclusion of jumps in the hazard rate of death is important for the individual agent and has an impact on the aggregate optimal wealth and controls as well. Furthermore, the presence of jump risk reduces accidental bequest significantly. In contrast, a diffusive component in the hazard rate of death has again only little impact on the results.

9 Conclusion

Mortality risk with jumps in the hazard rate of death fits mortality data best and is important for the individual agents, especially in middle and older years. Further research can focus on stochastic mortality risk with jumps and combine this with a deeper analysis of either products that are relevant in the retirement phase or mortality-related and health-related assets or insurance products.

In Kraft, Schendel, and Steffensen (2014), we consider a suchlike model where a family faces the risk of a health shock or an early death and has the possibility to contract a term life insurance to partially mitigate the risk of losing the wage earner's income. Similar analyses can be done by considering critical illness insurance or disability insurance.

Jump risk in the hazard rate of death is also important with regard to retirement products like annuities. Since the jump risk is important for the individual agent and significantly effects consumption and investment decisions, further research can analyze the effect of mortality jumps on the retirement planning and annuity decisions.

Another idea for further research is to choose a different type of preferences in order to avoid the risk neutrality with respect to the time of death.[8] Bommier (2006b) points out that mortality risk implicitly defines risk neutrality with respect to the time of death if it can be added to the time preference rate. He develops non-additive preferences that produce a constant absolute risk aversion with respect to the time of death and that are still time-consistent when considering utility as an age-dependent function. With this utility, he shows that risk aversion with respect to the time of death increases consumption early in life. Examining portfolio holdings in this setup may lead to new insights how risk aversion with respect to the time of death affects the optimal asset allocation over the life cycle.

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A Proof of Proposition 2

I start with the HJB (8) and set $\sigma_Y = 0$. Furthermore, π is redundant as a state variable. Therefore, I obtain the simpler HJB

$$\delta J = \sup_{c,\theta,\eta} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + J_t + J_x \left[x \left(r + \pi\eta + \theta\lambda\sigma_S \right) + y - c \right] + 0.5 J_{xx} x^2 \theta^2 \sigma_S^2 + J_y y \mu_Y \right. \\ \left. + \pi \left[J(\tau, (1-\eta)x, 0) - J(t,x,y) \right] \right\}. \tag{13}$$

Next, I use the following guess for the indirect utility:

$$J(t,x,y) = \mathbb{1}_{\{t<\tau\}} \left(\frac{1}{1-\gamma} (x + yf(t))^{1-\gamma} g(t)^\gamma \right) + \mathbb{1}_{\{t=\tau\}} \left(\epsilon(t) \frac{x^{1-\gamma}}{1-\gamma} \right).$$

I calculate the necessary values and derivatives using the guess and obtain

$$\begin{aligned} J(t\,|\,t<\tau,x,y) &= \frac{1}{1-\gamma}(x+yf(t))^{1-\gamma} g(t)^\gamma, \\ J_t(t\,|\,t<\tau,x,y) &= (x+yf)^{-\gamma} y f_t g^\gamma + \frac{\gamma}{1-\gamma}(x+yf)^{1-\gamma} g_t g^{\gamma-1}, \\ J_x(t\,|\,t<\tau,x,y) &= (x+yf)^{-\gamma} g^\gamma, \\ J_{xx}(t\,|\,t<\tau,x,y) &= -\gamma(x+yf)^{-\gamma-1} g^\gamma, \\ J_y(t\,|\,t<\tau,x,y) &= (x+yf)^{-\gamma} f g^\gamma, \\ J(t\,|\,t=\tau,(1-\eta)x,0) &= \epsilon(t) \frac{\left((1-\eta)x\right)^{1-\gamma}}{1-\gamma}. \end{aligned}$$

By substituting $\epsilon(t)\frac{((1-\eta)x)^{1-\gamma}}{1-\gamma}$ for $J(\tau,(1-\eta)x,0)$ the HJB (13) becomes

$$(\delta+\pi)J = \sup_{c,\theta,\eta} \left\{\frac{c^{1-\gamma}}{1-\gamma} + J_t + J_x\left[x\left(r+\pi\eta+\theta\lambda\sigma_S\right)+y-c\right] + 0.5J_{xx}x^2\theta^2\sigma_S^2 + J_y y\mu_Y \right.$$
$$\left. + \pi\epsilon\frac{\left((1-\eta)x\right)^{1-\gamma}}{1-\gamma}\right\}.$$

Next, taking the first order conditions for c,θ,η yields the optimal controls

$$c = J_x^{-\frac{1}{\gamma}},$$
$$\theta = -\frac{J_x}{J_{xx}x}\frac{\lambda}{\sigma_S},$$
$$\eta = 1-\frac{J_x^{-\frac{1}{\gamma}}}{x}\epsilon^{\frac{1}{\gamma}}.$$

Substituting the calculated derivatives, I express the optimal controls as

$$c(t,x,y) = \frac{x+yf(t)}{g(t)},$$
$$\theta(t,x,y) = \frac{\lambda}{\gamma x\sigma_S}(x+yf(t)),$$
$$\eta(t,x,y) = 1-\frac{x+yf(t)}{g(t)x}\epsilon(t)^{\frac{1}{\gamma}}.$$

Next, I insert these optimal controls, the guess and the corresponding derivatives into the HJB (13). After doing a zero addition with $\pm(x+fy)^{-\gamma}g^\gamma yf(r+\pi)$, I sort terms with $(x+yf)^{1-\gamma}$ and $(x+yf)^{-\gamma}$ to get the rearranged expression:

$$0 = (x+yf)^{-\gamma}\left[yg^\gamma f_t + g^\gamma y - g^\gamma yf(r+\pi) + fg^\gamma\mu_Y y\right]$$
$$+(x+yf)^{1-\gamma}\left[-\frac{\delta+\pi}{1-\gamma}g^\gamma + \frac{1}{1-\gamma}g^{\gamma-1}\left(1+\pi\epsilon^{\frac{1}{\gamma}}\right) + \frac{\gamma}{1-\gamma}g^{\gamma-1}g_t + g^\gamma(r+\pi)\right.$$
$$\left. -g^{\gamma-1}\epsilon^{\frac{1}{\gamma}}\pi + g^\gamma\frac{1}{\gamma}\lambda^2 - g^{\gamma-1} - \frac{1}{2}g^\gamma\frac{1}{\gamma}\lambda^2\right].$$

The equation is fulfilled, if both terms in the square brackets are zero. From the first square bracket, I get the following ordinary differential equation (ODE) for f:

$$f_t + f\left(\mu_Y(t)-r-\pi(t)\right)+1 = 0$$

with the solution (can be verified using Leibniz rule)

$$f(t) = \int_t^\infty \mathrm{e}^{\int_t^s \mu_Y(u)-r-\pi(u)du}\,ds.$$

Considering the remaining terms, I obtain the following ODE for g:

$$g_t + g\left(\frac{1-\gamma}{\gamma}r - \frac{\delta}{\gamma} - \pi(t) + \frac{1}{2}\frac{1-\gamma}{\gamma^2}\lambda^2\right) + 1 + \pi(t)\epsilon(t)^{\frac{1}{\gamma}} = 0$$

with the solution (verifiable via Leibniz rule again)

$$g(t) = \int_t^{\infty} e^{\int_t^s \frac{1-\gamma}{\gamma}r - \frac{1}{\gamma}\delta - \pi(u) + \frac{1}{2}\frac{1-\gamma}{\gamma^2}\lambda^2 \, du}\left(1 + \pi(s)\epsilon(s)^{\frac{1}{\gamma}}\right) ds.$$

This proves the proposition. □

B The Numerical Solution Approach

It is not possible to find an analytical solution for the incomplete market case with unspanned labor income and with stochastic mortality risk. Therefore, I use a numerical approach which is similar to the approach of Munk and Sørensen (2010). First, I simplify the optimization problem by reducing the number of state variables from four to three.

Lemma 3. *The optimization problem (6) with mortality risk can be simplified by reducing the number of state variables. The indirect utility (7) is rewritten for $t < \tau$ as*

$$J(t,x,y,\pi) = y^{1-\gamma} F(t,z,\pi),$$

where $z = \frac{x}{y}$. The HJB (8) simplifies to

$$\begin{aligned}
0 = \sup_{\widehat{c},\theta,\eta} \quad & \left\{ \frac{\widehat{c}^{1-\gamma}}{1-\gamma} + \epsilon\pi \frac{\left((1-\eta)z\right)^{1-\gamma}}{1-\gamma} + F_t \right. \\
& + F\left[-\delta - \pi - \kappa + (1-\gamma)\mu_Y - \frac{1}{2}\gamma(1-\gamma)\sigma_Y^2\right] \\
& + F_z\left[1 - \widehat{c} + z\left(r + \pi\eta + \theta\lambda\sigma_S - \mu_Y + \gamma\sigma_Y^2 - \gamma\sigma_S\sigma_Y\rho\theta\right)\right] \\
& + F_{zz}z^2\left[\frac{1}{2}\theta^2\sigma_S^2 + \frac{1}{2}\sigma_Y^2 - \sigma_S\sigma_Y\rho\theta\right] \\
& \left. + F_\pi \pi\mu_\pi + \frac{1}{2}F_{\pi\pi}\pi^2\sigma_\pi^2 + \kappa F(t,z,\pi+\beta) \right\}
\end{aligned}$$

with $\widehat{c} = \frac{c}{y}$. The optimal controls for $t \in [0,\tau)$ are

$$\begin{aligned}
\widehat{c}(t,z,\pi) &= F_z^{-\frac{1}{\gamma}}, \\
\theta(t,z,\pi) &= \frac{\sigma_Y\rho}{\sigma_S} + \frac{F_z}{F_{zz}z}\frac{\gamma\sigma_Y\rho - \lambda}{\sigma_S}, \\
\eta(t,z,\pi) &= 1 - \frac{F_z^{-\frac{1}{\gamma}}}{z}\epsilon(t)^{\frac{1}{\gamma}}.
\end{aligned}$$

Proof. Due to the power utility setup as well as the linearity in wealth dynamics (5) and labor income dynamics (1), I can use the homogeneity property of the indirect utility. I reduce the number of state variables for $t < \tau$ as follows

$$
\begin{aligned}
J(t,kx,ky,\pi) &= \sup_{\{c_s,\theta s,\eta_s\}_{s\in[t,\tau)}} \mathrm{E}_{t,x,y,\pi}\left[\int_t^\tau e^{-\delta(s-t)}\frac{(kc_s)^{1-\gamma}}{1-\gamma}ds + \epsilon(t)\mathrm{e}^{-\delta(\tau-t)}\frac{(kX_\tau)^{1-\gamma}}{1-\gamma}\right] \\
&= k^{1-\gamma}\sup_{\{c_s,\theta s,\eta_s\}_{s\in[t,\tau)}} \mathrm{E}_{t,x,y,\pi}\left[\int_t^\tau e^{-\delta(s-t)}\frac{c_s^{1-\gamma}}{1-\gamma}ds + \epsilon(t)\mathrm{e}^{-\delta(\tau-t)}\frac{X_\tau^{1-\gamma}}{1-\gamma}\right] \\
&= k^{1-\gamma}J(t,x,y,\pi).
\end{aligned}
$$

Thus, the value function is expressed as

$$J(t,x,y,\pi) = k^{\gamma-1}J(t,kx,ky,\pi).$$

Plugging in $k = \frac{1}{y}$ reduces the number of state variables by one. Hence, I get for $t < \tau$

$$
\begin{aligned}
J(t,x,y,\pi) &= y^{1-\gamma}J\left(t,\frac{x}{y},1,\pi\right) \\
&=: y^{1-\gamma}F(t,z,\pi),
\end{aligned}
$$

The partial derivatives of J can be rewritten in terms of F as follows:

$$
\begin{aligned}
J_t &= y^{1-\gamma}F_t, \\
J_x &= y^{1-\gamma}F_z\frac{1}{y}, \\
J_{xx} &= y^{1-\gamma}F_{zz}\frac{1}{y^2}, \\
J_y &= (1-\gamma)y^{-\gamma}F - y^{-\gamma}F_z\frac{x}{y}, \\
J_{yy} &= -\gamma(1-\gamma)y^{-\gamma-1}F + y^{-\gamma-1}F_{zz}\frac{x^2}{y^2} + 2\gamma y^{-\gamma-1}F_z\frac{x}{y}, \\
J_{xy} &= -y^{-\gamma-1}F_{zz}\frac{x}{y} - \gamma y^{-\gamma-1}F_z, \\
J_\pi &= y^{1-\gamma}F_\pi, \\
J_{\pi\pi} &= y^{1-\gamma}F_{\pi\pi}.
\end{aligned}
$$

Furthermore, I notice for $t = \tau$:

$$J(\tau,(1-\eta)x,0,\pi) = \epsilon(t)\frac{\big((1-\eta)x\big)^{1-\gamma}}{1-\gamma}.$$

In order to get the new normalized HJB, I rewrite the HJB (8) in terms of F by substituting the calculated derivatives. I substitute z for $\frac{x}{y}$, divide by $y^{1-\gamma}$ and choose $\widehat{c} = \frac{c}{y}$ as new control variable instead of c. Then, I get rid off all x, y, c in the HJB such that the agent maximizes over portfolio holdings, normalized consumption and insurance holdings now. The only state variables are time, normalized wealth and the hazard rate of death. The resulting rearranged normalized HJB is shown in Lemma 3.

I obtain the optimal (normalized) controls $\widehat{c}, \theta, \eta$ by taking the first order conditions with respect to $\widehat{c}, \theta, \eta$ from the normalized HJB. □

Basic Idea Using the results of Lemma 3, I solve the optimization problem (6) numerically. I use a backward iterative procedure. I discretize the setup and use finite difference approximations for the partial derivatives. Then, I guess the optimal normalized controls $\theta, \widehat{c}, \eta$ and use the guess to calculate the finite difference approximations for the partial derivatives. Afterwards, I calculate a new guess for the optimal normalized controls using the formulas from Lemma 3. I do this until the change in the normalized value function F between the iterations is very small. The solution technique is similar to the one used by Munk and Sørensen (2010).[9] The main difference is the infinite horizon setup of the model with uncertain time of death and the jump component. Due to the structure of the mortality risk, the survival probability of the agents to a specific high age is numerically not distinguishable from zero. I use this age as a start for the backward iterative procedure by assuming that agents die for sure if they would reach this age.

General Solution Technique I roughly present the solution technique here but I do not go into detail (e.g. boundary handling, numerical implementation of constraints) since this goes beyond the scope of the paper. First, I set up a grid about normalized wealth, hazard rate of death and time. The normalized indirect utility function in the grid point n, i, j is denoted by $F_{n,i,j}$ where i is the normalized wealth index, j the hazard rate of death index and n the time index. I denote the optimal controls on the grid by $\widehat{c}_{n,i,j}$, $\theta_{n,i,j}$, $\eta_{n,i,j}$ analogously. I start with the highest value of n and calculate backwards in time for all i and j simultaneously. First, I guess the optimal controls. A good guess is the previous value in the same grid point, e.g. $\eta_{n,i,j} = \eta_{n+1,i,j}$ since I do not expect optimal controls to vary much in a small time interval. After substituting the optimal controls into the normalized HJB of Lemma 3, the HJB becomes a partial differential equation (PDE) and can be expressed as

$$0 = K_1 + F_t + K_2 F + K_3 F_z + K_4 F_{zz} + K_5 F_\pi + K_6 F_{\pi\pi} + \kappa F(t, z, \pi + \beta)$$

[9] Munk and Sørensen (2010) provide details concerning the numerical approach in their appendix.

with state-dependent coefficients K_i (I use K_i as a short form for $K_i(t,z,\pi)$). Next, I approximate the partial derivatives with finite differences. I get for the grid point (n,i,j)

$$
\begin{aligned}
F_t: \quad & D_t^+ F_{n,i,j} = \frac{F_{n+1,i,j} - F_{n,i,j}}{\Delta_t}, \\
F_z: \quad & D_z^+ F_{n,i,j} = \frac{F_{n,i+1,j} - F_{n,i,j}}{\Delta_z}, \\
F_z: \quad & D_z^- F_{n,i,j} = \frac{F_{n,i,j} - F_{n,i-1,j}}{\Delta_z}, \\
F_{zz}: \quad & D_z^2 F_{n,i,j} = \frac{F_{n,i+1,j} - 2F_{n,i,j} + F_{n,i-1,j}}{(\Delta_z)^2}, \\
F_\pi: \quad & D_z^+ F_{n,i,j} = \frac{F_{n,i,j+1} - F_{n,i,j}}{\Delta_\pi}, \\
F_\pi: \quad & D_z^- F_{n,i,j} = \frac{F_{n,i,j} - F_{n,i,j-1}}{\Delta_\pi}, \\
F_{\pi\pi}: \quad & D_z^2 F_{n,i,j} = \frac{F_{n,i,j+1} - 2F_{n,i,j} + F_{n,i,j-1}}{(\Delta_\pi)^2},
\end{aligned}
$$

where Δ denotes the difference between two grid points (e.g. $\Delta_t = t_{n+1} - t_n$). I use an implicit approach and thus, I consider the forward looking finite difference for the time derivative. For z and π, I use both forward and backward looking differences depending on the coefficients at each grid point in order to ensure the stability of the solution approach. I approximate the jump term via linear interpolation with the nearest grid points, i.e. $F(t,z,\pi+\beta) = f_1^\beta F_{n,i,j+\widetilde{\beta_1}} + f_2^\beta F_{n,i,j+\widetilde{\beta_2}}$ where the $\widetilde{\beta}$ are the nearest grid points of $\pi+\beta$ and the f^β $(= f^\beta(n,i,j))$ are the factors from the linear interpolation.

After inserting the approximations and sorting terms, the PDE is represented by the equation

$$
\begin{aligned}
F_{n+1,i,j}\frac{1}{\Delta_t} + K_1 = {} & F_{n,i,j}\left(-K_2 + \frac{1}{\Delta_t} + \text{abs}\left(\frac{K_3}{\Delta_z}\right) + \text{abs}\left(\frac{K_5}{\Delta_\pi}\right) + 2\frac{K_4}{\Delta_z^2} + 2\frac{K_6}{\Delta_\pi^2}\right) \\
& + F_{n,i-1,j}\left(\frac{\min\{K_3,0\}}{\Delta_z} - \frac{K_4}{\Delta_z^2}\right) + F_{n,i+1,j}\left(-\frac{\max\{K_3,0\}}{\Delta_z} - \frac{K_4}{\Delta_z^2}\right) \\
& + F_{n,i,j-1}\left(\frac{\min\{K_5,0\}}{\Delta_\pi} - \frac{K_6}{\Delta_\pi^2}\right) + F_{n,i,j+1}\left(-\frac{\max\{K_5,0\}}{\Delta_\pi} - \frac{K_6}{\Delta_\pi^2}\right) \\
& - F_{n,i,j+\widetilde{\beta_1}}\kappa f_1^\beta - F_{n,i,j+\widetilde{\beta_2}}\kappa f_2^\beta,
\end{aligned}
$$

where K_i now depends on the specific grid point ($K_i = K_i(n,i,j)$). I use another shorthand notation where I suppress the n,i,j index again

$$
\begin{aligned}
F_{n+1,i,j}\frac{1}{\Delta_t} + C^0 = {} & F_{n,i-1,j}C^1 + F_{n,i+1,j}C^2 + F_{n,i,j-1}C^3 \\
& + F_{n,i,j+1}C^4 + F_{n,i,j}C^5 + F_{n,i,j+\widetilde{\beta_1}}C^6 + F_{n,i,j+\widetilde{\beta_2}}C^7.
\end{aligned}
$$

I vectorize the indirect utility for a fixed point in time such that I write

$$F_n = \begin{pmatrix} F_{n,1,1} & F_{n,1,2} & \dots & F_{n,1,N_\pi} & F_{n,2,1} & \dots & F_{n,2,N_\pi} & \dots & F_{n,N_z,N_\pi} \end{pmatrix}^T .$$

The ordering is mathematically unimportant but due to the jump component in π, this ordering results in an optimization problem with a smaller band width. Thus, this ordering increases the speed of solution algorithms dramatically. Then, I consider the equations above for all i, j for a fixed n. Resulting, I get a system of $N_\pi \times N_z$ equations that can be written in matrix form as follows

$$\begin{pmatrix} \dots \\ 0 \dots 0 C^1 0 \dots 0 C^3 C^5 C^4 0 \dots 0 C^6 C^7 0 \dots 0 C^2 0 \dots 0 \\ \dots \end{pmatrix} F_n = \begin{pmatrix} \vdots \\ F_{n+1,i,j} \frac{1}{\Delta_t} + C^0 \\ \vdots \end{pmatrix} .$$

The resulting system is characterized by a large, sparse, band matrix and can be solved relatively fast due to the fixed structure. With the solution F_n, I calculate a new guess using the finite difference approximations and the optimal control formulas of Lemma 3. A new iterative step starts using the new guess, or, if there is no significant change between old and new results, the algorithm continues with $n-1$ until the full system is solved.

With the solution, I simulate 100 000 life cycles for each model. I use the inverse transformation method to simulate the time of death and the mortality jumps.

Simulations of Section 6 and 7 In Section 6 and 7, π is redundant as a state variable. This simplifies the above approach significantly. The remaining matrix has a tridiagonal structure which allows a very fast solution. For the model D, I have specific boundary conditions for the last time period. In the model S, I start with $t = 110$ (agent aged 130). In more than 1 000 000 simulations no agent reached an age of 120 or more. An identical model with 200 as latest time of death yields the same results. I use 24 time steps for each year which yields overall 1 441 time steps in the D model and 2 641 time steps in the S model. For the z grid, I choose $z \in (0, 200]$ and use 10 001 z-grid points in both models. The lower bound follows directly from the included liquidity constraint. I choose the upper bound in a way such that the normalized wealth process does most likely not reach the bound. In the results, the average value of z is below 20 throughout the lifetime.

Simulations of Section 8 The solution approach is more time-intensive with a stochastic hazard rate of death. I use the same algorithm for the S, SD and SDJ model to avoid a numerical impact in the comparison graphs. I start with $t = 110$ in all models. I use 6 time steps for each year, this yields overall 661 time steps. I choose $z \in (0, 150]$ with 1 001 z-grid points. For the hazard rate of death, I use $\pi \in (0, 1]$ with 1 001 π-grid points in all models.

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