



No. 2005/18

**The Method of Endogenous Gridpoints
for Solving Dynamic Stochastic
Optimization Problems**

Christopher D. Carroll





Center for Financial Studies

The *Center for Financial Studies* is a nonprofit research organization, supported by an association of more than 120 banks, insurance companies, industrial corporations and public institutions. Established in 1968 and closely affiliated with the University of Frankfurt, it provides a strong link between the financial community and academia.

The CFS Working Paper Series presents the result of scientific research on selected topics in the field of money, banking and finance. The authors were either participants in the Center's Research Fellow Program or members of one of the Center's Research Projects.

If you would like to know more about the *Center for Financial Studies*, please let us know of your interest.

Prof. Dr. Jan Pieter Krahen

Prof. Volker Wieland, Ph.D.



CFS Working Paper No. 2005/18

**The Method of Endogenous Gridpoints
for Solving Dynamic Stochastic Optimization Problems***

Christopher D. Carroll¹

July 28, 2005

Abstract:

This paper introduces a method for solving numerical dynamic stochastic optimization problems that avoids rootfinding operations. The idea is applicable to many microeconomic and macroeconomic problems, including life cycle, buffer-stock, and stochastic growth problems. Software is provided.

JEL Classification: C6, D9, E2

Keywords: Dynamic optimization, precautionary saving, stochastic growth model, endogenous gridpoints, liquidity constraints

* I am grateful to Nicola Fuchs-Schündeln, Wouter den Haan, Michael Haliassos, Ken Judd, Albert Marcet, Dimitri Mavridis, Victor Rios-Rull, John Rust, Eric Young, and participants in the 2002 meetings of the Society for Computational Economics for discussions of the literature and existing solution techniques. Excellent research assistance has been provided by Zhou Lu and Marc Chan. I am especially grateful to Michael Reiter for improvements to the Matlab code. All errors are my own.

¹ Department of Economics, The Johns Hopkins University, Baltimore MD 21218-2685; <http://econ.jhu.edu/people/ccarroll>. Software archive at <http://econ.jhu.edu/people/ccarroll/EndogenousArchive.zip>.

1 The Problem

Consider a consumer whose goal is to maximize discounted utility from consumption

$$\max \sum_{s=t}^T \beta^{t-s} u(C_s) \quad (1)$$

for a CRRA utility function $u(C) = C^{1-\rho}/(1-\rho)$.¹

The consumer's problem will be specialized below to two cases: A standard microeconomic problem with uninsurable idiosyncratic shocks to labor income, and a standard representative agent problem with shocks to aggregate productivity (the 'micro' and the 'macro' models).² Versions of the model ('the j models') with costs of adjustment will also be considered.

M_t is 'market resources' (macro interpretation: capital plus current output) or 'cash-on-hand' (micro interpretation: net worth plus current income), while P_t is permanent labor productivity in both interpretations. In the j models, the stock of capital K_t is also a state variable.

The transition process for M_t is broken up, for convenience of analysis, into three steps. Financial assets at the end of the period are market resources minus consumption minus, in the j model, costs of investment spending I and costs of adjustment J . In those models (here and henceforth, equation(s) b), end-of-period capital is Z_t ,

$$A_t = M_t - C_t \quad (2-a)$$

$$A_t = M_t - C_t - J_t - I_t \quad (2-b)$$

$$Z_t = K_t + I_t,$$

and capital at the beginning of the next period is what remains after a depreciation factor $\bar{\tau}$ is applied,

$$K_{t+1} = A_t \bar{\tau}, \quad (3-a)$$

$$K_{t+1} = Z_t \bar{\tau}, \quad (3-b)$$

where $\bar{\tau} = (1 - \delta)$ in the usual macro notation and $\bar{\tau} = 1$ in the micro interpretation.

¹Putting leisure in the utility function is straightforward but would distract from the paper's point.

²Different aspects of the setup of the problem will strike micro and macroeconomists as peculiar; with patience, it should become clear how the problem as specified can be transformed into more familiar forms.

The final step can be thought of as the transition from the beginning of period $t + 1$, when capital K_{t+1} but has not yet been used to produce output, and the middle of that period, when output has been produced and incorporated into resources:

$$M_{t+1} = \overbrace{e_{t+1}\Theta_{t+1}P_{t+1}}^{\equiv L_{t+1}} \mathcal{W}_{t+1} + K_{t+1}\mathcal{R}_{t+1} \quad (4-a)$$

$$M_{t+1} = \overbrace{e_{t+1}\Theta_{t+1}P_{t+1}}^{\equiv L_{t+1}} \mathcal{W}_{t+1} + K_{t+1}r_{t+1} + A_t \underbrace{R}_{\equiv \mathcal{R}\Upsilon} \quad (4-b)$$

where \mathcal{W}_{t+1} is the wage rate; \mathcal{R}_{t+1} is the marginal product of capital (including return of capital); in the j models, r_{t+1} is the income produced by ownership of a unit of capital; R is the interest factor (including return of principal) for liquid assets kept in the financial asset; Θ_{t+1} is an iid transitory shock (e.g., unemployment) normalized to satisfy $E_t[\Theta_{t+n}] = 1 \forall n > 0$ (usually $\Theta_t = 1 \forall t$ in the macro interpretation); and e_t indicates labor effort (or labor supply), which for purposes of this paper is fixed at $e_t = 1$, but in general could be allowed to vary. The disarticulation of the flow of income into labor and capital components is useful in thinking separately about the effects of productivity growth (captured by ΘP) and capital accumulation (K).

Permanent labor productivity (in either interpretation) evolves according to

$$P_{t+1} = G_{t+1}P_t\Psi_{t+1} \quad (5)$$

for a permanent shock that satisfies $E_t[\Psi_{t+n}] = 1 \forall n > 0$ and G_t is exogenous and perfectly predictable (see below for varying interpretations of G).

Defining lower case variables as the upper-case variable scaled by the level of permanent labor productivity, e.g. $a_t = A_t/P_t$, we have

$$a_t = m_t - c_t \quad (6-a)$$

$$a_t = m_t - c_t - j_t - i_t \quad (6-b)$$

$$z_t = k_t + i_t$$

while with a bit of algebra the state transition becomes

$$m_{t+1} = \underbrace{e_t\Theta_{t+1}}_{\equiv l_{t+1}} \mathcal{W}_{t+1} + \underbrace{(a_t \Upsilon / G_{t+1} \Psi_{t+1})}_{=k_{t+1}} \mathcal{R}_{t+1} \quad (7-a)$$

$$m_{t+1} = \underbrace{e_t\Theta_{t+1}}_{\equiv l_{t+1}} \mathcal{W}_{t+1} + \underbrace{(z_t \Upsilon / G_{t+1} \Psi_{t+1})}_{=k_{t+1}} r_{t+1} + \mathcal{R} a_t \Upsilon / G_{t+1} \Psi_{t+1}. \quad (7-b)$$

Note that r_{t+1} is the marginal product of capital in period $t + 1$ and r_{t+1} rather than \mathcal{R}_{t+1} is the correct interest term with respect to z_t because m_{t+1} is monetary resources and the extra principal associated with an extra unit of z does not become part of m while the extra principal associated with an extra unit of a does become part of monetary resources. In other words, since m is monetary resources, this just says that physical capital yields money income at rate r .

The interest and wage factors are assumed not to depend on anything other than capital and productive labor input; together with the iid assumption about the structure of the shocks, this implies that the problem has a Bellman equation representation (henceforth boldface indicates functions)

$$\mathbf{V}_t(M_t, P_t) = \max_{C_t} \{u(C_t) + \beta E_t[\mathbf{V}_{t+1}(M_{t+1}, P_{t+1})]\} \quad (8-a)$$

$$\mathbf{V}_t(M_t, P_t, K_t) = \max_{C_t, I_t} \{u(C_t) + \beta E_t[\mathbf{V}_{t+1}(M_{t+1}, P_{t+1}, K_{t+1})]\} \quad (8-b)$$

subject to the transition equations.

Defining $\Lambda_{t+1} \equiv G_{t+1}\Psi_{t+1}$, consider the related problem

$$\mathbf{v}_t(m_t) = \max_{c_t} \left\{ u(c_t) + \beta E_t \left[\Lambda_{t+1}^{1-\rho} \mathbf{v}_{t+1} \left(\overbrace{\mathcal{W}_{t+1} l_{t+1} + \mathcal{R}_{t+1} \underbrace{a_t \mathcal{T} / \Lambda_{t+1}}_{k_{t+1}}}^{=m_{t+1}}} \right) \right] \right\} \quad (9-a)$$

$$\mathbf{v}_t(m_t, k_t) = \max_{c_t, i_t} \{u(c_t) + \beta E_t [\Lambda_{t+1}^{1-\rho} \mathbf{v}_{t+1}(m_{t+1}, k_{t+1})]\} \quad (9-b)$$

Assume that there is some last period T in which

$$\mathbf{V}_T(M_T, P_T) = P_T^{1-\rho} \mathbf{v}_T(M_T/P_T) \quad (10-a)$$

$$\mathbf{V}_T(M_T, P_T, K_T) = P_T^{1-\rho} \mathbf{v}_T(M_T/P_T) \quad (10-b)$$

for some well-behaved \mathbf{v}_T (we will be more specific about the terminal value function below). In this case it is easy to show that the solution to the ‘normalized’ problem defined by (9) yields the solution to the original problem via $\mathbf{V}_t = P_t^{1-\rho} \mathbf{v}_t$ for any $t < T$.³

We now specify the adjustment cost function as

$$\mathbf{j}(i, n) = (n/2)(i/n - \delta/\mathcal{T})^2 \omega \quad (11)$$

$$\mathbf{j}^i = (i/n - \delta/\mathcal{T}) \omega \quad (12)$$

$$\mathbf{j}^n = (i/n - \delta/\mathcal{T})(-i/n^2) n \omega + (1/2)(i/n - \delta/\mathcal{T})^2 \omega \quad (13)$$

$$= (1/2)(i/n - \delta/\mathcal{T})^2 \omega - (i/n)(i/n - \delta/\mathcal{T}) \omega \quad (14)$$

$$= \mathbf{j}/n - (i/n) \mathbf{j}^i. \quad (15)$$

³See Carroll (2004) for a proof.

where we will consider two alternative assumptions for how the scaling variable n is determined. The first, which is the standard assumption, we will call assumption N1; it is that

$$n_t = k_t, \quad (16)$$

which corresponds to the usual assumption that the costs of adjustment are proportional to the scale of the capital stock. The alternative assumption, which we will call N2, is that

$$n_t = (m_t - \mathcal{W})/\mathcal{R} \quad (17)$$

which is chosen so that if $\Theta_t = 1$ and $k_t = \bar{k}$ it implies that $n_t = \bar{k}$ where \bar{k} satisfies $1 + \varepsilon \bar{k}^{\varepsilon-1} = \mathcal{R}$. The purpose of this assumption is to be able to specify the problem in such a way that the costs of adjustment depend only on the realized outcome of production rather than on the capital input to production, which we will show will permit a specialization of the model to a version with only a single state variable in the middle of the period.

Note for future use that under N1 we have

$$\mathbf{n}^m(m, k) = 0 \quad (18)$$

$$\mathbf{n}^k(m, k) = 1 \quad (19)$$

while under N2 we have

$$\mathbf{n}^m(m, k) = \mathcal{R}^{-1} \quad (20)$$

$$\mathbf{n}^k(m, k) = 0. \quad (21)$$

Now define an end-of-period value function ‘Gothic \mathbf{v} ’ as

$$\mathbf{v}_t(a_t) = \beta E_t[\Lambda_{t+1}^{1-\rho} \mathbf{v}_{t+1}(\mathcal{W}_{t+1} l_{t+1} + \mathcal{T} \mathcal{R}_{t+1} a_t / \Lambda_{t+1})] \quad (22-a)$$

$$\mathbf{v}_t(a_t, z_t) = \beta E_t[\Lambda_{t+1}^{1-\rho} \mathbf{v}_{t+1}(\mathcal{W}_{t+1} l_{t+1} + \mathcal{T}(\mathcal{R} a_t + z_t r_{t+1}) / \Lambda_{t+1}, z_t \mathcal{T} / \Lambda_{t+1})] \quad (22-b)$$

with derivatives

$$\begin{aligned} \mathbf{v}_t^a(a_t) &= \beta E_t [\Lambda_{t+1}^{1-\rho} \mathbf{v}_{t+1}^m(\mathcal{W}_{t+1} l_{t+1} + \mathcal{R}_{t+1} a_t \mathcal{T} / \Lambda_{t+1}) \mathcal{R}_{t+1} \mathcal{T} / \Lambda_{t+1}] \\ &= \mathcal{T} \beta E_t [\Lambda_{t+1}^{-\rho} \mathbf{v}_{t+1}^m(\mathcal{W}_{t+1} l_{t+1} + \mathcal{R}_{t+1} a_t \mathcal{T} / \Lambda_{t+1}) \mathcal{R}_{t+1}] \end{aligned} \quad (23-a)$$

$$\begin{aligned} \mathbf{v}_t^a(a_t, z_t) &= \beta E_t \left[\Lambda_{t+1}^{1-\rho} \left(v_{t+1}^m \mathcal{R} \mathcal{T} / \Lambda_{t+1} + \mathbf{v}_{t+1}^k n_{t+1}^k \underbrace{dk_{t+1}/da_t}_{=0} \right) \right] \\ &= \mathcal{T} \beta E_t [\Lambda_{t+1}^{-\rho} (v_{t+1}^m \mathcal{R})] \end{aligned} \quad (23-b)$$

$$\begin{aligned} \mathbf{v}_t^z(a_t, z_t) &= \beta E_t \left[\Lambda_{t+1}^{1-\rho} \left((v_{t+1}^m + \mathbf{v}_{t+1}^k n_{t+1}^m) \underbrace{dm_{t+1}/dz_t}_{r_{t+1} \mathcal{T} / \Lambda_{t+1}} + \mathbf{v}_{t+1}^k n_{t+1}^k \underbrace{dk_{t+1}/dz_t}_{= \mathcal{T} / \Lambda_{t+1}} \right) \right] \\ &= \mathcal{T} \beta E_t [\Lambda_{t+1}^{-\rho} (v_{t+1}^m r_{t+1} + \mathbf{v}_{t+1}^k n_{t+1}^k)] \end{aligned}$$

and (22) and (6) imply that (9) can be rewritten using \mathbf{v}_t as

$$\mathbf{v}_t(m_t) = \max_{\{a_t\}} \{u(m_t - a_t) + \mathbf{v}_t(a_t)\} \quad (24\text{-a})$$

$$\mathbf{v}_t(m_t, k_t) = \max_{\{a_t, z_t\}} \{u(m_t - (z_t - k_t) - \mathbf{j}(z_t - n_t, n_t) - a_t) + \mathbf{v}_t(a_t, z_t)\}, \quad (24\text{-b})$$

and the envelope theorem can be applied

$$\mathbf{v}_t^m(m_t) = u'(c_t) \quad (25\text{-a})$$

$$\mathbf{v}_t^m(m_t, k_t) = u'(c_t) (1 + (j_t^i - j_t^n)n_t^m) \quad (25\text{-b})$$

$$\mathbf{v}_t^k(m_t, k_t) = u'(c_t) \left(1 + \underbrace{(j_t^i - j_t^n)n_t^k}_{\equiv \varsigma_t} \right)$$

while the first order conditions with respect to a_t and z_t yield the Euler equations

$$u'(c_t) = \mathbf{v}_t^a(a_t) \quad (26\text{-a})$$

$$= \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})\mathcal{R}_{t+1}]$$

$$u'(c_t) = \mathbf{v}_t^a(a_t, z_t) \quad (26\text{-b})$$

$$= \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})(1 + \varsigma_{t+1}n_{t+1}^m)\mathcal{R}]$$

$$u'(c_t)(1 + j_t^i) = \mathbf{v}_t^z(a_t, z_t)$$

$$= \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})((1 + \varsigma_{t+1}n_{t+1}^m)r_{t+1} + (1 + \varsigma_{t+1}n_{t+1}^k))]]$$

$$= \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1}) (\mathcal{R}_{t+1} + (r_{t+1}n_{t+1}^m + n_{t+1}^k)\varsigma_{t+1})].$$

Note that if costs of adjustment are zero so that $j_t^i = \varsigma_{t+1} = 0$ and there are no shocks, the two FOC's become

$$u'(c_t) = \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})\mathcal{R}] \quad (27)$$

$$u'(c_t) = \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})\mathcal{R}_{t+1}] \quad (28)$$

which obviously can both hold only if $\mathcal{R} = \mathcal{R}_{t+1}$; but this is just another way of saying that in the perfect foresight context with no costs of adjustment, the capital stock will be adjusted to the point where the marginal product of capital (net of depreciation), $\nabla \mathcal{R}_{t+1}$, matches the return available on cash.

Note further that the assumptions about costs of adjustment yield, under N1 in which $n_{t+1}^m = 0$ and $n_{t+1}^k = 1$,

$$u'(c_t) = \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})\mathcal{R}] \quad (29)$$

$$u'(c_t)(1 + j_t^i) = \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})(\mathcal{R}_{t+1} + \varsigma_{t+1})]$$

or under N2 in which $n_{t+1}^m = \mathcal{R}^{-1}$ and $n_{t+1}^k = 0$,

$$u'(c_t) = \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})(\mathcal{R} + \varsigma_{t+1})] \quad (30)$$

$$u'(c_t)(1 + j_t^i) = \nabla \beta E_t [u'(\Lambda_{t+1}c_{t+1})(\mathcal{R}_{t+1} + \varsigma_{t+1}r_{t+1}/\mathcal{R})].$$

2 Recursion

Generically, problems like this can be solved by specifying final-period decision rules \mathbf{c}_T and \mathbf{i}_T and a procedure for recursion (obtaining \mathbf{c}_t and \mathbf{i}_t from \mathbf{c}_{t+1} and \mathbf{i}_{t+1}). Here we specify the recursion; below we specify choices for the terminal decision rules.

2.1 A Standard Solution Method

The absence of a closed-form solution means that optimal decision functions (e.g. the consumption function) must be constructed by calculating their values at a finite grid of possible values of the state variables. Call some ordered set of such values $\mu_i \in \vec{\mu} \equiv \{\mu_1, \mu_2, \dots, \mu_I\}$ and $\kappa_j \in \{\kappa_1, \kappa_2, \dots, \kappa_J\} \equiv \vec{\kappa}$.

With \mathbf{c}_{t+1} and \mathbf{i}_{t+1} in hand, the usual solution procedure is to specify a $\vec{\mu}$ and $\vec{\kappa}$ and, for each element μ_i , to use a numerical rootfinding routine to find the χ and ι that satisfy (26),

$$u'(\chi_i) = \mathbf{v}_t^a(\mu_i - \chi_i) \quad (31\text{-a})$$

$$u'(\chi_{i,j}) = \mathbf{v}_t^a(\mu_i - \chi_{i,j} - \iota_{i,j} - \mathbf{j}(\iota_{i,j}, \mathbf{n}(\mu_i, \kappa_j)), \kappa_j + \iota_{i,j}) \quad (31\text{-b})$$

$$u'(\chi_{i,j})(1 + j_t^i) = \mathbf{v}_t^z(\mu_i - \chi_{i,j} - \iota_{i,j} - \mathbf{j}(\iota_{i,j}, \mathbf{n}(\mu_i, \kappa_j)), \kappa_j + \iota_{i,j}).$$

The points $\{\mu_i, \chi_i\}$ or, for the j models, $\{\mu_i, \kappa_j, \chi_{i,j}\}$ and $\{\mu_i, \kappa_j, \iota_{i,j}\}$ are then used to construct an interpolating approximation to \mathbf{c}_t and \mathbf{i}_t . (Choice of interpolation method is separable from the point of this paper; see Judd (1998) for a discussion of choices). Given the interpolated \mathbf{c}_t and \mathbf{i}_t functions the solution for earlier periods is found by recursion.

One of the most computationally burdensome steps in this approach is the numerical solution of (31) for each specified state gridpoint. Even if efficient methods are used for constructing the expectations (cf. the parameterized expectations method of den Haan and Marcet (1990)) and shrewd choices are made for the points to include in $\vec{\mu}$, for each gridpoint a numerical rootfinding operation still must evaluate a substantial number of candidate values for the control variables before finding values that satisfy (31) to an acceptable degree of precision.

2.2 Endogenous Gridpoints Solution Method

This paper's key contribution is to introduce an alternative approach that does not require numerical rootfinding. The trick is to begin with *end-of-period* assets a_t and capital z_t and to use the end-of-period marginal value functions \mathbf{v}_t^a and \mathbf{v}_t^z , the first order conditions, and the budget constraint to construct the unique values of middle-of-period m_t and k_t generated by those a_t and z_t values.

Specifically, define the exogenous, time-invariant ordered set of values of a_t collected in $\alpha_i \in \vec{\alpha} \equiv \{\alpha_1, \alpha_2, \dots, \alpha_I\}$ and z_t collected in $\zeta_j \in \vec{\zeta} \equiv \{\zeta_1, \zeta_2, \dots, \zeta_J\}$. For each end-of-period state $\{\alpha_i, \zeta_j\}$ the marginal values $\mathbf{v}_t^a(\alpha_i, \zeta_j)$ and $\mathbf{v}_t^z(\alpha_i, \zeta_j)$ are easy to calculate;

inverting the consumption first order condition, the gridpoints generate

$$\chi_i = u'^{-1}(\mathbf{v}_t^a(\alpha_i)) \quad (32\text{-a})$$

$$\chi_{i,j} = u'^{-1}(\mathbf{v}_t^a(\alpha_i, \zeta_j)). \quad (32\text{-b})$$

These equations can be used to think about the limiting behavior of the consumption function.

$$\lim_{\alpha_i \rightarrow 0} \chi_i = 0 \quad (33)$$

$$\lim_{\alpha_i \rightarrow 0} \chi_{i,j} = 0 \quad (34)$$

$$\lim_{\alpha_i \rightarrow \infty} \chi_i / \alpha_i = 1 \quad (35)$$

$$\lim_{\alpha_i \rightarrow \infty} \chi_{i,j} / \alpha_i = 0 \quad (36)$$

ERIC: Think about these limits - the first one is probably either 0 or 1, I'm not sure which. The second one comes from the formula for the perfect foresight level of consumption in a model with no human capital, see the handout from my first year class to refresh your memory, I'm implicitly assuming that because the marginal product of capital goes to zero as ε goes to infinity the marginal effect of extra capital goes to zero... but I haven't thought about this carefully at all.

In the j models, the first order condition with respect to z_t implies that

$$u'(\chi_{i,j})(1 + \mathbf{j}^i) = \mathbf{v}_t^z(\alpha_i, \zeta_j) \quad (37)$$

$$\mathbf{j}^i = \left(\frac{\mathbf{v}_t^z(\alpha_i, \zeta_j)}{u'(\chi_{i,j})} \right) - 1 \quad (38)$$

$$((\zeta_j - n_{i,j})/n_{i,j} - \delta/\Upsilon) \omega = \left(\frac{\mathbf{v}_t^z(\alpha_i, \zeta_j) - u'(\chi_{i,j})}{u'(\chi_{i,j})} \right) \quad (39)$$

$$\zeta_j - n_{i,j} = \left(\frac{\delta}{\Upsilon} + \left(\frac{\mathbf{v}_t^z(\alpha_i, \zeta_j) - u'(\chi_{i,j})}{\omega u'(\chi_{i,j})} \right) \right) n_{i,j} \quad (40)$$

$$n_{i,j} = \left[\Upsilon^{-1} + \left(\frac{\mathbf{v}_t^z(\alpha_i, \zeta_j) - u'(\chi_{i,j})}{\omega u'(\chi_{i,j})} \right) \right]^{-1} \zeta_j. \quad (41)$$

We can use this equation to think about the limits of the investment function as α and ζ go to zero ...

Note that the budget constraint implies that

$$\mu_i = \alpha_i + \chi_i \quad (42\text{-a})$$

$$\mu_{i,j} = \alpha_i + \chi_{i,j} + \underbrace{(\zeta_j - n_{i,j}) + \mathbf{j}(\zeta_j - n_{i,j}, n_{i,j})}_{\equiv j_{i,j}}. \quad (42\text{-b})$$

We have now obtained the values of both control variables from having specified the values of the two end-of-period state variables; we thus have implicitly defined the policy functions for those variables.

It is interesting to note at this point a distinction between the N1 and N2 models. For the N2 model, the initial value of capital at the beginning of the period, k_t , has no consequences for behavior once the middle of the period has been reached. And since n_t is a monotonic deterministic function of m_t in the N2 framework, the decision problem effectively is a single-state-variable problem. This completes the recursion.

The key distinction between this approach and the standard one is that the gridpoints for the policy functions are not predetermined; instead they are endogenously generated from a predetermined grid of values of end-of-period assets (hence the method's name). One reason the method is efficient is that expectations are never computed for any gridpoint not used in the final interpolating function; the standard method may compute expectations for many unused gridpoints.

3 Macro Specialization

We first specialize to a macroeconomic stochastic growth model. Assuming aggregate production is Cobb-Douglas in capital and labor $F(K, P) = K^\varepsilon P^{1-\varepsilon}$, after normalizing by productivity P (and assuming a constant value G for the labor productivity growth factor), under the usual assumptions of perfect competition etc. if there is no aggregate transitory shock ($\Theta_{t+1} = 1$) we have

$$\mathcal{R}_{t+1} = 1 + \varepsilon k_{t+1}^{\varepsilon-1} \quad (43)$$

$$\mathcal{W}_{t+1} = (1 - \varepsilon) k_{t+1}^\varepsilon \quad (44)$$

and market resources are the sum of capital and production,

$$m_{t+1} = k_{t+1} \mathcal{R}_{t+1} + \mathcal{W}_{t+1} \quad (45)$$

$$= k_{t+1} + k_{t+1}^\varepsilon. \quad (46)$$

We specify the terminal consumption function as

$$\mathbf{c}_T(m) = m, \quad (47)$$

which is very far from the converged infinite horizon consumption rule, but easy to verify as satisfying the assumption (10) imposed earlier. More efficient choices are available, but for our purposes simplicity trumps efficiency.

An arbitrary specification of the process for permanent productivity shocks is a three point distribution defined by $\vec{\Psi} = \{0.9, 1.0, 1.1\}$ with probabilities $\Pr(\vec{\Psi}) = \{0.25, 0.50, 0.25\}$.⁴

⁴With careful choice of points and weights, small-dimensional discrete representations like this do a good job of approximating commonly-used continuous distributions like a lognormal, cf. Judd (1998). An empirically realistic choice would have a much lower variance than the specification here.

The top panel of figure 1 plots the converged consumption function that emerges from this solution method for the benchmark set of parameter values specified in Table 1, along with the consumption function for the standard perfect foresight version of the model ($\vec{\Psi} = \Pr(\vec{\Psi}) = \{1\}$).

The steady state of the perfect foresight representative agent model occurs at the point where

$$u'(c) = \beta \Upsilon \mathcal{R} \Lambda^{-\rho} u'(c) \quad (48)$$

$$\mathcal{R} = G^\rho / \beta \Upsilon \quad (49)$$

$$1 + \varepsilon k^{\varepsilon-1} = G^\rho / \beta \Upsilon \quad (50)$$

$$k = ((G^\rho / \beta \Upsilon - 1) / \varepsilon)^{\frac{1}{\varepsilon-1}} \quad (51)$$

and defining the intertemporal interest rate as $\mathfrak{R}_{t+1} \equiv dM_{t+1}/dA_t = \mathcal{R} \Upsilon = \mathfrak{R}$, if wages grow from the current level \mathcal{W} by a factor G each year then the ratio of human wealth to current wages will be

$$h = \left(\frac{1}{1 - G/\mathfrak{R}} \right) \quad (52)$$

and the partial equilibrium perfect foresight model's marginal propensity to consume (note that this neglects the effect of consumption on the interest rate) is

$$\pi = (1 - (\mathfrak{R}\beta)^{1/\rho} / \mathfrak{R}) \quad (53)$$

leading to a steady-state level of consumption of

$$c = (k\mathcal{R} + h\mathcal{W})\pi \quad (54)$$

$$= (m + (h - 1)\mathcal{W})\pi \quad (55)$$

where in the second expression 1 must be subtracted from h to reflect the fact that current labor income \mathcal{W} has already been incorporated into m and must not be double-counted (as it otherwise would be since it was included in the infinite sum that led to the expression for h).

4 Micro Specialization

In the microeconomic literature, the usual approach is to take aggregate interest and wage rates as exogenous, and to focus on transitory (Θ) and permanent (Ψ) shocks to idiosyncratic labor productivity. We again start the recursion with $\mathbf{c}_T(m) = m$, and the permanent shocks are retained exactly as specified for the macro problem.⁵

⁵An empirically realistic calibration for micro data would exhibit a permanent variance perhaps 100 times greater than an appropriate macro calibration; but appropriate calibration is not the point of this paper.

4.1 Life Cycle Models

Life cycle models specify a stereotypical pattern of lifetime income growth defined by G_t where t is age rather than time and T is the maximum possible lifespan;⁶ mortality uncertainty can be accommodated by age-varying values of β .

4.2 Buffer Stock Models

If R, W, G and β are constant, $\bar{\gamma} = 1$, and the impatience condition

$$R\beta E[(G\Psi)^{-\rho}] < 1 \quad (56)$$

holds, Deaton (1991) and Carroll (2004) show that the problem defines a contraction mapping so that the consumption functions defined by the problem converge from any well-behaved initial starting function $\mathbf{c}_T(m)$; the converged function is defined as

$$\mathbf{c}(m) = \lim_{n \rightarrow \infty} \mathbf{c}_{T-n}(m). \quad (57)$$

We solve for the converged consumption function for two versions.

4.2.1 Version With Unemployment

Assume that in future periods there is a small probability φ that income will be zero (corresponding to a substantial spell of unemployment):

$$\Theta_{t+1} = \begin{cases} 0 & \text{with probability } \varphi > 0 \\ \Xi_{t+1}/(1 - \varphi) & \text{with probability } (1 - \varphi) \end{cases} \quad (58)$$

where $\Xi = \{0.9, 1.0, 1.1\}$ and $\Pr(\Xi|\Theta > 0) = \{0.25, 0.50, 0.25\}$ (the same structure of non-unemployment transitory shocks as for the permanent shocks).

Carroll (2004) shows that in this model,

$$\lim_{m_t \rightarrow 0} \mathbf{c}_t(m_t) = 0. \quad (59)$$

This implies that the minimum value in $\vec{\alpha}$ should be $\alpha_1 = 0$, which will generate $\{\mu_1, \chi_1\} = \{0., 0.\}$ as the first point in the set of interpolating points. The resulting converged $\mathbf{c}(m)$ is shown as the thin solid locus in the bottom panel of figure 1; see the software for details of how the remaining values in $\vec{\alpha}$ were chosen.

4.2.2 Version With Liquidity Constraints

Microeconomic models often include a liquidity constraint in addition to the usual transition equations, and capturing the constraint often induces much additional code.

⁶This is the context in which the assumption that $\mathbf{c}_T(m) = m$ actually makes economic sense, as distinct from merely providing a convenient starting point for recursion.

Dealing with a liquidity constraint using the method of endogenous gridpoints is simple. The key observation is that when the constraint is on the cusp of binding, the marginal value of consumption is equal to the marginal value of saving exactly zero (assuming the constraint is of the form that requires a to be nonnegative; generalization to more elaborate kinds of constraints is straightforward). If the first value in the ordered set $\vec{\alpha}$ is $\alpha_1 = 0$, then the method will produce

$$\chi_1 = \mu_1 = u'^{-1}(\mathbf{v}_t^a(0)), \quad (60)$$

and if we define $\hat{\mathbf{c}}_t(m)$ as the function produced by interpolation among the points generated by $\vec{\alpha}$, the consumption function imposing the constraint will be

$$\mathbf{c}_t(m) = \min(m, \hat{\mathbf{c}}_t(m)). \quad (61)$$

If the consumption function is defined as a piecewise linear spline interpolation among the $\{\mu, \chi\}$ points, the constraint can be handled simply by adding the point $\{\mu_0, \chi_0\} = \{0, 0\}$ to the set of points that constitute the interpolation data.

The converged solution is shown as the bold locus in the bottom panel of figure 1.

5 The Entrepreneurial Model

5.1 General Version

This is not so much a specialization as the description of a calibration of the j models. We retain all the shocks, and choose a specific value of the coefficient of relative risk aversion, for convenience $\rho = 2$. The transitory and permanent shocks are identical to what is assumed in the ‘micro’ model described above; the interpretation of $\Theta = 0$ might include a labor strike, a legal dispute that shuts the plant for a period, or any other temporary disruption to production that leaves the level of capital unchanged while affecting the output produced by that capital.

The combination of CRRA utility and strikes will prevent the agents from ever consuming an amount greater than or equal to their entire capital stock. That is, they never borrow, even when capital is far below its optimal steady state level.

The model is asymmetric in the sense that firms with “too much” capital have no similar constraint, except that they cannot sell off all their excess capital at once without incurring large adjustment costs. The quadratic specification of the adjustment cost function induces precisely this kind of gradual adjustment toward the optimal target.

In addition, we assume that

- The entrepreneur is impatient $\beta < \mathcal{R}$ and $G = 1$
- The cost of adjustment parameter is $\omega = 0.1$

Finally, we need to specify behavior in the terminal period, \mathbf{c}_T and \mathbf{i}_T . The idea is to assume that in period T the entrepreneur behaves according to the unconstrained perfect foresight model. In that model, investment decisions are conducted according to a pure

q model, which yields a present discounted value of the firm which the consumer then treats as pure wealth. Thus, defining $\mathbf{v}(k)$ as the value of the firm (the present discounted value of profits if the perfect-foresight entrepreneur behaves optimally from period T on), and if $b_T = \mathcal{R}k_T$, consumption will be given by

$$c_T = (1 - (\mathcal{R}\beta)^{1/\rho}\mathcal{R}^{-1})(b_T + \mathbf{v}(k_T)). \quad (62)$$

See <ftp://www.econ.jhu.edu/people/ccarroll/shared/Carroll-Maccini> for a fuller description, and programs that solve the perfect foresight version of the model.

5.2 Single State Variable Version

We noted above that assumption N2 effectively converts the problem into one in which only a single state variable, m_t , matters at the moment when decisions are made. We now elaborate on this point.

This is illustrated most transparently by combining the roles played by a_t and z_t in the j models; using a_t for the name of the combined variable, the revised accumulation equation becomes

$$a_t = m_t - c_t - j_t - i_t + i_t \quad (63)$$

$$= m_t - c_t - j_t \quad (64)$$

and, adopting assumption N2, the value function is

$$\mathbf{v}_t(m_t) = \max_{a_t} \{u(m_t - \mathbf{j}(a_t - (m_t - \mathcal{W})/\mathcal{R}, (m_t - \mathcal{W})/\mathcal{R}) - a_t) + \mathbf{v}_t(a_t)\} \quad (65)$$

where

$$\mathbf{v}_t(a_t) = \beta E_t[\Lambda_{t+1}^{1-\rho} \mathbf{v}_{t+1}(\mathcal{W}_{t+1} l_{t+1} + \mathcal{R}_{t+1} a_t \mathcal{T}/\Lambda_{t+1})] \quad (66)$$

$$\mathbf{v}_t^a(a_t) = \mathcal{T} \beta E_t[\mathcal{R}_{t+1} \Lambda_{t+1}^{-\rho} \mathbf{v}_{t+1}^m] \quad (67)$$

and the Envelope theorem says

$$\mathbf{v}_t^m(m_t) = u'(c_t) (1 + (j_t^i - j_t^n) \mathcal{R}^{-1}) \quad (68)$$

so that the first order condition implies

$$u'(c_t)(1 + j_t^i) = \mathcal{T} \beta E_t [\mathcal{R}_{t+1} u'(\Lambda_{t+1} c_{t+1}) (1 + (j_{t+1}^i - j_{t+1}^n) \mathcal{R}^{-1})] \quad (69)$$

or, substituting for $c_t = m_t - a_t - j_t$,

$$u'(\mathcal{R}n_i + \mathcal{W} - \mathbf{j}(\alpha_i - n_i, n_i) - \alpha_i) (1 + ((\alpha_i - n_i)/n_i - \delta/\mathcal{T})\omega) = \mathbf{v}_t^a(\alpha_i)$$

which can be solved using a rootfinding algorithm to find the n_i (and therefore the $\mu_i = \mathcal{R}n_i + \mathcal{W}$) consistent with any particular α_i . Note the somewhat surprising result that the nonlinearity of this equation requires solution via a numerical rootfinding operation even though the model above with two state variables permitted a solution without rootfinding.

It should be fairly straightforward to determine limiting behavior for this model; my guess is that the $\lim_{\alpha_i \rightarrow 0} \chi_i/\alpha_i = 1$ and that the limiting MPC as α goes to infinity is zero. Eric, please think about this.

6 q Specialization

The standard perfect foresight q model of investment is a specialization of the φ framework laid out above under the assumptions

$$\beta = R^{-1} \quad (70)$$

$$\rho = 0 \quad (71)$$

$$\mathcal{R}_t = 1 + \varepsilon k_t^{\varepsilon-1} \quad (72)$$

$$\mathcal{W}_t = (1 - \varepsilon)k_t^\varepsilon \quad (73)$$

$$\Theta_t = 1 \quad \forall t \quad (74)$$

$$P_t = 1 \quad \forall t \quad (75)$$

$$G_t = 1 \quad \forall t \quad (76)$$

These conditions permit several simplifications.

- The natural interpretation of C in this context is dividends paid to shareholders
- $\rho = 0$ makes utility linear, which is appropriate for the behavior of a risk neutral firm
- $\beta = R^{-1}$ makes the agent indifferent to the timing of dividends; the agent's goal is simply to maximize the present discounted value of dividends, discounting them at the riskfree interest factor $R = \mathcal{R}$
- Elimination of the transitory shocks $\Theta_t = 1$ means there is no reason to analyze the problem from the perspective of the middle of the period, as the agent's circumstances in the middle of the period are perfectly predictable from the middle of the period
- The assumptions on \mathcal{R} and \mathcal{W} reflect the assumption that the firm's total output is given by k^ε , which is naturally interpreted as gross profits (before investment expenses)
- Under these assumptions, the objective of the firm can be described as maximization of discounted profits, which is equivalent to maximization of discounted dividends
- For analysis of the q model it is useful to explicitly incorporate taxes. We will assume that there is an investment tax credit in the amount ϱ and a corporate tax rate of ϕ . Note that for simplicity we assume that the ITC applies both to the purchase price and the adjustment costs for investment.

Thus, designating after-tax profits as $\pi = (1 - \phi)f(k)$ and total expenditures on investment (including adjustment costs) as $\xi = (i + j)\hat{p}$ where $\hat{p} = (1 - \varrho)$ and ϱ is the investment tax credit, and calling the beginning of period value function as ν_t , the problem can be rewritten as

$$\nu_t(k_t) = \max_{\{i\}_t^T} \sum_{s=t}^T R^{t-s} (\pi_s - \xi_s) \quad (77)$$

and the problem can be written in Bellman equation form as

$$\nu_t(k_t) = \max_{a_t} \{ \pi(k_t) - (a_t - k_t) - \mathbf{j}(a_t - k_t, k_t) + \mathbf{v}_t(a_t) \} \quad (78)$$

where

$$\mathbf{v}_t(a_t) = \mathbf{\nabla} \beta E_t [\Lambda_{t+1}^{1-\rho} \nu_{t+1}(\mathbf{\nabla} a_t / \Lambda_{t+1})] \quad (79)$$

$$\mathbf{v}_t^a(a_t) = \mathbf{\nabla} \beta E_t [\Lambda_{t+1}^{-\rho} \nu_{t+1}^k(\mathbf{\nabla} a_t / \Lambda_{t+1})] \quad (80)$$

with FOC

$$1 + j_t^i = \mathbf{v}_t^a(a_t) \quad (81)$$

while the Envelope condition tells us

$$\nu_t^k(k_t) = (1 - \phi)r_t + 1 - j_t^k + j_t^i \quad (82)$$

so the FOC implies

$$1 + j_t^i = \overbrace{R^{-1}\beta}^{=1} \mathbf{\nabla} E_t [(1 - \phi)r_{t+1} + 1 + j_{t+1}^i - j_{t+1}^k] \quad (83)$$

7 Conclusion

The method of endogenous gridpoints can be extended to problems with multiple state variables and multiple controls, e.g. a micro consumer with a portfolio choice problem, or a labor supply decision; or a macro consumer with a utility function that exhibits habit formation (see Carroll (2000) for examples). The method is useful both because it is simpler than the standard method and because it reduces computational demands.

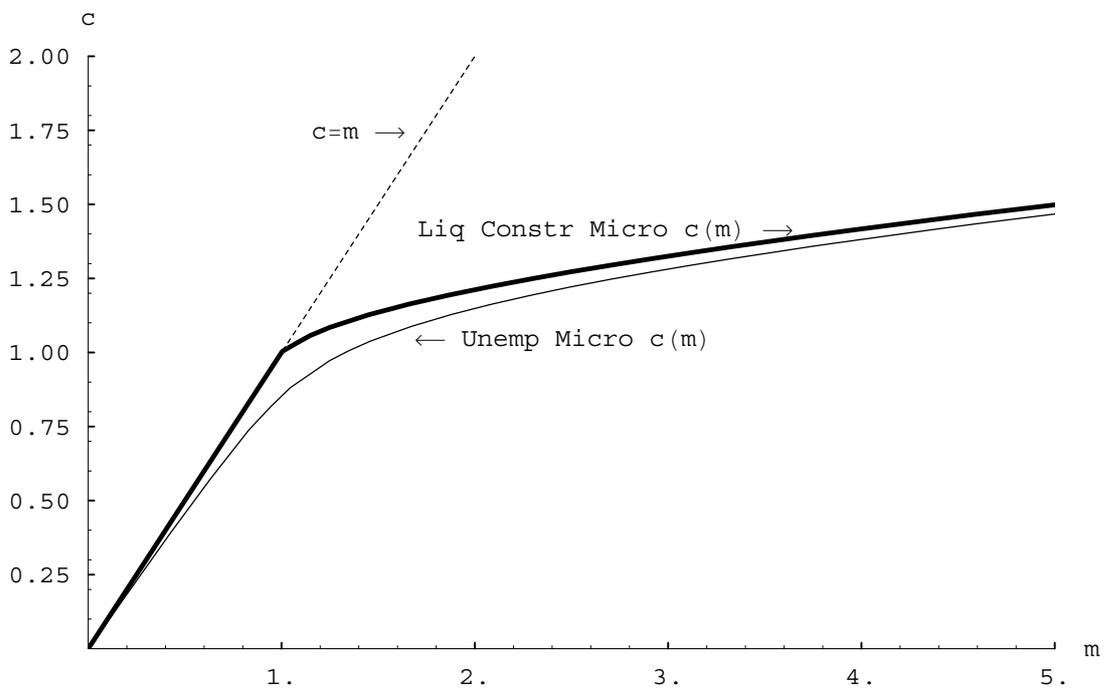
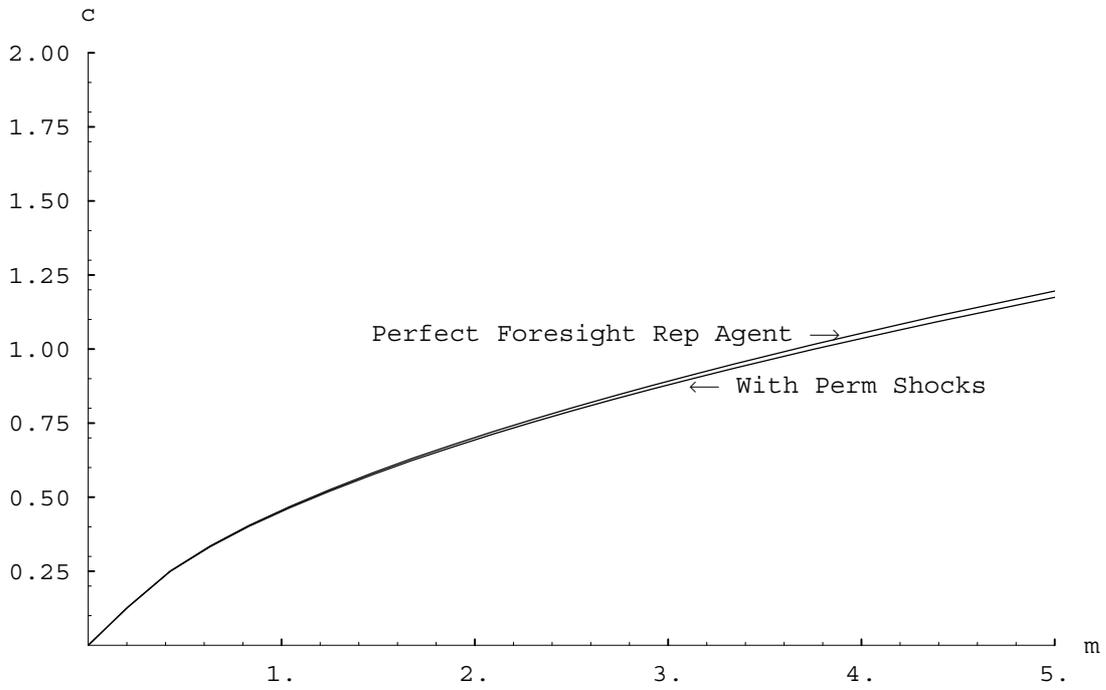
References

- Carroll, Christopher D., 2000, Lecture notes on solving microeconomic dynamic stochastic optimization problems, Manuscript, Johns Hopkins University. Available at <http://econ.jhu.edu/people/ccarroll/solvingmicrodsops.pdf>.
- Carroll, Christopher D., 2004, Theoretical foundations of buffer stock saving, NBER Working Paper Number 10867. <http://econ.jhu.edu/people/ccarroll/BufferStockProofsNew.pdf>.
- Deaton, Angus S., 1991, Saving and liquidity constraints, *Econometrica* 59, 1221–1248.
- denHaan, Wouter J, and Albert Marcet, 1990, Solving the stochastic growth model by parameterizing expectations, *Journal of Business and Economic Statistics* 8(1), 31–34. available at <http://ideas.repec.org/a/bes/jnlbes/v8y1990i1p31-34.html>.
- Judd, Kenneth L., 1998, *Numerical Methods in Economics* (The MIT Press, Cambridge, Massachusetts).

Table 1: Parameter Values

Parameters Common to All Models		
ρ	2	Relative Risk Aversion
β	0.96	Annual Discount Factor
e	1	Labor supply/effort (fixed)
$\vec{\Psi}$	{0.90, 1.00, 1.10}	Permanent Shock Realizations
$\Pr(\vec{\Psi})$	{0.25, 0.50, 0.25}	Permanent Shock Probabilities
Macro Model Parameters		
τ	0.90	Depreciation Factor
G	1.01	Exogenous Aggregate Productivity Growth Factor
ε	0.36	Capital Share in Production
Micro Model Parameters		
τ	1	Depreciation Factor
G	1.03	Trend Individual Wage Growth Factor
R	1.04	Real Interest Rate
W	1.00	Wage Rate
$\vec{\Xi}$	{0.90, 1.00, 1.10}	Transitory Shock Realizations for Employed
$\Pr(\vec{\Xi} \Theta > 0)$	{0.25, 0.50, 0.25}	Transitory Shock Probabilities for Employed
Parameter Unique to Unemployment Model		
\wp	0.005	Probability of Unemployment Spell

Figure 1: Macro and Micro Consumption Functions



Appendices: *Mathematica* Code

This appendix contains the core code used to generate the micro and macro model solutions graphed in the figures. `Common.nb` contains the parameters and code that are shared for both micro and macro solutions; `Micro.nb` and `Macro.nb` contain the specific parameterizations and specializations for the respective specific problems. The commands to execute the solutions and graph them are not of general interest and are not included, but are part of a downloadable package available on the author's website. Downloadable MATLAB code is also available on the author's webpage; Michael Haliassos and Dimitri Mavrides have written MATLAB and C++ code that solves a closely related problem; contact Haliassos for more information.

```

Common.nb
uP[c_] := If[c > 0, (*then*) c^-rho, (*else*) infinity];
nP[z_] := z^-(1/rho);
uP[at_] := T beta Sum[
  Ψtp1 = ΨVec[[ΨLoop]];
  Δtp1 = GΨtp1;
  Θtp1 = ΘVec[[ΘLoop]];
  ktp1 = T at/Δtp1;
  ltp1 = Θtp1 eEffort;
  mtp1 = If[MacroModel && ktp1 == 0, 0, ktp1 R[ktp1] + ltp1 W[ktp1]];
  ΨVecProb[[ΨLoop]] ΘVecProb[[ΘLoop]] R[ktp1] uP[Δtp1 Last[cInterpFunc][mtp1]]
, {ΨLoop, Length[ΨVec]}
, {ΘLoop, Length[ΘVec]}
]; (* End Sum *)
cInterpFunc = {Interpolation[{{0., 0.}, {1000., 1000.}], InterpolationOrder- > 1]};

SolveAnotherPeriod := Block[{},
AppendTo[cInterpFunc,
Interpolation[
Union[
Chop[
Prepend[
Table[
α = αVec[[αLoop]];
χ = nP[uP[α]];
μ = α + χ;
{μ, χ}
, {αLoop, Length[αVec]}]
, {0., 0.}] (* Prepending {0, 0} handles potential liquidity constraint *)
] (* Chop cuts off numerically insignificant digits *)
] (* Union removes duplicate entries *)
, InterpolationOrder- > 1] (* Piecewise linear interpolation *)
]; (* End of AppendTo *)
]; (* End of SolveAnotherPeriod *)
{β, ρ, n, eEffort, PeriodsToAdd} = {0.96, 2, 20, 1, 99};

```

Micro.nb

```

{T, G, p} = {1, 1.03, 0.005};
Λ = G;
MacroModel = False;
<< Common.nb;
(* Triple exponential growth to a = 10 picks a good set of values for α *)

αVec = Table[Exp[Exp[Exp[αLoop] - 1] - 1] - 1 \\ / N,
  {αLoop, 0, Log[Log[Log[10 + 1] + 1] + 1], Log[Log[Log[10 + 1] + 1] + 1] / (n - 1)}];

EVec = ΨVec = {0.9, 1., 1.1};
EVecProb = ΨVecProb = {0.25, 0.5, 0.25};
ΘVec = Prepend[EVec / (1 - p), 0.];
ΘVecProb = Prepend[EVecProb (1 - p), p];
R[k_] := 1.04;
W[k_] := 1.;

```

Macro.nb

```

{T, G, ε} = {0.9, 1.01, 0.36};

MacroModel = True;
<< Common.nb;
(* PFSS k *) kSS = (((G^ρ) / (β T)) - 1) / ε ^ (1 / (ε - 1));
(* PFSS a *) aSS = kSS G / T;
αVec = Table[Exp[αLoop] - 1, {αLoop, 0, Log[3 aSS], Log[3 aSS] / (n - 1)}];
ΨVec = {0.9, 1., 1.1};
ΨVecProb = {0.25, 0.5, 0.25};
ΘVec = {1.};
ΘVecProb = {1.};
R[k_] := If[k > 0, (*then*) 1 + ε k^ε (ε - 1), (*else*) ∞];
W[k_] := (1 - ε) k^ε;

```

CFS Working Paper Series:

No.	Author(s)	Title
2005/09	Markus Haas Stefan Mittnik Bruce Mizrach	Assessing Central Bank Credibility During the ERM Crises: Comparing Option and Spot Market-Based Forecasts
2005/10	Juan Carlo Conesa Dirk Krueger	On the Optimal Progressivity of the Income Tax Code
2005/11	Markus Haas Stefan Mittnik Marc S. Paoella	Modeling and Predicting Market Risk With Laplace-Gaussian Mixture Distributions
2005/12	Dirk Krueger Felix Kubler	Pareto Improving Social Security Reform when Financial Markets are Incomplete!?
2005/13	Keith Kuester Volker Wieland	Insurance Policies for Monetary Policy in the Euro Area
2005/14	Berc Rustem Volker Wieland Stan Zakovic	Stochastic Optimization and Worst-Case Analysis in Monetary Policy Design
2005/15	Dirk Krueger Fabrizio Perri	Does Income Inequality Lead to Consumption Inequality? Evidence and Theory
2005/16	Klaus Adam Roberto Billi	Discretionary Monetary Policy and the Zero Lower Bound on Nominal Interest Rates
2005/17	Roberto Billi	The Optimal Inflation Buffer with a Zero Bound on Nominal Interest Rates
2005/18	Christopher D. Carroll	The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems

Copies of working papers can be downloaded at <http://www.ifk-cfs.de>