# The Condensation Phase Transition and the Number of Solutions in Random Graph and Hypergraph Models

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# 1 Introduction

The study of random constraint satisfaction problems (CSPs) looks back on a long history and, during this time, has been approached from different points of view. Extensive investigations were undertaken in the mathematical field of combinatorics as well as in computer science and, more recently, in statistical mechanics. The motivation for this interdisciplinary research originates in a wide range of applications, namely, among others, in the fields of optimization, coding theory, artificial intelligence and spin glasses.

In a constraint satisfaction problem variables are related via constraints that determine which combinations of value assignments to the variables form a solution. The decision version of the problem aims at establishing whether or not an instance admits a solution. In the search version algorithms are applied to try and find concrete solutions. If the problem exhibits a solution, a canonical question will relate to the *total number* of solutions. Prominent examples of CSPs are the well-known $k$-SAT problem, the graph $k$-colouring problem and the hypergraph 2-colouring problem[1].

The focus of this thesis is on *random* constraint satisfaction problems, meaning that the underlying structures (the boolean formulas or (hyper)graphs) are generated randomly. Studying random problems is of great interest as random instances exhibit phenomena that deterministically constructed instances do not. Indeed, in many problems it seems to be impossible to generate deterministic instances that are as hard as random ones picked according to some appropriate distribution (cf. [BHvMW09, CM97] and the references therein for more details).
When speaking of the *evolution* of the random structures, we mostly refer to the setting where the constraint-to-variables density (the ratio between constraints and variables, often only called *constraint density* or *average degree*) increases, thus making it more and more unlikely for a random instance to exhibit a solution. Almost exclusively, the objects to be studied will be *sparse*, meaning that the average degree will be bounded when the number of variables tends to infinity.

The persistent study of random CSPs in different disciplines during the last three decades has led to a series of hypotheses and results, highlighting in particular their striking similarities. A prominent hypothesis states that when the constraint density passes through a critical threshold, the probability for a random instance of the problem to be solvable drops very rapidly from 1 to 0, thus the problem appears to undergo a *phase transition*. Although a wealth of research has been dedicated to understanding the behaviour of random CSPs, it has turned out very difficult to rigorously approach any

[1]In the colouring problems variables correspond to vertices and constraints to (hyper)edges. The expressions will be used synonymously further on.

of the hypotheses. In particular, for most random CSPs a proof of the existence, let alone the precise location, of the critical threshold remains elusive.

However, some progress has been made in shedding light on the various phenomena over the years. A great part of this success is owed to physicists from statistical mechanics who brought about new inspiring insights into the combinatorial nature of the problems. They developed non-rigorous but sophisticated methods to make very precise predictions about the location of the critical threshold (cf. [MM09, KMRTSZ07] for detailed information and references). Maybe even more importantly, they illuminated the impact of the geometry of the set of solutions, thereby explaining a variety of peculiarities that had been observed before, but had not been understood.

In the last years, several of the predictions could be proven by mathematicians from probabilistic combinatorics and up to now none has been falsified. Mathematicians benefited a lot from the physicists' insights and the knowledge of statements they had to prove. However, it still required developing some completely new techniques.

The results in this thesis take their place alongside a range of other contributions on the long way of solving this puzzle piece by piece. They pertain to two different random CSPs, namely random graph $k$-colouring and random hypergraph 2-colouring. On these models, they relate to two different objectives. The first is determining the distribution of the number of solutions in these CSPs in the limit when the number of vertices becomes large. The second consists in establishing the existence and location of yet another phase transition predicted by the physicists called "condensation".

The thesis will be structured as follows: The next two sections provide a brief overview of the historical evolution of the research in this area and a short outline of the physics approach to these problems. After that, a short summary of the results in this thesis will be given. Chapter 2 is devoted to formally introducing the models under consideration and defining essential concepts and the questions we are dealing with. In Chapter 3 the techniques and proof methods are explained. Chapter 4 presents the main results of the thesis and puts them in relation to other relevant work. The subsequent Chapters 5 up to 8 as well as Appendices A and B comprise the proofs of the results. Finally, Chapter 9 provides a conclusion and an outlook to further research questions and challenges.

## 1.1. Historical background

The graph $k$-colouring problem, asking whether it is possible to colour the vertices of a given graph with $k$ different colours such that no two adjacent vertices share the same colour, has been of central interest in discrete mathematics for more than one century. It had its beginnings in the "four colour problem" posed by De Morgan in 1852 and for randomly generated graphs it constitutes one of the longest-standing challenges in probabilistic combinatorics since the seminal paper [ER60] of Erdős and Rényi, which started the theory of random graphs (cf. [Bol01, JLR00] for a comprehensive survey of this field of research). This impressive paper laid the foundation for engagement in the theory of phase transitions as it illuminated many aspects of the evolution of random graphs and established the critical point for the emergence of a giant component as well as the one for a random graph being connected (which they had already investigated in [ER59]). From a number of intriguing questions posed in this paper, the one concerning the typical chromatic number of a random graph is the last that still remains unanswered.

Also the hypergraph 2-colouring problem has a long history: In the early 1900s, the mathematician Bernstein [Ber07] considered a question which can be rephrased in the following way: Is it possible to colour the vertices of a given hypergraph with two colours such that no hyperedge is monochromatic? A hypergraph for which this is possible possesses "Property B" as it was later called in honour of Bernstein. In the 1960s, Erdős popularized this problem [Erd63, Erd64] and proposed bounds on the smallest number of hyperedges in non-2-colourable $k$-uniform hypergraphs. Indeed, according to [AM06], determining this smallest number remains one of the most important problems in extremal graph theory up to these days.

The problems of $k$-colouring graphs and 2-colouring hypergraphs belong to the aforementioned set of constraint satisfaction problems, just as for example the well-known boolean satisfiability problem $k$-SAT or the independent set problem. In 1971, the renowned computer scientist and mathematician Cook [Coo71] proved that $k$-SAT is NP-complete for all $k \geq 3$. One year later, Karp [Kar72] showed that by reduction a whole bunch of combinatorial and graph theoretical computational problems, including $k$-colourability, can also be found to be NP-complete and thus cannot be solved by deterministic polynomial time algorithms unless the classes P and NP coincide. Lovász [Lov73] derived the same result for hypergraph 2-colouring.

Since the 1990s, random CSPs, involving randomly chosen constraints on the variables, have been intensely studied in the field of probabilistic combinatorics. The beginnings of this work were of experimental nature and the findings resulted in two hypotheses [CKT91, MSL92]: First, that in many random CSPs there exists a *satisfiability threshold*, a certain constraint-to-variables density below which random instances of the problem have solutions and above which they have not with high probability[2]. And second, that the difficulties of algorithmically computing a solution near this threshold go hand in hand with this threshold phenomenon.

While it turned out extremely difficult to verify any conjectures concerning the algorithmic performance, and until now we only have a very vague idea about the true connections, regarding the threshold behaviour some progress could be achieved. Indeed, in a breakthrough paper in 1999, Friedgut [Fri99] proved the existence of a *non-uniform satisfiability threshold sequence* in random $k$-SAT, i.e. a se-

[2]We say that a sequence of events $\mathcal{A}_n$ occurs with high probability (w.h.p.) if $\lim_{n\to\infty} \mathbb{P}[\mathcal{A}_n] = 1$.

quence depending on the number $n$ of variables that marks the point where the probability of being solvable drops from 1 to 0:

**Theorem 1.1.1.** *Let $F_k(n, dn)$ be a $k$-CNF [3] formula on $n$ variables and $dn$ constraints chosen uniformly at random from all such formulas. Then for each $k \geq 3$, there exists a sequence $d_{\text{sat}}(n)$, such that for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\left[F_k(n, dn) \text{ is satisfiable }\right] = \begin{cases} 1 & \text{if } d = (1-\varepsilon)d_{\text{sat}}(n), \\ 0 & \text{if } d = (1+\varepsilon)d_{\text{sat}}(n). \end{cases}$$

Achlioptas and Friedgut [AF99] could prove the same for random graph $k$-colouring for $k \geq 3$ and it also holds for random hypergraph 2-colouring and other monotone random CSPs [Fri05]. The non-uniformity of the threshold sequence left open the possibility that the threshold value might vary with growing $n$. Only for a very small number of problems, the existence of the limit $d_{\text{sat}} = \lim_{n\to\infty} d_{\text{sat}}(n)$ has been proven and its location been determined. The most prominent example presumably is the result for random $k$-SAT for large $k$ [DSS15]. However, it is widely conjectured that the sequence converges in other problems as well[4]. For this reason and as per common practice in the study of random CSPs, we will take the liberty of speaking of "the threshold" $d_{\text{sat}}$, or more specifically $d_{\text{col}}$ for the colouring problems. Proving this conjecture and determining the location of the threshold in random CSPs (as Theorem 1.1.1 is a pure existence result) is a major open problem.

A wealth of research has since been devoted to finding upper and lower bounds on the threshold in the different problems. While upper bounds can rather easily be derived via the first moment method, for a long period of time the best lower bounds were of algorithmic nature [FS96], later on they stemmed from the second moment method. However, in most cases the first and second moment method do not yield matching lower and upper bounds (cf. Section 3.2 for an explanation of the methods and Sections 4.2 and 4.3 for a discussion of their application in different problems). So, efforts were started to learn about the nature of this gap, but for a couple of years it was not clear how to get a handle on that.

Interestingly and - as might be said - fortunately, physicists doing research in the field of statistical mechanics have been working on random CSPs for the past decades as well. In the early 2000s, they put forward a "symmetry-breaking" version of the so-called *cavity method*, a non-rigorous but very sophisticated tool that allowed them to make very precise conjectures as to the location of the thresholds in

[3] CNF stands for "conjunctive normal form". In the $k$-SAT problem, the boolean formula is expressed in $k$-CNF, which is a conjunction of disjunctions, each encompassing $k$ literals.

[4] Not in all problems, however, as e.g. the problem of 2-colouring random graphs does not exhibit sharp threshold behaviour, because the probabilities of $G(n, dn)$ having an odd cycle and not having an odd cycle are both bounded away from 0 for every $d \in (0, 1)$.

different problems. But maybe even more intriguing were the insights into the combinatorial nature of the problems and the prediction of yet another phase transition called *condensation* that occurs shortly before the satisfiability threshold [KMRTSZ07] and that seems to be the reason why identifying the precise threshold for the existence of solutions is such a challenging task.

## 1.2. The physics perspective

In this section we want to outline the picture that has been painted by physicists from statistical mechanics about the combinatorial and structural properties of the solution space[5] in many random CSPs. This picture gives hints, albeit in a non-rigorous way, to questions such as why there seems to be a mysterious barrier in the constraint density that all rigorously analysed algorithms prove unable to pass or what is the nature of the gap between the first moment upper and the second moment lower bound in these kinds of problems. As will be described in detail in Section 4.3, some of the conjectures made by the physicists have meanwhile been proven, but a major part still evades a rigorous analysis.

In the language of statistical physics, random CSPs like hypergraph 2-colouring and graph $k$-colouring on sparse random (hyper)graphs are examples of *diluted mean-field models of disordered systems*. Resolving this term into its components reveals some very important common characteristics of these problems. The term *diluted* refers to the fact that the average degree in the underlying graph is bounded, while *mean-field* indicates that there is no underlying lattice geometry. Moreover, the concept of *disordered systems* reflects that the model involves randomness, which in our case comes in the form of the sparse random (hyper)graph that determines the geometry of interactions between individual "sites".

Diluted mean-field models have been studied thus intensely because they are considered a better approximation to "real" disordered systems than models where the underlying graph is complete, in the sense that they have a more realistic geometry. A prominent example for a model basing on a complete graph is the Sherrington-Kirkpatrick model [SK75], which is a fully-connected mean-field model, where each variable interacts with any other via randomly chosen couplings. Examples for these real disordered systems are glasses and spin-glasses, which attracted attention because of their peculiar magnetic properties. Already in the 1980s, Mézard and Parisi [MP85, MP87] as well as Fu and Anderson [FA86] made first attempts on adapting heuristics from the study of spin glasses to explain the CSP solution space. Unfortunately, unlike for example in the Sherrington-Kirkpatrick model, where the free energy is captured by the "Parisi formula" [Par80, Tal06], and in general in fully-connected

[5]To be concrete, the solution space of a distinct problem is a simple graph where every vertex represents a solution and vertices are connected if the solutions differ on exactly one variable. In the literature it is also common to connect vertices if the solutions differ only on a sub-linear number of variables. However, in most cases this yields asymptotically the same statements. The graph representing the solution space should not be confused with the underlying graph of the random CSP.

models, where every pair of vertices interacts in the same way, statistical mechanics models of disordered systems exhibit a non-trivial geometry and their analytical study turned out to be notoriously difficult.

However, more than 30 years ago, physicists introduced the so-called *replica method*, an analytic but non-rigorous approach for attacking these kinds of problems [MPV87]. It was originally developed to deal with the Sherrington-Kirkpatrick model and generalized former attempts of understanding its behaviour [SK75, Par79, Par80]. As an alternative approach, yet similar in spirit, the *cavity method* was presented around the same time. After having been applied to sparse random graphs [MP85] and coding theory, since the late 1990s the *replica symmetric (RS)* variants of these methods have been further developed into the more intricate *one-step replica symmetry breaking (1RSB)* versions [Mon98, MP01, MP03]. The one-step replica symmetric cavity method is a very sophisticated and powerful but still non-rigorous tool and originated in the context of spin glasses, where it was designed to work with models on locally tree-like graphs (cf. [MM09] for details and references).
As sparse random (hyper)graphs are locally tree-like and only possess a bounded number of short cycles with high probability, the cavity method can be used to put forward precise conjectures on diluted mean-field models of disordered systems. Its application to constraint satisfaction problems, first in [MPZ02], led to a huge amount of work in the physics literature (cf. [KMRTSZ07] for a survey).

The cavity method has been used to put forward conjectures in a variety of areas, during the last years mainly in compressive sensing and most recently in machine learning. Many of its predictions are given in terms of a distributional fixed point problem. Among the various predictions, perhaps the most exciting ones relate to the existence and location of phase transitions. Typically, the replica symmetric cavity method gives upper and lower bounds on the location, while the 1RSB version is conjectured to yield precise results. In particular, there exist conjectures on the exact location of the satisfiability threshold $d_{\mathrm{sat}}$ in many problems. What is more, according to the cavity method there occur other transitions prior to $d_{\mathrm{sat}}$ [KMRTSZ07] and when crossing them, the geometric properties of the solution space dramatically change. In the next paragraph an overview of this predicted development of the solution space will be given. The most important transition for our purposes in this thesis is the so-called *condensation phase transition*. It occurs very shortly before $d_{\mathrm{sat}}$ [KMSSZ12a] as the result of an "entropy crisis". It is a phenomenon that is ubiquitous in physics, holding the key to a variety of problems, for instance it seems to be closely related to the difficulty of proving precise results on the satisfiability threshold and in particular to the demise of the second moment method (cf. e.g. [COZ12]). Furthermore, it seems to be responsible for the difficulty of analysing the performance of certain message passing algorithms, although it turned out extremely challenging to rigorously get a handle on this prediction. In contrast to the satisfiability transition, the condensation phase transition is a genuine thermodynamic transition persisting in models with finite inverse temperature (that we introduce in Section 2.3). Its role in the context of structural glasses goes back to the work of Kauzmann in the 1940s [Kau48]. It has been established in a variety of models, ranging from the random energy model

[Der81] to the fully-connected $p$-spin-glass [Tal03, KT87]. However, there are only a few rigorous results on the condensation phase transition in diluted mean-field models.

The cavity method yields substantial insights into the geometry of the solution space and makes predictions on the *free entropy density* $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\ln Z\right]$.[6] The conjectured evolution of the geometry of the solution space is as follows:
For very low constraint densities, when the (hyper)graph is still very sparse and typically many solutions exist, the solution space is - more or less - a single connected component and is described as being *replica symmetric* [KMRTSZ07]. In this regime, in many problems the typical value or *quenched average* $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\ln Z\right]$ equals the so-called *annealed average* $\lim_{n\to\infty} \frac{1}{n}\ln\mathbb{E}\left[Z\right]$ (which is often a well-behaved analytical function) .
As the density increases, at some point called the *clustering* transition, which is quite a distance from the conjectured satisfiability threshold (for example for hypergraph 2-colouring it is about a factor of $k$ below $d_{\mathrm{col}}$), the set of solutions starts to "shatter" into a multitude of well-separated clusters and every cluster only contains an exponentially small fraction of all solutions. The clustering transition is called *dynamic one-step replica symmetry breaking* in physics language. It is purely combinatorial, i.e. it marks no phase transition in the sense defined later in Section 2.5 because still $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\ln Z\right] \sim \lim_{n\to\infty} \frac{1}{n}\ln\mathbb{E}\left[Z\right]$ holds. This clustering phenomenon has been rigorously proven [ART06, ACO08] for some of the most important random CSPs. After the clustering threshold, in a *typical* cluster (i.e. the cluster of a solution picked uniformly at random) all solutions agree on most variables, which are then called *frozen variables*. As the constraint density increases, a further transition takes place, the *freezing transition*, rigorously established by Molloy [Mol12]. After this transition, in *almost every* cluster a constant fraction (converging to one as $k$ tends to infinity) of variables take on the same value.
As the constraint density evolves further, both the overall number of solutions and the sizes of the clusters decrease. But, according to the prediction, the number of all solutions drops at a faster rate, a phenomenon referred to as "entropy crisis", and thus we end up at a point, typically only a constant factor below the satisfiability transition, where the number of solutions in the largest cluster equals (up to sub-exponential terms) the number of all solutions: the condensation phase transition $d_{\mathrm{cond}}$. This marks a further change in the geometry of the solution space, a sub-exponential number of "large" clusters now contain a constant fraction of the entire set of solutions. As a consequence, while in the clustering phase typical solutions can be considered as being nearly independent, according to the prediction they have non-trivial correlations in the condensation phase and thus the combinatorial nature of a typical solution becomes significantly more complicated. The condensation transition is a thermodynamic phase transition that is called *static one-step replica symmetry breaking* in physics terms and in the condensation phase it should be true that $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\ln Z\right] < \lim_{n\to\infty} \frac{1}{n}\ln\mathbb{E}\left[Z\right]$. At least some parts of this picture have been established rigorously, especially the existence and location of a

[6] Here, $Z$ is the number of solutions, or, more general, the partition function (cf. Sections 2.2 and 2.3) and the expectation is taken over the choice of the random (hyper)graph.

condensation phenomenon (cf. e.g. [COP12, DSS16, DSS16+, BCO15+] as well as Section 4.1). Finally, as the average degree approaches the satisfiability threshold, the number of clusters drops down, until none survives.

In general, 1RSB [KMRTSZ07, ZK07, MRTS08] can be understood as RS at cluster level and suggests that there is no extra structure in clusters. There are other problems, like for instance the Sherrington-Kirkpatrick model or the problem of finding extremal cuts of sparse random graphs, that are predicted to have a full RSB structure [Par80, DMS16+], meaning that inside the clusters the solutions shatter again into smaller clusters, which shatter again and so on and so forth. This phenomenon is, however, very far from being verified rigorously.

Concerning the failure of algorithms, there seems to be a connection between clustering and the computational difficulty of finding a solution [ACO08, Mol12, Zde09]: Efficient algorithms provably find solutions up to (roughly) the density at which frozen clusters come into existence. On the basis of insights from the cavity method, in the past years physicists have developed new *message passing algorithms*, the most prominent examples being Belief Propagation Guided Decimation and Survey Propagation Guided Decimation [BMZ05, MZ02]. They were originally developed to deal with the clustered geometry of the solution space [BMPWZ03, MPZ02] and experimental evaluation suggests that for small values of $k$ these algorithms yield good results even in a clustered phase. However, while a satisfactory analysis remains elusive, meanwhile there is some (rigorous) evidence that the algorithms break down below the clustering barrier for large $k$ in the limit of large $n$ [RTS09, CO11, Het16+] (cf. Subsection 4.3.2 for a more in-depth discussion).

Beside the algorithmic question, based on the cavity method a Survey Propagation-inspired first and second moment method have been developed [MS08, CO13, COP16]. The essence of these methods is that instead of determining the moments of the number of solutions, the arguments are executed for the number of clusters. So-called covers are used, such that each cluster corresponds to a single cover and the internal entropy of the clusters can completely be ignored. This yields improvements over the "classical" application of the first and second moment methods (cf. Section 4.2), as close to the satisfiability threshold the cluster sizes are conjectured to vary significantly. This phenomenon has in part been established rigorously [COP16, DSS15].

Apart from models of inherent physical interest, the cavity method has been applied to a wide variety of problems in probabilistic combinatorics, computer science, coding theory and, more recently, compressed sensing [KMSSZ12a, KMSSZ12b]. It seems to be crucial to deepen our understanding of the behaviour of random CSPs. Several of its most important predictions have been confirmed rigorously through alternative approaches [MM09]. In effect, it has become an important research endeavour to provide a rigorous mathematical foundation for the cavity method. The results in this thesis contribute to this effort.

## 1.3. Summary of results

This PhD thesis deals with two different types of questions on random graph and random hypergraph structures. One part is about the proof of the existence and the determination of the location of the condensation phase transition. This transition will be investigated for large values of $k$ in the problem of $k$-colouring random graphs and in the problem of 2-colouring random $k$-uniform hypergraphs, where in the latter case we investigate a more general model with finite inverse temperature. The other part deals with establishing the limiting distribution of the number of solutions in these structures in density regimes below the condensation threshold.

The thesis comprises four main results from four papers of which two are already published and the other two are submitted. This section provides a very short summary of the results of these papers as well as an assessment of the contribution of this thesis' author. A more detailed description and discussion of the results can be found in Sections 4.1 and 4.2.

The first main result is from the paper *The condensation phase transition in random graph coloring* by Bapst, Coja-Oghlan, Hetterich, Raßmann and Vilenchik published in *Communications in Mathematical Physics 341 (2016)* and cited in this thesis as [BCOHRV16]. In this paper we establish the existence and determine the precise location of the condensation phase transition in random graph $k$-colouring for large $k$. The result is in terms of a distributional fixed point problem and rigorously verifies the prediction of the cavity method. The detailed proof can be found in Chapter 5 and Appendix A. The author of this thesis contributed primarily to the analysis of the branching process presented in Section 5.2 as well as to the determination of the cluster size using Warning Propagation and to establishing a connection between the random tree process and the graph with planted colouring presented in Section 5.3.

The second result is from the paper *A positive temperature phase transition in random hypergraph 2-coloring* by Bapst, Coja-Oghlan and Raßmann published in the *Annals of Applied Probability 26 (2016)* and cited here as [BCOR16]. The main result in this paper proves the existence and determines the location of the condensation phase transition in random $k$-uniform hypergraph 2-colouring with additional temperature parameter $\beta$ for large values of $k$. The proof can be found in all details in Chapter 6. The author of this thesis contributed primarily to the investigation of the first and second moment presented in Section 6.2, to the calculations in the planted model performed in Section 6.3 and to the proof of the existence of $\Phi_{d,k}(\beta)$ in Section 6.5. Furthermore she carried out revision work of all the proofs and statements presented in Chapter 6.

The third result is from the paper *On the number of solutions in random hypergraph 2-colouring* by Raßmann submitted to *The Electronic Journal of Combinatorics* and cited as [Ras16a+]. In this paper, the asymptotic distribution of the logarithm of the number of 2-colourings of random $k$-uniform hypergraphs is determined for all $k \geq 3$, concentration of this number is established and the random colouring model is shown to be contiguous to the planted model. All proofs can be found in Chapter 7. As this is a single-author paper, the question regarding the contribution of this thesis' author does not arise.

The last result is from the paper *On the number of solutions in random graph $k$-colouring* by Raßmann submitted to *Combinatorics, Probability and Computing* and cited as [Ras16b+]. We determine the asymptotic distribution of the number of $k$-colourings for random graphs in a low density regime for all $k \geq 3$, and in a density up to the condensation transition for all $k \geq k_0$ for some constant $k_0$. The proof will be presented in Chapter 8 and Appendix B. As this is a single-author paper, the question regarding the contribution of this thesis' author does not arise.

# 2 Definition of the problems

[7] In this thesis the focus is on two random constraint satisfaction problems, namely **random graph $k$-colouring** and **random $k$-uniform hypergraph 2-colouring** (for $k \geq 3$). These models are famous benchmark problems in the study of random CSPs and stand out from other standard examples for different reasons:
As mentioned previously, random graph $k$-colouring has a long history and is one of the most popular random CSPs. In particular, it is the most famous model having $k$ spins. Random hypergraph 2-colouring is also a common CSP and one of the most widely studied models with 2 spins. It can be seen as the prototype of a symmetric CSP, where the inverse of each solution is a solution itself, and is closely related to NAE-$k$-SAT (cf. Section 4.3). Studying it offers the advantage of not having to deal with technically too involved calculations (e.g. in regards to the second moment calculations), yet it shares interesting qualitative phenomena with other commonly studied problems. The model can consequently be used to develop and test proof techniques that might also be applicable to models exhibiting more complicated combinatorics.

## 2.1. Graph and hypergraph models

There is a variety of different models for generating graph and hypergraph structures randomly. In this thesis, the focus will be on Erdős-Rényi random graphs and hypergraphs. To be precise, we consider three slightly different, but essentially very similar models, such that with the right choice of parameters the results proven for one model can be transferred easily to the other models.

The random graph models used to state the results are the Erdős-Rényi random graphs $G(n, p)$, which was originally introduced by Gilbert [Gil59], and $G(n, m)$. Both graphs are defined on the vertex set $[n] = \{1, \ldots, n\}$. $G(n, p)$ is obtained by connecting any two vertices with probability $p \in [0, 1]$ independently, while $G(n, m)$ is a graph chosen uniformly at random from all graphs with exactly $n$ vertices and $m$ edges. By setting $p = m/\binom{n}{2}$, these two models are equivalent with respect to monotone properties [Jan95, AF99].
Furthermore, for the sake of simplicity, we choose to prove most of the statements in Chapter 8 using the auxiliary random graph model $\mathcal{G}(n, m)$. This is a random (multi-)graph on the vertex set $[n]$ obtained by choosing exactly $m$ hyperedges $e_1, \ldots, e_m$ of the complete graph on $n$ vertices uniformly and independently at random (i.e. with replacement). This model yields the advantage of having mutually

[7] At some points in this chapter the phrasing is a verbatim copy of text passages from the papers included in this thesis: [BCOHRV16, BCOR16, Ras16a+, Ras16b+].

independent edges, which simplifies calculations significantly. In this model we may choose the same edge more than once, however, for sparse random graphs the probability of this event is bounded away from 1:

**Fact 2.1.1.** *Assume that* $m = m(n)$ *is a sequence such that* $m = O(n)$ *and let* $\mathcal{A}_n$ *be the event that* $\mathcal{G}(n, m)$ *has no multiple edges. Then there is a constant* $c > 0$ *such that* $\lim_{n\to\infty} \mathbb{P}[\mathcal{A}_n] > c$. [8]

Regarding hypergraph models, we consider the $k$-uniform random hypergraph $H_k(n, p)$ on the vertex set $[n]$, in which each of the $\binom{n}{k}$ possible hyperedges, comprising of $k \geq 3$ distinct vertices, is present with probability $p \in [0, 1]$ independently. Additionally, we let $H_k(n, m)$ denote the random $k$-uniform hypergraph on the vertex set $[n]$ with exactly $m$ hyperedges consisting of $k$ distinct vertices and chosen uniformly at random without replacement from all possible subsets of $[n]$ of size $k$.
For the proofs in Chapters 6 and 7, we use the auxiliary random hypergraph model $\mathcal{H}(n, m)$, a random $k$-uniform (multi-)hypergraph (with $k \geq 3$) on the vertex set $[n]$, obtained by choosing exactly $m$ hyperedges $e_1, \ldots, e_m$ of the complete hypergraph on $n$ vertices uniformly and independently at random (i.e. with replacement). This model yields the advantage of having mutually independent edges, which simplifies calculations. Although in this model we may choose the same edge more than once, the following analogue to Fact 2.1.1 shows that in the case of sparse random hypergraphs this is unlikely.

**Fact 2.1.2.** *Assume that* $m = m(n)$ *is a sequence such that* $m = O(n)$ *and let* $\mathcal{A}_n$ *be the event that* $\mathcal{H}(n, m)$ *has no multiple hyperedges. Then* $\mathbb{P}[\neg\mathcal{A}_n] = O(n^{2-k})$.

Throughout the thesis we consider the case $m = O(n)$ as $n \to \infty$, resulting in so-called *sparse* random graphs and hypergraphs. For these densities the phenomena described in the previous section are conjectured to happen. More explicitly, in $G(n, p)$ we set $p = d/n$ for a real number $d > 0$ that we call the *edge density* or *average degree*. In $H_k(n, p)$ we set $p = d/\binom{n-1}{k-1}$, where $d > 0$ is again a fixed real number. We refer to $d$ (or sometimes to $d/k$) as the *hyperedge density*. Analogously, in $G(n, m)$ and $\mathcal{G}(n, m)$ we let $d = 2m/n$ and in $H_k(n, m)$ and $\mathcal{H}(n, m)$ we set $d = km/n$. As for some of our results we need very precise computations (especially in Chapters 7 and 8 and Appendix B), we additionally introduce the parameter $d'$, which is such that $m = \lceil d'n/2 \rceil$ in the random graph models and $m = \lceil d'n/k \rceil$ in the random hypergraph models. We distinguish this quantity from $d$, which arises naturally in the computations of the first and second moment. We note that $d' \sim d$, although $d = d(n)$ might vary with $n$, whereas $d'$ is assumed to be fixed as $n \to \infty$.

As in the following chapters some results and phenomena will be stated in relative generality, in these cases we will use the symbol $G$ under the tacit assumption that it refers to either a random graph or a random hypergraph (from one of the models introduced above). Sometimes the statements are even

[8] This is the best we can hope for: $\mathbb{P}[\mathcal{A}]$ does not converge to 0 as there exist multiple edges with constant probability.

valid for other random structures (such as CNF formulas). In that case, we will not always explicitly state this fact.

Note that there exist other random graph models [JLR00] like for instance random graphs with non-uniform degree distributions or random regular graphs, which are usually generated via the configuration model. These graph models partly exhibit properties similar to the Erdős-Rényi random graphs and some of the results may be comparable to ours.[9]

## 2.2. Colouring (hyper)graphs

Having introduced the random graph and hypergraph models, the two random CSPs of interest can be formalized as follows: In the graph $k$-colouring problem we are interested in the number $Z_k(G(n,m))$ or $Z_k(G(n,p))$ of *$k$-colourings*, also called *solutions*, of $G(n,m)$ or $G(n,p)$ respectively. A $k$-colouring is a valid colouring of the vertices, i.e. a map $\sigma : [n] \to [k]$, such that for two adjacent vertices $v, w \in [n]$ we always have $\sigma(v) \neq \sigma(w)$. Analogously, in the hypergraph 2-colouring problem, we consider the number $Z(H_k(n,m))$ or $Z(H_k(n,p))$ of *2-colourings* of $H_k(n,m)$ and $H_k(n,p)$ respectively, which are maps $\sigma : [n] \to \{\pm 1\}$ that generate no *monochromatic* hyperedges (i.e. hyperedges $e$ such that $|\sigma(e)| = 1$).
In the following, we adopt the notion of just writing $Z$ for the number of solutions if the problem in question is obvious from the context or if we aim at making generic statements that are valid for all considered problems.

Often, to simplify calculations, we just consider a special type of colourings, namely *balanced colourings*. For the random graph $k$-colouring problem, we call a map $\sigma : [n] \to [k]$ balanced if $||\sigma^{-1}(i)| - \frac{n}{k}| \leq \sqrt{n}$ for $i \in [k]$. Most $k$-colourings of the random graph $G$ have this property with probability tending to 1 as $n \to \infty$ [AF99, CO13].[10] For the random hypergraph 2-colouring problem, we call $\sigma : [n] \to \{\pm 1\}$ balanced if $||\sigma^{-1}(i)| - \frac{n}{2}| \leq \sqrt{n}$ for $i \in \{\pm 1\}$.

A graph or hypergraph colouring problem admitting at least one solution instantly exhibits an exponential number of solutions w.h.p.. One reason for this is that in the sparse regime the (hyper)graph possesses a linear number of isolated vertices w.h.p.[11]: The degrees of the vertices[12] are approximately Poisson distributed with parameter $d$. For $d$ as defined in Section 2.1, the probability for each of them to take the value 0 is constant and independent of $n$.

[9] A short overview (without a claim to completeness) of some results on regular random graphs is given in Section 4.3. For graphs with general degree distributions, we are not aware of results concerning the study of phase transitions.

[10] This has been proven to hold in density regimes up to the condensation transition. For larger densities it might be suspected to be true but has to our knowledge not been proven yet.

[11] Of course, this is not the only reason as otherwise we could greatly simplify the problem by deleting all isolated vertices.

[12] When we speak of the *degree* of a vertex $v \in [n]$ in a (hyper)graph, we refer to the number of all (hyper)edges of this (hyper)graph that contain $v$.

Therefore, the correct scaling of $Z$ to obtain a limit consists in taking the $n$-th root. As we are always interested in asymptotic statements and as most proof techniques inherently require large values of $n$ anyway, we define the following quantity which we call the *free entropy density*: [13]

$$\Phi_k(d) = \lim_{n\to\infty} \mathbb{E}\left[Z^{1/n}\right] \tag{2.2.1}$$

The expectation is over the choice of the random (hyper)graph. With the $n$-th root sitting inside the expectation, $\Phi_k(d)$ is difficult to calculate for general values of $d$. It is widely conjectured that in most interesting random CSPs the limit $\Phi_k(d)$ exists for all $d$ and $k$, but this has not been proven in general. In fact, the existence of the limit for all $d$ and $k$ would imply that the sequence $d_{\mathrm{sat}}(n)$ from Theorem 1.1.1 converges, which is an open problem in the theory of random graphs. However, Theorems 4.1.5 and 4.1.9 presented in Section 4.1 determine the typical value of $\ln Z$ and show that it converges in a broad density regime.

Influenced by predictions from statistical physics [MM09], it has turned out that properties of *typical* colourings have a considerable impact on combinatorial and algorithmic aspects of the random (hyper)graph colouring problem. To make this precise, when speaking of a typical 2-colouring ($k$-colouring), we mean a 2-colouring ($k$-colouring) of the random hypergraph $H$ (the random graph $G$) chosen uniformly at random from the set of all its 2-colourings ($k$-colourings), provided that this set is non-empty.

## 2.3. Finite inverse temperatures

Particularly in the context of applications in physics, it is sometimes necessary to generalize the above framework and the definition of $Z$. Rather than only working with the (hyper)edge density $d$ as parameter, we introduce a second parameter $\beta$. Following physics diction, we refer to $\beta$ as the *inverse temperature*.
Theorem 4.1.4 is a result in terms of both of these parameters. As we only consider finite inverse temperatures in the hypergraph 2-colouring problem, we introduce the following notation solely in this context. However, we like to emphasize that an analogue definition would be possible as well for random graph $k$-colouring (which is called $k$-spin Potts antiferromagnet in the physics literature) and various other random CSPs.

In the following, $H$ is a $k$-uniform hypergraph and for a map $\sigma : [n] \to \{\pm 1\}$ we let $E_H(\sigma)$ be the number of monochromatic hyperedges $e$ of $H$ (i.e. either all vertices of $e$ are set to $-1$ or to 1 under $\sigma$). The *Hamiltonian* $E_H$ gives rise to the so-called *Boltzmann distribution* or *Gibbs measure* $\pi_{H,\beta}$ on

[13] In the physics literature the free entropy density is usually defined as $\Phi_k(d) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\ln Z\right]$ (cf. [MM09]), i.e. instead of taking the $n$-th root, the logarithm of $Z$ is taken and the whole expression is normalized by $n$. Here, we choose to take the $n$-th root as in general the random variable $Z$ may be zero and this is exactly the quantity considered in Chapter 5.

the set of all maps $\sigma : [n] \to \{\pm 1\}$ in the following way: We let

$$\pi_{H,\beta}[\sigma] = \frac{\exp\left[-\beta E_H(\sigma)\right]}{Z_\beta(H)}, \quad \text{where } Z_\beta(H) = \sum_{\tau:[n]\to\{\pm 1\}} \exp\left[-\beta E_H(\tau)\right], \tag{2.3.1}$$

where we note that the distribution is randomly generated as the underlying hypergraph $H$ is random. This means that in this model we deal with two layers of randomness, as in a first step the randomness comes in through the choice of the hypergraph and in a second step a random colour assignment for the chosen hypergraph is selected. The Boltzmann distribution weights every colour assignment $\sigma$ according to the number of monochromatic edges it generates. For every "violated" edge, a "penalty" of $\exp[-\beta]$ has to be paid. The parameter $\beta$ plays an important role in this definition because it determines the influence of the penalty imposed by $E_H(\sigma)$. If $\beta = 0$, the penalty factor vanishes and $\pi_{H,\beta}$ is just the uniform distribution over all colour assignments, regardless of their number of monochromatic edges. Clearly, as $\beta \to \infty$ the Boltzmann distribution $\pi_{H,\beta}$ will place more and more weight on maps $\sigma$ with fewer and fewer monochromatic edges. For infinite $\beta$, we recover the setting from the previous section because in this case $Z_\beta(H)$ equals the number of solutions $Z(H)$ and thus $\pi_{H,\beta}$ is the uniform distribution over all solutions. We call the normalisation constant $Z_\beta$ in (2.3.1) the *partition function*. In statistical mechanics, one of the main objectives is to study $\pi_{H,\beta}$ as $n \to \infty$ and to try and understand the behaviour of $Z_\beta$ as it supplies detailed information on basic properties of the system [MM09]. In general, however, computing $Z_\beta$ is #P-hard [Pap94].

Similar to (2.2.1), we also define the free entropy density for the partition function $Z_\beta$:

$$\Phi_{d,k}(\beta) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\ln Z_\beta(H)\right]. \tag{2.3.2}$$

Obviously, the question arises whether the limit (2.3.2) exists for all $d, k$ and $\beta$. Indeed this is the case, as follows from an application of the combinatorial interpolation method from [BGT13]. Details will be provided in Section 6.5. Furthermore, a standard application of Azuma's inequality shows that for any $d, k, \beta$ and $H$ as defined in Section 2.1, the sequence $\{\frac{1}{n} \ln Z_\beta(H)\}_n$ converges to $\Phi_{d,k}(\beta)$ in probability.

Naturally, the physics picture of the evolution of the solution space as well as the prediction that the condensation phase transition results from an "entropy crisis", as described in Section 1.2, are also valid in this extended scenario. We present it again, albeit from a slightly different point of view, namely instead of varying $d$, we keep varying $\beta$. From a "classical" statistical physics point of view, it seems less natural to vary the parameter $d$, which governs the geometry of the system, and fix $\beta$ than to fix $d$ and vary $\beta$. Thus, Theorem 4.1.4 encompasses the latter case. Our explanations concerning the evolution of the geometry will be a little more formal than in Section 1.2 because we build upon this intuition later in the proofs presented in Chapter 6.

Based on the cavity method, it is predicted that already for densities $d/k$ beyond about $2^{k-1} \ln k/k$

and for large enough $\beta$, the Boltzmann distribution can w.h.p. be approximated by a convex combination of probability measures corresponding to "clusters" of 2-colourings. That is, there exist sets $\mathcal{C}_{\beta,1},\ldots,\mathcal{C}_{\beta,N} \subset \{\pm 1\}^n$ and small numbers $0<\varepsilon<\delta$ such that

- if $\sigma,\tau \in \mathcal{C}_{\beta,i}$ for some $i$, then $\langle \sigma,\tau\rangle > (1-\varepsilon)n$,
- if $\sigma \in \mathcal{C}_{\beta,i}, \tau \in \mathcal{C}_{\beta,j}$ with $i \neq j$, then $|\langle \sigma,\tau\rangle| < \delta n$,

and if we denote by $Z_{\beta,i} = \sum_{\tau\in\mathcal{C}_{\beta,i}} \exp\left[-\beta E_H(\tau)\right]$ the volume of $\mathcal{C}_{\beta,i}$, we have

$$\left\| \pi_{H,\beta}[\,\cdot\,] - \sum_{i=1}^{N} \frac{Z_{\beta,i}}{Z_\beta(H)} \cdot \pi_{H,\beta}[\,\cdot\,|\mathcal{C}_{\beta,i}] \right\|_{\mathrm{TV}} < \exp\left[-\Omega(n)\right],$$

where $\|.\|_{\mathrm{TV}}$ is the total variation distance. Given a hypergraph, the construction of the "clusters" $\mathcal{C}_{\beta,i}$ will be formalised in Section 2.4.

With the cluster decomposition in place, the physics story of how the condensation phase transition comes about goes as follows. If $\beta$ is sufficiently small, we have $\max_{i\le N} \ln Z_{\beta,i} \le \ln Z_\beta(H) - \Omega(n)$ w.h.p.. That is, even the largest cluster only captures an exponentially small fraction of the overall mass $Z_\beta(H)$. Now, as we increase $\beta$ (while $d/k$ remains fixed), both $Z_\beta(H)$ and $\max_{i\le N} Z_{\beta,i}$ decrease. But in compliance with the the concept of the "entropy crisis", $Z_\beta(H)$ drops at a faster rate. In fact, for large enough densities $d/k$ there might be a critical value $\beta_{\mathrm{cond}}$ where the gap between $\max_{i\le N}\ln Z_{\beta,i}$ and $\ln Z_\beta(H)$ vanishes. This $\beta_{\mathrm{cond}}$ should mark a phase transition. This is because $\max_{i\le N}\ln Z_{\beta,i}$ and $\ln Z_\beta(H)$ cannot both extend analytically to $\beta > \beta_{\mathrm{cond}}$, as otherwise we would arrive at the absurd conclusion that $\max_{i\le N} Z_{\beta,i} > Z_\beta$.

To distinguish the refined version of the colouring problems from the simpler case where only solutions are considered, we will speak of *proper* graph colouring in case "$\beta=\infty$".

## 2.4. Clusters and cluster size

In this section we formally introduce the notion of *clusters*, which we already touched upon in Sections 1.2 and 2.3. With respect to random graph $k$-colouring, we again let $G$ be a graph on $n$ vertices. If $\sigma,\tau$ are $k$-colourings of $G$, we define their *overlap* as the $k\times k$-matrix $\rho(\sigma,\tau) = (\rho_{ij}(\sigma,\tau))_{i,j\in[k]}$ with entries

$$\rho_{ij}(\sigma,\tau) = \frac{|\sigma^{-1}(i)\cap\tau^{-1}(j)|}{n},$$

i.e. $\rho_{ij}(\sigma,\tau)$ is the fraction of vertices coloured $i$ under $\sigma$ and $j$ under $\tau$. Now, define the *cluster* of $\sigma$ in $G$ as

$$\mathcal{C}(G,\sigma) = \{\tau : \tau \text{ is a } k\text{-colouring of } G \text{ and } \rho_{ii}(\sigma,\tau) \ge 0.51/k \text{ for all } i\in[k]\}.$$

Suppose that $\sigma, \tau$ are balanced colourings. Then $\tau \in \mathcal{C}(G, \sigma)$ means that a little over $50\%$ of the vertices with colour $i$ under $\sigma$ also have colour $i$ under $\tau$. To this extent, $\mathcal{C}(G, \sigma)$ comprises of colourings "similar" to $\sigma$. In fact, for large $k$ and densities close to the condensation phase transition (formally introduced in Section 2.5), this definition exhibits w.h.p. the same asymptotics as other, more combinatorial concepts (e.g. colourings that can be reached from $\sigma$ by iteratively altering the colours of $o(n)$ vertices at a time) [Mol12].

That the clusters defined in this way are indeed well-separated in the interesting density regimes can be formalised by the notion of *separability*. Roughly speaking, separable colourings are defined by the property that two colour classes overlapping by little more than $50\%$ of their variables are nearly identical. This implies that the clusters of two separable colourings are either disjoint or identical. The notion has been used e.g. in [COV13], where it is essentially shown that balanced colourings are also separable.

With respect to random hypergraph 2-colouring a completely analogue definition is possible. However, as we are going to work with the finite temperature case of the problem and thus do not only consider solutions, but have to take into account all possible colour assignments, just *counting* the number of assignments "near" some specific colouring $\sigma$ does not make sense. Instead, for a hypergraph $H$ on $n$ vertices and a map $\sigma : [n] \to \{\pm 1\}$ we define the *cluster size* of $\sigma$ in $H$ as

$$\mathcal{C}_\beta(H, \sigma) = \sum_{\tau \in \{\pm 1\}^n : \langle \sigma, \tau \rangle \geq 2n/3} \exp\left[-\beta E_H(\tau)\right], \tag{2.4.1}$$

where $E_H(\tau)$ denotes the number of monochromatic hyperedges in $H$ under the colour assignment $\tau$. Thus, we sum up the contribution to the partition function of all $\tau$ whose "overlap" $\langle \sigma, \tau \rangle = \sum_{v \in [n]} \sigma(v)\tau(v)$ with the given $\sigma$ is big. Indeed, we will show in Chapter 6 that w.h.p. for typical $\sigma$ almost all the contribution comes from colourings with overlap $\langle \sigma, \tau \rangle \geq (1 - k^{-5})n$.

## 2.5. Phase transitions

In mathematical physics, a *phase transition* usually describes a point where the functions $\Phi_k(d)$ from (2.2.1) or $\Phi_{d,k}(\beta)$ from (2.3.2) are non-analytic. As already explained in detail in Section 1.2, the points where phase transitions occur play a very important role in understanding the evolution of the geometry of the set of solutions or, more generally, the set of weighted colour assignments.

As elaborated on in Section 2.2, the limit $\Phi_k(d)$ is currently not known to exist for all $d$ and $k$. In order to circumvent this problem, for a fixed $k \geq 3$ we call $d_0 \in (0, \infty)$ *smooth* if there exists $\varepsilon > 0$ such that

- for any $d \in (d_0 - \varepsilon, d_0 + \varepsilon)$ the limit $\Phi_k(d)$ exists, and

- the map $d \in (d_0 - \varepsilon, d_0 + \varepsilon) \mapsto \Phi_k(d)$ has an expansion as an absolutely convergent power series around $d_0$.

If $d_0$ fails to be smooth, we say that a *phase transition* occurs at $d_0$. Using a concentration result from [ACO08], it follows that for smooth $d_0$ the sequence of random variables $\{Z_k(G(n, d_0/n))^{1/n}\}_n$ converges to $\Phi_k(d_0)$ in probability. Thus, up to a sub-exponential factor, $\Phi_k(d)$ captures the “typical” value of the number $Z_k(G(n, d/n))$. A similar statement also holds for the number of 2-colourings of random hypergraphs.

The above definition of phase transitions is in compliance with its common use in combinatorics. For instance, the classical result of Erdős and Rényi [ER60] implies that the function that maps $d$ to the expected fraction of vertices belonging to the largest component of $G(n, d/n)$ (in the limit as $n \to \infty$) is non-analytic at $d = 1$. Similarly, if there actually is a sharp threshold $d_{\text{col}}$ for (hyper)graph colouring, then $d_{\text{col}}$ is a phase transition in the above sense. This can easily be understood: By definition, for $d < d_{\text{col}}$, the random (hyper)graph $G$ has a colouring w.h.p. and thus the number of colourings is, in fact, exponentially large in $n$ (as explained in Section 2.2). Hence, if $\Phi_k(d)$ exists for $d < d_{\text{col}}$, then $\Phi_k(d) > 0$. By contrast, for $d > d_{\text{col}}$ the random (hyper)graph $G$ fails to be colourable w.h.p. and therefore $\Phi_k(d) = 0$. Thus, $\Phi_k(d)$ cannot be analytic at $d_{\text{col}}$.

In the case of finite $\beta$, we choose an analogue definition: We call $\beta_0 > 0$ *smooth* if there exists $\varepsilon > 0$ such that the function $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \mapsto \Phi_{d,k}(\beta)$ admits an expansion as an absolutely convergent power series around $\beta_0$. Otherwise, we say that a *phase transition* occurs at $\beta_0$.

**The condensation phase transition**

The phase transition we will be mostly concerned with in this thesis is the condensation phase transition. As we noted in Section 2.2, $\Phi_k(d)$ is not known to exist for general values of $d$. However, for $d \in [0, 1)$ this quantity is easily understood.
With respect to random graphs, it is known that $G(n, d/n)$ decomposes for $d \in [0, 1)$ into tree components and a bounded number of connected components with precisely one cycle w.h.p. [ER60]. Moreover, the number of $k$-colourings of a tree with $\nu$ vertices and $\nu - 1$ edges is well-known to be $k^{\nu}(1 - 1/k)^{\nu-1}$ and thus w.h.p. we obtain

$$Z_k(G(n, d/n))^{1/n} \sim k(1 - 1/k)^{d/2} \qquad \text{for } d < 1. \tag{2.5.1}$$

As $Z_k(G)^{1/n} \leq k$ for any graph $G$ on $n$ vertices, (2.5.1) implies that

$$\Phi_k(d) = \lim_{n \to \infty} \mathbb{E}[Z_k(G(n, d/n))^{1/n}] = k(1 - 1/k)^{d/2} \qquad \text{for } d < 1. \tag{2.5.2}$$

Since $d \mapsto k(1 - 1/k)^{d/2}$ is analytic, the least $d > 0$ for which the limit $\Phi_k(d)$ either fails to exist or strays away from $k(1-1/k)^{d/2}$ is going to be a phase transition. Hence, for $k \geq 3$ we let

$$d_{\text{crit}} = \inf\left\{d \geq 0 : \text{the limit } \Phi_k(d) \text{ does not exist or } \Phi_k(d) < k(1-1/k)^{d/2}\right\}. \tag{2.5.3}$$

It will become evident in Chapter 5 that this is exactly the right definition for the condensation transition $d_{\text{cond}}$ non-formally introduced in Section 1.2. Furthermore, we show that $d_{\text{crit}}$ can also be expressed as $\sup\left\{d \geq 0 : \text{the limit } \Phi_k(d) \text{ exists and } \Phi_k(d) = k(1-1/k)^{d/2}\right\}$.

With respect to random hypergraphs, there is an analogue definition of the condensation transition $d_{\text{cond}}$ and it was shown in [COZ12] that indeed $\Phi_k(d)$ is non-analytic around $d_{\text{cond}}$ if the limit exists because $\Phi_k(d)$ coincides with the linear function $\lim_{n\to\infty} \mathbb{E}\left[Z\right]^{1/n}$ for $d < d_{\text{cond}}$.

For the case of finite $\beta$ in the random hypergraph 2-colouring problem, we show in Section 6.2 that for any $\beta$ we have

$$\Phi_{d,k}(\beta) \leq \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right) \tag{2.5.4}$$

and that there is a regime where equality holds in this equation. Since the function $\beta \in [0,\infty) \mapsto \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)$ is analytic, it follows that the least $\beta > 0$ for which the inequality in (2.5.4) is strict, marks a phase transition. Hence, we define

$$\beta_{\text{crit}}(d,k) = \inf\left\{\beta \geq 0 : \Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)\right\}. \tag{2.5.5}$$

In Chapter 6 we will show that indeed $\beta_{\text{crit}}(d,k)$ coincides with the condensation phase transition $\beta_{\text{cond}}$ that we non-formally introduced in Section 2.3.

## 2.6. Notation and further remarks

Throughout the thesis, we are concerned with asymptotic statements in the number $n$ of vertices. Therefore, we always tacitly assume that $n \geq n_0$ is sufficiently large for the various statements to hold. Moreover, to avoid floor and ceiling signs, we assume that $n$ is either even or divisible by $k$, depending on the situation. As mentioned above, we denote by $[n]$ the set $\{1,...,n\}$.

For $k$, the uniformity parameter or the number of colours respectively, it is sometimes necessary to have a lower bound to carry out sufficiently accurate analyses, especially in the proofs presented in Chapters 5 and 6. Hence, we often assume that $k \geq k_0$ for some large enough constant $k_0$. Thus, $k$ may be arbitrarily large but fixed while $n \to \infty$. In many cases it may not be impossible to optimize or at least calculate $k_0$, but so far no attempt has been made.
Furthermore, it might be interesting to note that for small values of $k$ various properties of random CSPs that are proven for big $k$, are not even conjectured to hold. In particular, the solution space is

expected to have a completely different structure, which may also be a reason why certain algorithms work well for small $k$ but can be proven to fail for larger $k$ (cf. Subsection 4.3.2). For example, as $k$ increases, typical satisfying assignments get closer and closer to being balanced as the number of occurences of the variables in the constraints approach their expectation.
In Chapter 7 and in most parts of Chapter 8, however, the statements and proofs do not require large values of $k$ and we assume that $k \geq 3$.

We use the standard $O$-notation when referring to the limit $n \to \infty$. Thus, $f(n) = O(g(n))$ means that there exist $C > 0$, $n_0 > 0$ such that for all $n > n_0$ we have $|f(n)| \leq C \cdot |g(n)|$. In addition, we use the standard symbols $o(\cdot), \Omega(\cdot), \Theta(\cdot)$. In particular, $o(1)$ stands for a term that tends to 0 as $n \to \infty$. We adopt the common notation that for the symbol $\Omega(\cdot)$ the sign matters, i.e. $f(n) = \Omega(g(n))$ means that there exist $C > 0$, $n_0 > 0$ such that for all $n > n_0$ we have $f(n) \geq C \cdot g(n)$, whereas $f(n) = -\Omega(g(n))$ implies $-f(n) \geq C \cdot g(n)$ for all $n > n_0$.

Additionally, we use asymptotic notation with respect to $k$. To make this explicit, we insert $k$ as an index. Thus, $f(k) = O_k(g(k))$ means that there exist $C > 0$ and $k_0 > 0$ such that for all $k > k_0$ we have $|f(k)| \leq C \cdot |g(k)|$. Further, we write $f(k) = \tilde{O}_k(g(k))$ to indicate that there exist $C > 0$ and $k_0 > 0$ such that for all $k > k_0$ we have $|f(k)| \leq k^C \cdot |g(k)|$. An analogous convention applies to $o_k(\cdot), \Omega_k(\cdot)$ and $\Theta_k(\cdot)$. Notice that here as well we have $\Omega_k(\cdot) \neq -\Omega_k(\cdot)$.

Furthermore, the notation $f(n) \sim g(n)$ stands for $\lim_{n\to\infty} f(n)/g(n) = 1$ or equivalently $f(n) = g(n)(1 + o(1))$. Besides taking the limit $n \to \infty$, at some point we need to consider the limit $\nu \to \infty$ for some number $\nu \in \mathbb{N}$. Thus, we additionally introduce $f(n, \nu) \sim_\nu g(n, \nu)$ meaning that $\lim_{\nu\to\infty} \lim_{n\to\infty} f(n, \nu)/g(n, \nu) = 1$.

Moreover, if $p = (p_1, \ldots, p_l)$ is a vector with entries $p_i \geq 0$, then we let

$$\mathcal{H}(p) = -\sum_{i=1}^{l} p_i \ln p_i.$$

Here and throughout, we use the conventions that $0 \ln 0 = 0$ and consistently $0 \ln \frac{0}{0} = 0$. Hence, if $\sum_{i=1}^{l} p_i = 1$, then $\mathcal{H}(p)$ is the entropy of the probability distribution $p$. As a special case, if $z \in [0, 1]$ is just a number, then the entropy function $\mathcal{H}(z)$ is defined as $\mathcal{H}(z) = -z \ln z - (1 - z) \ln(1 - z)$. Further, for a number $x$ and an integer $h > 0$, we let $(x)_h = x(x - 1) \cdots (x - h + 1)$ denote the $h$th falling factorial of $x$.

Concerning the distribution of random variables, if $X$ follows the Poisson distribution with parameter $\lambda$, we write $X \sim \mathrm{Po}(\lambda)$. If $X$ is Bernoulli-$p$-distributed, we denote this by $X \sim \mathrm{Be}(p)$ and if it is binomially distributed with parameters $n$ and $p$, we write $X \sim \mathrm{Bin}(n, p)$.

# 3 Techniques

[14] There is a variety of techniques that have become standard tools in the rigorous study of random CSPs over the years. Some of them will be introduced in detail in this chapter, namely the *planted model* and the *moment methods*. Furthermore, a short summary of *small subgraph conditioning* will be given. Others will be brought in “on the fly” in the following chapters when needed. Among them are the *core*, the *backbone* and the notion of *free vertices*. The core [PSW96] will be introduced in Chapter 6. It is, roughly speaking, a set of vertices such that every vertex has many neighbours[15], which belong to the core themselves. Vertices in the core do not contribute to the cluster size because they can only take a specific colour (cf. e.g. [COV13]). Otherwise, they would initiate an avalanche of colour changes ending up at a colouring outside the initial cluster.

## 3.1. Planted model

In many random CSPs and for a wide range of constraint densities (namely those where the number of solutions is sufficiently concentrated), it has turned out that typical properties of random solutions as well as the geometry of the solution space can be studied by way of the so-called *planted model*. This is an easily accessible distribution, often very convenient to work with. It can be used to study rigorously the various phase transitions in random CSPs, in particular it enables us to get a handle on the size of the cluster introduced in Section 2.4.

The idea of “planting” a property inside a random structure is very old and has for example been used to investigate the performance of algorithms [DF89, AK97]. Juels and Peinado [JP00] were, to our knowledge, the first to investigate the relationship between the “planted model” and the “random colouring model” for the clique problem in dense random graphs.

In this chapter, we present the planted model only in its common setup for the case of proper graph colouring (meaning that we do not have an additional parameter $\beta$). For finite inverse temperatures in random hypergraph 2-colouring, the planted model is refined in Chapter 6. Analogously, it can also be used for the study of other random CSPs, e.g. random $k$-SAT.

As already mentioned in Section 2.2, when investigating the properties of random CSPs, it is often

[14] At some points in this chapter the phrasing is a verbatim copy of text passages from the papers included in this thesis: [BCOHRV16, BCOR16, Ras16a+, Ras16b+].

[15] The set of neighbours of a vertex $v$ consists of all vertices which are connected to $v$ via an edge.

essential to have the notion of *typical* colourings at hand. To be precise, for a random (hyper)graph $G = G(n, m)$ let $\Lambda_{k,n,m}$ be the set of all pairs $(G, \sigma)$ with $\sigma$ being a colouring of $G$. Let $N = \binom{n}{2}$ for random graph $k$-colouring and $N = \binom{n}{k}$ for random $k$-uniform hypergraph 2-colouring. Now we define a probability distribution $\pi^{\mathrm{rc}}_{k,n,m}[G, \sigma]$ on $\Lambda_{k,n,m}$ by letting

$$\pi^{\mathrm{rc}}_{k,n,m}[G, \sigma] = \left[ Z(G) \binom{N}{m} \mathbb{P}\left[G \text{ is colourable}\right] \right]^{-1}.$$

We call this distribution the *random colouring model* or *Gibbs distribution*. It can also be described as the distribution produced by the following experiment.

**RC1** Generate a random (hyper)graph $G = G(n, m)$ given that $Z(G) > 0$.
**RC2** Choose a colouring $\sigma$ of $G$ uniformly at random. The result of the experiment is $(G, \sigma)$.

For densities below the colouring threshold, this experiment is key to studying the combinatorial nature of the (hyper)graph colouring problem as it corresponds to randomly picking solutions of random (hyper)graphs. However, up to now, there is no known method to implement this experiment efficiently for a wide range of (hyper)edge densities. In fact, the first step **RC1** is easy to process because we are only interested in values of $d$ where $G$ is colourable w.h.p. and consequently the conditioning on $Z(G) > 0$ does not cause problems because the probability $\mathbb{P}\left[G \text{ is colourable}\right]$ is close to 1. In fact, what turns the direct study of the distribution $\pi^{\mathrm{rc}}_{k,n,m}$ into a challenge is step **RC2** because in the interesting density regimes we cannot even find one colouring algorithmically, let alone sample one uniformly: The currently best-performing algorithms for sampling a colouring of $G$ are known to succeed up to a density about a factor of 2 below the colouring threshold for random graph $k$-colouring [AM97, GM75, KS98] and about a factor of $k$ below the threshold for random $k$-uniform hypergraph 2-colouring [AKKT02].

To circumvent these difficulties, we consider an alternative probability distribution on $\Lambda_{k,n,m}$ called the *planted model*, which is much easier to approach. To describe this experiment, for a colour assignment $\sigma$ let $\mathcal{F}(\sigma)$ be the number of (hyper)edges of the complete (hyper)graph that are monochromatic under $\sigma$.[16] Then the *planted distribution* is induced by the following experiment:

**PL1** Choose a colour assignment $\boldsymbol{\sigma}$ uniformly at random provided that $\mathcal{F}(\boldsymbol{\sigma}) \leq N - m$.
**PL2** Generate a (hyper)graph $\boldsymbol{G}$ on $[n]$ consisting of $m$ (hyper)edges that are bichromatic under $\boldsymbol{\sigma}$ uniformly at random. The result of the experiment is $(\boldsymbol{G}, \boldsymbol{\sigma})$.

[16] To be precise, for graph $k$-colouring we have $\sigma : [n] \to [k]$ and $\mathcal{F}(\sigma) = \sum_{i=1}^{k} \binom{|\sigma^{-1}(i)|}{2}$ and for $k$-uniform hypergraph 2-colouring we have $\sigma : [n] \to \{\pm 1\}$ and $\mathcal{F}(\sigma) = \binom{|\sigma^{-1}(-1)|}{k} + \binom{|\sigma^{-1}(1)|}{k}$.

Thus, the probability that the planted model assigns to a pair $(G,\sigma)$ is

$$\pi^{\mathrm{pl}}_{k,n,m}[G,\sigma] \sim \left[2^n \binom{N}{m} \mathbb{P}\left[\sigma \text{ is a colouring of } G\right]\right]^{-1}.$$

We observe that step **PL1** is easy to handle, as the conditioning on $\mathcal{F}(\boldsymbol{\sigma}) \leq N-m$ does not cause any difficulties. Additionally, in contrast to the "difficult" step **RC2**, step **PL2** is much easier to implement.

### 3.1.1. Quiet planting

Of course, the two probability distributions $\pi^{\mathrm{rc}}_{k,n,m}$ and $\pi^{\mathrm{pl}}_{k,n,m}$ differ. Under $\pi^{\mathrm{rc}}_{k,n,m}$, the (hyper)graph is chosen uniformly at random, whereas under $\pi^{\mathrm{pl}}_{k,n,m}$ its probability depends on its number of solutions in such a way that (hyper)graphs exhibiting many colourings are "favoured" by the planted model (or, put differently, in the planted model there exist more solutions because there is a solution - and its whole cluster - built into the problem).

However, the two models are related if $m = m(n)$ is such that w.h.p.

$$\ln Z(G) = \ln \mathbb{E}[Z(G)] + o(n). \tag{3.1.1}$$

For the problems of $k$-colouring random graphs and 2-colouring random hypergraphs, Coja-Oghlan and Achlioptas showed in [ACO08] that the following is true if (3.1.1) is satisfied:

If $(\mathcal{E}_n)$ is a sequence of events $\mathcal{E}_n \subset \Lambda_{k,n,m}$ such that $\pi^{\mathrm{pl}}_{k,n,m}[\mathcal{E}_n] \leq \exp[-\Omega(n)]$, then $\pi^{\mathrm{rc}}_{k,n,m}[\mathcal{E}_n] = o(1)$. (3.1.2)

The statement (3.1.2) was baptised *quiet planting* by Krzakala and Zdeborová [KZ09] and has ever since been used to study the behaviour of the set of colourings and its geometrical structure in various random constraint satisfaction problems [ACO08, BCOHRV16, Mol12, MR13, MRT11]. Although work has been greatly simplified by (3.1.2), yet a significant complication in its use is caused by the fact that $\mathcal{E}_n$ not only has to be unlikely but is required to be *exponentially* unlikely in the planted model. This has caused substantial difficulties in several applications (e.g. [BCOR16, BCOHRV16, Mol12]).

### 3.1.2. Contiguity and silent planting

In [ACO08] it has been proven that for random graph $k$-colouring and random hypergraph 2-colouring, in a certain density regime (well below the condensation transition) the number of solutions is concentrated around its expectation in the sense that for all $\varepsilon > 0$ w.h.p.

$$\frac{1}{n}|\ln Z - \ln \mathbb{E}\left[Z\right]| \leq \varepsilon.$$

This leaves open the possibility that $\ln Z$ has fluctuations of order e.g. $\sqrt{n}$, which appears plausible because the core fluctuates on this scale and it seems reasonable to expect that its behaviour influences the number of solutions. Rather surprisingly, Bapst, Coja-Oghlan and Efthymiou proved in [BCOE14+] that for the problem of $k$-colouring random graphs indeed $\ln Z$ fluctuates by less than $\omega(n)$ for *any* $\omega(n) \to \infty$, which is equivalent to saying that for all $\varepsilon > 0$ w.h.p.

$$\frac{1}{\omega} | \ln Z - \ln \mathbb{E}[Z] | \leq \varepsilon. \tag{3.1.3}$$

The key tool in [BCOE14+] is the method of small subgraph conditioning (cf. Section 3.3). The proof works because the fluctuations in the number of solutions are due to the fluctuations in the number of short cycles in the factor graph[17] and because this is the only important structure contributing to the fluctuations.

Our result Corollary 4.1.7 establishes this behaviour for random hypergraph 2-colouring. To obtain this result, it is an essential necessary condition that the number of colourings of an arbitrary tree does only depend on its number of vertices, as in sparse random (hyper)graphs most components either are trees or contain short cycles [ER60]. Thus, we have to make sure that the tree components do not contribute to the variance of the number of solutions. Indeed, in the random $k$-colouring problem, for every tree with $n$ nodes (and consequently $m = n - 1$ edges), the number of $k$-colourings of this tree is deterministic and given by $k^n(1 - 1/k)^m = k(k-1)^m$. For hypergraph 2-colouring, every $k$-uniform hypergraph being a tree with $m$ edges has exactly $k + (m-1)(k-1)$ vertices and its number of 2-colourings is, independently of the tree structure, given by $\left(2^k - 2\right)\left(2^{k-1} - 1\right)^m$.

A consequence of (3.1.3) concerns the following notion of *contiguity*. Suppose that $\boldsymbol{\mu} = (\mu_n)_{n \geq 1}$ and $\boldsymbol{\nu} = (\nu_n)_{n \geq 1}$ are two sequences of probability measures such that $\mu_n, \nu_n$ are defined on the same probability space $\Omega_n$ for every $n$. Then $(\mu_n)_{n \geq 1}$ is *contiguous* with respect to $(\nu_n)_{n \geq 1}$, in symbols $\boldsymbol{\mu} \triangleleft \boldsymbol{\nu}$, if for any sequence $(\mathcal{E}_n)_{n \geq 1}$ of events such that $\lim_{n \to \infty} \nu_n(\mathcal{E}_n) = 0$, we have $\lim_{n \to \infty} \mu_n(\mathcal{E}_n) = 0$.

Our result Corollary 4.1.8 establishes that the random colouring model is contiguous with respect to the planted model, a fact that we refer to as *silent planting*. Thus, instead of an exponentially small probability in (3.1.2) we only need a probability decaying to zero arbitrarily slowly in the planted model to obtain a probability decaying to zero in the random colouring model.

[17] The factor graph is an auxiliary graph, representing the original (hyper)graph in a slightly different way by directly internalising the constraints: It is a bipartite graph with variable nodes corresponding to the vertices in the original (hyper)graph and factor nodes corresponding to the (hyper)edges. In the factor graph a variable node is connected to a factor node if in the original (hyper)graph the variable was contained in the corresponding edge.

## 3.2. Moment methods

For several years, in many random constraint satisfaction problems the best bounds on the threshold for the existence of solutions derived from the first and second moment method (cf. Sections 4.2 and 4.3). These methods are non-constructive, meaning that they do not yield concrete solutions for the respective problem, but rather are probabilistic methods to prove the (non-)existence of a solution. In most cases, applied to random CSPs these simple techniques do not yield matching upper and lower bounds on the satisfiability threshold. However, one often obtains at least its exponential order and in many analyses the results from these methods form the basis for advanced calculations.

### 3.2.1. First moment method

The first moment method is a very simple technique to obtain an upper bound on the satisfiability threshold $d_{\mathrm{col}}$ by showing that above a certain density the first moment (which is nothing but the expectation) of the number of solutions tends to zero. If $G$ is a random (hyper)graph on $n$ vertices to be coloured and $Z$ its number of colourings, then Markov's inequality yields

$$\mathbb{P}\left[G \text{ is colourable}\right] = \mathbb{P}\left[Z \geq 1\right] \leq \mathbb{E}\left[Z\right]$$

and consequently if $\mathbb{E}\left[Z\right] = o(1)$ for some density $d$, then $d \geq d_{\mathrm{col}}$ for large enough $n$. For many random CSPs, it is easy to compute a critical density $d_{\mathrm{first}}$, such that $\mathbb{E}\left[Z\right] = o(1)$ for $d > d_{\mathrm{first}}$ while $\mathbb{E}\left[Z\right] = \exp\left[\Omega(n)\right]$ for $d < d_{\mathrm{first}}$.[18]

### 3.2.2. Second moment method

Unfortunately, having $\mathbb{E}\left[Z\right] = \exp\left[\Omega(n)\right]$ for $d < d_{\mathrm{first}}$ does not mean that in this density regime the random (hyper)graph admits a colouring w.h.p.. It could simply be the case that the first moment is pushed up by a small number of (hyper)graphs with excessively many solutions. To eliminate this possibility, a lower bound on the threshold $d_{\mathrm{col}}$ can be derived via the second moment method. The use of this method in the context of random CSPs was pioneered by Achlioptas and Moore [AM06] and Frieze and Wormald [FW05]. Based on the Paley-Zygmund inequality

$$\mathbb{P}\left[G \text{ is colourable}\right] = \mathbb{P}\left[Z \geq 1\right] \geq \frac{\mathbb{E}\left[Z\right]^2}{\mathbb{E}\left[Z^2\right]},$$

[18] As stated in Section 2.2, in sparse random graphs there exists a linear number of isolated vertices w.h.p. and thus, if the problem exhibits at least one solution, it immediately exhibits an exponentially (in $n$) large number of solutions.

where $G$ is again a random (hyper)graph on $n$ vertices, we conclude that if

$$\mathbb{E}\left[Z^2\right] \leq C \cdot \mathbb{E}\left[Z\right]^2 \tag{3.2.1}$$

for some density $d$ and some constant $C = C(k,d) > 0$, then $\mathbb{P}\left[G \text{ is colourable}\right] \geq 1/C$. Thus, the second moment lower bound $d_{\text{sec}}$ is a (hyper)edge density such that (3.2.1) holds for $d < d_{\text{sec}}$ but is violated for $d > d_{\text{sec}}$. Here, it is important that $C$ does not depend on $n$ because this ensures that the probability that $G$ is colourable is bounded away from 0 as $n$ tends to infinity. There exist different possibilities for pushing the probability from $1/C$ up to $1 - o(1)$ (to obtain $d_{\text{col}} > d$). The method of choice depends on the specific problem. For $k$-SAT and for models on Erdős-Rényi graphs, like e.g. the presented problems graph $k$-colouring and hypergraph 2-colouring, Friedgut's sharp threshold result Theorem 1.1.1 can be applied [Fri99, AF99, Fri05]. In other cases, the small subgraph conditioning technique (cf. Section 3.3) by Robinson and Wormald [RW94] can be used [KPGW10, COEH16, Wor99]. This includes scenarios like for example regular graph problems, where Friedgut's result does not hold, or other random CSPs, for which Friedgut's result has not been proven yet. In this case, it might seem more suitable to apply small subgraph conditioning because the required machinery is not as huge and a more precise bound on the required density can be obtained.

Applying the second moment method to the number $Z$ of solutions of the (hyper)graph problem is sometimes referred to as the "vanilla" application of the second moment method. In practice, often a random variable only counting the number of solutions with (nearly) balanced colour classes is used [AN05, AM06] with the goal of reducing the variance relative to the expectation. For asymmetric problems, this has been done by using a random variable that weights assignments cleverly [AP04] or only counts colourings whose complement is also satisfying [AM06]. In fact, we can use every random variable $\bar{Z}$ such that $\bar{Z}(G) > 0$ implies that $G$ is colourable.

Nevertheless, in all these cases (except for some very easy problems touched upon in Section 4.3), when comparing the best first moment upper bound $d_{\text{first}}$ and the best second moment lower bound $d_{\text{sec}}$, it turns out that they differ in the limit of large $k$ by at least a constant additive. For several problems, e.g. for hypergraph 2-colouring, it has been shown [AM06] that the second moment analysis is tight. That means just putting more effort into the calculations or increasing their accuracy does not help to squeeze out the missing constant. On the contrary, Achlioptas and Moore even proved that $\mathbb{E}\left[Z^2\right] > \exp\left[\Omega(n)\right]\mathbb{E}\left[Z\right]^2$ for $d > d_{\text{sec}}$, implying that the second moment method fails drastically.

### 3.2.3. To the condensation threshold and beyond

The reason for this failure is twofold, as has been explicitly shown for random hypergraph 2-colouring [COZ12] and random graph $k$-colouring [COV13]: For densities between $d_{\text{sec}}$ and $d_{\text{cond}}$ (the condensation transition introduced in Section 2.5), the random variable $Z$ is close to $\mathbb{E}\left[Z\right]$ w.h.p., but without being sufficiently concentrated for (3.2.1) to hold. In fact, it was shown that there is a constant $k_0 \geq 3$

and a sequence $\varepsilon_k \to 0$ such that for all $k \geq k_0$ and $d < d_{\text{cond}} - \varepsilon_k$ the random (hyper)graph is colourable w.h.p. and

$$\ln Z \sim \ln \mathbb{E}[Z].$$

Thus, below $d_{\text{cond}}$, $Z$ is w.h.p. of the same exponential order as $\mathbb{E}[Z]$.

For densities above $d_{\text{cond}}$, however, the expectation $\mathbb{E}[Z]$ is driven up by a very small number of (hyper)graphs possessing a vast number of colourings resulting in non-concentration of $Z$: W.h.p. there is some $\varepsilon_k \to 0$ such that for $d_{\text{cond}} + \varepsilon_k < d < d_{\text{col}}$ it is true that

$$\ln Z < \ln \mathbb{E}[Z] - \Omega(n).$$

This means that the *expected* number of 2-colourings exceeds the *actual* number by an exponential factor w.h.p.. In this case, this small number of "crazy" (hyper)graphs is responsible for an explosion of the second moment $\mathbb{E}\left[Z^2\right]$.

Zooming in on the reasons for this phenomenon reveals the following picture. By the definition of the second moment of $Z$, we have

$$\mathbb{E}\left[Z^2\right] = \sum_{\sigma,\tau} \mathbb{P}(\sigma \text{ is a colouring}) \, \mathbb{P}(\tau \text{ is a colouring} | \sigma \text{ is a colouring})$$

This implies that the second moment method works as long as, roughly speaking, the main contribution to the second moment comes from *uncorrelated* colourings because in this case $\mathbb{E}\left[Z^2\right]$ is of the same order of magnitude as $\mathbb{E}[Z]^2$.
Seen from another angle, calculating $\sum_\tau \mathbb{P}(\tau \text{ is a colouring} | \sigma \text{ is a colouring})$ for fixed $\sigma$ amounts to calculating the expected number of colourings in the planted model (cf. Section 3.1) with planted colouring $\sigma$. As we noticed above, under the distribution $\pi^{\text{pl}}_{k,n,m}$, (hyper)graphs exhibiting many colourings are chosen excessively often, or, in other words, the typically chosen (hyper)graph possesses a lot more solutions than the one chosen uniformly according to $\pi^{\text{rc}}_{k,n,m}$. As a consequence, the expected number of solutions is over-estimated. At $d_{\text{sec}}$ this over-estimation becomes significant in the second moment and the second moment method breaks down.

A slightly different perspective yields yet another explanation and paves the way for improving the second moment bound up to the condensation transition: It was shown in [AM06] that we can find a function $\psi : (0,1) \to \mathbb{R}$ such that $\psi(1/2) \sim \frac{1}{n} \ln \mathbb{E}[Z]$. Furthermore, if and only if $\psi(x)$ takes its global maximum at $x = 1/2$, then $\mathbb{E}\left[Z^2\right] \leq C \cdot \mathbb{E}[Z]^2$ for some constant $C > 0$ and the second moment method works. On the other hand, there exists a $0 < \alpha \ll 1/2$ such that the maximum of $\psi$ in $(0, \alpha)$ can be interpreted as the normalized logarithm of the *expected* size of the local cluster. Thus, the second moment argument breaks down at $d_{\text{sec}}$ because at this point the expected cluster size exceeds the total expected number of solutions.

It can, however, be proven that up to $d_{\mathrm{cond}}$ the *expected* size of the local cluster in the planted model exaggerates its *typical* size. Below $d_{\mathrm{cond}}$ this typical size is indeed not bigger than the total number of solutions (cf. [COZ12, Proposition 4.6]).
Thus, the second moment argument can be pushed up to the condensation transition by investigating the internal structure of the clusters and excluding solutions with huge clusters. This has for example been done in [COV13].

Ultimately, beyond $d_{\mathrm{cond}}$ the size of the typical local cluster in the planted model is by an exponential factor bigger than the expected number of 2-colourings. In this regime, the planted model fails to be a good approximation for the random colouring model as a pair chosen from the planted distribution corresponds to a pair chosen from the Gibbs distribution only with exponentially small probability. Two randomly chosen colourings strongly correlate as they belong to the same cluster with non-vanishing probability. This explains intuitively that the second moment method cannot be extended to densities beyond $d_{\mathrm{cond}}$, as a necessary condition for the second moment method to work is that a random pair of colourings decorrelate (cf. e.g. [ANP05]). Thus, it proves difficult to obtain mathematically precise results for densities beyond $d_{\mathrm{cond}}$, especially concerning the satisfiability threshold $d_{\mathrm{col}}$.

## 3.3. Small subgraph conditioning

Small subgraph conditioning is a method developed by Robinson and Wormald in [RW92, RW94]. It was originally used to show that random regular graphs of degree three or more are Hamiltonian w.h.p. and has since been applied in many different settings. Essentially, the method is used to study a sequence of random variables depending on a graph $G$, which are not concentrated around their means, but become concentrated conditioned on the presence of small sub-structures in $G$, in our case short cycles. Janson used the method in [Jan95] in order to obtain limiting distributions and to prove contiguity. In this process, small subgraph conditioning was developed into a comfortably applicable tool, only requiring the calculation of some joint moments and a very accurate analysis of the variance. In Section 4.2, we give a more thorough overview of work in this field (additionally we recommend the survey [Wor99] for a detailed discussion.) For the moment, we content ourselves with stating that the core idea of the method consists in showing the following:

When we consider the variance of the random variables in question, in our case the number of satisfying assignments with some additional properties, we can divide the set of all (hyper)graphs into groups according to the small cycle counts and decompose the variance into the variance of the group mean plus the expected value of the variance within a group.

We then proceed to show that conditioning on the number of small cycles reduces the variance significantly. More precisely, it can be proven that the contribution of the second summand is negligible and thus the limiting distribution of the logarithm of the number of satisfying assignments can be determined by the joint distribution of the number of short cycles.

The following theorem by Janson is typically the main tool when using small subgraph conditioning.

**Theorem 3.3.1** ([Jan95]). *Suppose that* $(\delta_l)_{l\geq 2}$ *and* $(\lambda_l)_{l\geq 2}$ *are sequences of real numbers such that* $\delta_l \geq -1$ *and* $\lambda_l > 0$ *for all l. Moreover, assume that* $(C_{l,n})_{l\geq 2, n\geq 1}$ *and* $(Z_n)_{n\geq 1}$ *are random variables such that each* $C_{l,n}$ *takes values in the non-negative integers. Additionally, suppose that for each* $n$ *the random variables* $C_{2,n}, \ldots, C_{n,n}$ *and* $Z_n$ *are defined on the same probability space. Moreover, let* $(X_l)_{l\geq 2}$ *be a sequence of independent random variables such that* $X_l$ *has distribution* $\mathrm{Po}(\lambda_l)$ *and assume that the following four conditions hold.*

***SSC1*** *for any integer* $L \geq 2$ *and any integers* $x_2, \ldots, x_L \geq 0$

$$\lim_{n\to\infty} \mathbb{P}\left[\forall 2 \leq l \leq L : C_{l,n} = x_l\right] = \prod_{l=2}^{L} \mathbb{P}\left[X_l = x_l\right].$$

***SSC2*** *for any integer* $L \geq 2$ *and any integers* $x_2, \ldots, x_L \geq 0$

$$\lim_{n\to\infty} \frac{\mathbb{E}[Z_n | \forall 2 \leq l \leq L : C_{l,n} = x_l]}{\mathbb{E}[Z_n]} = \prod_{l=2}^{L} (1+\delta_l)^{x_l} \exp\left[-\lambda_l \delta_l\right].$$

***SSC3*** $\sum_{l=2}^{\infty} \lambda_l \delta_l^2 < \infty$.

***SSC4*** $\lim_{n\to\infty} \mathbb{E}[Z_n^2]/\mathbb{E}[Z_n]^2 \leq \exp\left[\sum_{l=2}^{\infty} \lambda_l \delta_l^2\right]$.

*Then the sequence* $(Z_n/\mathbb{E}[Z_n])_{n\geq 1}$ *converges in distribution to* $\prod_{l=2}^{\infty}(1+\delta_l)^{X_l} \exp\left[-\lambda_l \delta_l\right]$.

The random variable $\prod_{l=2}^{\infty}(1+\delta_l)^{X_l} \exp\left[-\lambda_l \delta_l\right]$ has been studied in [Jan95], where it was shown that it has a bounded first and second moment. Unfortunately, in our context it is not possible to explicitly use Theorem 3.3.1 for reasons which will be elaborated on in Sections 4.2 as well as 7.1 and 8.1. We therefore have to refine the analysis similar to the one done in [COW16+].

# 4 Results and related work

## 4.1. Results

The results presented in this thesis are manifold and touch upon a variety of models and questions related to the study of random CSPs. Integrated are four papers, whose results will be presented in the chronological order of their creation. Two of them are already published, while the other two are submitted and preprints can be found online. Large parts of this chapter are a verbatim copy or a close adaption of the content of these papers.
The proofs of the results will be unfolded in full detail in Chapters 5 to 8 and in Appendices A and B.

### 4.1.1. Condensation in random graph $k$-colouring

The first result is from the paper

*The condensation phase transition in random graph coloring*

by Bapst, Coja-Oghlan, Hetterich, Raßmann and Vilenchik published in *Communications in Mathematical Physics 341 (2016)* [BCOHRV16]. It deals with proving the existence and exactly determining the location of the condensation phase transition in random graph $k$-colouring provided that $k$ exceeds a certain constant $k_0$. The solution is given in terms of a distributional fixed point problem and verifies the conjecture obtained via the cavity method.

To state the result, we need a bit of notation. Let $\Omega$ be the set of probability measures on $[k]$. We identify $\Omega$ with the $k$-simplex, i.e. the set of maps $\mu : [k] \to [0,1]$ such that $\sum_{h=1}^{k} \mu(h) = 1$, equipped with the topology and Borel algebra induced by $\mathbb{R}^k$. Moreover, we define a map $\mathcal{B} : \bigcup_{\gamma=1}^{\infty} \Omega^\gamma \to \Omega$, $(\mu_1, \ldots, \mu_\gamma) \mapsto \mathcal{B}[\mu_1, \ldots, \mu_\gamma]$ by letting

$$\mathcal{B}[\mu_1, \ldots, \mu_\gamma](i) = \begin{cases} 1/k & \text{if } \sum_{h \in [k]} \prod_{j=1}^{\gamma} (1 - \mu_j(h)) = 0, \\ \frac{\prod_{j=1}^{\gamma}(1-\mu_j(i))}{\sum_{h \in [k]} \prod_{j=1}^{\gamma}(1-\mu_j(h))} & \text{otherwise,} \end{cases} \qquad \text{for any } i \in [k]\,.$$

In physics language this would be called the *Belief Propagation operator*. Further, we let $\mathcal{P}$ be the set of all probability measures on $\Omega$ and for each $\mu \in \Omega$ we let $\delta_\mu \in \mathcal{P}$ denote the Dirac measure that puts mass one on the single point $\mu$. In particular, $\delta_{k^{-1}\mathbf{1}} \in \mathcal{P}$ is the measure putting mass one on the uniform distribution $k^{-1}\mathbf{1} = (1/k, \ldots, 1/k)$. For $\pi \in \mathcal{P}$ and $\gamma \geq 0$ let

$$Z_\gamma(\pi) = \sum_{h=1}^{k} \left(1 - \int_\Omega \mu(h) \mathrm{d}\pi(\mu)\right)^\gamma. \tag{4.1.1}$$

Further, define a map $\mathcal{F}_{d,k} : \mathcal{P} \to \mathcal{P}$, $\pi \mapsto \mathcal{F}_{d,k}[\pi]$ by letting

$$\mathcal{F}_{d,k}[\pi] = \exp[-d]\, \delta_{k^{-1}\mathbf{1}} + \sum_{\gamma=1}^{\infty} \frac{d^\gamma \exp[-d]}{\gamma! Z_\gamma(\pi)} \int_{\Omega^\gamma} \left[\sum_{h=1}^{k} \prod_{j=1}^{\gamma} (1 - \mu_j(h))\right] \delta_{\mathcal{B}[\mu_1,\ldots,\mu_\gamma]} \bigotimes_{j=1}^{\gamma} \mathrm{d}\pi(\mu_j). \tag{4.1.2}$$

Thus, in (4.1.2) we integrate a function with values in $\mathcal{P}$, viewed as a subset of the Banach space of signed measures on $\Omega$. The normalising term $Z_\gamma(\pi)$ from (4.1.1) ensures that $\mathcal{F}_{d,k}[\pi]$ really is a probability measure on $\Omega$. In physics terms, $\mathcal{F}_{d,k}$ represents a distributional version of the Belief Propagation operator.

The main theorem is given in terms of a fixed point of the map $\mathcal{F}_{d,k}$, i.e. a point $\pi^* \in \mathcal{P}$ such that $\mathcal{F}_{d,k}[\pi^*] = \pi^*$. In general, the map $\mathcal{F}_{d,k}$ has several fixed points. Hence, we need to single out the correct one. For $h \in [k]$ let $\delta_h \in \Omega$ denote the vector whose $h$th coordinate is 1 and whose other coordinates are 0 (i.e. the Dirac measure on $h$). We call a measure $\pi \in \mathcal{P}$ *frozen* if $\pi(\{\delta_1, \ldots, \delta_k\}) \geq 2/3$; in words, the total probability mass concentrated on the $k$ vertices of the simplex $\Omega$ is at least $2/3$.

As a final ingredient, we need a function $\phi_{d,k} : \mathcal{P} \to \mathbb{R}$. To streamline the notation, for $\pi \in \mathcal{P}$ and $h \in [k]$ we write $\pi_h$ for the measure $\mathrm{d}\pi_h(\mu) = k\mu(h)\mathrm{d}\pi(\mu)$. With this notation, $\phi_{d,k}$ is defined as

$$\phi_{d,k}(\pi) = \phi^e_{d,k}(\pi) + \frac{1}{k} \sum_{i \in [k]} \sum_{\gamma_1,\ldots,\gamma_k=0}^{\infty} \phi^v_{d,k}(\pi; i; \gamma_1, \ldots, \gamma_k) \prod_{h \in [k]} \left(\frac{d}{k-1}\right)^{\gamma_h} \frac{\exp[-d/(k-1)]}{\gamma_h!},$$

where

$$\phi^e_{d,k}(\pi) = -\frac{d}{2k(k-1)} \sum_{h_1=1}^{k} \sum_{h_2 \in [k]\setminus\{h_1\}} \int_{\Omega^2} \ln\left[1 - \sum_{h \in [k]} \mu_1(h)\mu_2(h)\right] \bigotimes_{i=1}^{2} \mathrm{d}\pi_{h_i}(\mu_i), \tag{4.1.3}$$

$$\phi^v_{d,k}(\pi; i; \gamma_1, \ldots, \gamma_k) = \begin{cases} \ln k & \text{if } \sum_{i=1}^k \gamma_i = 0, \\ \displaystyle\int_{\Omega^{\gamma_1+\cdots+\gamma_k}} \ln\left[\sum_{h=1}^{k} \prod_{h' \in [k]\setminus\{i\}} \prod_{j=1}^{\gamma_{h'}} 1 - \mu^{(j)}_{h'}(h)\right] \bigotimes_{h' \in [k]} \bigotimes_{j=1}^{\gamma_{h'}} \mathrm{d}\pi_{h'}(\mu^{(j)}_{h'}) & \text{if } \sum_{i=1}^k \gamma_i > 0. \end{cases} \tag{4.1.4}$$

The integrals in (4.1.3) and (4.1.4) are well-defined because the set where the argument of the logarithm vanishes has measure zero.

The above formulas are derived systematically via the cavity method [MM09]. The functional $\phi_{d,k}$ is an instalment of a generic formula, the so-called "Bethe free entropy". Generally speaking, the "Bethe free entropy" yields a good approximation of the free entropy of the system if we insert the right distribution - on trees e.g. this distribution can be determined as the fixed point of Belief Propagation.

Now, the main theorem can be stated.

**Theorem 4.1.1.** *There exists a constant $k_0 \geq 3$ such that for any $k \geq k_0$ the following holds. If $d \geq (2k-1)\ln k - 2$, then $\mathcal{F}_{d,k}$ has precisely one frozen fixed point $\pi^*_{d,k}$. Further, the function*

$$\Sigma_k : d \mapsto \ln k + \frac{d}{2}\ln(1 - 1/k) - \phi_{d,k}(\pi^*_{d,k}) \tag{4.1.5}$$

*has a unique zero $d_{\text{cond}}$ in the interval $[(2k-1)\ln k - 2, (2k-1)\ln k - 1]$. For this number $d_{\text{cond}}$, the following three statements are true.*

*(i) Any $0 < d < d_{\text{cond}}$ is smooth and $\Phi_k(d) = k(1-1/k)^{d/2}$.*
*(ii) There occurs a phase transition at $d_{\text{cond}}$.*
*(iii) If $d > d_{\text{cond}}$, then*

$$\limsup_{n\to\infty} \mathbb{E}[Z_k(G(n, d/n))^{1/n}] < k(1-1/k)^{d/2}.$$

*Thus, if $d$ is smooth, then $\Phi_k(d) < k(1-1/k)^{d/2}$.*

**Remark 4.1.2.** *We observe that the first part of Theorem 4.1.1 implies that $G(n, d/n)$ has a $k$-colouring w.h.p. for any $0 < d < d_{\text{cond}}$. Indeed, if $d < d_{\text{cond}}$, then $\Phi_k(d) = k(1-1/k)^{d/2} > 0$ and thus $Z_k(G(n, d/n)) > 0$ w.h.p. because $(Z_k(G(n, d/n))^{1/n})$ converges to $\Phi_k(d)$ in probability.*

The key strength of Theorem 4.1.1 is that we identify the *precise* location of the phase transition. Given the intricate combinatorics of the random graph colouring problem, it does not seem surprising that the answer is not exactly simple.

In the proof of Theorem 4.1.1 the nature of the condensation phase transition is brought to light. For instance, the fixed point $\pi^*_{d,k}$ turns out to have a nice combinatorial interpretation, and, perhaps surprisingly, $\pi^*_{d,k}$ emerges to be a *discrete* probability distribution. Furthermore, in the course of the proof the prediction of the evolution of the solution space up to $d_{\text{cond}}$ as described in Section 1.2 will be verified. We will present the proof in Chapters 5 and Appendix A.

With the definition of the cluster of a colouring $\sigma$ from Section 2.4, we obtain the following corollary:

**Corollary 4.1.3.** *With the notation and assumptions of Theorem 4.1.1, the function $\Sigma_k$ is continuous, strictly positive and monotonically decreasing on the interval $((2k-1)\ln k - 2, d_{\text{cond}})$, and*

$\lim_{d\to d_{\mathrm{cond}}} \Sigma_k(d) = 0$. *Further, given that* $Z_k(G(n, d/n)) > 0$, *let* $\boldsymbol{\tau}$ *be a uniformly chosen random* $k$*-colouring of this random graph. Then, for any* $d \in ((2k-1)\ln k - 2, d_{\mathrm{cond}})$,

$$\lim_{\varepsilon\searrow 0}\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln\frac{|\mathcal{C}(G(n,d/n),\boldsymbol{\tau})|}{Z_k(G(n,d/n))} \le \Sigma_k(d)+\varepsilon \,\Big|\, Z_k(G(n,d/n))>0\right] = 1, \quad \textit{and}$$

$$\lim_{\varepsilon\searrow 0}\limsup_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln\frac{|\mathcal{C}(G(n,d/n),\boldsymbol{\tau})|}{Z_k(G(n,d/n))} \ge \Sigma_k(d)-\varepsilon \,\Big|\, Z_k(G(n,d/n))>0\right] > 0.$$

We emphasise that conditioning on $Z_k(G(n, d/n)) > 0$ is necessary to speak of a random $k$-colouring $\boldsymbol{\tau}$ but otherwise harmless, as Theorem 4.1.1 implies that $G(n, d/n)$ is $k$-colourable w.h.p. for any $d < d_{\mathrm{cond}}$.

In other words, Corollary 4.1.3 shows that there is a certain function $\Sigma_k > 0$ such that the total number of $k$-colourings exceeds the number of $k$-colourings in the cluster of a randomly chosen $k$-colouring by at least a factor of $\exp[n(\Sigma_k(d) + o(1))]$ with probability tending to one. On the other hand, with a non-vanishing probability the total number of $k$-colourings surpasses the size of a single cluster by at most a factor of $\exp[n(\Sigma_k(d) + o(1))]$. As $d$ approaches $d_{\mathrm{cond}}$, the function $\Sigma_k(d)$ tends to 0 and thus the corollary formalizes the prediction of an entropy crisis (cf. Section 1.2).

### 4.1.2. Condensation in finite temperature random hypergraph 2-colouring

The second result is from the paper

*A positive temperature phase transition in random hypergraph 2-coloring*

by Bapst, Coja-Oghlan and Raßmann [BCOR16] published in the *Annals of Applied Probability 26 (2016)*. In this paper we establish the existence and approximate location of the condensation phase transition in random hypergraph 2-colouring for finite inverse temperatures $\beta$. More specifically, we obtain a formula that determines the location of the condensation phase transition up to an error $\varepsilon_k$ that tends to 0 for $k \to \infty$. This is the first (rigorous) result that determines the condensation phase transition within such accuracy in terms of finite $\beta$.

With the definition of a phase transition from Section 2.5, we have the following result.

**Theorem 4.1.4.** *For any fixed number* $C > 0$, *there exists a sequence* $\varepsilon_k > 0$ *with* $\lim_{k\to\infty} \varepsilon_k = 0$ *such that the following is true. Let*

$$\Sigma_{k,d}(\beta) = (\beta+1)\exp[-\beta + k\ln 2]\ln 2 - 2\left(\frac{d}{k} - 2^{k-1}\ln 2 + \ln 2\right).$$

1. *If $d/k < 2^{k-1} \ln 2 - \ln 2 - \varepsilon_k$, then any $\beta > 0$ is smooth and*

$$\Phi_{d,k}(\beta) = \ln 2 + \frac{d}{k} \ln \left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right). \tag{4.1.6}$$

2. *If $2^{k-1} \ln 2 - \ln 2 + \varepsilon_k < d/k < 2^{k-1} \ln 2 + C$, then $\Sigma_{k,d}(\beta)$ has a unique zero $\beta_{\mathrm{cond}}(d,k) \geq k \ln 2$ and*
   - *any $\beta \in (0, \beta_{\mathrm{cond}}(d,k) + \varepsilon_k)$ is smooth and $\Phi_{d,k}(\beta)$ is given by (4.1.6),*
   - *there occurs a phase transition at $\beta_{\mathrm{cond}}(d,k) + \varepsilon_k$*
   - *for $\beta > \beta_{\mathrm{cond}}(d,k) + \varepsilon_k$ we have*

$$\Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k} \ln \left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right).$$

In summary, Theorem 4.1.4 shows that in random hypergraphs with density $d/k$ less than about $2^{k-1} \ln 2 - \ln 2$ there does not occur a phase transition for any finite $\beta$. By contrast, for slightly larger densities there is a phase transition. Its approximate location is given by $\beta_{\mathrm{cond}}(d,k)$. While in Theorem 4.1.4 this value is determined implicitly as the zero of $\Sigma_{k,d}(\beta)$, it is not difficult to obtain the expansion

$$\beta_{\mathrm{cond}}(d,k) = (k-1)\ln 2 + \ln k + 2 \ln \ln 2 - \ln c + \delta_k,$$

where $c = d/k - 2^{k-1} \ln 2 + \ln 2$ and $\lim_{k \to \infty} \delta_k = 0$. Furthermore, the proof of Theorem 4.1.4 shows that there exists $c_1 > 0$ such that $\varepsilon_k \leq k^{c_1} 2^{-k}$. Thus, Theorem 4.1.4 determines the critical density from that on a phase transition starts to occur and the critical $\beta_{\mathrm{cond}}(d,k)$ up to an error term decaying exponentially with $k$.

The proof of the theorem is carried out in full detail in Chapter 6.

### 4.1.3. Number of solutions in random hypergraph 2-colouring

The third result is from the paper

*On the number of solutions in random hypergraph 2-colouring*

by Raßmann [Ras16a+] submitted to *The Electronic Journal of Combinatorics*. We determine the limiting distribution of the logarithm of the number of satisfying assignments in the random $k$-uniform hypergraph 2-colouring problem in a certain density regime essentially up to the second moment lower bound $d_{\mathrm{sec}}$ for all $k \geq 3$. As a direct consequence, we obtain that in this regime the random colouring model is contiguous with respect to the planted model, a result that helps simplifying the transfer of statements between these two models.

While studying random constraint satisfaction problems, for a long time a main focus has been on determining the expected value of the number of solutions and understanding how this number evolves when the constraint density changes. Despite the efforts, up to now the distribution of the number of solutions has remained elusive in any of the standard examples of random constraint satisfaction problems.

**Theorem 4.1.5.** *Let $k \geq 3$ and $d'/k \leq 2^{k-1}\ln 2 - 2$ as well as*

$$\lambda_l = \frac{[d(k-1)]^l}{2l} \quad \text{and} \quad \delta_l = \frac{(-1)^l}{(2^{k-1}-1)^l}$$

*for $l \geq 2$. Further let $(X_l)_l$ be a family of independent Poisson variables with $\mathbb{E}[X_l] = \lambda_l$, all defined on the same probability space. Then the random variable*

$$W = \sum_l [X_l \ln(1+\delta_l) - \lambda_l \delta_l]$$

*satisfies $\mathbb{E}|W| < \infty$ and $\ln Z(H_k(n,m)) - \ln \mathbb{E}[Z(H_k(n,m))]$ converges in distribution to $W$.*

**Remark 4.1.6.** *By definition, $W$ has an infinitely divisible distribution. It was shown in [Jan95] that the random variable $W' = \exp[W]$ converges almost surely and in $L^2$ with $\mathbb{E}[W'] = 1$ and $\mathbb{E}\left[W'^2\right] = \exp\left[\sum_l \lambda_l \delta_l^2\right]$. Thus, by Jensen's inequality it follows that $\mathbb{E}[W] \leq 0$. Furthermore, by basic calculations it is easy to verify that also $\mathbb{E}\left[W^2\right]$ is finite.*

As a direct consequence of Theorem 4.1.5, we obtain the following.

**Corollary 4.1.7.** *Assume that $k \geq 3$ and $d'/k \leq 2^{k-1}\ln 2 - 2$. Then*

$$\lim_{\omega\to\infty}\lim_{n\to\infty} \mathbb{P}\left[|\ln Z(H_k(n,m)) - \ln \mathbb{E}\left[Z(H_k(n,m)\right]| \leq \omega\right] = 1. \tag{4.1.7}$$

*On the other hand, for any fixed number $\omega > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left[|\ln Z(H_k(n,m)) - \ln \mathbb{E}[Z(H_k(n,m))]| \leq \omega\right] < 1.$$

The first part of Corollary 4.1.7 shows that for the covered range of $d'$ and $k$, $\ln Z_k(G(n,m))$ actually fluctuates w.h.p. by no more than $\omega = \omega(n)$ for *any* $\omega(n) \to \infty$. Moreover, the second part shows that this is best possible.

Furthermore, Theorem 4.1.5 enables us to establish a very strong connection between the random colouring model and the planted model. To state this, we recall the definition of contiguity from Subsection 3.1.2 and show that as a consequence of Theorem 4.1.5 the statement (3.1.2) can be sharpened in the strongest possible sense. Roughly speaking, we show that in a density regime nearly up to the second moment lower bound the random colouring model is contiguous with respect to the planted model, i.e. that in (3.1.2) it suffices that $\pi^{\mathrm{pl}}_{k,n,m}[\mathcal{E}_n] = o(1)$.

**Corollary 4.1.8.** *Assume that $d/k \leq 2^{k-1}\ln 2 - 2$. Then $(\pi^{\mathrm{rc}}_{k,n,m})_{n\geq 1} \triangleleft (\pi^{\mathrm{pl}}_{k,n,m})_{n\geq 1}$.*

As done in [BCOE14+], we refer to this contiguity statement as *silent planting*. We will elaborate on the proofs of Theorem 4.1.5 and Corollaries 4.1.7 and 4.1.8 in Chapter 7.

### 4.1.4. Number of solutions in random graph $k$-colouring

The last result is from the paper

*On the number of solutions in random graph $k$-colouring*

by Raßmann [Ras16b+] submitted to *Combinatorics, Probability and Computing*.

We show that under certain conditions the number $Z_k(G(n,m))$ of $k$-colourings of the random graph is concentrated tightly and determine the distribution of $\ln Z_k(G(n,m)) - \ln \mathbb{E}[Z_k(G(n,m))]$ asymptotically in a density regime up to the condensation transition.

**Theorem 4.1.9.** *There is a constant $k_0 > 3$ such that the following is true. Assume that either $k \geq 3$ and $d' \leq 2(k-1)\ln(k-1)$ or that $k \geq k_0$ and $d' < d_{\mathrm{cond}}$. Further, let*

$$\lambda_l = \frac{d^l}{2l} \quad \text{and} \quad \delta_l = \frac{(-1)^l}{(k-1)^{l-1}}$$

*for $l \geq 2$. Let $(X_l)_l$ be a family of independent Poisson variables with $\mathbb{E}[X_l] = \lambda_l$, all defined on the same probability space. Then the random variable*

$$W = \sum_{l\geq 3} [X_l \ln(1+\delta_l) - \lambda_l \delta_l] - d^2/(4(k-1))$$

*satisfies $\mathbb{E}|W| < \infty$ and $\ln Z_k(G(n,m)) - \ln \mathbb{E}[Z_k(G(n,m))]$ converges in distribution to $W$.*

Analogously to Remark 4.1.6, it is known that $W$ has a bounded first and second moment.

By obtaining an exact expression for the asymptotic distribution of the logarithm of the partition function up to the condensation threshold $d_{\mathrm{cond}}$, in the present paper we give a definite and complete answer to the question about the relationship between the planted model and the Gibbs distribution. Furthermore, we show that the fluctuations in the number of solutions can completely be attributed to the presence of short cycles, thereby eliminating the possibility of other influencing factors.

## 4.2. Discussion and former work

In this section, we discuss the relevance and impact of the results presented above, i.e. explain how they compare to other relevant work, relate to various questions that have come up in the literature and find their place alongside other existing results.

### 4.2.1. On phase transitions in random graph $k$-colouring

As already outlined in Section 1.1, graph colouring is one of the most fundamental problems in combinatorics and has attracted a great deal of attention since it was first posed by Erdős and Rényi [ER60]. Much effort has been devoted to studying the typical value of the chromatic number of the Erdős-Rényi random graph [Bol88, Luc91a, Mat87, AN05, COPS08, COV13] and its concentration [SS87, AK97, Luc91b]. With Theorem 4.1.1 we contribute to the endeavour of thoroughly understanding this problem by identifying the *precise* location of the condensation phase transition $d_{\mathrm{cond}}$ for the Erdős-Rényi random graph model. In effect, Theorem 4.1.1 is the first result that pins down the exact condensation phase transition in a diluted mean-field model, thereby verifying the prediction from the cavity method derived in [KMRTSZ07, ZK07].

A simple asymptotic expansion of $d_{\mathrm{cond}}$ in the limit of large $k$ yields

$$d_{\mathrm{cond}} = (2k-1)\ln k - 2\ln 2 + \varepsilon_k,$$

where $\varepsilon_k \to 0$ as $k \to \infty$. This asymptotic formula had already been obtained by Coja-Oghlan and Vilenchik in [COV13], although by means of a *much* simpler argument that does not quite get to the bottom of the condensation phenomenon and could therefore not be applied to establish the *exact* location of the condensation transition.

Essentially, our proof of Theorem 4.1.1 builds upon the second moment argument from [COV13]. Furthermore, it uses some of the techniques developed to study the geometry of the set of $k$-colourings of the random graph and adds to this machinery. Among the techniques that we use are the planted model introduced in Section 3.1, the notion of a core [ACO08, Mol12, COV13], techniques for proving the existence of “frozen variables” (or “hard fields” in physics jargon) [ACO08, CO13, Mol12], and a concentration argument from [COZ12]. Beyond that, the cornerstone of the present work is a novel argument that allows us to establish an explicit link between the combinatorics of the graph colouring problem and the cavity formalism, more precisely to connect the geometry of the set of $k$-colourings rigorously with the distributional fixed point problem from [ZK07].

Furthermore, Theorem 4.1.1 yields a small improvement over the best lower bound on the colouring threshold sequence $d_{\mathrm{col}}(n)$ from Theorem 1.1.1. Prior to Theorem 4.1.1, the best bounds on $d_{\mathrm{col}}(n)$ had been

$$(2k-1)\ln k - 2\ln 2 + \varepsilon_k \leq \liminf_{n\to\infty} d_{\mathrm{col}}(n) \leq \limsup_{n\to\infty} d_{\mathrm{col}}(n) \leq (2k-1)\ln k - 1 + \delta_k, \quad (4.2.1)$$

where $\varepsilon_k, \delta_k \to 0$ as $k \to \infty$. The upper bound in (4.2.1) was obtained by the first moment method [CO13], while the lower bound rests on a second moment argument [COV13], which improved a landmark result of Achlioptas and Naor [AN05]. In particular, the proofs of the bounds (4.2.1) exploit structural properties such as the "clustering" of the set of $k$-colourings and the emergence of "frozen variables".

Theorem 4.1.1 improves the lower bound in (4.2.1) by determining the precise "error term" $\varepsilon_k$. Indeed, Remark 4.1.2 implies that $\liminf_{n\to\infty} d_{\mathrm{col}}(n) \geq d_{\mathrm{cond}}$. In fact, $d_{\mathrm{cond}}$ is the best-possible lower bound that can be obtained via the kind of second moment argument developed in [AN05, COV13] because a necessary condition for the success of the second moment argument is that $\Phi_k(d) = k(1-1/k)^{d/2}$.

While Theorem 4.1.1 allows for the possibility that $d_{\mathrm{cond}}$ is equal to the $k$-colouring threshold $d_{\mathrm{col}}$ (if it exists), the physics prediction is that these two are different. More specifically, the cavity method yields a prediction as to the precise value of $d_{\mathrm{col}}$ in terms of another distributional fixed point problem. An asymptotic expansion in terms of $k$ leads to the conjecture $d_{\mathrm{col}} = (2k-1)\ln k - 1 + \eta_k$ with $\eta_k \to 0$ as $k \to \infty$ [KPW04]. Thus, the upper bound in (4.2.1) is conjectured to be asymptotically tight in the limit $k \to \infty$.

In effect, the predictions regarding the condensation phase transitions in other problems look very similar to the one in random graph colouring. Consequently, it seems reasonable to expect that the proof technique developed in [BCOHRV16] carries over to many other problems.

### 4.2.2. On phase transitions in random hypergraph 2-colouring for finite inverse temperatures

As discussed in Section 1.1 and Chapter 2, the problem of hypergraph 2-colouring stands out from other random CSPs because of the symmetry it exhibits and its technically not too involved calculations (for instance in the second moment analysis). With the work [BCOR16] we contribute to investigating an extension of this problem, namely its finite temperature version, meaning that we deal with a two-dimensional phase diagram governed by $d$ and, additionally, the inverse temperature $\beta$. Our result is the first to identify the condensation phase transition in such a finite temperature problem rigorously up to an error term that decays to 0 when $k \to \infty$.

The first rigorous result on a genuine condensation phase transition in a diluted mean field model is due to Coja-Oghlan and Zdeborová [COZ12], who dealt with proper hypergraph 2-colouring (i.e. the $\beta = \infty$ case of the problem considered here). Thus, the only parameter in [COZ12] is $d$. The main results of [COZ12] are that there occurs a condensation phase transition at $d/k = 2^{k-1}\ln 2 - \ln 2 + \gamma_k$, where $\lim_{k\to\infty} \gamma_k = 0$ and that the condensation phase is not empty. Up to the error term $\gamma_k$, the result confirms a prediction from [DRZ08]. Moreover, as Theorem 4.1.4 shows, the result from [COZ12] matches the smallest density for which a condensation phase transition occurs for finite $\beta$. In this sense, [COZ12] determines the intersection of the "condensation line" in the two-dimensional phase diagram of Theorem 4.1.4 with the $d$-axis.

As proper hypergraph 2-colouring has been an active area of research, there is a variety of rigorous results concerning the geometry and the evolution of the solution space [ACO08, AM02], for example the proof of the "shattering" of the solution space into small, well-separated clusters up to the condensation threshold [AM06, COZ12]. Although the existence and location of a sharp colouring threshold has not been proven yet, Friedgut's Theorem 1.1.1 can be applied. The best current bounds on the threshold sequence $d_{\mathrm{col}}(n)$ are

$$2^{k-1}\ln 2 - \ln 2 + \varepsilon_k \le \liminf_{n\to\infty} d_{\mathrm{col}}(n)/k \le \limsup_{n\to\infty} d_{\mathrm{col}}(n)/k \le 2^{k-1}\ln 2 - \ln 2/2 + \delta_k,$$

where $\varepsilon_k, \delta_k \to 0$ as $k \to \infty$. The upper bound was obtained by Achlioptas and Moore [AM06] via the first moment method. The lower bound is the location of the condensation phase transition shown in [COZ12], which represents an improvement over the former second moment lower bound from [AM06].

Furthermore, there is a prediction for the location of the colouring threshold $d_{\mathrm{col}}$ by statistical physicists [DRZ08, KMRTSZ07], suggesting that

$$d_{\mathrm{col}}/k = 2^{k-1}\ln 2 - \ln 2/2 - 1/4 + \varepsilon_k \qquad \text{with } \lim_{k\to\infty} \varepsilon_k \to 0.$$

This prediction was proven by Coja-Oghlan and Panagiotou [COP12] for the problem of NAE-$k$-SAT, which is almost equivalent to hypergraph 2-colouring, and it should be possible to transfer the result without major difficulties.

In a Paper by Ayre, Coja-Oghlan and Greenhill [ACOG15+], the generalized problem of random $k$-uniform hypergraph $q$-colouring has lately been investigated and a lower bound on the colouring threshold has been obtained. This bound matches the prediction for the condensation phase transition [KMRTSZ07].

Up to now, work on problems with finite $\beta$ has concentrated mostly on the $k$-spin Potts antiferromagnet at zero temperature, which is the physics name for the $k$-colouring problem. It has been studied on lattices [ZK08] and on the Erdős-Rényi random graph, where the condensation line at finite $\beta$ was investigated by Krzakala and Zdeborová [KZ08] by means of non-rigorous techniques. They predict the location of the condensation line in terms of an intricate fixed-point problem.

The only prior rigorous paper that explicitly deals with the positive temperature case is the recent work of Contucci, Dommers, Giardina and Starr [CDGS13]. They also study the $k$-spin Potts antiferromagnet on the Erdős-Rényi random graph with finite $\beta$ and show that for certain values of the average degree a condensation phase transition exists. To the extent to which the results are comparable, [CDGS13] is less precise than Theorem 4.1.4. Indeed, a direct application of the approach from [CDGS13] to the present problem would determine $\beta_{\text{cond}}(d,k)$ only up to an additive error of $\ln k$, rather than an error that diminishes with $k$. This is due to two technical differences between the present work and [CDGS13]. First, the second moment argument required in the case of the $k$-spin Potts antiferromagnet is technically *far* more challenging than in the present case. In effect, an enhanced version of the second moment argument along the lines of [COZ12] (with explicit conditioning on the cluster size) is not available in the Potts model. Second, [CDGS13] employs a conceptually less precise estimate of the cluster size than the one we derive. This originates from the fact that they essentially neglect the entropic contribution to the cluster size, with the consequence of under-estimating the typical cluster size significantly.
Very recently, Coja-Oghlan and Jaafari [COJ16+] determined the free entropy in the Potts antiferromagnet with finite $\beta$ rigorously for all temperatures and small average degrees and specified a regime where surely no phase transition occurs.

Theorem 4.1.4 is perfectly in line with the picture sketched by the non-rigorous cavity method. Indeed its proof is inspired by the physicists' notion that the condensation phase transition results from an "entropy crisis" [KMRTSZ07, MM09] (cf. Section 2.3). The proof of Theorem 4.1.4 is based on turning this scenario into a rigorous argument. To this end, we establish a rigorous version of the cluster decomposition summarized in Section 2.3 and, crucially, an estimate of the cluster volumes $Z_{\beta,i}$. The arguments that we develop for these problems partly build upon prior work from [ACO08, AM02, COZ12]. In particular, we provide a "finite-$\beta$" version of the second moment arguments from [ACO08, COZ12]. The argument that we develop for inferring the condensation transition from the second moment method and the estimate of the cluster size draws upon ideas developed for the $\beta = \infty$ case in [ACO08, BCOHRV16, COZ12]. Dealing with finite $\beta$ requires substantial additional work and ideas, especially with respect to the estimate of the cluster size.

### 4.2.3. On the asymptotic distribution of partition functions

Determining the distribution of the number of solutions in random graph $k$-colouring and random hypergraph 2-colouring has been an open problem for a very long time.

For colouring random *regular* graphs, where each vertex appears in exactly the same number of edges, it had been implicitly known for a while that the fluctuations in the number of colourings can be attributed to the presence of short cycles [Wor99]. As in the random $d$-regular graph for any fixed number $s$ the neighbourhood of depth $s$ of all but a bounded number of vertices is a $d$-regular tree, there are only extremely limited fluctuations in the local graph structure. Thus, it seemed reasonable to expect that the random variable $\ln Z$ is more tightly concentrated in random regular graphs than in the Erdős-Rényi model, where the depth-$s$ neighbourhoods can be of varying shapes and sizes (although all but a bounded number will be acyclic), and also the number of vertices and edges in the largest connected component as well as the core fluctuate. Thus, it did not seem obvious that small subgraph conditioning could be applied in the case of Erdős-Rényi random graphs. However, in [BCOE14+] it was established that also when $k$-colouring random Erdős-Rényi graphs, the fluctuations of $\ln Z$ are merely due to the appearance of short cycles.

The ideas for the proofs of the results from [Ras16a+] and [Ras16b+] follow the way beaten in [BCOE14+], where statements analogue to Corollary 4.1.7 and Corollary 4.1.8 are shown for the problem of $k$-colouring random graphs. However, Theorems 4.1.5 and 4.1.9 are stronger than the results obtained in [BCOE14+] because we determine the exact distribution of $\ln Z - \mathbb{E}[\ln Z]$ asymptotically. The proofs are mainly based on the observation that the variance in the logarithm of the number of 2-colourings can be attributed to the fluctuations in the number of cycles of bounded length and that conditioning on this number reduces the variance dramatically. The same phenomenon was observed in [BCOE14+] and also in [COW16+], where a combination of the second moment method and small subgraph conditioning was applied to derive a result similar to ours for the problem of random regular $k$-SAT.

Small subgraph conditioning was originally developed by Robinson and Wormald in [RW92, RW94] to investigate the Hamiltonicity of random regular graphs of degree at least three. Janson showed in [Jan95] that the method can be used to obtain limiting distributions. Small subgraph conditioning has frequently been used in random regular graph problems (see [Wor99] for an enlightening survey). In e.g. [KPGW10] and [COEH16] it was applied to upper-bound the chromatic number of the random $d$-regular graph, as the sharp threshold result Theorem 1.1.1 does not hold for this problem. More recently, is has also been used to establish a result on non-distinguishability of the Erdős-Rényi model and the stochastic block model [BMNN16] and to determine the satisfiability threshold for positive 1-in-$k$-SAT, a Boolean satisfiability problem, where each clause contains $k$ variables and demands that exactly one of them is true [Moo15+].

Similar to [Jan95], we aim at obtaining a limiting distribution. Unfortunately, Janson's result Theorem 3.3.1 does not apply directly in our case for the following reason. In contrast to [BCOE14+], where only bounds on the fluctuation of $\ln Z_k$ were proven, we aim at a statement about its asymptotic *distribution*. Thus, for our approach it does not suffice to consider colourings with *balanced* colour classes (with a deviation of $o(n^{-1/2})$ from their typical value), but we have to get a handle on all colourings providing a positive contribution. To this aim, we collect together colourings exhibiting similar colour class sizes. This results in the need to not only consider one random variable, but break it into a large number of smaller random variables. However, it is not evident how to apply Janson's result simultaneously to these variables, whose number grows with $n$. Instead, we choose to perform a variance analysis along the lines of [RW94]. The same approach was pursued in [COW16+], and thus our proof technique is similar to theirs in flavour, yet we have the advantage of only having to deal with a very moderately growing number of variables, which simplifies matters slightly.

We expect that it is possible to apply a combination of the second moment method and small subgraph conditioning to a variety of further random CSPs, such as e.g. random $k$-NAESAT, random $k$-XORSAT or random hypergraph $k$-colourability. However, for asymmetric problems like the well-known benchmark problem random $k$-SAT, we expect that the logarithm of the number of satisfying assignments exhibits stronger fluctuations and we doubt that a result similar to ours can be established.

## 4.3. Related work

In this section we provide a short overview of (mostly) rigorous work on random CSPs related to those we deal with, without raising a claim to completeness of the presented work.

### 4.3.1. Related constraint satisfaction problems

**Random $k$-XORSAT**, which is an ensemble of random linear systems over the field of integers modulo 2, is an example of a very simple random CSP which does not exhibit a condensation phase due to its algebraic nature: all clusters are simply translations of the kernel. The precise threshold for the existence of solutions is known [DM02, PS16] and is obtained by applying the second moment method to the number of solutions after "stripping" the instance down to a certain core, ending up with a set of variables independent of the assignment which the process started from (a fact which simplifies the second moment analysis substantially).

**Random $k$-SAT**, a special case of Boolean Satisfiability where (almost) all clauses have the same size $k$, has been one of the benchmark problems in computer science, ever since Cook proved that it is NP-complete in the worst case for all $k \geq 3$ [Coo71]. The problem can be stated as follows: Given a Conjunctive Normal Form (CNF) formula $F$, is it possible to assign truth values to the variables of $F$ so that it evaluates to true? While for instance hypergraph 2-colouring is a symmetric problem in the sense that the inverse of each solution is a solution again, random $k$-SAT is not. Satisfying assignments tend to "lean" towards the majority vote truth assignment: truth assignments satisfying many literal occurrences in the random formula have significantly greater probability of being satisfying. Furthermore, they tend to be correlated and to agree with each other and the majority truth assignment on more than half of the variables. It was shown in [AM06, AP04] that as a consequence in random $k$-SAT the bound $\mathbb{E}\left[Z^2\right] = O\left(\mathbb{E}\left[Z\right]^2\right)$ does not hold for *any* density.

Franco and Paull [FP83] were the first to mathematically investigate random $k$-SAT and to observe that the problem is w.h.p. unsatisfiable if $d > 2^k \ln 2$. In 1990, Chao and Franco [CF90] invented a simple algorithm called "unit clause" that finds satisfying truth assignments with uniformly positive probability[19] for $d < 2^k/k$. Frieze and Suen [FS96] later improved this lower bound to $d > c_k 2^k/k$ with $\lim_{k\to\infty} c_k = 1.817...$ and this remained the best lower bound for a long time.

In [AM06], Achlioptas and Moore tackled the problem by considering a special case of the $k$-SAT problem, namely the symmetric version NAE-$k$-SAT (see below), and by focusing on balanced assignments. By this means, they could significantly improve the lower bound by applying the second moment method and were able to determine the threshold for random $k$-SAT within a factor of two.
Later on, the second moment lower bound was improved by Achlioptas and Peres [AP04], matching the first moment upper bound up to an exponentially small second-order term, only leaving a gap of order $\Omega(k)$. Their result was the first rigorous proof of a replica method prediction for any NP-complete problem at zero temperature. They coped with the asymmetry and the tendency of the majority vote by cleverly weighting the truth assignments and concentrating the weight on balanced ones.
After that, the gap was narrowed to an additive constant (independent of $k$) via improved second moment arguments [COP13] and soon afterwards closed up to an error vanishing for $k \to \infty$ [COP16]. This was done by using a second moment argument inspired by the physicists' concept of "Survey Propagation", counting the number of clusters rather than solutions.
Finally, Ding, Sly and Sun [DSS15] could eliminate this last error term and exactly determine the satisfiability threshold in $k$-SAT for large $k$ via a second moment argument that fully rigorizes the notion of Survey Propagation.

Two special cases of the random $k$-SAT problem had been tackled before: For the 2-SAT problem, which was proven to belong to the class P, meaning that it is computationally tractable, Chvátal and Reed [CR92] and independently Goerdt [Goe96] found the $d_{2-\mathrm{SAT}}$ threshold to occur at the density $m/n \sim 1$. There is no condensation phase in this problem. Secondly, in random $k$-SAT with $k > \log_2 n$, where the clause length is growing with $n$, there too is no condensation phase. The precise threshold has been obtained via the second moment method [FW05, COF08].

**Random regular $k$-SAT** is a version of the $k$-SAT problem where each variable appears exactly $d$ times positively and $d$ times negatively in the random formula. This regularity condition leads to a relatively simple structure of the resulting factor graph because the neighbourhoods of all variables look structurally the same, and w.h.p. the total number of cycles of a fixed length is bounded. Rathi, Aurell, Rasmussen and Skoglund [RARS10] were the first to study instances of this problem. They applied the second moment method to prove that near $d_{\mathrm{sat}}$ random instances are satisfiable w.u.p.p.. A few years later, Coja-Oghlan and Panagiotou [COP16] used an enhancement of this method, namely a Survey Propagation-based second moment method, to exactly determine $d_{\mathrm{sat}}$. In [BCO15+], Bapst

[19] We say that a sequence of events $\mathcal{A}_n$ occurs with uniformly positive probability (w.u.p.p.) if $\liminf_{n\to\infty} \mathbb{P}[\mathcal{A}_n] > 0$.

and Coja-Oghlan determined the existence and location of the condensation phase transition in this problem.
In [COW16+], Coja-Oghlan and Wormald combined the second moment argument from [COP13] with small subgraph conditioning to obtain the asymptotic distribution of the number of solutions in random regular $k$-SAT.

**Random NAE-$k$-SAT** is the symmetric version of random $k$-SAT, where NAE stands for 'not all equal', meaning that a clause is satisfied if and only if it contains at least one satisfied and one unsatisfied literal, or, in other words, the inverse of a satisfying assignment is satisfying as well. Like random $k$-SAT, random NAE-$k$-SAT is known to be NP-complete in the worst case for any $k \geq 3$. It is closely related to hypergraph 2-colouring, which can be interpreted as a special case of NAE-$k$-SAT without negations. Thus, results, e.g. the ones from [COZ12] for hypergraph 2-colouring, carry over without much effort. Coja-Oghlan and Panagiotou determined the threshold for the existence of solutions in [COP12] up to an error that vanishes in the limit of large $k$ using a Survey Propagation-inspired second moment method.

**Random regular NAE-$k$-SAT** is again the regular version of random NAE-$k$-SAT, where each variable appears exactly $d$ times positively and $d$ times negatively. Ding, Sly and Sun [DSS16] investigated this problem via a second moment argument and determined the satisfiability threshold for large values of $k$. This was the first result exactly locating the threshold in a problem exhibiting condensation. Around the same time, they also determined the asymptotics of the independence number of random $d$-regular graphs [DSS16+] for large $d$. Very recently, Sly, Sun and Zhang [SSZ16+] showed that for large $k$ the free entropy is well defined and determined the number of solutions of a typical instance, thereby verifying the physicists 1RSB prediction.

**Random regular graph $k$-colouring** is the regular version of the graph colouring problem. Coja-Oghlan, Efthymiou and Hetterich [COEH16] determined the chromatic number on a set of density 1, thereby improving over a result of Kemkes, Péres-Giménez and Wormald [KPGW10], who had previously succeeded in locating the chromatic number for about half of all degrees $d$. In both papers, a second moment argument is used and combined with small subgraph conditioning. The enhanced result in [COEH16] matches the one obtained in [COV13] for Erdős-Rényi random graphs. Indeed, it uses the Survey Propagation-inspired second moment argument from [COV13] as a "black box". Combining this with small subgraph conditioning is crucial to obtain the required high probability as for regular graphs Friedgut's sharp threshold result Theorem 1.1.1 cannot be applied.

**Stochastic block models** were introduced in the 1980s as models for random graphs exhibiting community structures.
In these models the vertices are divided into different classes and edges are added between them with probabilities depending on the classes. This construction can be interpreted as a generalisation of the planted model corresponding to the Potts antiferromagnet with finite $\beta$. An extensively studied pro-

blem in stochastic block models is the *community detection* problem. It investigates in which regime it is possible to recover the allocation of vertices to classes by only looking at the structure of the graph. Related is the question of distinguishability, asking whether a stochastic block model can be distinguished from an Erdős-Rényi model with the same average degree.

While for a long time research mostly concentrated on dense models (with high average degree), recently sparse models received a lot of attention. This was triggered by conjectures from statistical physicists [DKMZ11] concerning the existence and location of different bounds separating regimes where community detection is possible, possible but computationally hard, or not possible at all.

These conjectures were rigorously proven in the special case of having two classes with equal sizes [MNS16+, MNS13+, Mas14]. Recently, [BMNN16] addressed the problem of community detection for more than two classes. They gave upper and lower bounds on the information-theoretic threshold, which corresponds to the condensation threshold in spin glasses and separates regimes where successful detection is possible from ones where it is not. Their bounds are tight for some values of the edge-probabilities. Additionally, they established contiguity of the stochastic block model and the Erdős-Rényi model in a certain density regime.

### 4.3.2. Algorithmic questions

A short overview of the literature concerning the algorithmic point of view in the presented problems will complete this chapter.

Since the second moment method is non-constructive, there is a separate algorithmic question: For which densities can solutions of random CSPs be constructed in polynomial time w.h.p.? An abundance of research has been invested in studying random CSPs by means of efficient algorithms. Unfortunately, the best known combinatorial algorithms asymptotically do not work better than extremely naive ones: For random $k$-SAT, the simple algorithms "unit clause" and "shortest clause" were analysed more than 20 years ago and could be proven to find solutions for densities up to $O\left(2^k/k\right)$ [CF90, CR92, FS96]. A few years ago, Coja-Oghlan [CO10] presented an algorithm provably succeeding up to $d = 2^k \ln k/k$. However, up to now, no polynomial-time algorithm is known to find satisfying assignments for $d = 2^k f(k)/k$ for any function $f(k) = \omega_k(\ln k)$. For the problem of colouring random graphs, analyses can be found in [AM97, GM75]. They show that a certain list-colouring algorithm finds colourings if $d \leq k \ln k$. Up to now, no polynomial time algorithms are known that are able to colour a random graph with average degree $(1+\varepsilon)\, k \ln k$ for some fixed $\varepsilon > 0$ and arbitrarily large $k$.

In [ACO08] Achlioptas and Coja-Oghlan proved that the point where the geometry of the solution space changes, the dynamical phase transition predicted by statistical physics [KMRTSZ07] and discussed in Section 1.2, coincides with the point where the best known analysed algorithms cease to work. In [ACO08] the “shattering” point for $k$-SAT, $k$-colouring and hypergraph 2-colouring was rigorously determined for large values of $k$. In general, however, no causal relation between the clustering of the solution space and the breakdown of the investigated algorithms could be proven to this day.

Following insights from the cavity method, new “message passing” algorithms for these problems have been developed. These algorithms are supposed to act more “far-sighted” than simpler combinatorial ones. When assigning a variable, instead of basing their decision only on the graph structure at a fixed small distance around this variable, they take into account constraints and variables in a much larger radius. The algorithms are called *Belief/Survey Propagation Guided Decimation* [BMPWZ03, MPZ02] and sequentially fix variables to satisfy the constraints while message passing is run after each step to provide a heuristic for the choice at the next step. During message passing vertices send messages back and forth, updating their belief about their marginals in a sequence of rounds. The key tool in this process is an approximate fixed point computation on a finite random graph. Essentially, the challenge when analysing these algorithms consists in investigating whether the fixed point computation provides a good approximation to the marginals of the Boltzmann distribution (in the case of the Belief Propagation algorithm) or a certain modified distribution (in the case of Survey Propagation).
Experiments on random graph $k$-colouring instances for small values of $k$ indicate an excellent performance [BMPWZ03, Zde09, ZK07]. However, while a comprehensive rigorous analysis remains elusive, sophisticated evidence is given in [RTS09] that for random $k$-SAT Belief Propagation Guided Decimation succeeds for $d = \Theta(2^k/k)$ but not for higher densities (and an analogue is also supposed to hold for the colouring problems). More precisely, the physics prediction is that the performance of Belief Propagation Guided Decimation hinges on the location of the “condensation line” in a two-dimensional phase diagram parametrised by $d$ and a value $t/n$ that measures the progress of the algorithm [RTS09]. This line promises to separate the regime where the algorithm succeeds in approximating the correct marginal distribution from the one where this is not possible. The idea behind it is that in each decimation step clauses are shortened and become more and more difficult to satisfy. In other words, successive decimation of variables has a similar effect as increasing the density of the formula.

However, this is far from being understood rigorously, although there are contributions attempting to analyse message passing algorithms along these lines. For the problem of random $k$-SAT, Coja-Oghlan [CO11] proved that a basic version of Belief Propagation Guided Decimation does not succeed for densities beyond $d = \Theta(2^k/k)$ for large $k$. Very recently, Hetterich [Het16+] proved that a similar basic version of Survey Propagation Guided Decimation cannot overcome the dynamical phase transition, i.e. $d = \Theta(2^k \ln k/k)$ for large $k$ in the limit of large $n$. Yet, it is not obvious how to generalize their results to more involved variants of the algorithms.

Thus, the rigorous understanding of the presented algorithms is still in its early stages and there is an amount of work to do in this field.

# 5 Condensation phase transition in random graph $k$-colouring

This chapter is dedicated to proving Theorem 4.1.1, which establishes the existence and determines the precise location of the condensation phase transition in random graph $k$-colouring for large values of $k$. The result is in terms of a distributional fixed point problem and rigorously verifies the prediction of the cavity method.

Large parts of this chapter are a verbatim copy or a close adaption of the content of the paper *The condensation phase transition in random graph coloring* [BCOHRV16] that is joint work with Victor Bapst, Amin Coja-Oghlan, Samuel Hetterich and Dan Vilenchik and is published in the *Communications in Mathematical Physics 341 (2016)*.

This chapter only presents parts of the proof, namely the parts where the author of this thesis mainly contributed. The other parts can be found in the appendix, Chapter A. The first section of this chapter describes an outline of the proof of Theorem 4.1.1 and gives a short introduction to the proof ideas. In Section 5.2, a first step to the analysis of a certain branching process is presented. Section 5.3 deals with determining the cluster size of a planted colouring in the random graph using Warning Propagation and establishing a connection between the random tree process and the random graph.

## 5.1. Outline of the proof

In this section, we sketch the steps of the proof of Theorem 4.1.1, thereby explaining the main ideas and introducing the most important concepts.

The proof of Theorem 4.1.1 is composed of two parallel threads. The first thread is to show that there exists a density, namely the density $d_{\mathrm{crit}}$ defined in (2.5.3), where a phase transition occurs and statements (i)-(iii) of the theorem are met. The second thread is to identify the frozen fixed point $\pi^*_{d,k}$ of $\mathcal{F}_{d,k}$ and to interpret it combinatorially. Finally, the two threads intertwine to show that $d_{\mathrm{crit}} = d_{\mathrm{cond}}$, i.e. that the "obvious" phase transition $d_{\mathrm{crit}}$ is indeed the unique zero of equation (4.1.5). The first thread is an extension of ideas developed in [COZ12] for random hypergraph 2-colouring to the (technically more involved) random graph colouring problem. The second thread and the intertwining of the two require novel arguments.

### 5.1.1. First thread

We recall the critical density $d_{\text{crit}}$ defined in (2.5.3):

$$d_{\text{crit}} = \inf\left\{d \geq 0 : \text{the limit } \Phi_k(d) \text{ does not exist or } \Phi_k(d) < k(1-1/k)^{d/2}\right\}.$$

We rather directly obtain the following bounds:

**Fact 5.1.1.** *We have* $d_{\text{crit}} \leq (2k-1)\ln k$.

*Proof.* The upper bound on the $k$-colouring threshold stated in (4.2.1) implies that $Z_k(G(n,d/n)) = 0$ w.h.p. for $d > (2k-1)\ln k$. By contrast, $k(1-1/k)^{d/2} > 0$ for any $d > 0$. □

Thus, $d_{\text{crit}}$ is a well-defined finite number, and there occurs a phase transition at $d_{\text{crit}}$. Moreover, the following proposition yields a lower bound on $d_{\text{crit}}$ and implies that $d_{\text{crit}}$ satisfies the first condition in Theorem 4.1.1. The proposition will be proven in Section A.1 via calculating the first moment of $Z_k$ and second moment of $Z_{k,\text{tame}}$, which is a random variable only counting separable $k$-colourings that have an appropriately bounded cluster size.

**Proposition 5.1.2.** *For any* $d > 0$*, we have*

$$\limsup_{n\to\infty} \mathbb{E}[Z_k(G(n,d/n))^{1/n}] \leq k(1-1/k)^{d/2}.$$

*Moreover,*

$$d_{\text{crit}} = \sup\left\{d \geq 0 : \liminf_{n\to\infty} \mathbb{E}[Z_k(G(n,d/n))^{1/n}] \geq k(1-1/k)^{d/2}\right\} \geq (2k-1)\ln k - 2.$$

Thus, we know that there *exists* a number $d_{\text{crit}}$ that satisfies conditions (i) and (ii) in Theorem 4.1.1. Of course, to actually calculate this number we need to unearth its combinatorial "meaning". As we saw in Section 1.2, if $d_{\text{crit}}$ really is the condensation phase transition, then the combinatorial interpretation should be as follows. For $d < d_{\text{crit}}$, the size of the cluster that a randomly chosen $k$-colouring $\boldsymbol{\tau}$ belongs to is smaller than $Z_k(G(n,d/n))$ by an exponential factor $\exp[\Omega(n)]$ w.h.p.. But as $d$ approaches $d_{\text{crit}}$, the gap between the cluster size and $Z_k(G(n,d/n))$ diminishes. Hence, $d_{\text{crit}}$ should mark the point where the cluster size has the same order of magnitude as $Z_k(G(n,d/n))$.

But how can we possibly get a handle on the size of the cluster that a randomly chosen $k$-colouring $\boldsymbol{\tau}$ of $G(n,d/n)$ belongs to? As explained in Section 3.1, no "constructive" method is known for obtaining a single $k$-colouring of $G(n,d/n)$ for $d$ anywhere close to $d_{\text{col}}$, let alone for sampling one uniformly at random. Nevertheless, in the case that $\Phi_k(d) = k(1-1/k)^{d/2}$, i.e. for $d < d_{\text{crit}}$, the experiment of first choosing the random graph $G(n, d/n)$ and then sampling a $k$-colouring $\boldsymbol{\tau}$ uniformly at random can be captured by the planted model: We *first* choose a map $\boldsymbol{\sigma} : [n] \to [k]$ uniformly at random, then we generate a graph $G(n, p', \boldsymbol{\sigma})$ on $[n]$ by connecting any two vertices $v, w \in [n]$ such that $\boldsymbol{\sigma}(v) \neq \boldsymbol{\sigma}(w)$ with probability $p'$ independently. If $p' = dk/(n(k-1))$ is chosen so that the expected number of edges is the same as in $G(n, d/n)$ and if $\Phi_k(d) = k(1-1/k)^{d/2}$, then the planted model should be a good approximation to the random colouring model. In particular, with respect to the cluster size we expect that

$$\mathbb{E}[|\mathcal{C}(G(n, p', \boldsymbol{\sigma}), \boldsymbol{\sigma})|^{1/n}] \sim \mathbb{E}[|\mathcal{C}(G(n, d/n), \boldsymbol{\tau})|^{1/n}],$$

i.e. that the suitably scaled cluster size in the planted model is about the same as the cluster size in $G(n, d/n)$. Hence, $d_{\text{crit}}$ should mark the point where $\mathbb{E}[|\mathcal{C}(G(n, p', \boldsymbol{\sigma}), \boldsymbol{\sigma})|^{1/n}]$ equals $k(1-1/k)^{d/2}$. The following proposition verifies that this is indeed so. Let us write $\boldsymbol{G} = G(n, p', \boldsymbol{\sigma})$ for the sake of brevity.

**Proposition 5.1.3.** *Assume that* $(2k-1)\ln k - 2 \leq d \leq (2k-1)\ln k$ *and set*

$$p' = d'/n \quad \textit{with } d' = \frac{dk}{k-1}. \tag{5.1.1}$$

1. *If*
$$\lim_{\varepsilon \searrow 0} \liminf_{n\to\infty} \mathbb{P}\left[|\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})|^{1/n} \leq k(1-1/k)^{d/2} - \varepsilon\right] = 1, \tag{5.1.2}$$
*then* $d \leq d_{\text{crit}}$.
2. *Conversely, if*
$$\lim_{\varepsilon \searrow 0} \liminf_{n\to\infty} \mathbb{P}\left[|\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})|^{1/n} \geq k(1-1/k)^{d/2} + \varepsilon\right] = 1, \tag{5.1.3}$$
*then* $\limsup_{n\to\infty} \mathbb{E}[Z_k(G(n, d/n))^{1/n}] < k(1-1/k)^{d/2}$. *In particular,* $d \geq d_{\text{crit}}$.

To show the first part of Proposition 5.1.3 we observe that below $d_{\text{crit}}$ a typical $k$-colouring is separable and has bounded cluster size and use this to bound $\mathbb{E}[Z_k]$ from below. The second part of the proof cleverly uses a variant of the "planting trick" argument from [ACO08] combined with temporarily introducing a finite temperature parameter in order to use concentration of the partition function. The proof can be found in Section A.2.

### 5.1.2. Second thread

Our next aim is to "solve" the fixed point problem for the map $\mathcal{F}_{d,k}$ to an extent that gives the fixed point an explicit combinatorial interpretation. This combinatorial interpretation is in terms of a certain random tree process, associated with a concept of "legal colourings". Specifically, we consider a multi-type Galton-Watson branching process. Its set of types is

$$\mathcal{T} = \{(i, \ell) : i \in [k], \ell \subset [k], i \in \ell\}.$$

The intuition is that $i$ is a "distinguished colour" and that $\ell$ is a set of "available colours". The branching process is further parameterized by a vector $\boldsymbol{q} = (q_1, \ldots, q_k) \in [0,1]^k$ such that $q_1 + \cdots + q_k \leq 1$. Let $d' = dk/(k-1)$ and

$$q_{i,\ell} = \frac{1}{k} \prod_{j \in \ell \setminus \{i\}} \exp\left[-q_j d'\right] \prod_{j \in [k] \setminus \ell} \left(1 - \exp\left[-q_j d'\right]\right) \qquad \text{for } (i, \ell) \in \mathcal{T}.$$

Then

$$\sum_{(i,\ell) \in \mathcal{T}} q_{i,\ell} = 1.$$

Further, for each $(i, \ell) \in \mathcal{T}$ such that $|\ell| > 1$, we define $\mathcal{T}_{i,\ell}$ as the set of all $(i', \ell') \in \mathcal{T}$ such that $\ell \cap \ell' \neq \emptyset$ and $|\ell'| > 1$. In addition, for $(i, \ell) \in \mathcal{T}$ such that $|\ell| = 1$ we set $\mathcal{T}_{i,\ell} = \emptyset$.

The branching process $\mathrm{GW}(d, k, \boldsymbol{q})$ starts with a single individual, whose type $(i, \ell) \in \mathcal{T}$ is chosen from the probability distribution $(q_{i,\ell})_{(i,\ell) \in \mathcal{T}}$. In the course of the process, each individual of type $(i, \ell) \in \mathcal{T}$ spawns a Poisson number $\mathrm{Po}(d' q_{i',\ell'})$ of offspring of type $(i', \ell')$ for each $(i', \ell') \in \mathcal{T}_{i,\ell}$. In particular, only the initial individual may have a type $(i, \ell)$ with $|\ell| = 1$, in which case it does not have any offspring. Let $1 \leq \mathcal{N} \leq \infty$ be the progeny of the process (i.e. the total number of individuals created).

We are going to view $\mathrm{GW}(d, k, \boldsymbol{q})$ as a distribution over trees endowed with some extra information. Let us define a *decorated graph* as a graph $T = (V, E)$ together with a map $\vartheta : V \to \mathcal{T}$ such that for each edge $e = \{v, w\} \in E$ we have $\vartheta(w) \in \mathcal{T}_{\vartheta(v)}$. Moreover, a *rooted decorated graph* is a decorated graph $(T, \vartheta)$ together with a distinguished vertex $v_0$, the *root*. Further, an *isomorphism* between two rooted decorated graphs $T$ and $T'$ is an isomorphism of the underlying graphs that preserves the root and the types of the vertices.

Given that $\mathcal{N} < \infty$, the branching process $\mathrm{GW}(d, k, \boldsymbol{q})$ canonically induces a probability distribution over isomorphism classes of rooted decorated trees. Indeed, we obtain a tree whose vertices are all the individuals created in the course of the branching process and where there is an edge between each individual and its offspring. The individual from which the process starts is the root. Moreover, by construction each individual $v$ comes with a type $\vartheta(v)$. We denote the (random) isomorphism class of this tree by $\boldsymbol{T}_{d,k,\boldsymbol{q}}$. (It is most natural to view the branching process as a probability distribution over *isomorphism classes* as the process does not specify the order in which offspring is created.)

To proceed, we define a *legal colouring* of a decorated graph $(G, \vartheta)$ as a map $\tau : V(G) \to [k]$ such that $\tau$ is a $k$-colouring of $G$ and such that for any type $(i, \ell) \in \mathcal{T}$ and for any vertex $v$ with $\vartheta(v) = (i, \ell)$

we have $\tau(v) \in \ell$. Let $\mathcal{Z}(G, \vartheta)$ denote the number of legal colourings.

Since $\mathcal{Z}(G, \vartheta)$ is isomorphism-invariant, we obtain the integer-valued random variable $\mathcal{Z}(\boldsymbol{T}_{d,k,\boldsymbol{q}})$. We have $\mathcal{Z}(\boldsymbol{T}_{d,k,\boldsymbol{q}}) \geq 1$ with certainty because a legal colouring $\tau$ can be constructed by colouring each vertex with its distinguished colour (i.e. setting $\tau(v) = i$ if $v$ has type $(i, \ell)$). Hence, $\ln \mathcal{Z}(\boldsymbol{T}_{d,k,\boldsymbol{q}})$ is a well-defined non-negative random variable. Additionally, we write $|\boldsymbol{T}_{d,k,\boldsymbol{q}}|$ for the number of vertices in $\boldsymbol{T}_{d,k,\boldsymbol{q}}$.

Finally, consider a rooted, decorated tree $(T, \vartheta, v_0)$ and let $\boldsymbol{\tau}$ be a legal colouring of $(T, \vartheta, v_0)$ chosen uniformly at random. Then the colour $\boldsymbol{\tau}(v_0)$ of the root is a random variable with values in $[k]$. Let $\mu_{T,\vartheta,v_0} \in \Omega$ denote its distribution. Clearly, $\mu_{T,\vartheta,v_0}$ is invariant under isomorphisms. Consequently, the distribution $\mu_{\boldsymbol{T}_{d,k,\boldsymbol{q}}}$ of the colour of the root of a tree in the random isomorphism class $\boldsymbol{T}_{d,k,\boldsymbol{q}}$ is a well-defined $\Omega$-valued random variable. Let $\pi_{d,k,\boldsymbol{q}} \in \mathcal{P}$ denote its distribution. Then we can characterise the frozen fixed point of $\mathcal{F}_{d,k}$ as follows.

**Proposition 5.1.4.** *Suppose that $d \geq (2k-1)\ln k - 2$.*

1. *The function*
$$q \in [0,1] \mapsto (1 - \exp[-dq/(k-1)])^{k-1} \tag{5.1.4}$$
*has a unique fixed point $q^*$ in the interval $[2/3, 1]$. Moreover, with*
$$\boldsymbol{q}^* = k^{-1}(q^*, \ldots, q^*) \in [0,1]^k \tag{5.1.5}$$
*the branching process $\mathrm{GW}(d, k, \boldsymbol{q}^*)$ is sub-critical. Thus, $\mathbb{P}[\mathcal{N} < \infty] = 1$.*
2. *The map $\mathcal{F}_{d,k}$ has precisely one frozen fixed point, namely $\pi_{d,k,\boldsymbol{q}^*}$.*
3. *We have $\phi_{d,k}(\pi_{d,k,\boldsymbol{q}^*}) = \mathbb{E}\left[\frac{\ln \mathcal{Z}(\boldsymbol{T}_{d,k,\boldsymbol{q}^*})}{|\boldsymbol{T}_{d,k,\boldsymbol{q}^*}|}\right]$.*
4. *The function $\Sigma_k$ from (4.1.5) is strictly decreasing and continuous on $[(2k-1)\ln k - 2, (2k-1)\ln k - 1]$ and has a unique zero $d_{\mathrm{cond}}$ in this interval.*

The function (5.1.4) and its fixed point explicitly occur in the physics work [ZK07]. The proof of Proposition 5.1.4 incorporates an analysis of the Galton Watson process GW and of the fixed points of $\mathcal{F}_{d,k}$. The main work consists in showing that indeed $\mathcal{F}_{d,k}$ has exactly one frozen fixed point $\pi_{d,k,\boldsymbol{q}^*}$ and that the Bethe free entropy $\phi_{d,k}$ evaluated at this fixed point is related to the number of legal colourings of $\boldsymbol{T}_{d,k,\boldsymbol{q}^*}$. The proof of Proposition 5.1.4 can be found in Sections 5.2 and A.3.

### 5.1.3. Tying up the threads

To prove that $d_{\mathrm{cond}} = d_{\mathrm{crit}}$, we establish a connection between the random tree $\boldsymbol{T}_{d,k,\boldsymbol{q}^*}$ and the random graph $\boldsymbol{G}$ with planted colouring $\boldsymbol{\sigma}$. We start by giving a recipe for computing the cluster size

$|\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})|$, and then show that the random tree process "cooks" it.

Computing the cluster size hinges on a close understanding of its combinatorial structure. As hypothesised in physics work [MM09] and established rigorously in [ACO08, CO13, Mol12], typically many vertices $v$ are "frozen" in $\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})$, i.e. $\tau(v) = \tau'(v)$ for any two colourings $\tau, \tau' \in \mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})$. More generally, we consider for each vertex $v$ the set

$$\ell(v) = \{\tau(v) : \tau \in \mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})\}$$

of colours that $v$ may take in colourings $\tau$ that belong to the cluster. Together with the "planted" colour $\boldsymbol{\sigma}(v)$, we can thus assign each vertex $v$ a type $\vartheta(v) = (\boldsymbol{\sigma}(v), \ell(v))$. This turns $\boldsymbol{G}$ into a decorated graph $(\boldsymbol{G}, \vartheta)$.

By construction, each colouring $\tau \in \mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})$ is a legal colouring of the decorated graph $\boldsymbol{G}$. Conversely, we will see that w.h.p. any legal colouring of $(\boldsymbol{G}, \vartheta)$ belongs to the cluster $\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})$. Hence, computing the cluster size $|\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})|$ amounts to calculating the number $\mathcal{Z}(\boldsymbol{G}, \vartheta)$ of legal colourings of $(\boldsymbol{G}, \vartheta)$.

This calculation is facilitated by the following observation. Let $\widetilde{\boldsymbol{G}}$ be the graph obtained from $\boldsymbol{G}$ by deleting all edges $e = \{v, w\}$ that join two vertices such that $\ell(v) \cap \ell(w) = \emptyset$. Then any legal colouring $\tau$ of $\widetilde{\boldsymbol{G}}$ is a legal colouring of $\boldsymbol{G}$, because $\tau(v) \in \ell(v)$ for any vertex $v$. Hence, $\mathcal{Z}(\boldsymbol{G}, \vartheta) = \mathcal{Z}(\widetilde{\boldsymbol{G}}, \vartheta)$.

Thus, we just need to compute $\mathcal{Z}(\widetilde{\boldsymbol{G}}, \vartheta)$. This task is much easier than computing $\mathcal{Z}(\boldsymbol{G}, \vartheta)$ directly because $\widetilde{\boldsymbol{G}}$ turns out to have *significantly* fewer edges than $\boldsymbol{G}$ w.h.p.. More precisely, w.h.p. $\widetilde{\boldsymbol{G}}$ (mostly) consists of connected components that are trees of bounded size. In fact, we shall see that in an appropriate sense the distribution of the tree components converges to that of the decorated random tree $\boldsymbol{T}_{d,k,\boldsymbol{q}_*}$. In effect, we obtain

**Proposition 5.1.5.** *Suppose that $d \geq (2k-1)\ln k - 2$ and let $p'$ be as in (5.1.1). Let $\mathbf{q}^*$ be as in (5.1.5). Then the sequence $\{\frac{1}{n}\ln|\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})|\}_n$ converges to $\mathbb{E}\left[\frac{\ln \mathcal{Z}(\boldsymbol{T}_{d,k,\boldsymbol{q}^*})}{|\boldsymbol{T}_{d,k,\boldsymbol{q}^*}|}\right]$ in probability.*

The proof of Proposition 5.1.5, that connects the geometry of the set of $k$-colourings rigorously with the distributional fixed point problem, is based on the precise analysis of a further, combinatorial fixed point problem called *Warning Propagation*. It can be found in all details in Section 5.3.

*Proof of Theorem 4.1.1.* Combining Propositions 5.1.3 and 5.1.5, we see that $d_{\text{crit}}$ is equal to $d_{\text{cond}}$, which is well-defined by Proposition 5.1.4. Further, (2.5.2) implies that $d_{\text{crit}} > 0$. Assume for contradiction that $d_{\text{crit}}$ is smooth. Then there is $\varepsilon > 0$ such that the limit $\Phi_k(d)$ exists for all $d \in (d_{\text{crit}} - \varepsilon, d_{\text{crit}} + \varepsilon)$ and such that the function $d \mapsto \Phi_k(d)$ is given by an absolutely convergent power series on this interval. Moreover, Proposition 5.1.2 implies that $\Phi_k(d) = k(1 - 1/k)^{d/2}$

for all $d \in (d_{\text{crit}} - \varepsilon, d_{\text{crit}})$. Consequently, the uniqueness of analytic continuations implies that $\Phi_k(d) = k(1 - 1/k)^{d/2}$ for all $d \in (d_{\text{crit}} - \varepsilon, d_{\text{crit}} + \varepsilon)$, in contradiction to the definition of $d_{\text{crit}}$. Thus, $d_{\text{crit}}$ is a phase transition. □

*Proof of Corollary 4.1.3.* Corollary 4.1.3 follows rather easily from the above and the following lemma establishing a connection between the planted model and the Boltzmann distribution on $G(n, d/n)$. As in Corollary 4.1.3, we let $\boldsymbol{\tau}$ denote a random $k$-colouring of $G(n, d/n)$.

**Lemma 5.1.6** ([BCOE14+]). *Assume that $d < d_{\text{cond}}$. Let $\mathcal{E}$ be a set of pairs $(G, \sigma)$, where $G$ is a graph and $\sigma$ is a $k$-colouring of $G$. Further, given that $Z_k(G(n, d/n)) > 0$, let $\boldsymbol{\tau}$ be a uniformly random $k$-colouring of $G(n, d/n)$.*
*Then $\mathbb{P}[(\boldsymbol{G}, \boldsymbol{\sigma}) \in \mathcal{E}] = o(1)$ implies that $\mathbb{P}[(G(n, d/n), \boldsymbol{\tau}) \in \mathcal{E} | Z_k(G(n, d/n)) > 0] = o(1)$.*

The statements about the properties of the function $\Sigma_k$ follow readily from Proposition 5.1.4. Now, assume that $d \in ((2k - 1)\ln k - 2, d_{\text{cond}})$. Propositions 5.1.4 and 5.1.5 show that $\frac{1}{n} \ln |\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})|$ converges to $\phi_{d,k}(\pi_{d,k,\boldsymbol{q}^*})$ in probability. Hence, Markov's inequality shows that for any $\varepsilon > 0$,

$$\mathbb{P}\left[\frac{1}{n} \ln |\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})| > \phi_{d,k}(\pi_{d,k,\boldsymbol{q}^*}) + \varepsilon\right] = o(1). \tag{5.1.6}$$

In combination with Lemma 5.1.6, (5.1.6) entails that

$$\mathbb{P}\left[\frac{1}{n} \ln |\mathcal{C}(G(n, d/n), \boldsymbol{\tau})| > \phi_{d,k}(\pi_{d,k,\boldsymbol{q}^*}) + \varepsilon | Z_k(G(n, d/n) > 0\right] = o(1). \tag{5.1.7}$$

Further, Propositions 5.1.4 and 5.1.5 imply that for any fixed $\varepsilon > 0$,

$$\mathbb{P}\left[\frac{1}{n} \ln |\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})| < \phi_{d,k}(\pi_{d,k,\boldsymbol{q}^*}) - \varepsilon\right] = o(1).$$

Hence, Lemma 5.1.6 yields

$$\mathbb{P}\left[\frac{1}{n} \ln |\mathcal{C}(G(n, d/n), \boldsymbol{\tau})| \geq \phi_{d,k}(\pi_{d,k,\boldsymbol{q}^*}) - \varepsilon\right] = \Omega(1). \tag{5.1.8}$$

Thus, Corollary 4.1.3 follows from (5.1.7), (5.1.8) and the fact that $Z_k(G(n, d/n))^{1/n}$ converges to $\Phi_k(d) = k(1 - 1/k)^{d/2}$ in probability. □

## 5.2. The fixed point problem

### 5.2.1. The branching process

Throughout this section we assume that $(2k-1)\ln k - 2 \le d \le (2k-1)\ln k$. Moreover, we recall that $d' = kd/(k-1)$.

**Lemma 5.2.1.** *Suppose that $d \ge (2k-1)\ln k - 2$.*

1. *The function*

$$F_{d,k} : [0,1]^k \to [0,1]^k, \qquad (q_1,\dots,q_k) \mapsto \left(\frac{1}{k}\prod_{j\in[k]\setminus\{i\}} 1-\exp\left(-d'q_j\right)\right)_{i\in[k]} \tag{5.2.1}$$

   *has a unique fixed point $\boldsymbol{q}^* = (q_1^*,\dots,q_k^*)$ such that $\sum_{j\in[k]} q_j^* \ge 2/3$. This fixed point has the property that $q_1^* = \dots = q_k^*$. Moreover, $q^* = kq_1^*$ is the unique fixed point of the function (5.1.4) in the interval $[2/3,1]$, and $q^* = 1 - O_k(1/k)$.*
2. *The branching process $\mathrm{GW}(d,k,\boldsymbol{q}^*)$ is sub-critical.*
3. *Furthermore, $\frac{\partial}{\partial d}\mathbb{E}\left[\frac{\ln \mathcal{Z}(\boldsymbol{T}_{d,k}(\boldsymbol{q}^*))}{|\boldsymbol{T}_{d,k}(\boldsymbol{q}^*)|}\right] = \tilde{O}_k(k^{-2})$.*

The proof of Lemma 5.2.1 requires several steps. We begin by studying the fixed points of $F_{d,k}$.

**Lemma 5.2.2.** *The function $F_{d,k}$ maps the compact set $[\frac{2}{3k},\frac{1}{k}]^k$ into itself and has a unique fixed point $\boldsymbol{q}^*$ in this set. Moreover, the function from (5.1.4) has a unique fixed point $q^*$ in the set $[2/3,1]$ and $\boldsymbol{q}^* = (q^*/k,\dots,q^*/k)$. Furthermore,*

$$q^* = 1 - 1/k + o_k(1/k). \tag{5.2.2}$$

*In addition, if $\boldsymbol{q} \in [0,1]^k$ is a fixed point of $F_{d,k}$, then*

$$q_1 = \dots = q_k. \tag{5.2.3}$$

*Proof.* Let $I = [\frac{2}{3k},\frac{1}{k}]^k$. As a first step, we show that $F_{d,k}(I) \subset I$. Indeed, let $\boldsymbol{q} \in I$. Then for any $i \in [k]$

$$(F_{d,k}(\boldsymbol{q}))_i = \frac{1}{k}\prod_{j\neq i} 1-\exp(-d'q_j) \le \frac{1}{k}.$$

On the other hand, as $d \geq (2k-1)\ln k - 2$ we see that $d' \geq 1.99k\ln k$. Hence,

$$(F_{d,k}(\boldsymbol{q}))_i = \frac{1}{k}\prod_{j\neq i} 1-\exp(-d'q_j) \geq \frac{1}{k}\left(1-\exp\left(-\frac{2d'}{3k}\right)\right)^{k-1}$$
$$\geq \frac{1}{k}(1-k^{-1.1})^k = \frac{1-o_k(1)}{k}.$$

Thus, $F_{d,k}(I) \subset I$.

In addition, we claim that $F_{d,k}$ is contracting on $I$. In fact, as $d' \geq 1.99k\ln k$ and $q_l \geq 2/3$ for all $l$, for any $i,j \in [k]$ we have

$$\frac{\partial}{\partial q_j}(F_{d,k}(\boldsymbol{q}))_i = \frac{\mathbf{1}_{i\neq j}}{k}\frac{\partial}{\partial q_j}\prod_{l\neq i}1-\exp(-d'q_l) = \frac{\mathbf{1}_{i\neq j}d'}{k\exp(d'q_j)}\cdot\prod_{l\neq i,j}1-\exp(-d'q_l)$$
$$= (1+o_k(1))\frac{\mathbf{1}_{i\neq j}d'}{k\exp(d'q_j)} \leq k^{-1.3}.$$

Therefore, for $\boldsymbol{q} \in I$ the Jacobi matrix $DF_{d,k}(\boldsymbol{q})$ satisfies

$$\|DF_{d,k}(\boldsymbol{q})\|^2 \quad \leq \quad \sum_{i,j\in[k]}\left(\frac{\partial}{\partial q_j}(F_{d,k}(\boldsymbol{q}))_i\right)^2 \leq k^2\cdot k^{-2.6} < 1.$$

Thus, $F_{d,k}$ is a contraction on the compact set $I$. Consequently, Banach's fixed point theorem implies that there is a unique fixed point $\boldsymbol{q}_* \in I$.

To establish (5.2.3), assume without loss that $\boldsymbol{q} = (q_1,\ldots,q_k) \in [0,1]^k$ is a fixed point such that $q_1 \leq \cdots \leq q_k$. For the trivial fixed point $q_1 = \ldots = q_k = 0$, the equation (5.2.3) obviously holds. So we assume $q_1 > 0$. Because $\boldsymbol{q}$ is a fixed point and as $q_1 \leq q_k$, we find that

$$\frac{q_k}{q_1} = \frac{(F_{d,k}(\boldsymbol{q}))_k}{(F_{d,k}(\boldsymbol{q}))_1} = \frac{1-\exp(-d'q_1)}{1-\exp(-d'q_k)} \leq 1,$$

whence (5.2.3) follows.

Further, we claim that the function $f_{d,k} : [0,1] \to [0,1]$, $q \mapsto (1-\exp(-dq/(k-1)))^{k-1}$ maps the interval $[2/3,1]$ into itself. This is because for $q \in [2/3,1]$ we have $0 \leq \exp(-dq/(k-1)) \leq k^{-1.3}$ due to our assumption on $d$. Moreover, the derivative of $f_{d,k}$ works out to be $f'_{d,k}(q) = d\exp(-dq/(k-1))(1-\exp(-dq/(k-1)))^{k-2}$. Thus, for $q \in [2/3,1]$ we find $0 \leq f'_{d,k}(q) < 1/2$. Hence, $f_{d,k}$ has a unique fixed point $q_* \in [2/3,1]$. Comparing the expressions $f_{d,k}(q)$ and $F_{d,k}(\boldsymbol{q})$, we see that $(q_*/k,\ldots,q_*/k)$ is a fixed point of $F_{d,k}$. Consequently, $\boldsymbol{q}_* = (q_*/k,\ldots,q_*/k)$.

Finally, since $f'_{d,k}(q) > 0$ for all $q$, the function $f_{d,k}$ is strictly increasing. Therefore, as $d = (2 -$

$o_k(1))k\ln k$,

$$q_* = f_{d,k}(q_*) \le f_{d,k}(1) = (1-\exp(-d/(k-1)))^{k-1} = 1 - 1/k + o_k(1/k). \tag{5.2.4}$$

Similarly, $q_* \ge f_{d,k}(2/3) \ge 1 - k^{-0.3}$. Hence, because $d \ge (2k-1)\ln k - 3$, we obtain

$$\begin{aligned} q_* &= f_{d,k}(q_*) \ge f_{d,k}(1-k^{-0.3}) = \left(1 - \exp\left[-\frac{d(1-k^{-0.3})}{k-1}\right]\right)^{k-1} \\ &= \left(1 - k^{-2} + O_k(k^{-2.1})\right)^{k-1} = 1 - 1/k + o_k(1/k). \end{aligned} \tag{5.2.5}$$

Combining (5.2.4) and (5.2.5), we conclude that $q_* = 1 - 1/k + o_k(1/k)$, as claimed. $\square$

**Remark 5.2.3.** *The proof of Lemma 5.2.2 directly incorporate parts of the calculations outlined in the physics work [ZK07] that predicted the existence and location of* $d_{\mathrm{cond}}$. *We redo these calculations here in detail to be self-contained and because not all steps are carried out in full detail in [ZK07].*

From here on out, we let $\boldsymbol{q}^*$ denote the fixed point of $F_{d,k}$ in $[2/(3k), 1/k]^k$ and we denote the fixed point of the function (5.1.4) in the interval $[2/3, 1]$ by $q^*$. Hence, $\boldsymbol{q}^* = (q^*/k, \dots, q^*/k)$. If we keep $k$ fixed, how does $q^*$ vary with $d$?

**Corollary 5.2.4.** *We have* $\frac{\mathrm{d}q^*}{\mathrm{d}d} = \Theta_k\left(k^{-2}\right)$.

*Proof.* The map $d \mapsto q^*$ is differentiable by the implicit function theorem. Moreover, differentiating (5.1.4) while keeping in mind that $q^* = q^*(d)$ is a fixed point, we find

$$\begin{aligned} \frac{\mathrm{d}q^*}{\mathrm{d}d} &= \frac{\mathrm{d}}{\mathrm{d}d}\left(1-\exp(-dq^*/(k-1))\right)^{k-1} \\ &= \frac{(k-1)\left(1-\exp\left(-dq^*/(k-1)\right)\right)^{k-2}}{\exp\left(dq^*/(k-1)\right)} \cdot \left(\frac{q^*}{k-1} + \frac{d}{k-1}\frac{\mathrm{d}q^*}{\mathrm{d}d}\right). \end{aligned}$$

Rearranging the above using $d = 2k\ln k + O_k(\ln k)$ and (5.2.2) yields the assertion. $\square$

**Corollary 5.2.5.** *We have* $q^*_{i,\ell} = \tilde{\Theta}_k\left(k^{-(2|\ell|-1)}\right)$ *for all* $(i,\ell) \in \mathcal{T}$. *Moreover,* $\frac{\mathrm{d}q^*_{i,\ell}}{\mathrm{d}d} = \tilde{O}_k\left(|\ell|k^{-2|\ell|}\right)$.

*Proof.* Lemma 5.2.2 shows that $q^*_j = q_*/k$ for all $j \in [k]$. Hence, due to (5.2.2) and because $d' = 2k\ln k + O_k(\ln k)$ we obtain

$$q^*_{i,\ell} = \frac{1}{k}\prod_{j\in[k]\setminus\ell} 1 - \exp\left(-d'q^*_j\right) \prod_{j\in\ell\setminus\{i\}} \exp\left(-d'q^*_j\right) = \tilde{\Theta}_k(k^{-(2|\ell|-1)}).$$

Furthermore, applying Corollary 5.2.4, we get

$$\begin{aligned}
\frac{\mathrm{d}q^*_{i,\ell}}{\mathrm{d}d} &= \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}d}\left[\prod_{j\in[k]\setminus\ell} 1-\exp\left(-d'q^*_j\right)\prod_{j\in\ell\setminus\{i\}}\exp\left(-d'q^*_j\right)\right] \\
&= \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}d}\left[(1-\exp\left(-d'q_*/k\right))^{k-|\ell|}\exp\left(-d'q_*/k\right)^{|\ell|-1}\right] \\
&= \frac{1}{k}\left(\frac{q_*}{k-1}+\frac{d'}{k}\frac{\mathrm{d}q_*}{\mathrm{d}d}\right)\left[\frac{k-|\ell|}{\exp(d'q_*/k)}(1-\exp(-d'q_*/k))^{k-|\ell|-1}\right. \\
&\qquad \left. -(|\ell|-1)(1-\exp\left(-d'q_*/k\right))^{k-|\ell|}\right]\exp\left(-d'(|\ell|-1)q_*/k\right) \\
&= |\ell|O_k(k^{-2})\exp\left(-d'(|\ell|-1)q_*/k\right) = \tilde{O}_k(|\ell|k^{-2|\ell|}).
\end{aligned}$$

□

**Lemma 5.2.6.** *The branching process* $\mathrm{GW}(d,k,\boldsymbol{q}_*)$ *is sub-critical.*

*Proof.* We introduce another branching process $\mathrm{GW}'(d,k,\boldsymbol{q}^*)$ with only three types $1,2,3$. The idea is that type 1 of the new process represents all types $(h,\{h\})\in\mathcal{T}$ with $h\in[k]$, that 2 represents all types $(h,\{j,h\})\in T$ with $h,j\in[k]$, $j\neq h$, and that 3 lumps together all of the remaining types. More specifically, in $\mathrm{GW}'(d,k,\boldsymbol{q}^*)$ an individual of type $i$ spawns a Poisson number $\mathrm{Po}(M_{ij})$ of offspring of type $j$ ($i,j\in\{1,2,3\}$), where $M=(M_{ij})$ is the following matrix: If either $i=1$ or $j=1$, then $M_{ij}=0$. Moreover,

$$M_{22} = \sum_{(i,\ell)\in\mathcal{T}_{(1,\{1,2\})}:|\ell|=2} q^*_{i,\ell}d' \qquad M_{23} = \sum_{(i,\ell)\in\mathcal{T}_{(1,\{1,2\})}:|\ell|>2} q^*_{i,\ell}d',$$

$$M_{32} = \sum_{(i,\ell)\in\mathcal{T}_{(1,[k])}:|\ell|=2} q^*_{i,\ell}d', \qquad M_{33} = \sum_{(i,\ell)\in\mathcal{T}_{(1,[k])}:|\ell|>2} q^*_{i,\ell}d'.$$

Due to the symmetry of the fixed point $\boldsymbol{q}^*$ (i.e. $\boldsymbol{q}^*=(q^*/k,\dots,q^*/k)$), $M_{22}$ is precisely the expected number of offspring of type $(i,\ell)$ with $|\ell|=2$ that an individual of type $(i_0,\ell_0)\in\mathcal{T}$ with $|\ell_0|=2$ spawns in the branching process $\mathrm{GW}(d,k,\boldsymbol{q}^*)$. Similarly, $M_{23}$ is just the expected offspring of type $(i,\ell)$ with $|\ell|>2$ of an individual with $|\ell_0|=2$. Furthermore, $M_{32}$ is an upper bound on the expected offspring of type $(i',\ell')$ with $|\ell'|=2$ of an individual of type $(i_0,\ell_0)$ with $|\ell_0|>2$. Indeed, $M_{32}$ is the expected offspring in the case that $\ell_0=[k]$, which is the case that yields the largest possible expectation. Similarly, $M_{33}$ is an upper bound on the expected offspring of type $(i',\ell')$ with $|\ell'|>2$ in the case $|\ell_0|>2$. Therefore, if $\mathrm{GW}'(d,k,\boldsymbol{q}^*)$ is sub-critical, then so is $\mathrm{GW}(d,k,\boldsymbol{q}^*)$.

To show that $\mathrm{GW}(d,k,\boldsymbol{q}^*)$ is sub-critical, we need to estimate the entries $M_{ij}$. Estimating the $q^*_{i,\ell}$ via

Corollary 5.2.5, we obtain

$$M_{22} \leq 2kq^*_{1,\{1,2\}}d' \leq \tilde{O}_k(k^{-1}), \qquad M_{23} \leq 2\sum_{l\geq 3} l\binom{k}{l-1}q^*_{1,[l]}d' \leq \tilde{O}_k(k^{-2}),$$

$$M_{32} \leq k(k-1)q^*_{1,\{1,2\}}d' \leq \tilde{O}_k(1), \qquad M_{33} \leq k\sum_{l\geq 3} l\binom{k}{l-1}q^*_{1,[l]}d' \leq \tilde{O}_k(k^{-1}).$$

The branching process $\mathrm{GW}'(d,k,\boldsymbol{q}^*)$ is sub-critical if and only if all eigenvalues of $M$ are less than 1 in absolute value. Because the first row and column of $M$ are 0, this is the case if and only if the eigenvalues of the $2\times 2$ matrix $M_* = (M_{ij})_{2\leq i,j\leq 3}$ are less than 1 in absolute value. Indeed, since the above estimates show that $M_*$ has trace $\tilde{O}_k(k^{-1})$ and determinant $\tilde{O}_k(k^{-2})$, both eigenvalues of $M_*$ are $\tilde{O}_k(k^{-1})$. □

**Lemma 5.2.7.** *We have* $\frac{\mathrm{d}}{\mathrm{d}d}\mathbb{E}[|\boldsymbol{T}_{d,k,\boldsymbol{q}^*}|^{-1}\ln \mathcal{Z}(\boldsymbol{T}_{d,k,\boldsymbol{q}^*})] = \tilde{O}_k(k^{-2})$.

*Proof.* Fix a number $d \in [(2k-1)\ln k - 2, (2k-1)\ln k]$ and a small number $\varepsilon > 0$ and let $\hat{d} = d+\varepsilon$. Let $\boldsymbol{q}^*$ be the unique fixed point of $F_{d,k}$ in $[2/(3k), 1/k]^k$ and let $\hat{\boldsymbol{q}}^*$ be the unique fixed point of $F_{\hat{d},k}$ in $[2/(3k), 1/k]^k$. Set $d' = dk/(k-1)$ and $\hat{d}' = \hat{d}k/(k-1)$. Moreover, let us introduce the shorthands $\boldsymbol{T} = \boldsymbol{T}_{d,k,\boldsymbol{q}^*}$ and $\hat{\boldsymbol{T}} = \boldsymbol{T}_{\hat{d},k,\hat{\boldsymbol{q}}^*}$. We aim to bound

$$\Delta = \left| \mathbb{E}\left[\frac{\ln \mathcal{Z}(\boldsymbol{T})}{|\boldsymbol{T}|}\right] - \mathbb{E}\left[\frac{\ln \mathcal{Z}(\hat{\boldsymbol{T}})}{|\hat{\boldsymbol{T}}|}\right] \right|.$$

To this end, we couple $\boldsymbol{T}$ and $\hat{\boldsymbol{T}}$ as follows:

- In $\boldsymbol{T}, \hat{\boldsymbol{T}}$ the type $(i_0, \ell_0)$ respectively $(\hat{i}_0, \hat{\ell}_0)$ of the root $v_0$ is chosen from the distribution

  $$Q = (q_{i,\ell})_{(i,\ell)\in\mathcal{T}} \quad \text{respectively} \quad \hat{Q} = (\hat{q}_{i,\ell})_{(i,\ell)\in\mathcal{T}}.$$

  We couple $(i_0, \ell_0)$, $(\hat{i}_0, \hat{\ell}_0)$ optimally.
- If $(i_0, \ell_0) \neq (\hat{i}_0, \hat{\ell}_0)$, then we generate $\boldsymbol{T}$, $\hat{\boldsymbol{T}}$ independently from the corresponding conditional distributions given the type of the root.
- If $(i_0, \ell_0) = (\hat{i}_0, \hat{\ell}_0)$, we generate a random tree $\widetilde{\boldsymbol{T}}$ by means of the following branching process.
  - Initially, there is one individual. Its type is $(i_0, \ell_0)$.
  - Each individual of type $(i, \ell)$ spawns a $\mathrm{Po}(\Lambda_{i',\ell'})$ number of offspring of each type $(i', \ell') \in \mathcal{T}_{i,\ell}$, where

    $$\Lambda_{i',\ell'} = \max\left\{q^*_{i',\ell'}d', \hat{q}^*_{i',\ell'}\hat{d}'\right\}.$$
  - Given that the total progeny is finite, we obtain $\widetilde{\boldsymbol{T}}$ by linking each individual to its offspring.

- For each type $(i,\ell)$, let
$$\lambda_{i,\ell} = 1 - \min\left\{d' q^*_{i,\ell}, \hat{d}' \hat{q}^*_{i,\ell}\right\}/\Lambda_{i,\ell}.$$
For every vertex $v$ of $\widetilde{\boldsymbol{T}}$, let $s_v$ be a random variable with distribution $\mathrm{Be}(\lambda_{i_v,\ell_v})$, where $(i_v,\ell_v)$ is the type of $v$. The random variables $(s_v)_v$ are mutually independent.
- Obtain $\boldsymbol{T}$ from $\widetilde{\boldsymbol{T}}$ by deleting all vertices $v$ such that $d' q^*_{i_v,\ell_v} < \hat{d}' \hat{q}^*_{i_v,\ell_v}$ and $s_v = 1$, along with the pending sub-tree.
- Similarly, obtain $\hat{\boldsymbol{T}}$ from $\widetilde{\boldsymbol{T}}$ by deleting all $v$ and their sub-trees such that $d' q^*_{i_v,\ell_v} > \hat{d}' \hat{q}^*_{i_v,\ell_v}$ and $s_v = 1$.

Let $\mathcal{A}$ be the event that the type of the root satisfies $\ell_0 = \{i_0\}$ and let $\hat{\mathcal{A}}$ be the event $\hat{\ell}_0 = \{\hat{i}_0\}$. If $\mathcal{A} \cap \hat{\mathcal{A}}$ occurs, then both $\boldsymbol{T}, \hat{\boldsymbol{T}}$ consist of a single vertex and have precisely one legal colouring. Thus, $|\boldsymbol{T}|^{-1} \ln \mathcal{Z}(\boldsymbol{T}) = |\hat{\boldsymbol{T}}|^{-1} \ln \mathcal{Z}(\hat{\boldsymbol{T}}) = 0$. Consequently,

$$\Delta \leq \mathbb{E}\left[\left|\frac{\ln \mathcal{Z}(\boldsymbol{T})}{|\boldsymbol{T}|} - \frac{\ln \mathcal{Z}(\hat{\boldsymbol{T}})}{|\hat{\boldsymbol{T}}|}\right| \,\middle|\, \neg\mathcal{A} \vee \neg\hat{\mathcal{A}}\right] \cdot \mathbb{P}\left[\neg\mathcal{A} \vee \neg\hat{\mathcal{A}}\right].$$

Further, since $|\boldsymbol{T}|^{-1} \ln \mathcal{Z}(\boldsymbol{T}), |\hat{\boldsymbol{T}}|^{-1} \ln \mathcal{Z}(\hat{\boldsymbol{T}}) \leq \ln k$ with certainty, we obtain

$$\begin{aligned}\Delta \leq &\left(\mathbb{P}\left[\neg\mathcal{A} \wedge \hat{\mathcal{A}}\right] + \mathbb{P}\left[\mathcal{A} \wedge \neg\hat{\mathcal{A}}\right]\right) \ln k \\ &+ \mathbb{E}\left[\left|\frac{\ln \mathcal{Z}(\boldsymbol{T})}{|\boldsymbol{T}|} - \frac{\ln \mathcal{Z}(\hat{\boldsymbol{T}})}{|\hat{\boldsymbol{T}}|}\right| \,\middle|\, \neg\mathcal{A} \wedge \neg\hat{\mathcal{A}}\right] \cdot \mathbb{P}\left[\neg\mathcal{A} \wedge \neg\hat{\mathcal{A}}\right].\end{aligned}$$

Because $(i_0,\ell_0)$ and $(\hat{i}_0,\hat{\ell}_0)$ are coupled optimally and $\mathbb{P}[\mathcal{A}] = kq^*_1$, $\mathbb{P}[\hat{\mathcal{A}}] = k\hat{q}^*_1$, Corollary 5.2.4 implies that $\mathbb{P}[\neg\mathcal{A} \wedge \hat{\mathcal{A}}], \mathbb{P}[\mathcal{A} \wedge \neg\hat{\mathcal{A}}] \leq \varepsilon\tilde{O}_k(k^{-2})$. Hence,

$$\Delta \leq \varepsilon\tilde{O}_k(k^{-2}) + \mathbb{E}\left[\left|\frac{\ln \mathcal{Z}(\boldsymbol{T})}{|\boldsymbol{T}|} - \frac{\ln \mathcal{Z}(\hat{\boldsymbol{T}})}{|\hat{\boldsymbol{T}}|}\right| \,\middle|\, \neg\mathcal{A} \wedge \neg\hat{\mathcal{A}}\right] \cdot \mathbb{P}\left[\neg\mathcal{A} \wedge \neg\hat{\mathcal{A}}\right]. \tag{5.2.6}$$

Now, let $\mathcal{E}$ be the event that $\ell_0 \neq \{i_0\}$, $\hat{\ell}_0 \neq \{\hat{i}_0\}$ and $(i_0,\ell_0) = (\hat{i}_0,\hat{\ell}_0)$. Due to Corollary 5.2.5 and because $(i_0,\ell_0)$, $(\hat{i}_0,\hat{\ell}_0)$ are coupled optimally, we see that

$$\mathbb{P}\left[\neg\mathcal{A} \wedge \neg\hat{\mathcal{A}} \wedge \neg\mathcal{E}\right] \leq \varepsilon\tilde{O}_k(k^{-2}). \tag{5.2.7}$$

Combining (5.2.6) and (5.2.7), we conclude that

$$\Delta \leq \varepsilon\tilde{O}_k(k^{-2}) + \mathbb{E}\left[\left|\frac{\ln \mathcal{Z}(\boldsymbol{T})}{|\boldsymbol{T}|} - \frac{\ln \mathcal{Z}(\hat{\boldsymbol{T}})}{|\hat{\boldsymbol{T}}|}\right| \,\middle|\, \mathcal{E}\right] \cdot \mathbb{P}\left[\neg\mathcal{A} \wedge \neg\hat{\mathcal{A}}\right]. \tag{5.2.8}$$

Further, since $\mathbb{P}\left[\neg\mathcal{A}\wedge\neg\hat{\mathcal{A}}\right]\leq\mathbb{P}\left[\neg\mathcal{A}\right]\leq 1-kq_1^*\leq O_k(1/k)$ by Lemma 5.2.1, (5.2.8) yields

$$\begin{aligned}\Delta &\leq \varepsilon\tilde{O}_k(k^{-2})+O_k(1/k)\cdot\mathbb{E}\left[\left|\frac{\ln\mathcal{Z}(\boldsymbol{T})}{|\boldsymbol{T}|}-\frac{\ln\mathcal{Z}(\hat{\boldsymbol{T}})}{|\hat{\boldsymbol{T}}|}\right|\,\middle|\mathcal{E}\right]\\ &\leq \varepsilon\tilde{O}_k(k^{-2})+O_k(\ln k/k)\cdot\mathbb{P}\left[\boldsymbol{T}\neq\hat{\boldsymbol{T}}|\mathcal{E}\right].\end{aligned}\tag{5.2.9}$$

Thus, we are left to estimate the probability that $\boldsymbol{T}\neq\hat{\boldsymbol{T}}$, given that both trees have a root of the same type $(i_0,\ell_0)$ with $|\ell_0|>1$. Our coupling ensures that this event occurs if and only if $s_v=1$ for some vertex $v$ of $\widetilde{\boldsymbol{T}}$. To estimate the probability of this event, we observe that by Corollary 5.2.5

$$\lambda_{i,\ell}\leq\begin{cases}\varepsilon\tilde{O}_k(1/k) & \text{if } |\ell|=2,\\ \varepsilon\tilde{O}_k(1) & \text{if } |\ell|>2.\end{cases}\tag{5.2.10}$$

Now, let $\mathcal{N}_1$ be the number of vertices $v\neq v_0$ of $\widetilde{\boldsymbol{T}}$ such that $|\ell_v|=2$, and let $\mathcal{N}_2$ be the number of $v\neq v_0$ such that $|\ell_v|>2$. Then (5.2.9), (5.2.10) and the construction of the coupling yield

$$\Delta/\varepsilon\leq\tilde{O}_k(k^{-2})+\tilde{O}_k(k^{-1})\left(k^{-1}\mathbb{E}[\mathcal{N}_1|\mathcal{E}]+\cdot\mathbb{E}[\mathcal{N}_2|\mathcal{E}]\right).\tag{5.2.11}$$

To complete the proof, we claim that

$$\mathbb{E}[\mathcal{N}_1|\mathcal{E}]\leq\tilde{O}_k(k^{-1}),\qquad\mathbb{E}[\mathcal{N}_2|\mathcal{E}]\leq\tilde{O}_k(k^{-2}).\tag{5.2.12}$$

Indeed, consider the matrix $\tilde{M}=(\tilde{M}_{ij})_{i,j=1,2}$ with entries

$$\tilde{M}_{11}=\sum_{(i,\ell)\in\mathcal{T}_{1,\{1,2\}}:|\ell|=2}\Lambda_{i,\ell},\qquad\tilde{M}_{12}=\sum_{(i,\ell)\in\mathcal{T}_{1,\{1,2\}}:|\ell|>2}\Lambda_{i,\ell},$$

$$\tilde{M}_{21}=\sum_{(i,\ell)\in\mathcal{T}_{1,[k]}:|\ell|=2}\Lambda_{i,\ell},\qquad\tilde{M}_{22}=\sum_{(i,\ell)\in\mathcal{T}_{1,[k]}:|\ell|>2}\Lambda_{i,\ell}.$$

Then Corollary 5.2.5 entails that

$$\tilde{M}_{11}=\tilde{O}_k(k^{-1}),\qquad\tilde{M}_{12}=\tilde{O}_k(k^{-2}),\qquad\tilde{M}_{21}=\tilde{O}_k(1),\qquad\tilde{M}_{22}=\tilde{O}_k(k^{-1}).\tag{5.2.13}$$

In addition, let $\xi=\binom{\xi_1}{\xi_2}$, where $\xi_1=1-\xi_2=\mathbb{P}\left[|\ell_0|=2|\mathcal{E}\right]$. Then Corollary 5.2.5 shows that $\xi_2=\tilde{O}_k(k^{-2})$. Furthermore, by the construction of the branching process and (5.2.13) we have

$$\begin{pmatrix}\mathbb{E}\left[\mathcal{N}_1|\mathcal{E}\right]\\ \mathbb{E}\left[\mathcal{N}_2|\mathcal{E}\right]\end{pmatrix}\leq\sum_{t=1}^{\infty}\tilde{M}^t\xi=\begin{pmatrix}\tilde{O}_k(k^{-1})\\ \tilde{O}_k(k^{-2})\end{pmatrix},$$

which implies (5.2.12).

Finally, (5.2.11) and (5.2.12) imply that $\Delta \leq \varepsilon\tilde{O}_k(k^{-2})$. Taking $\varepsilon \to 0$ completes the proof. □

*Proof of Lemma 5.2.1.* The first assertion is immediate from Lemma 5.2.2. The second claim follows from Lemma 5.2.6, and the third one from Lemma 5.2.7. □

## 5.3. The cluster size

The objective in this section is to prove Proposition 5.1.5. For technical reasons, we consider a variant of the "planted model" $G(n, p', \boldsymbol{\sigma})$ in which the number of vertices is not exactly $n$ but $n - o(n)$. This is necessary because we are going to perform inductive arguments in which small parts of the random graph get removed. Thus, let $\eta = \eta(n) = o(n)$ be a non-negative integer sequence. Throughout the section, we write $n' = n - \eta(n)$. Moreover, we let $\boldsymbol{G} = G(n', p', \boldsymbol{\sigma})$, where $p' = d'/n'$ with $d' = kd/(k-1)$ as in (5.1.1). By a slight abuse of notation we do not distinguish between $\boldsymbol{\sigma}$ and its restriction to the vertices in $[n']$. Unless specified otherwise, all statements in this section are understood to hold for *any* sequence $\eta = o(n)$.

### 5.3.1. Preliminaries

Assume that $G = (V, E)$, let $\sigma$ be a $k$-colouring of $G$, let $v \in V$ and let $\omega \geq 1$ be an integer. We write $\partial_G^\omega(v)$ for the subgraph of $G$ consisting of all vertices at distance at most $\omega$ from $v$. Moreover, $|\partial_G^\omega(v)|$ signifies the number of vertices of $\partial_G^\omega(v)$. Where the reference to $G$ is clear from the context, we omit it. We begin with the following standard fact about the random graph $\boldsymbol{G}$.

**Lemma 5.3.1.** *Let* $\omega = 10\lceil \ln\ln\ln n \rceil$.

1. *With probability* $1 - \exp(-\Omega(\ln^2 n))$ *the random graph* $\boldsymbol{G}$ *is such that* $|\partial_{\boldsymbol{G}}^\omega(v)| \leq n^{0.01}$ *for all vertices* $v$.
2. *W.h.p. all but* $o(n)$ *vertices* $v$ *of* $\boldsymbol{G}$ *are such that* $\partial_{\boldsymbol{G}}^\omega(v)$ *is acyclic.*

In addition, we need to know that the "local structure" of the random graph $\boldsymbol{G}$ endowed with the colouring $\boldsymbol{\sigma}$ enjoys the following concentration property.

**Lemma 5.3.2.** *Let* $\mathcal{S}$ *be a set of triples* $(G_0, \sigma_0, v_0)$ *such that* $G_0$ *is a graph,* $\sigma_0$ *is a* $k$*-colouring of* $G_0$*, and* $v_0$ *is a vertex of* $G_0$*. Let* $\omega = 10\lceil \ln\ln\ln n \rceil$ *and define a random variable* $S_v = S_v(\boldsymbol{G}, \boldsymbol{\sigma})$ *by letting*

$$S_v = \mathbf{1}_{(\partial_{\boldsymbol{G}}^\omega(v), \boldsymbol{\sigma}|_{\partial_{\boldsymbol{G}}^\omega(v)}, v) \in \mathcal{S}}.$$

*Further, let* $S = \sum_v S_v$*. Then* $S = \mathbb{E}[S] + o(n)$ *w.h.p..*

The proof of Lemma 5.3.2 is based on standard arguments. The full details can be found in Subsection 5.3.4.

### 5.3.2. Warning Propagation

The goal in this section is to prove Proposition 5.1.5, i.e. to determine the cluster size $|\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})|$. A key step in this endeavor will be to determine the sets

$$\ell(v) = \{\tau(v) : \tau \in \mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})\}$$

of colours that vertex $v$ may take under a $k$-colouring in $\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})$. In particular, we called a vertex *frozen* in $\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})$ if $\ell(v) = \{\boldsymbol{\sigma}(v)\}$. To establish Proposition 5.1.5, we will first show that the sets $\ell(v)$ can be determined by means of a process called *Warning Propagation*, which hails from the physics literature (see [MM09] and the references therein). More precisely, we will see that Warning Propagation yields colour sets $L(v)$ such that $L(v) = \ell(v)$ for all but $o(n)$ vertices w.h.p.. Crucially, by tracing Warning Propagation we will be able to determine the number of vertices of any type $(i, \ell)$. Moreover, we will show that the cluster $\mathcal{C}(\boldsymbol{\sigma})$ essentially consists of all $k$-colourings $\tau$ of $\boldsymbol{G}$ such that $\tau(v) \in L(v)$ for all $v$. In addition, the number of such colourings $\tau$ can be calculated by considering a certain reduced graph $\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})$. This graphs turns out to be a forest (possibly after the removal of $o(n)$ vertices), and the final step of the proof consists in arguing that, informally speaking, w.h.p. the statistics of the trees in this forest are given by the distribution of the multi-type branching process from Section 5.1.

Let us begin by describing Warning Propagation on a general graph $G$ endowed with a $k$-colouring $\sigma$. For each edge $e = \{v, w\}$ of $G$ and any colour $i$, we define a sequence $(\mu_{v\to w}(i, t|G, \sigma))_{t\geq 1}$ such that $\mu_{v\to w}(i, t|G, \sigma) \in \{0, 1\}$ for all $i, v, w$. The idea is that $\mu_{v\to w}(i, t|G, \sigma) = 1$ indicates that in the $t$th step of the process vertex $v$ "warns" vertex $w$ that the other neighbours $u \neq w$ of $v$ force $v$ to take colour $i$. We initialize this process by having each vertex $v$ emit a warning about its original colour $\sigma(v)$ at $t = 0$, i.e.

$$\mu_{v\to w}(i, 0|G, \sigma) = \mathbf{1}_{i=\sigma(v)} \tag{5.3.1}$$

for all edges $\{v, w\}$ and all $i \in [k]$. Letting $\partial v = \partial_G(v)$ denote the neighbourhood of $v$ in $G$, for $t \geq 0$ we let

$$\mu_{v\to w}(i, t+1|G, \sigma) = \prod_{j\in[k]\setminus\{i\}} \max\{\mu_{u\to v}(j, t|G, \sigma) : u \in \partial v \setminus \{w\}\}. \tag{5.3.2}$$

That is, $v$ warns $w$ about colour $i$ in step $t + 1$ if and only if at step $t$ it received warnings from its other neighbours $u$ (*not* including $w$) about all colours $j \neq i$. Further, for a vertex $v$ and $t \geq 0$ we let

$$L(v,t|G,\sigma) = \left\{ j \in [k] : \max_{u \in \partial v} \mu_{u \to v}(j,t|G,\sigma) = 0 \right\} \quad \text{and}$$

$$L(v|G,\sigma) = \bigcup_{t=0}^{\infty} L(v,t|G,\sigma).$$

Thus, $L(v,t|G,\sigma)$ is the set of colours that vertex $v$ receives no warnings about at step $t$. To unclutter the notation, we omit the reference to $G, \sigma$ where it is apparent from the context.

To understand the semantics of this process, observe that by construction the list $L(v,t|G,\sigma)$ only depends on the vertices at distance at most $t+1$ from $v$. Further, if we assume that the $t$th neighbourhood $\partial^t v$ in $G$ is a tree, then $L(v,t|G,\sigma)$ is precisely the set of colours that $v$ may take in $k$-colourings $\tau$ of $G$ such that $\tau(w) = \sigma(w)$ for all vertices $w$ at distance greater than $t$ from $v$, as can be verified by a straightforward induction on $t$. As we will see, this observation together with the fact that the random graph $\boldsymbol{G}$ contains only few short cycles (cf. Lemma 5.3.1) allows us to show that for most vertices $v$ we have $\ell(v) = L(v|\boldsymbol{G},\boldsymbol{\sigma})$ w.h.p.. In effect, the number of $k$-colourings $\tau$ of $\boldsymbol{G}$ with $\tau(v) \in L(v|\boldsymbol{G},\boldsymbol{\sigma})$ for all $v$ will emerge to be a very good approximation to the cluster size $|\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})|$.

Counting these $k$-colourings is greatly facilitated by the following observation. For a graph $G$ together with a $k$-colouring $\sigma$, let us denote by $G_{\mathrm{WP}}(t|\sigma)$ the graph obtained from $G$ by removing all edges $\{v,w\}$ such that either $|L(v,t)| < 2$, $|L(w,t)| < 2$ or $L(v,t) \cap L(w,t) = \emptyset$. Furthermore, obtain $G_{\mathrm{WP}}(\sigma)$ from $G$ by removing all edges $\{v,w\}$ such that $L(v) \cap L(w) = \emptyset$. We view $G_{\mathrm{WP}}(t|\sigma)$ and $G_{\mathrm{WP}}(\sigma)$ as decorated graphs in which each vertex $v$ is endowed with the colour list $L(v,t)$ and $L(v)$ respectively. As before, we let $\mathcal{Z}$ denote the number of legal colourings of a decorated graph. Thus, $\mathcal{Z}(G_{\mathrm{WP}}(\sigma))$ is the number of colourings $\tau$ of $G_{\mathrm{WP}}(\sigma)$ such that $\tau(v) \in L(v|G,\sigma)$ for all $v$. The key statement in this section is

**Proposition 5.3.3.** *W.h.p. we have* $\ln \mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})) = \ln |\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})| + o(n)$.

We begin by proving that $\mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma}))$ is a lower bound on the cluster size w.h.p.. To this end, let us highlight a few elementary facts.

**Fact 5.3.4.** *The following statements hold for any* $G, \sigma$.

1. *For all* $v, w, i$ *and all* $t \geq 0$, *we have* $\mu_{v \to w}(i, t+1) \leq \mu_{v \to w}(i,t)$.
2. *We have* $\sigma(v) \in L(v,t)$ *for all* $v,t$. *Moreover, if* $\mu_{v \to w}(i,t) = 1$ *for some* $w \in \partial v$, *then* $i = \sigma(v)$.
3. *There is a number* $t^*$ *such that for any* $t > t^*$ *we have* $\mu_{v \to w}(i,t) = \mu_{v \to w}(i,t^*)$ *for all* $v, w, i$.

*Proof.* We prove (1) and (2) by induction on $t$. In the case $t = 0$ both statements are immediate from (5.3.1). Now, assume that $t \geq 1$ and $\mu_{v\to w}(i,t) = 0$. Then there is a colour $j \neq i$ and a neighbour $u \neq w$ of $v$ such that $\mu_{u\to v}(j,t-1) = 0$. By induction, we have $\mu_{u\to v}(j,t) = 0$. Hence, (5.3.2) implies that $\mu_{v\to w}(i,t+1) = 0$. Furthermore, if $\mu_{v\to w}(i,t+1) = 1$ for some $i \neq \sigma(v)$, then $v$ has a neighbour $u \neq w$ such that $\mu_{u\to v}(\sigma(v),t) = 1$. But since $\sigma(u) \neq \sigma(v)$ because $\sigma$ is a $k$-colouring, this contradicts the induction hypothesis. Thus, we have established (1) and (2). Finally, (3) is immediate from (1). □

**Fact 5.3.5.** *If for some $t \geq 0$, $\tau$ is a colouring of $G_{\mathrm{WP}}(t|\sigma)$ such that $\tau(v) \in L(v,t)$ for all $v$, then $\tau$ is a $k$-colouring of $G$. Moreover, if $\tau$ is a $k$-colouring of $G_{\mathrm{WP}}(\sigma)$ such that $\tau(v) \in L(v)$ for all $v$, then $\tau$ is a $k$-colouring of $G$.*

*Proof.* Let $\{v,w\}$ be an edge of $G$. Clearly, if $L(v,t) \cap L(w,t) = \emptyset$, then $\tau(v) \neq \tau(w)$. Thus, assume that $L(v,t) \cap L(w,t) \neq \emptyset$. Then $|L(v,t)| > 1$. Indeed, if $|L(v,t)| = 1$, then by Fact 5.3.4 we have $L(v,t) = \{\sigma(v)\}$ and thus $\sigma(v) \notin L(w,t)$ by (5.3.2). Similarly, $|L(w,t)| > 1$. Hence, the edge $\{v,w\}$ is present in $G_{\mathrm{WP}}(t|\sigma)$, and thus $\tau(v) \neq \tau(w)$. This implies the first assertion. The second assertion follows from the first assertion and Fact 5.3.4, which shows that there is a finite $t$ such that $L(v,t) = L(v)$ for all $v$. □

To turn Fact 5.3.5 into a lower bound on the cluster size, we are going to argue that in $\boldsymbol{G}$ there are *a lot* of frozen vertices w.h.p.. In fact, w.h.p. the number of such frozen vertices will turn out to be so large that *all* colourings $\tau$ as in Fact 5.3.5 belong to the cluster $\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})$.

To exhibit frozen vertices, we consider an appropriate notion of a "core". More precisely, assume that $\sigma$ is a $k$-colouring of a graph $G$. We denote by $\mathrm{core}(G,\sigma)$ the largest set $V'$ of vertices with the following property.

$$\text{If } v \in V' \text{ and } j \neq \sigma(v)\text{, then } |V' \cap \sigma^{-1}(j) \cap \partial v| \geq 100.$$

In words, any vertex in the core has at least 100 neighbours of any colour $j \neq \sigma(v)$ that also belong to the core. The core is well-defined: If $V', V''$ are two sets with this property, then so is $V' \cup V''$. The following is immediate from the definition of the core.

**Fact 5.3.6.** *Assume that $v \in \mathrm{core}(G,\sigma)$. Then $L(v,t) = \{\sigma(v)\}$ for all $t$.*

The core has become a standard tool in the theory of random structures in general and in random graph colouring in particular. Indeed, standard arguments show that w.h.p. $\boldsymbol{G}$ has a very large core. More precisely, we have

**Proposition 5.3.7** ([COV13]). *W.h.p. $\boldsymbol{G}, \boldsymbol{\sigma}$ are such that the following two properties hold for all sets $S \subset [n']$ of size $|S| \leq \sqrt{n}$.*

1. *Let $\boldsymbol{G}'$ be the subgraph obtained from $\boldsymbol{G}$ by removing the vertices in $S$. Then*
$$|\mathrm{core}(\boldsymbol{G}', \boldsymbol{\sigma}) \cap \boldsymbol{\sigma}^{-1}(i)| \geq \frac{n}{k}(1 - k^{-2/3}) \quad \textit{for all } i \in [k]. \tag{5.3.3}$$
2. *If $v \in \mathrm{core}(\boldsymbol{G}', \boldsymbol{\sigma})$, then $\boldsymbol{\sigma}(v) = \tau(v)$ for all $\tau \in \mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})$.*

**Corollary 5.3.8.** *W.h.p. we have $|\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})| \geq \mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma}))$.*

*Proof.* By Proposition 5.3.7 we may assume that (5.3.3) is true for $S = \emptyset$. Let $\tau$ be a $k$-colouring of $\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})$ such that $\tau(v) \in L(v)$ for all $v$. Then Fact 5.3.5 implies that $\tau$ is a $k$-colouring of $\boldsymbol{G}$. Furthermore, Fact 5.3.6 implies that $\tau(v) = \sigma(v)$ for all $v \in \mathrm{core}(\boldsymbol{G}, \boldsymbol{\sigma})$. Hence, (5.3.3) entails that $\rho_{ii}(\sigma, \tau) \geq 1 - k^{-2/3} > 0.51$ for all $i \in [k]$. Thus, $\tau \in \mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})$. □

While $\mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma}))$ provides a lower bound on the cluster size, the two numbers do not generally coincide. This is because for a few vertices $v$, the list $L(v)$ produced by Warning Propagation may be a proper subset of $\ell(v)$. For instance, assume that the vertices $v_1, v_2, v_3, v_4$ induce a cycle of length four such that $\sigma(v_1) = \sigma(v_3) = 1$ and $\sigma(v_2) = \sigma(v_4) = 2$, while $v_1, v_2, v_3, v_4$ are not adjacent to any further vertices of colour 1 or 2. Moreover, suppose that for each colour $j \in \{3, 4, \ldots, k\}$, each of $v_1, \ldots, v_4$ has at least one neighbour of colour $j$ that belongs to the core. Then Warning Propagation yields $L(v_1) = L(v_3) = \{1\}$ and $L(v_2) = L(v_4) = \{2\}$. However, $v_1, v_2, v_3, v_4$ are actually unfrozen as we might as well give colour 2 to $v_1, v_3$ and colour 1 to $v_2, v_4$. (A bipartite sub-structure of this kind is known as a "Kempe chain", cf. [Mol12].)

The reason for this problem is, roughly speaking, that we launched Warning Propagation from the initialization (5.3.1), which is the obvious choice but may be too restrictive. Thus, to obtain an upper bound on the cluster size we will start Warning Propagation from a different initialization. Ideally, this starting point should be such that only vertices that are frozen emit warnings. By Proposition 5.3.7, the vertices in the core meet this condition w.h.p.. Thus, we are going to compare the above installment of Warning Propagation with the result of starting Warning Propagation from an initialization where only the vertices in the core send out warnings.

Thus, given a graph $G$ together with a $k$-colouring $\sigma$, we let

$$\begin{aligned}
\mu'_{v \to w}(i, 0 | G, \sigma) &= \mathbf{1}_{i = \sigma(v)} \cdot \mathbf{1}_{v \in \mathrm{core}(G, \sigma)}, \\
\mu'_{v \to w}(i, t+1 | G, \sigma) &= \prod_{j \in [k] \setminus \{i\}} \max\left\{\mu'_{u \to v}(j, t | G, \sigma) : u \in \partial v \setminus \{w\}\right\}
\end{aligned}$$

for all edges $\{v, w\}$ of $G$, all $i \in [k]$ and all $t \geq 0$. Furthermore, let

$$L'(v, t|G, \sigma) = \left\{ j \in [k] : \max_{u \in \partial v} \mu'_{u \to v}(j, t|G, \sigma) = 0 \right\} \quad \text{and}$$

$$L'(v|G, \sigma) = \bigcap_{t=0}^{\infty} L'(v, t|G, \sigma).$$

As before, we drop $G, \sigma$ from the notation where possible.

Similarly as before, we can use the lists $L'(v, t)$ to construct a decorated reduced graph. Indeed, let $G'_{\mathrm{WP}}(t|\sigma)$ be the graph obtained from $G$ by removing all edges $\{v, w\}$ such that $|L'(v, t)| < 2$ or $|L'(w, t)| < 2$ or $L'(v, t) \cap L'(w, t) = \emptyset$. We decorate each vertex in this graph with the list $L'(v, t)$. In addition, let $G'_{\mathrm{WP}}(\sigma)$ be the graph obtain from $G$ by removing all edges $\{v, w\}$ such that $L'(v) \cap L'(w) = \emptyset$ endowed with the lists $L'(v)$.

**Fact 5.3.9.** *The following statements hold for all $G, \sigma$.*

1. *For all $v$, we have $\sigma(v) \in L'(v)$. Moreover, if there are $j, t, w$ such that $\mu'_{v \to w}(j, t) = 1$, then $j = \sigma(v)$.*
2. *If $v \in \mathrm{core}(G, \sigma)$, then $L'(v, t) = \{\sigma(v)\}$ for all $t$.*
3. *We have $\mu'_{v \to w}(i, t+1) \geq \mu'_{v \to w}(i, t)$.*
4. *There is a number $t^*$ such that for any $t > t^*$ we have $\mu'_{v \to w}(i, t) = \mu'_{v \to w}(i, t^*)$ for all $v, w, i$.*

*Proof.* This follows by induction on $t$ (cf. the proof of Fact 5.3.4). □

**Lemma 5.3.10.** *W.h.p. for all vertices $v$ we have $\ell(v) = \{\tau(v) : \tau \in \mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})\} \subset L'(v|\boldsymbol{G}, \boldsymbol{\sigma})$.*

*Proof.* Proposition 5.3.7 shows that w.h.p.

$$\tau(v) = \boldsymbol{\sigma}(v) \quad \text{for all } v \in \mathrm{core}(\boldsymbol{G}, \boldsymbol{\sigma}). \tag{5.3.4}$$

Assuming (5.3.4), we are going to prove by induction on $t$ that

$$\ell(v) \subset L'(v, t) \qquad \text{for all } v \in [n], t \geq 0. \tag{5.3.5}$$

By construction, for any vertex $v$ and any colour $j$ we have $j \in L'(v, 0)$, unless $v$ has a neighbour $w \in \mathrm{core}(\boldsymbol{G}, \boldsymbol{\sigma})$ such that $\boldsymbol{\sigma}(w) = j$. Moreover, if such a neighbour $w$ exists, (5.3.4) implies that w.h.p. $\tau(w) = j$ and thus $\tau(v) \neq j$ for all $\tau \in \mathcal{C}(\boldsymbol{\sigma})$. Hence, (5.3.5) is true for $t = 0$.

Now, assume that (5.3.5) holds for $t$. Suppose that $j \notin L'(v, t+1)$. Then $v$ has a neighbour $u$ such that $\mu'_{u \to v}(j, t+1) = 1$. Therefore, for each $l \neq j$ there is $w_l \neq v$ such that $\mu'_{w_l \to u}(l, t) = 1$. Consequently,

$L'(u,t) = \{j\}$. Hence, by induction we have $\tau(u) = j$ and thus $\tau(v) \neq j$ for all $\tau \in \mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})$. $\square$

As an immediate consequence of Lemma 5.3.10, we obtain

**Corollary 5.3.11.** *W.h.p. we have* $|\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})| \leq \mathcal{Z}(\boldsymbol{G}'_{\mathrm{WP}}(\boldsymbol{\sigma}))$.

Combining Corollary 5.3.8 and Corollary 5.3.11, we see that w.h.p.

$$\mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})) \leq |\mathcal{C}(\boldsymbol{G}, \boldsymbol{\sigma})| \leq \mathcal{Z}(\boldsymbol{G}'_{\mathrm{WP}}(\boldsymbol{\sigma})).$$

To complete the proof of Proposition 5.3.3, we are going to argue that w.h.p. $\ln \mathcal{Z}(\boldsymbol{G}'_{\mathrm{WP}}(\boldsymbol{\sigma})) = \ln \mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})) + o(n)$.

To this end, we need one more general construction. Let $G$ be a graph and let $\sigma$ be a $k$-colouring of $G$. Let $t \geq 0$ be an integer. For each vertex $v$ of $G$, we define a rooted, decorated graph $T(v,t|G,\sigma)$ as follows.

- The graph underlying $T(v,t|G,\sigma)$ is the connected component of $v$ in $G_{\mathrm{WP}}(v,t|G,\sigma)$.
- The root of $T(v,t|G,\sigma)$ is $v$.
- The type of each vertex $w$ of $T(v,t|G,\sigma)$ is $(\sigma(w), L(w,t|G,\sigma))$.

Analogously we obtain rooted, decorated graphs $T(v|G,\sigma)$ from $G_{\mathrm{WP}}(\sigma)$ as well as $T'(v,t|G,\sigma)$ from $G'_{\mathrm{WP}}(t|\sigma)$ and $T'(v|G,\sigma)$ from $G'_{\mathrm{WP}}(\sigma)$.

Of course, the total number $\mathcal{Z}(G_{\mathrm{WP}}(\sigma))$ of legal colourings of $G_{\mathrm{WP}}(\sigma)$ is just the product of the number of legal colourings of all the connected components of $G_{\mathrm{WP}}(\sigma)$. The following lemma shows that w.h.p. for all but $o(n)$ vertices the components in $\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})$ and $\boldsymbol{G}'_{\mathrm{WP}}(\boldsymbol{\sigma})$ coincide.

**Lemma 5.3.12.** *W.h.p.* $\boldsymbol{G}, \boldsymbol{\sigma}$ *is such that* $T(v|\boldsymbol{G}, \boldsymbol{\sigma}) = T'(v|\boldsymbol{G}, \boldsymbol{\sigma})$ *for all but* $o(n)$ *vertices* $v$.

The main technical step towards the proof of Lemma 5.3.12 is to show that w.h.p. most of the components $T'(v|\boldsymbol{G}, \boldsymbol{\sigma})$ are “small” by comparison to $n$. Technically, it is easier to establish this statement for $T'(v,0|\boldsymbol{G}, \boldsymbol{\sigma})$, which contains $T'(v|\boldsymbol{G}, \boldsymbol{\sigma})$ as a subgraph due to the monotonicity property Fact 5.3.9, (3).

**Lemma 5.3.13.** *For any* $\varepsilon > 0$*, there is a number* $\omega = \omega(\varepsilon) > 0$ *such that w.h.p. for at least* $(1-\varepsilon)n$ *vertices* $v$ *the component* $T'(v,0|\boldsymbol{G}, \boldsymbol{\sigma})$ *contains no more than* $\omega$ *vertices.*

The proof of Lemma 5.3.13, which we defer to Subsection 5.3.4, is a bit technical but based on known arguments. Lemma 5.3.1 shows that w.h.p. for most vertices $v$ such that $T'(v,0|\boldsymbol{G}, \boldsymbol{\sigma})$ contains at most, say, $\omega = \lceil \ln\ln\ln n \rceil$ vertices, $T'(v, 0|\boldsymbol{G}, \boldsymbol{\sigma})$ is a tree. In this case, the following observation applies.

**Lemma 5.3.14.** *Let $G$ be a graph and let $\sigma$ be a $k$-colouring of $G$. Assume that $T'(v, 0|G, \sigma)$ is a tree on $\omega$ vertices for some integer $\omega \geq 1$. Then for any vertex $y$ in $T'(v, 0|G, \sigma)$ we have $L(y|G, \sigma) = L'(y|G, \sigma)$. Moreover, if $T'(v, 0|G, \sigma)$ has $\omega$ vertices, then $L(y|G, \sigma) = L(y, \omega + 1|G, \sigma)$ and $L'(y|G, \sigma) = L'(y, \omega + 1|G, \sigma)$.*

*Proof.* To get started, let us recall some basic properties of the warnings:

- **P1** If for an edge $\{x, y\}$ in $G$ we have $\mu_{x\to y}(i, 0) = 1$ or $\mu'_{x\to y}(i, 0) = 1$ then $i = \sigma(x)$.
- **P2** For each vertex $v \in G$, we have $\sigma(v) \in L(v, t)$ and $\sigma(v) \in L'(v, t)$ for all $t \geq 0$.
- **P3** For all edges $\{x, y\}$ in $G$, we have $\mu_{x\to y}(i, t) \geq \mu'_{x\to y}(i, t)$ for all $i \in [k]$.

As a first step we are going to show that for each edge $\{x, y\}$ in $T'(v, 0|G, \sigma)$ we have

$$\mu_{x\to y}(i, t) = \mu'_{x\to y}(i, t) = 0 \quad \text{for all } t > \omega \text{ and all } i \in [k]. \tag{5.3.6}$$

To do so, pick and fix an arbitrary vertex $y$ in $T'(v, 0)$. We define the *$y$-height* $h_y(x)$ of a vertex $x \neq y$ in $T'(v, 0)$ as follows. Since $T'(v, 0)$ is a tree, there is a unique path from $x$ to $y$ in $T'(v, 0)$. Let $P_y(x)$ be the neighbour of $x$ on this path. Then $h_y(x)$ is the maximum distance from $x$ to a leaf of $T'(v, 0)$ that belongs to the component of $x$ in the subgraph of $T'(v, 0)$ obtained by removing the edge $\{x, P_y(x)\}$.

Let $U$ be the set of all neighbours $u$ of $x$ that do not belong to $T'(v, 0)$, and let $U'$ be the set of all neighbours $u' \neq P_y(x)$ of $x$ in $T'(v, 0)$. We compute

$$\mu'_{x\to P_y(x)}(i, 1) = \prod_{j\in[k]\setminus\{i\}} \max\left\{\mu'_{u\to x}(j, 0) : u \in U\right\} = 0 \quad \text{for all } i \in [k]$$

where we omitted the vertices in $U'$ since by construction of $\mathrm{core}(G, \sigma)$ we conclude that for all $u' \in U'$ we get $\mu'_{u'\to x}(i, 0) = 0$ for all $i \in [k]$. For each $j \in [k] \setminus L'(x, 0)$, there exists a neighbour $u \in U$ such that $\sigma(u) = j$ and $\mu'_{u\to x}(i, 0) = \mathbf{1}_{i=j}$ and let $U_C$ be the set of all such neighbours. By Fact 5.3.9 and **P3** for all $u \in U_C$, we find

$$\mu_{u\to x}(i, t) = \mu'_{u\to x}(i, t) = \mathbf{1}_{i=\sigma(u)} \quad \text{for all } i \in [k] \text{ for all } t \geq 0. \tag{5.3.7}$$

By construction of $T'(v, 0)$, for all $u \in U$ the lists $L'(x, 0)$ and $L'(u, 0)$ are disjoint and by **P1**, **P2** and (5.3.7), we obtain

$$\text{For any } u \in U \setminus U_C \text{ we find } \sigma(u) \in L'(u,0) \subset [k] \setminus L'(x,0) \text{ and thus there exists a } u' \in U_C \text{ such that } \mu_{u\to x}(i,0) = \mu_{u'\to x}(i,0) = \mathbf{1}_{i=\sigma(u)} \text{ and in particular } \mu_{u\to x}(i,t) \leq \mu_{u'\to x}(i,0) = \mathbf{1}_{i=\sigma(u)} \text{ for all } t \geq 0. \tag{5.3.8}$$

We conclude by (5.3.8) that

$$\mu'_{x\to P_y(x)}(i,1) = \prod_{j\in[k]\setminus\{i\}} \max\left\{\mu'_{u\to x}(j,0) : u \in U_C\right\} = 0 \quad \text{for all } i \in [k]. \tag{5.3.9}$$

To prove (5.3.6), we show by induction on $h_y(x)$ that for all $i \in [k]$

$$\mu_{x\to P_y(x)}(i,t) = \mu'_{x\to P_y(x)}(i,t) = 0 \quad \text{for all } t \geq h_y(x)+1. \tag{5.3.10}$$

To get started, suppose that $h_y(x) = 0$. Then $x$ is a leaf of $T'(v,0)$. We compute

$$\begin{aligned}
\mu_{x\to P_y(x)}(i,1) &= \prod_{j\in[k]\setminus\{i\}} \max\left\{\mu_{u\to x}(j,0) : u \in U\right\} \\
&= \prod_{j\in[k]\setminus\{i\}} \max\left\{\mu_{u\to x}(j,0) : u \in U_C\right\} && \text{[by (5.3.8)]} \\
&= \prod_{j\in[k]\setminus\{i\}} \max\left\{\mu'_{u\to x}(j,0) : u \in U_C\right\} && \text{[by (5.3.7)]} \\
&= \mu'_{x\to P_y(x)}(i,1) = 0 && \text{[by (5.3.9)]}
\end{aligned}$$

for all $i \in [k]$. By Fact 5.3.4 and **P3**, we conclude that $\mu_{x\to P_y(x)}(i,t) = \mu'_{x\to P_y(x)}(i,t) = 0$ for all $t \geq 1$.

Now, assume that $h_y(x) > 0$. Then all $u' \in U'$ satisfy $h_y(u') < h_y(x)$. Moreover, $P_y(u') = x$. Therefore, by induction

$$\mu_{u'\to x}(i,t) = \mu_{u'\to x}(i,h_y(x)) = 0 = \mu'_{u'\to x}(i,h_y(x)) = \mu'_{u'\to x}(i,t) \tag{5.3.11}$$

for all $u' \in U'$, $i \in [k]$, $t > h_y(x)$.

We compute

$$\begin{aligned}\mu_{x\to P_y(x)}(i,t) &= \prod_{j\in[k]\setminus\{i\}} \max\left\{\mu_{u\to x}(j,t-1) : u \in U\cup U'\right\}\\ &= \prod_{j\in[k]\setminus\{i\}} \max\left\{\mu_{u\to x}(j,0) : u\in U_C\right\} \qquad \text{[by (5.3.8) and (5.3.11)]}\\ &= \prod_{j\in[k]\setminus\{i\}} \max\left\{\mu'_{u\to x}(j,0) : u\in U_C\right\} \qquad \text{[by (5.3.7)]}\\ &= \mu'_{x\to P_y(x)}(i,1) = 0 \text{for all } i\in[k], t\geq h_y(x)+1.\end{aligned}$$

Again by Fact 5.3.4 and **P3**, we conclude that $\mu_{x\to P_y(x)}(i,t) = \mu'_{x\to P_y(x)}(i,t) = 0$ for all $i\in[k]$ and $t\geq h_y(x)+1$.

Finally, we observe that $h_y(x)\leq\omega = |T'(v,0)|$ for all $x$. Hence, applying (5.3.10) to the neighbours $x$ of $y$ in $T'(v,0)$, we obtain $\mu_{x\to y}(j,t) = \mu_{x\to y}(i,\omega+1) = \mu'_{x\to y}(i,\omega+1) = 0 = \mu'_{x\to y}(i,t)$ for all $i\in[k]$ and all $t>\omega$. Together with (5.3.7) which states that for any $x\in T'(v,0)$ and for any $j\in[k]\setminus L'(x,0)$ there exists a vertex $u\notin T'(v,0)$ that is adjacent to $x$ in $G$ such that $\mu_{u\to x}(j,t) = \mu'_{u\to x}(j,t) = 1$ for all $t\geq 0$ and with (5.3.8) which states that for any $j\in L'(x,0)$ there exists no vertex $u\notin T'(v,0)$ that is adjacent to $x$ in $G$ such that $\mu_{u\to x}(j,t) = \mu'_{u\to x}(j,t) = 1$ for any $t\geq 0$ we conclude that $L(x) = L(x,\omega+1) = L'(x,\omega+1) = L'(x)$ as desired. □

*Proof of Lemma 5.3.12.* Lemma 5.3.13 implies that for all but $o(n)$ vertices $v$ we have $|T'(v,0)|\leq \ln\ln\ln n$ w.h.p.. Together with Lemma 5.3.1, this implies that w.h.p. $T'(v,0)$ is a tree for all but $o(n)$ vertices $v$. Thus, assume in the following that $v$ is such that $T'(v,0)$ is a tree.

It is immediate from Facts 5.3.4, 5.3.6 and 5.3.9 that $L(w)\subset L'(w)\subset L'(w,0)$ for all vertices $w$. Therefore, $\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})\subset \boldsymbol{G}'_{\mathrm{WP}}(\boldsymbol{\sigma})\subset \boldsymbol{G}'_{\mathrm{WP}}(0|\boldsymbol{\sigma})$ and thus

$$T(v)\subset T'(v)\subset T'(v,0). \tag{5.3.12}$$

Conversely, Lemma 5.3.14 shows that $L(x) = L'(x)$ for all vertices $x$ in $T'(v,0)$. Together with equation (5.3.12), this implies that $T(v) = T'(v)$. □

*Proof of Proposition 5.3.3.* By Corollary 5.3.8 and Corollary 5.3.11, w.h.p. we have $\mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma}))\leq |\mathcal{C}(\boldsymbol{G},\boldsymbol{\sigma})|\leq \mathcal{Z}(\boldsymbol{G}'_{\mathrm{WP}}(\boldsymbol{\sigma}))$. Thus, it suffices to show that $\ln\mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})) = \ln\mathcal{Z}(\boldsymbol{G}'_{\mathrm{WP}}(\boldsymbol{\sigma})) + o(n)$ w.h.p.. Indeed, because the various connected components of $\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})$ can be coloured independently, we find that

$$\ln \mathcal{Z}(\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})) = \sum_{v\in[n']} \frac{\ln \mathcal{Z}(T(v|\boldsymbol{G},\boldsymbol{\sigma}))}{|T(v|\boldsymbol{G},\boldsymbol{\sigma})|},$$
$$\ln \mathcal{Z}(\boldsymbol{G}'_{\mathrm{WP}}(\boldsymbol{\sigma})) = \sum_{v\in[n']} \frac{\ln \mathcal{Z}(T'(v|\boldsymbol{G},\boldsymbol{\sigma}))}{|T'(v|\boldsymbol{G},\boldsymbol{\sigma})|}. \tag{5.3.13}$$

Clearly, for any vertex $v$ we have $\frac{\ln \mathcal{Z}(T(v|\boldsymbol{G},\boldsymbol{\sigma}))}{|T(v|\boldsymbol{G},\boldsymbol{\sigma})|}, \frac{\ln \mathcal{Z}(T'(v|\boldsymbol{G},\boldsymbol{\sigma}))}{|T'(v|\boldsymbol{G},\boldsymbol{\sigma})|} \leq \ln k$. Hence, Lemma 5.3.12 shows that w.h.p.

$$\sum_{v\in[n']} \frac{\ln \mathcal{Z}(T(v|\boldsymbol{G},\boldsymbol{\sigma}))}{|T(v|\boldsymbol{G},\boldsymbol{\sigma})|} \sim \sum_{v\in[n']} \frac{\ln \mathcal{Z}(T'(v|\boldsymbol{G},\boldsymbol{\sigma}))}{|T'(v|\boldsymbol{G},\boldsymbol{\sigma})|}. \tag{5.3.14}$$

Finally, the assertion follows from (5.3.13) and (5.3.14). □

### 5.3.3. Counting legal colourings

Proposition 5.3.3 reduces the proof of Proposition 5.1.5 to the problem of counting the legal colourings of the reduced graph $\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})$. Lemma 5.3.13 implies that w.h.p. $\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})$ is a forest consisting mostly of trees of size, say at most $\ln\ln\ln n$. In this section we are going to show that w.h.p. the "statistics" of these trees follows the distribution of the random tree generated by the branching process from Section 5.1. To formalise this, let $\boldsymbol{T} = \boldsymbol{T}_{d,k,\boldsymbol{q}^*}$ with $\boldsymbol{q}^*$ from (5.1.5) denote the random isomorphism class of rooted, decorated trees produced by the process $\mathrm{GW}(d,k,\boldsymbol{q}^*)$. Moreover, for a rooted, decorated tree $T$ let $H_T$ be the number of vertices $v$ in $\boldsymbol{G}_{\mathrm{WP}}(\boldsymbol{\sigma})$ such that $T(v|\boldsymbol{G},\boldsymbol{\sigma}) \cong T$. In this section we prove:

**Proposition 5.3.15.** *If $T$ is such that $\mathbb{P}[T \in \boldsymbol{T}] > 0$, then $(\frac{1}{n}H_T)_{n\geq 1}$ converges to $\mathbb{P}[T \in \boldsymbol{T}]$ in probability.*

We begin by showing that the fixed point problem $\boldsymbol{q}^* = F_{d,k}(\boldsymbol{q}^*)$ with $F_{d,k}$ from (5.2.1) provides a good approximation to the number of vertices $v$ such that $L(v|\boldsymbol{G},\boldsymbol{\sigma}) = \{i\}$ for any $i$. To this end, we let

$$\boldsymbol{q}^0 = (1/k,\ldots,1/k) \quad \text{and} \quad \boldsymbol{q}^t = F_{d,k}(\boldsymbol{q}^{t-1}) \quad \text{for } t \geq 1.$$

In addition, let $Q_i(t|\boldsymbol{G},\boldsymbol{\sigma})$ be the set of vertices $v$ of $\boldsymbol{G}$ such that $L(v,t|\boldsymbol{G},\boldsymbol{\sigma}) = \{i\}$.

**Lemma 5.3.16.** *For any $i \in [k]$ and any fixed $t > 0$, we have $\frac{1}{n}|Q_i(t|\boldsymbol{G},\boldsymbol{\sigma})| = q_i^t + o(1)$ w.h.p..*

*Proof.* We proceed by induction on $t$. To get started, we set $Q_i(-1|\boldsymbol{G},\boldsymbol{\sigma}) = \boldsymbol{\sigma}^{-1}(i)$ and $q_i^{-1} = 1/k$. Then w.h.p. $\frac{1}{n}|Q_i(-1|\boldsymbol{G},\boldsymbol{\sigma})| = q_i^{-1} + o(1)$.

Now, assuming that $t \geq 0$ and that the assertion holds for $t-1$, we are going to argue that

$$\mathbb{E}[|Q_i(t|\boldsymbol{G}, \boldsymbol{\sigma})|/n] = q_i^t + o(1). \tag{5.3.15}$$

Indeed, let $v = n'$ be the last vertex of the random graph, and let us condition on the event that $\boldsymbol{\sigma}(v) = i$. By symmetry and the linearity of expectation, it suffices to show that

$$\mathbb{P}[L(v, t|\boldsymbol{G}, \boldsymbol{\sigma}) = \{i\} \, |\boldsymbol{\sigma}(v) = i] = kq_i^t + o(1). \tag{5.3.16}$$

To show (5.3.16), let $\widetilde{\boldsymbol{G}}$ signify the subgraph obtained from $\boldsymbol{G}$ by removing $v$. Moreover, let $\mathcal{Q}^{t-1}(\varepsilon)$ be the event that

$$|n^{-1}|Q_j(t-1|\widetilde{\boldsymbol{G}}, \boldsymbol{\sigma})| - q_j^{t-1}| < \varepsilon \quad \text{for all } j \in [k].$$

Since $\widetilde{\boldsymbol{G}}$ is nothing but a random graph $G(n'-1, p', \boldsymbol{\sigma})$ with one less vertex and as $n'-1 = n - o(n)$, by induction we have

$$\mathbb{P}[\mathcal{Q}^{t-1}(\varepsilon)] = 1 - o(1) \qquad \text{for any } \varepsilon > 0. \tag{5.3.17}$$

Let $\mathcal{A}(i)$ be the event that for each $j \in [k] \setminus \{i\}$ there is $w \in \partial_{\boldsymbol{G}}(v)$ such that $L(w, t-1|\widetilde{G}, \boldsymbol{\sigma}) = \{j\}$. Given $\boldsymbol{\sigma}(v) = i$, we can obtain $\boldsymbol{G}$ from $\widetilde{\boldsymbol{G}}$ by connecting $v$ with each vertex $w \in [n'-1]$ such that $\boldsymbol{\sigma}(w) \neq i$ with probability $p'$ independently. Therefore,

$$\begin{aligned}
\mathbb{P}\left[\mathcal{A}(i)|\widetilde{\boldsymbol{G}}, \boldsymbol{\sigma}(v) = i\right] &= \prod_{j \neq i} 1 - (1-p')^{|Q_j(t-1|\widetilde{\boldsymbol{G}}, \boldsymbol{\sigma})|} \\
&\sim \prod_{j \neq i} 1 - \exp(-p'|Q_j(t-1|\widetilde{\boldsymbol{G}}, \boldsymbol{\sigma})|) \\
&= \prod_{j \neq i} 1 - \exp\left[-\frac{kd}{k-1} \cdot n^{-1}|Q_j(t-1|\widetilde{\boldsymbol{G}}, \boldsymbol{\sigma})|\right].
\end{aligned}$$

Furthermore, for any fixed $\delta > 0$ there is an ($n$-independent) $\varepsilon > 0$ such that given that $\mathcal{Q}^{t-1}(\varepsilon)$ occurs, we have

$$\left| q_i^t - \prod_{j \neq i} 1 - \exp\left(-\frac{kd}{k-1} \cdot n^{-1}|Q_j(t-1|\widetilde{\boldsymbol{G}}, \boldsymbol{\sigma})|\right) \right| < \delta. \tag{5.3.18}$$

Combining (5.3.17) and (5.3.18), we see that for any fixed $\delta > 0$ we have

$$\left| \mathbb{P}\left[\mathcal{A}(i)|\boldsymbol{\sigma}(v) = i\right] - kq_i^t \right| < \delta + o(1). \tag{5.3.19}$$

If $v$ is acyclic and $\boldsymbol{\sigma}(v) = i$ as well as $\mathcal{A}(i)$ occurs, then $L(v, t|\boldsymbol{G}, \boldsymbol{\sigma}) = \{i\}$. Therefore, (5.3.16) follows from (5.3.19) and Lemma 5.3.1.

Finally, the random variable $|Q_i^t(\boldsymbol{G}, \boldsymbol{\sigma})|$ satisfies the assumptions of Lemma 5.3.2. Indeed, the event $v \in Q_i(t|\boldsymbol{G}, \boldsymbol{\sigma})$ is determined solely by the sub-graph of $\boldsymbol{G}$ encompassing those vertices at distance at most $t$ from $v$. Thus, (5.3.15) and Lemma 5.3.2 imply that $\frac{1}{n}|Q_i(t|\boldsymbol{G},\boldsymbol{\sigma})| = q_i^t + o(1)$ w.h.p., as desired. $\square$

As a next step, we consider the statistics of the trees $T(v,\omega|\boldsymbol{G},\boldsymbol{\sigma})$ with $\omega \geq 0$ large but fixed as $n \to \infty$. Thus, for an isomorphism class $T$ of rooted, decorated graphs we let $H_{T,\omega}$ be the number of vertices $v$ in $\boldsymbol{G}_{\mathrm{WP}}(\omega|\boldsymbol{\sigma})$ such that $T(v,\omega|\boldsymbol{G},\boldsymbol{\sigma}) \in T$.

**Lemma 5.3.17.** *Assume that $T$ is an isomorphism class of rooted decorated trees with $\mathbb{P}[\boldsymbol{T} = T] > 0$. Then for any $\varepsilon > 0$ there is $\omega > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\mathbb{P}[\boldsymbol{T} = T] - \frac{1}{n} H_{T,\omega}\right| > \varepsilon\right] = 0.$$

*Proof.* We observe that $\mathbb{P}[\boldsymbol{T} = T]$ is a number that depends on $T$ but not on $n$. Furthermore, if $T_*$ is the isomorphism class of a rooted sub-tree of $T$, then $\mathbb{P}[\boldsymbol{T} = T_*] \geq \mathbb{P}[\boldsymbol{T} = T]$.

The proof is by induction on the height of the trees in $T$. In the case that $T$ consists of a single vertex $v$ of type $(i, \{i\})$ for some $i \in [k]$, the assertion readily follows from Lemma 5.3.16.

Let $(i_0, \ell_0)$ be the type of the root and $v = n'$. To this end, consider the graph $\widetilde{\boldsymbol{G}}$ obtained by removing $v$. By Lemma 5.3.16 the number of vertices $w$ of $\widetilde{\boldsymbol{G}}$ with $L(w,\omega|\widetilde{\boldsymbol{G}},\boldsymbol{\sigma}) = \{j\}$ is $n(q_j + o_\omega(1))$ w.h.p. for all $j$, where $o_\omega(1)$ signifies a term that tends to 0 in the limit of large $\omega$. Let $\mathcal{A}$ be the event that this is indeed the case. Moreover, let $\mathcal{B}$ be the following event:

- $\boldsymbol{\sigma}(v) = i_0$.
- for each colour $j \notin \ell_0$, vertex $v$ has a neighbour $w$ in $\widetilde{\boldsymbol{G}}$ such that $L(w,\omega|\widetilde{\boldsymbol{G}},\boldsymbol{\sigma}) = \{j\}$.
- $v$ does not have a neighbour $w$ with $L(w,\omega|\widetilde{\boldsymbol{G}},\boldsymbol{\sigma}) = \{h\}$ for any $h \in \ell_0$.

Then

$$\begin{aligned}\mathbb{P}[\mathcal{B}|\mathcal{A}] &= \frac{1}{k}\prod_{j\notin\ell_0} \mathbb{P}\left[\mathrm{Bin}(n(q_j^* + o_\omega(1)), p') > 0\right] \cdot \prod_{j\in\ell_0\setminus\{i_0\}} \mathbb{P}\left[\mathrm{Bin}(n(q_j^* + o_\omega(1)), p') = 0\right] \\ &\sim \frac{1}{k}\prod_{j\notin\ell_0} \mathbb{P}\left[\mathrm{Po}(np'(q_j^* + o_\omega(1))) > 0\right] \cdot \prod_{j\in\ell_0\setminus\{i_0\}} \mathbb{P}\left[\mathrm{Po}(np'(q_j^* + o_\omega(1))) = 0\right] \\ &= q_{i_0,\ell_0}^* + o_\omega(1).\end{aligned}$$

Since $\mathbb{P}[\mathcal{A}] \sim 1$, we find

$$\mathbb{P}[\mathcal{B}] = q_{i_0,\ell_0}^* + o_\omega(1). \tag{5.3.20}$$

Let $T_{v_0}$ be the unique tree of the isomorphism class of rooted decorated trees consisting only of the root $v_0$. Let $\mathcal{Y}_v$ be the event that $v$ has no neighbour of any type $(i', \ell') \in \mathcal{T}_{i_0,\ell_0}$. Therefore let $q_\emptyset^0 = \sum_{(i',\ell')\in\mathcal{T}_{i,\ell}} q_{i,\ell}$. We find

$$\begin{aligned}\mathbb{P}\left[\mathcal{Y}_v|\mathcal{B}\right] &= (1-p')^{n(q_\emptyset+o_\omega(1))} = o_\omega(1) + \exp(-np'q_\emptyset) \\ &= o_\omega(1) + \exp(-d'q_\emptyset) = o_\omega(1) + \mathbb{P}\left[T_{v_0} \in \boldsymbol{T}_{i_0,\ell_0}\right]. \end{aligned} \tag{5.3.21}$$

Combining (5.3.20) and (5.3.21), we find that

$$\mathbb{P}\left[\mathcal{B} \cap \mathcal{Y}_v\right] = \mathbb{P}\left[T_{v_0} \in \boldsymbol{T}\right] + o_\omega(1).$$

As for the inductive step, pick and fix one representative $T_0 \in T$. If we remove the root $v_0$ from $T_0$, then we obtain a decorated forest $T_0 - v_0$. Each tree $T'$ in this forest contains precisely one neighbour of the root of $T_0$, which we designate as the root of $T'$. Let $\mathcal{V}(T)$ be the set of all isomorphism classes of rooted decorated trees $T'$ obtained in this way. Furthermore, for each $\hat{T} \in \mathcal{V}(T)$ let $y(\hat{T})$ be the number of components of the forest $T_0 - v_0$ that belong to the isomorphism class $\hat{T}$.

We are going to show that for $v = n'$ and for $\omega = \omega(T, \varepsilon)$ sufficiently large we have

$$|\, \mathbb{P}\left[T(v, \omega|\boldsymbol{G}, \boldsymbol{\sigma}) \cong T_0\right] - \mathbb{P}\left[\boldsymbol{T} = T\right] | < \varepsilon.$$

Furthermore, for each tree $T' \in \mathcal{V}(T)$ we let $\widetilde{Q}(T')$ be the set of all vertices $w$ of $\widetilde{\boldsymbol{G}}$ such that $T(w, \omega|\widetilde{\boldsymbol{G}}, \boldsymbol{\sigma}) \cong T'$. In addition, let $\widetilde{Q}_\emptyset$ be the set of all vertices $w$ of $\widetilde{\boldsymbol{G}}$ that satisfy none of the following conditions:

- $w \in \bigcup_{T'\in\mathcal{V}(T)} Q(T')$.
- $\vartheta(w) \notin \mathcal{T}_{i_0,\ell_0}$.
- $L(w, \omega|\widetilde{\boldsymbol{G}}, \boldsymbol{\sigma}) = \{j\}$ for some $j \in [k]$.

Further, let $q(T') = \mathbb{P}\left[\boldsymbol{T} = T'\right]$ and let

$$q_\emptyset(T) = q_\emptyset^0 - \sum_{T'\in\mathcal{V}(T)} q(T').$$

Let $\mathcal{Q}$ be the event that $|\widetilde{Q}(T')|/n = q(T') + o_\omega(1)$ for all $T' \in \mathcal{V}(T)$ and that $|\widetilde{Q}_\emptyset|/n = q_\emptyset(T) + o_\omega(1)$. Then

$$\mathbb{P}\left[\mathcal{Q}\right] \sim 1$$

by induction. Letting again $\partial v = \partial_{\boldsymbol{G}}(v)$ and $\mathcal{Y}$ be the event that for each $T' \in \mathcal{V}(T)$ we have

$y(T') = |\partial v \cap \widetilde{Q}(T')|$ and $\partial v \cap \widetilde{Q}_\emptyset = \emptyset$. Then

$$\begin{aligned}\mathbb{P}[\mathcal{Y}|\mathcal{B}] &\sim \mathbb{P}[\mathcal{Y}|\mathcal{B},\mathcal{Q}]\\ &= (1-p')^{n(q_\emptyset + o_\omega(1))} \prod_{T'\in\mathcal{V}(T)} \mathbb{P}\left[\mathrm{Bin}(n(q(T')+o_\omega(1)),p') = y(T')\right]\\ &= o_\omega(1) + \exp(-np'q_\emptyset) \prod_{T'\in\mathcal{V}(T)} \mathbb{P}\left[\mathrm{Po}(np'q(T')) = y(T')\right]\\ &= o_\omega(1) + \exp(-d'q_\emptyset) \prod_{T'\in\mathcal{V}(T)} \mathbb{P}\left[\mathrm{Po}(d'q(T')) = y(T')\right]\\ &= o_\omega(1) + \mathbb{P}\left[T_0 \in \boldsymbol{T}_{i_0,\ell_0}\right]. \end{aligned} \tag{5.3.22}$$

The last equality sign follows from the fact that in tree $\boldsymbol{T}_{i_0,\ell_0}$, the root has a Poisson number of children of possible "shape" $T'$. Combining (5.3.20) and (5.3.22), we find that

$$\mathbb{P}[\mathcal{B}\cap\mathcal{Y}] = \mathbb{P}[T_0 \in \boldsymbol{T}] + o_\omega(1). \tag{5.3.23}$$

Let $\mathcal{R}$ be the event that $\partial^\omega_{\boldsymbol{G}}(v)$ is acyclic. By Lemma 5.3.1 we have $\mathbb{P}[\mathcal{R}] \sim 1$. Furthermore, given $\mathcal{R}$, we have $T(v,\omega|\boldsymbol{G},\boldsymbol{\sigma}) \in T$ if and only if the event $\mathcal{B}\cap\mathcal{Y}$ occurs. Thus, (5.3.23) implies that

$$\mathbb{P}[T(v,\omega|\boldsymbol{G},\boldsymbol{\sigma}) \in T] = \mathbb{P}[\mathcal{B}\cap\mathcal{Y}] + o(1) = \mathbb{P}[\boldsymbol{T} = T] + o_\omega(1). \tag{5.3.24}$$

Moreover, (5.3.24) shows that

$$\frac{1}{n}\mathbb{E}[H_{T,\omega}] = \mathbb{P}[\boldsymbol{T} = T] + o_\omega(1). \tag{5.3.25}$$

Finally, because the event $T(v,\omega|\boldsymbol{G},\boldsymbol{\sigma}) \in T$ is governed by the vertices at distance at most $|T| + \omega$ from $v$, Lemma 5.3.2 implies together with (5.3.25) that for any $\varepsilon > 0$ there is $\omega$ such that

$$\mathbb{P}\left[|H_{T,\omega} - \mathbb{P}[\boldsymbol{T} = T]\,| < \varepsilon n\right] = 1 - o(1).$$

This completes the induction. □

**Lemma 5.3.18.** *For any $\varepsilon > 0$, there is $\omega > 0$ such that w.h.p. all but $\varepsilon n$ vertices $v$ satisfy $T(v|\boldsymbol{G},\boldsymbol{\sigma}) = T(v,\omega|\boldsymbol{G},\boldsymbol{\sigma})$.*

*Proof.* Lemma 5.3.14 implies that $T(v|\boldsymbol{G},\boldsymbol{\sigma}) = T(v,\omega+2|\boldsymbol{G},\boldsymbol{\sigma})$, unless $T'(v,0|\boldsymbol{G},\boldsymbol{\sigma})$ contains at least $\omega$ vertices. Furthermore, Lemma 5.3.13 implies that for any fixed $\varepsilon > 0$ there is $\omega = \omega(\varepsilon)$ such that this holds for no more than $\varepsilon n$ vertices w.h.p.. □

Finally, Proposition 5.3.15 is immediate from Lemmas 5.3.17 and 5.3.18 and Proposition 5.1.5 follows from Propositions 5.3.3 and 5.3.15.

### 5.3.4. Remaining proofs

**Proof of Lemma 5.3.13.**

Set $\theta = \lceil \ln \ln n \rceil$. Moreover, for a set $S \subset V$ let $C_S$ denote the $\boldsymbol{\sigma}$-core of the subgraph of $\boldsymbol{G}$ obtained by removing the vertices in $S$. Further, for any vertex $w \in S$ let $\Lambda(w, S)$ be the set of colours $j \in [k]$ such that in $\boldsymbol{G}$ vertex $w$ does not have a neighbour in $\boldsymbol{\sigma}^{-1}(j) \cap C_S$. In addition, let us call $S$ *wobbly* in $\boldsymbol{G}$ if the following conditions are satisfied.

**W1** $|S| = \theta$.
**W2** We have $|\Lambda(w, S)| \geq 2$ for all $w \in S$.
**W3** The subgraph of $\boldsymbol{G}$ induced on $S$ has a spanning tree $T$ such that

$$\Lambda(u, S) \cap \Lambda(w, S) \neq \emptyset \quad \text{for each edge } \{u, w\} \text{ of } T.$$

Assume that $T'(v, 0|\boldsymbol{G}, \boldsymbol{\sigma})$ contains at least $\theta$ vertices. If $T = (S, E_T)$ is a sub-tree on $\theta$ vertices contained in $T'(v, 0|\boldsymbol{G}, \boldsymbol{\sigma})$, then $S$ is wobbly. Therefore, it suffices to prove that the total number $W$ of vertices that are contained in a wobbly set $S$ satisfies

$$\mathbb{E}[W] \leq \sum_{S \subset V : |S| = \theta} \theta \cdot \mathbb{P}[S \text{ is wobbly}] = o(n). \tag{5.3.26}$$

To prove (5.3.26), we need a bit of notation. For a set $S$, let $\mathcal{E}_S$ be the event that

$$|C_S \cap \boldsymbol{\sigma}^{-1}(i)| \geq \frac{n}{k}(1 - k^{-2/3}) \quad \text{for all } i \in [k]\,.$$

Then Proposition 5.3.7 implies that for any set $S$ of size $\theta$ we have

$$\mathbb{P}[\mathcal{E}_S] \geq 1 - \exp(-\Omega(n)). \tag{5.3.27}$$

Further, for a vertex $w \in S$ and a set $J_w \subset [k] \setminus \{\boldsymbol{\sigma}(w)\}$, let $\mathcal{L}(w, J_w)$ be the event that $\Lambda(w, S) \supset J_w$. Crucially, the core $C_S$ of the subgraph of $\boldsymbol{G}$ obtained by removing $S$ is independent of the edges between $S$ and $C_S$. Therefore, $w$ is adjacent to a vertex $x$ in $C_S$ with $\boldsymbol{\sigma}(x) \neq \boldsymbol{\sigma}(w)$ with probability $p'$, independently for all such vertices $x$. Consequently,

$$\mathbb{P}[\mathcal{L}(w, J_w) | \mathcal{E}_S] \leq \prod_{j \in J_w} (1 - p')^{\frac{n}{k}(1 - k^{-2/3})} \leq k^{-1.99|J_w|}. \tag{5.3.28}$$

Moreover, due to the independence of the edges in $\boldsymbol{G}$, the events $\mathcal{L}(w, J_w)$ are independent for all $w \in S$.

Let $S \subset V$ be a set of size $\theta$. Let us call a vertex $w \in S$ *rich* if $|\Lambda(w,S)| \geq \sqrt{k}$. Further, let $R_S$ be the set of rich vertices in $S$. To estimate the probability that $S$ is wobbly, we consider the following events.

- Let $\mathcal{A}_S$ be the event that $|R_S| \geq k^{-1/3}\theta$ and that $\boldsymbol{G}$ contains a tree $T$ with vertex set $S$.
- Let $\mathcal{A}'_S$ be the event that and that $\boldsymbol{G}$ contains a tree $T$ with vertex set $S$ such that

$$\sum_{w \in R_S} |\partial_T^1(w)| \geq \theta/2.$$

  (In words, the sum of the degrees of the rich vertices in $T$ is at least $\theta/2$.)
- Let $\mathcal{A}''_S$ be the event that $\boldsymbol{G}$ contains a tree $T$ with vertex set $S$ such that

$$\sum_{w \in R_S} |\partial_T^1(w)| < \theta/2.$$

- Let $\mathcal{W}_S$ be the event that condition **W2** is satisfied.
- For a given tree $T$ with vertex set $S$, let $\mathcal{W}'_{S,T}$ be the event that condition **W3** is satisfied.

If $S$ is wobbly, then the event $\mathcal{A}_S \cup (\mathcal{W}_S \cap \mathcal{A}'_S) \cup (\mathcal{W}_S \cap \mathcal{W}'_{S,T} \cap \mathcal{A}''_S)$ for a tree $T$ occurs. Therefore,

$$\mathbb{P}[S \text{ is wobbly}] \leq \mathbb{P}[\mathcal{A}_S] + \mathbb{P}\left[\mathcal{W}_S \cap \mathcal{A}'_S \setminus \mathcal{A}_S\right] + \mathbb{P}\left[\mathcal{W}_S \cap \mathcal{W}'_{S,T} \cap \mathcal{A}''_S \setminus (\mathcal{A}_S \cup \mathcal{A}'_S)\right]. \quad (5.3.29)$$

In the following, we are going to estimate the three probabilities on the r.h.s. separately.

With respect to the probability of $\mathcal{A}_S$, (5.3.27) and (5.3.28) yield

$$\begin{aligned}\mathbb{P}\left[|R_S| \geq k^{-1/3}\theta\right] &\leq \mathbb{P}[\neg\mathcal{E}_S] + \mathbb{P}\left[\exists R \subset S, |R| = \lceil k^{-1/3}\theta\rceil : \forall w \in R : |\Lambda(w,S)| \geq \sqrt{k}|\mathcal{E}_S\right] \\ &\leq \exp(-\Omega(n)) + \binom{\theta}{k^{-1/3}\theta}\left[\binom{k}{\sqrt{k}}k^{-1.9\sqrt{k}}\right]^{k^{-1/3}\theta} \\ &\leq \exp(-\sqrt{k}\theta).\end{aligned}$$

Furthermore, by Cayley's formula there are $\theta^{\theta-2}$ possible trees with vertex set $S$. Since any two vertices in $S$ are connected in $\boldsymbol{G}$ with probability at most $p'$, and because edges occur independently, we obtain

$$\mathbb{P}[\mathcal{A}_S] \leq \theta^{\theta-2}p'^{\theta-1} \cdot \mathbb{P}\left[|R_S| \geq k^{-1/3}\theta\right] \leq \theta^{\theta-2}p'^{\theta-1}\exp(-\sqrt{k}\theta). \quad (5.3.30)$$

To bound the probability of $\mathcal{W}_S \cap \mathcal{A}'_S \setminus \mathcal{A}_S$, let $R \subset S$. Moreover, let $e(S)$ denote the total number of edges spanned by $S$ in $\boldsymbol{G}$, and let $e(R,S)$ denote the number of edges that join a vertex in $R$ with another vertex in $S$. Let $\mathcal{A}'_S(R,t)$ be the event $e(S) \geq \theta - 1$ and $e(R,S) = t$. If $\mathcal{A}'_S \setminus \mathcal{A}_S$ occurs, then there exist $R \subset S$, $|R| \leq r = \lfloor k^{-1/3}\theta \rfloor$, and $t \geq \theta/4$ such that $\mathcal{A}'_S(R,t)$ occurs. Therefore, by the union bound,

$$\mathbb{P}\left[\mathcal{W}_S \cap \mathcal{A}'_S \setminus \mathcal{A}_S\right] \quad \leq \quad \sum_{R \subset S : |R| \leq r} \sum_{t \geq \theta/4} \mathbb{P}\left[\mathcal{W}_S \cap \mathcal{A}'_S(R,t)\right]. \tag{5.3.31}$$

Further, because the event $\mathcal{W}_S$ is independent of the subgraph of $\boldsymbol{G}$ induced on $S$, (5.3.31) yields

$$\mathbb{P}\left[\mathcal{W}_S \cap \mathcal{A}'_S \setminus \mathcal{A}_S\right] \quad \leq \quad \mathbb{P}\left[\mathcal{W}_S\right] \cdot \sum_{R \subset S : |R| \leq r} \sum_{t \geq \theta/4} \mathbb{P}\left[\mathcal{A}'_S(R,t)\right]. \tag{5.3.32}$$

Because any two vertices in $S$ are connected with probability at most $p'$ independently, the random variable $e(R,S)$ is stochastically dominated by a binomial distribution $\mathrm{Bin}(r\theta, p')$. Therefore,

$$\mathbb{P}\left[e(R,S) = t\right] \leq \mathbb{P}\left[\mathrm{Bin}(r\theta, p') = t\right] \leq \binom{r\theta}{t} p'^{t}. \tag{5.3.33}$$

Similarly, we find

$$\begin{aligned} \mathbb{P}\left[e(S) \geq \theta - 1 | e(R,S) = t\right] &\leq \mathbb{P}\left[\mathrm{Bin}\left(\binom{\theta}{2}, p'\right) \geq \theta - t - 1\right] \\ &\leq \binom{\theta^2/2}{\theta - t - 1} p'^{\theta - t - 1}. \end{aligned} \tag{5.3.34}$$

Combining (5.3.33) and (5.3.34), we get

$$\mathbb{P}\left[\mathcal{A}'_S(R,t)\right] \quad \leq \quad \binom{r\theta}{t} \binom{\theta^2/2}{\theta - t - 1} p^{\theta - 1}. \tag{5.3.35}$$

Further, plugging (5.3.35) into (5.3.32), we obtain

$$\begin{aligned} \mathbb{P}\left[\mathcal{W}_S \cap \mathcal{A}'_S \setminus \mathcal{A}_S\right] &\leq \mathbb{P}\left[\mathcal{W}_S\right] \cdot 2^{\theta} p^{\theta - 1} \sum_{t \geq \theta/4} \binom{r\theta}{t} \binom{\theta^2/2}{\theta - t - 1} \\ &\leq 2^{1+\theta} p^{\theta - 1} \, \mathbb{P}\left[\mathcal{W}_S\right] \binom{r\theta}{\theta/4} \binom{\theta^2/2}{3\theta/4 - 1} \\ &\leq 2^{1+\theta} p^{\theta - 1} \, \mathbb{P}\left[\mathcal{W}_S\right] \left(\frac{\mathrm{e} r\theta}{\theta/4}\right)^{\theta/4} \left(\frac{\mathrm{e}\theta^2/2}{3\theta/4}\right)^{3\theta/4} \\ &\leq \theta^{\theta} p^{\theta - 1} k^{-\theta/13} \, \mathbb{P}\left[\mathcal{W}_S\right]. \end{aligned} \tag{5.3.36}$$

Finally, if the event $\mathcal{W}_S$ occurs, then for each $w \in S$ there is $j \in [k] \setminus \{\boldsymbol{\sigma}(w)\}$ such that $j \in \Lambda(w, S)$. Thus, (5.3.27) and (5.3.28) yield

$$\mathbb{P}[\mathcal{W}_S] \le \mathbb{P}[\neg\mathcal{E}_S] + \prod_{w \in S} \sum_{j \neq \boldsymbol{\sigma}(w)} \mathbb{P}[\mathcal{L}(w, \{j\}) | \mathcal{E}_S]$$
$$\le \exp(-\Omega(n)) + k^{-0.99\theta} \le k^{-0.98\theta}. \tag{5.3.37}$$

Combining (5.3.36) and (5.3.37), we arrive at

$$\mathbb{P}\left[\mathcal{W}_S \cap \mathcal{A}'_S \setminus \mathcal{A}_S\right] \quad \le \quad \theta^\theta p^{\theta-1} k^{-1.02\theta}. \tag{5.3.38}$$

To bound the probability of $\mathcal{A}''_S$, suppose that $T$ is a tree with vertex set $S$, let $U \subset S$ and denote by $\mathcal{A}''_S(T, U)$ the event that the following statements are true:

(i) $T$ is contained as a subgraph in $\boldsymbol{G}$.
(ii) Let $s_0 = \min S$ and consider $s_0$ the root of $T$. Then for each $u \in U$ the parent $P(u)$ satisfies $P(u) \notin R_S$.

If the event $\mathcal{A}''_S \setminus (\mathcal{A}_S \cup \mathcal{A}'_S)$ occurs, then there exist a tree $T$ and a set $U$ of size $|U| \ge \theta/3$ such that $\mathcal{A}''_S(T, U)$ occurs. Therefore,

$$\mathbb{P}\left[\mathcal{W}_S \cap \mathcal{W}'_{S,T} \cap \mathcal{A}''_S \setminus (\mathcal{A}_S \cup \mathcal{A}'_S)\right] \le \sum_T \sum_{U:|U| \ge \theta/3} \mathbb{P}\left[\mathcal{W}_S \cap \mathcal{W}'_{S,T} \cap \mathcal{A}''_S(T, U)\right]. \tag{5.3.39}$$

Fix a tree $T$ on $S$ and a set $U \subset S$, $|U| \ge \theta/3$. Since any two vertices are connected in $\boldsymbol{G}$ with probability at most $p'$ independently, the probability that (i) occurs is bounded by ${p'}^{\theta-1}$. Furthermore, if (ii) occurs and $u \in U$, then $|\Lambda(P(u), S)| \le \sqrt{k}$ because $P(u)$ is not rich. In addition, **W3** requires that $\Lambda(P(u), S) \cap \Lambda(u, S) \neq \emptyset$. There are two ways how this can come about: first, it could be that $\Lambda(P(u), S) \cap \Lambda(u, S) \setminus \{\boldsymbol{\sigma}(u)\} \neq \emptyset$. Then the event $\mathcal{L}(u, \{j\})$ occurs for some $j \in \Lambda(P(u), S) \setminus \{\boldsymbol{\sigma}(u)\}$. Hence, due to (5.3.28)

$$\mathbb{P}\left[\Lambda(P(u), S) \cap \Lambda(u, S) \setminus \{\boldsymbol{\sigma}(u)\} \neq \emptyset | \mathcal{E}_S, |\Lambda(P(u), S)| \le \sqrt{k}\right] \le k^{-1.49} \tag{5.3.40}$$

for any $u \in U$.

Alternatively, it could be that $\boldsymbol{\sigma}(u) \in \Lambda(P(u), S)$. Given that $\Lambda(P(u), S)$ has size at most $\sqrt{k}$, the probability of this event is bounded by $k^{-1/2}$ because $\boldsymbol{\sigma}(u)$ is random. Additionally, by **W2** there is another colour $j \in \Lambda(u)$, $j \neq \boldsymbol{\sigma}(u)$. Hence, the event $\mathcal{L}(u, \{j\})$ occurs and (5.3.28) yields

$$\mathbb{P}\left[\boldsymbol{\sigma}(u) \in \Lambda(P(u), S), \Lambda(u, S) \setminus \{\boldsymbol{\sigma}(u)\} \neq \emptyset | \mathcal{E}_S, |\Lambda(P(u), S)| \leq \sqrt{k}\right] \leq k^{-1.49} \tag{5.3.41}$$

for any $u \in U$.

Combining (5.3.27), (5.3.40) and (5.3.41), we find

$$\mathbb{P}\left[\forall u \in U : \Lambda(P(u), S) \cap \Lambda(u, S) \neq \emptyset \wedge |\Lambda(P(u), S)| \leq \sqrt{k}\right] \leq \exp(-\Omega(n)) + k^{-1.48|U|}. \tag{5.3.42}$$

In addition, if $w \in S \setminus U$, then **W2** requires that the event $\mathcal{L}(w, \{j\})$ occurs for some $j \neq \boldsymbol{\sigma}(w)$ and (5.3.28) yields

$$\mathbb{P}\left[\forall w \in S \setminus U : \exists j \in [k] \setminus \{\boldsymbol{\sigma}(w)\} : \mathcal{L}(w, j) | \mathcal{E}_S\right] \leq k^{-0.99|S \setminus U|}. \tag{5.3.43}$$

Combining (5.3.42) and (5.3.43), we obtain

$$\mathbb{P}\left[\mathcal{W}_S \cap \mathcal{W}'_{S,T} \cap \mathcal{A}''_S(T, U) | T \subset \boldsymbol{G}\right] \leq \exp(-\Omega(n)) + k^{-0.99(\theta - |U|)} \cdot k^{-1.48|U|} \leq k^{-1.1\theta}. \tag{5.3.44}$$

Further, the probability that $T$ is contained in $\boldsymbol{G}$ is bounded by ${p'}^{\theta-1}$. Thus, (5.3.44) implies

$$\mathbb{P}\left[\mathcal{W}_S \cap \mathcal{W}'_{S,T} \cap \mathcal{A}''_S(T, U)\right] \leq k^{-1.1\theta} {p'}^{\theta-1}. \tag{5.3.45}$$

Finally, combining (5.3.39) and (5.3.45) and using Cayley's formula, we obtain

$$\begin{aligned} \mathbb{P}\left[\mathcal{W}_S \cap \mathcal{W}'_S \cap \mathcal{A}''_S \setminus (\mathcal{A}_S \cup \mathcal{A}'_S)\right] &\leq 2^\theta \theta^{\theta-2} k^{-1.1\theta} {p'}^{\theta-1} \\ &\leq \theta^{\theta-2} {p'}^{\theta-1} k^{-1.09\theta}. \end{aligned} \tag{5.3.46}$$

Plugging (5.3.30), (5.3.38) and (5.3.46) into (5.3.29), we see that

$$\theta\, \mathbb{P}\left[S \text{ is wobbly}\right] \leq 2\theta^{\theta+1} p^{\theta-1} k^{-1.02\theta}.$$

Hence, (5.3.26) yields

$$\begin{aligned} \mathbb{E}\left[W\right] &\leq 2\theta^{\theta+1} {p'}^{\theta-1} k^{-1.02\theta} \cdot \binom{n}{\theta} \leq 2\left(\frac{\mathrm{e}n}{\theta}\right)^\theta \theta^{\theta+1} {p'}^{\theta-1} k^{-1.02\theta} \\ &\leq n(3np')^\theta k^{-1.02\theta} \leq n(7k \ln k)^\theta k^{-1.02\theta} = o(n), \end{aligned}$$

as desired.

**Proof of Lemma 5.3.2.**

The following large deviations inequality known as *Warnke's inequality* facilitates the proof of Lemma 5.3.2.

**Lemma 5.3.19** ([War16]). *Let $X_1, \dots, X_N$ be independent random variables with values in a finite set $\Lambda$. Assume that $f : \Lambda^N \to \mathbb{R}$ is a function, that $\Gamma \subset \Lambda^N$ is an event and that $c, c' > 0$ are numbers such that the following is true.*

> *If $x, x' \in \Lambda^N$ are such that there is $k \in [N]$ such that $x_i = x'_i$ for all $i \neq k$, then*
> $$|f(x) - f(x')| \leq \begin{cases} c & \text{if } x \in \Gamma, \\ c' & \text{if } x \notin \Gamma. \end{cases} \tag{5.3.47}$$

*Then for any $\gamma \in (0, 1]$ and any $t > 0$ we have*

$$\mathbb{P}\left[|f(X_1, \dots, X_N) - \mathbb{E}[f(X_1, \dots, X_N)]| > t\right] \leq 2\exp\left(-\frac{t^2}{2N(c + \gamma(c' - c))^2}\right) + \frac{2N}{\gamma}\,\mathbb{P}\left[(X_1, \dots, X_N) \notin \Gamma\right].$$

*Proof of Lemma 5.3.2.* The proof is based on Lemma 5.3.19. Of course, we can view $(\boldsymbol{G}, \boldsymbol{\sigma})$ as chosen from a product space $X_2, \dots, X_N$ with $N = 2n'$ where $X_i$ is a $0/1$ vector of length $i - 1$ whose components are independent $\mathrm{Be}(p')$ variables for $2 \leq i \leq n'$ and where $X_i \in [k]$ is uniformly distributed for $i > \binom{n'}{2}$ ("vertex exposure"). Let $\Gamma$ be the event that $|N^{\omega}(v)| \leq \lambda = n^{0.01}$ for all vertices $v$. Then by Lemma 5.3.1 we have

$$\mathbb{P}[\Gamma] \quad \geq \quad 1 - \exp(-\Omega(\ln^2 n)). \tag{5.3.48}$$

Furthermore, let $\boldsymbol{G}'$ be the graph obtained from $\boldsymbol{G}$ by removing all edges $e$ that are incident with a vertex $v$ such that $|\partial^{\omega}_{\boldsymbol{G}}(v)| > \lambda$ and let

$$S' = \sum_v S_v(\boldsymbol{G}', \boldsymbol{\sigma}) = \left|\left\{v \in [n'] : \partial^{\omega}_{\boldsymbol{G}'}(v), \boldsymbol{\sigma}|_{\partial^{\omega}_{\boldsymbol{G}'}(v)}, v) \in \mathcal{S}\right\}\right|.$$

If $\Gamma$ occurs, then $S = S'$. Hence, (5.3.48) implies that

$$\mathbb{E}[S'] \quad = \quad \mathbb{E}[S] + o(1). \tag{5.3.49}$$

Moreover, the random variable $S' = f(X_2, \dots, X_N)$ satisfies (5.3.47) with $c = \lambda$ and $c' = n'$. Indeed, altering either the colour of one vertex $u$ or its set of neighbours can only affect those vertices $v$ that are at distance at most $\omega$ from $u$, and in $\boldsymbol{G}'$ there are no more than $\lambda$ such vertices. Thus, Lemma 5.3.19 applied with, say, $t = n^{2/3}$ and $\gamma = 1/n$ and (5.3.48) yields

$$\mathbb{P}\left[|S' - \mathbb{E}[S']| > t\right] \leq \exp(-\Omega(\ln^2 n)) = o(1). \tag{5.3.50}$$

Finally, the assertion follows from (5.3.49) and (5.3.50). □

# 6 Condensation phase transition in random hypergraph 2-colouring for finite inverse temperatures

This chapter is dedicated to proving Theorem 4.1.4, which establishes the existence and determines the location of the condensation phase transition in random $k$-uniform hypergraph 2-colouring with additional temperature parameter $\beta$ for large values of $k$.

Large parts of this chapter are a verbatim copy or a close adaption of the content of the paper *A positive temperature phase transition in random hypergraph 2-coloring* [BCOR16] that is joint work with Victor Bapst and Amin Coja-Oghlan and is published in the *Annals of Applied Probability 26 (2016)*.

The first section of this chapter presents an outline of the proof of Theorem 4.1.4 and gives a short introduction to the proof ideas. Subsequently, the first and second moment of $Z_\beta$ are determined in Section 6.2. Calculations in Section 6.3 are performed in the planted model and the expected cluster size is established in Section 6.4. The last section can be seen as a kind of appendix where we prove the existence of the free entropy density $\Phi_{d,k}(\beta)$ for finite $\beta$.

The author of this thesis contributed primarily to the investigation of the first and second moment presented in Section 6.2, to the calculations in the planted model performed in Section 6.3 and to the proof of the existence of $\Phi_{d,k}(\beta)$ in Section 6.5. Furthermore she carried out revision work of all presented proofs and statements.

*Throughout the whole chapter we assume that* $0 \leq d/k \leq 2^{k-1}\ln 2 + O_k(1)$. *We let* $m = \lceil dn/k \rceil$.

## 6.1. Outline of the proof

The proof of Theorem 4.1.4 is based on establishing the physicists' notion of an "entropy crisis" (cf. Section 2.3) rigorously. To this end, we are going to trace two key quantities. First, the free entropy density $\Phi_{d,k}(\beta)$ defined in (2.3.2), which we examine here for the random hypergraph $H_k(n,p)$, i. e.

$$\Phi_{d,k}(\beta) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\ln Z_\beta(H_k(n,p))\right].$$

Thus, $\Phi_{d,k}(\beta)$ mirrors the typical value of the partition function $Z_\beta(H_k(n,p))$. Second, the size of the cluster of a typical $\sigma$ chosen from the Boltzmann distribution. More specifically, we are going to argue that it is sufficient to study the cluster size defined in (2.4.1) in the planted model. Ultimately, it will emerge that the condensation phase transition marks the point where the cluster size in the planted model equals the typical value of $Z_\beta(H_k(n,p))$.

To implement this strategy, we begin by deriving upper and lower bounds on $\Phi_{d,k}(\beta)$ via the first and the second moment method. More precisely, in Subsection 6.2.1 we are going to prove the following.

**Proposition 6.1.1.** *For any $\beta$, we have*

$$\Phi_{d,k}(\beta) \leq \ln 2 + \frac{d}{k} \ln\Big(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\Big).$$

*Moreover, if either $d/k \leq 2^{k-1}\ln 2 - 2$ and $\beta \geq 0$ or $d/k > 2^{k-1}\ln 2 - 2$ and $\beta \leq k\ln 2 - \ln k$, we have*

$$\Phi_{d,k}(\beta) = \ln 2 + \frac{d}{k} \ln\Big(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\Big).$$

We remember the quantity $\beta_{\mathrm{crit}}(d,k)$ defined in (2.5.5). Then Proposition 6.1.1 readily implies the following lower bounds on $\beta_{\mathrm{crit}}(d,k)$.

**Corollary 6.1.2.** *We have $\beta_{\mathrm{crit}}(d,k) \geq k\ln 2 - \ln k$. If $d/k \leq 2^{k-1}\ln 2 - 2$, then $\beta_{\mathrm{crit}}(d,k) = \infty$.*

It is well-known that $\ln Z_\beta$ enjoys the following "Lipschitz property".

**Fact 6.1.3.** *Let $H$ be a hypergraph and $H'$ obtained from $H$ by either adding or removing a single edge. Then $|\ln Z_\beta(H) - \ln Z_\beta(H')| \leq \beta$.*

This Lipschitz property implies the following concentration bound for $\ln Z_\beta(H_k(n,p))$.

**Lemma 6.1.4.** *For any $\alpha > 0$ there is $\delta = \delta(\alpha) > 0$ such that*

$$\mathbb{P}\left[|\ln Z_\beta(H_k(n,p)) - \mathbb{E}[\ln Z_\beta(H_k(n,p))]| > \alpha n\right] < \exp\left[-\delta n\right].$$

*Proof.* This is immediate from Fact 6.1.3 and McDiarmid's inequality [McD98, Theorem 3.8]. □

The second main component of the proof of Theorem 4.1.4 is the analysis of the cluster size in the planted model. First, we observe that for the cluster size in $H_k(n,p)$ we have a concentration bound analogous to Lemma 6.1.4:

**Lemma 6.1.5.** *For any $\sigma : [n] \to \{\pm 1\}$ and $\alpha > 0$, there is $\delta = \delta(\alpha, \sigma) > 0$ such that*

$$\mathbb{P}\left[|\ln \mathcal{C}_\beta(H_k(n,p),\sigma) - \mathbb{E}[\ln \mathcal{C}_\beta(H_k(n,p),\sigma)]| > \alpha n\right] < \exp\left[-\delta n\right].$$

*Proof.* This follows from McDiarmid's inequality [McD98, Theorem 3.8] and because it holds that $|\ln \mathcal{C}_\beta(H,\sigma) - \ln \mathcal{C}_\beta(H',\sigma)| \leq \beta$ for any $\sigma$ if the hypergraph $H'$ is obtained from the hypergraph $H$ by either adding or removing a single edge. □

Ideally, we would like to compare the cluster size of an assignment $\sigma$ chosen from the Boltzmann distribution on $H_k(n,p)$ with the partition function $Z_\beta(H_k(n,p))$. Then according to the physicists' prediction of the "entropy crisis", the condensation phase transition should mark the point $\beta$ where $\mathcal{C}_\beta(H_k(n,p),\sigma)$ is of the same order of magnitude as $Z_\beta(H_k(n,p))$. However, it seems difficult to calculate $\mathcal{C}_\beta(H_k(n,p),\sigma)$ directly, as the Boltzmann distribution on a randomly generated hypergraph is a very difficult object to approach directly.

We explained this phenomenon in detail in Section 3.1, where we introduced the planted model. It will emerge that the planted model is sufficient to pin down the condensation phase transition. However, we have to refine the definitions from Section 3.1 in the following way to adapt them to the case of finite $\beta$:

Let $\boldsymbol{\sigma} : [n] \to \{\pm 1\}$ be a map chosen uniformly at random. Moreover, given $d, k, \beta$, set

$$p_1 = \frac{\exp\left[-\beta\right]}{1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)} \cdot \frac{d}{\binom{n-1}{k-1}}, \qquad p_2 = \frac{1}{1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)} \cdot \frac{d}{\binom{n-1}{k-1}}.$$

Now, obtain a random $k$-uniform hypergraph $\boldsymbol{H}$ by inserting each hyperedge that is monochromatic under $\boldsymbol{\sigma}$ with probability $p_1$ and each hyperedge that is bichromatic under $\boldsymbol{\sigma}$ with probability $p_2$ independently. In symbols, for any hypergraph $H$ with vertex set $[n]$ we have

$$\mathbb{P}\left[\boldsymbol{H} = H | \boldsymbol{\sigma}\right] = p_1^{E_H(\boldsymbol{\sigma})}(1-p_1)^{m_1} p_2^{e(H)-E_H(\boldsymbol{\sigma})}(1-p_2)^{m_2},$$

where $e(H)$ denotes the total number of hyperedges of $H$ and $m_1$ (respectively $m_2$) the numbers of hyperedges that are monochromatic (respectively bichromatic) under $\boldsymbol{\sigma}$ and are *not* in $H$.

The following proposition, which we will prove in Section 6.3, reduces the problem of determining $\beta_{\text{crit}}(d,k)$ to that of calculating $\mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma})$.

**Proposition 6.1.6.** *Assume that $d/k = 2^{k-1}\ln 2 + O_k(1)$ and $\beta_0 \geq k\ln 2 - \ln k$. If for all $k \ln 2 - \ln k \leq \beta \leq \beta_0$ we have*

$$\lim_{\varepsilon \searrow 0} \liminf_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) \leq \ln 2 + \frac{d}{k}\ln\Big(1 - 2^{1-k}\left(1-\exp\left[-\beta\right]\right)\Big) - \varepsilon\right] = 1, \tag{6.1.1}$$

*then* $\beta_0 \leq \beta_{\text{crit}}(d,k)$. *Conversely, if*

$$\lim_{\varepsilon\searrow 0}\liminf_{n\to\infty}\mathbb{P}\left[\frac{1}{n}\ln\mathcal{C}_{\beta_0}(\boldsymbol{H},\boldsymbol{\sigma}) \geq \ln 2 + \frac{d}{k}\ln\left(1-2^{1-k}\left(1-\exp\left[-\beta_0\right]\right)\right)+\varepsilon\right]=1, \tag{6.1.2}$$

*then* $\beta_0 \geq \beta_{\text{crit}}(d,k)$.

Finally, in Section 6.4 we are going to estimate the cluster size $\mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma})$ to derive the following result.

**Proposition 6.1.7.** *Assume that* $d/k = 2^{k-1}\ln 2 + O_k(1)$ *and* $\beta \geq k\ln 2 - \ln k$. *Then w.h.p. the cluster size in the planted model satisfies*

$$\frac{1}{n}\ln\mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) = \frac{\ln 2}{2^k} - \frac{\beta\ln 2}{\exp\left[\beta\right]} + \tilde{O}_k(4^{-k}).$$

*Proof of Theorem 4.1.4.* The result of the theorem in the case $d/k \leq 2^{k-1}\ln 2 - 2$ follows from Corollary 6.1.2. Let us thus assume that $d/k = 2^{k-1}\ln 2 + O_k(1)$. Because we will use Proposition 6.1.6, we can also assume that $\beta \geq k\ln 2 - \ln k$. We write $c_k = d/k - 2^{k-1}\ln 2 + \ln 2$ and $b_k = \beta - k\ln 2$. With Proposition 6.1.7, we have w.h.p.

$$\begin{aligned}
&\frac{1}{n}\ln\mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) - \left(\ln 2 + \frac{d}{k}\ln\left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)\right)\\
=&\left(\frac{\ln 2}{2^k} - (k\ln 2 + b_k)\ln 2\frac{\exp\left[-b_k\right]}{2^k}\right) - \left(\frac{\ln 2}{2^k} - \frac{c_k}{2^{k-1}} + \frac{\ln 2\exp\left[-b_k\right]}{2^k}\right) + \tilde{O}_k(4^{-k})\\
=&\frac{1}{2^k}\left(2c_k - (k\ln 2 + b_k + 1)\ln 2\exp\left[-b_k\right]\right) + \tilde{O}_k(4^{-k})\\
=&\frac{1}{2^k}\left(-\Sigma_{k,d}(\beta) + \tilde{O}_k(2^{-k})\right).
\end{aligned}$$

The equation $\Sigma_{k,d}(\beta) = 0$ has exactly one solution $\beta_{\text{cond}}(d,k) \geq k\ln 2 - \ln k$ for $d/k > 2^{k-1}\ln 2 - \ln 2$, and no such solution for $d/k < 2^{k-1}\ln 2 - \ln 2$. Moreover $\Sigma_{k,d}(\beta)$ is smooth for $d/k > 2^{k-1}\ln 2 - \ln 2 + 2^{-k}$, with derivatives of order $\Omega(k^{-4})$. Consequently there is $\varepsilon_k = \tilde{O}_k(2^{-k})$ such that the following is true.

1. If $d/k < 2^{k-1}\ln 2 - \ln 2 - \varepsilon_k$, then w.h.p. for all $\beta \geq k\ln 2 - \ln k$,

$$\frac{1}{n}\ln\mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) \leq \left(\ln 2 + \frac{d}{k}\ln\left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)\right) - \Omega(1)$$

2. If $d/k > 2^{k-1}\ln 2 - \ln 2 + \varepsilon_k$, then w.h.p. for all $\beta \geq k\ln 2 - \ln k$:

- if $\beta \leq \beta_{\text{cond}}(d,k) - \varepsilon_k$ then

$$\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) \leq \left(\ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)\right) - \Omega(1)$$

- if $\beta \geq \beta_{\text{cond}}(d,k) + \varepsilon_k$ then

$$\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) \geq \left(\ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)\right) + \Omega(1).$$

The proof of the theorem is completed by using Proposition 6.1.6. □

## 6.2. The first and the second moment

In this section we prove Proposition 6.1.1 and also lay the foundations for the proof of Proposition 6.1.6. We let $m = \lceil dn/k \rceil$ and recall that $H_k(n,m)$ signifies the hypergraph on $[n]$ obtained by choosing $m$ edges uniformly at random without replacement while to create $\mathcal{H}(n,m)$ we choose $m$ edges $\boldsymbol{e}_1, ..., \boldsymbol{e}_m$ with replacement uniformly and independently at random, thereby allowing for multiple edges.

### 6.2.1. The first moment

We begin with the following estimate of the first moment of $Z_\beta$ in $\mathcal{H}(n,m)$.

**Lemma 6.2.1.** *We have* $\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right] = \Theta\left(2^n\left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)^m\right)$.

The proof of Lemma 6.2.1 is straightforward, but we carry it out at leisure to introduce some notation that will be used throughout. For a map $\sigma : [n] \to \{\pm 1\}$, let

$$\mathcal{F}(\sigma) = \binom{|\sigma^{-1}(-1)|}{k} + \binom{|\sigma^{-1}(1)|}{k}$$

be the number of "forbidden $k$-sets" of vertices that are identically coloured under $\sigma$. The function $x \mapsto \binom{x}{k} + \binom{n-x}{k}$ is convex and takes its minimal value at $x = \frac{n}{2}$. Therefore,

$$\mathcal{F}(\sigma) \geq 2\binom{n/2}{k} = 2^{1-k}N(1+O(1/n)) = 2^{1-k}N + O(N/n), \qquad \text{with } N = \tbinom{n}{k}. \tag{6.2.1}$$

As introduced in Section 2.2, we call $\sigma$ *balanced* if $\left||\sigma^{-1}(1)| - \frac{n}{2}\right| \leq \sqrt{n}$. Let $\text{Bal} = \text{Bal}(n)$ be the set of all balanced maps $\sigma : [n] \to \{\pm 1\}$. Stirling's formula yields $|\mathrm{Bal}| = \Omega(2^n)$. If $\sigma \in \mathrm{Bal}$, then

$$\mathcal{F}(\sigma) \leq \binom{n/2+\sqrt{n}}{k} + \binom{n/2-\sqrt{n}}{k} = 2^{1-k}N + O(N/n). \tag{6.2.2}$$

For a hypergraph $H$, let

$$Z_{\beta,\mathrm{bal}}(H) = \sum_{\sigma\in\mathrm{Bal}} \exp\left[-\beta E_H(\sigma)\right].$$

*Proof of Lemma 6.2.1.* By the independence of edges in the random hypergraph $\mathcal{H}(n,m)$ we have

$$\begin{aligned}\mathbb{E}\left[\exp\left[-\beta E_{\mathcal{H}(n,m)}(\sigma)\right]\right] &= \mathbb{E}\left[\prod_{i=1}^{m}\exp\left[-\beta\mathbf{1}_{\boldsymbol{e}_i\in\mathcal{F}(\sigma)}\right]\right] = \prod_{i=1}^{m}\mathbb{E}\left[\exp\left[-\beta\mathbf{1}_{\boldsymbol{e}_i\in\mathcal{F}(\sigma)}\right]\right]\\ &= \left(1-N^{-1}\mathcal{F}(\sigma)\left(1-\exp\left[-\beta\right]\right)\right)^m\\ &\leq \left(1-2^{1-k}(1+O(1/n))\left(1-\exp\left[-\beta\right]\right)\right)^m.\end{aligned}$$

Consequently,

$$\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right] = O\left(2^n\left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)^m\right). \tag{6.2.3}$$

If $\sigma\in\mathrm{Bal}$, by (6.2.2) we have $\mathbb{E}\left[\exp\left[-\beta E_{\mathcal{H}(n,m)}(\sigma)\right]\right] = \Omega\left(\left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)^m\right)$. Therefore,

$$\begin{aligned}\mathbb{E}[Z_\beta(\mathcal{H}(n,m))] &\geq |\mathrm{Bal}|\cdot\Omega\left(\left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)^m\right)\\ &= \Omega\left(2^n\left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)^m\right).\end{aligned} \tag{6.2.4}$$

Thus, Lemma 6.2.1 follows from (6.2.3) and (6.2.4). □

The following lemma relates the expectation of the partition functions of the models $H_k(n,m)$ and $\mathcal{H}(n,m)$.

**Lemma 6.2.2.** *We have* $\mathbb{E}\left[Z_\beta(H_k(n,m))\right] = \Theta\left(\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right]\right)$.

*Proof.* Let $\mathcal{A}$ be the event that $\mathcal{H}(n,m)$ has no multiple edges. Then, using Fact 2.1.2 we get

$$\mathbb{E}[Z_\beta(\mathcal{H}(n,m))] \geq \mathbb{E}[Z_\beta(\mathcal{H}(n,m))|\mathcal{A}]\,\mathbb{P}\left[\mathcal{A}\right] \geq \mathbb{E}[Z_\beta(H_k(n,m))](1-o(1)),$$

implying that

$$\mathbb{E}[Z_\beta(H_k(n,m))] \leq O(1)\mathbb{E}[Z_\beta(\mathcal{H}(n,m))]. \tag{6.2.5}$$

On the other hand let $m_0 = \frac{2^{1-k}\exp[-\beta]}{1-2^{1-k}(1-\exp[-\beta])} m$ and

$$f(x) = -x\beta - x\ln x - (1-x)\ln(1-x) + x\ln(2^{1-k}) + (1-x)\ln(1-2^{1-k}).$$

We observe that $f$ is strictly concave and attains its maximum at $x = \frac{m_0}{m}$ where it is equal to $\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)$. For $\sigma \in \mathrm{Bal}$, we get with Stirling's formula

$$\begin{aligned}
\mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right]\right] &= \sum_{\mu} \mathbb{P}\left[E_{H_k(n,m)} = \mu\right] \exp\left[-\beta\mu\right] \\
&\geq \sum_{\mu \in [m_0 - \sqrt{m}, m_0 + \sqrt{m}]} \exp\left[-\beta\mu\right] \frac{\binom{m}{\mu}(\mathcal{F}(\sigma))^{\mu}(N - \mathcal{F}(\sigma))^{m-\mu}}{N^m} \\
&= \sum_{\mu \in [m_0 - \sqrt{m}, m_0 + \sqrt{m}]} \Theta_m\left(\frac{1}{\sqrt{m}}\right) \exp\left[mf\left(\frac{m_0}{m}\right)\right] \Theta(1) \\
&= \Theta\left(\left(1 - 2^{1-k}(1 - \exp\left[-\beta\right]\right)^m\right)
\end{aligned}$$

Therefore,

$$\mathbb{E}[Z_\beta(H_k(n,m))] \geq |\mathrm{Bal}| \cdot \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right]\right] = \Omega\left(2^n\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)^m\right)\right). \tag{6.2.6}$$

Combining (6.2.5), Lemma 6.2.1 and (6.2.6) proves the assertion. □

As a further consequence of Lemma 6.2.1, we obtain

**Corollary 6.2.3.** *1. We have $\Phi_{d,k}(\beta) \leq \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)$ for all $d, \beta$.*

*2. Assume that $d, \beta$ are such that*

$$\limsup_{n\to\infty} \frac{1}{n}\mathbb{E}[\ln Z_\beta(\mathcal{H}(n,m))] < \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right).$$

*Then $\Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)$.*

To prove this corollary, we need the following installment of the Chernoff bound on the tails of a binomially distributed random variable.

**Lemma 6.2.4** ([JLR00, p.29]). *Assume that $X_1, \ldots, X_n$ are independent random variables such that $X_i$ has a Bernoulli distribution with mean $p_i$. Let $\lambda = \mathbb{E}[X]$ and set $\phi(x) = (1+x)\ln(1+x) - x$. Then*

$$\mathbb{P}\left[X \geq \lambda + t\right] \leq \exp\left[-\lambda\phi(t/\lambda)\right], \quad \mathbb{P}\left[X \leq \lambda - t\right] \leq \exp\left[-\lambda\phi(-t/\lambda)\right] \quad \text{for any } t > 0.$$

*In particular,* $\mathbb{P}\left[X \geq t\lambda\right] \leq \exp\left[-t\lambda \ln(t/\mathrm{e})\right]$ *for any* $t > 1$.

*Proof of Corollary 6.2.3.* Let $\mathcal{E}$ be the event that $|e(H_k(n,p)) - m| \leq \sqrt{n}\ln n$. Then we can couple the random hypergraphs $\mathcal{H}(n,m)$ and $H_k(n,p)$ given $\mathcal{E}$ as follows.

1. Choose a random hypergraph $H_0 = \mathcal{H}(n,m)$.
2. Let $e = \mathrm{Bin}\left(\binom{n}{k}, p\right)$ be a binomial random variable given that $|e - m| \leq \sqrt{n}\ln n$.
3. Obtain a random hypergraph $H_1$ from $H_0$ as follows.
   - If $e \geq m$, choose a set of $e - m$ random edges from all edges not present in $H_0$ and add them to $H_0$.
   - If $e < m$, remove $m - e$ randomly chosen edges from $H_0$.

The outcome $H_1$ has the same distribution as $H_k(n,p)$ given $\mathcal{E}$, and $H_0, H_1$ differ in at most $\sqrt{n}\ln n$ edges. Therefore, noting that $\frac{1}{n}|\ln Z_\beta| \leq \frac{d}{k}\beta + \ln 2$ with certainty, we obtain with Fact 6.1.3

$$\begin{aligned}\frac{1}{n}\mathbb{E}\ln Z_\beta(H_k(n,p)) &\leq \frac{1}{n}\mathbb{E}[\ln Z_\beta(H_1)] + \left(\frac{d}{k}\beta + \ln 2\right)\mathbb{P}\left[\neg\mathcal{E}\right] \\ &\leq \frac{1}{n}\mathbb{E}[\ln Z_\beta(H_0)] + \frac{\beta \ln n}{\sqrt{n}} + \left(\frac{d}{k}\beta + \ln 2\right)\mathbb{P}\left[\neg\mathcal{E}\right] \\ &= \frac{1}{n}\mathbb{E}[\ln Z_\beta(\mathcal{H}(n,m))] + \left(\frac{d}{k}\beta + \ln 2\right)\mathbb{P}\left[\neg\mathcal{E}\right] + o(1). \qquad (6.2.7)\end{aligned}$$

Since $e(H_k(n,p))$ is a binomial random variable with mean $m + O(1)$, Lemma 6.2.4 implies that $\mathbb{P}\left[\neg\mathcal{E}\right] = o(1)$. Thus, by (6.2.7) and Jensen's inequality,

$$\frac{1}{n}\mathbb{E}\ln Z_\beta(H_k(n,p)) \leq \frac{1}{n}\mathbb{E}[\ln Z_\beta(\mathcal{H}(n,m))] + o(1) \leq \frac{1}{n}\ln \mathbb{E}[Z_\beta(\mathcal{H}(n,m))] + o(1).$$

Thus, the assertions follow by Lemmas 6.2.1 and 6.2.2 and by taking $n \to \infty$. □

We conclude this section by observing that the contribution to $Z_\beta$ of certain "exotic" $\sigma$ is negligible. We begin with $\sigma$ that are very imbalanced.

**Lemma 6.2.5.** *For any* $\varepsilon > 0$*, there is* $\delta > 0$ *such that the following is true. Let* $\bar{B}_\varepsilon$ *be the set of all* $\sigma : [n] \to \{\pm 1\}$ *such that* $\left||\sigma^{-1}(1)| - \frac{n}{2}\right| > \varepsilon n$*. Moreover, let*

$$Z_{\beta,\bar{B}_\varepsilon}(H) = \sum_{\sigma \in \bar{B}_\varepsilon} \exp\left[-\beta E_H(\sigma)\right].$$

*Then* $\mathbb{E}[Z_{\beta,\bar{B}_\varepsilon}(H_k(n,m))] \leq \exp\left[-\delta n\right]\mathbb{E}\left[Z_\beta(H_k(n,m))\right]$.

*Proof.* Stirling's formula implies that for any $\varepsilon > 0$ there is $\delta > 0$ such that $\frac{1}{n}\ln|\bar{B}_\varepsilon| < \ln 2 - \delta$.

Hence, (6.2.1) and the independence of the edges imply that

$$\begin{aligned}\mathbb{E}\left[Z_{\beta,\bar{B}_\varepsilon}(\mathcal{H}(n,m))\right] &= \sum_{\sigma\in\bar{B}_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{\mathcal{H}(n,m)}(\sigma)\right]\right] \\ &\leq |\bar{B}_\varepsilon| \left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)^m \\ &\leq \exp\left[-\delta n\right] 2^n \left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)^m .\end{aligned}$$

The assertion then follows from Lemma 6.2.2 and by observing that (as in (6.2.5))

$$\mathbb{E}[Z_{\beta,\bar{B}_\varepsilon}(H_k(n,m))] = O(\mathbb{E}[Z_{\beta,\bar{B}_\varepsilon}(\mathcal{H}(n,m))]).$$

□

Next, we consider $\sigma$ having an untypically high number of monochromatic edges.

**Lemma 6.2.6.** *For any $\varepsilon > 0$, there is $\delta > 0$ such that the following is true. Let*

$$m_0 = \frac{2^{1-k}\exp\left[-\beta\right]}{1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)}, \quad Z_{\beta,\varepsilon}(H) = \sum_{\sigma:[n]\to\{\pm1\}} \exp\left[-\beta E_H(\sigma)\right]\cdot \mathbf{1}_{|E_H(\sigma)-m_0|>\varepsilon m}.$$

*Then $\mathbb{E}[Z_{\beta,\varepsilon}(\mathcal{H}(n,m))] \leq \exp\left[-\delta n\right]\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right]$.*

*Proof.* Let $M_0 = \{\mu \in [m] : |\mu - m_0| > \varepsilon m\}$. Moreover for $\alpha > 0$ let $B_\alpha$ be the set of all $\sigma : [n] \to \{\pm1\}$ such that $\left||\sigma^{-1}(1)| - \frac{n}{2}\right| < \alpha n$. By Lemma 6.2.5 there exists $\delta > 0$ such that

$$\begin{aligned}&\mathbb{E}[Z_{\beta,\varepsilon}(\mathcal{H}(n,m))] \\ &\quad\leq \exp\left[-\delta n\right]\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right] + \sum_{\mu\in M_0}\sum_{\sigma\in B_\alpha}\exp\left[-\beta\mu\right]\mathbb{P}\left[E_{\mathcal{H}(n,m)}(\sigma)=\mu\right]. \end{aligned} \tag{6.2.8}$$

As in the proof of Lemma 6.2.2 we define $f(x) = -x\beta - x\ln x - (1-x)\ln(1-x) + x\ln(2^{1-k}) + (1-x)\ln(1-2^{1-k})$ and find that for any $\gamma > 0$ we can choose $\alpha > 0$ small enough so that

$$\frac{1}{m}\ln\left(\exp\left[-\beta\mu\right]\mathbb{P}\left[E_{\mathcal{H}(n,m)}(\sigma)=\mu\right]\right) \leq \gamma + f\left(\frac{\mu}{m}\right) \qquad \text{for all } \sigma\in B_\alpha.$$

Because $f$ is strictly concave and attains its maximum at $x = \frac{m_0}{m}$, there is $\delta' > 0$ such that

$$\sum_{\mu\in M_0}\sum_{\sigma\in B_\alpha}\exp\left[-\beta\mu\right]\mathbb{P}[E_{\mathcal{H}(n,m)}(\sigma)=\mu] \leq \exp\left[-\delta' n\right]\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right]. \tag{6.2.9}$$

Finally, the assertion follows from (6.2.8) and (6.2.9). □

### 6.2.2. The second moment

In Subsection 6.2.1 we derived an upper bound on $\Phi_{d,k}(\beta)$ by calculating the expectation value of $Z_\beta(\mathcal{H}(n,m))$. Here we obtain a matching lower bound for certain values of $\beta$ and $d$ by estimating the second moment $\mathbb{E}[Z_{\beta,\mathrm{bal}}(\mathcal{H}(n,m))^2]$. To this end, we define for $\alpha \in [-1,1]$,

$$Z_\beta(\alpha) = \sum_{\sigma,\tau\in\mathrm{Bal}:\langle\sigma,\tau\rangle=\alpha n} \exp\left[-\beta\left(E_{\mathcal{H}(n,m)}(\sigma) + E_{\mathcal{H}(n,m)}(\tau)\right)\right]. \tag{6.2.10}$$

Thus, in (6.2.10) we sum over balanced pairs $\sigma,\tau : [n] \to \{\pm 1\}$ that agree on precisely $n((1+\alpha)/2)$ vertices. Hence, we can express the second moment as

$$\begin{aligned}\mathbb{E}\left[Z_{\beta,\mathrm{bal}}(\mathcal{H}(n,m))^2\right] &= \sum_{\sigma,\tau\in\mathrm{Bal}} \mathbb{E}\left[\exp\left[-\beta\left(E_{\mathcal{H}(n,m)}(\sigma) + E_{\mathcal{H}(n,m)}(\tau)\right)\right]\right]\\ &= \sum_{\nu=0}^{n} \mathbb{E}\left[Z_\beta(2\nu/n - 1)\right].\end{aligned}$$

Consequently, we need to bound $Z_\beta(\alpha)$ for $\alpha \in [-1,1]$. To this aim, recall the function $\mathcal{H}(z) = -z\ln z - (1-z)\ln(1-z)$ from Section 2.6.

**Lemma 6.2.7.** *For $\alpha \in [\pm 1]$, we have*

$$\frac{1}{n}\ln \mathbb{E}[Z_\beta(\alpha)] = \ln 2 + \Lambda_\beta(\alpha) - \frac{\ln n}{2n} + O(1/n), \qquad \text{where}$$

$$\Lambda_\beta(\alpha) = \mathcal{H}\left(\frac{1+\alpha}{2}\right) + \frac{d}{k}\ln\left[1 - 2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right.$$
$$\left.\cdot\left[2 - \left(1-\exp\left[-\beta\right]\right)\frac{(1+\alpha)^k + (1-\alpha)^k}{2^k}\right]\right].$$

*Proof.* Let $e$ be a randomly chosen edge of $\mathcal{H}(n,m)$. Let $\sigma,\tau : [n] \to \{\pm 1\}$ be two balanced maps with overlap $\langle\sigma,\tau\rangle = \alpha n$. Let us write $\sigma \vDash e$ if $e \notin \mathcal{F}(\sigma)$ (i.e. $e$ is bichromatic under $\sigma$). By inclusion-exclusion,

$$\begin{aligned}&\mathbb{P}\left[\sigma \vDash e\right], \mathbb{P}\left[\tau \vDash e\right] = 1 - 2^{1-k} + O(1/n),\\ &\qquad \mathbb{P}\left[\sigma,\tau \vDash e\right] = 1 - 2^{2-k} + 2^{1-2k}\left((1+\alpha)^k + (1-\alpha)^k\right) + O(1/n).\end{aligned}$$

Hence, by the independence of edges,

$$\begin{aligned}
\mathbb{E}\left[Z_\beta(\alpha)\right] &= \sum_{\sigma,\tau:\langle\sigma,\tau\rangle=\alpha n} \mathbb{E}\prod_{i=1}^{m} \exp\left[-\beta(\mathbf{1}_{\sigma\not\models e_i} + \mathbf{1}_{\tau\not\models e_i})\right] \\
&= \sum_{\sigma,\tau:\langle\sigma,\tau\rangle=\alpha n} \left(\mathbb{E}\left[\exp\left[-\beta(\mathbf{1}_{\sigma\not\models e_1} + \mathbf{1}_{\tau\not\models e_1})\right]\right]\right)^m \\
&= 2^n\binom{n}{(1+\alpha)n/2}\left(1\cdot\mathbb{P}\left[\sigma,\tau\models e_1\right] + \exp\left[-\beta\right]\right. \\
&\quad \left.\cdot\left(\mathbb{P}\left[\sigma\models e_1,\tau\not\models e_1\right] + \mathbb{P}\left[\sigma\not\models e_1,\tau\models e_1\right]\right) + \exp\left[-2\beta\right]\cdot\mathbb{P}\left[\sigma,\tau\not\models e_1\right]\right)^m \\
&= 2^n\binom{n}{(1+\alpha)n/2}(1+O(1/n))\left[1-2^{2-k}\left(1-\exp\left[-\beta\right]\right)\right. \\
&\qquad \left. +2^{1-2k}\left(1-\exp\left[-\beta\right]\right)^2\left((1+\alpha)^k + (1-\alpha)^k\right)\right]^m. \qquad (6.2.11)
\end{aligned}$$

Furthermore, by Stirling's formula,

$$\binom{n}{(1+\alpha)n/2} = O(n^{-1/2})\exp\left[n\mathcal{H}\left(\frac{1+\alpha}{2}\right)\right]. \qquad (6.2.12)$$

The assertion follows by combining (6.2.11) and (6.2.12). □

Hence, we need to study the function $\Lambda_\beta$. Since $\Lambda_\beta(\alpha) = \Lambda_\beta(-\alpha)$, $\alpha = 0$ is a stationary point. Moreover, with

$$s = s(\alpha,\beta) = 1 - 2^{1-k}\left(1-\exp\left[-\beta\right]\right)\left[2 - \left(1-\exp\left[-\beta\right]\right)\frac{(1+\alpha)^k + (1-\alpha)^k}{2^k}\right]$$

the first two derivatives of $\Lambda_\beta$ work out to be

$$\Lambda_\beta'(\alpha) = \frac{\ln(1-\alpha) - \ln(1+\alpha)}{2} + \frac{2d}{4^k s}\ln\exp\left[-\beta\right] - 1^2((1+\alpha)^{k-1} - (1-\alpha)^{k-1}), \qquad (6.2.13)$$

$$\begin{aligned}
\Lambda_\beta''(\alpha) &= \frac{1}{\alpha^2-1} + \frac{2d(k-1)\left(\exp\left[-\beta\right]-1\right)^2}{4^k s}\left((1+\alpha)^{k-2} + (1-\alpha)^{k-2}\right) \\
&\quad - \frac{dk\left(1-\exp\left[-\beta\right]\right)^4}{2^{4k-2}s^2}\left[(1+\alpha)^{k-1} - (1-\alpha)^{k-1}\right]^2. \qquad (6.2.14)
\end{aligned}$$

In particular,

$$\Lambda_\beta''(0) = -1 + \tilde{O}_k(2^{-k}) < 0. \qquad (6.2.15)$$

Hence, there is a local maximum at $\alpha = 0$. As a consequence, if $\Lambda_\beta$ takes its strict *global* maximum at $\alpha = 0$, then $\mathbb{E}[Z_\beta(\mathcal{H}(n,m))^2] = O(\mathbb{E}[Z_\beta(\mathcal{H}(n,m))]^2)$. More generally, we have

**Lemma 6.2.8.** *Assume that $\beta \geq 0$ and $J \subset [-1,1]$ is a compact set such that $\Lambda_\beta(\alpha) < \Lambda_\beta(0)$ for all $\alpha \in J \setminus \{0\}$. Then*

$$\sum_{\nu=0}^{n} \mathbb{E}\left[Z_\beta(2\nu/n-1)\right] \mathbf{1}_{2\nu/n-1\in J} = O(\mathbb{E}[Z_\beta(\mathcal{H}(n,m))]^2).$$

*Proof.* We start by observing that $\frac{\ln 2+\Lambda_\beta(0)}{2} = \ln 2 + \frac{d}{k}\ln\left(1-2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)$. Hence, Lemma 6.2.1 yields

$$\exp\left[n(\ln 2+\Lambda_\beta(0))\right] = O(\mathbb{E}[Z_\beta(\mathcal{H}(n,m))]^2). \tag{6.2.16}$$

Now, by (6.2.15), there exist $\eta, c > 0$ such that $\Lambda_\beta(\alpha) \leq \Lambda_\beta(0) - c\alpha^2$ for all $\alpha \in J_0 = J \cap (-\eta,\eta)$. Hence, by Lemma 6.2.7 and (6.2.16),

$$\begin{aligned}\sum_{\nu=0}^{n} \mathbb{E}\left[Z_\beta(2\nu/n-1)\right] \mathbf{1}_{2\nu/n-1\in J_0} &= O(n^{-1/2}2^n)\sum_{\nu=0}^{n} \exp\left[n\Lambda_\beta(2\nu/n-1)\right] \mathbf{1}_{2\nu/n-1\in J_0}\\ &= O\left(2^n \exp\left[n\Lambda_\beta(0)\right]\right) \sum_{\nu:|2\nu/n-1|<\eta} \frac{\exp\left[-nc(2\nu/n-1)^2\right]}{\sqrt{n}}\\ &= O\left(2^n \exp\left[n\Lambda_\beta(0)\right]\right) = O\left(\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right]^2\right).\end{aligned} \tag{6.2.17}$$

Further, let $J_1 = J \setminus (-\eta,\eta)$. Then $J_1$ is compact. Hence, there exists $\delta > 0$ such that $\Lambda_\beta(\alpha) < \Lambda_\beta(0) - \delta$ for all $\alpha \in J_1$. Therefore, Lemma 6.2.7 and (6.2.16) yield

$$\begin{aligned}\sum_{\nu=0}^{n} \mathbb{E}\left[Z_\beta(2\nu/n-1)\right] \mathbf{1}_{2\nu/n-1\in J_1} &= O(n2^n)\sup_{\alpha\in J_1} \exp\left[n\Lambda_\beta(\alpha)\right]\\ &= O(n2^n)\exp\left[n\left(\Lambda_\beta(0)-\delta\right)\right] = O(\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right]^2).\end{aligned} \tag{6.2.18}$$

Finally, the assertion follows from (6.2.17) and (6.2.18). □

Now we prove that $[-1+2^{-3k/4}, 1-2^{-3k/4}] \subset J$ for all $\beta \geq 0$ and $J$ as defined in Lemma 6.2.8.

**Lemma 6.2.9.** *For $d/k = 2^{k-1}\ln 2 + O_k(1)$ and $\beta \geq 0$, we have $\Lambda_\beta(\alpha) < \Lambda_\beta(0)$ for all $\alpha \neq 0$ with $|\alpha| \leq 1-2^{-3k/4}$.*

*Proof.* We know that there is a local maximum at $\alpha = 0$. Moreover, we read off of (6.2.14) that $\Lambda''_\beta(\alpha) < 0$ if $|\alpha| < 1 - 6\ln k/k$, and thus

$$\Lambda_\beta(0) > \Lambda_\beta(\alpha) \qquad \text{for all } \alpha \in (-(1-6\ln k/k), 1-6\ln k/k).$$

Further, for $|\alpha| \geq 1 - 6\ln k/k$ we obtain from (6.2.13)

$$\Lambda'_\beta(\alpha) \leq \frac{\ln(1-\alpha)}{2} + \frac{2d\,(1-\exp[-\beta])^2\,(1+\alpha)^{k-1}}{4^k(1+O_k(2^{-k}))}$$
$$\leq \frac{\ln(1-\alpha)}{2} + \frac{d(1-\exp[-\beta])^2\exp[(1+\alpha)(k-1)/2]}{2^k(1+O_k(2^{-k}))}.$$

Hence, $\Lambda'_\beta(\alpha) < 0$ if $|\alpha| < 1 - 2.01\ln k/k$ and $k$ large enough and a similar estimate yields

$$\Lambda'_\beta(\alpha) > 0 \quad \text{if } |\alpha| > 1 - 1.99\ln k/k.$$

Thus, to proceed we need to evaluate $\Lambda_\beta$ at $|\alpha| = 1 - \gamma\ln k/k$ for $\gamma \in [1.99, 2.01]$ and at $|\alpha| = 1 - 2^{-3k/4}$. We find

$$\Lambda_\beta(\alpha) = -\ln 2 + o_k(1)$$

for $|\alpha| = 1-\gamma\ln k/k$ with $\gamma \in [1.99, 2.01]$ and $\Lambda_\beta(\alpha) = -\ln 2 + o_k(1)$ for $|\alpha| = 1-2^{-3k/4}$ proving the assertion. □

**Lemma 6.2.10.** *The function $\beta \mapsto \Lambda_\beta(\alpha) - \Lambda_\beta(0)$ is non-decreasing for $\alpha \neq 0$. In particular, if $d > 0$ and $\beta_0 \geq 0$ are such that $\Lambda_{\beta_0}(\alpha) < \Lambda_{\beta_0}(0)$ for all $\alpha \neq 0$, then $\Lambda_\beta(\alpha) < \Lambda_\beta(0)$ for all $\alpha \neq 0, 0 \leq \beta < \beta_0$.*

*Proof.* The derivative of $\Lambda_\beta$ with respect to $\beta$ works out to be

$$\frac{\partial\Lambda_\beta}{\partial\beta} = \frac{d}{k}\cdot\frac{2^{2-2k}((1+\alpha)^k+(1-\alpha)^k)\exp[-\beta]\,(1-\exp[-\beta]) - 2^{2-k}\exp[-\beta]}{1-2^{2-k}\,(1-\exp[-\beta]) + 2^{1-2k}\,(1-\exp[-\beta])^2\,((1+\alpha)^k+(1-\alpha)^k)}.$$

Substituting $z = (1+\alpha)^k + (1-\alpha)^k$ and $b = 1-\exp[-\beta]$ in the above, we obtain

$$g(z) = \frac{d}{k}\cdot\frac{2^{2-2k}b(1-b)z - 2^{2-k}(1-b)}{1-2^{2-k}b+2^{1-2k}b^2z}.$$

Because the function $z \mapsto \frac{az-b}{cz+d}$ with $a, b, c, d \geq 0$ is non-decreasing, this completes the proof. □

With these instruments in hand, we identify regimes of $d$ and $\beta$ where $\Lambda_\beta(\alpha)$ takes its global maximum at $\alpha = 0$.

**Lemma 6.2.11.** *Assume that $d/k = 2^{k-1}\ln 2 + O_k(1)$ and $\beta \leq k\ln 2 - \ln k$. Then $\Lambda_\beta(0) > \Lambda_\beta(\alpha)$ for all $\alpha \in [-1,1]\setminus\{0\}$.*

*Proof.* For $|\alpha| \leq 1 - 2^{-3k/4}$, this is the statement of Lemma 6.2.9. We write $\alpha = 1-\delta$ with

$\delta \in [0, 2^{-3k/4}]$. Let

$$f_\beta(\delta) = (1 - \exp[-\beta]) \left[2 - (1 - \exp[-\beta]) \frac{(2-\delta)^k + \delta^k}{2^k}\right] \in [0, 2].$$

For $\beta = k \ln 2 - \ln k$, we have the expansion

$$f_\beta(\delta) = \left(1 - \frac{k}{2^k}\right) \left[2 - \left(1 - \frac{k}{2^k}\right) \left(1 - k\frac{\delta}{2} + \tilde{O}_k(4^{-k})\right)\right] = 1 + k\frac{\delta}{2} + \tilde{O}_k(4^{-k}).$$

Therefore,

$$\begin{aligned}\Lambda_\beta(\alpha) &= -\frac{\delta}{2} \ln\left(\frac{\delta}{2}\right) - \left(1 - \frac{\delta}{2}\right) \ln\left(1 - \frac{\delta}{2}\right) \\ &\qquad + \left(2^{k-1} \ln 2 + O_k(1)\right) \ln\left(1 - 2^{1-k}\left(1 + k\frac{\delta}{2} + \tilde{O}_k(4^{-k})\right)\right) \\ &= -\ln 2 - \frac{\delta}{2} \ln \delta + \frac{\delta}{2} - (k-1)\frac{\delta}{2} \ln 2 + O_k(2^{-k}).\end{aligned}$$

The function $\delta \mapsto -\frac{\delta}{2} \ln \delta + \frac{\delta}{2} - (k-1)\frac{\delta}{2} \ln 2$ is easily studied: it takes its maximum at $\delta_0 = 2^{1-k}$ for which it is equal to $2^{-k}$. Hence for $\alpha = 1 - \delta$ with $\delta \in [0, 2^{-3k/4}]$,

$$\Lambda_\beta(\alpha) \leq -\ln 2 + O_k(2^{-k}).$$

By symmetry this also holds for $\alpha = -1 + \delta$ with $\delta \in [0, 2^{-3k/4}]$. By comparison,

$$\begin{aligned}\Lambda_\beta(0) &= \ln 2 + \left(2^{k-1} \ln 2 + O_k(1)\right) \ln\left(1 - 2^{2-k} + \frac{4k}{4^k} + O_k(4^{-k})\right) \\ &= -\ln 2 + 2^{1-k} k \ln 2 + O_k(2^{-k}).\end{aligned}$$

Therefore $\Lambda_\beta(0) > \Lambda_\beta(\alpha)$ for all $\alpha \neq 0$ if $\beta = k \ln 2 - \ln k$. Using Lemma 6.2.10 we can expand the result to all $\beta \leq k \ln 2 - \ln k$. □

**Lemma 6.2.12.** *Assume that* $d/k \leq 2^{k-1} \ln 2 - 2$ *and* $\beta \geq 0$. *Then* $\Lambda_\beta(0) > \Lambda_\beta(\alpha)$ *for all* $\alpha \in [-1, 1] \setminus \{0\}$.

*Proof.* Let $r_k = O_k(1)$ such that $d/k = 2^{k-1} \ln 2 + r_k$. Define the function

$$\Lambda_\infty : [-1, 1] \to \mathbb{R}, \quad \alpha \mapsto \mathcal{H}\left(\frac{1+\alpha}{2}\right) + \frac{d}{k} \ln\left(1 - 2^{2-k} + 2^{1-2k}\left((1+\alpha)^k + (1-\alpha)^k\right)\right).$$

Analogously to the proof of Lemma 6.2.11 we get $\Lambda_\infty(\alpha) \leq -\ln 2 - (\ln 2 + 2r_k - 1)2^{-k} + \tilde{O}_k(4^{-k})$

for all $\alpha$ and $\Lambda_\infty(0) = -\ln 2 - 2(\ln 2 + 2r_k)2^{-k} + \tilde{O}_k(4^{-k})$, which implies that for $r_k \leq -2$ we have $\Lambda_\infty(\alpha) < \Lambda_\infty(0)$ for all $\alpha \in [-1,1] \setminus \{0\}$. Because the continuous functions $\Lambda_\beta$ converge uniformly to $\Lambda_\infty$ as $\beta \to \infty$, we conclude that there is $\beta_0 \geq 0$ such that for all $\beta > \beta_0$,

$$\Lambda_\beta(\alpha) < \Lambda_\beta(0) \qquad \text{for all } \alpha \in [-1,1] \setminus \{0\}\,. \tag{6.2.19}$$

Hence, Lemma 6.2.10 implies that (6.2.19) holds for all $\beta \geq 0$, as desired. □

*Proof of Proposition 6.1.1.* The first assertion follows directly from Corollary 6.2.3. Moreover, if $d, \beta$ are such that for some $n$-independent number $C > 0$ we have

$$\mathbb{E}[Z_\beta(\mathcal{H}(n,m))^2] \leq C \cdot \mathbb{E}[Z_\beta(\mathcal{H}(n,m))]^2, \tag{6.2.20}$$

then the Paley-Zygmund inequality implies that

$$\mathbb{P}\left[Z_\beta(\mathcal{H}(n,m)) \geq \mathbb{E}[Z_\beta(\mathcal{H}(n,m))]/2\right] \geq \frac{\mathbb{E}[Z_\beta(\mathcal{H}(n,m))]^2}{4\mathbb{E}[Z_\beta(\mathcal{H}(n,m))^2]} \geq \frac{1}{4C} > 0. \tag{6.2.21}$$

Let $\mathcal{A}$ be the event that $\mathcal{H}(n,m)$ has no multiple edges. Since $\mathcal{A}$ occurs w.h.p. by Fact 2.1.2, equation (6.2.21) implies that

$$\mathbb{P}\left[Z_\beta(\mathcal{H}(n,m)) \geq \mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right]/2 | \mathcal{A}\right] \geq \frac{1-o(1)}{4C}. \tag{6.2.22}$$

Further, since the number $e(H_k(n,p))$ of edges in $H_k(n,p)$ is binomially distributed with mean $m + O(1)$, Stirling's formula implies that $\mathbb{P}\left[e(H_k(n,p)) = m\right] \geq \Omega(n^{-1/2})$. As $H_k(n,p)$ is identically distributed to $\mathcal{H}(n,m)$ given $e(H_k(n,p)) = m$ and $\mathcal{A}$, (6.2.22) implies that

$$\mathbb{P}\left[Z_\beta(H_k(n,p)) \geq \mathbb{E}[Z_\beta(\mathcal{H}(n,m))]/2\right] \geq \Omega(n^{-1/2}). \tag{6.2.23}$$

Thus, the concentration bound from Lemma 6.1.4 and (6.2.23) yield

$$\ln \mathbb{E}[Z_\beta(\mathcal{H}(n,m))] - \mathbb{E}[\ln Z_\beta(H_k(n,p))] - \ln 2 = o(n).$$

Hence, if (6.2.20) is true, then

$$\frac{1}{n}\mathbb{E}\left[\ln Z_\beta(H_k(n,p))\right] \geq \frac{1}{n} \ln \mathbb{E}[Z_\beta(\mathcal{H}(n,m))] - o(1). \tag{6.2.24}$$

Finally, Lemma 6.2.8 and Lemma 6.2.12 imply that (6.2.20) holds for all $\beta \geq 0$ and $d/k \leq 2^{k-1}\ln 2 - 2$. Moreover, by Lemma 6.2.8 and Lemma 6.2.11 the bound (6.2.20) is true if $d/k = 2^{k-1}\ln 2 + O_k(1)$ and $\beta \leq k \ln 2 - \ln k$. Thus, the assertion follows from (6.2.24). □

## 6.3. The planted model

The aim of this section is to prove Proposition 6.1.6. Throughout the section we let $m = \lceil dn/k \rceil$. For $\varepsilon > 0$, we let $B_\varepsilon$ be the set of all $\sigma : [n] \to \{\pm 1\}$ such that $\left||\sigma^{-1}(1)| - \frac{n}{2}\right| < \varepsilon n$. Further, the map $\boldsymbol{\sigma} : [n] \to \{\pm 1\}$ is assumed to be a map chosen uniformly at random and $\boldsymbol{H}$ the random hypergraph obtained by inserting each edge that is monochromatic under $\boldsymbol{\sigma}$ with probability $p_1$ and each edge that is bichromatic with probability $p_2$.

### 6.3.1. Quiet planting

We begin with the second part of Proposition 6.1.6. The following statement relates the planted model to the random hypergraph $H_k(n, m)$. A similar statement has been obtained independently by Achlioptas and Theodoropoulos [AT+].

**Lemma 6.3.1.** *Let $d > 0$ and $\beta \geq 0$. Assume that there is a sequence $(\mathcal{E}_n)_{n \geq 1}$ of events such that $\limsup_{n\to\infty} \mathbb{P}\left[\boldsymbol{H} \in \mathcal{E}_n\right]^{1/n} < 1$. Then $\mathbb{E}[Z_\beta(H_k(n,m))\mathbf{1}_{\mathcal{E}_n}] \leq \exp\left[-\Omega(n)\right]\mathbb{E}[Z_\beta(H_k(n,m))]$.*

*Proof.* Fix $\alpha > 0$ such that $\limsup_{n\to\infty} \mathbb{P}\left[\boldsymbol{H} \in \mathcal{E}_n\right]^{1/n} \leq \exp\left[-\alpha\right]$. For any $\varepsilon > 0$, we have the decomposition

$$\begin{aligned}\mathbb{E}[Z_\beta(H_k(n,m))\mathbf{1}_{\mathcal{E}_n}] &= \sum_{\sigma:[n]\to\{\pm1\}} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right]\mathbf{1}_{\mathcal{E}_n}\right] \\ &\leq \sum_{\sigma\in B_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right]\mathbf{1}_{\mathcal{E}_n}\right] + \sum_{\sigma\notin B_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right]\right].\end{aligned} \tag{6.3.1}$$

To bound the first summand in (6.3.1), we let $m_0 = \frac{2^{1-k}\exp[-\beta]}{1-2^{1-k}(1-\exp[-\beta])} m$ and define

$$M_\varepsilon = \left\{\mu \in [m] : |\mu - m_0| < \varepsilon n\right\}.$$

Now, for any $\mu \in [m]$ we have

$$\begin{aligned}&\sum_{\sigma\in B_\varepsilon} \mathbb{P}[\{E_{H_k(n,m)}(\sigma) = \mu\} \cap \{H_k(n,m) \in \mathcal{E}_n\}] \\ &\qquad= \sum_{\sigma\in B_\varepsilon} \mathbb{P}[H_k(n,m) \in \mathcal{E}_n | E_{H_k(n,m)}(\sigma) = \mu]\,\mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right].\end{aligned}$$

Under the conditions $e(\boldsymbol{H}) = m$ and $E_{H_k(n,m)}(\sigma) = E_{\boldsymbol{H}}(\sigma)$ for $\sigma : [n] \to \{\pm 1\}$, the two random hypergraphs $H_k(n,m)$ and $\boldsymbol{H}$ are identically distributed. Therefore,

$$\mathbb{P}[H_k(n,m) \in \mathcal{E}_n | E_{H_k(n,m)}(\sigma) = \mu] = \mathbb{P}[\boldsymbol{H} \in \mathcal{E}_n | E_{\boldsymbol{H}}(\sigma) = \mu, e(\boldsymbol{H}) = m] \\ \leq \frac{\mathbb{P}[\boldsymbol{H} \in \mathcal{E}_n]}{\mathbb{P}\left[E_{\boldsymbol{H}}(\sigma) = \mu, e(\boldsymbol{H}) = m\right]}.$$

By standard concentration results there is $\alpha > 0$ such that

$$\mathbb{P}\left[E_{\boldsymbol{H}}(\sigma) = \mu, e(\boldsymbol{H}) = m\right] \geq \exp\left[-\frac{\alpha}{2}\, n\right] \qquad \text{for any } \sigma \in B_\varepsilon, \mu \in M_\varepsilon.$$

Hence, for any $\mu \in M_\varepsilon$,

$$\sum_{\sigma \in B_\varepsilon} \mathbb{P}[\{E_{H_k(n,m)}(\sigma) = \mu\} \cap \{H_k(n,m) \in \mathcal{E}_n\}] \\ \leq \exp\left[\frac{\alpha}{2} n\right] \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\boldsymbol{H} \in \mathcal{E}_n]\, \mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right]$$

and therefore, letting $A = 2^n \left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)^m$, we get

$$\sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right] \mathbf{1}_{\mathcal{E}_n}\right] \\ = \sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp\left[-\beta\mu\right] \mathbb{P}[\{E_{H_k(n,m)}(\sigma) = \mu\} \cap \{H_k(n,m) \in \mathcal{E}_n\}] \\ \leq \exp\left[-\frac{\alpha}{2} n\right] \sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp\left[-\beta\mu\right] \mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right] \\ \leq A \exp\left[-\frac{\alpha}{2} n\right]. \tag{6.3.2}$$

Furthermore, Lemma 6.2.6 shows that there is $\delta > 0$ such that

$$\sum_{\mu \notin M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp\left[-\beta\mu\right] \mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right] \leq A \exp\left[-\delta n\right]. \tag{6.3.3}$$

To bound the second summand in (6.3.1) we get from Lemma 6.2.5 that there is $\delta' > 0$ such that

$$\sum_{\sigma \notin B_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right]\right] \leq A \exp\left[-\delta' n\right]. \tag{6.3.4}$$

Combining the estimates (6.3.2), (6.3.3) and (6.3.4) in the decomposition (6.3.1) yields

$$\mathbb{E}[Z_\beta(H_k(n,m)) \mathbf{1}_{\mathcal{E}_n}] \leq A \exp\left[-\max\left(\alpha/2, \delta, \delta'\right) n\right].$$

Thus, the assertion follows with Lemmas 6.2.1 and 6.2.2. $\square$

**Corollary 6.3.2.** *Let $d > 0$ and $\beta \geq 0$. Assume that there exists a sequence $(\mathcal{E}_n)_{n\geq 1}$ of events such that*

$$\lim_{n\to\infty} \mathbb{P}\left[H_k(n,m) \in \mathcal{E}_n\right] = 1 \quad \textit{while} \quad \limsup_{n\to\infty} \mathbb{P}\left[\boldsymbol{H} \in \mathcal{E}_n\right]^{1/n} < 1.$$

*Then* $\Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k} \ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)$.

*Proof.* Since $Z_\beta(H_k(n,m))^{1/n} \leq 2$ and $\mathbb{P}\left[H_k(n,m) \in \mathcal{E}_n\right] = 1 - o(1)$, Jensen's inequality yields

$$\mathbb{E}\left[Z_\beta(H_k(n,m))^{1/n}\right] = \mathbb{E}\left[Z_\beta(H_k(n,m))^{1/n}\mathbf{1}_{\mathcal{E}_n}\right] + o(1) \leq \mathbb{E}\left[Z_\beta(H_k(n,m))\mathbf{1}_{\mathcal{E}_n}\right]^{1/n} + o(1).$$

Hence, under the assumptions of the corollary we obtain with Jensen's inequality and Lemma 6.3.1

$$\Phi_{d,k}(\beta) \leq \limsup_{n\to\infty} \ln \mathbb{E}\left[Z_\beta(H_k(n,m))^{1/n}\right] \leq \exp\left[-\Omega(1)\right] \limsup_{n\to\infty} \ln \mathbb{E}\left[Z_\beta(H_k(n,m))\right]^{1/n}.$$

The result then follows from Lemmas 6.2.1 and 6.2.2. □

### 6.3.2. An unlikely event

As a next step, we establish the following.

**Lemma 6.3.3.** *Assume that (6.1.2) holds for some $\beta \geq k \ln 2 - \ln k$. Then there exists $z > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(H_k(n,m)) \leq z\right] = 1 \quad \textit{while} \quad \limsup_{n\to\infty} \mathbb{P}\left[\frac{1}{n} \ln Z_\beta(\boldsymbol{H}) \leq z\right]^{1/n} < 1.$$

The proof of Lemma 6.3.3, to which the rest of this subsection is dedicated, is an extension of the argument from [BCOHRV16, Section 6] to the case of finite $\beta$. We need the following concentration result.

**Lemma 6.3.4.** *For any fixed $d > 0$, $\beta \geq 0$, $\alpha > 0$, there are $\delta > 0$, $\delta' > 0$ such that the following is true. Suppose that $(\sigma_n)_{n\geq 1}$ is a sequence of maps $[n] \to \{\pm 1\}$. Then for all large enough $n$,*

$$\mathbb{P}\left[|\ln(Z_\beta(\boldsymbol{H})) - \mathbb{E}[\ln Z_\beta(\boldsymbol{H})|\boldsymbol{\sigma} = \sigma_n]| > \alpha n | \boldsymbol{\sigma} = \sigma_n\right] \leq \exp\left[-\delta n\right]$$

*and*

$$\mathbb{P}\left[|\ln(\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma})) - \mathbb{E}\left[\ln \mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma})|\boldsymbol{\sigma} = \sigma_n\right]| > \alpha n | \boldsymbol{\sigma} = \sigma_n\right] \leq \exp\left[-\delta' n\right].$$

*Proof.* This is immediate from the Lipschitz property and McDiarmid's inequality [McD98, Theorem 3.8]. □

We further need several statements about quantities in the planted model conditioned on $\boldsymbol{\sigma}$ being some fixed (balanced) colouring.

**Lemma 6.3.5.** *Assume that (6.1.2) is true for some $\beta \geq k \ln 2 - \ln k$. Then there exist a fixed number $\varepsilon > 0$ and a sequence $\sigma_n$ of balanced maps $[n] \to \{\pm 1\}$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) > \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right) + \varepsilon | \boldsymbol{\sigma} = \sigma_n\right] = 1.$$

*Proof.* By Stirling's formula there is an $n$-independent number $\delta > 0$ such that for sufficiently large $n$ we have

$$\mathbb{P}\left[\boldsymbol{\sigma} \in \mathrm{Bal}\right] \geq \delta. \tag{6.3.5}$$

Let $A = \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1-\exp\left[-\beta\right]\right)\right)$. Using (6.1.2) we know there is $\varepsilon > 0$ such that $\liminf_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) > A + 3\varepsilon\right] \geq 0.9$. With the concentration bound from Lemma 6.1.5 we get

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) > A + 2\varepsilon\right] = 1.$$

Thus, setting $p_n = \liminf_{n\to\infty} \max_{\sigma_n \in \mathrm{Bal}} \mathbb{P}\left[\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) > A + 2\varepsilon | \boldsymbol{\sigma} = \sigma_n\right]$ and using (6.3.5) implies

$$\begin{aligned} 1 &\leq \liminf_{n\to\infty}\left(\sum_{\sigma_n \in \mathrm{Bal}} \mathbb{P}\left[\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) > A + 2\varepsilon | \boldsymbol{\sigma} = \sigma_n\right]\mathbb{P}[\boldsymbol{\sigma} = \sigma_n] + \sum_{\sigma_n \notin \mathrm{Bal}} \mathbb{P}[\boldsymbol{\sigma} = \sigma_n]\right) \\ &\leq \liminf_{n\to\infty} p_n \, \mathbb{P}[\boldsymbol{\sigma} \in \mathrm{Bal}] + \mathbb{P}[\boldsymbol{\sigma} \notin \mathrm{Bal}] \leq \liminf_{n\to\infty} p_n + 1 - \delta \end{aligned}$$

and consequently $\liminf_{n\to\infty} p_n \geq \delta$. Thus the concentration bound from Lemma 6.3.4 yields

$$\lim_{n\to\infty} \max_{\sigma_n \in \mathrm{Bal}} \mathbb{P}\left[\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) > A + \varepsilon | \boldsymbol{\sigma} = \sigma_n\right] = 1,$$

thereby completing the proof. □

**Lemma 6.3.6.** *For any $\eta > 0$, there is $\delta > 0$ such that*

$$\limsup_{n\to\infty} \frac{1}{n}\ln \mathbb{P}\left[||\boldsymbol{\sigma}^{-1}(1)| - n/2| > \eta n\right] \leq -\delta.$$

*Proof.* This is immediate from the Chernoff bound. □

For a set $S \subset V$, let $\mathrm{Vol}(S|H)$ be the sum of the degrees of the vertices in $S$ in the hypergraph $H$.

**Lemma 6.3.7.** *For any $\gamma > 0$, there is $\alpha > 0$ such that for any set $S \subset [n]$ of size $|S| \leq \alpha n$ and any*

$\sigma : [n] \to \{\pm 1\}$ *we have* $\limsup \frac{1}{n} \ln \mathbb{P}\left[\mathrm{Vol}(S|\boldsymbol{H}) \geq \gamma n | \boldsymbol{\sigma} = \sigma\right] \leq -\alpha$.

*Proof.* Let $(X_v)_{v\in[n]}$ be a family of independent random variables with distribution $\mathrm{Bin}\left(\binom{n-1}{k-1}, 2p\right)$. Then for any $\sigma$ and any set $S \subset [n]$ the volume $\mathrm{Vol}(S|\boldsymbol{H})$ is stochastically dominated by $X_S = 2k\sum_{v\in S} X_v$. Furthermore, $\mathbb{E}[X_S] = 4dk|S|$. Thus, for any $\gamma > 0$ we can choose an $n$-independent $\alpha > 0$ such that for any $S \subset [n]$ of size $|S| \leq \alpha n$ we have $\mathbb{E}[X_S] \leq \gamma n/2$. In fact, the Chernoff bound shows that by picking $\alpha > 0$ sufficiently small, we can ensure that $\mathbb{P}\left[\mathrm{Vol}(S|\boldsymbol{H}) \geq \gamma n | \boldsymbol{\sigma} = \sigma\right] \leq \mathbb{P}\left[X_S \geq \gamma n\right] \leq \exp\left[-\alpha n\right]$, as desired. □

**Lemma 6.3.8.** *Let* $d > 0$ *and* $\beta \geq 0$. *Assume that there exist numbers* $z > 0$, $\varepsilon > 0$ *and a sequence* $(\sigma_n)_{n\geq 1}$ *of balanced maps* $[n] \to \{\pm 1\}$ *such that*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\ln Z_\beta(\boldsymbol{H}) | \boldsymbol{\sigma} = \sigma_n\right] > z + \varepsilon.$$

*Then* $\limsup_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}) \leq z\right]^{1/n} < 1$.

*Proof.* Suppose that $n$ is large enough so that $\frac{1}{n}\mathbb{E}\left[\ln Z_\beta(\boldsymbol{H}) | \boldsymbol{\sigma} = \sigma_n\right] > z + \varepsilon/2$. Set $n_i = |\sigma_n^{-1}(i)|$ and let $T$ be the set of all $\tau : [n] \to \{\pm 1\}$ such that $|\tau^{-1}(i)| = n_i$ for $i = \pm 1$. As $Z_\beta$ is invariant under permutations of the vertices, we have

$$\frac{1}{n}\mathbb{E}\left[\ln Z_\beta(\boldsymbol{H}) | \boldsymbol{\sigma} = \tau\right] = \frac{1}{n}\mathbb{E}\left[\ln Z_\beta(\boldsymbol{H}) | \boldsymbol{\sigma} = \sigma_n\right] > z + \varepsilon/2 \quad \text{for any } \tau \in T. \tag{6.3.6}$$

Let $\gamma = \varepsilon/(4\beta) > 0$. By Lemma 6.3.7 there exists $\alpha > 0$ such that for large enough $n$ for any set $S \subset V$ of size $|S| \leq \alpha n$ and any $\sigma : [n] \to \{\pm 1\}$ we have

$$\mathbb{P}\left[\mathrm{Vol}(S|\boldsymbol{H}) < \frac{\gamma n}{2} | \boldsymbol{\sigma} = \sigma\right] \geq 1 - \exp\left[-\alpha n\right]. \tag{6.3.7}$$

Fix such an $\alpha > 0$ and pick and fix a small $0 < \eta < \alpha/3$. By Lemma 6.3.6 there exists an ($n$-independent) number $\delta = \delta(\beta, \varepsilon, \eta) > 0$ such that

$$\mathbb{P}\left[\boldsymbol{\sigma} \in B_\eta\right] \geq 1 - \exp\left[-\delta n\right]. \tag{6.3.8}$$

As $\sigma_n$ is balanced, we have $|n_i - n/2| \leq \sqrt{n}$ for $i = \pm 1$. Therefore, if $\boldsymbol{\sigma} \in B_\eta$, a map $\tau_{\boldsymbol{\sigma}} \in T$ can be obtained from $\boldsymbol{\sigma}$ by changing the colours of at most $2\eta n$ vertices. Hence, if $\boldsymbol{\sigma} \in B_\eta$, we let $\boldsymbol{H}_{\tau_{\boldsymbol{\sigma}}}$ be the random hypergraph with planted colouring $\tau_{\boldsymbol{\sigma}}$. Further, let $\boldsymbol{H}_{\boldsymbol{\sigma}}$ be the hypergraph obtained by removing from $\boldsymbol{H}_{\tau_{\boldsymbol{\sigma}}}$ each edge that is monochromatic under $\boldsymbol{\sigma}$ but not under $\tau_{\boldsymbol{\sigma}}$ with probability $1 - \exp\left[-\beta\right]$ independently and inserting each edge that is monochromatic under $\tau_{\boldsymbol{\sigma}}$ but not under $\boldsymbol{\sigma}$ with probability $(1 - \exp\left[-\beta\right])\, p_2$ independently. Then $\boldsymbol{H}_{\boldsymbol{\sigma}} = \boldsymbol{H}$ in distribution.

For the set $S_{\boldsymbol{\sigma}}$ of vertices $v$ with $\boldsymbol{\sigma}(v) \neq \tau_{\boldsymbol{\sigma}}(v)$, our choice of $\eta$ ensures that $|S_{\boldsymbol{\sigma}}| < \alpha n$. Let $\Delta$

be the number of edges present in $\boldsymbol{H}_{\tau_{\boldsymbol{\sigma}}}$ but not in $\boldsymbol{H}_{\boldsymbol{\sigma}}$ or vice versa. Then $\Delta \leq \mathrm{Vol}(S_{\boldsymbol{\sigma}}|\boldsymbol{H}_{\tau_{\boldsymbol{\sigma}}}) + \mathrm{Vol}(S_{\boldsymbol{\sigma}}|\boldsymbol{H}_{\boldsymbol{\sigma}})$. Hence, with (6.3.7) there exists a constant $c > 0$ such that

$$\mathbb{P}\left[\Delta \leq \gamma n | \boldsymbol{\sigma} \in B_\eta\right] \geq 1 - c\exp\left[-\alpha n\right]. \tag{6.3.9}$$

Using (6.3.8), (6.3.9) and the fact that removing a single edge can reduce $\frac{1}{n}\ln Z_\beta$ by at most $\beta/n$, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}) \leq z\right] \\
&\quad = \mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}_{\boldsymbol{\sigma}}) \leq z\right] \leq \exp\left[-\delta n\right] + \mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}_{\boldsymbol{\sigma}}) \leq z | \boldsymbol{\sigma} \in B_\eta\right] \\
&\quad \leq \exp\left[-\delta n\right] + c\exp\left[-\alpha n\right] + \mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}_{\boldsymbol{\sigma}}) \leq z | \boldsymbol{\sigma} \in B_\eta, \Delta \leq \gamma n\right] \\
&\quad \leq \exp\left[-\delta n\right] + c\exp\left[-\alpha n\right] + \mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}_{\tau_{\boldsymbol{\sigma}}}) - \gamma\beta \leq z | \boldsymbol{\sigma} \in B_\eta, \Delta \leq \gamma n\right].
\end{aligned} \tag{6.3.10}$$

By the choice of $\gamma$, (6.3.8), (6.3.9) and (6.3.6) we have

$$\begin{aligned}
&\mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}_{\tau_{\boldsymbol{\sigma}}}) - \gamma\beta \leq z | \boldsymbol{\sigma} \in B_\eta, \Delta \leq \gamma n\right] \\
&\qquad \leq 2\,\mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}_{\tau_{\boldsymbol{\sigma}}}) \leq z + \varepsilon/4 | \boldsymbol{\sigma} \in B_\eta\right] \\
&\qquad \leq 3\,\mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}) \leq z + \varepsilon/4 | \boldsymbol{\sigma} = \sigma_n\right] \\
&\qquad \leq 3\,\mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}) \leq \frac{1}{n}\mathbb{E}[\ln Z_\beta(\boldsymbol{H})|\boldsymbol{\sigma} = \sigma_n] - \varepsilon/4 | \boldsymbol{\sigma} = \sigma_n\right].
\end{aligned} \tag{6.3.11}$$

The assertion follows by combining (6.3.10) and (6.3.11) with Lemma 6.3.4. $\square$

*Proof of Lemma 6.3.3.* Lemma 6.3.5 shows that there exist $\varepsilon > 0$ and balanced maps $\sigma_n : [n] \to \{\pm 1\}$ such that

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) \geq \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right) + \varepsilon | \boldsymbol{\sigma} = \sigma_n\right] = 1. \tag{6.3.12}$$

Clearly, (6.3.12) implies that

$$\lim_{n\to\infty}\mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}) \geq \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right) + \varepsilon | \boldsymbol{\sigma} = \sigma_n\right] = 1. \tag{6.3.13}$$

Hence, with $z = \ln 2 + \frac{d}{k} \ln \left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right) + \varepsilon/2$, Lemma 6.3.8 and (6.3.13) yield

$$\limsup_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln Z_\beta(\boldsymbol{H}) \leq z\right]^{1/n} < 1. \tag{6.3.14}$$

By comparison, Lemma 6.2.1 and Lemma 6.2.2 imply

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln Z_\beta(H_k(n,m)) \leq z\right] = 1 \tag{6.3.15}$$

and the assertion follows from (6.3.14) and (6.3.15). □

### 6.3.3. Tame colourings

To facilitate the proof of the first part of Proposition 6.1.6 we introduce a random variable that explicitly controls the "cluster size" $\mathcal{C}_\beta(H_k(n,m),\sigma)$. The idea of explicitly controlling the cluster size was introduced in [COZ12] in the "zero temperature" case, and here we generalise it to the case of finite $\beta$. More precisely, we call $\sigma : [n] \to \{\pm 1\}$ *tame* in $H$ if $\sigma$ is balanced and if $\mathcal{C}_\beta(H,\sigma) \leq \mathbb{E}[Z_\beta(H)]$. Now, let

$$Z_{\beta,\mathrm{tame}}(H_k(n,m)) = \sum_{\sigma:[n]\to\{\pm1\}} \exp\left[-\beta E_{H_k(n,m)}(\sigma)\right] \cdot \mathbf{1}_{\sigma \text{ is tame}}.$$

**Lemma 6.3.9.** *Let* $0 \leq d/k \leq 2^{k-1}\ln 2 + O_k(1)$ *is such that* $\liminf_{n\to\infty} \frac{\mathbb{E}[Z_{\beta,\mathrm{tame}}(H_k(n,m))]}{\mathbb{E}[Z_\beta(H_k(n,m))]} > 0$. *Then*

$$\liminf_{n\to\infty} \frac{\mathbb{E}[Z_{\beta,\mathrm{tame}}(H_k(n,m))]^2}{\mathbb{E}[Z_{\beta,\mathrm{tame}}(H_k(n,m))^2]} > 0.$$

*Proof.* The proof is based on a second moment argument. Mimicking the notation of Section 6.2.2, we let

$$Z_{\beta,\mathrm{tame}}(\alpha) = \sum_{\sigma,\tau:\langle\sigma,\tau\rangle=\alpha n} \exp\left[-\beta\left(E_{H_k(n,m)}(\sigma) + E_{H_k(n,m)}(\tau)\right)\right] \cdot \mathbf{1}_{\sigma \text{ is tame}} \cdot \mathbf{1}_{\tau \text{ is tame}}.$$

Then it is clear that

$$\mathbb{E}[Z_{\beta,\mathrm{tame}}(H_k(n,m))^2] = \sum_{\nu=0}^{n} \mathbb{E}\left[Z_{\beta,\mathrm{tame}}(2\nu/n - 1)\right].$$

Furthermore, we have $Z_{\beta,\mathrm{tame}}(\alpha) \leq Z_\beta(\alpha)$ for any $\alpha$. Let $I = [-1 + 2^{-3k/4}, 1 - 2^{-3k/4}]$. Then Lemma 6.2.9 and Lemma 6.2.8 yield

$$\sum_{\alpha\in I} \mathbb{E}\left[Z_\beta(\alpha)\right] = O(\mathbb{E}[Z_\beta(H_k(n,m))]^2). \tag{6.3.16}$$

By the definition of “tame” we have

$$\sum_{\alpha>1-2^{-3k/4}} \mathbb{E}\left[Z_{\beta,\text{tame}}(\alpha)\right] \leq \mathbb{E}\left[\sum_{\sigma} \exp\left[-\beta E_{H_k(n,m)}(\sigma)\right] \cdot \mathbf{1}_{\sigma \text{ is tame}} \cdot \mathcal{C}_\beta(H_k(n,m),\sigma)\right]$$
$$\leq \mathbb{E}\left[\sum_{\sigma} \exp\left[-\beta E_{H_k(n,m)}(\sigma)\right] \cdot \mathbb{E}\left[Z_{\beta,\text{tame}}(H_k(n,m))\right]\right]$$
$$= O(\mathbb{E}\left[Z_{\beta,\text{tame}}(H_k(n,m))\right]^2). \tag{6.3.17}$$

Moreover, $\sum_{\alpha<-1+2^{-3k/4}} \mathbb{E}\left[Z_{\beta,\text{tame}}(\alpha)\right] = \sum_{\alpha>1-2^{-3k/4}} \mathbb{E}\left[Z_{\beta,\text{tame}}(\alpha)\right]$ by symmetry. Hence, equations (6.3.16) and (6.3.17) yield $\mathbb{E}[Z_{\beta,\text{tame}}(H_k(n,m))^2] = O(\mathbb{E}[Z_\beta(H_k(n,m))]^2)$.

Finally, the assertion follows from our assumption $\mathbb{E}[Z_{\beta,\text{tame}}(H_k(n,m))] = \Omega(\mathbb{E}[Z_\beta(H_k(n,m))])$. □

**Lemma 6.3.10.** *Let $d > 0$ and $\beta \geq 0$ and assume that $\limsup_{n\to\infty} \mathbb{P}\left[\boldsymbol{\sigma} \text{ is not tame in } \boldsymbol{H}\right]^{1/n} < 1$. Then there is $c > 0$ such that $\mathbb{E}[Z_{\beta,\text{tame}}(H_k(n,m))] \geq \mathbb{E}[Z_\beta(H_k(n,m))]/c$.*

*Proof.* The proof is very similar to the proof of Lemma 6.3.1. We fix $\alpha > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\left[\boldsymbol{\sigma} \text{ is not tame in } \boldsymbol{H}\right]^{1/n} \leq \exp\left[-\alpha\right] < 1.$$

For any $\varepsilon > 0$, we have

$$\mathbb{E}[Z_\beta(H_k(n,m)) - Z_{\beta,\text{tame}}(H_k(n,m))]$$
$$= \sum_{\sigma:[n]\to\{\pm1\}} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right] \mathbf{1}_{\sigma \text{ is not tame in } H_k(n,m)}\right]$$
$$\leq \sum_{\sigma\in B_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right] \mathbf{1}_{\sigma \text{ is not tame in } H_k(n,m)}\right] + \sum_{\sigma\notin B_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right]\right].$$

We set $m_0$ and $M_\varepsilon$ as in the proof of Lemma 6.3.1 and let $\mathcal{A}(\sigma,\mu)$ be the event $\{E_{\boldsymbol{H}}(\sigma) = \mu, e(\boldsymbol{H}) = m, |\sigma^{-1}(1)| = |\boldsymbol{\sigma}^{-1}(1)|\}$. Further, we fix an $\varepsilon > 0$ such that $\mathbb{P}[\mathcal{A}(\sigma,\mu)] > \exp\left[-\frac{\alpha}{2}n\right]$ for all

$\sigma \in B_\varepsilon, \mu \in M_\varepsilon$. Then for any $\mu \in M_\varepsilon$:

$$\begin{aligned}
&\sum_{\sigma \in B_\varepsilon} \mathbb{P}[\{E_{H_k(n,m)}(\sigma) = \mu\} \cap \{\sigma \text{ is not tame in } H_k(n,m)\}] \\
&\qquad = \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\sigma \text{ is not tame in } H_k(n,m) | E_{H_k(n,m)}(\sigma) = \mu]\, \mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right] \\
&\qquad = \sum_{\sigma \in B_\varepsilon} \mathbb{P}[\boldsymbol{\sigma} \text{ is not tame in } \boldsymbol{H} | \mathcal{A}(\sigma, \mu)]\, \mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right] \\
&\qquad \le \sum_{\sigma \in B_\varepsilon} \frac{\mathbb{P}[\boldsymbol{\sigma} \text{ is not tame in } \boldsymbol{H}]}{\mathbb{P}(\mathcal{A}(\sigma, \mu))}\, \mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right] \\
&\qquad \le \exp\left[-\frac{\alpha}{2} n\right] \sum_{\sigma \in B_\varepsilon} \mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right].
\end{aligned}$$

Letting $A = 2^n \left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)^m$, we get

$$\begin{aligned}
&\sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right] \mathbf{1}_{\sigma \text{ is not tame in } H_k(n,m)}\right] \\
&= \sum_{\mu \in M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp\left[-\beta \mu\right] \mathbb{P}\left[\{E_{H_k(n,m)}(\sigma) = \mu\} \cap \{\sigma \text{ is not tame in } H_k(n,m)\}\right] \le A \exp\left[-\frac{\alpha}{2} n\right].
\end{aligned} \tag{6.3.18}$$

Furthermore, Lemma 6.2.6 shows that there is $\delta > 0$ such that

$$\sum_{\mu \notin M_\varepsilon} \sum_{\sigma \in B_\varepsilon} \exp\left[-\beta \mu\right] \mathbb{P}\left[E_{H_k(n,m)}(\sigma) = \mu\right] \le A \exp\left[-\delta n\right] \tag{6.3.19}$$

and Lemma 6.2.5 implies that there is $\delta' > 0$ such that

$$\sum_{\sigma \notin B_\varepsilon} \mathbb{E}\left[\exp\left[-\beta E_{H_k(n,m)}(\sigma)\right]\right] \le A \exp\left[-\delta' n\right]. \tag{6.3.20}$$

Combining the estimates (6.3.18), (6.3.19) and (6.3.20) and using Lemmas 6.2.1 and 6.2.2 yields

$$\begin{aligned}
\mathbb{E}[Z_\beta(H_k(n,m)) - Z_{\beta,\text{tame}}(H_k(n,m))] &\le A \exp\left[-\max(\alpha/2, \delta, \delta')n\right] \\
&\le \exp\left[-\Omega(n)\right] \mathbb{E}\left[Z_\beta(H_k(n,m))\right],
\end{aligned}$$

which proves the assertion. □

**Corollary 6.3.11.** *Assume that* $d/k = 2^{k-1}\ln 2 + O_k(1)$ *and that* $\beta_0 \ge k \ln 2 - \ln k$ *is such that (6.1.1) holds for all* $k \ln 2 - \ln k \le \beta \le \beta_0$. *Then* $\beta_{\text{crit}}(d,k) \ge \beta_0$.

The proof of this corollary extends a "zero temperature" argument from [BCOHRV16, Section 5] to the case of $\beta \in [0, \infty)$.

*Proof.* Assume for contradiction that $\beta_0$ is such that (6.1.1) holds for all $k \ln 2 - \ln k \leq \beta \leq \beta_0$ but $\beta_{\text{crit}}(d,k) < \beta_0$. By Corollary 6.1.2 we have $\beta_{\text{crit}}(d,k) \geq k \ln 2 - \ln k$. We pick and fix a number $\beta_{\text{crit}}(d,k) < \beta < \beta_0$. If we let $A = \ln 2 + \frac{d}{k} \ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right)$, then there exists $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}[\ln Z_\beta H_k(n,m)] < A - \varepsilon. \tag{6.3.21}$$

On the other hand, (6.1.1) and Lemma 6.1.5 ensure that we can apply Lemma 6.3.10 and find a number $c > 0$ such that

$$\mathbb{E}[Z_{\beta,\text{tame}}(H_k(n,m))] \geq c \cdot \mathbb{E}[Z_\beta(H_k(n,m))]. \tag{6.3.22}$$

Hence, Lemma 6.3.9 implies that $\mathbb{E}[Z_{\beta,\text{tame}}(H_k(n,m))^2] = O(\mathbb{E}[Z_{\beta,\text{tame}}(H_k(n,m))]^2)$. Using the Paley-Zygmund inequality there is a number $C > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}\left[Z_{\beta,\text{tame}}(H_k(n,m)) \geq \mathbb{E}[Z_{\beta,\text{tame}}(H_k(n,m))]/2\right] \geq 1/C > 0.$$

With (6.3.22) and because $c/2 \cdot \mathbb{E}[Z_\beta(H_k(n,m))] > \exp\left[nA - n\varepsilon/3\right]$, we see that

$$\liminf_{n\to\infty} \mathbb{P}\left[Z_{\beta,\text{tame}}(H_k(n,m)) \geq \exp\left[nA - n\varepsilon/3\right]\right] > 0.$$

With Lemma 6.1.4 it follows that

$$\lim_{n\to\infty} \mathbb{P}\left[Z_{\beta,\text{tame}}(H_k(n,m)) \geq \exp\left[nA - 2n\varepsilon/3\right]\right] = 1.$$

With (6.3.21) we get the contradiction

$$A - \varepsilon > \liminf_{n\to\infty} \frac{1}{n} \mathbb{E}[\ln Z_{\beta,\text{tame}}(H_k(n,m))] \geq A - 2\varepsilon/3,$$

which refutes our assumption that $\beta_{\text{crit}}(d,k) < \beta_0$. □

*Proof of Proposition 6.1.6.* The assertion is immediate from Corollary 6.3.2 combined with Lemma 6.3.3 and from Corollary 6.3.11. □

## 6.4. The cluster size

In this section we prove Proposition 6.1.7. Throughout the section we assume that $d/k = 2^{k-1} \ln 2 + O_k(1)$ and that $\beta \geq k \ln 2 - \ln k$.

In order to analyse the cluster size, we will show that there is a large set of vertices (the "core") whose value cannot be changed without creating a large number of monochromatic edges. Hence, the contribution of these vertices to the cluster size can be controlled. Then we analyse the contribution of the remaining vertices.

The proof strategy broadly follows the argument for estimating the cluster size in the "zero temperature" case from [COZ12]. However, the fact that we are dealing with a finite $\beta$ causes significant complications. More precisely, one of the key features of the "zero temperature" case is the existence of "frozen variables", i.e. vertices that take the same colour in all colourings in the cluster. Indeed, in the zero temperature case the problem of estimating the cluster size basically reduces to estimating the number of "frozen variables". By contrast, in the case of finite $\beta$, frozen variables do not exist. In effect, we need to take a much closer look.

We let $\boldsymbol{\sigma} : [n] \to \{\pm 1\}$ be a map chosen uniformly at random conditioned on the event that $\boldsymbol{\sigma} \in \mathrm{Bal}$ and $\boldsymbol{H}$ be the random hypergraph obtained by inserting each edge that is monochromatic under $\boldsymbol{\sigma}$ with probability $p_1$ and each edge that is bichromatic with probability $p_2$.

We say that a vertex $v$ *supports* an edge $e \ni v$ under $\boldsymbol{\sigma}$ if $\boldsymbol{\sigma}(e \setminus \{v\}) = \{-\boldsymbol{\sigma}(v)\}$. In this case, we call $e$ *critical*. Moreover, if $U \subset [n]$, then we say that an edge $e$ of $\boldsymbol{H}$ is $U$*-endangered* if $|\boldsymbol{\sigma}(U \cap e)| = 1$ (i.e. the vertices in $U \cap e$ all have the same colour).

For the first three subsections of this section, it will be convenient to introduce a slightly more general construction. Let $\omega \geq 0$ be fixed and let $v_1, \ldots, v_\omega$ be vertices chosen uniformly at random without replacement from all vertices in $\boldsymbol{H}$. Let $\boldsymbol{H}'$ be the hypergraph obtained from $\boldsymbol{H}$ by removing $v_1, \ldots, v_\omega$ and edges $e$ involving one of these vertices. Without loss of generality we can assume that $\{v_1, \ldots, v_\omega\} = \{n - \omega + 1, \ldots, n\}$. The edge set of $\boldsymbol{H}'$ is thus $[n']$, with $n' = n - \omega$.

### 6.4.1. The core

Let $\mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma})$ be the maximal set $V' \subset [n]$ of vertices such that the following two conditions hold.

**CR1** Each vertex $v \in V'$ supports at least 100 edges that consist of vertices from $V'$ only,
**CR2** No vertex $v \in V'$ occurs in more than 10 edges that are $V'$-endangered under $\boldsymbol{\sigma}$.

If $V', V''$ are sets that satisfy **CR1–CR2**, then so does $V' \cup V''$. Hence, the core is well-defined.

**Proposition 6.4.1.** *W.h.p.* $|\mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma})| = n(1 - \tilde{O}_k(2^{-k}))$

To prove this proposition, we consider the following *whitening process* on the graph $\boldsymbol{H}'$ whose result $U$ is such that its complement $\bar{U} = [n'] \setminus U$ is a subset of $\mathrm{core}(\boldsymbol{H}', \boldsymbol{\sigma})$.

**WH1** Let $W$ contain all vertices of $\boldsymbol{H}'$ that either support fewer than 200 edges or occur in more than 2 edges that are monochromatic under $\boldsymbol{\sigma}$.

**WH2** Let $U = W$ initially. While there is a vertex $v \in [n'] \setminus U$ such that

- $v$ occurs in more than 5 edges that are $[n'] \setminus U$-endangered and contain a vertex from $U$, or
- $v$ supports fewer than 150 edges containing vertices in $[n'] \setminus U$ only,

add $v$ to $U$.

Proposition 6.4.1 will be a consequence of the following lemma by taking $\omega = 0$ and noticing that $\mathrm{core}(\boldsymbol{H}', \boldsymbol{\sigma})$ is a superset of the set $\bar{U}$.

**Lemma 6.4.2.** *Let $U$ be the outcome of the process **WH1–WH2** on $\boldsymbol{H}'$. Then $|U| = n'\tilde{O}_k(2^{-k})$ w.h.p..*

The rest of this subsection is dedicated to the proof of this lemma. We first bound the size of the set $W$ generated by **WH1**.

**Lemma 6.4.3.** *W.h.p. the set $W$ contains $n'\tilde{O}_k(2^{-k})$ vertices.*

*Proof.* Our assumptions on $\beta$ and $d$ ensure that the number of monochromatic edges containing a fixed vertex $v$ is binomially distributed with mean $\tilde{O}_k(2^{-k})$. Therefore, the probability that $v$ occurs in more than 2 monochromatic edges is bounded by $\tilde{O}_k(2^{-2k})$. Furthermore, the number of edges supported by $v$ is binomially distributed with mean $k \ln 2 + O_k(1)$. Hence, by the Chernoff bound the probability that $v$ supports fewer than 200 edges is bounded by $\tilde{O}_k(2^{-k})$. Consequently,

$$\mathbb{E}\left[|W|\right] = n'\tilde{O}_k(2^{-k}). \tag{6.4.1}$$

Finally, either adding or removing a single edge from the hypergraph can alter the size of $W$ by at most $k$. Therefore, (6.4.1) and Azuma's inequality imply that $|W| = n'\tilde{O}_k(2^{-k})$ w.h.p., as desired. □

In the next step we state two results excluding some properties of small sets of vertices in $\boldsymbol{H}'$.

**Lemma 6.4.4.** *W.h.p. the random hypergraph $\boldsymbol{H}'$ enjoys the following property.*

*There is no set $T \neq \emptyset$ of vertices with $|T| \leq n'/k^8$ such that at least $0.9|T|$ vertices from $T$ occur in two or more $[n'] \setminus T$-endangered edges that contain another vertex from $T$.* (6.4.2)

*Proof.* For a set $T \subset [n']$, we define $\varepsilon = |T|/n'$ and we let $X_i(T)$ for $i \in \{2, ...k\}$ be the number of edges that are $[n'] \setminus T$-endangered and contain exactly $i$ vertices from $T$. Then $X_i(T)$ is stochastically dominated by a binomial random variable $\mathrm{Bin}\left((1+o(1))2^{i+1-k}\binom{\varepsilon n'}{i}\binom{n'}{k-i}, 2p\right)$. Indeed, there are $\binom{\varepsilon n'}{i}$ ways to choose $i$ vertices from $T$ and at most $\binom{(1-\varepsilon)n'}{k-i} \leq \binom{n'}{k-i}$ ways to choose $k-i$ vertices from $[n'] \setminus T$. Moreover, these $k-i$ vertices are required to have the same colour and because we assumed that $\boldsymbol{\sigma}$ is balanced, this gives rise to the $(1+o(1))2^{i+1-k}$-factor. Let $X(T) = \sum_{i=2}^{k} X_i(T)$ be the total number of edges that are $[n'] \setminus T$-endangered and contain at least two vertices from $T$. Then using the rough upper bound $\binom{n}{k}2p \leq n2^k \ln 2$ we obtain

$$\mathbb{E}[X(T)] = \sum_{i=2}^{k} \mathbb{E}[X_i(T)] \leq k\mathbb{E}[X_2(T)] \leq 3.6k^3\varepsilon^2 n'. \tag{6.4.3}$$

Let $\mathcal{E}(T)$ be the event that $X(T) \geq 1.8|T|$. If the set $T$ satisfies (6.4.2) then $\mathcal{E}(T)$ occurs. The Chernoff bound from Lemma 6.2.4 and the above upper bound (6.4.3) on $\mathbb{E}[X(T)]$ yield

$$\mathbb{P}[\mathcal{E}(T)] \leq \exp\left[-1.8\varepsilon n' \ln\left(\frac{1}{2ek^3\varepsilon}\right)\right].$$

Hence, the probability of the event $\mathcal{E}$ that there is a set $T$ of size $|T| \leq n'/k^8$ such that $\mathcal{E}(T)$ occurs is bounded by

$$\begin{aligned}\mathbb{P}[\mathcal{E}] &\leq \sum_{T:|T|\leq n'/k^8} \mathbb{P}[\mathcal{E}(T)] \leq \sum_{1/n'\leq\varepsilon\leq 1/k^8} \binom{n'}{\varepsilon n'} \exp\left[-1.8\varepsilon n' \ln\left(\frac{1}{2ek^3\varepsilon}\right)\right] \\ &\leq \sum_{1/n'\leq\varepsilon\leq 1/k^8} \left(\frac{2en'}{\varepsilon n'}\right)^{\varepsilon n'} \exp\left[-1.8\varepsilon n' \ln\left(\frac{1}{2ek^3\varepsilon}\right)\right] \\ &\leq \sum_{1/n'\leq\varepsilon\leq 1/k^8} \exp\left[\varepsilon n'\left(5 + 5.6\ln(k) + 0.8\ln(\varepsilon)\right)\right] = o(1),\end{aligned}$$

as claimed. □

**Lemma 6.4.5.** *W.h.p. the random hypergraph $\boldsymbol{H}'$ enjoys the following property.*

*There is no set $T \neq \emptyset$ of vertices of size $|T| \leq n'/k^6$ such that at least $0.09|T|$ vertices from $T$ support at least 20 edges that contain another vertex from $T$.* (6.4.4)

*Proof.* For a set $T \subset [n']$ and a set $Q \subset [T]$, we let $\mathcal{E}(T,Q)$ be the event that each vertex $v \in Q$ supports at least 20 edges that contain another vertex from $T$. Let $\varepsilon = |T|/n'$. Then for each vertex $v$ the number $X_v$ of edges that $v$ supports and that contain another vertex from $T$ is stochastically dominated by a binomial random variable $\mathrm{Bin}\left((1+o(1))2^{2-k}\varepsilon n'\binom{n'}{k-2}, p_2\right)$. Indeed, there are $\varepsilon n' - 1$ ways to choose another vertex $v' \neq v$ from $T$, and at most $\binom{n'}{k-2}$ ways to choose $k-2$ further vertices to complete the edges. Moreover, these $k-2$ vertices are required to have colour $-\boldsymbol{\sigma}(v)$, and because we assumed that $\boldsymbol{\sigma}$ is balanced this gives rise to the $(1+o(1))2^{2-k}$-factor. Furthermore, the random variables $X_v$ are mutually independent, because the edges in question are distinct as they are supported by the distinguished vertex $v$. Therefore, using the rough upper bound $\binom{n}{k}p_2 \leq n2^k \ln 2$, we obtain

$$\begin{aligned}\mathbb{P}\left[\mathcal{E}(T,Q)\right] &\leq \prod_{v\in Q}\mathbb{P}\left[X_v \geq 20\right] \\ &\leq \mathbb{P}\left[\mathrm{Bin}\left((1+o(1))2^{2-k}\varepsilon n'\binom{n'}{k-2}, p_2\right) \geq 20\right]^{|Q|} \leq (k^2\varepsilon)^{20|Q|}. \quad (6.4.5)\end{aligned}$$

Now, let $\mathcal{E}(T)$ be the event that there is a set $Q \subset [T]$ of size $|Q| \geq 0.09|T|$ such that $\mathcal{E}(T,Q)$ occurs. Then (6.4.5) implies that

$$\mathbb{P}\left[\mathcal{E}(T)\right] \leq 2^{|T|}(k^2|T|/n')^{1.8|T|}.$$

Hence, the probability of the event $\mathcal{E}$ that there is a set $T$ of size $|T| \leq n'/k^6$ such that $\mathcal{E}(T)$ occurs is bounded by

$$\begin{aligned}\mathbb{P}\left[\mathcal{E}\right] &\leq \sum_{T:|T|\leq n'/k^6}\mathbb{P}\left[\mathcal{E}(T)\right] \leq \sum_{1\leq t\leq n'/k^6}\binom{n'}{t}2^t(k^2t/n')^{1.8t} \\ &\leq \sum_{1\leq t\leq n'/k^6}\left(\frac{2\mathrm{e}n'}{t}\right)^t(k^2t/n')^{1.8t} \leq \sum_{1\leq t\leq n'/k^6}\left[2\mathrm{e}(t/n')^{0.8}k^{3.6}\right]^t = o(1),\end{aligned}$$

as claimed. □

*Proof of Lemma 6.4.2.* By Lemmas 6.4.4 and 6.4.5 we may assume that $\boldsymbol{H}'$ enjoys the properties (6.4.2) and (6.4.4). We are going to argue that $|U| \leq k|W|$ w.h.p.. Indeed, assume for contradiction that $|U| > k|W|$ and let $U'$ be the set obtained by **WH2** when precisely $(k-1)|W|$ vertices have been added to $U$; thus, $|U'| = k|W|$. Then by construction each vertex $v \in U'$ has one of the following properties.

1. $v$ belongs to $W$, or
2. $v$ occurs in two or more $[n'] \setminus U'$-endangered edges, or
3. $v$ supports at least 20 edges that contain another vertex from $U'$.

Let $U_0 \subset U'$ be the set of all $v \in U'$ that satisfy (1), let $U_1 \subset U' \setminus U_0$ be the set of all $v \in U' \setminus U_0$ that satisfy (2) and let $U_2 = U' \setminus (U_0 \cup U_1)$. There are two cases to consider.

**Case 1:** $|U_1| \geq 0.9|U'|$ then (6.4.2) implies that $|U'| > n'/k^8$.

**Case 2:** $|U_1| < 0.9|U'|$ then $|U_0| + |U_2| \geq 0.1|U'|$ and since $|U_0| = |W|$ and $|U'| = k|W|$ we have $|U_2| \geq 0.09|U'|$ for $k$ large enough. Thus, (6.4.4) entails that $|U'| > n'/k^6$.

Hence, in either case we have $k|W| = |U'| > n'/k^8$ and thus $|W| > n'/k^9$. But by Lemma 6.4.3 we have $|W| = n'\tilde{O}_k(2^{-k})$ w.h.p.. Thus, we conclude that $|U| \le k|W| = n'\tilde{O}_k(2^{-k})$ w.h.p.. □

### 6.4.2. The backbone

We define the *backbone* $\mathrm{back}(\boldsymbol{H}, \boldsymbol{\sigma})$ as the set of all vertices $v \in [n] \setminus \mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma})$ such that the following two conditions hold.

**BB1** $v$ supports at least one edge $e$ such that $e \setminus \{v\} \subset \mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma})$ and
**BB2** $v$ does not occur in a $\{v\} \cup \mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma})$-endangered edge.

Given $\boldsymbol{H}'$, we simply reconstruct $\boldsymbol{H}$ (in distribution) by adding for each $i \in [\omega]$ each monochromatic edge involving $v_i$ with probability $p_1$, and each bichromatic edge involving $v_i$ with probability $p_2$. We let $\mathcal{A}$ be the event that

- no vertex $v \in [n']$ is incident with more than one edge containing a vertex from $\{v_1, \ldots, v_\omega\}$, and
- there is no edge containing two vertices from $\{v_1, \ldots, v_\omega\}$.

With the notation from the previous subsection we let $\bar{U}$ be the complement of the set of vertices produced by the whitening process **WH1–WH2** applied to the hypergraph $\boldsymbol{H}'$. We note that $|\bar{U}| = n'(1 - \tilde{O}_k(2^{-k}))$ w.h.p. by Lemma 6.4.2. In addition, if $\mathcal{A}$ occurs, then $\bar{U} \subset \mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma})$. In this case the following lemma states the probabilities for some events concerning the vertices $v_i, i \in [\omega]$.

**Lemma 6.4.6.** *Assume that $\mathcal{A}$ holds. Let $l \ge 0$ be fixed. Then the following statements are true for all $i \in [\omega]$:*

1. *The probability that $v_i$ supports exactly $l$ edges is $(1 + o(1))\frac{\lambda^l}{l! \exp[\lambda]}$ where*

$$\lambda = \frac{d}{2^{k-1} - 1 + \exp[-\beta]} = k \ln 2 + \tilde{O}_k(2^{-k}).$$

2. *The probability that $v_i$ occurs in exactly $l$ monochromatic edges is $(1 + o(1))\frac{(\lambda')^l}{l! \exp[\lambda']}$ where $\lambda' = \tilde{O}_k(2^{-k})$.*
3. *The probability that there exist exactly $l$ edges blocking $v_i$ and containing at least one vertex outside $\{v_i\} \cup \bar{U}$ is $(1 + o(1))\frac{(\lambda'')^l}{l! \exp[\lambda'']}$ where $\lambda'' = \tilde{O}_k(2^{-k})$.*
4. *The probability that exactly $l$ edges are $\{v_i\} \cup \bar{U}$-endangered is $(1 + o(1))\frac{(\lambda''')^l}{l! \exp[\lambda''']}$ where $\lambda''' = \tilde{O}_k(2^{-k})$.*

*Proof.* For each $i \in [\omega]$, the number of edges supported by $v_i$ is $\mathrm{Bin}\left(\binom{n-1}{k-1}(1 + o(1))2^{1-k}, p_2\right)$

distributed and the number of monochromatic edges involving $v_i$ is $\mathrm{Bin}\left(\binom{n-1}{k-1}(1+o(1))2^{1-k}, p_1\right)$ distributed. Indeed, because we assumed that $\boldsymbol{\sigma}$ is balanced, there are $\binom{n-1}{k-1}(1+o(1))2^{1-k}$ edges $e$ involving $v_i$ such that $\boldsymbol{\sigma}(v) = -\boldsymbol{\sigma}(v_i)$ (respectively $\boldsymbol{\sigma}(v) = \boldsymbol{\sigma}(v_i)$) for all $v \in e \setminus \{v_i\}$ and each of them is added independently at random with probability $p_2$ (respectively $p_1$). Hence the Poisson approximation of the binomial distribution shows that the probability that $v_i$ supports precisely $l$ edges is $(1+o(1))\frac{\lambda^l}{l!\exp[\lambda]}$ with

$$\lambda = \binom{n-1}{k-1}\frac{p_2}{2^{k-1}} = \frac{d}{2^{k-1}-1+\exp\left[-\beta\right]},$$

which proves assertion (1). Moreover, since $\beta = \Omega_k(k\ln 2)$ and $d = \tilde{O}_k(2^k)$, the probability that $v_i$ occurs in precisely $l$ monochromatic edges is $(1+o(1))\frac{(\lambda')^l}{l!\exp[\lambda']}$ with

$$\lambda' = \binom{n-1}{k-1}\frac{p_1}{2^{k-1}} = \lambda\tilde{O}_k(2^{-k}) = \tilde{O}_k(2^{-k}).$$

This implies assertion (2).

The probability that in an edge blocking $v_i$ at least one of the vertices is outside $\{v_i\} \cup \bar{U}$ is $\tilde{O}_k(2^{-k})$ by Lemma 6.4.2. Using (1), the number of edges blocking $v_i$ and containing at least one vertex outside $\{v_i\} \cup \bar{U}$ is stochastically dominated by a $\mathrm{Bin}\left(\binom{n-1}{k-1}\tilde{O}_k(4^{-k}), p_2\right)$ random variable. (3) then follows by the Poisson approximation.

If an edge $e$ is $\{v_i\} \cup \bar{U}$-endangered it is either monochromatic or such that $|(e \setminus \{v_i\}) \cap \bar{U}| \leq k-2$. Given $\boldsymbol{H}'$, these two events are independent and the numbers of edges of each type are binomially distributed. The expected number of edges of the first type is $\tilde{O}_k(2^{-k})$ by (2). The expected number of edges of the second type is $\tilde{O}_k(2^{-k})$ by Lemma 6.4.3. Thus (4) follows again from the Poisson approximation. □

### 6.4.3. The rest

Let $\mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma}) = [n] \setminus (\mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma}) \cup \mathrm{back}(\boldsymbol{H}, \boldsymbol{\sigma}))$.

**Proposition 6.4.7.** *W.h.p.* $|\mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma})| = n2^{-k}(1 + \tilde{O}_k(2^{-k}))$

*Proof.* $\mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma})$ contains at least all vertices that do not support an edge. As the number of edges that a vertex supports is binomially distributed with mean $k\ln 2 + O_k(1)$, by the Chernoff bound we have $|\mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma})| \geq n2^{-k}(1 + \tilde{O}_k(2^{-k}))$ w.h.p.. Now let $Y = \mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma})$ and let $\omega = \omega(n)$ be a slowly diverging function. Let $\varepsilon = \tilde{O}_k(2^{-k})$. We are going to show that

$$\mathbb{E}[Y(Y-1)\cdot\ldots\cdot(Y-\omega+1)] \le \left(\frac{(1+\varepsilon+o(1))n}{2^k}\right)^{\omega}. \tag{6.4.6}$$

This bound implies the assertion; indeed,

$$\begin{aligned}\mathbb{P}\left[Y > (1+2\varepsilon)n2^{-k}\right] \le \quad & \mathbb{P}\left[Y(Y-1)\cdot\ldots\cdot(Y-\omega+1) > ((1+2\varepsilon-o(1))n2^{-k})^{\omega}\right] \\ \le \quad & \frac{\mathbb{E}[Y(Y-1)\cdot\ldots\cdot(Y-\omega+1)]}{((1+2\varepsilon-o(1))n2^{-k})^{\omega}} \le \left(\frac{1+\varepsilon+o(1)}{1+2\varepsilon-o(1)}\right)^{\omega} = o(1).\end{aligned}$$

To prove (6.4.6), we observe that $Y(Y-1)\cdot\ldots\cdot(Y-\omega+1)$ is just the number of ordered $\omega$-tuples of vertices belonging to neither the core nor the backbone – that is, belonging to $Y$. Hence, by symmetry and the linearity of expectation,

$$\mathbb{E}[Y(Y-1)\cdot\ldots\cdot(Y-\omega+1)] \le n^{\omega}\,\mathbb{P}\left[v_1,\ldots,v_\omega \in Y\right].$$

Thus, we are left to estimate $\mathbb{P}\left[v_1,\ldots,v_\omega \in Y\right]$. If $\mathcal{A}$ occurs, then $\bar{U} \subset \mathrm{core}(\boldsymbol{H},\boldsymbol{\sigma})$. Furthermore, if $\bar{U} \subset \mathrm{core}(\boldsymbol{H},\boldsymbol{\sigma})$ and $v_1,\ldots,v_\omega \in Y$, then for any $i \in [\omega]$ one of the following must occur.

1. There is no edge blocking $v_i$ that consists of vertices in $\{v_i\} \cup \bar{U}$ only.
2. $v_i$ occurs in more than 10 edges that are $\{v_i\} \cup \bar{U}$-endangered.
3. There are at least 200 edges blocking $v_i$ but fewer than 100 of them consist of vertices in $\{v_i\}\cup\bar{U}$ only.
4. There are at most 200 edges blocking $v_i$ and one edge $e$ such that $v_i \in e$ and that is $\{v_i\} \cup \bar{U}$-endangered.

Indeed, if a vertex $v_i$ is in $\mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma})$ then it violates one of the conditions **CR1** and **CR2** and one of **BB1** and **BB2**. Therefore we have to consider several cases. If $v_i$ violates **BB1**, then (1) is true. If it violates **CR1** and **BB2**, then either (3) or (4) is true. If $v_i$ violates **CR2** and one of **BB1** and **BB2**, then (2) is true.

Let $\mathcal{B}_i$ be the event that one of the above is true for $i \in [\omega]$. By the principle of deferred decisions we have $\mathbb{P}\left[\mathcal{A}\right] = 1 - O(\omega^2/n)$ and therefore we get

$$\mathbb{P}\left[v_1,\ldots,v_\omega \in Y\right] \le \mathbb{P}\left[v_1,\ldots,v_\omega \in Y|\mathcal{A}\right] + o(1) \le \mathbb{P}\left[\cap_{i=1}^{\omega}\mathcal{B}_i|\mathcal{A}\right] + o(1).$$

Given that there is no edge containing two vertices from $v_1, ..., v_\omega$, the events $\mathcal{B}_1,\ldots,\mathcal{B}_\omega$ are mutually independent. Therefore, $\mathbb{P}\left[\cap_{i=1}^{\omega}\mathcal{B}_i|\mathcal{A}\right] = \mathbb{P}\left[\mathcal{B}_1|\mathcal{A}\right]^{\omega}$. Given that $\mathcal{A}$ occurs, by Lemma 6.4.6 the probability of event (1) is asymptotically equal to $2^{-k} + \tilde{O}_k(4^{-k})$ and the probabilities of events (2), (3) and

(4) are asymptotically equal to $\tilde{O}_k(4^{-k})$. Hence, $\mathbb{P}[\mathcal{B}_1|\mathcal{A}] = 2^{-k} + \tilde{O}_k(4^{-k})$ and $\mathbb{P}[v_1, \ldots, v_\omega \in Y] \le (2^{-k} + \tilde{O}_k(4^{-k}) + o(1))^\omega = ((1 + \varepsilon + o(1))2^{-k})^\omega$. □

We define $\mathrm{free}(\boldsymbol{H}, \boldsymbol{\sigma})$ as the set of all vertices $v \in \mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma})$ such that $v$ occurs only in edges $e$ such that $e \cap \mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma})$ is bichromatic.

**Proposition 6.4.8.** *W.h.p.* $|\mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma}) \setminus \mathrm{free}(\boldsymbol{H}, \boldsymbol{\sigma})| = n\tilde{O}_k(4^{-k})$. *In particular,* $|\mathrm{free}(\boldsymbol{H}, \boldsymbol{\sigma})| = n(2^{-k} + \tilde{O}_k(4^{-k}))$.

*Proof.* We introduce $Y = |\mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma}) \setminus \mathrm{free}(\boldsymbol{H}, \boldsymbol{\sigma})|$ and proceed just as in the proof of Propositi-on 6.4.7. To estimate $\mathbb{P}[v_1, \ldots, v_\omega \in Y]$ we observe that if $\bar{U} \subset \mathrm{core}(\boldsymbol{H}, \boldsymbol{\sigma})$ and $v_1, \ldots, v_\omega \in Y$ then for any $i \in [\omega]$ one of the following must occur.

1. There is no edge blocking $v_i$ that consists of vertices in $\{v_i\} \cup \bar{U}$ only and $v_i$ occurs in at least one edge that is $\{v_i\} \cup \bar{U}$-endangered.
2. $v_i$ occurs in more than 10 edges that are $\{v_i\} \cup \bar{U}$-endangered.
3. There are at least 200 edges blocking $v_i$ but fewer than 100 of them consist of vertices in $\{v_i\} \cup \bar{U}$ only.
4. There are at most 200 edges blocking $v_i$ and one edge $e$ such that $v_i \in e$ and that is $\{v_i\} \cup \bar{U}$-endangered.

Events (2), (3) and (4) are as in the proof of Proposition 6.4.7 and their probabilities are asymptotically equal to $\tilde{O}_k(4^{-k})$. By Lemma 6.4.6 the probability of (1) is $\tilde{O}_k(4^{-k})$ and the assertion follows. □

*In the following three subsections we calculate the cluster size* $\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma})$ *up to a small error term. We proceed by first eliminating the contribution of the vertices in the core and in a second step the contribution of the vertices in the backbone. Finally we calculate the contribution of the vertices in* $\mathrm{rest}(\boldsymbol{H}, \boldsymbol{\sigma})$.

### 6.4.4. Rigidity of the core

In the following we let $x = k^{-5}$. We first show that the cluster of $\boldsymbol{\sigma}$ under $\boldsymbol{H}$ mostly consists of configurations at distance less than $2x$ from $\boldsymbol{\sigma}$.

**Lemma 6.4.9.** *W.h.p.*

$$\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma}) \sim \sum_{\tau \in \{\pm 1\}^n : \langle \boldsymbol{\sigma}, \tau \rangle \ge (1-x)n} \exp\left[-\beta E_{\boldsymbol{H}}(\tau)\right]$$

To prove this result we recall the notation from Section 6.2. We need the following technical lemma:

**Lemma 6.4.10.** *Let* $d/k = 2^{k-1}\ln 2 + O_k(1)$ *and* $\beta \geq k\ln 2 - \ln k$. *Then* $\sup_{\alpha\in[2/3,1-k^{-5}]}\Lambda_\beta(\alpha) < \Lambda_\beta(1) - \Omega_k(k^{-5})$.

*Proof.* We observe that for $\alpha \in [1-k^{-5}, 1-k^{-7}]$,

$$\Lambda'_\beta(\alpha) = \frac{\ln(1-\alpha)}{2} + \frac{d}{2^k} + \tilde{O}_k(2^{-k}) = k\ln 2 + O_k(\ln k) \geq 1. \tag{6.4.7}$$

An expansion of $\Lambda_\beta(\alpha)$ near $\alpha = 1$ gives $\Lambda_\beta(1-k^{-7}) \leq \Lambda_\beta(1) + O_k(k^{-6})$ and together with (6.4.7) this implies

$$\Lambda_\beta(1-k^{-5}) \leq \Lambda_\beta(1) - \Omega_k(k^{-5}). \tag{6.4.8}$$

Further, using that $\Lambda'_\beta(\alpha) > 0$ if $\alpha > 1 - 1.99\ln k/k$ (as in the proof of Lemma 6.2.9) and (6.4.8) we obtain

$$\sup_{\alpha\in[1-1.99\ln k/k,1-k^{-5}]}\Lambda_\beta(\alpha) \leq \Lambda_\beta(1-k^{-5}) \leq \Lambda_\beta(1) - \Omega_k(k^{-5}). \tag{6.4.9}$$

A study of $\Lambda_\beta(\alpha)$ also gives

$$\sup_{\gamma\in[1.99,2.01]}\Lambda_\beta(1-\gamma\ln k/k) \leq \Lambda_\beta(1) - \Omega_k(k^{-5}) \tag{6.4.10}$$

and $\Lambda_\beta(\alpha) - \Lambda_\beta(1-2.01\ln k/k) = \mathcal{H}\left(\frac{1+\alpha}{2}\right) + \tilde{O}_k\left(\left(\frac{2}{2.01}\right)^k\right) \leq 0$ for $\alpha \in [2/3, 1-2.01\ln k/k]$, which leads to

$$\begin{aligned}\sup_{\alpha\in[2/3,1-2.01\ln k/k]}\Lambda_\beta(\alpha) &\leq \mathcal{H}\left(\frac{1+\alpha}{2}\right) + \tilde{O}_k\left(\left(\frac{2}{2.01}\right)^k\right) + \Lambda_\beta(1-2.01\ln k/k)\\ &\leq \Lambda_\beta(1) - \Omega_k(k^{-5}).\end{aligned} \tag{6.4.11}$$

Combining (6.4.9), (6.4.10) and (6.4.11) completes the proof of the assertion. □

*Proof of Lemma 6.4.9.* Given $\boldsymbol{\sigma}$ and $\alpha \in [-1,1]$ and using Lemma 6.2.2 we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{\tau\in\{\pm1\}^n:\langle\boldsymbol{\sigma},\tau\rangle=\alpha n}\exp\left[-\beta E_{\boldsymbol{H}}(\tau)\right]\middle|\, |e(\boldsymbol{H})-m| \leq m^{2/3}\right]\\ &\quad= \frac{\mathbb{E}\left[\sum_{\tau:\langle\boldsymbol{\sigma},\tau\rangle=\alpha n}\exp\left[-\beta E_{H_k(n,p)}(\boldsymbol{\sigma})\right]\exp\left[-\beta E_{H_k(n,p)}(\tau)\right]\middle|\, |e(H_k(n,p))-m| \leq m^{2/3}\right]}{\mathbb{E}\left[\exp\left[-\beta E_{H_k(n,p)}(\boldsymbol{\sigma})\right]\middle|\, |e(H_k(n,p))-m| \leq m^{2/3}\right]}\\ &\quad\leq \frac{\mathbb{E}\left[Z_\beta(\alpha)\right]}{\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right]}\exp\left[O\left(m^{2/3}\right)\right].\end{aligned}$$

In order to derive the last line, we used an observation similar to equation (6.2.5) and Lemma 6.2.2. We observe that we have w.h.p. $\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma}) \geq \exp\left[-\beta E_{\boldsymbol{H}}(\boldsymbol{\sigma})\right] \sim \exp\left[-n\tilde{O}_k(2^{-k})\right]$ by Lemma 6.2.6. Hence,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{\tau \in \{\pm 1\}^n : 2/3n \leq \langle \boldsymbol{\sigma}, \tau \rangle < (1-x)n} \exp\left[-\beta E_{\boldsymbol{H}}(\tau)\right] \middle| |e(\boldsymbol{H}) - m| \leq m^{2/3}\right] \\
&\qquad \leq \sum_{\nu=0}^{n} \frac{\mathbb{E}\left[Z_\beta(2\nu/n - 1)\right]}{\mathbb{E}\left[Z_\beta(\mathcal{H}(n,m))\right]} \mathbf{1}_{2\nu/n - 1 \in [2/3, (1-x)]} \exp\left[O\left(m^{2/3}\right)\right] \\
&\qquad \leq \exp\left[n\left(\sup_{\alpha \in [2/3, 1-x]} \Lambda_\beta(\alpha) - \Lambda_\beta(1) + \tilde{O}_k(2^{-k})\right)\right] \mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma}) \\
&\qquad \leq \exp\left[-n\Omega_k(k^{-5})\right] \mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma})
\end{aligned}$$

by Lemmas 6.2.7 and 6.4.10. It follows from Markov's inequality that w.h.p.

$$\sum_{\tau \in \{\pm 1\}^n : 2/3n \leq \langle \boldsymbol{\sigma}, \tau \rangle < (1-x)n} \exp\left[-\beta E_{\boldsymbol{H}}(\tau)\right] = o(\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma})).$$

□

We now approximate $\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma})$ based on the previous decomposition of the vertex set $V$. Given a $k$-uniform hypergraph $\boldsymbol{H}$, $\boldsymbol{\sigma} : [n] \to \{\pm 1\}$, and three maps $\tau_{\text{core}} : \text{core}(\boldsymbol{H}, \boldsymbol{\sigma}) \to \{\pm 1\}$, $\tau_{\text{back}} : \text{back}(\boldsymbol{H}, \boldsymbol{\sigma}) \to \{\pm 1\}$ and $\tau_{\text{rest}} : \text{rest}(\boldsymbol{H}, \boldsymbol{\sigma}) \to \{\pm 1\}$, we define $E_{\boldsymbol{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$ as $E_{\boldsymbol{H}}(\tau)$ for the unique $\tau$ whose restriction to $\text{core}(\boldsymbol{H}, \boldsymbol{\sigma})$ (respectively $\text{back}(\boldsymbol{H}, \boldsymbol{\sigma}), \text{rest}(\boldsymbol{H}, \boldsymbol{\sigma})$) is given by $\tau_{\text{core}}$ (respectively $\tau_{\text{back}}, \tau_{\text{rest}}$).

We introduce the "restricted" cluster size

$$\mathcal{C}_\beta^{\text{back+rest}}(\boldsymbol{H}, \boldsymbol{\sigma}) = \sum_{\tau_{\text{back}}, \tau_{\text{rest}}} \exp\left[-\beta E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})\right].$$

The summation is over $\tau_{\text{back}} : \text{back}(\boldsymbol{H}, \boldsymbol{\sigma}) \to \{\pm 1\}$ and $\tau_{\text{rest}} : \text{rest}(\boldsymbol{H}, \boldsymbol{\sigma}) \to \{\pm 1\}$. The aim of this section is to prove the following.

**Proposition 6.4.11.** *W.h.p.*

$$\frac{1}{n} \ln \mathcal{C}_\beta^{\text{back+rest}}(\boldsymbol{H}, \boldsymbol{\sigma}) \leq \frac{1}{n} \ln \mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma}) \leq \frac{1}{n} \ln \mathcal{C}_\beta^{\text{back+rest}}(\boldsymbol{H}, \boldsymbol{\sigma}) + \exp\left[-88\beta\right]$$

In order to proceed we first need a few additional results. We introduce the set $\mathcal{E}_{\boldsymbol{H}}(\tau, \boldsymbol{\sigma})$ of edges that

- are supported by a vertex $v$ such that $\tau_{\text{core}}(v) \neq \boldsymbol{\sigma}_{\text{core}}(v)$ and
- contain two or more vertices $v'$ such that $\tau_{\text{core}}(v') \neq \boldsymbol{\sigma}_{\text{core}}(v')$.

The following lemma is reminiscent of [COZ12, Lemma 5.9].

**Lemma 6.4.12.** *W.h.p. for all $\tau : [n] \to \{\pm 1\}$ satisfying $\langle \boldsymbol{\sigma}, \tau \rangle \geq (1-x)n$ it holds that*

$$|\mathcal{E}_{\boldsymbol{H}}(\tau, \boldsymbol{\sigma})| \leq 2|\{v : \boldsymbol{\sigma}_{\text{core}}(v) \neq \tau_{\text{core}}(v)\}|.$$

*Proof.* We claim that w.h.p. $\boldsymbol{H}$ has the following property. Let $T \subset V$ be a set of size $|T| \leq n/(2e^3k^2\lambda^2)$. Then there are no more than $2|T|$ edges that are supported by a vertex in $T$ and contain a second vertex from $T$. Indeed, by a first moment argument, with $|T| = tn$ the probability that there is a set $T$ that violates the above property is bounded by

$$\binom{n}{tn}\binom{(1+o(1))\lambda n}{2tn}(kt^2)^{2tn} \leq \left[(1+o(1))\frac{e}{t}\left(\frac{\lambda e}{2t}\right)^2 \left(kt^2\right)^2\right]^{tn}$$
$$\leq \left((1+o(1))t\left(e^3\lambda^2k^2\right)\right)^{tn} = o(1).$$

With $T = \{v : \boldsymbol{\sigma}_{\text{core}}(v) \neq \tau_{\text{core}}(v)\}$ and $x = k^{-5}$, we have $|T| \leq 2xn < n/(2e^3k^2\lambda^2)$, which completes the proof. □

**Lemma 6.4.13.** *W.h.p. for all $\tau : [n] \to \{\pm 1\}$ satisfying $\langle \boldsymbol{\sigma}, \tau \rangle \geq (1-x)n$ it holds that*

$$E_{\boldsymbol{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \geq E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) + 88 \operatorname{dist}(\tau_{\text{core}}, \boldsymbol{\sigma}_{\text{core}}).$$

*Proof.* Denote for a vertex $v \in V$ and $\tau : [n] \to \{\pm 1\}$ by

- $X(v)$ the number of critical (under $\boldsymbol{\sigma}$) edges $e$ supported by $v$ such that $e \setminus \{v\} \subset \operatorname{core}(\boldsymbol{H}, \boldsymbol{\sigma})$,
- $Y(v)$ the number of $\operatorname{core}(\boldsymbol{H}, \boldsymbol{\sigma})$-endangered edges containing $v$,
- $M_\tau(v)$ the number of edges containing $v$ that are monochromatic under $(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$.

We can lower bound $E_{\boldsymbol{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$ in terms of $E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$ as

$$E_{\boldsymbol{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \geq E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) + \sum_{v : \tau_{\text{core}}(v) \neq \boldsymbol{\sigma}_{\text{core}}(v)} (X(v) - M_\tau(v)) - |\mathcal{E}_{\boldsymbol{H}}(\tau, \boldsymbol{\sigma})|. \tag{6.4.12}$$

Only edges that were $\operatorname{core}(\boldsymbol{H}, \boldsymbol{\sigma})$-endangered can be monochromatic under $(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})$, implying that $M_\tau(v) \leq Y(v)$. In particular

$$\forall v \in \operatorname{core}(\boldsymbol{H}, \boldsymbol{\sigma}), \quad X(v) - M_\tau(v) \geq 90. \tag{6.4.13}$$

On the other hand, we can upper bound $|\mathcal{E}_{\boldsymbol{H}}(\tau, \boldsymbol{\sigma})|$ with Lemma 6.4.12. Replacing in (6.4.12) and using (6.4.13) gives

$$E_{\boldsymbol{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \geq E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) + 88 \operatorname{dist}(\tau_{\text{core}}, \boldsymbol{\sigma}_{\text{core}})$$

w.h.p., thereby completing the proof. □

*Proof of Proposition 6.4.11.* We first establish the lower bound on $\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma})$. With Proposition 6.4.1 we have $\langle \boldsymbol{\sigma}, (\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}}) \rangle \geq (1-x)n$ w.h.p. for all $(\tau_{\text{back}}, \tau_{\text{rest}})$. Hence with Lemma 6.4.9 w.h.p.

$$\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma}) \geq \sum_{\tau_{\text{back}}, \tau_{\text{rest}}} \exp\left[-\beta E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})\right] = \mathcal{C}_\beta^{\text{back+rest}}(\boldsymbol{H}, \boldsymbol{\sigma}).$$

To derive the upper bound we write

$$\begin{aligned}\mathcal{C}_\beta(\boldsymbol{H}, \boldsymbol{\sigma}) &\leq \sum_{\substack{\tau_{\text{core}}\,: \\ \langle \boldsymbol{\sigma}_{\text{core}}, \tau_{\text{core}} \rangle \geq (1-x)n}} \sum_{\tau_{\text{back}}, \tau_{\text{rest}}} \left[-\beta E_{\boldsymbol{H}}(\tau_{\text{core}}, \tau_{\text{back}}, \tau_{\text{rest}})\right] \\ &\leq \sum_{\substack{\tau_{\text{core}}\,: \\ \langle \boldsymbol{\sigma}_{\text{core}}, \tau_{\text{core}} \rangle \geq (1-x)n}} \exp\left[-88\beta \operatorname{dist}(\boldsymbol{\sigma}_{\text{core}}, \tau_{\text{core}})\right] \mathcal{C}_\beta^{\text{back+rest}}(\boldsymbol{H}, \boldsymbol{\sigma}), \end{aligned} \tag{6.4.14}$$

where the second inequality holds w.h.p. by Lemma 6.4.13. Finally

$$\begin{aligned}\sum_{\substack{\tau_{\text{core}}\,: \\ \langle \boldsymbol{\sigma}_{\text{core}}, \tau_{\text{core}} \rangle \geq (1-x)n}} \exp\left[-88\beta \operatorname{dist}(\boldsymbol{\sigma}_{\text{core}} \tau_{\text{core}})\right] &= \sum_{i=0}^{xn/2} \binom{n}{i} \exp\left[-88\beta i\right] \leq \sum_{i=0}^{n} \binom{n}{i} \exp\left[-88\beta i\right] \\ &= (1 + \exp\left[-88\beta\right])^n \leq \exp\left[n \exp\left[-88\beta\right]\right]. \end{aligned} \tag{6.4.15}$$

Replacing with (6.4.15) in (6.4.14) completes the proof. □

### 6.4.5. Rigidity of the backbone

We proceed one step further by eliminating the vertices in the backbone and consequently comparing $\mathcal{C}_\beta^{\text{back+rest}}(\boldsymbol{H}, \boldsymbol{\sigma})$ to $\mathcal{C}_\beta^{\text{rest}}(\boldsymbol{H}, \boldsymbol{\sigma})$, where

$$\mathcal{C}_\beta^{\text{rest}}(\boldsymbol{H}, \boldsymbol{\sigma}) = \sum_{\tau_{\text{rest}}} \exp\left[-\beta E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\text{core}}, \boldsymbol{\sigma}_{\text{back}}, \tau_{\text{rest}})\right].$$

The sum is over $\tau_{\text{rest}} : \text{rest}(\boldsymbol{H}, \boldsymbol{\sigma}) \to \{\pm 1\}$. We prove the following result.

**Proposition 6.4.14.** *W.h.p.*

$$\frac{1}{n}\ln \mathcal{C}_\beta^{\mathrm{rest}}(\boldsymbol{H},\boldsymbol{\sigma}) \leq \frac{1}{n}\ln \mathcal{C}_\beta^{\mathrm{back+rest}}(\boldsymbol{H},\boldsymbol{\sigma}) \leq \frac{1}{n}\ln \mathcal{C}_\beta^{\mathrm{rest}}(\boldsymbol{H},\boldsymbol{\sigma}) + \tilde{O}_k(4^{-k})$$

*Proof.* The left inequality is obvious. To prove the right inequality we observe that, by definition of the backbone, for any $\tau_{\mathrm{back}} : \mathrm{back}(\boldsymbol{H},\boldsymbol{\sigma}) \to \{\pm 1\}$ and $\tau_{\mathrm{rest}} : \mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma}) \to \{\pm 1\}$, the following is true:

$$E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\mathrm{core}},\tau_{\mathrm{back}},\tau_{\mathrm{rest}}) \geq E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\mathrm{core}},\boldsymbol{\sigma}_{\mathrm{back}},\tau_{\mathrm{rest}}) + \mathrm{dist}(\boldsymbol{\sigma}_{\mathrm{back}},\tau_{\mathrm{back}}). \tag{6.4.16}$$

Indeed for any vertex $v \in \mathrm{back}(\boldsymbol{H},\boldsymbol{\sigma})$ with $\boldsymbol{\sigma}_{\mathrm{back}}(v) \neq \tau_{\mathrm{back}}(v)$ and any edge $e \ni v$,

- either $v$ supports $e$ and $e \setminus \{v\} \subset \mathrm{core}(\boldsymbol{H},\boldsymbol{\sigma})$, in which case it is bichromatic under the assignment $(\boldsymbol{\sigma}_{\mathrm{core}},\boldsymbol{\sigma}_{\mathrm{back}},\tau_{\mathrm{rest}})$ and monochromatic under $(\boldsymbol{\sigma}_{\mathrm{core}},\tau_{\mathrm{back}},\tau_{\mathrm{rest}})$,
- or $e$ is not $\{v\}\cup\mathrm{core}(\boldsymbol{H},\boldsymbol{\sigma})$-endangered and is bichromatic both under $(\boldsymbol{\sigma}_{\mathrm{core}},\boldsymbol{\sigma}_{\mathrm{back}},\tau_{\mathrm{rest}})$ and under $(\boldsymbol{\sigma}_{\mathrm{core}},\tau_{\mathrm{back}},\tau_{\mathrm{rest}})$.

Moreover, by the definition of $\mathrm{back}(\boldsymbol{H},\boldsymbol{\sigma})$, there is at least one edge of the first type for any $v \in \mathrm{back}(\boldsymbol{H},\boldsymbol{\sigma})$ with $\boldsymbol{\sigma}_{\mathrm{back}}(v) \neq \tau_{\mathrm{back}}(v)$.

Using the definition of $\mathcal{C}_\beta^{\mathrm{back+rest}}(\boldsymbol{H},\boldsymbol{\sigma})$ and (6.4.16) yields

$$\begin{aligned}\mathcal{C}_\beta^{\mathrm{back+rest}}(\boldsymbol{H},\boldsymbol{\sigma}) &\leq \sum_{\tau_{\mathrm{back}},\tau_{\mathrm{rest}}} \exp\left[-\beta\mathrm{dist}(\boldsymbol{\sigma}_{\mathrm{back}},\tau_{\mathrm{back}})\right]\exp\left[-\beta E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\mathrm{core}},\boldsymbol{\sigma}_{\mathrm{back}},\tau_{\mathrm{rest}})\right]\\ &\leq \sum_{\tau_{\mathrm{back}}} \exp\left[-\beta\mathrm{dist}(\boldsymbol{\sigma}_{\mathrm{back}},\tau_{\mathrm{back}})\right]\mathcal{C}_\beta^{\mathrm{rest}}(\boldsymbol{H},\boldsymbol{\sigma}).\end{aligned} \tag{6.4.17}$$

The remaining sum can easily be upper-bounded:

$$\begin{aligned}\sum_{\tau_{\mathrm{back}}} \exp\left[-\beta\mathrm{dist}(\boldsymbol{\sigma}_{\mathrm{back}},\tau_{\mathrm{back}})\right] &= \sum_{i=0}^{|\mathrm{back}(\boldsymbol{H},\boldsymbol{\sigma})|}\binom{|\mathrm{back}(\boldsymbol{H},\boldsymbol{\sigma})|}{i}\exp\left[-\beta i\right]\\ &= (1+\exp\left[-\beta\right])^{|\mathrm{back}(\boldsymbol{H},\boldsymbol{\sigma})|} \leq \exp\left[\exp\left[-\beta\right]|\mathrm{back}(\boldsymbol{H},\boldsymbol{\sigma})|\right]\end{aligned} \tag{6.4.18}$$

The upper bound of Proposition 6.4.14 then follows from (6.4.17) and (6.4.18) combined with Proposition 6.4.1. □

### 6.4.6. The remaining vertices

We finally deal with the vertices that belong neither to the core nor to the backbone. As anticipated in Proposition 6.4.8, most of them are free. This yields the following result.

**Proposition 6.4.15.** *W.h.p.*

$$\frac{1}{n}\ln \mathcal{C}_\beta^{\mathrm{rest}}(\boldsymbol{H},\boldsymbol{\sigma}) = \frac{\ln 2}{2^k} - \beta\frac{E_{\boldsymbol{H}}(\boldsymbol{\sigma})}{n} + \tilde{O}_k(4^{-k})$$

To prove this, we need the following result. Let $M'_{\boldsymbol{\sigma}}(v)$ be the number of monochromatic edges involving $v$ in the configuration $\boldsymbol{\sigma}$.

**Lemma 6.4.16.** *W.h.p.*

$$\sum_{v\in \mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma})\setminus \mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})} M'_{\boldsymbol{\sigma}}(v) = n\tilde{O}_k(4^{-k})$$

*Proof.* We start with the following observation:

$$\sum_{v\in \mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma})\setminus \mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})} M'_{\boldsymbol{\sigma}}(v) \leq \sum_{v\in V: M'_{\boldsymbol{\sigma}}(v)>2} M'_{\boldsymbol{\sigma}}(v) + 2|\mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma}) \setminus \mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})|$$

The number of monochromatic edges involving a vertex $v$ is a $\mathrm{Bin}\left(\binom{n-1}{k-1}(1+o(1))2^{k-1}, p_1\right)$ random variable. Hence $\sum_{v\in V: M'_{\boldsymbol{\sigma}}(v)>2} M'_{\boldsymbol{\sigma}}(v) = n\tilde{O}_k(4^{-k})$. Applying Proposition 6.4.8 completes the proof. □

*Proof of Proposition 6.4.15.* By the definition of $\mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})$, the number of monochromatic edges $E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\mathrm{core}}, \boldsymbol{\sigma}_{\mathrm{back}}, \tau_{\mathrm{rest}})$ does not depend on the values $\tau_{\mathrm{rest}}(v)$ for $v \in \mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})$. Consequently,

$$\mathcal{C}_\beta^{\mathrm{rest}}(\boldsymbol{H},\boldsymbol{\sigma}) \geq 2^{|\mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})|}\exp\left[-\beta E_{\boldsymbol{H}}(\boldsymbol{\sigma})\right].$$

Together with Proposition 6.4.8, this gives the lower bound on $\frac{1}{n}\ln \mathcal{C}_\beta^{\mathrm{rest}}(\boldsymbol{H},\boldsymbol{\sigma})$. For the upper bound, we start with the general inequality

$$\frac{1}{n}\ln \mathcal{C}_\beta^{\mathrm{rest}}(\boldsymbol{H},\boldsymbol{\sigma}) \leq \frac{\ln 2}{n}|\mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma})| - \frac{\beta}{n}\inf_{\tau_{\mathrm{rest}}} E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\mathrm{core}}, \boldsymbol{\sigma}_{\mathrm{back}}, \tau_{\mathrm{rest}}).$$

As the number of monochromatic edges does not depend on the values of the vertices in $\mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})$, we have

$$\inf_{\tau_{\mathrm{rest}}} E_{\boldsymbol{H}}(\boldsymbol{\sigma}_{\mathrm{core}}, \boldsymbol{\sigma}_{\mathrm{back}}, \tau_{\mathrm{rest}}) \geq E_{\boldsymbol{H}}(\boldsymbol{\sigma}) - \sum_{v\in \mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma})\setminus \mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})} M'_{\boldsymbol{\sigma}}(v).$$

Hence, we obtain

$$\frac{1}{n}\ln \mathcal{C}_\beta^{\mathrm{rest}}(\boldsymbol{H},\boldsymbol{\sigma}) \leq \frac{\ln 2}{n}|\mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma})| - \beta\frac{E_{\boldsymbol{H}}(\boldsymbol{\sigma})}{n} + \frac{\beta}{n}\sum_{v\in \mathrm{rest}(\boldsymbol{H},\boldsymbol{\sigma})\setminus \mathrm{free}(\boldsymbol{H},\boldsymbol{\sigma})} M'_{\boldsymbol{\sigma}}(v). \tag{6.4.19}$$

The upper bound follows by combining (6.4.19) with Proposition 6.4.7 and Lemma 6.4.16. □

*Proof of Proposition 6.1.7.* Combining Propositions 6.4.11, 6.4.14 and 6.4.15 we obtain that w.h.p.

$$\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) = \frac{\ln 2}{2^k} - \beta\frac{E_{\boldsymbol{H}}(\boldsymbol{\sigma})}{n} + \tilde{O}_k(4^{-k}). \tag{6.4.20}$$

The number of monochromatic edges in the planted model is tightly concentrated by Chernoff bounds. Therefore, we get w.h.p.

$$E_{\boldsymbol{H}}(\boldsymbol{\sigma}) = \binom{n}{k}2^{1-k}p_1(1+o(1)) \sim \frac{\exp[-\beta]}{2^{k-1}-1+\exp[-\beta]}\frac{d}{k}n.$$

For $d/k = 2^{k-1}\ln 2 + O_k(1)$ and $\beta \geq k\ln 2 - \ln k$, we have $E_{\boldsymbol{H}}(\boldsymbol{\sigma}) = \ln 2\exp[-\beta]\,n + \tilde{O}_k(4^{-k})n$. Inserting this in (6.4.20) yields w.h.p.

$$\frac{1}{n}\ln \mathcal{C}_\beta(\boldsymbol{H},\boldsymbol{\sigma}) = \frac{\ln 2}{2^k} - \beta\ln 2\exp[-\beta] + \tilde{O}_k(4^{-k}),$$

thereby proving Proposition 6.1.7. □

## 6.5. Existence of $\Phi_{d,k}(\beta)$

**Theorem 6.5.1.** *The limit $\Phi_{d,k}(\beta)$ exists for any $d > 0$, $k \geq 3$, $\beta \geq 0$.*

We prove the existence of the limit using the so-called interpolation method. The proof is very similar to and adapted from [BGT13]. Let us first shortly summarize the idea of the interpolation method. Given $H_k(n,m)$ and $n_1, n_2$ such that $n = n_1 + n_2$ and $\mathcal{M}_1 \overset{d}{=} \mathrm{Bin}(m, n_1/n)$, we can construct a sequence of hypergraphs interpolating between $H_k(n,m)$ and a disjoint union of $H_k(n_1,\mathcal{M}_1)$ and $H_k(n_2, m-\mathcal{M}_1)$, where we have split the set of nodes $[n]$ into two sets $[n_1] = \{1,...,n_1\}$ and $\{n_1+1,...,n\}$ which we denote, with some abuse of notation, as $[n_2]$.

To realize this interpolation, for any $0 \leq r \leq m$, let $H_k(n,m,r)$ be the random graph on nodes $[n]$ obtained as follows: It contains $m$ hyperedges, where the first $r$ hyperdeges $e_1,...,e_r$ are selected independently and uniformly at random from all possible hyperedges on $H_k(n,m)$. The remaining $m-r$ hyperdeges are generated independently and uniformly at random from all possible hyperedges on nodes $[n_1]$ with probability $n_1/n$ and from all possible hyperedges on nodes $[n_2]$ with probability $n_2/n$ .

We observe that $H_k(n,m,m) = H_k(n,m)$ and that $H_k(n,m,0)$ is a disjoint union of the graphs $H_k(n_1,\mathcal{M}_1)$, $H_k(n_2,\mathcal{M}_2)$ conditioned on $\mathcal{M}_1+\mathcal{M}_2 = m$, where $\mathcal{M}_j = \mathrm{Bin}(m, n_j/n)$.

A centerpiece of the interpolation method is the following lemma:

**Lemma 6.5.2.** *For every $r = 1, ..., m$,*

$$\mathbb{E}[\ln Z_\beta(H_k(n,m,r))] \geq \mathbb{E}[\ln Z_\beta(H_k(n,m,r-1))].$$

*Proof.* We let $H_k(n,m,r-1)$ be obtained from $H_k(n,m,r)$ by deleting a hyperdege chosen uniformly at random from all $e_1, ..., e_r$ and adding a hyperedge $e$ to nodes $[n_1]$ or $[n_2]$ with the appropriate probabilities. Let $H_k^0$ be the hypergraph obtained in this interpolation process from $H_k(n,m,r)$ after deleting but before inserting a hyperedge. We let $Z_\beta^0$ and $\pi_\beta^0$ be the corresponding partition function and Gibbs measure respectively.

We now show that conditional on any realization of the graph $H_k^0$, we have

$$\mathbb{E}[\ln Z_\beta(H_k(n,m,r))|H_k^0] \geq \mathbb{E}[\ln Z_\beta(H_k(n,m,r-1))|H_k^0].$$

Note that since we fix $H_k^0$, the only randomness underlying the expectation arises from choosing the hyperedge $e = (n_{i_1}, ..., n_{i_k})$. We have

$$\begin{aligned}
&\mathbb{E}[\ln Z_\beta(H_k(n,m,r))|H_k^0] - \ln Z_\beta^0 \\
&= \mathbb{E}\left[\ln \frac{Z_\beta(H_k(n,m,r))}{Z_\beta^0} - |H_k^0\right] \\
&= \mathbb{E}\left[\ln \frac{\sum_\sigma \mathbf{1}_{\{\text{not all } \sigma_{i_j} \text{ identical}\}} \exp[-\beta E(\sigma)] + \exp[-\beta] \sum_\sigma \mathbf{1}_{\{\sigma_{i_1}=...=\sigma_{i_k}\}} \exp[-\beta E(\sigma)]}{\sum_x \exp[-\beta E(x)]} |H_k^0\right].
\end{aligned}$$

Since $\beta < \infty$, we have $0 \leq (1 - \exp[-\beta])\, \pi_\beta^0(\sigma_{i_1} = ... = \sigma_{i_k}) < 1$. Using the expansion $\ln(1-x) = -\sum_{j\geq 1} x^j/j$, we get

$$\begin{aligned}
&\mathbb{E}[\ln Z_\beta(H_k(n,m,r))|H_k^0] - \ln Z_\beta^0 \\
&= -\mathbb{E}\left[\sum_{j=1}^{\infty} \frac{(1-\exp[-\beta])^j\, \pi_\beta^0(\sigma_{i_1} = ... = \sigma_{i_k})^j}{j} |H_k^0\right] \\
&= -\sum_{j=1}^{\infty} \frac{(1-\exp[-\beta])^j}{j} \mathbb{E}\left[\sum_{\sigma^{(1)},...,\sigma^{(j)}} \frac{\exp\left[-\beta \sum_{s=1}^{j} E(\sigma^{(j)})\right]}{(Z_\beta^0)^j} \mathbf{1}_{\{\sigma_{i_1}^{(s)} = ... = \sigma_{i_k}^{(s)} \forall s \in [j]\}} |H_k^0\right] \\
&= -\sum_{j=1}^{\infty} \frac{(1-\exp[-\beta])^j}{j} \sum_{\sigma^{(1)},...,\sigma^{(j)}} \frac{\exp\left[-\beta \sum_{s=1}^{j} E(\sigma^{(j)})\right]}{(Z_\beta^0)^j} \mathbb{E}\left[\mathbf{1}_{\{\sigma_{i_1}^{(s)} = ... = \sigma_{i_k}^{(s)} \forall s \in [j]\}} |H_k^0\right],
\end{aligned}$$

where the sum $\sum_{\sigma^{(1)},...,\sigma^{(j)}}$ is over $j$-tupels of colour assignments. We now introduce equivalency classes on $[n]$ for each such $j$-tupel for all $j \in \{1, ..., \infty\}$. For $t, r \in [n]$, we say that $t$ is equivalent to $r$, denoted by $t \sim r$, if $\sigma_t^{(s)} = \sigma_r^{(s)}$ for all $s \in [j]$. Let $O_l, 1 \leq l \leq J$ be the corresponding equivalency classes. For an edge $(n_{i_1}, ..., n_{i_k})$ generated uniformly at random, it follows that $\mathbb{E}\left[\mathbf{1}_{\{\sigma_{i_1}^{(s)}=...=\sigma_{i_k}^{(s)} \forall s \in [j]\}} | H_k^0\right] = \sum_{l=2}^{J} \left(\frac{|O_l|}{n}\right)^k$ and thus

$$\begin{aligned}
&\mathbb{E}\left[\ln Z_\beta(H_k(n,m,r))|H_k^0\right] - \ln Z_\beta^0 \\
&\qquad = -\sum_{j=1}^{\infty} \frac{(1-\exp[-\beta])^j}{j} \sum_{\sigma^{(1)},...,\sigma^{(j)}} \frac{\exp\left[-\beta \sum_{s=1}^{j} E(\sigma^{(j)})\right]}{(Z_\beta^0)^j} \sum_{l=2}^{J} \left(\frac{|O_l|}{n}\right)^k.
\end{aligned}$$

A similar calculation for $\mathbb{E}[\ln Z_\beta(H_k(n,m,r-1))|H_k^0]$ obtained by adding a hyperedge to nodes $[n_1]$ with probability $n/n_1$ or to nodes $[n_2]$ with probability $n/n_2$ gives

$$\begin{aligned}
\mathbb{E}[\ln Z_\beta(H_k(n,m,r-1))|H_k^0] - \ln Z_\beta^0 &= -\sum_{j=1}^{\infty} \frac{(1-\exp[-\beta])^j}{j} \sum_{\sigma^{(1)},...,\sigma^{(j)}} \frac{\exp\left[-\beta \sum_{s=1}^{j} E(\sigma^{(j)})\right]}{(Z_\beta^0)^j} \\
&\quad \cdot \sum_{l=2}^{J} \left(\frac{n_1}{n}\left(\frac{|O_l \cap [n_1]|}{n_1}\right)^k + \frac{n_2}{n}\left(\frac{|O_l \cap [n_2]|}{n_2}\right)^k\right).
\end{aligned}$$

Using the convexity of the function $f(x) = x^k$, the claim follows. □

**Lemma 6.5.3.** *For every $1 \leq n_1, n_2 \leq n-1$ such that $n_1 + n_2 = n$ and every $\beta < \infty$,*

$$\mathbb{E}[\ln Z_\beta(H_k(n,m))] \geq \mathbb{E}[\ln Z_\beta(H_k(n_1, \mathcal{M}_1))] + \mathbb{E}[\ln Z_\beta(H_k(n_2, \mathcal{M}_2))]$$

*where $\mathcal{M}_1 \stackrel{d}{=} \mathrm{Bin}(m, n_1/n)$ and $\mathcal{M}_2 = m - \mathcal{M}_1 \stackrel{d}{=} \mathrm{Bin}(m, n_2/n)$.*

*Proof.* For a disjoint union of two graphs $H = H_1 + H_2$ with $H = (V, E)$ and $H_1 = (V_1, E_1), H_2 = (V_2, E_2)$, we always have $\ln Z_\beta(H) = \ln Z_\beta(H_1) + \ln Z_\beta(H_2)$. Knowing that, the claim follows from Lemma 6.5.2. □

**Lemma 6.5.4** ([BGT13], Proposition 5)**.** *Given $\alpha \in (0,1)$, suppose a non-negative sequence $(a_n)_{n\geq 1}$ satisfies*

$$a_n \geq a_{n_1} + a_{n_2} - O(n^\alpha)$$

*for every $n_1, n_2$ such that $n = n_1 + n_2$. Then the limit $\lim_{n\to\infty} \frac{a_n}{n}$ exists.*

*Proof of Theorem 6.5.1.* Since $\mathcal{M}_1, \mathcal{M}_2$ have a binomial distribution, we obtain

$$\mathbb{E}[\ln Z_\beta(H_k(n,m))] \geq \mathbb{E}[\ln Z_\beta(H_k(n_1, \mathcal{M}_1))] + \mathbb{E}[\ln Z_\beta(H_k(n_2, \mathcal{M}_2))] - O(\sqrt{n}).$$

by Fact 6.1.3 and Lemma 6.5.3. With Lemma 6.5.4, we conclude that $\frac{1}{n}\lim_{n\to\infty}\mathbb{E}[\ln Z_\beta(H_k(n,m))]$ exists. Completely analogue to the proof of Corollary 6.2.3, we find that

$$\frac{1}{n}\mathbb{E}[\ln Z_\beta(H_k(n,p))] \leq \frac{1}{n}\mathbb{E}[\ln Z_\beta(\mathcal{H}(n,m))] + o(1)$$

and with Lemma 6.2.2 the assertion follows. □

# 7 Number of solutions in random hypergraph 2-colouring

This chapter contains the proof of Theorem 4.1.5, where the asymptotic distribution of the logarithm of the number of 2-colourings of random $k$-uniform hypergraphs is determined for all $k \geq 3$. Further the proofs of Corollaries 4.1.7 and 4.1.8 are presented, in which concentration of this number is established and the planted model is shown to be contiguous to the random colouring model.

Large parts of this chapter are a verbatim copy or a close adaption of the content of the paper *On the number of solutions in random hypergraph 2-colouring* [Ras16a+] submitted to *The Electronic Journal of Combinatorics*.

The first section of this chapter contains an outline of the proof of Theorem 4.1.5, introduces important notation and states the main steps of the proof. In the following section the first moment of the number of solutions is explicitly calculated. After that, the number of short cycles is determined and in Section 7.4 the second moment is calculated very precisely. The last section contains the sketch of an alternative approach for tackling the second moment calculation.

As the paper is a single-author paper, the question of the contribution of this thesis' author does not arise.

*From here on out we always assume that $m = \lceil d'n/k \rceil$, where $d'$ remains fixed as $n \to \infty$. We also require that $k \geq 3$.*

## 7.1. Outline of the proof

We classify the 2-colourings according to their proportion of assigned colours: For a map $\sigma : [n] \to \{\pm 1\}$, we define

$$\rho(\sigma) = |\sigma^{-1}(1)|/n \tag{7.1.1}$$

and call this value the *colour density* of $\sigma$. We let $\mathcal{A}(n)$ signify the set of all possible colour densities $\rho(\sigma)$ for $\sigma : [n] \to \{\pm 1\}$. We will later show that when bounding the moments of $Z(\mathcal{H}(n, m))$ we can confine ourselves to colourings such that the proportion of the two colours does not deviate too much from 1/2. Formally, we say that $\rho \in [0, 1]$ is $(\omega, n)$-*balanced* for $\omega \in \mathbb{N}$ if

$$\rho \in \left[\frac{1}{2} - \frac{\omega}{\sqrt{n}}, \frac{1}{2} + \frac{\omega}{\sqrt{n}}\right)$$

and we denote by $\mathcal{A}_\omega(n)$ the set of all $(\omega, n)$-balanced colour densities $\rho \in \mathcal{A}(n)$. For a hypergraph $H$ on $[n]$, we let $Z_\omega(H)$ signify the number of $(\omega, n)$-*balanced* colourings, which are 2-colourings $\sigma$ such that $\rho(\sigma) \in \mathcal{A}_\omega(n)$. As we will see, it will turn out useful to split up the set $\mathcal{A}_\omega(n)$ into smaller sets in the following way. For $\nu \in \mathbb{N}$ and $s \in [\omega\nu]$, let

$$\rho_{\omega,\nu}^s = \frac{1}{2} - \frac{\omega}{\sqrt{n}} + \frac{2s-1}{\nu\sqrt{n}}. \tag{7.1.2}$$

Let $\mathcal{A}_{\omega,\nu}^s(n)$ be the set of all colour densities $\rho \in \mathcal{A}(n)$ such that

$$\rho \in \left[\rho_{\omega,\nu}^s - \frac{1}{\nu\sqrt{n}}, \rho_{\omega,\nu}^s + \frac{1}{\nu\sqrt{n}}\right).$$

For a hypergraph $H$, let $Z_{\omega,\nu}^s(H)$ denote the number of 2-colourings $\sigma$ of $H$ such that $\rho(\sigma) \in \mathcal{A}_{\omega,\nu}^s(n)$. The strategy is to apply small subgraph conditioning to the random variables $Z_{\omega,\nu}^s$ rather than directly to $Z$. We observe that for each fixed $\nu$ we have $Z_\omega = \sum_{s=1}^{\omega\nu} Z_{\omega,\nu}^s$. In Section 7.2 we will calculate the first moments of $Z$ and $Z_\omega$ to obtain the following.

**Proposition 7.1.1.** *Let $k \geq 3, d' \in (0, \infty)$ and $\omega > 0$. Then*

$$\mathbb{E}\left[Z(\mathcal{H}(n,m))\right] = \Theta\left(2^n\left(1-2^{1-k}\right)^m\right) \quad \text{and} \quad \lim_{\omega\to\infty}\liminf_{n\to\infty}\frac{\mathbb{E}\left[Z_\omega(\mathcal{H}(n,m))\right]}{\mathbb{E}\left[Z(\mathcal{H}(n,m))\right]} = 1.$$

As outlined in Section 4.2, our basic strategy is to show that the fluctuations of $\ln Z$ can be attributed to fluctuations in the number of cycles of a bounded length. Hence, for an integer $l \geq 2$ we let $C_{l,n}$ denote the number of cycles of length (exactly) $l$ in $\mathcal{H}(n, m)$. Let

$$\lambda_l = \frac{[d(k-1)]^l}{2l} \quad \text{and} \quad \delta_l = \frac{(-1)^l}{\left(2^{k-1}-1\right)^l}. \tag{7.1.3}$$

We will see that $\lambda_l$ denotes the expected number of cycles of length $l$ in a random $k$-uniform hypergraph, whereas $\delta_l$ is a correction factor that takes into account that we only allow for bichromatic edges. It is well-known that $C_{2,n}, \ldots$ are asymptotically independent Poisson variables [Bol01, Theorem 5.16]. More precisely, we have the following.

**Fact 7.1.2.** *If $c_2, \ldots, c_L$ are non-negative integers, then*

$$\lim_{n\to\infty}\mathbb{P}\left[\forall 2 \leq l \leq L : C_{l,n} = c_l\right] = \prod_{l=2}^{L}\mathbb{P}\left[\mathrm{Po}(\lambda_l) = c_l\right].$$

Next, we investigate the impact of the cycle counts $C_{l,n}$ on the first moment of $Z_{\omega,\nu}^s$. In Section 7.3 we prove the following.

**Proposition 7.1.3.** *Assume that $k \geq 3$ and $d' \in (0, \infty)$. Then*

$$\sum_{l=2}^{\infty} \lambda_l \delta_l^2 < \infty. \tag{7.1.4}$$

*Moreover, let $\omega, \nu \in \mathbb{N}$. If $c_2, \ldots, c_L$ are non-negative integers, then for any $s \in [\omega\nu]$:*

$$\frac{\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m)) | \forall 2 \leq l \leq L : C_{l,n} = c_l\right]}{\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))\right]} \sim \prod_{l=2}^{L} \left[1 + \delta_l\right]^{c_l} \exp\left[-\delta_l \lambda_l\right]. \tag{7.1.5}$$

Additionally, we need to know the second moment of $Z^s_{\omega,\nu}$ very precisely. The following proposition is the key result of our approach and the one that requires the most technical work. Its proof can be found at the end of Section 7.4.

**Proposition 7.1.4.** *Assume that $k \geq 3$ and $d'/k < 2^{k-1} \ln 2 - 2$ and let $\omega, \nu \in \mathbb{N}$. Then for every $s \in [\omega\nu]$ we have*

$$\frac{\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))^2\right]}{\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))\right]^2} \sim_\nu \exp\left[\sum_{l \geq 2} \lambda_l \delta_l^2\right].$$

We now derive Theorem 4.1.4 from Propositions 7.1.1-7.1.4. The key observation we will need is that the variance of the random variables $Z^s_{\omega,\nu}$ can almost entirely be attributed to the fluctuations of the number of short cycles. As done in [COW16+], the arguments we use are similar to the small subgraph conditioning from [Jan95, RW94]. But we do not refer to any technical statements from [Jan95, RW94] directly because instead of working only with the random variable $Z$ we need to control all $Z^s_{\omega,\nu}$ for fixed $\omega, \nu \in \mathbb{N}$ simultaneously. In fact, ultimately we have to take $\nu \to \infty$ and $\omega \to \infty$ as well. Our line of argument follows the path beaten in [COW16+] and the following three lemmas are an adaption of the ones there.

For $L > 2$, let $\mathcal{F}_L = \mathcal{F}_{L,n}(d, k)$ be the $\sigma$-algebra generated by the random variables $C_{l,n}$ with $2 \leq l \leq L$. For each $L \geq 2$, the standard decomposition of the variance yields

$$\mathrm{Var}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))\right] = \mathrm{Var}\left[\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m)) | \mathcal{F}_L\right]\right] + \mathbb{E}\left[\mathrm{Var}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m)) | \mathcal{F}_L\right]\right].$$

The term $\mathrm{Var}\left[\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m)) | \mathcal{F}_L\right]\right]$ accounts for the amount of variance induced by the fluctuations of the number of cycles of length at most $L$. The strategy when using small subgraph conditioning is to bound the second summand, which is the expected conditional variance

$$\mathbb{E}\left[\mathrm{Var}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m)) | \mathcal{F}_L\right]\right] = \mathbb{E}\left[\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))^2 | \mathcal{F}_L\right] - \mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m)) | \mathcal{F}_L\right]^2\right].$$

In the following lemma we show that in fact in the limit of large $L$ and $n$ this quantity is negligible.

This implies that conditioned on the number of short cycles the variance vanishes and thus the limiting distribution of $\ln Z^s_{\omega,\nu}$ is just the limit of $\ln \mathbb{E}\left[Z^s_{\omega,\nu}|\mathcal{F}_L\right]$ as $n, L \to \infty$. This limit is determined by the joint distribution of the number of short cycles.

**Lemma 7.1.5.** *For $d' \in (0,\infty)$ and any $\omega, \nu \in \mathbb{N}$ and $s \in [2\omega\nu]$, we have*

$$\limsup_{L\to\infty} \limsup_{n\to\infty} \mathbb{E}\left[\frac{\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))|\mathcal{F}_L\right]^2}{\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))\right]^2}\right] = 0.$$

*Proof.* Fix $\omega, \nu \in \mathbb{N}$ and set $Z_s = Z^s_{\omega,\nu}(\mathcal{H}(n,m))$. Using Fact 7.1.2 and equation (7.1.5) from Proposition 7.1.3 we can choose for any $\varepsilon > 0$ a constant $B = B(\varepsilon)$ and $L \geq L_0(\varepsilon)$ large enough such that for each large enough $n \geq n_0(\varepsilon, B, L)$ we have for any $s \in [\omega\nu]$:

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right] &\geq \sum_{c_1,\ldots,c_L\leq B} \mathbb{E}\left[Z_s|\forall 2\leq l\leq L: C_{l,n}=c_l\right]^2 \mathbb{P}\left[\forall 2\leq l\leq L: C_{l,n}=c_l\right] \\
&\geq \exp\left[-\varepsilon\right]\mathbb{E}\left[Z_s\right]^2 \sum_{c_1,\ldots,c_L\leq B}\prod_{l=2}^{L}\left[(1+\delta_l)^{c_l}\exp\left[-\lambda_l\delta_l\right]\right]^2 \mathbb{P}\left[\mathrm{Po}(\lambda_l)=c_l\right] \\
&= \exp\left[-\varepsilon\right]\mathbb{E}\left[Z_s\right]^2 \sum_{c_1,\ldots,c_L\leq B}\prod_{l=2}^{L}\frac{\left[(1+\delta_l)^2\lambda_l\right]^{c_l}}{c_l!\exp\left[2\lambda_l\delta_l+\lambda_l\right]} \\
&\geq \mathbb{E}\left[Z_s\right]^2\exp\left[-2\varepsilon+\sum_{l=2}^{L}\delta_l^2\lambda_l\right]. \qquad (7.1.6)
\end{aligned}$$

The tower property for conditional expectations and the standard formula for the decomposition of the variance yields

$$\mathbb{E}\left[Z_s^2\right] = \mathbb{E}\left[\mathbb{E}\left[Z_s^2|\mathcal{F}_L\right]\right] = \mathbb{E}\left[\mathbb{E}\left[Z_s^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right] + \mathbb{E}\left[\mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right]$$

and thus, using (7.1.6) we have

$$\frac{\mathbb{E}\left[\mathbb{E}\left[Z_s^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right]}{\mathbb{E}\left[Z_s\right]^2} \leq \frac{\mathbb{E}\left[Z_s^2\right]}{\mathbb{E}\left[Z_s\right]^2} - \exp\left[-2\varepsilon+\sum_{l=2}^{L}\delta_l^2\lambda_l\right]. \qquad (7.1.7)$$

Finally, the estimate $\exp[-x] \geq 1-x$ for $|x| < 1/8$ combined with (7.1.7) and Proposition 7.1.4 implies that for large enough $\nu, n, L$ and each $s \in [\omega\nu]$ we have

$$\frac{\mathbb{E}\left[\mathbb{E}\left[Z_s^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right]}{\mathbb{E}\left[Z_s\right]^2} \leq 2\varepsilon\exp\left[\sum_{l=2}^{\infty}\delta_l^2\lambda_l\right].$$

As this holds for any $\varepsilon > 0$ and by equation (7.1.4) the expression $\exp\left[\sum_{l=2}^{\infty} \delta_l^2 \lambda_l\right]$ is bounded, the proof of the lemma is completed by first taking $n \to \infty$ and then $L \to \infty$. $\square$

**Lemma 7.1.6.** *For $d \in (0, \infty)$ and any $\alpha > 0$, we have*

$$\limsup_{L\to\infty} \limsup_{n\to\infty} \mathbb{P}\left[|Z(\mathcal{H}(n,m)) - \mathbb{E}\left[Z(\mathcal{H}(n,m))|\mathcal{F}_L\right]| > \alpha \mathbb{E}\left[Z(\mathcal{H}(n,m))\right]\right] = 0.$$

*Proof.* To unclutter the notation, we set $Z = Z(\mathcal{H}(n,m))$ and $Z_\omega = Z_\omega(\mathcal{H}(n,m))$. First we observe that Proposition 7.1.1 implies that for any $\alpha > 0$ we can choose $\omega \in \mathbb{N}$ large enough such that

$$\liminf_{n\to\infty} \mathbb{E}\left[Z_\omega\right] > (1-\alpha^2)\mathbb{E}\left[Z\right]. \tag{7.1.8}$$

We let $\nu \in \mathbb{N}$. To prove the statement, we need to get a handle on the cases where the random variables $Z^s_{\omega,\nu}(\mathcal{H}(n,m))$ deviate strongly from their conditional expectation $\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))|\mathcal{F}_L\right]$. We let $Z_s = Z^s_{\omega,\nu}(\mathcal{H}(n,m))$ and define

$$X_s = |Z_s - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]| \cdot \mathbf{1}_{\{|Z_s - \mathbb{E}[Z_s|\mathcal{F}_L]| > \alpha\mathbb{E}[Z_s]\}}$$

and $X = \sum_{s=1}^{\omega\nu} X_s$. Then these definitions directly yield

$$\mathbb{P}\left[X < \alpha\mathbb{E}\left[Z_\omega\right]\right] \leq \mathbb{P}\left[|Z_\omega - \mathbb{E}\left[Z_\omega|\mathcal{F}_L\right]| < 2\alpha\mathbb{E}\left[Z_\omega\right]\right]. \tag{7.1.9}$$

By the definition of the $X_s$'s and Chebyshev's inequality it is true for every $s$ that

$$\mathbb{E}\left[X_s|\mathcal{F}_L\right] \leq \sum_{j\geq 0} 2^{j+1}\alpha\mathbb{E}\left[Z_s\right]\mathbb{P}\left[|Z_s - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]| > 2^j\alpha\mathbb{E}\left[Z_s\right]\right] \leq \frac{4\mathrm{Var}\left[Z_s|\mathcal{F}_L\right]}{\alpha\mathbb{E}\left[Z_s\right]}.$$

Hence, using that with Proposition 7.1.1 there is a number $\beta = \beta(\alpha,\omega)$ such that $\mathbb{E}\left[Z_s\right]/\mathbb{E}\left[Z\right] \leq \beta/(\omega\nu)$ for all $s \in [\omega\nu]$ and $n$ large enough, we have

$$\mathbb{E}\left[X|\mathcal{F}_L\right] \leq \sum_{s=1}^{\omega\nu} \frac{4\mathrm{Var}\left[Z_s|\mathcal{F}_L\right]}{\alpha\mathbb{E}\left[Z_s\right]} \leq \frac{2\beta\mathbb{E}\left[Z\right]}{\alpha\nu\omega} \sum_{s=1}^{\omega\nu} \frac{\mathrm{Var}\left[Z_s|\mathcal{F}_L\right]}{\mathbb{E}\left[Z_s\right]^2}.$$

Taking expectations, choosing $\varepsilon = \varepsilon(\alpha,\beta,\omega)$ small enough and applying Lemma 7.1.5, we obtain

$$\mathbb{E}\left[X\right] = \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{F}_L\right]\right] \leq \frac{2\beta\mathbb{E}\left[Z\right]}{\alpha\nu\omega} \sum_{s=1}^{\omega\nu} \frac{\mathbb{E}\left[\mathrm{Var}\left[Z_s|\mathcal{F}_L\right]\right]}{\mathbb{E}\left[Z_s\right]^2} \leq \frac{4\beta\varepsilon\mathbb{E}\left[Z\right]}{\alpha} \leq \alpha^2\mathbb{E}\left[Z\right]. \tag{7.1.10}$$

Using (7.1.9), Markov's inequality, (7.1.10) and (7.1.8), it follows that

$$\mathbb{P}\left[|Z_\omega - \mathbb{E}\left[Z_\omega|\mathcal{F}_L\right]| < 2\alpha\mathbb{E}\left[Z_\omega\right]\right] \geq 1 - 2\alpha. \tag{7.1.11}$$

Finally, the triangle inequality combined with Markov's inequality and equations (7.1.8) and (7.1.11) yields

$$\begin{aligned}&\mathbb{P}\left[|Z - \mathbb{E}\left[Z|\mathcal{F}_L\right]| > \alpha\mathbb{E}\left[Z\right]\right]\\&\quad\leq \mathbb{P}\left[|Z - Z_\omega| + |Z_\omega - \mathbb{E}\left[Z_\omega|\mathcal{F}_L\right]| + |\mathbb{E}\left[Z_\omega|\mathcal{F}_L\right] - \mathbb{E}\left[Z|\mathcal{F}_L\right]| > \alpha\mathbb{E}\left[Z\right]\right]\\&\quad\leq 3\alpha + \alpha/3 + 3\alpha < 7\alpha,\end{aligned}$$

which proves the statement. □

**Lemma 7.1.7.** *Let*

$$U_L = \sum_{l=2}^{L} C_{l,n}\ln(1+\delta_l) - \lambda_l\delta_l. \tag{7.1.12}$$

*Then* $\limsup_{L\to\infty}\limsup_{n\to\infty}\mathbb{E}\left[|U_L|\right] < \infty$ *and further for any* $\varepsilon > 0$ *we have*

$$\limsup_{L\to\infty}\limsup_{n\to\infty}\mathbb{P}\left[|\ln\mathbb{E}\left[Z(\mathcal{H}(n,m))|\mathcal{F}_L\right] - \ln\mathbb{E}\left[Z(\mathcal{H}(n,m))\right] - U_L| > \varepsilon\right] = 0 \tag{7.1.13}$$

*Proof.* In a first step we show that $\mathbb{E}\left[|U_L|\right]$ is uniformly bounded. As $x - x^2 \leq \ln(1+x) \leq x$ for $|x| \leq 1/8$ we have for every $l \leq L$:

$$\mathbb{E}\left[|C_{l,n}\ln(1+\delta_l) - \lambda_l\delta_l|\right] \leq \delta_l\mathbb{E}\left[|C_{l,n} - \lambda_l|\right] + \delta_l^2\mathbb{E}\left[C_{l,n}\right].$$

Therefore, Fact 7.1.2 implies that

$$\mathbb{E}\left[|U_L|\right] \leq \sum_{l=2}^{L}\delta_l\sqrt{\lambda_l} + \delta_l^2\lambda_l. \tag{7.1.14}$$

Proposition 7.1.3 ensures that $\sum_l \delta_l^2\lambda_l < \infty$. Furthermore, as we are in the regime $d'/k \leq 2^{k-1}\ln 2$, we have $\sum_l \delta_l\sqrt{\lambda_l} \leq \sum_l k^l 2^{-(k-1)l/2} < \infty$ and thus (7.1.14) shows that $\mathbb{E}\left[|U_L|\right]$ is uniformly bounded.

To prove (7.1.13), for given $n$ and a constant $B > 0$ we let $\mathcal{C}_B$ be the event that $C_{l,n} < B$ for all $l \leq L$. Referring to Fact 7.1.2, we can find for each $L, \varepsilon > 0$ a $B > 0$ such that

$$\mathbb{P}\left[\mathcal{C}_B\right] > 1 - \varepsilon. \tag{7.1.15}$$

To simplify the notation we set $Z = Z(\mathcal{H}(n,m))$ and $Z_\omega = Z_\omega(\mathcal{H}(n,m))$. By Proposition 7.1.1 we can choose for any $\alpha > 0$ a $\omega > 0$ large enough such that $\mathbb{E}[Z_\omega] > (1-\alpha)\mathbb{E}[Z]$ for large enough $n$. Then Propositions 7.1.1 and 7.1.3 combined with Fact 7.1.2 imply that for any $c_1, ..., c_L \leq B$ and small enough $\alpha = \alpha(\varepsilon, L, B)$ we have for $n$ large enough:

$$\mathbb{E}[Z|\forall 2 \leq l \leq L : C_{l,n} = c_l] \geq \mathbb{E}[Z_\omega|\forall 2 \leq l \leq L : C_{l,n} = c_l] \\ \geq \exp[-\varepsilon]\,\mathbb{E}[Z]\prod_{l=2}^{L}(1+\delta_l)^{c_l}\exp[-\delta_l\lambda_l]\,. \tag{7.1.16}$$

On the other hand, for $\alpha$ sufficiently small and large enough $n$ we have

$$\mathbb{E}[Z|\forall 2 \leq l \leq L : C_{l,n} = c_l] = \mathbb{E}[Z - Z_\omega|\forall 2 \leq l \leq L : C_{l,n} = c_l] + \mathbb{E}[Z_\omega|\forall 2 \leq l \leq L : C_{l,n} = c_l] \\ \leq \frac{2\alpha\mathbb{E}[Z]}{\prod_{l=2}^{L}\mathbb{P}[\mathrm{Po}(\lambda_l) = c_l]} + \mathbb{E}[Z_\omega|\forall 2 \leq l \leq L : C_{l,n} = c_l] \\ \leq \exp[\varepsilon]\,\mathbb{E}[Z]\prod_{l=2}^{L}(1+\delta_l)^{c_l}\exp[-\delta_l\lambda_l] \tag{7.1.17}$$

Thus, the proof of (7.1.13) is completed by combining (7.1.15), (7.1.16), (7.1.17) and taking logarithms. □

*Proof of Theorem 4.1.5.* For $L \geq 2$, we define

$$W_L = \sum_{l=2}^{L} X_l \ln(1+\delta_l) - \lambda_l\delta_l.$$

Then Fact 7.1.2 implies that for each $L$ the random variables $U_L$ defined in (7.1.12) converge in distribution to $W_L$ as $n \to \infty$. Furthermore, because $\sum_l \delta_l\sqrt{\lambda_l}, \sum_l \delta_l^2\lambda_l < \infty$, the martingale convergence theorem implies that $W$ is well-defined and that the $W_L$ converge to $W$ almost surely as $L \to \infty$. Therefore, from Lemmas 7.1.7 and 7.1.6 it follows that $\ln Z(\mathcal{H}(n,m)) - \ln\mathbb{E}[Z(\mathcal{H}(n,m))]$ converges to $W$ in distribution, meaning that for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty}\mathbb{P}[|\ln Z(\mathcal{H}(n,m)) - \ln\mathbb{E}[Z(\mathcal{H}(n,m))] - W| > \varepsilon] = 0. \tag{7.1.18}$$

To derive Theorem 4.1.5 from (7.1.18) let $S$ be the event that $\mathcal{H}(n,m)$ consists of $m$ distinct edges. Given that $S$ occurs, $\mathcal{H}(n,m)$ is identical to $H_k(n,m)$. Furthermore, Fact 2.1.2 implies that $\mathbb{P}[S] = \Omega(1)$. Consequently, (7.1.18) yields

$$0 = \lim_{n\to\infty}\mathbb{P}[|\ln Z(\mathcal{H}(n,m)) - \ln\mathbb{E}[Z(\mathcal{H}(n,m))] - W| > \varepsilon|S] \\ = \lim_{n\to\infty}\mathbb{P}[|\ln Z(H_k(n,m)) - \ln\mathbb{E}[Z(\mathcal{H}(n,m))] - W| > \varepsilon]\,. \tag{7.1.19}$$

Furthermore, Lemma 7.2.1 implies that $\mathbb{E}\left[Z(\mathcal{H}(n,m))\right], \mathbb{E}\left[Z(H_k(n,m)\right] = \Theta\left(2^n\left(1-2^{1-k}\right)^m\right)$. Thus, it holds that $\mathbb{E}\left[Z(\mathcal{H}(n,m))\right] = \Theta(\mathbb{E}\left[Z(H_k(n,m)\right])$ and with (7.1.19) it follows that

$$\lim_{n\to\infty} \mathbb{P}\left[|\ln Z(H_k(n,m)) - \ln \mathbb{E}\left[Z(H_k(n,m)))\right] - W| > \varepsilon\right] = 0,$$

which proves Theorem 4.1.5. □

*Proof of Corollary 4.1.7.* The first part of the proof follows directly from Theorem 4.1.5 and the properties of $W$. By the definition of convergence in distribution and Markov's inequality we have

$$\lim_{n\to\infty} \mathbb{P}\left[|\ln Z(H_k(n,m)) - \ln \mathbb{E}\left[Z(H_k(n,m)\right]| \le \omega\right] = \mathbb{P}\left[|W| \le \omega\right] \ge 1 - \frac{\mathbb{E}|W|}{\omega}$$

and (4.1.7) follows.

To prove the second part, we construct an event whose probability is bounded away from 0 and that is such that conditioned on this event, the number of solutions of the random hypergraph $H_k(n,m)$ is not concentrated very strongly.
We consider the event $\mathcal{T}_t$ that the random hypergraph $H_k(n,m)$ contains $t$ isolated triangles, i.e. $t$ connected components such that each component consists of $3k-3$ vertices and 3 edges and the intersection of each pair of edges contains exactly one vertex. It is well-known that for $t \ge 0$ there exists $\varepsilon = \varepsilon(d,t) > 0$ such that

$$\liminf_{n\to\infty} \mathbb{P}\left[\mathcal{T}_t\right] > \varepsilon. \tag{7.1.20}$$

Given $\mathcal{T}_t$, we let $H_k^*(n,m)$ denote the random hypergraph obtained by choosing a set of $t$ isolated triangles randomly and removing them. Then $H_k^*(n,m)$ is identical to $H_k(n-(3k-3)t, m-3t)$ and with Proposition 7.1.1 there exists a constant $C = C(d,k)$ such that

$$\mathbb{E}\left[Z(H_k^*(n,m))\right] = \mathbb{E}\left[Z(H_k(n-(3k-3)t, m-3t))\right] \le C \cdot 2^{n-(3k-3)t}\left(1-2^{1-k}\right)^{m-3t}.$$

A very accurate calculation of the number of 2-colourings of a triangle in a hypergraph yields that this number is given by $\left(2^{k-2}-1\right)\left(2^{2k-1}-2^k+2\right)$. Thus, we obtain

$$\begin{aligned}
\mathbb{E}\left[Z(H_k(n,m))|\mathcal{T}_t\right] &\le \mathbb{E}\left[Z(H_k(n-(3k-3)t, m-3t))\right]\left(\left(2^{k-2}-1\right)\left(2^{2k-1}-2^k+2\right)\right)^t \\
&\le C \cdot 2^n\left(1-2^{1-k}\right)^{m-3t}\left(1-2^{2-k}\right)^t\left(1-2^{1-k}+2^{2-2k}\right)^t \\
&\le C \cdot 2^n\left(1-2^{1-k}\right)^m\left(1-8\left(2^k-2\right)^{-3}\right) \\
&\le O\left(\mathbb{E}\left[Z(H_k(n,m))\right]\right)\left(1-8\left(2^k-2\right)^{-3}\right),
\end{aligned}$$

implying that for any $\omega > 0$ we can choose $t$ large enough so that

$$\mathbb{E}\left[Z(H_k(n,m))|\mathcal{T}_t\right] \leq \mathbb{E}\left[Z(H_k(n,m))\right]/(2\exp\left[\omega\right]).$$

Using Markov's inequality, we obtain

$$\begin{aligned}\mathbb{P}\left[\ln Z(H_k(n,m)) \geq \ln \mathbb{E}\left[Z(H_k(n,m))\right] - \omega|\mathcal{T}_t\right]\\ = \mathbb{P}\left[Z(H_k(n,m))/\mathbb{E}\left[Z(H_k(n,m))\right] \geq \exp\left[-\omega\right]|\mathcal{T}_t\right] \leq 1/2.\end{aligned} \tag{7.1.21}$$

Thus, combining (7.1.20) and (7.1.21) yields that for any finite $\omega > 0$ there is $\varepsilon > 0$ such that for large enough $n$ we have

$$\begin{aligned}\mathbb{P}\left[\left|\ln Z(H_k(n,m)) - \mathbb{E}\left[\ln Z(H_k(n,m))\right]\right| > \omega\right]\\ \geq \mathbb{P}\left[\ln Z(H_k(n,m)) < \mathbb{E}\left[\ln Z(H_k(n,m))\right] - \omega\right]\\ \geq \mathbb{P}\left[\ln Z(H_k(n,m)) \geq \ln \mathbb{E}\left[Z(H_k(n,m))\right] - \omega|\mathcal{T}_t\right]\mathbb{P}\left[\mathcal{T}_t\right]\\ > \varepsilon/2,\end{aligned}$$

thereby completing the proof of the second claim. □

*Proof of Corollary 4.1.8.* This proof is nearly identical to the one in [BCOE14+]. Assume for contradiction that $(\mathcal{A}_n)_{n\geq 1}$ is a sequence of events such that for some fixed number $0 < \varepsilon < 1/2$ we have

$$\lim_{n\to\infty} \pi^{\mathrm{pl}}_{k,n,m}\left[\mathcal{A}_n\right] = 0 \quad \text{while} \quad \limsup_{n\to\infty} \pi^{\mathrm{rc}}_{k,n,m}\left[\mathcal{A}_n\right] > \varepsilon. \tag{7.1.22}$$

Let $H_k(n,m,\sigma)$ denote a $k$-uniform hypergraph on $[n]$ with precisely $m$ edges chosen uniformly at random from all edges that are bichromatic under $\sigma$. Let $\mathcal{V}(\sigma)$ be the event that $\sigma$ is a 2-colouring of

$H_k(n, m)$. Then

$$\begin{aligned}
\mathbb{E}\left[Z(H_k(n,m))\mathbf{1}_{\mathcal{A}_n}\right] &= \sum_{\sigma:[n]\to\{\pm 1\}} \mathbb{P}\left[\mathcal{V}(\sigma) \text{ and } (H_k(n,m),\sigma)\in\mathcal{A}_n\right] \\
&= \sum_{\sigma:[n]\to\{\pm 1\}} \mathbb{P}\left[(H_k(n,m),\sigma)\in\mathcal{A}_n|\mathcal{V}(\sigma)\right]\mathbb{P}\left[\mathcal{V}(\sigma)\right] \\
&= \sum_{\sigma:[n]\to\{\pm 1\}} \mathbb{P}\left[H_k(n,m,\sigma)\in\mathcal{A}_n\right]\mathbb{P}\left[\mathcal{V}(\sigma)\right] \\
&\le O\left(\left(1-2^{1-k}\right)^m\right) \sum_{\sigma:[n]\to\{\pm 1\}} \mathbb{P}\left[H_k(n,m,\sigma)\in\mathcal{A}_n\right] \\
&= O\left(2^n\left(1-2^{1-k}\right)^m\right)\mathbb{P}\left[H_k(n,m,\boldsymbol{\sigma})\in\mathcal{A}_n\right] = o\left(2^n\left(1-2^{1-k}\right)^m\right).
\end{aligned} \tag{7.1.23}$$

By Theorem 4.1.4, for any $\varepsilon > 0$ there is $\delta > 0$ such that for all large enough $n$ we have

$$\mathbb{P}\left[Z(H_k(n,m)) < \delta\mathbb{E}\left[Z(H_k(n,m))\right]\right] < \varepsilon/2. \tag{7.1.24}$$

Now, let $\mathcal{E}$ be the event that $Z(H_k(n,m)) \ge \delta\mathbb{E}[Z(H_k(n,m)]$ and let $q = \pi^{\mathrm{rc}}_{k,n,m}\left[\mathcal{A}_n|\mathcal{E}\right]$. Then

$$\begin{aligned}
\mathbb{E}\left[Z(H_k(n,m))\mathbf{1}_{\mathcal{A}_n}\right] &\ge \delta\mathbb{E}[Z(H_k(n,m))]\,\mathbb{P}\left[((H_k(n,m),\boldsymbol{\sigma})\in\mathcal{A}_n,\mathcal{E}\right] \\
&\ge \delta q\mathbb{E}[Z(H_k(n,m))]\,\mathbb{P}\left[\mathcal{E}\right] \ge \delta q\mathbb{E}[Z(H_k(n,m))]/2 \\
&= \frac{\delta q}{2}\cdot\Omega\left(2^n\left(1-2^{1-k}\right)^m\right).
\end{aligned} \tag{7.1.25}$$

Combining (7.1.23) and (7.1.25), we obtain $q = o(1)$. Hence, (7.1.24) implies that

$$\pi^{\mathrm{rc}}_{k,n,m}\left[\mathcal{A}_n\right] = \pi^{\mathrm{rc}}_{k,n,m}\left[\mathcal{A}_n|\neg\mathcal{E}\right]\cdot\mathbb{P}\left[\neg\mathcal{E}\right] + q\cdot\mathbb{P}\left[\mathcal{E}\right] \le \mathbb{P}\left[\neg\mathcal{E}\right] + q \le \varepsilon/2 + o(1),$$

in contradiction to (7.1.22). □

## 7.2. The first moment

The aim in this section is to prove Proposition 7.1.1 and a result that we need for Proposition 7.1.4. For a hypergraph $H$, let $Z_\rho(H)$ be its number of 2-colourings with colour density $\rho$. We set $\bar{\rho} = \frac{1}{2}$. For $\rho \in [0,1]$, we define

$$f_1 : \rho \mapsto \mathcal{H}(\rho) + g_1(\rho) \qquad \text{with} \quad g_1(\rho) = \frac{d}{k}\ln\left(1-\rho^k-(1-\rho)^k\right). \tag{7.2.1}$$

The next lemma shows that $f_1(\rho)$ is the function we need to analyse in order to determine the expectation of $Z_\rho$.

**Lemma 7.2.1.** *Let $d' \in (0,\infty)$. There exist numbers $C_1 = C_1(k,d), C_2 = C_2(k,d) > 0$ such that for any colour density $\rho$:*

$$C_1 n^{-1/2} \exp\left[n f_1(\rho)\right] \leq \mathbb{E}\left[Z_\rho(\mathcal{H}(n,m))\right] \leq C_2 \exp\left[n f_1(\rho)\right]. \tag{7.2.2}$$

*Moreover, if $|\rho - \bar{\rho}| = o(1)$, then*

$$\mathbb{E}\left[Z_\rho(\mathcal{H}(n,m))\right] \sim \sqrt{\frac{2}{\pi n}} \exp\left[\frac{d(k-1)}{2^k-2}\right] \exp\left[n f_1(\rho)\right]. \tag{7.2.3}$$

*Proof.* The edges in the random hypergraph $\mathcal{H}(n,m)$ are independent by construction, so the expected number of solutions with colour density $\rho$ can be written as

$$\mathbb{E}[Z_\rho(\mathcal{H}(n,m))] = \binom{n}{\rho n}\left(1 - \frac{\binom{\rho n}{k} + \binom{(1-\rho)n}{k}}{N}\right)^m, \quad \text{where } N = \binom{n}{k}. \tag{7.2.4}$$

Further, the number of “forbidden” edges is given by

$$\begin{aligned}
&\binom{\rho n}{k} + \binom{(1-\rho)n}{k} \\
&\qquad = \frac{1}{k!}\left(n^k\left(\rho^k + (1-\rho)^k\right) - \frac{k(k-1)}{2} n^{k-1}\left(\rho^{k-1} + (1-\rho)^{k-1}\right) + \Theta\left(n^{k-2}\right)\right) \\
&\qquad = N\left(\rho^k + (1-\rho)^k\right) - \frac{k(k-1)}{2k!} n^{k-1}\left(\rho^{k-1}(1-\rho) + \rho(1-\rho)^{k-1}\right) + \Theta\left(n^{k-2}\right)
\end{aligned}$$

yielding

$$1 - \frac{\binom{\rho n}{k} + \binom{(1-\rho)n}{k}}{N} = 1 - \rho^k - (1-\rho)^k + \frac{k(k-1)}{2n}\left(\rho^{k-1}(1-\rho) + \rho(1-\rho)^{k-1}\right) + \Theta\left(n^{-2}\right).$$

To proceed we observe that $\ln\left(x + \frac{y}{n}\right) = \ln(x) + \ln\left(1 + \frac{y}{xn}\right)$ for $x > 0, y < xn$ and consequently

$$\begin{aligned}
&m \ln\left(1 - \frac{\binom{\rho n}{k} + \binom{(1-\rho)n}{k}}{N}\right) \\
&= \frac{dn}{k}\left(\ln\left(1 - \rho^k - (1-\rho)^k\right) + \ln\left(1 - \frac{k(k-1)}{2n}\frac{\rho^{k-1}(1-\rho) + \rho(1-\rho)^{k-1}}{1 - \rho^k - (1-\rho)^k} + \Theta\left(n^{-2}\right)\right)\right) \\
&\sim \frac{dn}{k}\ln\left(1 - \rho^k - (1-\rho)^k\right) + \frac{d(k-1)}{2}\left(\frac{\rho^{k-1}(1-\rho) + \rho(1-\rho)^{k-1}}{1 - \rho^k - (1-\rho)^k}\right) + \Theta\left(n^{-1}\right).
\end{aligned} \tag{7.2.5}$$

Equation (7.2.2) follows from (7.2.4), (7.2.5) and Stirling's formula applied to $\binom{n}{\rho n}$. Moreover, equation (7.2.3) follows from (7.2.4) and (7.2.5) because $|\rho - \bar{\rho}| = o(1)$ implies that

$$\binom{n}{\rho n} \sim \sqrt{\frac{2}{\pi n}} \exp\left[n\mathcal{H}(\rho)\right] \qquad \text{and} \qquad \frac{\rho^{k-1}(1-\rho)+\rho(1-\rho)^{k-1}}{1-\rho^k-(1-\rho)^k} \sim \frac{1}{2^{k-1}-1}.$$

□

The following corollary states an expression for $\mathbb{E}\left[Z(\mathcal{H}(n,m))\right]$. Additionally, it shows that when $\omega \to \infty$, this value can be approximated by $\mathbb{E}\left[Z_\omega(\mathcal{H}(n,m))\right]$.

**Corollary 7.2.2.** *Let $d' \in (0,\infty)$. Then*

$$\mathbb{E}\left[Z(\mathcal{H}(n,m))\right] \sim \exp\left[\frac{d(k-1)}{2^k-2} + nf_1(\bar{\rho})\right]\left(1+\frac{d(k-1)}{2^{k-1}-1}\right)^{-\frac{1}{2}}. \tag{7.2.6}$$

*Furthermore, for $\omega > 0$ we have*

$$\lim_{\omega\to\infty}\lim_{n\to\infty}\frac{\mathbb{E}\left[Z_\omega(\mathcal{H}(n,m))\right]}{\mathbb{E}\left[Z(\mathcal{H}(n,m))\right]} = 1. \tag{7.2.7}$$

*Proof.* The functions $\rho \mapsto \mathcal{H}(\rho)$ and $\rho \mapsto g_1(\rho)$ are both concave and attain their maximum at $\rho = \bar{\rho}$. Consequently, setting $B(d,k) = 4\left(1+\frac{d(k-1)}{2^{k-1}-1}\right)$ and expanding around $\bar{\rho}$, we obtain

$$f_1(\bar{\rho}) - \frac{B(d,k)}{2}(\rho-\bar{\rho})^2 - O\left((\rho-\bar{\rho})^3\right) \le f_1(\rho) \le f_1(\bar{\rho}) - \frac{B(d,k)}{2}(\rho-\bar{\rho})^2. \tag{7.2.8}$$

Plugging the upper bound from (7.2.8) into (7.2.2) and observing that the number of all colour densities for maps $\sigma : [n] \to \{\pm 1\}$ is bounded from above by $n = \exp[o(n)]$, we find

$$S_1 = \sum_{\rho\colon |\rho-\bar{\rho}|>n^{-3/8}} \mathbb{E}\left[Z_\rho(\mathcal{H}(n,m))\right] \le C_2 \exp\left[nf_1(\bar{\rho}) - \frac{B(d,k)}{2}n^{1/4}\right]. \tag{7.2.9}$$

On the other hand, equation (7.2.3) implies that

$$\begin{aligned} S_2 &= \sum_{\rho\colon |\rho-\bar{\rho}|\le n^{-3/8}} \mathbb{E}\left[Z_\rho(\mathcal{H}(n,m))\right] \\ &\sim \sqrt{\frac{2}{\pi n}} \exp\left[\frac{d(k-1)}{2^k-2}\right] + \exp\left[nf_1(\bar{\rho})\right]\sum_\rho \exp\left[-n\frac{B(d,k)}{2}(\rho-\bar{\rho})^2\right]. \end{aligned} \tag{7.2.10}$$

The last sum is in the standard form of a Gaussian summation. Using $\int_{-\infty}^{\infty} \exp\left[-a(x+b)^2\right]dx = \sqrt{\frac{a}{\pi}}$, we get

$$\sum_{\rho\in\mathcal{A}(n)} \exp\left[-n\frac{B(d,k)}{2}(\rho-\bar\rho)^2\right] \sim n\int \exp\left[-n\frac{B(d,k)}{2}(\rho-\bar\rho)^2\right] d\rho$$

$$\sim n\sqrt{\frac{2\pi}{nB(d,k)}} = \sqrt{\frac{\pi n}{2}}\left(1+\frac{d(k-1)}{2^{k-1}-1}\right)^{-\frac{1}{2}} \tag{7.2.11}$$

Plugging (7.2.11) into (7.2.10), we obtain

$$S_2 \sim \exp\left[\frac{d(k-1)}{2^k-2}+nf_1(\bar\rho)\right]\left(1+\frac{d(k-1)}{2^{k-1}-1}\right)^{-\frac{1}{2}}. \tag{7.2.12}$$

Finally, comparing (7.2.9) and (7.2.12), we see that $S_1 = o(S_2)$. Thus, $S_1 + S_2 \sim S_2$ and (7.2.6) follows from (7.2.12).

To prove (7.2.7), we find that analogously to (7.2.9), (7.2.10) and the calculation leading to (7.2.12), it holds that

$$S_1' = \sum_{\rho:\ |\rho-\bar\rho|>\omega n^{-1/2}} \mathbb{E}\left[Z_\rho(\mathcal{H}(n,m))\right] \le C_2\exp\left[nf(\bar\rho)-\frac{B(d,k)}{2}\omega\right].$$

and

$$S_2' = \sum_{\rho:\ |\rho-\bar\rho|\le\omega n^{-1/2}} \mathbb{E}\left[Z_\rho(\mathcal{H}(n,m))\right] \sim \exp\left[\frac{d(k-1)}{2^k-2}+nf_1(\bar\rho)\right]\left(1+\frac{d(k-1)}{2^{k-1}-1}\right)^{-\frac{1}{2}}.$$

Thus, we have $\lim_{\omega\to\infty}\lim_{n\to\infty}\frac{S_1'+S_2'}{S_2'} = 1$, yielding (7.2.7). □

*Proof of Proposition 7.1.1.* The statements are immediate by Corollary 7.2.2 and the fact that

$$f_1(\bar\rho) = \ln 2 + \frac{d}{k}\ln\left(1-2^{1-k}\right).$$

□

Finally, we derive an expression for $\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))\right]$ that we will need to prove Proposition 7.1.4.

**Lemma 7.2.3.** *Let $d' \in (0,\infty), \omega,\nu\in\mathbb{N}, s\in[\omega\nu]$ and $\rho\in\mathcal{A}^s_{\omega,\nu}(n)$. Then with $\rho^s_{\omega,\nu}$ as defined in (7.1.2) we have*

$$\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))\right] \sim_\nu |\mathcal{A}^s_{\omega,\nu}(n)|\sqrt{\frac{2}{\pi n}}\exp\left[\frac{d(k-1)}{2^k-2}\right]\exp\left[nf_1\left(\rho^s_{\omega,\nu}\right)\right].$$

*Proof.* Using a Taylor expansion of $f_1(\rho)$ around $\rho = \rho^s_{\omega,\nu}$, we get

$$f_1(\rho) = f_1(\rho^s_{\omega,\nu}) + \Theta\left(\frac{\omega}{\sqrt{n}}\right)|\rho - \rho^s_{\omega,\nu}| + \Theta\left(\left(\rho - \rho^s_{\omega,\nu}\right)^2\right). \tag{7.2.13}$$

As $|\rho - \rho^s_{\omega,\nu}| \leq \frac{1}{\nu\sqrt{n}}$ for $\rho \in \mathcal{A}^s_{\omega,\nu}(n)$, we conclude that $f_1(\rho) = f_1(\rho^s_{\omega,\nu}) + O\left(\frac{\omega}{\nu n}\right)$ and as this is independent of $\rho$ the assertion follows by inserting (7.2.13) in (7.2.3) and multiplying with $|\mathcal{A}^s_{\omega,\nu}(n)|$. □

## 7.3. Counting short cycles

We recall that for $l \in \{2,\dots,L\}$ we denote by $C_{l,n}$ the number of cycles of length $l$ in $\mathcal{H}(n,m)$. Further we let $c_2,\dots,c_L$ be a sequence of non-negative integers and $S$ be the event that $C_{l,n} = c_l$ for $l = 2,\dots,L$. Additionally, for an assignment $\sigma : [n] \to \{\pm 1\}$ we let $\mathcal{V}(\sigma)$ be the event that $\sigma$ is a colouring of the random graph $\mathcal{H}(n,m)$. We also recall $\lambda_l, \delta_l$ from (7.1.3).

*Proof of Proposition 7.1.3.* First observe that from the definition of $\lambda_l$ and $\delta_l$ in (7.1.3) and the fact that $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x)$ we get

$$\exp\left[\sum_{l\geq 2}\lambda_l\delta_l^2\right] = \exp\left[-\frac{d(k-1)}{2}\frac{1}{\left(2^{k-1}-1\right)^2}\right]\left(1 - \frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right)^{-1/2}. \tag{7.3.1}$$

Together with (7.3.1), Proposition 7.1.3 readily follows from the following lemma about the distribution of the random variables $C_{l,n}$ given $\mathcal{V}(\sigma)$.

**Lemma 7.3.1.** *Let* $\mu_l = \frac{(d(k-1))^l}{2l}\left[1 + \frac{(-1)^l}{\left(2^{k-1}-1\right)^l}\right]$. *Then* $\mathbb{P}\left[S|\mathcal{V}(\sigma)\right] \sim \prod_{l=2}^{L}\frac{\exp[-\mu_l]}{c_l!}\mu_l^{c_l}$ *for any* $\sigma$ *with* $\rho(\sigma) \in \mathcal{A}_\omega(n)$.

Before we establish Lemma 7.3.1, let us point out how it implies Proposition 7.1.3. By Bayes' rule, we have

$$\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))|S\right] = \frac{1}{\mathbb{P}[S]}\sum_{\tau\in\mathcal{A}^s_{\omega,\nu}(n)}\mathbb{P}[\mathcal{V}(\tau)]\,\mathbb{P}[S|\mathcal{V}(\tau)]. \tag{7.3.2}$$

Inserting the result from Lemma 7.3.1 into (7.3.2) yields

$$\begin{aligned}\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))|S\right] &\sim \frac{\prod_{l=2}^{L}\frac{\exp[-\mu_l]}{c_l!}\mu_l^{c_l}}{\mathbb{P}[S]}\sum_{\tau\in\mathcal{A}^s_{\omega,\nu}(n)}\mathbb{P}[\mathcal{V}(\tau)]\\ &\sim \frac{\prod_{l=2}^{L}\frac{\exp[-\mu_l]}{c_l!}\mu_l^{c_l}}{\mathbb{P}[S]}\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))\right].\end{aligned}$$

From Lemma 7.3.1 and Fact 7.1.2 we get that

$$\frac{\prod_{l=2}^{L}\frac{\exp[-\mu_l]}{c_l!}\mu_l^{c_l}}{\mathbb{P}[S]} \sim \prod_{l=2}^{L}[1+\delta_l]^{c_l}\exp\left[-\delta_l\lambda_l\right]$$

and Proposition 7.1.3 follows. □

*Proof of Lemma 7.3.1.* We are going to show that for any fixed sequence of integers $m_1,\ldots,m_L \geq 0$, the joint factorial moments satisfy

$$\mathbb{E}\left[(C_{2,n})_{m_2}\cdots(C_{L,n})_{m_L}|\mathcal{V}(\sigma)\right] \sim \prod_{l=2}^{L}\mu_l^{m_l}. \tag{7.3.3}$$

Then Lemma 7.3.1 follows from [Bol01, Theorem 1.23].

We consider the number of sequences of $m_2+\cdots+m_L$ distinct cycles such that $m_2$ corresponds to the number of cycles of length 2, and so on. Clearly this number is equal to $(C_{2,n})_{m_2}\cdots(C_{L,n})_{m_L}$.

We call a cycle *good*, if it does not contain edges that overlap on more than one vertex. We call a sequence of good cycles *good sequence* if for any two cycles $C$ and $C'$ in this sequence, there are no vertices $v\in C$ and $v'\in C'$ such that $v$ and $v'$ are contained in the same edge. Let $Y$ be the number of good sequences and $\bar{Y}$ be the number of sequences that are not good. Then it holds that

$$\mathbb{E}\left[(C_{2,n})_{m_2}\cdots(C_{L,n})_{m_L}|\mathcal{V}(\sigma)\right] = \mathbb{E}[Y|\mathcal{V}(\sigma)]+\mathbb{E}[\bar{Y}|\mathcal{V}(\sigma)]. \tag{7.3.4}$$

The following claim states that the contribution of $\mathbb{E}[\bar{Y}|\mathcal{V}(\sigma)]$ is negligible. Its proof follows at the end of this section.

**Claim 7.3.2.** *We have* $\mathbb{E}\left[\bar{Y}|\mathcal{V}(\sigma)\right] = O\left(n^{-1}\right)$.

Thus it remains to count good sequences given $\mathcal{V}(\sigma)$. We let $\sigma\in\mathcal{A}_\omega(n)$ and first consider the number $D_{l,n}$ of rooted, directed, good cycles of length $l$. This will introduce a factor of $2l$ for the number of all good cycles of length $l$, thus $D_{l,n} = 2lC_{l,n}$. For a rooted, directed, good cycle of length $l$, we need to pick $l$ vertices $(v_1, ..., v_l)$ as roots, introducing a factor $(1+o(1))\left(\frac{n}{2}\right)^l$, and there have to exist edges between them which generates a factor $\left[\frac{m}{\binom{n}{k}(1-2^{1-k})}\right]^l$. To choose the remaining vertices in the participating edges we have to distinguish between pairs of vertices $(v_i, v_{i+1})$ that are assigned the same colour and those that are not, because if $\sigma(v_i) = \sigma(v_{i+1})$ we have to make sure that at least one of the other $k-2$ vertices participating in this edge is assigned the opposite colour. This gives rise to the third factor in the following calculation.

$$\begin{aligned}
&\mathbb{E}\left[D_{l,n}|\mathcal{V}(\sigma)\right] \\
&\quad\sim \left(\frac{n}{2}\right)^l \left[\frac{m}{\binom{n}{k}(1-2^{1-k})}\right]^l \cdot 2\sum_{i=0}^{l}\left[\binom{l}{i}\binom{n-2}{k-2}^i\left[\binom{n-2}{k-2}-\binom{n/2}{k-2}\right]^{l-i} 1_{\{i \text{ is even}\}}\right] \\
&\quad= \left(\frac{n}{2}\right)^l \left[\frac{m}{\binom{n}{k}(1-2^{1-k})}\right]^l \cdot \left[\left[2\binom{n-2}{k-2}-\binom{n/2}{k-2}\right]^l + \left[-\binom{n/2}{k-2}\right]^l\right] \\
&\quad\sim \left(\frac{n}{2}\right)^l \left[\frac{k!dn}{kn^k(1-2^{1-k})}\right]^l \cdot \left[\frac{\left[\left(2^{k-1}-1\right)n^{k-2}\right]^l + \left(-n^{k-2}\right)^l}{\left[2^{k-2}(k-2)!\right]^l}\right] \\
&\quad= \left[d(k-1)\right]^l\left(1+\frac{(-1)^l}{\left(2^{k-1}-1\right)^l}\right)
\end{aligned}$$

Hence, recalling that $C_{l,n} = \frac{1}{2l}D_{l,n}$, we get

$$\mathbb{E}\left[C_{l,n}|\mathcal{V}(\sigma)\right] \sim \frac{\left[d(k-1)\right]^l}{2l}\left(1+\frac{(-1)^l}{\left(2^{k-1}-1\right)^l}\right). \tag{7.3.5}$$

In fact, since $Y$ considers only good sequences and $l, m_2, \ldots, m_L$ remain fixed as $n \to \infty$, (7.3.5) yields

$$\mathbb{E}[Y|\mathcal{V}(\sigma)] \sim \prod_{l=2}^{L}\left(\frac{\left[d(k-1)\right]^l}{2l}\left(1+\frac{(-1)^l}{\left(2^{k-1}-1\right)^l}\right)\right)^{m_l}.$$

Plugging the above relation and Claim 7.3.2 into (7.3.4) we get (7.3.3). The proposition follows. □

*Proof of Claim 7.3.2:* The idea of the proof is to find an event, namely that there exists an induced subgraph with too many edges, that always occurs if $\bar{Y} > 0$ and whose probability we can bound from above. To this aim let $A = \{i \in \mathbb{R} | i = (l-1)(k-1)+j \text{ for some } l \leq L, j \in \{0, ..., k-2\}\}$. For every subset $R$ of $(l-1)(k-1)+j$ vertices, where $l \leq L$ and $j \in \{0, ..., k-2\}$ let $\mathbb{I}_R$ be equal to 1 if the number of edges that only consist of vertices in $R$ is at least $l$. Let the $H_L$ be the event that

$\sum_{R:|R|\in A} \mathbb{I}_R > 0$. It is direct to check that if $\bar{Y} > 0$ then $H_L$ occurs. This implies that

$$\mathbb{P}\left[\bar{Y} > 0|\mathcal{V}(\sigma)\right] \leq \mathbb{P}\left[H_L|\mathcal{V}(\sigma)\right].$$

The claim follows by appropriately bounding $\mathbb{P}[H_L|\mathcal{V}(\sigma)]$. For this, we are going to use Markov's inequality, i.e.

$$\mathbb{P}[H_L|\mathcal{V}(\sigma)] \leq \mathbb{E}\left[\sum_{R:|R|\in A} \mathbb{I}_R|\mathcal{V}(\sigma)\right] = \sum_{l=2}^{L}\sum_{j=0}^{k-2}\sum_{R:|R|=(l-1)(k-1)+j} \mathbb{E}\left[\mathbb{I}_R|\mathcal{V}(\sigma)\right].$$

For any set $R$ such that $|R| = (l-1)(k-1)+j$, we can put $l$ edges inside the set in at most $\binom{\binom{(l-1)(k-1)+j}{k}}{l}$ ways, which obviously gets largest if $j = k-2$ and thus $(l-1)(k-1)+j = l(k-1)-1$. Clearly conditioning on $\mathcal{V}(\sigma)$ can only reduce the number of different placings of the edges.

We observe that for a colouring $\sigma$ and two fixed vertices $v$ and $v'$ with $\sigma(v) \neq \sigma(v')$ the probability that $e(v, v')$ does not exist is $\left(1 - \frac{1}{N-\mathcal{F}(\sigma)}\right)^m$. Using inclusion/exclusion and the binomial theorem, with $N = \binom{n}{k}$ and $\mathcal{F}(\sigma) \sim 2^{1-k}N$, for a fixed set $R$ of cardinality $(l-1)(k-1)+j$ we get that

$$\begin{aligned}
\mathbb{E}\left[\mathbb{I}_R|\mathcal{V}(\sigma)\right] &\leq \binom{\binom{l(k-1)-1}{k}}{l} \sum_{i=0}^{l} \binom{l}{i} (-1)^i \left(1 - \frac{i}{N - \mathcal{F}(\sigma)}\right)^m \\
&\leq \binom{\binom{l(k-1)-1}{k}}{l} \left(\frac{m}{N - \mathcal{F}(\sigma)}\right)^l \sim \binom{\binom{l(k-1)-1}{k}}{l} \left(\frac{m}{\binom{n}{k}\left(1 - 2^{1-k}\right)}\right)^l.
\end{aligned}$$

With $m = \frac{dn}{k}$ and since $\binom{i}{j} \leq (ie/j)^j$, it holds that

$$\begin{aligned}
\mathbb{P}[H_L|\mathcal{V}(\sigma)] &\leq (1+o(1)) \sum_{l=2}^{L} \binom{n}{l(k-1)-1} \binom{\binom{l(k-1)-1}{k}}{l} \left(\frac{m}{\binom{n}{k}\left(1 - 2^{1-k}\right)}\right)^l \\
&= (1+o(1)) \sum_{l=2}^{L} \left(\frac{ne}{l(k-1)-1}\right)^{l(k-1)-1} \left(\frac{e^{k+1}(l(k-1)-1)^k}{k^k l}\right)^l \left(\frac{mk^k}{n^k e^k\left(1 - 2^{1-k}\right)}\right)^l \\
&= (1+o(1)) \sum_{l=2}^{L} \frac{m^l e^{kl-1}(l(k-1)-1)^{l+1}}{n^{l+1} l^l \left(1 - 2^{1-k}\right)^l} \\
&= \frac{1+o(1)}{n} \sum_{l=2}^{L} \left(\frac{e^k d(l(k-1)-1)}{l\left(1 - 2^{1-k}\right)}\right)^l \frac{l(k-1)-1}{e} = O\left(n^{-1}\right),
\end{aligned}$$

where the last equality follows since $L$ is a fixed number. □

## 7.4. The second moment

In this section we prove Proposition 7.1.4. To this end, we need to derive an expression for the second moment of the random variables $Z^s_{\omega,\nu}$ for $s \in [\omega\nu]$ that is asymptotically tight. As a consequence, we need to put more effort into the calculations than done in prior work on hypergraph-2-colouring (e.g.[COZ12]), where the second moment of $Z$ is only determined up to a constant factor. Part of the proof is based on ideas from [BCOE14+], but as we aim for a stronger result, the arguments are extended and adapted to our situation.

### 7.4.1. The overlap

For two colour assignments $\sigma, \tau : [n] \to \{\pm 1\}$, we define the *overlap matrix*

$$\rho(\sigma,\tau) = \begin{pmatrix} \rho_{1,1}(\sigma,\tau) & \rho_{1,-1}(\sigma,\tau) \\ \rho_{-1,1}(\sigma,\tau) & \rho_{-1,-1}(\sigma,\tau) \end{pmatrix}$$

with entries

$$\rho_{i,j}(\sigma,\tau) = \frac{1}{n} \cdot |\sigma^{-1}(i) \cap \tau^{-1}(j)| \quad \text{for } i,j \in \{\pm 1\}.$$

Obviously, it holds that

$$\rho_{1,1}(\sigma,\tau) + \rho_{1,-1}(\sigma,\tau) + \rho_{-1,1}(\sigma,\tau) + \rho_{-1,-1}(\sigma,\tau) = 1.$$

If we further remember the definition from (7.1.1), we can alternatively represent $\rho(\sigma,\tau)$ as

$$\rho(\sigma,\tau) = \begin{pmatrix} \rho_{1,1}(\sigma,\tau) & \rho(\sigma) - \rho_{1,1}(\sigma,\tau) \\ \rho(\tau) - \rho_{1,1}(\sigma,\tau) & 1 - \rho(\sigma) - \rho(\tau) + \rho_{1,1}(\sigma,\tau) \end{pmatrix}.$$

To simplify the notation, for a $2 \times 2$-matrix $\rho = (\rho_{ij})$ we introduce the shorthands

$$\rho_{i,\star} = \rho_{i,1} + \rho_{i,-1}, \qquad \rho_{\cdot,\star} = (\rho_{1,\star}, \rho_{-1,\star}), \qquad \rho_{\star,j} = \rho_{1,j} + \rho_{-1,j}, \qquad \rho_{\star,\cdot} = (\rho_{\star,1}, \rho_{\star,-1}).$$

We let $\mathcal{B}(n)$ be the set of all overlap matrices $\rho(\sigma,\tau)$ for $\sigma, \tau : [n] \to \{\pm 1\}$ and $\mathcal{B}$ denote the set of all probability distributions $\rho = (\rho_{i,j})_{i,j \in \{\pm 1\}}$ on $\{\pm 1\}^2$. Further, we let $\bar{\rho}$ be the $2 \times 2$-matrix with all entries equal to $1/4$.

For a given hypergraph $H$ on $[n]$, let $Z^{(2)}_\rho(H)$ be the number of pairs $(\sigma,\tau)$ of 2-colourings of $H$ whose overlap matrix is $\rho$. Analogously to (7.2.1), we define the functions $f_2, g_2 : \mathcal{B} \mapsto \mathbb{R}$ as

$$f_2 : \rho \mapsto \mathcal{H}(\rho) + g_2(\rho) \qquad \text{with} \quad g_2(\rho) = \frac{d}{k} \ln\left(1 - \sum \rho^k_{i,\star} - \sum \rho^k_{\star,j} + \sum \rho^k_{i,j}\right).$$

The following lemma states a formula for $\mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right]$ for $\rho \in \mathcal{B}(n)$ in terms of $f_2(\rho)$.

**Lemma 7.4.1.** *Let $d' \in (0,\infty)$ and set*

$$C_n(d,k) = \sqrt{\frac{32}{(\pi n)^3}} \exp\left[\frac{d(k-1)}{2}\frac{2^k-3}{(2^{k-1}-1)^2}\right]. \tag{7.4.1}$$

*Then for $\rho \in \mathcal{B}(n)$ we have*

$$\mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right] \sim \sqrt{\frac{2\pi}{n^3}} \prod_{i,j=1}^{2} (2\pi\rho_{i,j})^{-1/2} \exp\left[nf_2(\rho)\right]$$
$$\exp\left[\frac{d(k-1)}{2}\frac{\sum\rho_{i,\star}^{k-1} - \sum\rho_{i,\star}^{k} + \sum\rho_{\star,j}^{k-1} - \sum\rho_{\star,j}^{k} - \sum\rho_{i,j}^{k-1} + \sum\rho_{i,j}^{k}}{1-\sum\rho_{i,\star}^{k} - \sum\rho_{\star,j}^{k} + \sum\rho_{i,j}^{k}}\right]. \tag{7.4.2}$$

*Moreover, if $\rho \in \mathcal{B}(n)$ satisfies $\|\rho-\bar{\rho}\|_2^2 = o(1)$, then*

$$\mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right] \sim C_n(d,k)\exp\left[nf_2(\rho)\right]. \tag{7.4.3}$$

*Proof.* Let $\rho = \begin{pmatrix} \rho_{1,1} & \rho_{1,-1} \\ \rho_{-1,1} & \rho_{-1,-1} \end{pmatrix} \in \mathcal{B}(n)$. Then

$$\mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right] = \sum_{\sigma,\tau:\rho(\sigma,\tau)=\rho} \mathbb{P}\left[\sigma,\tau \text{ are colourings of } \mathcal{H}(n,m)\right] = \sum_{\sigma,\tau:\rho(\sigma,\tau)=\rho} \left(1-\frac{\mathcal{F}(\sigma,\tau)}{N}\right)^m$$
$$= \binom{n}{\rho_{1,1}n,\rho_{1,-1}n,\rho_{-1,1}n,\rho_{-1,-1}n}\left(1-\frac{\mathcal{F}(\sigma,\tau)}{N}\right)^m. \tag{7.4.4}$$

where $N=\binom{n}{k}$ and $\mathcal{F}(\sigma,\tau)$ is the total number of possible monochromatic edges under either $\sigma$ or $\tau$. In the last line, $\sigma$ and $\tau$ are just two arbitrary fixed 2-colourings with overlap $\rho$ and the equation is valid because the following computation shows that $\mathcal{F}(\sigma,\tau)$ only depends on $\rho$:

$$\begin{aligned}\mathcal{F}(\sigma,\tau) &= \sum_{i\in\{\pm1\}}\binom{\rho_{i,\star}n}{k} + \sum_{j\in\{\pm1\}}\binom{\rho_{\star,j}n}{k} - \sum_{i,j\in\{\pm1\}}\binom{\rho_{i,j}n}{k}\\ &= N\left[\sum_{i\in\{\pm1\}}\rho_{i,\star}^k + \sum_{j\in\{\pm1\}}\rho_{\star,j}^k - \sum_{i,j\in\{\pm1\}}\rho_{i,j}^k\right] + \frac{k(k-1)}{2k!}n^{k-1}\cdot\\ &\quad\left[\sum_{i\in\{\pm1\}}\rho_{i,\star}^k - \sum_{i\in\{\pm1\}}\rho_{i,\star}^{k-1} + \sum_{j\in\{\pm1\}}\rho_{\star,j}^k - \sum_{j\in\{\pm1\}}\rho_{\star,j}^{k-1} - \sum_{i,j\in\{\pm1\}}\rho_{i,j}^k + \sum_{i,j\in\{\pm1\}}\rho_{i,j}^{k-1}\right]\\ &\quad + \Theta\left(n^{k-2}\right),\end{aligned}$$

yielding

$$\begin{aligned}1-\frac{F(\sigma,\tau)}{N} &= 1 - \sum_{i\in\{\pm1\}}\rho_{i,\star}^k - \sum_{j\in\{\pm1\}}\rho_{\star,j}^k + \sum_{i,j\in\{\pm1\}}\rho_{i,j}^k - \frac{k(k-1)}{2n}\\ &\quad\cdot\left[\sum_{i\in\{\pm1\}}\rho_{i,\star}^k - \sum_{i\in\{\pm1\}}\rho_{i,\star}^{k-1} + \sum_{j\in\{\pm1\}}\rho_{\star,j}^k - \sum_{j\in\{\pm1\}}\rho_{\star,j}^{k-1} - \sum_{i,j\in\{\pm1\}}\rho_{i,j}^k + \sum_{i,j\in\{\pm1\}}\rho_{i,j}^{k-1}\right]\\ &\quad + \Theta\left(n^{-2}\right).\end{aligned}$$

We proceed as in the proof of Lemma 7.2.1 by using that $\ln\left(x-\frac{y}{n}\right) = \ln(x) + \ln\left(1-\frac{y}{xn}\right)$ for $x>0$, $\frac{y}{n}<x$ and consequently

$$\begin{aligned}&m\ln\left(1-\frac{F(\sigma,\tau)}{N}\right)\\ &= \frac{dn}{k}\Bigg[\ln\left(1-\sum\rho_{i,\star}^k - \sum\rho_{\star,j}^k + \sum\rho_{i,j}^k\right)\\ &\quad + \ln\left(1-\frac{k(k-1)}{2n}\frac{\sum\rho_{i,\star}^k - \sum\rho_{i,\star}^{k-1} + \sum\rho_{\star,j}^k - \sum\rho_{\star,j}^{k-1} - \sum\rho_{i,j}^k + \sum\rho_{i,j}^{k-1}}{1-\sum\rho_{i,\star}^k - \sum\rho_{\star,j}^k + \sum\rho_{i,j}^k} + \Theta\left(n^{-2}\right)\right)\Bigg]\\ &\sim \frac{dn}{k}\ln\left(1-\sum\rho_{i,\star}^k - \sum\rho_{\star,j}^k + \sum\rho_{i,j}^k\right)\\ &\quad + \frac{d(k-1)}{2}\frac{\sum\rho_{i,\star}^{k-1} - \sum\rho_{i,\star}^k + \sum\rho_{\star,j}^{k-1} - \sum\rho_{\star,j}^k - \sum\rho_{i,j}^{k-1} + \sum\rho_{i,j}^k}{1-\sum\rho_{i,\star}^k - \sum\rho_{\star,j}^k + \sum\rho_{i,j}^k} + \Theta\left(n^{-1}\right). \qquad (7.4.5)\end{aligned}$$

As $\mathcal{F}(\sigma,\tau)$ does only depend on $\rho$, (7.4.4) becomes Using Stirling's formula, we get the following approximation for the number of colour assignments with overlap $\rho$:

$$\binom{n}{\rho_{1,1}n, \rho_{1,-1}n, \rho_{-1,1}n, \rho_{-1,-1}n} \sim \sqrt{2\pi}n^{-3/2} \prod_{i,j\in\{\pm 1\}} (2\pi\rho_{i,j})^{-1/2} \exp\left[n\mathcal{H}(\rho)\right]. \tag{7.4.6}$$

Inserting (7.4.5) and (7.4.6) into (7.4.4) completes the proof of (7.4.2). Equation (7.4.3) follows from (7.4.2) because if $\|\rho - \bar{\rho}\|_2^2 = o(1)$, then

$$\prod_{i,j=1}^{2} (2\pi\rho_{i,j})^{-1/2} \sim \frac{4}{\pi^2}$$

and

$$\frac{\sum \rho_{i,\star}^{k-1} - \sum \rho_{i,\star}^{k} + \sum \rho_{\star,j}^{k-1} - \sum \rho_{\star,j}^{k} - \sum \rho_{i,j}^{k-1} + \sum \rho_{i,j}^{k}}{1 - \sum \rho_{i,\star}^{k} - \sum \rho_{\star,j}^{k} + \sum \rho_{i,j}^{k}} \sim \frac{2^k - 3}{\left(2^{k-1} - 1\right)^2}.$$

□

### 7.4.2. Dividing up the interval

Let $\omega, \nu \in \mathbb{N}$ and $s \in [\omega\nu]$. Analogously to the notation in Section 7.1 we introduce the sets

$$\mathcal{B}_\omega(n) = \left\{\rho \in \mathcal{B}(n) : \rho_{i,\star}, \rho_{\star,i} \in \left[\frac{1}{2} - \frac{\omega}{\sqrt{n}}, \frac{1}{2} + \frac{\omega}{\sqrt{n}}\right) \quad \text{for } i \in \{\pm 1\}\right\}$$

and

$$\mathcal{B}_{\omega,\nu}^{s}(n) = \left\{\rho \in \mathcal{B}_\omega(n) : \rho_{i,\star}, \rho_{\star,i} \in \left[\rho_{\omega,\nu}^{s} - \frac{1}{\nu\sqrt{n}}, \rho_{\omega,\nu}^{s} + \frac{1}{\nu\sqrt{n}}\right) \quad \text{for } i \in \{\pm 1\}\right\},$$

imposing constraints on the overlap matrix $\rho$ insofar as the colour densities resulting from its projection on each colouring must not deviate too much from $1/2$ in the set $\mathcal{B}_\omega(n)$ and from $\rho_{\omega,\nu}^{s}$ in the set $\mathcal{B}_{\omega,\nu}^{s}(n)$. By the linearity of expectation, for any $s \in [\omega\nu]$ we have

$$\mathbb{E}\left[Z_{\omega,\nu}^{s}(\mathcal{H}(n,m))^2\right] = \sum_{\rho \in \mathcal{B}_{\omega,\nu}^{s}(n)} \mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right].$$

We are going to show that the expression on the right hand side of this equation is dominated by the contributions with $\rho$ "close to" $\bar{\rho}$ in terms of the euclidian norm. More precisely, for $\eta > 0$ we introduce the set

$$\mathcal{B}_{\omega,\nu,\eta}^{s}(n) = \left\{\rho \in \mathcal{B}_{\omega,\nu}^{s}(n) : \|\rho - \bar{\rho}\|_2 \le \eta\right\}$$

and define

$$Z^{s\,(2)}_{\omega,\nu,\eta}(\mathcal{H}(n,m)) = \sum_{\rho \in \mathcal{B}^s_{\omega,\nu,\eta}(n)} Z^{(2)}_{\rho}(\mathcal{H}(n,m)).$$

The following proposition reveals that it suffices to consider overlap matrices $\rho$ such that $\|\rho - \bar{\rho}\|_2 \leq n^{-3/8}$. Here, the number $3/8$ is somewhat arbitrary, any number smaller than $1/2$ would do.

**Proposition 7.4.2.** *Let $k \geq 3$ and $\omega, \nu \in \mathbb{N}$. If $d'/k < 2^{k-1} \ln 2 - 2$, than for every $s \in [\omega\nu]$ we have*

$$\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))^2\right] \sim \mathbb{E}\left[Z^{s\,(2)}_{\omega,\nu,n^{-3/8}}(\mathcal{H}(n,m))\right].$$

To prove this proposition, we need the following lemma.

**Lemma 7.4.3.** *Let $d/k < 2^{k-1} \ln 2 - 2$ and $C_n(d,k)$ as defined in Lemma 7.4.1. Set*

$$B(d,k) = 4\left(1 - \frac{d(k-1)}{2^{k-1}-1}\right).$$

1. *If $\rho \in \mathcal{B}_\omega(n)$ satisfies $\|\rho - \bar{\rho}\|_2 \leq n^{-3/8}$ then*

$$\mathbb{E}\left[Z^{(2)}_{\rho}(\mathcal{H}(n,m))\right] \sim C_n(d,k) \exp\left[n f_2(\bar{\rho}) - n\frac{B(d,k)}{2}\|\rho - \bar{\rho}\|_2^2\right]. \tag{7.4.7}$$

2. *There exists $A = A(d,k) > 0$ such that if $\rho \in \mathcal{B}_\omega(n)$ satisfies $\|\rho - \bar{\rho}\|_2 > n^{-3/8}$, then*

$$\mathbb{E}\left[Z^{(2)}_{\rho}(\mathcal{H}(n,m))\right] = O\left(\exp\left[n f_2(\bar{\rho}) - A n^{1/4}\right]\right). \tag{7.4.8}$$

*Proof.* To prove (7.4.7), we observe that if $\rho \in \mathcal{B}_\omega(n)$ satisfies $\|\rho - \bar{\rho}\|_2 \leq n^{-3/8}$, by Taylor expansion around $\bar{\rho}$ (where $\mathcal{H}$ and $g_2$ are maximized) we obtain

$$\mathcal{H}(\rho) = \mathcal{H}(\bar{\rho}) - 2\|\rho - \bar{\rho}\|_2^2 + o\left(n^{-1}\right) \text{ and} \tag{7.4.9}$$

$$g_2(\rho) = g_2(\bar{\rho}) - \frac{2d(k-1)}{2^{k-1}-1}\|\rho - \bar{\rho}\|_2^2 + o\left(n^{-1}\right). \tag{7.4.10}$$

Inserting this into (7.4.3) yields (7.4.7).

To prove (7.4.8), we distinguish two cases.

**Case 1:** $\|\rho - \bar{\rho}\|_2 = o(1)$: We observe that similarly to (7.4.9) and (7.4.10) there exists a constant $A = A(d,k) > 0$ such that

$$f_2(\rho) \leq f_2(\bar{\rho}) - A\|\rho - \bar{\rho}\|_2^2.$$

Hence, if $\|\rho - \bar{\rho}\|_2 > n^{-3/8}$ and $\|\rho - \bar{\rho}\|_2 = o(1)$, then

$$\mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right] = O\left(n^{-3/2}\right)\exp\left[nf_2(\rho)\right] \leq \exp\left[nf_2\left(\bar{\rho}\right) - An^{1/4}\right]. \tag{7.4.11}$$

**Case 2:** $\|\rho - \bar{\rho}\|_2 = c$ where $c > 0$ is a constant independent of $n$: We consider the function $\bar{f}_2 : \left[0, \frac{1}{2}\right] \mapsto \mathbb{R}$ that results from $f_2$ by setting $\rho_{i,\star} = \rho_{\star,i} = 1/2$. This function was introduced by Achlioptas and Moore [AM06] and has been studied at different places in the literature on random hypergraph 2-colouring. The following lemma quantifies the largest possible deviation of $f_2$ and $\bar{f}_2$.

**Lemma 7.4.4.** *Let $\bar{f}_2 : [0,1] \to \mathbb{R}$ be defined as*

$$\bar{f}_2(\rho) = \ln 2 + \mathcal{H}\left(2\rho\right) + \frac{d}{k}\ln\left(1 - 2^{2-k} + 2\rho^k + 2\left(\frac{1}{2} - \rho\right)^k\right).$$

*Then for $\rho = (\rho_{i,j}) \in \mathcal{B}_\omega(n)$ we have*

$$\exp\left[nf_2(\rho)\right] \sim \exp\left[n\bar{f}_2(\rho_{1,1}) + O\left(\omega^2\right)\right].$$

*Proof.* For $\rho \in \mathcal{B}_\omega(n)$, we consider the function

$$\zeta(\rho) = f_2(\rho) - \bar{f}_2(\rho_{1,1})$$

and approximate $\zeta(\rho)$ by a Taylor expansion around $\rho = \bar{\rho}$. As $f_2(\bar{\rho}) = \bar{f}_2(\bar{\rho}_{1,1})$ and $\frac{\partial f_2}{\partial \rho_{i,j}}(\bar{\rho}) = 0$ for $i,j \in \{0,1\}$ and $\bar{f}_2'(\bar{\rho}_{1,1}) = 0$, we have $\zeta(\rho) = C \cdot \|\rho - \bar{\rho}\|_2^2 = O\left(\frac{\omega}{\sqrt{n}}\right)$ for some constant $C$. Thus,

$$\max_{\rho \in \mathcal{B}_\omega(n)} |\zeta(\rho)| = O\left(\frac{\omega^2}{n}\right),$$

yielding the assertion. □

In [BCOR16, Lemma 4.11] the function $\bar{f}_2$ is analysed and it is shown that in the regime $d/k \leq 2^{k-1}\ln 2 - 2$ it takes its *global* maximum at $\rho = \bar{\rho}$ and $\bar{f}_2(\rho) < \bar{f}_2(\bar{\rho})$ for all $\rho \in \left[0, \frac{1}{2}\right]$ with $\rho \neq \bar{\rho}$ independent of $n$. Combining this with Lemma 7.4.4 we find that there exists a constant $A' = A'(d,k) > 0$ such that

$$f_2(\rho) = f_2\left(\bar{\rho}\right) - A' + O\left(\frac{\omega^2}{n}\right),$$

where we used that $f_2(\bar{\rho}) = \bar{f}_2(\bar{\rho})$.

Thus,

$$\mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right] = O\left(n^{-3/2}\right)\exp\left[nf_2(\rho)\right] \leq \exp\left[nf_2\left(\bar{\rho}\right) - A'n + O\left(\omega^2\right)\right]. \tag{7.4.12}$$

As $\exp\left[nf_2\left(\bar{\rho}\right) - A'n + O\left(\omega^2\right)\right] = o\left(\exp\left[nf_2\left(\bar{\rho}\right) - An^{1/4}\right]\right)$, equation (7.4.12) together with (7.4.11) completes the proof of (7.4.8). □

*Proof of Proposition 7.4.2.* We let $s \in [\omega\nu]$. For a $\hat{\rho} \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)$, we have $\|\hat{\rho} - \bar{\rho}\|_2 = O\left(\frac{\omega}{\sqrt{n}}\right)$ and obtain from the first part of Lemma 7.4.3 that

$$\mathbb{E}\left[Z^{s\,(2)}_{\omega,\nu,n^{-3/8}}(\mathcal{H}(n,m))^2\right] \geq \mathbb{E}\left[Z_{\hat{\rho}}^{(2)}(\mathcal{H}(n,m))\right] \sim C_n(d,k)\exp\left[nf_2\left(\bar{\rho}\right) + O(\omega^2)\right]. \tag{7.4.13}$$

On the other hand, because $|\mathcal{B}^s_{\omega,\nu}(n)|$ is bounded by a polynomial in $n$, the second part of Lemma 7.4.3 yields

$$\sum_{\rho \in \mathcal{B}^s_{\omega,\nu}(n) : \|\rho - \bar{\rho}\|_2 > n^{-3/8}} \mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right] = O\left(\exp\left[nf_2\left(\bar{\rho}\right) - An^{1/4} + O(\ln n)\right]\right). \tag{7.4.14}$$

Combining (7.4.13) and (7.4.14), we obtain

$$\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))^2\right] \sim \sum_{\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)} \mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right] = \mathbb{E}\left[Z^{s\,(2)}_{\omega,\nu,n^{-3/8}}(\mathcal{H}(n,m))\right]$$

as claimed. □

### 7.4.3. The leading constant

In this section we compute the contribution of overlap matrices $\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)$. In a first step we show that for $\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)$ we can approximate $f_2$ by a function $f_2^s$ that results from $f_2$ by (approximately) fixing the marginals $\rho_{i,\star}, \rho_{\star,j}$ for $i,j \in \{\pm 1\}$.

**Lemma 7.4.5.** *Let $k \geq 3, \omega, \nu \in \mathbb{N}$ and $C_n(d,k)$ as in (7.4.1). For $s \in [\omega\nu]$, remember $\rho^s_{\omega,\nu}$ from (7.1.2). Let $f_2^s : \mathcal{B} \to \mathbb{R}$ be defined as*

$$f_2^s : \rho \mapsto \mathcal{H}(\rho) + \frac{d}{k}\ln\left(1 - 2{\rho^s_{\omega,\nu}}^k - 2(1-\rho^s_{\omega,\nu})^k + \sum_{i,j\in\{\pm 1\}} \rho_{i,j}\right).$$

*Then for $\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)$ it holds that*

$$\mathbb{E}\left[Z_\rho^{(2)}(\mathcal{H}(n,m))\right] \sim C_n(d,k)\exp\left[nf_2^s(\rho) + O\left(\frac{\omega}{\nu}\right)\right].$$

*Proof.* Equation (7.4.3) of Lemma 7.4.1 yields that

$$\mathbb{E}\left[Z_{\rho}^{(2)}(\mathcal{H}(n,m))\right] \sim C_n(d,k)\exp\left[nf_2(\rho)\right]. \tag{7.4.15}$$

Analogously to the proof of Lemma 7.4.4 we define

$$\zeta^s(\rho) = f_2(\rho) - f_2^s(\rho).$$

To bound $\zeta^s(\rho)$ from above for all $\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)$, we observe that we can express the function $f_2$ by setting $\rho_{1,\star} = \rho^s_{\omega,\nu} + \alpha$ and $\rho_{\star,1} = \rho^s_{\omega,\nu} + \beta$, where $|\alpha|, |\beta| \leq \frac{1}{\nu\sqrt{n}}$ and thus

$$\begin{aligned} &f_2 : \rho \mapsto \mathcal{H}(\rho) \\ &\quad + \frac{d}{k}\ln\left(1 - \left(\rho^s_{\omega,\nu} + \alpha\right)^k - \left(\rho^s_{\omega,\nu} + \beta\right)^k - \left(1 - \rho^s_{\omega,\nu} - \alpha\right)^k - \left(1 - \rho^s_{\omega,\nu} - \beta\right)^k + \sum_{i,j\in\{\pm 1\}} \rho_{i,j}\right). \end{aligned}$$

As we are only interested in the difference between $f_2$ and $f_2^s$, we can reparametrise $\zeta^s$ as

$$\begin{aligned} &\zeta^s(\alpha,\beta) \\ &= \frac{d}{k}\ln\left(\frac{1 - \left(\rho^s_{\omega,\nu} + \alpha\right)^k - \left(\rho^s_{\omega,\nu} + \beta\right)^k - \left(1 - \rho^s_{\omega,\nu} - \alpha\right)^k - \left(1 - \rho^s_{\omega,\nu} - \beta\right)^k + \sum_{i,j\in\{\pm 1\}} \rho_{i,j}}{1 - 2{\rho^s_{\omega,\nu}}^k - 2(1-\rho^s_{\omega,\nu})^k + \sum_{i,j\in\{\pm 1\}} \rho_{i,j}}\right). \end{aligned}$$

Differentiating and simplifying the expression yields $\frac{\partial \zeta^s}{\partial \alpha}(\alpha,\beta), \frac{\partial \zeta^s}{\partial \beta}(\alpha,\beta) = O\left(\frac{\omega}{\sqrt{n}}\right)$. As we are interested in $\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)$ and $|\mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)| \leq \frac{2}{\nu\sqrt{n}}$ according to the fundamental theorem of calculus it follows for every $s \in [\omega\nu]$ that

$$\max_{\rho\in\mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)} |\zeta^s(\rho)| = \int_{-\left(\nu\sqrt{n}\right)^{-1}}^{\left(\nu\sqrt{n}\right)^{-1}} O\left(\frac{\omega}{\sqrt{n}}\right) d\alpha = O\left(\frac{\omega}{n\nu}\right).$$

Combining this with (7.4.15) yields the assertion. □

**Proposition 7.4.6.** *Let $k \geq 3, \omega, \nu \in \mathbb{N}$ and $d'(k-1) < \left(2^{k-1}-1\right)^2$. Then for all $s \in [\omega\nu]$ we have*

$$\begin{aligned} \mathbb{E}\left[Z^{s\,(2)}_{\omega,\nu,n^{-3/8}}(\mathcal{H}(n,m))\right] \sim_\nu \left(|\mathcal{A}^s_{\omega,\nu}(n)|\sqrt{\frac{2}{\pi n}}\exp\left[nf_1\left(\rho^s_{\omega,\nu}\right)\right]\right)^2 \cdot \\ \exp\left[\frac{d(k-1)}{2}\frac{2^k-3}{\left(2^{k-1}-1\right)^2}\right]\left(1 - \frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right)^{-1/2}. \end{aligned}$$

*Proof.* By Lemma 7.4.5 we know that for $\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n)$ we have

$$\mathbb{E}\left[Z^{(2)}_\rho(\mathcal{H}(n,m))\right] \sim C_n(d,k)\exp\left[nf^s_2(\rho) + O\left(\frac{\omega}{\nu}\right)\right]. \tag{7.4.16}$$

A Taylor expansion of $f^s_2(\rho)$ around

$$\rho^s = \begin{pmatrix} {\rho^s_{\omega,\nu}}^2 & \rho^s_{\omega,\nu}(1-\rho^s_{\omega,\nu}) \\ \left(1-\rho^s_{\omega,\nu}\right)\rho^s_{\omega,\nu} & \left(1-\rho^s_{\omega,\nu}\right)^2 \end{pmatrix}$$

while setting $D(d,k) = 4\left(1 - \frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right)$ yields

$$f^s_2(\rho) = f^s_2\left(\rho^s\right) + \Theta\left(\frac{\omega}{n}\right)\|\rho-\rho^s\|_2 - \frac{D(d,k)}{2}\|\rho-\rho^s\|^2_2 + o\left(n^{-1}\right).$$

Combining this with (7.4.16) we find that

$$\mathbb{E}\left[Z^{(2)}_\rho(\mathcal{H}(n,m))\right] \sim C_n(d,k)\exp\left[nf^s_2\left(\rho^s\right) + \Theta\left(\omega\right)\|\rho-\rho^s\|_2 - n\frac{D(d,k)}{2}\|\rho-\rho^s\|^2_2 + O\left(\frac{\omega}{\nu}\right)\right]. \tag{7.4.17}$$

For $\rho^0, \rho^1 \in \mathcal{B}^s_{\omega,\nu}(n)$, we introduce the set of overlap matrices

$$\mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n,\rho^0,\rho^1) = \{\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n) : \rho_{\cdot,\star} = \rho^0, \rho_{\star,\cdot} = \rho^1\}.$$

In particular, $\mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n,\rho^0,\rho^1)$ contains the "product" overlap $\rho^0 \otimes \rho^1$ defined by $\left(\rho^0 \otimes \rho^1\right)_{ij} = \rho^0_i\rho^1_j$. With these definitions we see that

$$\mathbb{E}\left[Z^{s\,(2)}_{\omega,\nu,n^{-3/8}}(\mathcal{H}(n,m))\right] = \sum_{\rho^0,\rho^1 \in \mathcal{B}^s_{\omega,\nu}(n)}\ \sum_{\rho \in \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n,\rho^0,\rho^1)} \mathbb{E}\left[Z^{(2)}_\rho(\mathcal{H}(n,m))\right]. \tag{7.4.18}$$

Let us fix from now on two colour densities $\rho^0, \rho^1 \in \mathcal{B}^s_{\omega,\nu}(n)$. We simplify the notation by setting

$$\widehat{\mathcal{B}} = \mathcal{B}^s_{\omega,\nu,n^{-3/8}}(n,\rho^0,\rho^1), \qquad \widehat{\rho} = \rho^0 \otimes \rho^1.$$

Thus, we are going to evaluate

$$\mathcal{S}_1 = \sum_{\rho \in \widehat{\mathcal{B}}} \mathbb{E}\left[Z^{(2)}_\rho(\mathcal{H}(n,m))\right].$$

We define the set $\mathcal{E}_n = \left\{\boldsymbol{\varepsilon} = (\varepsilon, -\varepsilon, -\varepsilon, \varepsilon), \varepsilon \in \frac{1}{n}\mathbb{Z}, 0 \le \varepsilon \le 1\right\}$. Then for each $\rho \in \widehat{\mathcal{B}}$ we can find $\boldsymbol{\varepsilon} \in \mathcal{E}_n$ such that

$$\rho = \widehat{\rho} + \boldsymbol{\varepsilon}$$

Hence, this gives $\|\rho - \rho^s\|_2 = \|\widehat{\rho} + \boldsymbol{\varepsilon} - \rho^s\|_2$ and the triangle inequality yields

$$\|\boldsymbol{\varepsilon}\|_2 - \|\widehat{\rho} - \rho^s\|_2 \le \|\widehat{\rho} + \boldsymbol{\varepsilon} - \rho^s\|_2 \le \|\boldsymbol{\varepsilon}\|_2 + \|\widehat{\rho} - \rho^s\|_2 .$$

As $\|\widehat{\rho} - \rho^s\|_2 \le \frac{1}{\nu\sqrt{n}}$ and for $\nu \to \infty$ it holds that $\frac{1}{\nu\sqrt{n}} = o(n^{-1/2})$, in this case we have

$$\|\rho - \rho^s\|_2 = \|\boldsymbol{\varepsilon}\|_2 + o(n^{-1/2}). \tag{7.4.19}$$

Observing that $f_2^s(\rho^s) = \left(f_1(\rho^s_{\omega,\nu})\right)^2$ and inserting (7.4.19) into (7.4.17), we find

$$\begin{aligned} \mathcal{S}_1 &\sim_\nu \sum_{\rho \in \widehat{\mathcal{B}}} C_n(d,k) \exp\left[n f_2^s(\rho^s) - n\frac{D(d,k)}{2}\|\boldsymbol{\varepsilon}\|_2^2 + o(n^{1/2})\|\boldsymbol{\varepsilon}\|_2 + o(1)\right] \\ &\sim_\nu C_n(d,k) \exp\left[2n f_1^s\left(\rho^s_{\omega,\nu}\right)\right] \sum_{\rho \in \widehat{\mathcal{B}}} \exp\left[-n\frac{D(d,k)}{2}\|\boldsymbol{\varepsilon}\|_2^2 + o(n^{1/2})\|\boldsymbol{\varepsilon}\|_2\right]. \end{aligned} \tag{7.4.20}$$

It follows from the definition of $\widehat{\mathcal{B}}$ that

$$\left\{\widehat{\rho} + \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \in \mathcal{E}_n, \|\boldsymbol{\varepsilon}\|_2 \le n^{-3/8}/2\right\} \subset \left\{\rho \in \widehat{\mathcal{B}}\right\} \subset \{\widehat{\rho} + \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \in \mathcal{E}_n\}.$$

As

$$\begin{aligned} \mathcal{S}_2 &\sim_\nu C_n(d,k) \exp\left[n f_2^s(\rho^s)\right] \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_n,\, \|\boldsymbol{\varepsilon}\|_2 > n^{-3/8}/2} \exp\left[-n\frac{D(d,k)}{2}\|\boldsymbol{\varepsilon}\|_2^2(1+o(1))\right] \\ &\le C_n(d,k) \exp\left[n f_2^s(\rho^s)\right] O(n) \exp\left[-\frac{D(d,k)}{8} n^{1/4}\right], \end{aligned}$$

equation (7.4.20) yields $\lim_{\nu\to\infty}\lim_{n\to\infty} \mathcal{S}_2/\mathcal{S}_1 = 0$ and we see that $\boldsymbol{\varepsilon} \in \mathcal{E}_n$ with $\|\boldsymbol{\varepsilon}\|_2 > n^{-3/8}/2$ do only contribute negligibly. Thus, we conclude, using the formula of Euler-Maclaurin and a Gaussian integration, that

$$\begin{aligned}\mathcal{S}_1 &\sim_\nu C_n(d,k)\exp\left[2nf_1^s\left(\rho^s_{\omega,\nu}\right)\right]\sum_{\varepsilon\in\mathcal{E}_n}\exp\left[-n\frac{D(d,k)}{2}\left\|\varepsilon\right\|_2^2+o(n^{1/2})\left\|\varepsilon\right\|_2\right]\\ &\sim_\nu C_n(d,k)\exp\left[2nf_1^s\left(\rho^s_{\omega,\nu}\right)\right]n\int\exp\left[-n\frac{D(d,k)}{8}\varepsilon^2+o(n^{1/2})\varepsilon\right]d\varepsilon\\ &\sim_\nu C_n(d,k)\exp\left[2nf_1^s\left(\rho^s_{\omega,\nu}\right)\right]\sqrt{\frac{\pi n}{8}}\left(1-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right)^{-1/2}. \qquad (7.4.21)\end{aligned}$$

In particular, the last expression is independent of the choice of the vectors $\rho^0,\rho^1$ that defined $\widehat{\mathcal{B}}$. Therefore, substituting (7.4.21) in the decomposition (7.4.18) completes the proof. □

*Proof of Proposition 7.1.4.* From (7.3.1) we remember that

$$\exp\left[\sum_{l\geq 2}\lambda_l\delta_l^2\right]=\exp\left[-\frac{d(k-1)}{2}\frac{1}{\left(2^{k-1}-1\right)^2}\right]\left(1-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right)^{-1/2}. \qquad (7.4.22)$$

To prove Proposition 7.1.4 we combine Lemma 7.2.3 with Propositions 7.4.2 and 7.4.6 yielding

$$\begin{aligned}\frac{\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))^2\right]}{\mathbb{E}\left[Z^s_{\omega,\nu}(\mathcal{H}(n,m))\right]^2} &\sim_\nu \exp\left[\frac{d(k-1)}{2}\left(\frac{2^k-3}{\left(2^{k-1}-1\right)^2}-\frac{2}{2^{k-1}-1}\right)\right]\left(1-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right)^{-1/2}\\ &=\exp\left[-\frac{d(k-1)}{2}\frac{1}{\left(2^{k-1}-1\right)^2}\right]\left(1-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right)^{-1/2}. \qquad (7.4.23)\end{aligned}$$

Combining equations (7.4.22) and (7.4.23) completes the proof. □

## 7.5. Excursion: Colour patterns - A different approach

In the course of proving Proposition 7.1.4, it was not clear from the beginning that we could guarantee the second moment of the total number of solutions to be small enough for small subgraph conditioning to work. An idea going beyond a straightforward calculation of the second moment was to split the number of all colourings and to group colourings exhibiting the same "pattern", i.e. colourings satisfying the edges of the hypergraph in the same prescribed way. The purpose behind that was to be able to get a handle on the "cross-terms" emerging from pairs of colour assignments that colour the edges of a hypergraph in different ways, because we suspected pairs of colourings having "untypical" patterns to push up the variance.

Fortunately, it turned out that we did not need to pursue this more complicated approach. Nevertheless, as it might be interesting and might potentially be useful in further applications, we shortly sketch it here without going into too much detail.

To begin, we decompose the number of solutions $Z$ into a sum of contributions that are tractable. To this aim, let $\Theta = \{\pm1\}^k \backslash \{(1,...,1) \cup (-1,...,-1)\}$ be the set of all $2^k - 2$ valid combinations to colour a $k$-uniform hyperedge. We call a vector $\boldsymbol{\vartheta} = (\vartheta_1, ..., \vartheta_m)$ with $\vartheta_i \in \Theta$ for all $i \in [m]$ a *colour pattern*. Given $\mathcal{H}(n,m)$ and a colouring $\sigma$, let $\mu(\vartheta) = \mu_{\mathcal{H}(n,m),\sigma}(\vartheta)$ for $\vartheta \in \Theta$ denote the number of edges $e$ of $\mathcal{H}(n,m)$ such that $\sigma_{|e} = \vartheta$ (the number of occurences of $\vartheta$ in $\mathcal{H}(n,m)$ under $\sigma$). Let $M$ be the set of all vectors $\boldsymbol{\mu} = (\mu(\vartheta))_{\vartheta\in\Theta}$ such that $\sum_{\vartheta\in\Theta}\mu(\vartheta) = m$. Finally, let $Z_{\boldsymbol{\mu}}(\mathcal{H}(n,m))$ be the number of colourings $\sigma$ of $\mathcal{H}(n,m)$ "fitting" $\boldsymbol{\mu}$. Then obviously we have

$$Z(\mathcal{H}(n,m)) = \sum_{\boldsymbol{\mu}\in M} Z_{\boldsymbol{\mu}}(\mathcal{H}(n,m)).$$

The strategy is to apply small subgraph conditioning to the random variables $Z_{\boldsymbol{\mu}}$ rather than directly to $Z$. To calculate the second moment of $Z_{\boldsymbol{\mu}}$, the key tool will be the following result of Hoeffding [Hoe51] establishing a limiting normal distribution for the sum of real functions of random permutations.

Let $(Y_{n1}, ..., Y_{nn})$ be a random vector which takes on the $n!$ permutations of $(1, ..., n)$ with equal probabilities. Let $c_n(i,j)$ for $i, j = 1, ..., n$ be real numbers and $S_n = \sum_{i=1}^n c_n(i, Y_{ni})$. We say that $S_n$ is *asymptotically normal distributed* if

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - \mathbb{E}[S_n]}{\mathrm{Var}[S_n]}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2}y^2\right] dy$$

for $-\infty < x < \infty$. Then the following holds.

**Theorem 7.5.1** ([Hoe51]). *The mean and variance of $S_n = \sum_{i=1}^n c_n(i, Y_{in})$ are*

$$\mathbb{E}[S_n] = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} c_n(i,j) \quad \textit{and} \quad \mathrm{Var}[S_n] = \frac{1}{n-1}\sum_{i=1}^{n}\sum_{j=1}^{n} d_n^2(i,j)$$

*with $d_n(i,j) = c_n(i,j) - \frac{1}{n}\sum_{g=1}^n c_n(g,j) - \frac{1}{n}\sum_{h=1}^n c_n(i,h) + \frac{1}{n^2}\sum_{g=1}^n\sum_{h=1}^n c_n(g,h)$. Furthermore, the distribution of $S_n$ is asymptotically normal if*

$$\lim_{n\to\infty} \frac{\max_{1\le i,j\le n} d_n^2(i,j)}{\sum_{i=1}^n\sum_{j=1}^n d_n^2(i,j)} = 0.$$

A main observation is that only $Z_{\boldsymbol{\mu}}$ with $\boldsymbol{\mu}$ close to some "canonical" $\bar{\boldsymbol{\mu}}$ contribute to $Z$. We assume that $2^k - 2$ divides $m$ and let $\bar{\boldsymbol{\mu}} = \big(m/(2^k-2), ..., m/(2^k-2)\big)$ and $M_\omega$ be the set of all $\boldsymbol{\mu} \in M$

with $\|\boldsymbol{\mu} - \bar{\boldsymbol{\mu}}\|_2 \leq \omega m^{-1/2}$. Then it can be shown that

$$\lim_{\omega\to\infty} \liminf_{n\to\infty} \sum_{\boldsymbol{\mu}\in M_\omega} \frac{\mathbb{E}\left[Z_{\boldsymbol{\mu}}\right]}{\mathbb{E}\left[Z\right]} = 1.$$

The proof will not be stated here as it is very similar to the proof of 7.1.1.

The rest of this section deals with giving an idea how to prove the following statement.

**Proposition 7.5.2.** *For every $\omega > 0$, we have*

$$\limsup_{n\to\infty} \max_{\boldsymbol{\mu}\in M_\omega} \frac{\mathbb{E}\left[Z_{\boldsymbol{\mu}}^2\right]}{\mathbb{E}\left[Z_{\boldsymbol{\mu}}\right]^2} = \left[1 - \frac{d(k-1)}{(2^{k-1}-1)^2}\right]^{-1/2}.$$

However, we will leave out some of the technical details and just perform the computations for certain canonical choices of $\boldsymbol{\mu}$ and under certain conditions on the colourings.

### 7.5.1. Random permutations

In a first step we show that the distribution of the overlap of two random colour patterns satisfying some balanced condition is asymptotically normal.

We let $p = (p_1, \ldots, p_m)$ with $p_i \in \Theta$ for $i \in [m]$. Additionally , we let $\pi$ be a random permutation of $[m]$ and $p^\pi$ be the permuted sequence, i.e. $p^\pi_r = p_{\pi(r)}$. For $r, s \in [m]$, we define

$$c(p_r, p_s) = \sum_{i=1}^{k} 1_{\{p_{ri}=1\}} 1_{\{p_{si}=1\}} \qquad \text{and} \qquad c(p_r) = \sum_{s=1}^{m} c(p_r, p_s).$$

Further, let

$$X = \sum_{r=1}^{m} c(p_r, p^\pi_r)$$

be the *overlap* of $p$ and $p^\pi$. Then $X = X(p)$ is a random variable and its distribution depends on the choice of $p$. We let $\alpha(i, j)$ for all $i, j \in [k]$ be defined as

$$\alpha(i, j) = \sum_{r=1}^{m} 1_{\{p_{ri}=1\}} 1_{\{p_{rj}=1\}}.$$

In order to simplify calculations, in the following we choose $p$ such that

$$\alpha(i,j) = \begin{cases} \frac{m}{2} & \text{if } i = j \\ \frac{m(2^{k-2}-1)}{2^k-2} & \text{if } i \neq j. \end{cases} \tag{7.5.1}$$

This condition is for instance satisfied if $\sum_{i=1}^{m} 1_{\{p_i=\vartheta\}} = m/(2^k-2)$ for all $\vartheta \in \Theta$. What is the asymptotic distribution of $X$ in this case?

**Proposition 7.5.3.** *The random variable $X$ is asymptotically normal with*

$$\mathbb{E}[X] = \frac{km}{4} \qquad \textit{and} \qquad \mathrm{Var}[X] = \frac{km}{16}\left[\frac{k-1}{(2^{k-1}-1)^2}+1\right].$$

*Proof.* Using Theorem 7.5.1 we can calculate the expected value of $X$ as

$$\begin{aligned} \mathbb{E}[X] &= \frac{1}{m}\sum_{r=1}^{m}\sum_{s=1}^{m} c(p_r,p_s) = \frac{1}{m}\sum_{r=1}^{m}\sum_{s=1}^{m}\left[\sum_{i=1}^{k} 1_{\{p_{ri}=1\}}1_{\{p_{si}=1\}}\right] \\ &= \frac{1}{m}\sum_{i=1}^{k}\left[\sum_{r=1}^{m} 1_{\{p_{ri}=1\}}\sum_{s=1}^{m} 1_{\{p_{si}=1\}}\right] = \frac{1}{m}\sum_{i=1}^{k}\frac{m^2}{4} = \frac{km}{4}. \end{aligned} \tag{7.5.2}$$

To calculate the variance of $X$, for $r, s \in [m]$ we define $d(p_r, p_s)$ as

$$d(p_r,p_s) = c(p_r,p_s) - \frac{1}{m}\left[\sum_{u=1}^{m} c(p_u,p_s) + \sum_{v=1}^{m} c(p_r,p_v)\right] + \frac{1}{m}\mathbb{E}[X]$$

and the symmetry of the functional $c$ combined with (7.5.2) yields

$$d(p_r,p_s) = c(p_r,p_s) - \frac{2}{m}c(p_r) + \frac{k}{4}. \tag{7.5.3}$$

According to Theorem 7.5.1, the variance of $X$ is then given by

$$\mathrm{Var}[X] = \frac{1}{m-1}\sum_{r=1}^{m}\sum_{s=1}^{m} d(p_r,p_s)^2. \tag{7.5.4}$$

With the decomposition of $d(p_r, p_s)$ from (7.5.3) we have

$$\sum_{r,s=1}^{m} d(p_r,p_s)^2 = \sum_{r,s=1}^{m} \Bigg[ c(p_r,p_s)^2 - \frac{2}{m} c(p_r,p_s)\left[c(p_r)+c(p_s)\right] + \frac{1}{m^2}\left[c(p_r)+c(p_s)\right]^2$$
$$+\frac{k}{2}c(p_r,p_s) - \frac{k}{2m}\left[c(p_r)+c(p_s)\right] + \frac{k^2}{16}\Bigg]$$
$$= \sum_{r,s=1}^{m} \Bigg[ c(p_r,p_s)^2 - \frac{4}{m} c(p_r,p_s)c(p_r) + \frac{2}{m^2}c(p_r)^2 + \frac{2}{m^2}c(p_r)c(p_s)$$
$$+\frac{k}{2}c(p_r,p_s) - \frac{k}{m}c(p_r) + \frac{k^2}{16}\Bigg]$$
$$= \sum_{r,s=1}^{m} c(p_r,p_s)^2 - \frac{2}{m}\sum_{r=1}^{m} c(p_r)^2 + \frac{(km)^2}{16} \tag{7.5.5}$$

because $\sum_{r=1}^{m} c(p_r) = \frac{km^2}{4}$ and thus $\sum_{r,s=1}^{m} c(p_r)c(p_s) = \frac{k^2m^4}{16}$. As we chose $p$ such that (7.5.1) holds, we have

$$\sum_{r=1}^{m}\sum_{s=1}^{m} c(p_r,p_s)^2 = \sum_{r=1}^{m}\sum_{s=1}^{m}\left[\sum_{i,j=1}^{k} 1_{\{p_{ri}=1\}}1_{\{p_{rj}=1\}}1_{\{p_{si}=1\}}1_{\{p_{sj}=1\}}\right]$$
$$= \sum_{i,j=1}^{k}\left[\sum_{r=1}^{m} 1_{\{p_{ri}=1\}}1_{\{p_{rj}=1\}}\right]^2 = k\alpha(1,1)^2 + k(k-1)\alpha(1,2)^2$$
$$= \frac{km^2}{16\left(2^k-2\right)^2}\left[k\left(2^k-4\right)^2 + 2^k\left(3\cdot 2^k - 8\right)\right]. \tag{7.5.6}$$

Moreover,

$$\frac{2}{m}\sum_{r=1}^{m} c(p_r)^2 = \frac{2}{m}\sum_{r=1}^{m}\left[\sum_{s=1}^{m}\sum_{i=1}^{k} 1_{\{p_{ri}=1\}}1_{\{p_{si}=1\}}\right]^2 = \frac{m}{2}\sum_{r=1}^{m}\left[\sum_{i=1}^{k} 1_{\{p_{ri}=1\}}\right]^2$$
$$= \frac{m^2}{2}\left[k\alpha(1,1) + k(k-1)\alpha(1,2)\right] = \frac{km^2}{8\left(2^k-2\right)}\left(\left(2^k-4\right)k + 2^k\right). \tag{7.5.7}$$

Combining (7.5.5) and (7.5.4) and the equations (7.5.6) and (7.5.7) yields

$$\begin{aligned}\mathrm{Var}\,[X] &= \frac{1}{m-1}\left[\sum_{r,s=1}^{m} c(p_r,p_s)^2 - \frac{2}{m}\sum_{r=1}^{m} c(p_r)^2 + \frac{(km)^2}{16}\right] \\ &= \frac{1}{m-1}\left[\frac{km^2}{16\,(2^k-2)^2}\left[k\left(2^k-4\right)^2 + 2^k\left(3\cdot 2^k - 8\right)\right]\right. \\ &\qquad\left. - \frac{km^2}{8\,(2^k-2)}\left(\left(2^k-4\right)k + 2^k\right) + \frac{(km)^2}{16}\right] \\ &= \frac{km^2}{m-1}\left[\frac{k-1}{4\,(2^k-2)^2} + \frac{1}{16}\right] \approx \frac{km}{16}\left[\frac{k-1}{(2^{k-1}-1)^2} + 1\right]. \end{aligned} \tag{7.5.8}$$

Theorem 7.5.1 provides that the distribution of $X$ is asymptotically normal because

$$\lim_{m\to\infty} \frac{\max_{1\le r,s\le m} d(p_r,p_s)^2}{\frac{1}{m}\sum_{r=1}^{m}\sum_{s=1}^{m} d(p_s,p_s)^2} = 0.$$

Together with (7.5.2) and (7.5.8) this completes the proof of the proposition. □

### 7.5.2. The configuration model

To proceed, we introduce the so-called *configuration model*, which is an alternative model to create random hypergraphs. For each $i \in \{1,...,n\}$ independently, we consider a $\mathrm{Po}(d)$-distributed random variable and collect the realisations in a vector $\boldsymbol{d} = (d_1,...,d_n)$, to which we refer as the *degree sequence* of the hypergraph. We then proceed as follows:

- Create $d_i$ 'clones' of each vertex $i$:

$$i \rightsquigarrow (i,1),...,(i,d_i)$$

  for all $i \in [n]$. Let $\mathcal{D} = \{(i,1),...,(i,d_i), 1 \le i \le n\}$.
- Choose a random bijection $\pi : [m]\times[k] \to \mathcal{D}$.
- Set $\mu_{ij} = \pi(i,j)$ where $\mu_{ij}$ is the $j$'th vertex of the $i$'th hyperedge.

This model actually generates random hypergraphs where $d_i$ is the degree of vertex $v_i$. We can think of the clones as a deck of cards. To create the hypergraph, we just shuffle the deck randomly and put the cards down in the resulting order to "fill in" the $k$-hyperedges one by one.

### 7.5.3. Entropy

In a next step we show that the distribution of the overlap of two random assignments satisfying a certain balanced condition in the configuration model is asymptotically normal.

For a given hypergraph, we let $p_\varrho$ be the number of vertices of degree $\varrho$ for $\varrho \in [0, \infty)$. To simplify calculations, in the following we only consider 2-colourings $\sigma$ that are balanced on every degree, meaning that for all $\varrho \in [0, \infty)$ we have

$$|\{v : \sigma(v) = 1, \deg(v) = \varrho\}| = |\{v : \sigma(v) = -1, \deg(v) = \varrho\}|,$$

where $\deg(v)$ denotes the degree of vertex $v$. For two randomly chosen colourings $\sigma$ and $\tau$, which are balanced on every degree, we let $Y$ be the "overlap" in the configuration model, i.e. the number of vertices where both colourings evaluate to 1 weighted with their degree:

$$Y = \sum_{v \in [n]} \deg(v) 1_{\{\sigma(v)=\tau(v)=1\}}.$$

Then the following holds.

**Proposition 7.5.4.** *The random variable $Y$ is asymptotically normal with*

$$\mathbb{E}[Y] = \frac{dn}{4} \quad \textit{and} \quad \mathrm{Var}[Y] = \frac{dn(d+1)}{16}.$$

*Proof.* The strategy is to prove this statement by applying Theorem 7.5.1. For two colourings $\sigma$ and $\tau$ and a vertex $v$, we define

$$c(\sigma(v), \tau(w)) = 1_{\{\sigma(v)=1\}} 1_{\{\tau(w)=1\}} \qquad \text{and} \qquad c(\sigma(v)) = 1_{\{\sigma(v)=1\}}.$$

We decompose $Y$ into a sum of contributions $Y_\varrho$ for $\varrho \in [0, \infty)$. We let

$$Y_\varrho = \sum_{v : \deg(v) = \varrho} 1_{\{\sigma(v)=1 \text{ and } \tau(v)=1\}} = \sum_{v : \deg(v) = \varrho} c(\sigma(v), \tau(v)).$$

Thus, $Y_\varrho$ can be interpreted as the overlap of $\sigma$ and $\tau$ restricted to the vertices of degree $\varrho$. Then Theorem 7.5.1 implies that the $Y_\varrho$ are asymptotically normal distributed.

We have

$$Y = \sum_{\varrho=0}^{\infty} \varrho Y_\varrho. \tag{7.5.9}$$

Therefore, also $Y$ is asymptotically normal and we use Theorem 7.5.1 to calculate the expectation and variance of $Y$. We begin with the expectation. Conditioned on the vertices of degree $\varrho$, we interpret $\tau$ as a random permutation of $\sigma$. We have

$$\begin{aligned}\mathbb{E}\left[Y_\varrho\right] &= \frac{1}{np_\varrho}\sum_{v:\deg(v)=\varrho}\ \sum_{w:\deg(w)=\varrho} c(\sigma(v),\tau(w)) = \frac{1}{np_\varrho}\left(\sum_{v:\deg(v)=\varrho} c(\sigma(v))\right)^2 \\ &= \frac{1}{np_\varrho}\left(\frac{np_\varrho}{2}\right)^2 = \frac{np_\varrho}{4}.\end{aligned}$$

Inserting this into (7.5.9) and applying the linearity of the expectation as well as the fact that the degrees of the vertices are $\mathrm{Po}(d)$ distributed, yields

$$\mathbb{E}\left[Y\right] = \sum_{\varrho=0}^{\infty}\varrho\mathbb{E}\left[Y_\varrho\right] = \sum_{\varrho=0}^{\infty}\varrho\frac{np_\varrho}{4} = \frac{dn}{4}.$$

As $Y_\varrho$ are independent random variables, the variance of $Y$ decomposes in the following way:

$$\mathrm{Var}\left[Y\right] = \sum_{\varrho=0}^{\infty}\varrho^2\mathrm{Var}\left[Y_\varrho\right]. \tag{7.5.10}$$

Thus, analogously to (7.5.5) we find that

$$\begin{aligned}\mathrm{Var}\left[Y_\varrho\right] &= \frac{1}{np_\varrho - 1}\left[\sum_{v:\deg(v)=\varrho}\ \sum_{w:\deg(w)=\varrho} c(\sigma(v),\tau(w))^2 - \frac{2}{np_\varrho}\sum_{v:\deg(v)=\varrho} c(\sigma(v))^2 + \frac{(np_\varrho)^2}{16}\right] \\ &= \frac{1}{16}\frac{(np_\varrho)^2}{np_\varrho - 1}.\end{aligned} \tag{7.5.11}$$

Inserting (7.5.11) into (7.5.10) that the degrees of the vertices are $\mathrm{Po}(d)$ distributed yields

$$\mathrm{Var}\left[Y\right] = \sum_{\varrho=0}^{\infty}\frac{\varrho^2}{16}\frac{(np_\varrho)^2}{np_\varrho - 1} \approx \frac{n}{16}\sum_{\varrho} p_\varrho\varrho^2 = \frac{dn(d+1)}{16},$$

thereby completing the proof. □

### 7.5.4. Matchings

The penultimate step consists in connecting the number of colourings in the configuration model to the number of patterns. More specifically, we count in how many ways two random colourings having overlap $\rho$ in the configuration model can be mapped to a bi-pattern also having overlap $\rho$. This number is given by

$$
\begin{aligned}
f(\rho) &= \frac{(km\rho)!^2\left(km\left(\frac{1}{2}-\rho\right)\right)!^2}{(km)!} \\
&\approx \frac{2\pi km\rho\cdot 2\pi km\left(\frac{1}{2}-\rho\right)(km\rho)^{2km\rho}\left(km\left(\frac{1}{2}-\rho\right)\right)^{2km\left(\frac{1}{2}-\rho\right)}}{\sqrt{2\pi km}(km)^{km}} \\
&= \left(\sqrt{2\pi km}\right)^3\rho\left(\frac{1}{2}-\rho\right)\rho^{2km\rho}\left(\frac{1}{2}-\rho\right)^{2km\left(\frac{1}{2}-\rho\right)} \\
&= \left(\sqrt{2\pi km}\right)^3\rho\left(\frac{1}{2}-\rho\right)\exp\left[km\left(2\rho\ln(\rho)+2\left(\frac{1}{2}-\rho\right)\ln\left(\frac{1}{2}-\rho\right)\right)\right],
\end{aligned}
$$

where we used Stirling's approximation. We let $g(\rho)=km\left(2\rho\ln(\rho)+2\left(\frac{1}{2}-\rho\right)\ln\left(\frac{1}{2}-\rho\right)\right)$ denote the exponential part of this function. As can be easily verified, we have

$$
g'(\rho)=km\left(2\ln(\rho)-2\ln\frac{1}{2}-\rho\right)\quad\text{and}\quad g''(\rho)=km\left(\frac{2}{\rho}+\frac{2}{\frac{1}{2}-\rho}\right).
$$

Thus, expanding $f$ around $\rho=\frac{1}{4}$ gives

$$
\begin{aligned}
f(\rho) &= \frac{\left(\sqrt{2\pi km}\right)^3}{16}\exp\left[g\left(\frac{1}{4}\right)+\frac{1}{2}\cdot g''\left(\frac{1}{4}\right)\left(\rho-\frac{1}{4}\right)^2\right] \\
&= \frac{\left(\sqrt{2\pi km}\right)^3}{16}4^{-km}\exp\left[\frac{16km}{2}\left(\rho-\frac{1}{4}\right)^2\right].
\end{aligned}
\tag{7.5.12}
$$

### 7.5.5. Putting things together

We now have all the pieces in place to give a sketch of the proof of Proposition 7.5.2. More precisely, we are going to prove the following:

$$
\limsup_{n\to\infty}\frac{\mathbb{E}[\left(Z_{\boldsymbol{\mu}}^{\mathrm{bal}}\right)^2]}{\mathbb{E}[Z_{\boldsymbol{\mu}}^{\mathrm{bal}}]^2}=\left[1-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right]^{-1/2}, \tag{7.5.13}
$$

where $Z_{\boldsymbol{\mu}}^{\mathrm{bal}}$ is the number of colourings that are balanced on every degree and whose colour pattern fits $\boldsymbol{\mu}$ and $\boldsymbol{\mu}$ is chosen such that (7.5.1) is satisfied. If we write $\boldsymbol{\mu}=(\mu(\vartheta))_{\vartheta\in\Theta}$, the total number of allowed colour patterns is equal to

$$
\binom{m}{(\mu_\vartheta)_\vartheta}.
$$

Furthermore, such patterns can be sampled simply by permuting some "canonical pattern" randomly. Analogously, in order to create a legal "bi-pattern" (two rows of patterns), we merely choose two permutations independently. Of course, the total number of pairs of patterns is $\binom{m}{(\mu_\vartheta)_\vartheta}^2$.

Let $\lambda(\rho)$ be the probability that a legal bi-pattern has overlap $\rho$. Proposition 7.5.3 yields that for $n \to \infty$ we have

$$\lambda(\rho) = \frac{1}{\sqrt{2\pi\xi}} \exp\left[-\frac{1}{2\xi}\left(\rho - \frac{1}{4}\right)^2 (km)^2\right], \tag{7.5.14}$$

where $\xi = \frac{km}{16}\left[\frac{k-1}{(2^{k-1}-1)^2} + 1\right]$. Further, let $\zeta(\rho)$ be the probability that two randomly chosen colourings have overlap $\rho$ in the configuration model. Then Proposition 7.5.4 yields that for $n \to \infty$ we have

$$\zeta(\rho) = \frac{1}{\sqrt{2\pi\chi}} \exp\left[-\frac{1}{2\chi}\left(\rho - \frac{1}{4}\right)^2 (km)^2\right], \tag{7.5.15}$$

where $\chi = \frac{km(d+1)}{16}$. Then the number of triples $(\mathcal{H}(n,m), \sigma, \tau)$ of hypergraphs $\mathcal{H}(n,m)$ and colourings $\sigma, \tau$ with overlap $\rho$ comes to

$$\Lambda(\rho) = 4^n \zeta(\rho) \binom{m}{(\mu_\vartheta)_\vartheta}^2 \lambda(\rho) f(\rho).$$

The first two factors account for the entropy (number of ways of choosing the assignments $\sigma, \tau$). The next two factors are the number of patterns as desired. The last factor is the number of ways of matching the vertex clones to the edges. By comparison, the number of pairs $(\mathcal{H}(n,m), \sigma)$ of hypergraphs and colourings $\sigma$ comes to

$$\mathbb{E}[Z_{\boldsymbol{\mu}}^{\mathrm{bal}}] \sim 2^n \binom{m}{(\mu_\vartheta)_\vartheta} \binom{km}{km/2}^{-1}.$$

Therefore, integrating over all "possible" $\rho$ gives

$$\frac{\mathbb{E}[\left(Z_{\boldsymbol{\mu}}^{\mathrm{bal}}\right)^2]}{\mathbb{E}[Z_{\boldsymbol{\mu}}^{\mathrm{bal}}]^2} \sim \sum_\rho \frac{\Lambda(\rho)\binom{km}{km/2}^2}{4^n\binom{m}{(\mu_\vartheta)_\vartheta}^2} \sim km \int_\rho \frac{\Lambda(\rho)\binom{km}{km/2}^2}{4^n\binom{m}{(\mu_\vartheta)_\vartheta}^2} d\rho = km \binom{km}{km/2}^2 \int_\rho \zeta(\rho)\lambda(\rho)f(\rho)d\rho. \tag{7.5.16}$$

By Stirling,

$$\binom{km}{km/2} \sim 2^{km}\sqrt{\frac{2}{\pi km}}. \tag{7.5.17}$$

Furthermore, inserting (7.5.14), (7.5.15) and (7.5.12) gives

$$\int_\rho \zeta(\rho)\lambda(\rho)f(\rho)d\rho$$
$$\sim \frac{\left(\sqrt{2\pi km}\right)^3}{16}4^{-km}\int \frac{1}{2\pi\sqrt{\xi\chi}}\exp\left[-\frac{(km)^2}{2}\left(\rho-\frac{1}{4}\right)^2\left(\frac{1}{\xi}+\frac{1}{\chi}-\frac{16}{km}\right)\right]d\rho$$
$$= \frac{\left(\sqrt{2\pi km}\right)^3}{16}4^{-km}\int \frac{1}{2\pi\sqrt{\xi\chi}}\exp\left[-\frac{(km)^2}{2}\left(\rho-\frac{1}{4}\right)^2\left(\frac{\xi+\chi-\frac{16\xi\chi}{km}}{\xi\chi}\right)\right]d\rho$$

Using the formula $\int_{-\infty}^{\infty}\exp\left[-a\,(x+b)^2\right]dx=\sqrt{\frac{\pi}{a}}$ for a Gaussian integral, this transforms to

$$\int_\rho \zeta(\rho)\lambda(\rho)f(\rho)d\rho$$
$$\sim \frac{\left(\sqrt{2\pi km}\right)^3}{16km\sqrt{2\pi}}4^{-km}\left[\xi+\chi-\frac{16\xi\chi}{km}\right]^{-\frac{1}{2}}$$
$$= \frac{\pi}{2}4^{-km}\left[\frac{k-1}{\left(2^{k-1}-1\right)^2}+1+(d+1)-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}-d-\frac{k-1}{\left(2^{k-1}-1\right)^2}-1\right]^{-\frac{1}{2}}$$
$$= \frac{\pi}{2}4^{-km}\left[1-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right]^{-\frac{1}{2}} \tag{7.5.18}$$

Plugging (7.5.18) and (7.5.17) into (7.5.16), we get

$$\frac{\mathbb{E}[\left(Z_{\boldsymbol{\mu}}^{\text{bal}}\right)^2]}{\mathbb{E}[Z_{\boldsymbol{\mu}}^{\text{bal}}]^2}\sim km4^{km}\frac{2}{\pi km}\frac{\pi}{2}4^{-km}\left[1-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right]^{-\frac{1}{2}}=\left[1-\frac{d(k-1)}{\left(2^{k-1}-1\right)^2}\right]^{-\frac{1}{2}}.$$

and thus we have proven (7.5.13).

# 8 Number of solutions in random graph $k$-colouring

This chapter contains the proof of Theorem 4.1.9 establishing the limiting distribution of the logarithm of the number of $k$-colourings of a random graph. The result is obtained up to the condensation threshold for large values of $k$ and in lower density regimes for all $k \geq 3$.

Large parts of this chapter are a verbatim copy or a close adaption of the content of the paper *On the number of solutions in random graph k-colouring* [Ras16b+] submitted to *Combinatorics, Probability and Computing*.

The first section of this chapter presents an outline of the proof of Theorem 4.1.9 and gives a short introduction to the proof ideas. In Section 8.2 the first moment of the number of solutions is explicitly calculated and further on the number of short cycles is determined in Section B.3. To apply small subgraph conditioning, the second moment of some auxiliary random variables is calculated very precisely in different density regimes. This is done in Section 8.3.

As the paper is a single-author paper, the question of the contribution of this thesis' author does not arise.

*Throughout the chapter we assume that $m = \lceil d'n/2 \rceil$, where $d'$ remains fixed as $n \to \infty$. We also require that $k \geq 3$.*

## 8.1. Outline of the proof

To determine the distribution of $\ln Z_k(\mathcal{G}(n,m))$, it will be necessary to control the size of the colour classes. To formalize this, we introduce the following notation. For a map $\sigma : [n] \to [k]$, we define

$$\rho(\sigma) = (\rho_1(\sigma), \ldots, \rho_k(\sigma)), \quad \text{where } \rho_i(\sigma) = |\sigma^{-1}(i)|/n \quad \text{for } i = 1 \ldots k.$$

Thus, $\rho(\sigma)$ is a probability distribution on $[k]$, to which we refer as the *colour density* of $\sigma$.

Let $\mathcal{A}_k(n)$ signify the set of all possible colour densities $\rho(\sigma)$ for $\sigma : [n] \to [k]$. Further, let $\mathcal{A}_k$ be the set of all probability distributions $\rho = (\rho_1, \ldots, \rho_k)$ on $[k]$, and let $\rho^\star = (1/k, \ldots, 1/k)$ signify the barycentre of $\mathcal{A}_k$.

In order to simplify the notation, for the rest of this chapter we assume that $\omega, \nu$ are odd natural numbers, formally we define $N = \{2i - 1 : i \in \mathbb{N}\}$ and let $\omega, \nu \in N$. We say that $\rho = (\rho_1, \ldots, \rho_k) \in$

$\mathcal{A}_k(n)$ is $(\omega, n)$-*balanced* if

$$\rho_i \in \left[\frac{1}{k} - \frac{\omega}{\sqrt{n}}\,,\; \frac{1}{k} + \frac{\omega}{\sqrt{n}}\right) \qquad \text{for all } i \in [k]$$

and let $\mathcal{A}_{k,\omega}(n)$ denote the set of all $(\omega, n)$-balanced $\rho \in \mathcal{A}_k(n)$. As we will see, in order to prove statements about the number $Z_k$ of all solutions, it suffices to consider solutions $\sigma$ with $\rho(\sigma) \in \mathcal{A}_{k,\omega}(n)$. We let $Z_{k,\omega}(G)$ signify the number of $(\omega, n)$-*balanced* $k$-*colourings* of a graph $G$ on $[n]$, i.e. $k$-colourings $\sigma$ such that $\rho(\sigma) \in \mathcal{A}_{k,\omega}(n)$.

Since verifying the required properties to apply small subgraph conditioning directly for the random variable $Z_\omega$ is very intricate, we break $Z_\omega$ down into smaller contributions, for which we determine the first and second moment in the following sections.

To this aim, we decompose the set $\mathcal{A}_{k,\omega}(n)$ into smaller sets. We define

$$S_{k,\omega,\nu} = \left\{ s \in \mathbb{Z}^k : \|s\|_1 = 2i, i \in \mathbb{N}, i \leq \frac{\omega\nu - 1}{2} \right\}. \tag{8.1.1}$$

$S_{k,\omega,\nu}$ contains vectors that we use as centres of disjoint 'balls' to partition the set $\mathcal{A}_{k,\omega}(n)$: For $s = (s_1, ..., s_k) \in S_{k,\omega,\nu}$, we let $\rho^{k,\omega,\nu,s} \in \mathbb{R}^k$ be the vector with components

$$\rho_i^{k,\omega,\nu,s} = \frac{1}{k} + \frac{s_i}{\nu\sqrt{n}}. \tag{8.1.2}$$

Further, we let $\mathcal{A}^s_{k,\omega,\nu}(n)$ be the set of all colour densities $\rho \in \mathcal{A}_{k,\omega}(n)$ such that

$$\rho_i \in \left[\rho_i^{k,\omega,\nu,s} - \frac{1}{\nu\sqrt{n}}\,,\; \rho_i^{k,\omega,\nu,s} + \frac{1}{\nu\sqrt{n}}\right).$$

For a graph $G$, we denote by $Z^s_{k,\omega,\nu}(G)$ the number of 2-colourings $\sigma$ such that $\rho(\sigma) \in \mathcal{A}^s_{k,\omega,\nu}(n)$. For each fixed $\nu$, we have $Z_{k,\omega} = \sum_{s \in S_{k,\omega,\nu}} Z^s_{k,\omega,\nu}$ and our strategy is to apply small subgraph conditioning to the random variables $Z^s_{k,\omega,\nu}$ rather than directly to $Z_k$. But first, we will calculate the first moments of $Z_k$ and $Z_{k,\omega}$ in Section 8.2 to obtain the following.

**Proposition 8.1.1.** *Fix an integer $k \geq 3$ and a number $d' \in (0, \infty)$. Let $\omega > 0$. Then*

$$\mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right] = \Theta(k^n (1 - 1/k)^m) \quad \textit{and} \quad \lim_{\omega\to\infty} \liminf_{n\to\infty} \frac{\mathbb{E}\left[Z_{k,\omega}(\mathcal{G}(n,m))\right]}{\mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right]} = 1.$$

The key observation the proof is based on is that the fluctuations of $Z_k(\mathcal{G}(n,m))$ can be attributed to fluctuations in the number of cycles of a bounded length. Hence, for an integer $l \geq 2$ we let $C_{l,n}$

denote the number of cycles of length exactly $l$ in $\mathcal{G}(n,m)$. Let

$$\lambda_l = \frac{d^l}{2l} \quad \text{and} \quad \delta_l = \frac{(-1)^l}{(k-1)^{l-1}}. \tag{8.1.3}$$

The following fact shows that $C_{2,n},\ldots$ are asymptotically independent Poisson variables (e.g. [Bol01, Theorem 5.16]):

**Fact 8.1.2.** *If $c_2,\ldots,c_L$ are non-negative integers, then*

$$\lim_{n\to\infty} \mathbb{P}\left[\forall 2 \le l \le L : C_{l,n} = c_l\right] = \prod_{l=2}^{L} \mathbb{P}\left[\mathrm{Po}(\lambda_l) = c_l\right].$$

In Section B.3 the impact of the cycle counts $C_{l,n}$ on the first moment of $Z^s_{\omega,\nu}(\mathcal{G}(n,m))$ is investigated. As this was already done in [BCOE14+], we carry it out in the present work only for the sake of completeness. The result is the following:

**Proposition 8.1.3.** *Assume that $k \ge 3$ and $d' \in (0,\infty)$. Then*

$$\sum_{l=2}^{\infty} \lambda_l \delta_l^2 < \infty.$$

*Moreover, let $\omega,\nu \in N$ and $c_2,\ldots,c_L$ be non-negative integers. Then*

$$\frac{\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\forall 2 \le l \le L : C_{l,n} = c_l\right]}{\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]} \sim \prod_{l=2}^{L} \left[1+\delta_l\right]^{c_l} \exp\left[-\delta_l\lambda_l\right]. \tag{8.1.4}$$

Additionally, to apply small subgraph conditioning, we have to determine the second moment of $Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))$ very precisely. This step constitutes the main technical work in this chapter. We consider two regimes of $d'$ and $k$ separately. In the simpler case, based on the second moment argument from [AN05], we obtain the following result.

**Proposition 8.1.4.** *Assume that $k \ge 3$ and $d' < 2(k-1)\ln(k-1)$. Then*

$$\frac{\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2\right]}{\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]^2} \sim \exp\left[\sum_{l\ge 2} \lambda_l \delta_l^2\right].$$

The second regime of $d'$ and $k$ is that $k \ge k_0$ for a certain constant $k_0 \ge 3$ and $d' < d_{\mathrm{cond}}$ (with $d_{\mathrm{cond}} = d_{\mathrm{crit}}$ the number defined in (2.5.3)). In this case, we replace $Z^s_{k,\omega,\nu}$ by the slightly tweaked random variable $\widetilde{Z}^s_{k,\omega,\nu}$ used in the second moment arguments from [BCOHRV16, COV13].

**Proposition 8.1.5.** *There is a constant $k_0 \geq 3$ such that the following is true. Assume that $k \geq k_0$ and $2(k-1)\ln(k-1) \leq d' < d_{\text{cond}}$. Then for each $\omega, \nu \in N$ and $s \in S_{k,\omega,\nu}$ there exists an integer-valued random variable $0 \leq \widetilde{Z}^s_{k,\omega,\nu} \leq Z^s_{k,\omega,\nu}$ such that*

$$\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right] \sim \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right] \qquad \textit{and} \tag{8.1.5}$$

$$\frac{\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2\right]}{\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]^2} \leq (1+o(1))\exp\left[\sum_{l\geq 2}\lambda_l\delta_l^2\right].$$

The proofs of Propositions 8.1.4 and 8.1.5 appear at the end of Section 8.3. In order to apply small subgraph conditioning to the random variable $\widetilde{Z}^s_{k,\omega,\nu}$, we need to investigate the impact of $C_{l,n}$ on the first moment of $\widetilde{Z}^s_{k,\omega,\nu}$. Thus, we need a similar result as Proposition 8.1.3 for $\widetilde{Z}^s_{k,\omega,\nu}$. Fortunately, instead of having to reiterate the proof of Proposition 8.1.3, we obtain the following by combining Proposition 8.1.3 with (8.1.5):

**Corollary 8.1.6.** *Let $c_2, \ldots, c_L$ be non-negative integers. With the assumptions and notation of Proposition 8.1.5 we have*

$$\frac{\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\forall 2\leq l\leq L: C_{l,n}=c_l\right]}{\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]} \sim \prod_{l=2}^{L}\left[1+\delta_l\right]^{c_l}\exp\left[-\delta_l\lambda_l\right].$$

As the proof is nearly identical to the analogous proof in [BCOE14+], we defer it to Appendix B.

The aim is now to derive Theorem 4.1.9 from Propositions 8.1.1-8.1.4. The key observation is that the variance of the random variables $Z^s_{k,\omega,\nu}$ is affected by the presence of cycles of bounded length and that this is the only significant influence. As a consequence, conditioning on the small cycle counts up to some preselected length reduces the variance of $Z^s_{k,\omega,\nu}$. What is maybe surprising is that conditioning on the number of enough small cycles reduces the variance to any desired fraction of $\mathbb{E}[Z^s_{k,\omega,\nu}]^2$.

As done in [COW16+, Ras16a+], the arguments we use are similar to the small subgraph conditioning from [Jan95, RW94]. But we do not refer to any technical statements from [Jan95, RW94] directly because instead of working only with the random variable $Z_k$ we need to control all $Z^s_{k,\omega,\nu}$ for fixed $\omega, \nu \in N$ simultaneously. In fact, ultimately we have to take $\nu \to \infty$ and $\omega \to \infty$ as well. Our line of argument follows the path beaten in [COW16+, Ras16a+] and the following three lemmas are nearly identical to the ones derived there.

For $L > 2$, let $\mathcal{F}_L = \mathcal{F}_{L,n}(d,k)$ be the $\sigma$-algebra generated by the random variables $C_{l,n}$ with $2 \leq l \leq L$. The set of all graphs can be divided into groups according to the small cycle counts: For each $L \geq 2$, the decomposition of the variance of $Z^s_{k,\omega,\nu}$ yields

$$\mathrm{Var}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right] = \mathrm{Var}\left[\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{F}_L\right]\right] + \mathbb{E}\left[\mathrm{Var}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{F}_L\right]\right],$$

meaning that the variance can be written as the variance of the group mean plus the expected value of the variance within a group. The term $\mathrm{Var}\left[\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{F}_L\right]\right]$ accounts for the amount of variance induced by the fluctuations of the number of cycles of length at most $L$. The strategy when using small subgraph conditioning is to bound the second summand, which is the expected conditional variance

$$\mathbb{E}\left[\mathrm{Var}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{F}_L\right]\right] = \mathbb{E}\left[\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{F}_L\right]^2\right].$$

In the following lemma we show that in fact in the limit of large $L$ and $n$ this quantity is negligible. This implies that conditioned on the number of short cycles the variance vanishes and thus the limiting distribution of $\ln Z^s_{k,\omega,\nu}$ is just the limit of $\ln \mathbb{E}\left[Z^s_{k,\omega,\nu}|\mathcal{F}_L\right]$ as $n, L \to \infty$. This limit is determined by the joint distribution of the number of short cycles.

**Lemma 8.1.7.** *Let $k \geq 3$ and $d' \in (0,\infty)$. For any $\omega, \nu \in N$ and $s \in S_{k,\omega,\nu}$, we have*

$$\limsup_{L\to\infty} \limsup_{n\to\infty} \mathbb{E}\left[\frac{\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{F}_L\right]^2}{\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]^2}\right] = 0.$$

*Proof.* Fix $\omega, \nu \in N$ and set $Z_s = Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))$. Using Fact 8.1.2 and equation (8.1.3) from Proposition 8.1.3 we can choose for any $\varepsilon > 0$ a constant $B = B(\varepsilon)$ and $L \geq L_0(\varepsilon)$ large enough such that for each large enough $n \geq n_0(\varepsilon, B, L)$ we have for any $s \in S_{k,\omega,\nu}$:

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right] &\geq \sum_{c_1,\ldots,c_L \leq B} \mathbb{E}\left[Z_s|\forall 2 \leq l \leq L : C_{l,n} = c_l\right]^2 \mathbb{P}\left[\forall 2 \leq l \leq L : C_{l,n} = c_l\right] \\
&\geq \exp\left[-\varepsilon\right] \mathbb{E}\left[Z_s\right]^2 \sum_{c_1,\ldots,c_L \leq B} \prod_{l=2}^{L} \left[(1+\delta_l)^{c_l} \exp\left[-\lambda_l \delta_l\right]\right]^2 \mathbb{P}\left[\mathrm{Po}(\lambda_l) = c_l\right] \\
&= \exp\left[-\varepsilon\right] \mathbb{E}\left[Z_s\right]^2 \sum_{c_1,\ldots,c_L \leq B} \prod_{l=2}^{L} \frac{\left[(1+\delta_l)^2 \lambda_l\right]^{c_l}}{c_l! \exp\left[2\lambda_l \delta_l + \lambda_l\right]} \\
&\geq \mathbb{E}\left[Z_s\right]^2 \exp\left[-2\varepsilon + \sum_{l=2}^{L} \delta_l^2 \lambda_l\right]. \qquad (8.1.6)
\end{aligned}$$

The tower property for conditional expectations and the standard formula for the decomposition of the variance yields

$$\mathbb{E}\left[Z_s^2\right] = \mathbb{E}\left[\mathbb{E}\left[Z_s^2|\mathcal{F}_L\right]\right] = \mathbb{E}\left[\mathbb{E}\left[Z_s^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right] + \mathbb{E}\left[\mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right]$$

and thus, using (8.1.6) we have

$$\frac{\mathbb{E}\left[\mathbb{E}\left[Z_s^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right]}{\mathbb{E}\left[Z_s\right]^2} \le \frac{\mathbb{E}\left[Z_s^2\right]}{\mathbb{E}\left[Z_s\right]^2} - \exp\left[-2\varepsilon + \sum_{l=2}^{L} \delta_l^2 \lambda_l\right]. \tag{8.1.7}$$

Finally, the estimate $\exp[-x] \ge 1 - x$ for $|x| < 1/8$ combined with (8.1.7) and Proposition 8.1.4 implies that for large enough $\nu, n, L$ and each $s \in S_{k,\omega,\nu}$ we have

$$\frac{\mathbb{E}\left[\mathbb{E}\left[Z_s^2|\mathcal{F}_L\right] - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]^2\right]}{\mathbb{E}\left[Z_s\right]^2} \le 2\varepsilon \exp\left[\sum_{l=2}^{\infty} \delta_l^2 \lambda_l\right].$$

As this holds for any $\varepsilon > 0$ and by equation (8.1.3) the expression $\exp\left[\sum_{l=2}^{\infty} \delta_l^2 \lambda_l\right]$ is bounded, the proof of the lemma is completed by first taking $n \to \infty$ and then $L \to \infty$. □

**Lemma 8.1.8.** *For any $\alpha > 0$, we have*

$$\limsup_{L\to\infty} \limsup_{n\to\infty} \mathbb{P}\left[\left|Z_k(\mathcal{G}(n,m)) - \mathbb{E}\left[Z_k(\mathcal{G}(n,m))|\mathcal{F}_L\right]\right| > \alpha \mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right]\right] = 0.$$

*Proof.* To unclutter the notation, we set $Z_k = Z_k(\mathcal{G}(n,m))$ and $Z_{k,\omega} = Z_{k,\omega}(\mathcal{G}(n,m))$. First we observe that Proposition 8.1.1 implies that for any $\alpha > 0$ we can choose $\omega \in N$ large enough such that

$$\liminf_{n\to\infty} \mathbb{E}\left[Z_{k,\omega}\right] > (1-\alpha^2)\mathbb{E}\left[Z_k\right]. \tag{8.1.8}$$

We let $\nu \in N$. To prove the statement, we need to get a handle on the cases where the variables $Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))$ deviate strongly from their conditional expectation $\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{F}_L\right]$. We let $Z_s = Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))$ and define

$$X_s = \left|Z_s - \mathbb{E}\left[Z_s|\mathcal{F}_L\right]\right| \cdot \mathbf{1}_{\{|Z_s - \mathbb{E}[Z_s|\mathcal{F}_L]| > \alpha \mathbb{E}[Z_s]\}}$$

and $X = \sum_{s \in S_{k,\omega,\nu}} X_s$. Then these definitions directly yield

$$\mathbb{P}\left[X < \alpha \mathbb{E}\left[Z_{k,\omega}\right]\right] \le \mathbb{P}\left[\left|Z_{k,\omega} - \mathbb{E}\left[Z_{k,\omega}|\mathcal{F}_L\right]\right| < 2\alpha \mathbb{E}\left[Z_{k,\omega}\right]\right]. \tag{8.1.9}$$

By the definition of the $X_s$'s and Chebyshev's inequality it is true for every $s$ that

$$\mathbb{E}\left[X_s|\mathcal{F}_L\right] \leq \sum_{j\geq 0} 2^{j+1}\alpha\mathbb{E}\left[Z_s\right]\mathbb{P}\left[|Z_s-\mathbb{E}\left[Z_s|\mathcal{F}_L\right]|>2^j\alpha\mathbb{E}\left[Z_s\right]\right] \leq \frac{4\mathrm{Var}\left[Z_s|\mathcal{F}_L\right]}{\alpha\mathbb{E}\left[Z_s\right]}.$$

Hence, using that with Proposition 8.1.1 there is a number $\beta=\beta(\alpha,\omega)$ such that $\mathbb{E}\left[Z_s\right]/\mathbb{E}\left[Z_k\right] \leq \beta/(|S_{k,\omega,\nu}|)$ for all $s\in S_{k,\omega,\nu}$ and $n$ large enough, we have

$$\mathbb{E}\left[X|\mathcal{F}_L\right] \leq \sum_{s\in S_{k,\omega,\nu}} \frac{4\mathrm{Var}\left[Z_s|\mathcal{F}_L\right]}{\alpha\mathbb{E}\left[Z_s\right]} \leq \frac{4\beta\mathbb{E}\left[Z_k\right]}{\alpha|S_{k,\omega,\nu}|}\sum_{s\in S_{k,\omega,\nu}} \frac{\mathrm{Var}\left[Z_s|\mathcal{F}_L\right]}{\mathbb{E}\left[Z_s\right]^2}.$$

Taking expectations, choosing $\varepsilon=\varepsilon(\alpha,\beta,\omega)$ small enough and applying Lemma 8.1.7, we obtain

$$\mathbb{E}\left[X\right]=\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{F}_L\right]\right] \leq \frac{4\beta\mathbb{E}\left[Z_k\right]}{\alpha|S_{k,\omega,\nu}|}\sum_{s\in S_{k,\omega,\nu}} \frac{\mathbb{E}\left[\mathrm{Var}\left[Z_s|\mathcal{F}_L\right]\right]}{\mathbb{E}\left[Z_s\right]^2} \leq \frac{4\beta\varepsilon\mathbb{E}\left[Z_k\right]}{\alpha} \leq \alpha^2\mathbb{E}\left[Z_k\right]. \quad (8.1.10)$$

Using (8.1.9), Markov's inequality, (8.1.10) and (8.1.8), it follows that

$$\mathbb{P}\left[|Z_{k,\omega}-\mathbb{E}\left[Z_{k,\omega}|\mathcal{F}_L\right]|<2\alpha\mathbb{E}\left[Z_{k,\omega}\right]\right] \geq 1-2\alpha. \quad (8.1.11)$$

Finally, the triangle inequality combined with Markov's inequality and equations (8.1.8) and (8.1.11) yields

$$\begin{aligned}&\mathbb{P}\left[|Z_k-\mathbb{E}\left[Z_k|\mathcal{F}_L\right]|>\alpha\mathbb{E}\left[Z_k\right]\right]\\ &\quad\leq \mathbb{P}\left[|Z_k-Z_{k,\omega}|+|Z_{k,\omega}-\mathbb{E}\left[Z_{k,\omega}|\mathcal{F}_L\right]|+|\mathbb{E}\left[Z_{k,\omega}|\mathcal{F}_L\right]-\mathbb{E}\left[Z_k|\mathcal{F}_L\right]|>\alpha\mathbb{E}\left[Z_k\right]\right]\\ &\quad\leq 3\alpha+\alpha/3+3\alpha<7\alpha,\end{aligned}$$

which proves the statement. □

**Lemma 8.1.9.** *Let*

$$U_L=\sum_{l=2}^{L} C_{l,n}\ln(1+\delta_l)-\lambda_l\delta_l. \quad (8.1.12)$$

*Then* $\limsup_{L\to\infty}\limsup_{n\to\infty}\mathbb{E}\left[|U_L|\right]<\infty$ *and further for any* $\varepsilon>0$ *we have*

$$\limsup_{L\to\infty}\limsup_{n\to\infty}\mathbb{P}\left[|\ln\mathbb{E}\left[Z_k(\mathcal{G}(n,m))|\mathcal{F}_L\right]-\ln\mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right]-U_L|>\varepsilon\right]=0 \quad (8.1.13)$$

*Proof.* In a first step we show that $\mathbb{E}\left[|U_L|\right]$ is uniformly bounded. As $x-x^2\leq\ln(1+x)\leq x$ for

$|x| \leq 1/8$ we have for every $l \leq L$:

$$\mathbb{E}\left[|C_{l,n}\ln(1+\delta_l) - \lambda_l\delta_l|\right] \leq \delta_l \mathbb{E}\left[|C_{l,n} - \lambda_l|\right] + \delta_l^2 \mathbb{E}\left[C_{l,n}\right].$$

Therefore, Fact 8.1.2 implies that

$$\mathbb{E}\left[|U_L|\right] \leq \sum_{l=2}^{L} \delta_l \sqrt{\lambda_l} + \delta_l^2 \lambda_l. \tag{8.1.14}$$

Proposition 8.1.3 ensures that $\sum_l \delta_l^2 \lambda_l < \infty$. Furthermore, as $d' \leq (2k-1)\ln k$, we have $\sum_l \delta_l \sqrt{\lambda_l} \leq \sum_l k^l 2^{-(k-1)l/2} < \infty$ and thus (8.1.14) shows that $\mathbb{E}\left[|U_L|\right]$ is uniformly bounded.

To prove (8.1.13), for given $n$ and a constant $B > 0$ we let $\mathcal{C}_B$ be the event that $C_{l,n} < B$ for all $l \leq L$. Referring to Fact 8.1.2, we can find for each $L, \varepsilon > 0$ a $B > 0$ such that

$$\mathbb{P}\left[\mathcal{C}_B\right] > 1 - \varepsilon. \tag{8.1.15}$$

To simplify the notation we set $Z_k = Z_k(\mathcal{G}(n,m))$ and $Z_{k,\omega} = Z_{k,\omega}(\mathcal{G}(n,m))$. By Proposition 8.1.1 we can choose for any $\alpha > 0$ a $\omega > 0$ large enough such that $\mathbb{E}\left[Z_{k,\omega}\right] > (1-\alpha)\mathbb{E}\left[Z_k\right]$ for large enough $n$. Then Propositions 8.1.1 and 8.1.3 combined with Fact 8.1.2 imply that for any $c_1, ..., c_L \leq B$ and small enough $\alpha = \alpha(\varepsilon, L, B)$ we have for $n$ large enough:

$$\begin{aligned}\mathbb{E}\left[Z_k | \forall 2 \leq l \leq L : C_{l,n} = c_l\right] &\geq \mathbb{E}\left[Z_{k,\omega} | \forall 2 \leq l \leq L : C_{l,n} = c_l\right] \\ &\geq \exp\left[-\varepsilon\right] \mathbb{E}\left[Z_k\right] \prod_{l=2}^{L} (1+\delta_l)^{c_l} \exp\left[-\delta_l \lambda_l\right].\end{aligned} \tag{8.1.16}$$

On the other hand, for $\alpha$ sufficiently small and large enough $n$ we have

$$\begin{aligned}&\mathbb{E}\left[Z_k | \forall 2 \leq l \leq L : C_{l,n} = c_l\right] \\ &\quad = \mathbb{E}\left[Z_k - Z_{k,\omega} | \forall 2 \leq l \leq L : C_{l,n} = c_l\right] + \mathbb{E}\left[Z_{k,\omega} | \forall 2 \leq l \leq L : C_{l,n} = c_l\right] \\ &\quad \leq \frac{2\alpha \mathbb{E}\left[Z_k\right]}{\prod_{l=2}^{L} \mathbb{P}\left[\mathrm{Po}(\lambda_l) = c_l\right]} + \mathbb{E}\left[Z_{k,\omega} | \forall 2 \leq l \leq L : C_{l,n} = c_l\right] \\ &\quad \leq \exp\left[\varepsilon\right] \mathbb{E}\left[Z_k\right] \prod_{l=2}^{L} (1+\delta_l)^{c_l} \exp\left[-\delta_l \lambda_l\right]\end{aligned} \tag{8.1.17}$$

Thus, the proof of (8.1.13) is completed by combining (8.1.15), (8.1.16), (8.1.17) and taking logarithms. □

*Proof of Theorem 4.1.9.* For $L \geq 2$, we define

$$W_L = \sum_{l=2}^{L} X_l \ln(1+\delta_l) - \lambda_l \delta_l \quad \text{and} \quad W' = \sum_{l \geq 2} X_l \ln(1+\delta_l) - \lambda_l \delta_l.$$

Then Fact 8.1.2 implies that for each $L$ the random variables $U_L$ defined in (8.1.12) converge in distribution to $W_L$ as $n \to \infty$. Furthermore, because $\sum_l \delta_l \sqrt{\lambda_l}, \sum_l \delta_l^2 \lambda_l < \infty$, the martingale convergence theorem implies that $W'$ is well-defined and that the $W_L$ converge to $W'$ almost surely as $L \to \infty$. Hence, from Lemmas 8.1.9 and 8.1.8 it follows that $\ln Z_k(\mathcal{G}(n,m)) - \ln \mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right]$ converges to $W'$ in distribution, meaning that for any $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\ln Z_k(\mathcal{G}(n,m)) - \ln \mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right] - W'\right| > \varepsilon\right] = 0. \tag{8.1.18}$$

To derive Theorem 4.1.9 from (8.1.18), we denote by $S$ the event that $\mathcal{G}(n,m)$ consists of $m$ distinct edges, or, equivalently, that no cycles of length 2 exist in $\mathcal{G}(n,m)$. Given that $S$ occurs, $\mathcal{G}(n,m)$ is identical to $G(n,m)$ and $W'$ is identical to $W$. Furthermore, Fact 2.1.1 implies that $\mathbb{P}[S] = \Omega(1)$. Consequently, (8.1.18) yields

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mathbb{P}\left[\left|\ln Z_k(\mathcal{G}(n,m)) - \ln \mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right] - W'\right| > \varepsilon \middle| S\right] \\ &= \lim_{n\to\infty} \mathbb{P}\left[\left|\ln Z_k(G(n,m)) - \ln \mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right] - W\right| > \varepsilon\right]. \end{aligned} \tag{8.1.19}$$

As Lemma 8.2.1 implies that $\mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right], \mathbb{E}\left[Z_k(G(n,m)\right] = \Theta\left(k^n \left(1-1/k\right)^m\right)$, we have $\mathbb{E}\left[Z_k(\mathcal{G}(n,m))\right] = \Theta(\mathbb{E}\left[Z_k(G(n,m)\right])$ and with (8.1.19) it follows that

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\ln Z_k(G(n,m)) - \ln \mathbb{E}\left[Z_k(G(n,m)))\right] - W\right| > \varepsilon\right] = 0,$$

which proves Theorem 4.1.9. □

## 8.2. The first moment

The aim in this section is to prove Proposition 8.1.1. The calculations that have to be done follow the path beaten in [AN05, COV13, KPGW10, Ras16a+] and are in fact very similar to [BCOE14+]. Thus, most of the proofs are deferred to Section B.1. Furthermore, at the end of the section we state a result that we need for Proposition 8.1.4.

Let $Z_{k,\rho}(G)$ be the number of $k$-colourings of the graph $G$ with colour density $\rho$. Let $\rho^\star$ be a $k$-dimensional vector with all entries set to $1/k$. We define

$$f_1 : \rho \in \mathcal{A}_k \mapsto \mathcal{H}(\rho) + \frac{d}{2} \ln\left(1 - \sum_{i=1}^{k} \rho_i^2\right).$$

In order to determine the expectation of $Z_{k,\rho}$, we have to analyse the function $f_1(\rho)$. The following lemma was already obtained in [BCOE14+] and its proof can be found in Section B.1.

**Lemma 8.2.1.** *Let $k \geq 3$ and $d' \in (0, \infty)$. Then there exist numbers $C_1 = C_1(k, d), C_2 = C_2(k, d) > 0$ such that for any $\rho \in \mathcal{A}_k(n)$ we have*

$$C_1 n^{\frac{1-k}{2}} \exp\left[n f_1(\rho)\right] \leq \mathbb{E}\left[Z_{k,\rho}(\mathcal{G}(n, m))\right] \leq C_2 \exp\left[n f_1(\rho)\right]. \tag{8.2.1}$$

*Moreover, if $\|\rho - \rho^\star\|_2 = o(1)$ and $d = 2m/n$, then*

$$\mathbb{E}\left[Z_{k,\rho}(\mathcal{G}(n, m))\right] \sim (2\pi n)^{\frac{1-k}{2}} k^{k/2} \exp\left[d/2 + n f_1(\rho)\right]. \tag{8.2.2}$$

We can now state the expectation of $Z_k$. The proof will be carried out in detail in Section B.1.

**Corollary 8.2.2.** *For any $k \geq 3, d' \in (0, \infty)$ and $d = 2m/n$, we have*

$$\mathbb{E}\left[Z_k(\mathcal{G}(n, m))\right] \sim \exp\left[d/2 + n f_1\left(\rho^\star\right)\right] \left(1 + \frac{d}{k-1}\right)^{-\frac{k-1}{2}}.$$

*Proof of Proposition 8.1.1.* The first assertion is immediate from Corollary 8.2.2. Moreover, the second assertion follows from Corollary 8.2.2 and the second part of Lemma 8.2.1. □

Finally, as our approach requires the analysis of the random variables $Z^s_{k,\omega,\nu}(\mathcal{G}(n, m))$, we derive an expression for $\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n, m))\right]$ that we will need to prove Proposition 8.1.4.

**Lemma 8.2.3.** *Let $k \geq 3, \omega, \nu \in N, d' \in (0, \infty)$ and $d = 2m/n$. For $s \in S_{k,\omega,\nu}$ and $\rho^{k,\omega,\nu,s}$ as defined in (8.1.2), we have*

$$\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n, m))\right] \sim_\nu |\mathcal{A}^s_{k,\omega,\nu}(n)| \, (2\pi n)^{\frac{1-k}{2}} k^{k/2} \exp\left[d/2 + n f_1(\rho^{k,\omega,\nu,s})\right].$$

*Proof.* Using a Taylor expansion of $f_1(\rho)$ around $\rho = \rho^{k,\omega,\nu,s}$, we get

$$f_1(\rho) = f_1(\rho^{k,\omega,\nu,s}) + \Theta\left(\frac{\omega}{\sqrt{n}}\right) \|\rho - \rho^{k,\omega,\nu,s}\|_1 + \Theta\left(\|\rho - \rho^{k,\omega,\nu,s}\|_2^2\right). \tag{8.2.3}$$

As $\|\rho - \rho^{k,\omega,\nu,s}\|_1 = O\left(\frac{1}{\nu\sqrt{n}}\right)$ for $\rho \in \mathcal{A}^s_{k,\omega,\nu}(n)$ and $\|\rho - \rho^{k,\omega,\nu,s}\|_2^2 = O\left(\frac{1}{\nu^2 n}\right)$, we conclude that $f_1(\rho) = f_1(\rho^{k,\omega,\nu,s}) + O\left(\frac{\omega}{\nu n}\right)$ and as this is independent of $\rho$, the assertion follows by inserting (8.2.3) in (8.2.2) and multiplying by $|\mathcal{A}^s_{k,\omega,\nu}(n)|$. □

## 8.3. The second moment

The aim of this section is to prove Proposition 8.1.4, which constitutes the main technical contribution of this work and Proposition 8.1.5, which is done in the last subsection and is based on and an enhancement of results derived in [AN05]. The crucial points in our analysis are that, similar to [BCOE14+, Ras16a+], we need an asymptotically tight expression for the second moment and instead of confining ourselves to the case of colourings whose colour densities are $(O(1), n)$-balanced, as done in most of prior work [AN05, BCOHRV16, COV13, KPGW10], we need to deal with $(\omega, n)$-balanced colour densities for a diverging function $\omega = \omega(n) \to \infty$. However, our work has to extend the calculations from [BCOE14+] following the example of [Ras16a+], because we aim for a statement about the whole distribution of $\ln Z_k(G(n,m))$. Our line of argument follows that of [Ras16a+], where analogue statements are proven for the problem of hypergraph 2-colouring.

### 8.3.1. Classifying the overlap

To standardise the notation, we define the *overlap matrix* $\rho(\sigma,\tau) = (\rho_{ij}(\sigma,\tau))_{i,j\in[k]}$ for two colour assignments $\sigma, \tau : [n] \to [k]$ as the doubly stochastic $k \times k$-matrix with entries

$$\rho_{ij}(\sigma,\tau) = \frac{1}{n} \cdot |\sigma^{-1}(i) \cap \tau^{-1}(j)|.$$

We let $\mathcal{B}_k(n)$ denote the set of all overlap matrices and $\mathcal{B}_k$ denote the set of all probability measures $\rho = (\rho_{ij})_{i,j\in[k]}$ on $[k]\times[k]$. Moreover, we let $\bar\rho$ signify the $k \times k$-matrix with all entries equal to $k^{-2}$, the barycentre of $\mathcal{B}_k$. For a $k \times k$-matrix $\rho = (\rho_{ij})$, we introduce the shorthands

$$\rho_{i\star} = \sum_{j=1}^{k} \rho_{ij}, \qquad \rho_{\cdot\star} = (\rho_{i\star})_{i\in[k]}, \qquad\qquad \rho_{\star j} = \sum_{i=1}^{k} \rho_{ij}, \qquad \rho_{\star\cdot} = (\rho_{\star i})_{i\in[k]}.$$

With the notation from Section 8.1, we observe that for any $\sigma,\tau : [n] \to [k]$ we have $\rho_{\cdot\star}, \rho_{\star\cdot} \in \mathcal{A}_k(n)$. We introduce the set

$$\mathcal{B}_{k,\omega}(n) = \left\{\rho \in \mathcal{B}_k(n) : \rho_{i\star}, \rho_{\star i} \in \left[\frac{1}{k} - \frac{w}{\sqrt{n}}, \frac{1}{k} + \frac{w}{\sqrt{n}}\right) \text{ for all } i \in [k]\right\},$$

which corresponds to $\mathcal{A}_{k,\omega}(n)$ insofar as for $\rho \in \mathcal{B}_{k,\omega}(n)$ we have $\rho_{i\star}, \rho_{\star i} \in \mathcal{A}_{k,\omega}(n)$ for all $i \in [k]$. We remember $S_{k,\omega,\nu}$ from (8.1.1). Then for $s \in S_{k,\omega,\nu}$ we define

$$\mathcal{B}^s_{k,\omega,\nu}(n) = \left\{\rho \in \mathcal{B}_{k,\omega}(n) : \rho_{i\star}, \rho_{\star i} \in \left[\rho_i^{k,\omega,\nu,s} - \frac{1}{\nu\sqrt{n}}, \rho_i^{k,\omega,\nu,s} + \frac{1}{\nu\sqrt{n}}\right) \text{ for all } i \in [k]\right\}.$$

Thus, for any fixed $\nu$, $\mathcal{B}_{k,\omega}(n)$ is a disjoint union of all $\mathcal{B}^s_{k,\omega,\nu}(n)$ for $s \in S_{k,\omega,\nu}$. For a given graph $G$ on $[n]$, we let $Z^{(2)}_{k,\rho}(G)$ be the number of pairs $(\sigma,\tau)$ of $k$-colourings of $G$ whose overlap is $\rho$. By the linearity of expectation,

$$\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2\right] = \sum_{\rho\in\mathcal{B}^s_{k,\omega,\nu}(n)} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right]. \tag{8.3.1}$$

To proceed calculating this quantity, we first need the following elementary estimates whose proofs can be found in Section B.2.

**Fact 8.3.1.** *For any $k \geq 3$, $d' \in (0,\infty)$ and $d = 2m/n$, the following estimates are true.*

1. *Let $\rho \in \mathcal{B}_k(n)$. Then*

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \sim \frac{\sqrt{2\pi}n^{\frac{1-k^2}{2}}}{\prod_{i,j=1}^k \sqrt{2\pi\rho_{ij}}} \exp\left[d/2 + n\mathcal{H}(\rho) + m\ln(1 - \|\rho_{\cdot\star}\|_2^2 - \|\rho_{\star\cdot}\|_2^2 + \|\rho\|_2^2)\right]. \tag{8.3.2}$$

2. *For any $\rho \in \mathcal{B}_k(n)$ with $\|\rho - \bar\rho\|_2^2 = o(1)$, we have*

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \sim k^{k^2}(2\pi n)^{\frac{1-k^2}{2}} \exp\left[d/2 + n\mathcal{H}(\rho) + m\ln(1 - \|\rho_{\cdot\star}\|_2^2 - \|\rho_{\star\cdot}\|_2^2 + \|\rho\|_2^2)\right]. \tag{8.3.3}$$

To simplify the notation, we introduce the function $f_2 : \mathcal{B}_k \to \mathbb{R}$ defined as

$$f_2(\rho) = \mathcal{H}(\rho) + \frac{d}{2}\ln(1 - \|\rho_{\cdot\star}\|_2^2 - \|\rho_{\star\cdot}\|_2^2 + \|\rho\|_2^2). \tag{8.3.4}$$

A direct consequence of Fact 8.3.1 that will be used in the sequel is that for every $\rho \in \mathcal{B}_k(n)$ we have

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] = \exp\left[nf_2(\rho) + O(\ln n)\right]. \tag{8.3.5}$$

### 8.3.2. Dividing up the hypercube

To proceed, we refine equation (8.3.1). For each $\omega, \nu \in N$, $s \in S_{k,\omega,\nu}$ and $\eta > 0$, we introduce

$$\mathcal{B}^s_{k,\omega,\nu,\eta}(n) = \left\{\rho \in \mathcal{B}^s_{k,\omega,\nu}(n) : \|\rho - \bar\rho\|_2 \leq \eta\right\}.$$

We are going to show that the r.h.s. of (8.3.1) is dominated by the contributions with $\rho$ "close to" $\bar\rho$ in terms of the euclidean norm. More precisely, for a graph $G$ let

$$Z^{s\,(2)}_{k,\omega,\nu,\eta}(G) = \sum_{\rho \in \mathcal{B}^s_{k,\omega,\nu,\eta}(n)} Z^{(2)}_{k,\rho}(G) \qquad \text{for any } \eta > 0.$$

Then the second moment argument performed in [AN05] fairly directly yields the following statement showing that overlap matrices that are far apart from $\bar{\rho}$ do asymptotically not contribute to the second moment.

**Proposition 8.3.2.** *Assume that $k \geq 3$ and $d' < 2(k-1)\ln(k-1)$. Further, let $\omega, \nu \in N$. Then for any fixed $\eta > 0$ and any $s \in S_{k,\omega,\nu}$, it holds that*

$$\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2\right] \sim \mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,\eta}(\mathcal{G}(n,m))\right].$$

To prove this proposition, we first define a function

$$\bar{f}_2 : \rho \in \mathcal{B}_{k,\omega}(n) \to \mathbb{R}, \quad \rho \mapsto \mathcal{H}(\rho) + \frac{d}{2}\ln\left(1 - \frac{2}{k} + \|\rho\|_2^2\right).$$

The following lemma shows how $f_2$ defined in (8.3.4) relates to $\bar{f}_2$.

**Lemma 8.3.3.** *For $\rho = (\rho_{ij}) \in \mathcal{B}_{k,\omega}(n)$, we have*

$$\exp\left[n f_2(\rho)\right] \sim \exp\left[n \bar{f}_2(\rho) + O\left(\omega^2\right)\right].$$

*Proof.* We define the function

$$\zeta(\rho) = f_2(\rho) - \bar{f}_2(\rho)$$

and derive an upper bound on $\zeta(\rho)$. By definition, for each $\rho \in \mathcal{B}_{k,\omega}(n)$ there exist $\alpha = (\alpha_i)_{i \in [k]}$ and $\beta = (\beta_j)_{j \in [k]}$ such that $\rho_{i\star} = \frac{1}{k} + \alpha_i$ and $\rho_{\star j} = \frac{1}{k} + \beta_j$ for all $i, j \in [k]$ with $|\alpha_i|, |\beta_j| \leq \frac{\omega}{\sqrt{n}}$. Thus,

$$f_2(\rho) = \mathcal{H}(\rho) + \frac{d}{2}\ln\left(1 - \|\bar{\rho}_{\cdot\star} + \alpha\|_2^2 - \|\bar{\rho}_{\star\cdot} + \beta\|_2^2 + \|\rho\|_2^2\right).$$

As we are only interested in the difference between $f_2$ and $\bar{f}_2$, we can reparametrise $\zeta$ as

$$\zeta(\alpha, \beta) = \frac{d}{2}\ln\left(\frac{1 - \|\bar{\rho}_{\cdot\star} + \alpha\|_2^2 - \|\bar{\rho}_{\star\cdot} + \beta\|_2^2 + \|\rho\|_2^2}{1 - \frac{2}{k} + \|\rho\|_2^2}\right).$$

Differentiating and simplifying the expression yields $\frac{\partial \zeta}{\partial \alpha_i}(\alpha, \beta), \frac{\partial \zeta}{\partial \beta_j}(\alpha, \beta) = O\left(\frac{\omega}{\sqrt{n}}\right)$ for all $i, j \in [k]$. According to the fundamental theorem of calculus, it follows that

$$\max_{\rho \in \mathcal{B}_{k,\omega}(n)} |\zeta(\rho)| = \int_{-\omega/\sqrt{n}}^{\omega/\sqrt{n}} O\left(\frac{\omega}{\sqrt{n}}\right) d\alpha_1 = O\left(\frac{\omega^2}{n}\right),$$

completing the proof. □

*Proof of Proposition 8.3.2.* Equation (8.3.5) combined with Lemma 8.3.3 reduces our task to studying the function $\bar{f}_2(\rho)$. For the range of $d$ covered by Proposition 8.3.2, this analysis is the main technical achievement of [AN05], where (essentially) the following statement is proved.

**Lemma 8.3.4.** *Assume that $k \geq 3, \omega \in N$ as well as $d' \leq 2(k-1)\ln(k-1)$ and $d = 2m/n$. For any $n > 0$ and any overlap matrix $\rho \in \mathcal{B}_{k,\omega}(n)$, we have*

$$\bar{f}_2(\rho) \leq \bar{f}_2(\bar{\rho}) - \frac{2(k-1)\ln(k-1) - d}{4(k-1)^2}\left(k^2\|\rho\|_2^2 - 1\right) + o(1). \tag{8.3.6}$$

*Proof.* For $\rho$ such that $\sum_{i=1}^k \rho_{ij} = \sum_{i=1}^k \rho_{ji} = 1/k$, the bound (8.3.6) is proved in [AN05, Section 3]. This implies that (8.3.6) also holds for $\rho \in \mathcal{B}_{k,\omega}(n)$, because $\bar{f}_2$ is uniformly continuous on the compact set $\mathcal{B}_{k,\omega}(n)$. □

Now, assume that $k$ and $d$ satisfy the assumptions of Proposition 8.3.2 and let $\nu \in N$ and $\eta > 0$ be any fixed number. Then, for any $\hat{\rho} \in \mathcal{B}^s_{k,\omega,\nu}(n)$, we have $\|\hat{\rho} - \bar{\rho}\|_2 = O\left(\frac{\omega}{\sqrt{n}}\right)$. Consequently, we obtain with (8.3.5) that

$$\sum_{\substack{\rho \in \mathcal{B}^s_{k,\omega,\nu}(n) \\ \|\rho - \bar{\rho}\|_2 \leq \eta}} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \geq \mathbb{E}\left[Z^{(2)}_{k,\hat{\rho}}(\mathcal{G}(n,m))\right] \geq \exp\left[n f_2(\bar{\rho}) + O(\ln n)\right]. \tag{8.3.7}$$

On the other hand, the function $\mathcal{B} \to \mathbb{R}$, $\rho \to k^2\|\rho\|_2$ is smooth, strictly convex and attains its global minimum of 1 at $\rho = \bar{\rho}$. Consequently, there exist $(c_k)_k > 0$ such that if $\|\rho - \bar{\rho}\|_2 > \eta$, then $\left(k^2\|\rho\|_2 - 1\right) \geq c_k$. Hence, Fact 8.3.1, Lemma 8.3.3 and Lemma 8.3.4 yield

$$\sum_{\substack{\rho \in \mathcal{B}^s_{k,\omega,\nu}(n) \\ \|\rho - \bar{\rho}\|_2 > \eta}} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \leq \exp\left[n f_2(\bar{\rho}) - n c_k d_k + o(n)\right], \tag{8.3.8}$$

where $d_k = \frac{2(k-1)\ln(k-1) - d}{4(k-1)^2} > 0$.

Combining (8.3.8) and (8.3.7), we conclude that $\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2\right] \sim \mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,\eta}(\mathcal{G}(n,m))\right]$, thereby completing the proof of Proposition 8.3.2. □

Having reduced our task to studying overlaps $\rho$ such that $\|\rho - \bar{\rho}\|_2 \leq \eta$ for a small but fixed $\eta > 0$, in this section we are going to argue that, in fact, it suffices to consider $\rho$ such that $\|\rho - \bar{\rho}\|_2 \leq n^{-3/8}$ (where the constant $3/8$ is somewhat arbitrary; any number smaller than $1/2$ would do). More precisely, we have

**Proposition 8.3.5.** *Assume that $k \geq 3$ and that $d' < d_{\text{cond}}$. Let $\nu, \omega \in N$ and $s \in S_{k,\omega,\nu}$. There exists a number $\eta_0 = \eta_0(d', k)$ such that for any $0 < \eta < \eta_0$ we have*

$$\mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,\eta}(\mathcal{G}(n,m))\right] \sim \mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,n^{-3/8}}(\mathcal{G}(n,m))\right].$$

The key to proving this proposition is the following lemma. It specifies the expected number of pairs of solutions in the cases where the overlap matrices $\rho \in \mathcal{B}^s_{k,\omega,\nu}(n)$ satisfy $\|\rho - \bar{\rho}\|_2 \leq n^{-3/8}$ or $\|\rho - \bar{\rho}\|_2 \in (n^{-3/8}, \eta)$.

**Lemma 8.3.6.** *Let $k \geq 3, d' < (k-1)^2$ and $d = 2m/n$. Set*

$$C_n(d,k) = \exp\left[d/2\right] k^{k^2} (2\pi n)^{\frac{1-k^2}{2}} \quad \text{and} \quad D(d,k) = k^2\left(1 - \frac{d}{(k-1)^2}\right). \tag{8.3.9}$$

- *If $\rho \in \mathcal{B}^s_{k,\omega,\nu,\eta}(n)$ satisfies $\|\rho - \bar{\rho}\|_2 \leq n^{-3/8}$, then*

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \sim C_n(d,k)\exp\left[2nf_1(\rho^\star) - n\frac{D(d,k)}{2}\|\rho - \bar{\rho}\|_2^2\right]. \tag{8.3.10}$$

- *There exist numbers $\eta = \eta(d,k) > 0$ and $A = A(d,k) > 0$ such that if $\rho \in \mathcal{B}^s_{k,\omega,\nu,\eta}(n)$ satisfies $\|\rho - \bar{\rho}\|_2 \in (n^{-3/8}, \eta)$, then*

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] = \exp\left[2nf_1(\rho^\star) - An^{1/4}\right]. \tag{8.3.11}$$

*Proof.* As Fact 8.3.1 yields $\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \sim C_n(d,k)\exp\left[nf_2(\rho)\right]$, we have to analyse $f_2$. Expanding this function around $\bar{\rho}$ yields

$$f_2(\rho) = f_2(\bar{\rho}) - \frac{D(d,k)}{2}\|\rho - \bar{\rho}\|_2^2 + O(\|\rho - \bar{\rho}\|_2^3). \tag{8.3.12}$$

Consequently, for $\|\rho - \bar{\rho}\|_2 \leq n^{-3/8}$,

$$\exp\left[nf_2(\rho)\right] = \exp\left[nf_2(\bar{\rho}) - n\frac{D(d,k)}{2}\|\rho - \bar{\rho}\|_2^2 + O(n^{-1/8})\right].$$

As $f_2$ satisfies $f_2(\bar{\rho}) = 2f_1(\rho^\star)$, the statement in (8.3.10) follows.

To prove (8.3.11), we observe that similarly to (8.3.12) and because $f_2$ is smooth in a neighbourhood of $\bar{\rho}$, there exist $\eta > 0$ and $A > 0$ such that for $\|\rho - \bar{\rho}\|_2 \leq \eta$,

$$f_2(\rho) \leq f_2(\bar{\rho}) - A\|\rho - \bar{\rho}\|_2^2.$$

Hence, if $\|\rho - \bar{\rho}\|_2 \in (n^{-3/8}, \eta)$, then

$$\mathbb{E}\left[Z_{k,\rho}^{(2)}(\mathcal{G}(n,m))\right] = O\left(n^{\frac{1-k^2}{2}}\right) \exp\left[nf_2(\rho)\right] \leq \exp\left[2nf_1(\rho^\star) - An^{1/4}\right],$$

as claimed. □

*Proof of Proposition 8.3.5.* We fix $s \in S_{k,\omega,\nu}$. Further, we fix $\eta > 0$ and $A > 0$ as given by Lemma 8.3.6. For each $\hat{\rho} \in \mathcal{B}^s_{k,\omega,\nu,\eta}(n)$, we have $\|\hat{\rho} - \bar{\rho}\|_2 = O\left(\frac{\omega}{\sqrt{n}}\right)$ and obtain from the first part of Lemma 8.3.6 that

$$\mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,n^{-3/8}}(\mathcal{G}(n,m))\right] \geq \mathbb{E}\left[Z^{(2)}_{k,\rho_0}(\mathcal{G}(n,m))\right] \sim C_n(d,k)\exp\left[2nf_1(\rho^\star) + O\left(\omega^2\right)\right]. \quad (8.3.13)$$

On the other hand, because $|\mathcal{B}^s_{k,\omega,\nu,\eta}(n)|$ is bounded by a polynomial in $n$, the second part of Lemma 8.3.6 yields

$$\sum_{\substack{\rho \in \mathcal{B}^s_{k,\omega,\nu,\eta}(n) \\ \|\rho - \bar{\rho}\|_2 > n^{-3/8}}} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \leq \exp\left[2nf_1(\rho^\star) - An^{1/6} + O(\ln n)\right]. \quad (8.3.14)$$

Combining (8.3.13) and (8.3.14), we obtain

$$\mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,\eta}(\mathcal{G}(n,m))\right] \sim \sum_{\rho \in \mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n)} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \sim \mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,n^{-3/8}}(\mathcal{G}(n,m))\right],$$

as claimed. □

### 8.3.3. Calculating the constant

This section is dedicated to computing the contribution of the overlap matrices $\rho \in \mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n)$. To this aim, we first show that in each region of the hypercube we can approximate $f_2$ by a function where the marginals are set to those of the centre of this region as defined in (8.1.2). More formally, let $f_2^s : \mathcal{B}_k \to \mathbb{R}$ be defined as

$$f_2^s : \rho \mapsto \mathcal{H}(\rho) + \frac{d}{2}\ln\left(1 - 2\|\rho^{k,\omega,\nu,s}\|_2^2 + \|\rho\|_2^2\right).$$

Then the following is true

**Lemma 8.3.7.** *Let $k \geq 3, \omega, \nu \in N$ and $C_n(d,k)$ as in* (8.3.9). *Then for $\rho \in \mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n)$ it holds that*

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \sim C_n(d,k)\exp\left[nf_2^s(\rho) + O\left(\frac{\omega}{\nu}\right)\right].$$

*Proof.* Equation (8.3.3) of Fact 8.3.1 yields that

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \sim C_n(d,k)\exp\left[nf_2(\rho)\right]. \tag{8.3.15}$$

For $s \in S_{k,\omega,\nu}$, we define the function

$$\zeta^s(\rho) = f_2(\rho) - f_2^s(\rho).$$

To derive an upper bound on $\zeta^s(\rho)$ for all values $\rho \in \mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n)$, we first we observe that there exist $\alpha = (\alpha_i)_{i\in[k]}$ and $\beta = (\beta_j)_{j\in[k]}$ such that the function $f_2$ can be expressed by setting $\rho_{i\star} = \rho_i^{k,\omega,\nu,s} + \alpha_i$ and $\rho_{\star j} = \rho_j^{k,\omega,\nu,s} + \beta_j$ for all $i,j \in [k]$ with $|\alpha_i|, |\beta_j| \leq \frac{1}{\nu\sqrt{n}}$. Thus,

$$f_2 : \rho \mapsto \mathcal{H}(\rho) + \frac{d}{2}\ln\left(1 - \|\rho^{k,\omega,\nu,s} + \alpha\|_2^2 - \|\rho^{k,\omega,\nu,s} + \beta\|_2^2 + \|\rho\|_2^2\right).$$

As we are only interested in the difference between $f_2$ and $f_2^s$, we can reparametrise $\zeta^s$ as

$$\zeta^s(\alpha,\beta) = \frac{d}{2}\ln\left(\frac{1 - \|\rho^{k,\omega,\nu,s} + \alpha\|_2^2 - \|\rho^{k,\omega,\nu,s} + \beta\|_2^2 + \|\rho\|_2^2}{1 - 2\|\rho^{k,\omega,\nu,s}\|_2^2 + \|\rho\|_2^2}\right).$$

Differentiating and simplifying the expression yields $\frac{\partial\zeta^s}{\partial\alpha_i}(\alpha,\beta), \frac{\partial\zeta^s}{\partial\beta_j}(\alpha,\beta) = O\left(\frac{\omega}{\sqrt{n}}\right)$ for all $i,j \in [k]$. According to the fundamental theorem of calculus it follows for every $s \in S_{k,\omega,\nu}$ that

$$\max_{\rho\in\mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n)} |\zeta^s(\rho)| = \int_{-(\nu\sqrt{n})^{-1}}^{(\nu\sqrt{n})^{-1}} O\left(\frac{\omega}{\sqrt{n}}\right) d\alpha_1 = O\left(\frac{\omega}{n\nu}\right).$$

Combining this with (8.3.15) yields the assertion. □

Now we are able to give a very precise expression for the second moment.

**Proposition 8.3.8.** *Assume that $k \geq 3, \omega, \nu \in N, d' < (k-1)^2$ and $d = 2m/n$. Let $s \in S_{k,\omega,\nu}$. Then*

$$\mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,n^{-3/8}}(\mathcal{G}(n,m))\right]$$
$$\sim_\nu \left(|\mathcal{A}_{k,\omega}(n)|\,(2\pi n)^{\frac{1-k}{2}}\,k^{k/2}\exp\left[nf_1(\rho^{k,\omega,\nu,s})\right]\right)^2 \exp\left[d/2\right]\left(1 - \frac{d}{(k-1)^2}\right)^{-\frac{(k-1)^2}{2}}.$$

The rest of this subsection will be dedicated to proving this proposition. In due course we are going to need the set of matrices with coefficients in $\frac{1}{n}\mathbb{Z}$ whose lines and columns sum to zero:

$$\mathcal{E}_n = \left\{ (\epsilon_{i,j})_{\substack{1\le i\le k\\ 1\le j\le k}},\ \ \forall i,j\in[k],\ \epsilon_{i,j}\in\frac{1}{n}\mathbb{Z},\ \ \forall j\in[k],\ \sum_{i=1}^{k}\epsilon_{ij}=\sum_{i=1}^{k}\epsilon_{ji}=0 \right\}. \tag{8.3.16}$$

The following result regards Gaussian summations over matrices in $\mathcal{E}_n$.

**Lemma 8.3.9.** *Let $k\ge 2$, $d' < (k-1)^2$ and $D>0$ be fixed. Then*

$$\sum_{\epsilon\in\mathcal{E}_n}\exp\left[-n\frac{D}{2}\|\epsilon\|_2^2 + o(n^{1/2})\|\epsilon\|_2\right] \sim \left(\sqrt{2\pi n}\right)^{(k-1)^2} D^{-\frac{(k-1)^2}{2}} k^{-(k-1)}.$$

Lemma 8.3.9 and its proof are very similar to an argument used in [KPGW10, Section 3]. In fact, Lemma 8.3.9 follows from

**Lemma 8.3.10** ([KPGW10, Lemma 6 (b) and 7 (c)])**.** *There is a $(k-1)^2\times(k-1)^2$-matrix $\mathcal{H} = (\mathcal{H}_{(i,j),(k,l)})_{i,j,k,l\in[k-1]}$ such that for any $\varepsilon = (\varepsilon_{ij})_{i,j\in[k]}\in\mathcal{E}_n$ we have*

$$\sum_{i,j,i',j'\in[k-1]}\mathcal{H}_{(i,j),(i',j')}\varepsilon_{ij}\varepsilon_{i'j'} = \|\varepsilon\|_2^2\,.$$

*This matrix $\mathcal{H}$ is positive definite and* $\det\mathcal{H} = k^{2(k-1)}$.

*Proof of Lemma 8.3.9.* Together with the Euler-Maclaurin formula and Lemma 8.3.10, a Gaussian integration yields

$$\begin{aligned}
&\sum_{\epsilon\in S_n}\exp\left[-n\frac{D}{2}\|\epsilon\|_2^2 + o(n^{1/2})\|\epsilon\|_2\right]\\
&\quad= \sum_{\epsilon\in(\mathbb{Z}/n)^{(k-1)^2}}\exp\left[-n\frac{D}{2}\sum_{i,j,i',j'\in[k-1]}\mathcal{H}_{(i,j),(i',j')}\varepsilon_{ij}\varepsilon_{i'j'} + o(n^{1/2})\|\epsilon\|_2\right]\\
&\quad\sim n^{(k-1)^2}\int\dots\int\exp\left[-n\frac{D}{2}\sum_{i,j,i',j'\in[k-1]}\mathcal{H}_{(i,j),(i',j')}\varepsilon_{ij}\varepsilon_{i'j'}\right]\mathrm{d}\varepsilon_{11}\cdots\mathrm{d}\varepsilon_{(k-1)(k-1)}\\
&\quad\sim\left(\sqrt{2\pi n}\right)^{(k-1)^2} D^{\frac{-(k-1)^2}{2}}(\det\mathcal{H})^{-1/2} \sim \left(\sqrt{2\pi n}\right)^{(k-1)^2} D^{\frac{-(k-1)^2}{2}} k^{-(k-1)},
\end{aligned}$$

as desired. □

Now we are ready to prove Proposition 8.3.8.

*Proof of Proposition 8.3.8.* Lemma 8.3.7 states that for every $\rho \in \mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n)$ we have

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \sim C_n(d,k) \exp\left[n f^s_2(\rho) + O\left(\frac{\omega}{\nu}\right)\right]. \tag{8.3.17}$$

Thus, all we have to do is analysing the function $f^s_2$ for $s \in S_{k,\omega,\nu}$. To this aim, we expand $f^s_2(\rho)$ around $\rho = \rho^s$ where $\rho^s = (\rho^s_{ij})_{i,j}$ with $\rho_{ij} = \rho^{k,\omega,\nu,s}_i \cdot \rho^{k,\omega,\nu,s}_j$. Then with $D(d,k)$ as defined in (8.3.9) we have

$$f^s_2(\rho) = f^s_2(\rho^s) + \Theta\left(\frac{\omega}{n}\right)\|\rho - \rho^s\|_2 - \frac{D(d,k)}{2}\|\rho - \rho^s\|_2^2 + o(n^{-1}). \tag{8.3.18}$$

Combining (8.3.18) with (8.3.17), we find that

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \\ \sim C_n(d,k) \exp\left[n f^s_2(\rho^s) + \Theta(\omega)\|\rho - \rho^s\|_2 - n\frac{D(d,k)}{2}\|\rho - \rho^s\|_2^2 + O\left(\frac{\omega}{\nu}\right)\right]. \tag{8.3.19}$$

For two vectors of "marginals" $\rho^0, \rho^1 \in \mathcal{B}^s_{k,\omega,\nu}(n)$, we introduce the set of overlap matrices

$$\mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n,\rho^0,\rho^1) = \{\rho \in \mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n) : \rho_{\cdot\star} = \rho^0, \rho_{\star\cdot} = \rho^1\}.$$

and observe that with this definition we have

$$\mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,n^{-3/8}}(\mathcal{G}(n,m))\right] = \sum_{\rho^0,\rho^1 \in \mathcal{B}^s_{k,\omega,\nu}(n)} \sum_{\rho \in \mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n,\rho^0,\rho^1)} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right]. \tag{8.3.20}$$

In particular, the set $\mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n,\rho^0,\rho^1)$ contains the "product" overlap $\rho^0 \otimes \rho^1$ defined by $(\rho^0 \otimes \rho^1)_{ij} = \rho^0_i \rho^1_j$ for $i,j \in [k]$. To proceed, we fix two colour densities $\rho^0, \rho^1 \in \mathcal{B}^s_{k,\omega,\nu}(n)$ and simplify the notation by writing

$$\widehat{\mathcal{B}} = \mathcal{B}^s_{k,\omega,\nu,n^{-3/8}}(n,\rho^0,\rho^1), \qquad \widehat{\rho} = \rho^0 \otimes \rho^1.$$

Thus, the inner sum from (8.3.20) simplifies to

$$\mathcal{S}_1 = \sum_{\rho \in \widehat{\mathcal{B}}} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right].$$

and we are going to evaluate this quantity. We observe that with $\mathcal{E}_n$ as defined in (8.3.16), for each

$\rho \in \widehat{\mathcal{B}}$ we can find $\varepsilon \in \mathcal{E}_n$ such that

$$\rho = \widehat{\rho} + \varepsilon.$$

Hence, this gives $\|\rho - \rho^s\|_2 = \|\widehat{\rho} + \varepsilon - \rho^s\|_2$ and the triangle inequality yields

$$\|\varepsilon\|_2 - \|\widehat{\rho} - \rho^s\|_2 \leq \|\widehat{\rho} + \varepsilon - \rho^s\|_2 \leq \|\varepsilon\|_2 + \|\widehat{\rho} - \rho^s\|_2 .$$

By definition of $\widehat{\rho}$ and $\rho^s$, we have $\|\widehat{\rho} - \rho^s\|_2 \leq \frac{1}{\nu\sqrt{n}}$ and consequently

$$\|\rho - \rho^s\|_2 = \|\boldsymbol{\varepsilon}\|_2 + O\left(\frac{1}{\nu\sqrt{n}}\right). \tag{8.3.21}$$

Observing that $f_2^s(\rho^s) = \left(f_1(\rho^{k,\omega,\nu,s})\right)^2$ and inserting (8.3.21) into (8.3.19) while taking first $n \to \infty$ and afterwards $\nu \to \infty$, we obtain

$$\mathcal{S}_1 \sim_\nu C_n(d,k) \exp\left[2nf_1^s\left(\rho^{k,\omega,\nu,s}\right)\right] \sum_{\rho \in \widehat{\mathcal{B}}} \exp\left[-n\frac{D(d,k)}{2}\|\varepsilon\|_2^2 + o(n^{1/2})\|\varepsilon\|_2\right]. \tag{8.3.22}$$

To apply Lemma 8.3.9, we have to relate $\rho \in \widehat{\mathcal{B}}$ to $\varepsilon \in \mathcal{E}_n$. From the definitions we obtain

$$\left\{\widehat{\rho} + \varepsilon : \varepsilon \in \mathcal{E}_n, \|\varepsilon\|_2 \leq n^{-3/8}/2\right\} \subset \left\{\rho \in \widehat{\mathcal{B}}\right\} \subset \left\{\widehat{\rho} + \varepsilon : \varepsilon \in \mathcal{E}_n\right\}.$$

We show that the contribution of $\varepsilon \in \mathcal{E}_n$ with $\|\varepsilon\|_2 > n^{-3/8}/2$ is negligible:

$$\begin{aligned}
\mathcal{S}_2 &= C_n(d,k) \exp\left[2nf_1^s\left(\rho^{k,\omega,\nu,s}\right)\right] \sum_{\substack{\epsilon \in \mathcal{S}_n \\ \|\epsilon\|_2 > n^{-3/8}/2}} \exp\left[-n\frac{D(d,k)}{2}\|\epsilon\|_2^2(1+o(1))\right] \\
&= C_n(d,k) \exp\left[2nf_1^s\left(\rho^{k,\omega,\nu,s}\right)\right] \sum_{\substack{l \in \mathbb{Z}/n \\ l > n^{-3/8}/2}} \sum_{\substack{\epsilon \in S_n \\ \|\epsilon\|_2 = l}} \exp\left[-nl^2\frac{D(d,k)}{2}(1+o(1))\right] \\
&= C_n(d,k) \exp\left[2nf_1^s\left(\rho^{k,\omega,\nu,s}\right)\right] O\left(n^{k^2}\right) \exp\left[-\frac{D(d,k)}{2}n^{1/4}\right]
\end{aligned}$$

Consequently, (8.3.22) yields $\Sigma_2 = o(\Sigma_1)$. Thus, we obtain from Lemma 8.3.9 that

$$\begin{aligned}
\mathcal{S}_1 &\sim_\nu C_n(d,k) \exp\left[2nf_1^s\left(\rho^{k,\omega,\nu,s}\right)\right] \sum_{\rho \in \widehat{\mathcal{B}}} \exp\left[-n\frac{D(d,k)}{2}\|\epsilon\|_2^2 + o(n^{1/2})\|\epsilon\|_2\right]. \\
&\sim_\nu C_n(d,k) \exp\left[2nf_1^s\left(\rho^{k,\omega,\nu,s}\right)\right] \left(\sqrt{2\pi n}\right)^{(k-1)^2} k^{-k(k-1)} \left(1 - \frac{d}{(k-1)^2}\right)^{-\frac{(k-1)^2}{2}}.
\end{aligned} \tag{8.3.23}$$

In particular, the last expression is independent of the choice of the vectors $\rho^0, \rho^1$ that defined $\widehat{\mathcal{B}}$. Therefore, substituting (8.3.23) in the decomposition (8.3.20) completes the proof of Proposition 8.3.8.

□

*Proof of Proposition 8.1.4.* First observe that

$$\exp\left[\sum_{l\geq 2}\lambda_l\delta_l^2\right] = \left(1-\frac{d}{(k-1)^2}\right)^{-\frac{(k-1)^2}{2}}\exp\left[-\frac{d}{2}\right].$$

Proposition 8.1.4 is immediately obtained by combining Lemma 8.2.3 with Propositions 8.3.2, 8.3.5 and 8.3.8. □

### 8.3.4. Up to the condensation threshold

In this last subsection we prove Proposition 8.1.5. In the regime $2(k-1)\ln(k-1) \leq d' < d_{\mathrm{cond}}$ for $k \geq k_0$ for some big constant $k_0$, we consider random variables $\widetilde{Z}^s_{k,\omega,\nu}$ instead of $Z^s_{k,\omega,\nu}$. To prove the proposition we show the following result by adapting our setting in a way that we can apply the second moments argument from [COV13] and [BCOHRV16].

**Proposition 8.3.11.** *Let $\omega, \nu \in N$. There is a constant $k_0 > 3$ such that for $k \geq k_0$ and $2(k-1)\ln(k-1) \leq d' < d_{\mathrm{cond}}$ the following is true. For each $s \in S_{k,\omega,\nu}$, there exists an integer-valued random variable $0 \leq \widetilde{Z}^s_{k,\omega,\nu} \leq Z^s_{k,\omega,\nu}$ that satisfies*

$$\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right] \sim \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]$$

*and such that for any fixed $\eta > 0$ we have* $\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2\right] \leq (1+o(1))\mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,\eta}(\mathcal{G}(n,m))\right]$.

In this section we work with the Erdős-Rényi random graph model $G(n,p)$, which is a random graph on $[n]$ vertices where every possible edge is present with probability $p = d/n$ independently. We further assume from now on that $k$ divides $n$.

The use of results from [COV13, BCOHRV16] is complicated by the fact that we are dealing with $(\omega, n)$-balanced $k$-colourings that allow a larger discrepancy between the colour classes than [COV13, BCOHRV16], where balanced colourings are defined such that in each color class only a deviation of at most $\sqrt{n}$ from the typical value $n/k$ is allowed. To circumvent this problem, we introduce the following:

Choose a map $\sigma : [n] \to [k]$ uniformly at random and generate a graph $G(n, p', \sigma)$ on $[n]$ by connecting any two vertices $v, w \in [n]$ such that $\sigma(v) \neq \sigma(w)$ with probability $p' = dk/(n(k-1))$ independently.

Given $\sigma$ and $G(n, p', \sigma)$, we define

$$\alpha_i = |\sigma^{-1}(i) - n/k| \qquad \text{for } i \in [k]$$

and let $\alpha = \max_{i \in [k]} \alpha_i$. Thus, by definition $\alpha \leq \omega\sqrt{n}$. We set $n' = n + k\lceil\alpha\rceil$. Further, we let

$$\beta_i = |\sigma^{-1}(i) - (n + k\lceil\alpha\rceil)/k| \qquad \text{for } i \in [k].$$

We then construct a coloured graph $G'_{n',p',\sigma'}$ from $G(n, p', \sigma)$ in the following way:

- Add $k\lceil\alpha\rceil$ vertices to $G(n, p)$ and denote them by $n + 1, n + 2, ..., n + k\lceil\alpha\rceil$.
- Define a colouring $\sigma' : [n'] \to [k]$ by setting $\sigma'(i) = \sigma(i)$ for $i \in [n]$, $\sigma(i) = 1$ for $i \in n + 1, ..., n + \beta_1$ and $\sigma(i) = j$ for $j \in \{2, ..., k\}$ and $i \in n + \beta_{j-1} + 1, ..., n + \beta_j$.
- Add each possible edge $(i, j)$ with $\sigma'(i) \neq \sigma'(j)$ involving a vertex $i \in \{n + 1, ..., n + k\lceil\alpha\rceil\}$ with probability $p' = dk/(n(k - 1))$.

We call a colouring $\tau : [n] \to [k]$ of a graph $G$ on $[n]$ *perfectly balanced* if $|\tau^{-1}(i)| = |\tau^{-1}(j)|$ for all $i, j \in [k]$ and we denote the set of all such perfectly balanced colourings by $\widetilde{\mathcal{B}}_k(n)$. Then the following holds by construction:

**Fact 8.3.12.** *$G'_{n',p',\sigma'}$ has the same distribution as $G(n', p', \tau)$ conditioned on the event that $\tau : [n'] \to k$ is perfectly balanced.*

Let $G''_{n,p',\sigma'|[n]}$ denote the graph obtained from $G'_{n',p',\sigma'}$ by deleting the vertices $n + 1, ..., n + k\lceil\alpha\rceil$ and the incident edges.

**Fact 8.3.13.** *$G''_{n,p',\sigma'|n}$ has the same distribution as $G(n, p', \tau)$ conditioned on the event that $\tau$ is $(\omega, n)$-balanced.*

To proceed, we adopt the following notation from [COV13]: Let $\rho \in \mathcal{B}_k$ be called *$s$-stable* if it has precisely $s$ entries bigger than $0.51/k$. Further, let $\bar{\mathcal{B}}_k$ be the set of all $\rho \in \mathcal{B}_k$ such that

$$\sum_{j=1}^{k} \rho_{ij} = \sum_{j=1}^{k} \rho_{ji} = 1/k \quad \text{for all } i \in [k].$$

Then any $\rho \in \bar{\mathcal{B}}_k$ is $s$-stable for some $s \in \{0, 1, \ldots, k\}$. In addition, let $\kappa = \ln^{20} k/k$ and let us call $\rho \in \mathcal{B}_k$ *separable* if $k\rho_{ij} \notin (0.51, 1 - \kappa)$ for all $i, j \in [k]$. A $k$-colouring $\sigma$ of a graph $G$ on $[n]$ is called *separable* if for any other $k$-colouring $\tau$ of $G$ the overlap matrix $\rho(\sigma, \tau)$ is separable. We have the following result:

**Lemma 8.3.14.** *Let $s \in S_{k,\omega,\nu}$. There is $k_0 > 0$ such that for all $k > k_0$ and all $d'$ such that $2(k-1)\ln(k-1) \leq d' \leq (2k-1)\ln k$ the following is true. Let $\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))$ denote the number of $(\omega,n)$-balanced $k$-colourings of $\mathcal{G}(n,m)$ that fail to be separable. Then $\mathbb{E}[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))] = o(\mathbb{E}[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))])$.*

To prove this lemma, we combine Fact 8.3.12 with [COV13, Lemma 3.3]. This yields the following[20].

**Lemma 8.3.15** ([COV13])**.** *There is $k_0 > 0$ such that for all $k \geq k_0$ and all $d'$ with $2(k-1)\ln(k-1) \leq d' \leq (2k-1)\ln k$ each $\tau \in \widetilde{\mathcal{B}}_k(n')$ is separable in $G'_{n',p',\tau}$ w.h.p..*

*Proof of Lemma 8.3.14.* Choose a map $\sigma : [n] \to [k]$ uniformly at random and generate a graph $G(n,p',\sigma)$ on $[n]$ by connecting any two vertices $v,w \in [n]$ such that $\sigma(v) \neq \sigma(w)$ with probability $p'$ independently. Construct $G'_{n',p',\sigma'}$ from $G(n,p',\sigma)$ in the way defined above. Then $\sigma' \in \widetilde{\mathcal{B}}_k(n)$. By Lemma 8.3.15, $\sigma'$ is separable in $G'_{n',p',\sigma'}$ w.h.p.. Thus, $\sigma$ is separable in $G''_{n,p',\sigma'|n}$ if we define separability using $\kappa' = \frac{\ln^{21} k}{k}$. By choosing $k_0$ large enough and applying Fact 8.3.13, the assertion follows. $\square$

For the next ingredient to the proof of Proposition 8.3.11, we need the following definition. For a graph $G$ on $[n]$ and a $k$-colouring $\sigma$ of $G$, we let $\mathcal{C}(G,\sigma)$ be the set of all $\tau \in \mathcal{B}_k$ that are $k$-colourings of $G$ such that $\rho(\sigma,\tau)$ is $k$-stable.

**Lemma 8.3.16.** *Let $s \in S_{k,\omega,\nu}$. There is $k_0 > 0$ such that for all $k > k_0$ and all $d'$ such that $(2k-1)\ln k - 2 \leq d' \leq d_{\mathrm{cond}}$ the following is true. There exists an $\varepsilon > 0$ such that if $\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))$ denotes the number of $(\omega,n)$-balanced $k$-colourings $\sigma$ of $\mathcal{G}(n,m)$ satisfying $|\mathcal{C}(\mathcal{G}(n,m),\sigma)| > \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right] / \exp\left[\varepsilon n\right]$, then $\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right] = o\left(\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]\right)$.*

To prove this lemma, we combine 8.3.12 with [BCOHRV16, Corollary 1.1] and obtain the following:

**Lemma 8.3.17** ([BCOHRV16])**.** *Let $s \in S_{k,\omega,\nu}$. There is $k_0 > 0$ such that for all $k > k_0$ and all $d'$ such that $(2k-1)\ln k - 2 \leq d' \leq d_{\mathrm{cond}}$ the following is true. Let $\tau \in \widetilde{\mathcal{B}}_k(n')$ be a perfectly balanced colour assignment. Then there exists $\varepsilon > 0$ such that if $\widetilde{Z}^s_{k,\omega,\nu}(G'_{n',p',\tau})$ denotes the number of $(\omega,n)$-balanced $k$-colourings $\tau$ of $G'_{n',p',\tau}$ satisfying $|\mathcal{C}(G'_{n',p',\tau},\tau)| > \mathbb{E}\left[Z^s_{k,\omega,\nu}(G'_{n',p',\tau})\right] / \exp\left[\varepsilon n\right]$, then $\mathbb{E}\left[\widetilde{Z}^s_{k,\omega,\nu}(G'_{n',p',\tau})\right] = o\left(\mathbb{E}\left[Z^s_{k,\omega,\nu}(G'_{n',p',\tau})\right]\right)$.*

[20] As a matter of fact, Lemma 3.2 in [COV13] also holds for densities $2(k-1)\ln(k-1) \leq d' \leq 2(k-1)\ln k - 2$, as all steps in the proof are also valid in this regime.

*Proof of Lemma 8.3.16.* Choose a map $\sigma : [n] \to [k]$ uniformly at random and generate a graph $G(n, p', \sigma)$ on $[n]$ by connecting any two vertices $v, w \in [n]$ such that $\sigma(v) \neq \sigma(w)$ with probability $p'$ independently. Construct $G'_{n',p',\sigma'}$ from $G(n, p', \sigma)$ in the way defined above. To construct $G''_{n,p',\sigma'_{|[n]}}$ from $G'_{n',p',\sigma'}$, we have to delete $O(\sqrt{n})$ many vertices. By [BCOHRV16, Section 6], for each of these vertices $v$ we can bound the logarithm of the number of colourings that emerge when deleting $v$ by $O(\ln n)$. Thus,

$$\ln |\mathcal{C}(G''_{n,p',\sigma'_{|[n]}}, \sigma'_{|[n]})| = \ln |\mathcal{C}(G'_{n',p',\sigma'}, \sigma')| + O(\sqrt{n} \ln n) = \ln |\mathcal{C}(G'_{n',p',\sigma'}, \sigma')| + o(n). \quad (8.3.24)$$

Then Lemma 8.3.16 follows by combining Lemma 8.3.17 with (8.3.24) and Fact 8.3.13. □

To complete the proof, we have to analyse the function $f_2$ defined in (8.3.4), as we know from (8.3.5) that

$$\mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] = \exp\left[n f_2(\rho) + O(\ln n)\right].$$

The following lemma shows that we can confine ourselves to the investigation of the function $\bar{f}_2$ defined in (8.3.2).

**Lemma 8.3.18.** *Let* $\lim_{n\to\infty}(\rho_n)_n = \rho_0$. *Then* $\lim_{n\to\infty} \ln \mathbb{E}\left[Z^{(2)}_{k,\rho_n}(\mathcal{G}(n,m))\right] \leq \bar{f}_2(\rho_0)$.

*Proof.* Lemma 8.3.3 yields that

$$\exp\left[n f_2(\rho)\right] \sim \exp\left[n \bar{f}_2(\rho) + O\left(\omega^2\right)\right].$$

Together with the uniform continuity of $\bar{f}_2$ this proves the assertion. □

We use results from [COV13] where an analysis of $\bar{f}_2$ was performed. The following lemma summarizes this analysis from [COV13, Section 4]. The same result was used in [BCOE14+].

**Lemma 8.3.19.** *For any* $c > 0$, *there is* $k_0 > 0$ *such that for all* $k > k_0$ *and all* $d$ *such that* $(2k-1)\ln k - c \leq d' \leq (2k-1)\ln k$ *the following statements are true.*

1. *If* $1 \leq s < k$, *then for all separable* $s$*-stable* $\rho \in \bar{\mathcal{B}}_k$ *we have* $\bar{f}_2(\rho) < \bar{f}_2(\bar{\rho})$.
2. *If* $\rho \in \bar{\mathcal{B}}_k$ *is* $0$*-stable and* $\rho \neq \bar{\rho}$, *then* $\bar{f}_2(\rho) < \bar{f}_2(\bar{\rho})$.
3. *If* $d' = (2k-1)\ln k - 2$, *then for all separable,* $k$*-stable* $\rho \in \bar{\mathcal{B}}_k$ *we have* $\bar{f}_2(\rho) < \bar{f}_2(\bar{\rho})$.

*Proof of Proposition 8.3.11.* Assume that $k \geq k_0$ for a large enough number $k_0$ and that $d' \geq 2(k-1)\ln(k-1)$. We consider two different cases.

**Case 1:** $d' \leq (2k-1)\ln k - 2$**:** Let $\widetilde{Z}^s_{k,\omega,\nu}$ be the number of $(\omega, n)$-balanced separable $k$-colourings of $\mathcal{G}(n,m)$. Then Lemma 8.3.15 implies that $\mathbb{E}[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))] \sim \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]$. Furthermore, in the case that $d' = (2k-1)\ln k - 2$, the combination of the statements of Lemma 8.3.19 imply that $\bar{f}_2(\rho) < \bar{f}_2(\bar{\rho})$ for any separable $\rho \in \bar{\mathcal{B}}_k \setminus \{\bar{\rho}\}$. As $\bar{f}_2(\rho)$ is the sum of the concave function $\rho \mapsto \mathcal{H}(\rho)$ and the convex function $\rho \mapsto \frac{d}{2}\ln(1-2/k\,\|\rho\|_2^2)$, this implies that, in fact, for any $d' \le (2k-1)\ln k - 2$ we have $\bar{f}_2(\rho) < \bar{f}_2(\bar{\rho})$ for any separable $\rho \in \bar{\mathcal{B}}_k \setminus \{\bar{\rho}\}$. Hence, the uniform continuity of $\bar{f}_2$ on $\mathcal{B}_k$ and (8.3.5) yield

$$\mathbb{E}[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))^2] \le (1+o(1)) \sum_{\substack{\rho\in\mathcal{B}^s_{k,\omega,\nu}(n)\\ \rho \text{ is 0-stable}}} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right]. \tag{8.3.25}$$

Additionally, as $\bar{\mathcal{B}}_k$ is a compact set, with the second statement of Lemma 8.3.19 it follows that for any $\eta > 0$ there exists $\varepsilon > 0$ such that

$$\max_{\substack{\rho\in\mathcal{B}^s_{k,\omega,\nu}(n)\\ \rho \text{ is 0-stable}\\ \|\rho-\bar{\rho}\|_2>\eta}} \exp\left[n\bar{f}_2(\rho)\right] \le \exp\left[n(\bar{f}_2(\bar{\rho})-\varepsilon)\right]. \tag{8.3.26}$$

As on the other hand it holds that

$$\mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,\eta}(\mathcal{G}(n,m))\right] \ge \exp\left[n\bar{f}_2(\bar{\rho})\right]/\mathrm{poly}(n), \tag{8.3.27}$$

combining (8.3.26) and (8.3.27) with (8.3.5) and the observation that $|\mathcal{B}^s_{k,\omega,\nu}(n)| \le n^{k^2}$, we see that for any $\eta > 0$,

$$\sum_{\substack{\rho\in\mathcal{B}^s_{k,\omega,\nu}(n)\\ \rho \text{ is 0-stable}\\ \|\rho-\bar{\rho}\|_2>\eta}} \mathbb{E}\left[Z^{(2)}_{k,\rho}(\mathcal{G}(n,m))\right] \le \sum_{\substack{\rho\in\mathcal{B}^s_{k,\omega,\nu}(n)\\ \rho \text{ is 0-stable}\\ \|\rho-\bar{\rho}\|_2>\eta}} \exp\left[n\bar{f}_2(\rho)+O(\ln n)\right] = o\left(\mathbb{E}\left[Z^{s\,(2)}_{k,\omega,\nu,\eta}(\mathcal{G}(n,m))\right]\right). \tag{8.3.28}$$

**Case 2:** $(2k-1)\ln k - 2 < d' < d_{\mathrm{cond}}$**:** For an appropriately chosen $\varepsilon > 0$ , we let $\widetilde{Z}^s_{k,\omega,\nu}$ be the number of $(\omega,n)$-balanced separable $k$-colourings $\sigma$ of $\mathcal{G}(n,m)$ satisfying $|\mathcal{C}(\mathcal{G}(n,m),\sigma)| \le \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]/\exp[\varepsilon n]$. Then Lemmas 8.3.15 and 8.3.16 imply that $\mathbb{E}[\widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))] \sim \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]$. Furthermore, the first part of Lemma 8.3.19 and equation (8.3.5) entail that (8.3.25) holds for this random variable $\widetilde{Z}^s_{k,\omega,\nu}$. Moreover, as in the previous case (8.3.26), (8.3.27), (8.3.5) and the second part of Lemma 8.3.19 show that (8.3.28) holds true for any fixed $\eta > 0$.

In either case the assertion follows by combining (8.3.25) and (8.3.28). □

*Proof of Proposition 8.1.5.* The assertion is obtained by combining Proposition 8.1.1 with Propositions 8.3.11, 8.3.5 and 8.3.8. □

# 9 Conclusion and open questions

With the results presented in this thesis we provide an important contribution to the endeavour of understanding and rigorously verifying the various phenomena arising in random constraint satisfaction problems.
As to condensation, a phase transition thoroughly changing the geometry of the solution space and posing a serious obstacle to locating the satisfiability threshold, we rigorously established its existence and exactly determined its location in random graph colouring. It was the first time that the condensation phase transition could be located in a rigorous manner within such accuracy in a random CSP. Our result verifies predictions made by the cavity method, a sophisticated tool of statistical physics.
Furthermore, we also investigated a non-zero temperature model, which is commonly used in the physics literature but has so far been given only scant attention to in mathematics. In this model, instead of only considering solutions, each colour assignment is weighted according to its number of monochromatic edges. We located the condensation transition in finite inverse temperature $k$-uniform random hypergraph 2-colouring up to an error tending to 0 when the uniformity $k$ grows to infinity. This is the first result pinning down the condensation phase transition within such accuracy in terms of the finite temperature parameter.
Apart from this, we investigated the distribution of the number of solutions in a regime where it can be proven that w.h.p. solutions exist. We determined this distribution asymptotically in the limit of large $n$ for random graph $k$-colouring and random hypergraph 2-colouring using the method of small subgraph conditioning. From this it follows that in the covered problems the random colouring model is contiguous with respect to the planted model, a statement that simplifies transferring results between these two models.

We expect that it is possible to apply similar methods and techniques to a variety of further random constraint satisfaction problems.
In particular, it seems reasonable to expect that the proof technique developed for locating the condensation phase transition in random graph $k$-colouring carries over to many other problems, especially because the physics predictions look very similar in many of them.
It would furthermore be interesting to explore to what extent the approach for determining the condensation phase transition for finite inverse temperatures can be transferred from random hypergraph 2-colouring to other random CSPs. Indeed, Coja-Oghlan and Jaafari [COJ16+] already started investigating non-zero random graph $k$-colouring.
It still is an open question whether the method for determining the condensation phase transition as precisely as we did for random graph $k$-colouring can be applied to models with finite inverse tem-

perature. One problem occuring for these models is that the "cut up" decorated graph we investigated in order to determine the cluster size in Section 5.3 does not essentially consist of bounded tree components in the case of finite inverse temperatures. To us it is not clear how to solve this problem.
As a matter of course, apart from the condensation transition, it is also of considerable interest to obtain results on the actual satisfiability threshold in zero temperature problems. Up to now, there only exist rigorous proofs of its location for large values of $k$ in random $k$-SAT [DSS15], in random regular $k$-SAT [COP16], in random regular NAE-$k$-SAT [DSS16] and for large values of $d$ in the independent set problem on $d$-regular random graphs [DSS16+]. It would complete the picture to also establish its location in further random CSPs. In many problems, it is even still not verified that the satisfiability threshold is different from the condensation threshold as predicted by the cavity method. In any way, it remains an important research endeavour and an outstanding mathematical challenge to fully rigorize the predictions made by this method.
Concerning the distribution of the number of solutions, we believe that a combination of the second moment method and small subgraph conditioning could be used to obtain the limiting distribution of the number of solutions in e.g. random NAE-$k$-SAT, random $k$-XORSAT, random hypergraph $k$-colouring or in random regular models. However, for asymmetric problems like the well-known benchmark problem random $k$-SAT, we expect that the logarithm of the number of satisfying assignments exhibits stronger fluctuations and we doubt that a result similar to ours can be established.
In general, a complete description of all problems for which a limiting distribution can be found might be achievable and it possibly covers all models where the partition function on a tree on $n$ vertices is constant. In this case, the proof technique might be generalized to develop a generic proof suitable for all these models.
Going in a slightly different direction, the investigation could be extended to regimes beyond the condensation transition. [SSZ16+] enhanced the second moment method and analysed a certain Survey Propagation model in the case of random regular NAE-$k$-SAT. In this way, they were able to determine the total number of solutions for a typical instance in the whole satisfiable regime.

Additionally, it would certainly be of considerable interest to advance the rigorous study of algorithms, especially of certain message passing algorithms, as there is plenty of experimental work, but so far precise rigorous results are scarce.

To summarise, we can say that in the last decades much has been achieved in thoroughly understanding the various aspects and phenomena in random constraint satisfaction problems. The results in this thesis contribute to this endeavour. But investigation is in every respect far from being complete and the process will go on, offering new and exciting perspectives along the way.

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# A Complementary proofs: Condensation phase transition in random graph $k$-colouring

This chapter presents the remaining parts of the proof of Theorem 4.1.1. They are not part of this thesis' author's achievement and are only included here for the sake of completeness.

In section Section A.1 the first and second moment method are applied to prove bounds on $d_{\text{crit}}$. In Section A.2 calculations are performed in the planted model to prove Proposition 5.1.3. The last section Section A.3 is devoted to determining the frozen fixed point $\pi_{d,k,\boldsymbol{q}^*}$ of $\mathcal{F}_{d,k}$, to show that it is unique and that it describes the expected number of vertices in a certain tree process.

The chapter is a verbatim copy of parts of the paper *The condensation phase transition in random graph coloring* [BCOHRV16] that is joint work with Victor Bapst, Amin Coja-Oghlan, Samuel Hetterich and Dan Vilenchik and is published in the *Communications in Mathematical Physics 341 (2016)*.

## A.1. Groundwork: the first and the second moment method

In this section we prove Proposition 5.1.2 and also lay the foundations for the proof of Proposition 5.1.3.

### A.1.1. The first moment

We start by deriving an upper bound on $\Phi_k(d)$ by computing the expected number of $k$-colourings. To avoid fluctuations of the total number of edges, we work with the $\mathcal{G}(n, m)$ model.

**Lemma A.1.1.** *We have* $\mathbb{E}[Z_k(\mathcal{G}(n, m))] = \Theta(k^n(1 - 1/k)^m)$.

Lemma A.1.1 is folklore. We carry the proof out regardless to make a few observations that will be important later. For a map $\sigma : [n] \to [k]$, let

$$\mathcal{F}(\sigma) = \sum_{i=1}^{k} \binom{|\sigma^{-1}(i)|}{2}$$

be the number of "forbidden pairs" of vertices that are coloured the same under $\sigma$. By convexity,

$$N - \mathcal{F}(\sigma) \geq (1 - 1/k)N, \qquad \text{with } N = \binom{n}{2}. \tag{A.1.1}$$

Hence, using Stirling's formula, we find

$$\mathbb{P}\left[\sigma \text{ is a } k\text{-colouring of } G(n,m)\right] = \binom{N - \mathcal{F}(\sigma)}{m} / \binom{N}{m} \leq O((1 - 1/k)^m). \tag{A.1.2}$$

As there are $k^n$ possible maps $\sigma$ in total, the linearity of expectation and (A.1.2) imply

$$\mathbb{E}[Z_k(\mathcal{G}(n,m))] = O(k^n(1 - 1/k)^m).$$

To bound $\mathbb{E}[Z_k(\mathcal{G}(n,m))]$ from below, call $\sigma : [n] \to [k]$ *balanced* if $|\sigma^{-1}(i) - \frac{n}{k}| \leq \sqrt{n}$ for all $i \in [k]$. Let $\mathrm{Bal} = \mathrm{Bal}_{n,k}$ be the set of all balanced $\sigma : [n] \to [k]$. For $\sigma \in \mathrm{Bal}$, we verify easily that $N - \mathcal{F}(\sigma) = (1 - 1/k)N + O(n)$. Thus, (A.1.2) and Stirling's formula yield

$$\mathbb{P}\left[\sigma \text{ is a } k\text{-colouring of } G(n,m)\right] = \Omega((1 - 1/k)^m) \qquad \text{for any } \sigma \in \mathrm{Bal}. \tag{A.1.3}$$

As $|\mathrm{Bal}| = \Omega(k^n)$ by Stirling, the linearity of expectation and (A.1.3) imply $\mathbb{E}[Z_k(\mathcal{G}(n,m))] = \Omega(k^n(1 - 1/k)^m)$, whence Lemma A.1.1 follows.

Letting $Z_{k,\mathrm{bal}}$ denote the number of balanced $k$-colourings, we obtain from the above argument

**Corollary A.1.2.** *For any $d \geq 0$, we have $\mathbb{E}[Z_{k,\mathrm{bal}}(G(n,m))] = \Theta(k^n(1 - 1/k)^m)$.*

As a further consequence of Lemma A.1.1, we obtain

**Corollary A.1.3.** *For any $c > 0$, we have*

$$\limsup_{n \to \infty} \mathbb{E}[Z_k(G(n, c/n))^{1/n}] \leq k(1 - 1/k)^{c/2}.$$

*Proof.* Lemma A.1.1 and Jensen's inequality yield

$$\mathbb{E}[Z_k(G(n,m))^{1/n}] \leq \mathbb{E}[Z_k(G(n,m))]^{1/n} \leq k(1 - 1/k)^{d/2} + o(1). \tag{A.1.4}$$

Now, let $c > 0$ and set $d = c - \varepsilon$ for some $\varepsilon > 0$. The number of edges in $G(n, c/n)$ is binomially distributed with mean $(1 + o(1))cn/2 = m + \Omega(n)$. Hence, by the Chernoff bound the probability of the event $\mathcal{A}$ that $G(n, c/n)$ has at least $m$ edges tends to 1 as $n \to \infty$. Because adding further edges can only decrease the number of $k$-colourings and since the number of $k$-colourings is trivially bounded by $k^n$, we obtain from (A.1.4) that

$$\begin{aligned}\mathbb{E}[Z_k(G(n,c/n))^{1/n}] \leq & \mathbb{E}[Z_k(G(n,c/n))^{1/n} \cdot \mathbf{1}_{\mathcal{A}}] + \mathbb{P}[\mathcal{A} \text{ does not occur}] \cdot k \\ \leq & \mathbb{E}[Z_k(G(n,m))^{1/n}] + o(1) \leq k(1-1/k)^{d/2} + o(1).\end{aligned}$$

Consequently, $\limsup \mathbb{E}[Z_k(G(n,c/n))^{1/n}] \leq k(1-1/k)^{d/2}$. This holds for any $d > c$. Hence, letting $\varepsilon = d - c \to 0$, we see that

$$\limsup \mathbb{E}[Z_k(G(n,c/n))^{1/n}] \leq k(1-1/k)^{c/2},$$

as desired. □

We conclude this subsection with the following crucial observation.

**Lemma A.1.4.** *Let*

$$\begin{aligned}D_* &= \left\{d > 0 : \liminf \mathbb{E}[Z_k(G(n,d/n))^{1/n}] < k(1-1/k)^{d/2}\right\}, \\ D^* &= \left\{d > 0 : \limsup \mathbb{E}[Z_k(G(n,d/n))^{1/n}] < k(1-1/k)^{d/2}\right\}.\end{aligned}$$

*If* $d_1 \in D_*$ *and* $d_2 > d_1$, *then* $d_2 \in D_*$. *Similarly, if* $d_1 \in D^*$ *and* $d_2 > d_1$, *then* $d_2 \in D^*$.

*Proof.* Let $0 < d_1 < d_2$ and let $q \sim (d_2 - d_1)/n$ be such that $d_1/n + (1 - d_1/n)q = d_2/n$. Let us denote the random graph $G(n, d_1/n)$ by $G_1$. Furthermore, let $G_2$ be a random graph obtained from $G_1$ by joining any two vertices that are not already adjacent in $G_1$ with probability $q$ independently. Then $G_2$ is identical to $G(n, d_2/n)$, because in $G_2$ any two vertices are adjacent with probability $d_1/n + (1 - d_1/n)q = d_2/n$ independently. Set $N = \binom{n}{2}$.

Let $e(G_i)$ signify the number of edges in $G_i$ for $i = 1, 2$. Because $e(G_i)$ is a binomial random variable with mean $\mu_i = \frac{d_i}{n} \cdot N = nd_i/2 + O(1)$, the Chernoff bound implies that

$$\begin{aligned}\mathbb{P}\left[|e(G_1) - \mu_1| > n^{2/3}\right] &= o(1), \\ p_{\mu_1,\mu_2}(G_1, G_2) = \mathbb{P}\left[|e(G_2) - e(G_1) - (\mu_2 - \mu_1)| > n^{2/3}\right] &= o(1).\end{aligned} \tag{A.1.5}$$

Further, since $Z_k^{1/n} \leq k$ with certainty, (A.1.5) implies that

$$\begin{aligned}\mathbb{E}[Z_k(G_2)^{1/n} \mid Z_k(G_1)] &= \mathbb{E}[Z_k(G_2)^{1/n} \mid Z_k(G_1),\ |e(G_2) - e(G_1) - (\mu_2 - \mu_1)| \\ &\leq n^{2/3}](1 - p_{\mu_1,\mu_2}(G_1, G_2)) + k \cdot p_{\mu_1,\mu_2}(G_1, G_2) \\ &\leq \mathbb{E}[Z_k(G_2)^{1/n}\mathbf{1}_{|e(G_2)-e(G_1)-(\mu_2-\mu_1)|\leq n^{2/3}} \mid Z_k(G_1)] + o(1). \qquad \text{(A.1.6)}\end{aligned}$$

Suppose that we condition on $e(G_1), e(G_2)$ and $|e(G_1) - \mu_1| \leq n^{2/3}$, $|e(G_2) - e(G_1) - (\mu_2 - \mu_1)| \leq n^{2/3}$. Assume that $\sigma$ is a $k$-colouring of $G_1$. What is the probability that $\sigma$ remains a $k$-colouring of $G_2$? For this to happen, none of the $e(G_2) - e(G_1)$ additional edges must be among the $\mathcal{F}(\sigma)$ pairs of vertices with the same colour under $\sigma$. Using Stirling's formula, we see that the probability of $\sigma$ remaining a $k$-colouring in $G_2$ is bounded by

$$\gamma = \binom{N - \mathcal{F}(\sigma) - e(G_1)}{e(G_2) - e(G_1)} \Big/ \binom{N - e(G_1)}{e(G_2) - e(G_1)} \leq (1 - 1/k)^{(d_2 - d_1 + o(1))n/2}. \qquad \text{(A.1.7)}$$

Hence, by (A.1.6), Jensen's inequality and (A.1.7)

$$\begin{aligned}\mathbb{E}[Z_k(G_2)^{1/n} \mid Z_k(G_1)] &\leq \mathbb{E}\left[Z_k(G_2) \cdot \mathbf{1}_{|e(G_2)-e(G_1)-(\mu_2-\mu_1)|\leq n^{2/3}} \,\Big|\, Z_k(G_1)\right]^{1/n} + o(1) \\ &\leq \gamma^{1/n} Z_k(G_1)^{1/n} + o(1) \leq (1 - 1/k)^{(d_2 - d_1)/2} Z_k(G_1)^{1/n} + o(1). \qquad \text{(A.1.8)}\end{aligned}$$

Averaging (A.1.8) over $G_1$, we obtain

$$\begin{aligned}\mathbb{E}[Z_k(G(n, d_2/n))^{1/n}] &= \mathbb{E}[Z_k(G_2)^{1/n}] \\ &\leq (1 - 1/k)^{(d_2 - d_1)/2}\mathbb{E}[Z_k(G_1)^{1/n} \cdot \mathbf{1}_{|e(G_1)-\mu_1|\leq n^{2/3}}] \\ &\qquad + k \cdot \mathbb{P}\left[\mathbf{1}_{|e(G_1)-\mu_1|>n^{2/3}}\right] + o(1) \\ &\leq (1 - 1/k)^{(d_2 - d_1)/2}\mathbb{E}[Z_k(G(n, d_1/n))^{1/n}] + o(1) \quad \text{[due to (A.1.5)]}.\end{aligned}$$

Thus, if $\mathbb{E}[Z_k(G(n, d_1/n))^{1/n}] < k(1 - 1/k)^{d_1/2} - \delta + o(1)$, then

$$\mathbb{E}[Z_k(G(n, d_2/n)^{1/n}] \leq k(1 - 1/k)^{d_2/2} - \varepsilon + o(1)$$

for some $\varepsilon = \varepsilon(\delta, k, d_1, d_2) > 0$. Taking $n \to \infty$ yields the assertion. □

### A.1.2. The second moment

The main technical step in the article [COV13] that yields the lower bound (4.2.1) on $d_{\mathrm{col}}$ is a second moment argument for a random variable $Z_{k,\mathrm{tame}}$ related to the number of $k$-colourings. We are going to employ this second moment estimate to bound $Z_k(G(n,d/n))$ from below.

The random variable $Z_{k,\mathrm{tame}}$ counts $k$-colourings with some additional properties. Suppose that $\sigma$ is a balanced $k$-colouring of a graph $G$ on $[n]$. We call $\sigma$ *separable* if for any balanced $\tau \in \mathcal{C}(G,\sigma)$ and any $i \in [k]$ we have

$$\rho_{ii}(\sigma,\tau) \geq (1-\kappa)/k, \text{ where } \kappa = \ln^{20} k/k.$$

Thus, if $\sigma$ is a balanced, separable $k$-colouring, then for any colour $i$ and for any other balanced $k$-colouring $\tau$ in the cluster of $\sigma$, a $1-\kappa+o(1)$-fraction of the vertices coloured $i$ under $\sigma$ are coloured $i$ under $\tau$ as well. In particular, the clusters of any two such colourings are either disjoint or identical.

**Definition A.1.5.** *Let $G$ be a graph with $n$ vertices and $m$ edges. A $k$-colouring $\sigma$ of $G$ is* tame *if*

***T1*** *$\sigma$ is balanced,*
***T2*** *$\sigma$ is separable, and*
***T3*** $|\mathcal{C}(G,\sigma) \cap \mathrm{Bal}| \leq k^n(1-1/k)^m$.

Let $Z_{k,\mathrm{tame}}(G)$ denote the number of tame $k$-colourings of $G$.

**Lemma A.1.6** ([COV13])**.** *Assume that $d > 0$ is such that*

$$\liminf_{n\to\infty} \frac{\mathbb{E}[Z_{k,\mathrm{tame}}(G(n,m))]}{k^n(1-1/k)^m} > 0. \tag{A.1.9}$$

*Then*

$$\liminf_{n\to\infty} \frac{\mathbb{E}[Z_{k,\mathrm{tame}}(G(n,m))]^2}{\mathbb{E}[Z_{k,\mathrm{tame}}(G(n,m))^2]} > 0.$$

*Furthermore, there exists $\varepsilon_k = o_k(1)$ such that (A.1.9) is satisfied if $d \leq (2k-1)\ln k - 2\ln 2 - \varepsilon_k$.*

As fleshed out in [COV13], together with the sharp threshold result from [AF99], Lemma A.1.6 implies that $G(n,d/n)$ is $k$-colourable w.h.p. if $d \leq (2k-1)\ln k - 2\ln 2 - \varepsilon_k$. Here we are going to combine Lemma A.1.6 with the following variant of that sharp threshold result to obtain a lower bound on the number of $k$-colourings.

**Lemma A.1.7** ([ACO08])**.** *For any $k \geq 3$ and for any real $\xi > 0$, there is a sequence $d_{k,\xi}(n)$ such that for any $\varepsilon > 0$ the following holds.*

1. *If* $p(n) < (1-\varepsilon)d_{k,\xi}(n)/n$*, then* $Z_k(G(n,p(n))) \geq \xi^n$ *w.h.p..*
2. *If* $p(n) > (1+\varepsilon)d_{k,\xi}(n)/n$*, then* $Z_k(G(n,p(n))) < \xi^n$ *w.h.p..*

Lemmas A.1.6 and A.1.7 entail the following lower bound on $d_{\text{crit}}$.

**Lemma A.1.8.** *Assume that* $d^* > 0$ *and* $\varepsilon > 0$ *are such that (A.1.9) holds for any* $d \in (d^* - \varepsilon, d^*)$*. Then* $d_{\text{crit}} \geq d^*$*.*

*Proof.* Assume for contradiction that $d^*$ is such that (A.1.9) holds for all $d \in (d^* - \varepsilon, d^*)$ but $d_{\text{crit}} < d^*$. Pick and fix a number

$$\max\{d^* - \varepsilon, d_{\text{crit}}\} < d_* < d^*.$$

Corollary A.1.3 implies that $\limsup \mathbb{E}[Z_k(G(n, d_*/n))^{1/n}] \leq k(1-1/k)^{d_*/2}$. Therefore, since $d_* > d_{\text{crit}}$, there exists $\varepsilon_* > 0$ such that

$$\liminf_{n\to\infty} \mathbb{E}[Z_k(G(n, d_*/n))^{1/n}] < k(1-1/k)^{d_*/2} - \varepsilon_*. \tag{A.1.10}$$

Further, pick and fix $d_* < \hat{d} < d^*$ such that $k(1-1/k)^{\hat{d}/2} > k(1-1/k)^{d_*/2} - \varepsilon_*$ and $\xi$ such that

$$k(1-1/k)^{d_*/2} - \varepsilon_* < \xi < k(1-1/k)^{\hat{d}/2}. \tag{A.1.11}$$

We are going to use Lemmas A.1.6 and A.1.7 to establish a lower bound on $Z_k(G(n, d_*/n))$ that contradicts (A.1.10). By the Paley-Zygmund inequality and because (A.1.9) holds for any $d^* - \varepsilon < d < d^*$,

$$\mathbb{P}\left[Z_{k,\text{tame}}(\mathcal{G}(n,m)) \geq \frac{1}{2}\mathbb{E}[Z_{k,\text{tame}}(\mathcal{G}(n,m))]\right] \geq \frac{\mathbb{E}[Z_{k,\text{tame}}(\mathcal{G}(n,m))]^2}{4 \cdot \mathbb{E}[Z_{k,\text{tame}}(\mathcal{G}(n,m))^2]} \tag{A.1.12}$$

for any $d^* - \varepsilon < d < d^*$. Moreover, Lemma A.1.6 and (A.1.12) imply

$$\liminf_{n\to\infty} \mathbb{P}\left[Z_{k,\text{tame}}(\mathcal{G}(n,m)) \geq \frac{1}{2}\mathbb{E}[Z_{k,\text{tame}}(\mathcal{G}(n,m))\right] > 0 \tag{A.1.13}$$

for any $d^* - \varepsilon < d < d^*$. Further, because (A.1.9) is true for any $d^* - \varepsilon < d < d^*$ and $\xi < k(1-1/k)^{d/2}$ for any $d < \hat{d} < d^*$, we see that

$$\frac{1}{2}\mathbb{E}[Z_{k,\text{tame}}(\mathcal{G}(n,m))] = \Omega(k^n(1-1/k)^m) > \xi^n \qquad \text{for any } d < \hat{d}.$$

Hence, (A.1.13) implies

$$\liminf_{n\to\infty} \mathbb{P}\left[Z_{k,\text{tame}}(\mathcal{G}(n,m)) \geq \xi^n\right] > 0 \qquad \text{for any } d < \hat{d}. \tag{A.1.14}$$

Since the number of edges in $G(n, d/n)$ has a binomial distribution with mean $m$, with probability at least $1/3$ the number of edges in $G(n, d/n)$ does not exceed $m$. Therefore, (A.1.14) implies that

$$\liminf_{n\to\infty} \mathbb{P}\left[Z_k(G(n, d/n)) \geq \xi^n\right] \geq \frac{1}{3} \liminf_{n\to\infty} \mathbb{P}\left[Z_{k,\text{tame}}(\mathcal{G}(n, m)) \geq \xi^n\right] > 0 \quad \text{for any } d < \hat{d}. \tag{A.1.15}$$

Moreover, (A.1.15) entails that the sequence $d_{k,\xi}(n)$ from Lemma A.1.7 satisfies $\liminf d_{k,\xi}(n) \geq \hat{d}$. Therefore,

$$\lim_{n\to\infty} \mathbb{P}\left[Z_k(G(n, d/n)) \geq \xi^n\right] = 1 \quad \text{for any } d < \hat{d}. \tag{A.1.16}$$

Since $d_* < \hat{d}$, (A.1.16) entails that

$$\liminf_{n\to\infty} \mathbb{E}\left[Z_{k,\text{tame}}(G(n, d_*/n))^{1/n}\right] \geq \xi. \tag{A.1.17}$$

Combining (A.1.10), (A.1.11) and (A.1.17) yields a contradiction, which refutes our assumption that $d_{\text{crit}} < d^*$. □

*Proof of Proposition 5.1.2.* The first assertion follows from Corollary A.1.3. Hence, the second assertion

$$d_{\text{crit}} = \sup\left\{d \geq 0 : \liminf_{n\to\infty} \mathbb{E}[Z_k(G(n, d/n))^{1/n}] \geq k(1 - 1/k)^{d/2}\right\}.$$

is immediate from Lemma A.1.4. The third assertion follows from Lemma A.1.6 and Lemma A.1.8. □

## A.2. The planted model

### A.2.1. Overview

The aim in this section is to prove Proposition 5.1.3. The proof of the first part is fairly straightforward. More precisely, in Subsection A.2.2 we are going to establish

**Lemma A.2.1.** *Assume that* $(2k - 1)\ln k - 2 \leq d \leq (2k - 1)\ln k$ *is such that (5.1.2) holds. Then* $d_{\text{crit}} \geq d$.

The more challenging claim is that $d \geq d_{\text{crit}}$ if typically the cluster in the planted model is "too big". To prove this, we consider a variant of the planted model in which the number of edges is fixed. More precisely, for a map $\sigma : [n] \to [k]$ we let $G(n, m, \sigma)$ denote a graph on the vertex set $V = [n]$ with precisely $m$ edges that do not join vertices $v, w$ with $\sigma(v) = \sigma(w)$ chosen uniformly at random. In other words, $G(n, m, \sigma)$ is just the random graph $G(n, m)$ conditioned on the event that $\sigma$ is a $k$-

colouring. The following lemma, which is a variant of the "planting trick" from [ACO08], establishes a general relationship between $G(n,m)$ and $G(n,m,\sigma)$.

**Lemma A.2.2.** *Let $d > 0$. Assume that there exists a sequence $(\mathcal{E}_n)_{n\geq 1}$ of events such that*

$$\lim_{n\to\infty} \mathbb{P}\left[G(n,m) \in \mathcal{E}_n\right] = 1 \quad \textit{while} \quad \limsup_{n\to\infty} \mathbb{P}\left[G(n,m,\boldsymbol{\sigma}) \in \mathcal{E}_n\right]^{1/n} < 1. \tag{A.2.1}$$

*Then for any $c > d$ we have $\limsup \mathbb{E}[Z_k(G(n,c/n))^{1/n}] < k(1-1/k)^{c/2}$. In particular, $d_{\mathrm{crit}} \leq d$.*

We prove Lemma A.2.2 in Subsection A.2.2. Hence, assuming that the typical cluster size in the planted model is "too big" w.h.p., we need to exhibit events $\mathcal{E}_n$ such that (A.2.1) holds. An obvious choice seems to be

$$\mathcal{E}_n(\varepsilon) = \left\{ Z_k^{1/n} \leq k(1-1/k)^{d/2} + \varepsilon \right\}.$$

But (A.2.1) requires that the probability that $\mathcal{E}_n$ occurs in $G(n,m,\sigma)$ is *exponentially* small, and neither the cluster size nor $Z_k$ are known to be sufficiently concentrated to obtain such an exponentially small probability.

Therefore, we define the events $\mathcal{E}_n$ by means of another random variable. For a graph $G = (V,E)$ and a map $\sigma : V \to [k]$, let $\mathcal{H}_G(\sigma)$ be the number of edges $\{v,w\}$ of $G$ such that $\sigma(v) = \sigma(w)$. In words, $\mathcal{H}_G(\sigma)$ is the number of edges of $G$ that are monochromatic under $\sigma$. Furthermore, given $\beta > 0$ let

$$Z_{\beta,k}(G) = \sum_{\sigma:V\to[k]} \exp(-\beta \cdot \mathcal{H}_G(\sigma)),$$

the partition function of the $k$-spin Potts antiferromagnet on $G$ at inverse temperature $\beta$.

For large $\beta$, there is a stiff "penalty factor" of $\exp(-\beta)$ for any monochromatic edge. Thus, we expect that $Z_{\beta,k}$ becomes a good proxy for $Z_k$ as $\beta \to \infty$. At the same time, $\ln Z_{\beta,k}$ enjoys a Lipschitz property. Namely, suppose that we obtain a graph $G'$ from $G$ by either adding or removing a single edge. Then

$$|\ln(Z_{\beta,k}(G)) - \ln(Z_{\beta,k}(G'))| \leq \beta. \tag{A.2.2}$$

Due to this Lipschitz property, one can easily show that $\ln Z_{\beta,k}$ is tightly concentrated. More precisely, we have

**Lemma A.2.3.** *For any fixed $d > 0$, $\varepsilon > 0$ there is $\alpha > 0$ such that the following is true. Suppose that $(\sigma_n)_{n\geq 1}$ is a sequence of maps $[n] \to [k]$. Then for all large enough $n$,*

$$\mathbb{P}\left[|\ln(Z_{\beta,k}(G(n,p',\sigma_n))) - \mathbb{E}[\ln Z_{\beta,k}(G(n,p',\sigma_n))]| > \varepsilon n\right] \leq \exp(-\alpha n).$$

*Proof.* This is immediate from the Lipschitz property (A.2.2) and McDiarmid's inequality [McD98, Theorem 3.8]. □

Furthermore, in Subsection A.2.2 we show that Lemma A.2.3 implies

**Lemma A.2.4.** *Assume that $d$ is such that (5.1.3) holds. Then there exist $z, \beta > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln Z_{\beta,k}(\mathcal{G}(n,m)) \leq z\right] = 1$$

$$\textit{while} \quad \limsup_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln Z_{\beta,k}(G(n,m,\boldsymbol{\sigma})) \leq z\right]^{1/n} < 1.$$

Finally, Proposition 5.1.3 is immediate from Lemmas A.2.1, A.2.2 and A.2.4.

### A.2.2. Remaining proofs

#### Proof of Lemma A.2.1

We use the following observation from [COV13].

**Lemma A.2.5** ([COV13])**.** *Suppose that $(2k-1)\ln k - 2 \leq d \leq (2k-1)\ln k$. Let $p'$ be as in (5.1.1). Then the planted colouring $\boldsymbol{\sigma}$ is separable in $G(n,p',\boldsymbol{\sigma})$ w.h.p..*

*Proof of Lemma A.2.1.* If (5.1.2) holds, then there exists $\varepsilon > 0$ such that with $p'$ from (5.1.1) we have

$$\lim_{n\to\infty} \mathbb{P}\left[|\mathcal{C}(G(n,p',\boldsymbol{\sigma}),\boldsymbol{\sigma})| \leq k^n(1-1/k)^m \exp(-\varepsilon n)\right] = 1. \tag{A.2.3}$$

Pick a number $d^* > d$ such that with $m^* = \lceil d^* n/2 \rceil$ we have

$$k^n(1-1/k)^{m^*} \geq k^n(1-1/k)^m \exp(-\varepsilon n/2).$$

We claim that if we choose $\boldsymbol{\sigma} : [n] \to [k]$ uniformly at random and independently a random graph $G(n,m^*)$, then

$$\liminf_{n\to\infty} \mathbb{P}\left[\boldsymbol{\sigma} \text{ is tame} | \boldsymbol{\sigma} \text{ is a } k\text{-colouring of } G(n,m^*)\right] > 0. \tag{A.2.4}$$

To see this, let $\mathcal{E}$ be the event that the random graph $G(n,p',\boldsymbol{\sigma})$ has no more than $m^*$ edges. Because the number of edges in $G(n,p',\boldsymbol{\sigma})$ is binomially distributed with mean $m < m^* - \Omega(n)$, the Chernoff bound implies that $\mathbb{P}[\mathcal{E}] = 1 - o(1)$. Therefore, (A.2.3) implies

$$\lim_{n\to\infty} \mathbb{P}\left[|\mathcal{C}(G(n,p',\boldsymbol{\sigma}),\boldsymbol{\sigma})| \le k^n(1-1/k)^m \exp(-\varepsilon n) \,|\, \mathcal{E}\right] = 1. \tag{A.2.5}$$

Further, set $d'' = kd^*/(k-1)$ and let $p'' = d''/n > p'$. Then we can think of $G(n,p'',\boldsymbol{\sigma})$ as being obtained from $G(n,p',\boldsymbol{\sigma})$ by adding further random edges. More precisely, let $\mathcal{A}$ be the event that $G(n,p'',\boldsymbol{\sigma})$ contains precisely $m^*$ edges and set

$$p'_n = \mathbb{P}\left[|\mathcal{C}(G(n,p'',\boldsymbol{\sigma}),\boldsymbol{\sigma})| \le k^n(1-1/k)^{m^*} \,|\, \mathcal{A}\right].$$

Since adding edges can only decrease the cluster size, (A.2.5) entails

$$\lim_{n\to\infty} p'_n \ge \lim_{n\to\infty} \mathbb{P}\left[|\mathcal{C}(G(n,p',\boldsymbol{\sigma}),\boldsymbol{\sigma})| \le k^n(1-1/k)^m \exp(-\varepsilon n) \,|\, \mathcal{E}\right] = 1. \tag{A.2.6}$$

Similarly, let $p''_n = \mathbb{P}[\boldsymbol{\sigma} \text{ is separable in } G(n,p'',\boldsymbol{\sigma}) \,|\, \mathcal{A}]$. Then Lemma A.2.5 implies

$$\lim_{n\to\infty} p''_n \ge \lim_{n\to\infty} \mathbb{P}\left[\boldsymbol{\sigma} \text{ is separable in } G(n,p',\boldsymbol{\sigma}) \,|\, \mathcal{E}\right] = 1.$$

Further, consider $p'''_n = \mathbb{P}[\boldsymbol{\sigma} \text{ is balanced}]$. Then by Stirling's formula,

$$\liminf_{n\to\infty} p'''_n \quad > \quad 0. \tag{A.2.7}$$

Finally, let $p_n = \mathbb{P}[\boldsymbol{\sigma} \text{ is a tame } k\text{-colouring of } G(n,p'',\boldsymbol{\sigma})|\mathcal{A}]$. Given the event $\mathcal{A}$, $G(n,p'',\boldsymbol{\sigma})$ is just a uniformly random graph with $m^*$ edges in which $\boldsymbol{\sigma}$ is a $k$-colouring. Hence,

$$p_n = \mathbb{P}[\boldsymbol{\sigma} \text{ is tame}|\boldsymbol{\sigma} \text{ is a } k\text{-colouring of } G(n,m^*)].$$

As (A.2.6)–(A.2.7) yield $\liminf_{n\to\infty} p_n > 0$, we obtain (A.2.4).

The estimate (A.2.4) enables us to bound $\mathbb{E}[Z_{k,\mathrm{tame}}(G(n,m^*))]$ from below. Indeed, by the linearity of expectation

$$\begin{aligned}
&\mathbb{E}[Z_{k,\mathrm{tame}}(G(n,m^*))] \\
&\quad = \sum_{\sigma:[n]\to[k]} \mathbb{P}[\sigma \text{ is a tame } k\text{-colouring of } G(n,m^*)] \\
&\quad = k^n \cdot \mathbb{P}[\boldsymbol{\sigma} \text{ is a tame } k\text{-colouring of } G(n,m^*)] \\
&\quad = k^n\, \mathbb{P}[\boldsymbol{\sigma} \text{ is a } k\text{-colouring of } G(n,m^*)]\, \mathbb{P}[\boldsymbol{\sigma} \text{ is tame}|\, \boldsymbol{\sigma} \text{ is a } k\text{-colouring of } G(n,m^*)] \\
&\quad = k^n\, \mathbb{P}[\boldsymbol{\sigma} \text{ is a } k\text{-colouring of } G(n,m^*)] \cdot p_n \\
&\quad = \mathbb{E}[Z_k(G(n,m^*))] \cdot p_n.
\end{aligned}$$

Thus, Lemma A.1.1 and (A.2.4) yield

$$\liminf_{n\to\infty} \frac{\mathbb{E}[Z_{k,\mathrm{tame}}(G(n,m^*)]}{k^n(1-1/k)^{m^*}} > 0.$$

As this holds for all $d^*$ in an interval $(d+\eta, d+2\eta)$ with $\eta > 0$, Lemma A.1.8 implies that $d_{\mathrm{crit}} \geq d$. □

#### Proof of Lemma A.2.2

**Lemma A.2.6.** *Assume that* $d > 0$ *is such that* $\limsup \mathbb{E}[Z_k(\mathcal{G}(n,m))^{1/n}] < k(1-1/k)^{d/2}$. *Then for any* $c > d$ *we have* $\limsup \mathbb{E}[Z_k(G(n,c/n))^{1/n}] < k(1-1/k)^{c/2}$.

*Proof.* Assume that $d, \delta > 0$ are such that $\limsup \mathbb{E}[Z_k(\mathcal{G}(n,m))^{1/n}] < k(1-1/k)^{d/2} - \delta$. We claim that

$$d^* \in D^* = \left\{c > 0 : \limsup \mathbb{E}[Z_k(G(n,c/n))^{1/n}] < k(1-1/k)^{c/2}\right\} \quad \text{for any } d^* > d. \qquad \text{(A.2.8)}$$

Indeed, the number $e(G(n,d^*/n))$ of edges of $G(n,d^*/n)$ is binomially distributed with mean $(1+o(1))d^*n/2$. Since $d, d^*$ are independent of $n$ and $d^* > d$, the Chernoff bound implies that

$$\mathbb{P}\left[e(G(n,d^*/n)) \leq m\right] \leq \exp(-\Omega(n)). \qquad \text{(A.2.9)}$$

Further, if we condition on the event that $m^* = e(G(n,d^*/n)) > m$, then we can think of $G(n,d^*/n)$ as follows: first, create a random graph $\mathcal{G}(n,m)$; then, add another $m^* - m$ random edges. Since the addition of further random edges cannot increase the number of $k$-colourings, by (A.2.9) we find that

$$\begin{aligned}\mathbb{E}[Z_k(G(n,d^*/n))^{1/n}] &\leq \mathbb{E}[Z_k(G(n,d^*/n))^{1/n}|m^* > m] + k\cdot\mathbb{P}\left[e(G(n,d^*/n)) \leq m\right] \\ &\leq \mathbb{E}[Z_k(G(n,m))^{1/n}] + o(1).\end{aligned}$$

Taking $n \to \infty$, and assuming that $d^* > d$ is sufficiently close to $d$, we conclude that

$$\limsup_{n\to\infty} \mathbb{E}[Z_k(G(n,d^*/n))^{1/n}] \leq k(1-1/k)^{d/2} - \delta < k(1-1/k)^{d_*/2}.$$

Hence, for any $\varepsilon > 0$ there is $d^* \in (d, d+\varepsilon)$ such that $d^* \in D^*$. Thus, (A.2.8) follows from Lemma A.1.4. □

*Proof of Lemma A.2.2.* Assuming the existence of $d$ and $(\mathcal{E}_n)_{n\geq 1}$ as in Lemma A.2.2, we are going to argue that

$$\limsup \mathbb{E}[Z_k(\mathcal{G}(n,m))^{1/n}] < k(1-1/k)^{d/2}. \qquad \text{(A.2.10)}$$

Then the assertion follows from Lemma A.2.6.

Since $Z_k^{1/n} \leq k$ with certainty and $\mathbb{P}[\mathcal{G}(n,m) \in \mathcal{E}_n] = 1 - o(1)$, Jensen's inequality yields

$$\mathbb{E}[Z_k(\mathcal{G}(n,m))^{1/n}] = \mathbb{E}[Z_k(\mathcal{G}(n,m))^{1/n} \cdot \mathbf{1}_{\mathcal{E}_n}] + o(1) \leq \mathbb{E}[Z_k(\mathcal{G}(n,m)) \cdot \mathbf{1}_{\mathcal{E}_n}]^{1/n} + o(1).$$

Furthermore, by the linearity of expectation,

$$\begin{aligned}\mathbb{E}[Z_k(\mathcal{G}(n,m)) \cdot \mathbf{1}_{\mathcal{E}_n}] &= \sum_{\sigma:[n]\to[k]} \mathbb{P}\left[\mathcal{E}_n \text{ occurs and } \sigma \text{ is a } k\text{-colouring of } \mathcal{G}(n,m)\right] \\ &= \sum_{\sigma:[n]\to[k]} \mathbb{P}\left[\mathcal{E}_n | \sigma \text{ is a } k\text{-colouring of } \mathcal{G}(n,m)\right] \\ &\qquad \cdot \mathbb{P}\left[\sigma \text{ is a } k\text{-colouring of } \mathcal{G}(n,m)\right] \\ &= \sum_{\sigma:[n]\to[k]} \mathbb{P}\left[G(n,m,\sigma) \in \mathcal{E}_n\right] \\ &\qquad \cdot \mathbb{P}\left[\sigma \text{ is a } k\text{-colouring of } \mathcal{G}(n,m)\right]. \end{aligned} \tag{A.2.11}$$

To estimate the last factor, we use (A.1.1) and Stirling's formula, which yield

$$\mathbb{P}\left[\sigma \text{ is a } k\text{-colouring of } \mathcal{G}(n,m)\right] \leq \binom{\binom{n}{2} - \mathcal{F}(\sigma)}{m} / \binom{\binom{n}{2}}{m} \leq O((1-1/k)^m).$$

Plugging this estimate into (A.2.11) and recalling that $\boldsymbol{\sigma}$ is a random map $[n] \to [k]$, we obtain

$$\begin{aligned}\mathbb{E}[Z_k(\mathcal{G}(n,m)) \cdot \mathbf{1}_{\mathcal{E}_n}] &\leq O((1-1/k)^m) \sum_{\sigma:[n]\to[k]} \mathbb{P}\left[G(n,m,\sigma) \in \mathcal{E}_n\right] \\ &= O((1-1/k)^m) \cdot k^n \, \mathbb{P}\left[G(n,m,\boldsymbol{\sigma}) \in \mathcal{E}_n\right] \\ &= O(\mathbb{E}[Z_k(G(n,m))]) \cdot \mathbb{P}\left[G(n,m,\boldsymbol{\sigma}) \in \mathcal{E}_n\right]. \end{aligned} \tag{A.2.12}$$

Finally, using our assumption that $\limsup \mathbb{P}\left[G(n,m,\boldsymbol{\sigma}) \in \mathcal{E}_n\right]^{1/n} < 1$ and combining (A.2.11) and (A.2.12), we see that

$$\begin{aligned}\limsup \mathbb{E}[Z_k(\mathcal{G}(n,m))^{1/n}] &\leq k(1-1/k)^{d/2} \cdot \limsup \mathbb{P}\left[G(n,m,\boldsymbol{\sigma}) \in \mathcal{E}_n\right]^{1/n} \\ &< k(1-1/k)^{d/2},\end{aligned}$$

thereby completing the proof of (A.2.10). □

**Proof of Lemma A.2.4**

**Lemma A.2.7.** *Let $d > 0$. For any $\varepsilon > 0$, there exists $\beta > 0$ such that*

$$\frac{1}{n}\ln \mathbb{E}[Z_{\beta,k}(\mathcal{G}(n,m))] \leq \ln k + \frac{d}{2}\ln(1-1/k) + \varepsilon.$$

*Proof.* For any fixed number $\gamma > 0$, we can choose $\beta(\gamma) > 0$ so large that $\ln k - \beta\gamma < 0$. Now, let $\mathcal{M}(\mathcal{G}(n,m))$ be the set of all $\sigma : [n] \to [k]$ such that at least $\gamma n$ edges are monochromatic under $\sigma$, and let $\overline{\mathcal{M}}(\mathcal{G}(n,m))$ contain all $\sigma \notin \mathcal{M}(\mathcal{G}(n,m))$. Then

$$\begin{aligned} Z_{\beta,k}(\mathcal{G}(n,m)) &\leq |\mathcal{M}(\mathcal{G}(n,m))| \cdot \exp(-\beta\gamma n) + |\overline{\mathcal{M}}(\mathcal{G}(n,m))| \\ &\leq k^n \cdot \exp(-\beta\gamma n) + |\overline{\mathcal{M}}(\mathcal{G}(n,m))| \leq 1 + |\overline{\mathcal{M}}(\mathcal{G}(n,m))|. \end{aligned} \tag{A.2.13}$$

Further, if $\sigma \in \overline{\mathcal{M}}(\mathcal{G}(n,m))$, then $\sigma$ is a $k$-colouring of a subgraph of $\mathcal{G}(n,m)$ containing $m - \gamma n$ edges. Hence, we obtain from Stirling's formula that for $\gamma = \gamma(\varepsilon) > 0$ small enough,

$$\begin{aligned} \mathbb{P}\left[\sigma \in \overline{\mathcal{M}}(\mathcal{G}(n,m))\right] &\leq \binom{\binom{n}{2}}{\gamma n} \cdot \binom{\binom{n}{2} - \mathcal{F}(\sigma)}{m - \gamma n} \Big/ \binom{\binom{n}{2}}{m} \\ &\leq (1-1/k)^m \cdot \exp(\varepsilon n/2). \end{aligned}$$

Hence,

$$\mathbb{E}[\overline{\mathcal{M}}(\mathcal{G}(n,m))] \leq k^n(1-1/k)^m \cdot \exp(\varepsilon n/2). \tag{A.2.14}$$

Combining (A.2.13) and (A.2.14), we obtain

$$\mathbb{E}[Z_{\beta,k}(\mathcal{G}(n,m))] \leq 1 + k^n(1-1/k)^m \cdot \exp(\varepsilon n/2) < k^n(1-1/k)^m \cdot \exp(\varepsilon n).$$

Taking logarithms completes the proof. □

**Lemma A.2.8.** *Assume that (5.1.3) is true. Then there exist a fixed number $\varepsilon > 0$, a sequence $\sigma_n$ of balanced maps $[n] \to [k]$ and a sequence $\mu_n$ of numbers satisfying $|\mu_n - dn/2| \leq \sqrt{n}$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left[|\mathcal{C}(G(n,\mu_n,\sigma_n),\sigma_n)|^{1/n} > k(1-1/k)^{d/2} + \varepsilon\right] = 1.$$

*Proof.* Let $\mathcal{A}$ be the event that the number of edges in the random graph $G(n,p',\boldsymbol{\sigma})$ differs from $dn/2$ by at most $\sqrt{n}$. Let $N = \binom{n}{2}$. For any balanced $\sigma : [n] \to [k]$, the expected number of edges in $G(n,p',\boldsymbol{\sigma})$ is

$$(N - \mathcal{F}(\sigma))p' = (1-1/k)Np' + O(1) = dn/2 + O(1). \tag{A.2.15}$$

Since the number of edges in $G(n,p',\boldsymbol{\sigma})$ is a binomial random variable, (A.2.15) shows together with the central limit theorem that there exists a fixed $\gamma > 0$ such that for sufficiently large $n$

$$\mathbb{P}\left[G(n, p', \sigma) \in \mathcal{A}\right] \geq \gamma \qquad \text{for all balanced } \sigma. \tag{A.2.16}$$

Furthermore, by Stirling's formula there is an $n$-independent number $\delta > 0$ such that for sufficiently large $n$ we have

$$\mathbb{P}\left[\boldsymbol{\sigma} \in \text{Bal}\right] \geq \delta. \tag{A.2.17}$$

Combining (A.2.16) and (A.2.17), we see that

$$\mathbb{P}\left[\boldsymbol{\sigma} \in \text{Bal},\ G(n, p', \boldsymbol{\sigma}) \in \mathcal{A}\right] = \mathbb{P}\left[\boldsymbol{\sigma} \in \text{Bal}\right] \cdot \mathbb{P}\left[G(n, p', \boldsymbol{\sigma}) \in \mathcal{A} | \boldsymbol{\sigma} \in \text{Bal}\right] \geq \gamma\delta > 0. \tag{A.2.18}$$

Thus, pick $\sigma_n \in \text{Bal}$ and $\mu_n \in [dn/2 - \sqrt{n}, dn/2 + \sqrt{n}]$ that maximize

$$p(\sigma_n, \mu_n) = \mathbb{P}\left[|\mathcal{C}(G(n, \mu_n, \sigma_n), \sigma_n)|^{1/n} > k(1 - 1/k)^{d/2} + \varepsilon\right].$$

Then (5.1.3) and (A.2.18) imply that $\lim_{n\to\infty} p(\sigma_n, \mu_n) = 1$. □

**Lemma A.2.9.** *For any $\eta > 0$, there is $\delta > 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left[\sum_{i=1}^{k} ||\boldsymbol{\sigma}^{-1}(i)| - n/k| > \eta n\right] \leq -\delta.$$

*Proof.* For each $i \in [k]$, the number $|\boldsymbol{\sigma}^{-1}(i)|$ is a binomially distributed random variable with mean $n/k$. Moreover, if $\sum_{i=1}^{k} ||\boldsymbol{\sigma}^{-1}(i)| - n/k| > \eta n$, then there is some $i \in [k]$ such that $||\boldsymbol{\sigma}^{-1}(i)| - n/k| > \eta n/k$. Thus, the assertion is immediate from the Chernoff bound. □

Let $\text{Vol}_G(S)$ be the sum of the degrees of the vertices in $S$ in the graph $G$.

**Lemma A.2.10.** *For any $\gamma > 0$, there is $\alpha > 0$ such that for any set $S \subset [n]$ of size $|S| \leq \alpha n$ and any $\sigma : [n] \to [k]$ we have*

$$\limsup \frac{1}{n} \ln \mathbb{P}\left[\text{Vol}_{G(n,p',\sigma)}(S) > \gamma n\right] \leq -\alpha.$$

*Proof.* Let $(X_v)_{v\in[n]}$ be a family of independent random variables with distribution $\text{Bin}(n, p')$. Then for any set $S$ the volume $\text{Vol}(S)$ in $G(n, p', \sigma)$ is stochastically dominated by $X_S = 2\sum_{v\in S} X_v$. Indeed, for each vertex $v \in S$ the degree is a binomial random variable with mean at most $np'$, and the only correlation amongst the degrees of the vertices in $S$ is that each edge joining two vertices in $S$ contributes two to $\text{Vol}(S)$. Furthermore, $\mathbb{E}[X_S] \leq 2d'|S|$. Thus, for any $\gamma > 0$ we can choose an $n$-independent $\alpha > 0$ such that for any $S \subset [n]$ of size $|S| \leq \alpha n$ we have $\mathbb{E}[X_S] \leq \gamma n/2$. In fact, the

Chernoff bound shows that by picking $\alpha > 0$ sufficiently small, we can ensure that

$$\mathbb{P}\left[\mathrm{Vol}(S) \geq \gamma n\right] \leq \mathbb{P}\left[X_S \geq \gamma n\right] \leq \exp(-\alpha n),$$

as desired. □

**Lemma A.2.11.** *Assume that there exist numbers $z > 0$, $\varepsilon > 0$ and a sequence $(\sigma_n)_{n\geq 1}$ of balanced maps $[n] \to [k]$ such that*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\ln Z_{\beta,k}(G(n, p', \sigma_n))\right] > z + \varepsilon.$$

*Then* $\limsup_{n\to\infty} \mathbb{P}\left[\ln Z_{\beta,k}(G(n, p', \boldsymbol{\sigma})) \leq nz\right]^{1/n} < 1$.

*Proof.* Let $Y = \frac{1}{n}\ln Z_{\beta,k}$ for the sake of brevity. Suppose that $n$ is large enough so that we have $\mathbb{E}\left[Y(G(n, p', \sigma_n))\right] > z + \varepsilon/2$. Set $n_i = |\sigma_n^{-1}(i)|$ and let $T$ be the set of all $\tau : [n] \to [k]$ such that $|\tau^{-1}(i)| = n_i$ for $i = 1, \ldots, k$. As $Z_{\beta,k}$ is invariant under permutations of the vertices, we have

$$\mathbb{E}\left[Y(G(n, p', \tau))\right] = \mathbb{E}\left[Y(G(n, p', \sigma_n))\right] > z + \varepsilon/2 \quad \text{for any } \tau \in T. \tag{A.2.19}$$

Let $\gamma = \varepsilon/(4\beta) > 0$. By Lemma A.2.10 there exists $\alpha > 0$ such that for large enough $n$ for any set $S \subset V$ of size $|S| \leq \alpha n$ and any $\sigma : [n] \to [k]$ we have

$$\mathbb{P}\left[\mathrm{Vol}_{G(n,p',\sigma)}(S) > \gamma n\right] \geq 1 - \exp(-\alpha n). \tag{A.2.20}$$

Pick and fix a small $0 < \eta < \alpha/3$ and let $\mathcal{A}$ be the event that $\sum_{i=1}^{k} ||\boldsymbol{\sigma}^{-1}(i)| - n/k| \leq \eta n$. Then by Lemma A.2.9 there exists an ($n$-independent) number $\delta = \delta(\beta, \varepsilon, \eta) > 0$ such that for $n$ large enough

$$\mathbb{P}\left[\mathcal{A}\right] \geq 1 - \exp(-\delta n). \tag{A.2.21}$$

Because $\sigma_n$ is balanced, we have $|n_i - n/k| \leq \sqrt{n}$ for all $i \in [k]$. Therefore, if $\mathcal{A}$ occurs, then it is possible to obtain from $\boldsymbol{\sigma}$ a map $\tau_{\boldsymbol{\sigma}} \in T$ by changing the colours of at most $2\eta n$ vertices. If $\mathcal{A}$ occurs, we let $\boldsymbol{G}_1 = G(n, p', \tau_{\boldsymbol{\sigma}})$. Further, let $\boldsymbol{G}_2$ be the random graph obtained by removing from $\boldsymbol{G}_1$ all edges that are monochromatic under $\boldsymbol{\sigma}$. Finally, let $\boldsymbol{G}_3$ be the random graph obtained from $\boldsymbol{G}_2$ by inserting an edge between any two vertices $v, w$ with $\boldsymbol{\sigma}(v) \neq \boldsymbol{\sigma}(w)$ but $\tau_{\boldsymbol{\sigma}}(v) = \tau_{\boldsymbol{\sigma}}(w)$ with probability $p'$ independently. Thus, the bottom line is that in $\boldsymbol{G}_3$, we connect any two vertices that are coloured differently under $\boldsymbol{\sigma}$ with probability $p'$ independently. That is, $\boldsymbol{G}_3 = G(n, p', \boldsymbol{\sigma})$.

Let $S_{\boldsymbol{\sigma}}$ be the set of vertices $v$ with $\boldsymbol{\sigma}(v) \neq \tau_{\boldsymbol{\sigma}}(v)$ and let $\Delta$ be the number of edges we removed to obtain $\boldsymbol{G}_2$ from $\boldsymbol{G}_1$. Then $\Delta$ is bounded by the volume of $S_{\boldsymbol{\sigma}}$ in $\boldsymbol{G}_1 = G(n, p', \tau_{\boldsymbol{\sigma}})$. Hence, (A.2.20) implies that

$$\mathbb{P}\left[\Delta \leq \gamma n | \mathcal{A}\right] \geq 1 - \exp(-\alpha n). \tag{A.2.22}$$

Since removing a single edge can reduce $Y$ by at most $\beta/n$, we obtain

$$\begin{aligned}
&\mathbb{P}[Y(G(n,p',\boldsymbol{\sigma})) \leq z] \\
&\quad = \mathbb{P}[Y(\boldsymbol{G}_3) \leq z] \leq \exp(-\delta n) + \mathbb{P}[Y(\boldsymbol{G}_3) \leq z|\mathcal{A}] \qquad \text{[by (A.2.21)]} \\
&\quad \leq \exp(-\delta n) + \exp(-\alpha n) + \mathbb{P}[Y(\boldsymbol{G}_3) \leq z|\mathcal{A}, \Delta \leq \gamma n] \qquad \text{[by (A.2.22)]} \\
&\quad \leq \exp(-\delta n) + \exp(-\alpha n) + \mathbb{P}[Y(\boldsymbol{G}_1) - \gamma\beta \leq z|\mathcal{A}, \Delta \leq \gamma n] \\
&\quad \leq \exp(-\delta n) + \exp(-\alpha n) + 2\,\mathbb{P}[Y(\boldsymbol{G}_1) \leq z + \varepsilon/4|\mathcal{A}] \qquad \text{[by the choice of } \gamma \text{ and (A.2.22)]} \\
&\quad \leq \exp(-\delta n) + \exp(-\alpha n) + 3\,\mathbb{P}[Y(G(n,p',\sigma_n)) \leq z + \varepsilon/4] \qquad \text{[by (A.2.21)]} \\
&\quad \leq \exp(-\delta n) + \exp(-\alpha n) + 3\,\mathbb{P}[Y(G(n,p',\sigma_n)) \leq \mathbb{E}[Y(G(n,p',\sigma_n))] - \varepsilon/4] \ \text{[by (A.2.19)]}.
\end{aligned}$$

Finally, the assertion follows from Lemma A.2.3. □

*Proof of Lemma A.2.4.* Lemma A.2.8 shows that there exist $\varepsilon > 0$, balanced maps $\sigma_n : [n] \to [k]$ and a sequence $\mu_n$ satisfying $|\mu_n - dn/2| \leq \sqrt{n}$ such that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln|\mathcal{C}(G(n,\mu_n,\sigma_n),\sigma_n)| \geq \ln k + \frac{d}{2}\ln(1-1/k) + \varepsilon\right] = 1. \tag{A.2.23}$$

By the definition of $Z_{\beta,k}$, (A.2.23) implies that

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln Z_{\beta,k}(G(n,\mu_n,\sigma_n)) \geq \ln k + \frac{d}{2}\ln(1-1/k) + \varepsilon\right] = 1 \quad \text{for all } \beta > 0. \tag{A.2.24}$$

By comparison, Lemma A.2.7 yields $\beta > 0$ such that with $z = \ln k + \frac{d}{2}\ln(1-1/k) + \varepsilon/8$ we have

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln Z_{\beta,k}(G(n,m)) \leq z\right] = 1.$$

Thus, we aim to prove that there is $\alpha > 0$ such that for sufficiently large $n$

$$\mathbb{P}\left[\frac{1}{n}\ln Z_{\beta,k}(G(n,m,\boldsymbol{\sigma})) \leq z + \varepsilon/8\right] \leq \exp(-\alpha n). \tag{A.2.25}$$

Indeed, since $|\ln Z_{\beta,k}(G(n,\mu_n,\sigma_n))| \leq \beta\mu_n = O(n)$, (A.2.24) implies that for large enough $n$

$$\frac{1}{n}\mathbb{E}[\ln Z_{\beta,k}(G(n,\mu_n,\sigma_n))] \geq \ln k + \frac{d}{2}\ln(1-1/k) + \varepsilon - o(1)$$
$$\geq \ln k + \frac{d}{2}\ln(1-1/k) + \varepsilon/2.$$

Thus, since the number of edges in $G(n,p',\sigma_n))$ is binomially distributed with expectation $dn/2 + O(1)$, equation (A.2.25) follows from Lemma A.2.11. □

## A.3. Determining the fixed point

### A.3.1. The "hard fields"

In this section we make the first step towards proving that $\pi_{d,k,\boldsymbol{q}^*}$ is the unique frozen fixed point of $\mathcal{F}_{d,k}$. More specifically, identifying the set $\Omega$ with the $k$-simplex, we show that every face of $\Omega$ carries the same probability mass under any frozen fixed point of $\mathcal{F}_{d,k}$ as under the measure $\pi_{d,k,\boldsymbol{q}^*}$. Formally, let us denote the extremal points of $\Omega$ by $\delta_h = (\mathbf{1}_{i=h})_{i\in[k]}$, i.e. $\delta_h$ is the probability measure on $[k]$ that puts mass 1 on the single point $h \in [k]$. In addition, let $\Omega_\ell$ be the set of all $\mu \in \Omega$ with support $\ell$ (i.e. $\mu(i) > 0$ for all $i \in \ell$ and $\mu(i) = 0$ for all $i \in [k] \setminus \ell$). Further, for a probability measure $\pi \in \mathcal{P}$ we let $\rho_h(\pi) = \pi(\{\delta_h\})$ denote the probability mass of $\delta_h$ under $\pi$. In physics jargon, the numbers $\rho_h(\pi)$ are called the "hard fields" of $\pi$. In addition, recalling that $\mathrm{d}\pi_i(\mu) = k\mu(i)\mathrm{d}\pi(\mu)$, we set $\rho_{i,\ell}(\pi) = \pi_i(\Omega_\ell)$ for any $(i,\ell) \in \mathcal{T}$. The main result of this section is

**Lemma A.3.1.** *Suppose that $d \geq (2k-1)\ln k - 2$. Let $q^* \in [2/3, 1]$ be the fixed point of (5.1.4). If $\pi \in \mathcal{P}$ is a frozen fixed point of $\mathcal{F}_{d,k}$, then $\rho_i(\pi) = q^*/k$ and $\rho_{i,\ell}(\pi) = kq^*_{i,\ell}$ for all $(i,\ell) \in \mathcal{T}$.*

**Remark A.3.2.** *The proofs of several statements in this section (Lemmas A.3.1, A.3.3, A.3.4 and Corollary A.3.5) directly incorporate parts of the calculations outlined in the physics work [ZK07] that predicted the existence and location of $d_{\mathrm{cond}}$. We redo these calculations here in detail to be self-contained and because not all steps are carried out in full detail in [ZK07].*

To avoid many case distinctions, we introduce the following convention when working with product measures. Let us agree that $\Omega^0 = \{\emptyset\}$. Hence, if $B : \Omega^0 \to \Omega$ is a map, then $B(\emptyset) \in \Omega$. Furthermore, there is precisely one probability measure $\pi_0$ on $\Omega^0$, namely the measure that puts mass one on the point $\emptyset \in \Omega^0$. Thus, the integral $\int_{\Omega^\emptyset} B(\mu)\mathrm{d}\pi_0(\mu)$ is simply equal to $B(\emptyset)$. If $\pi_1, \pi_2, \ldots$ are probability measures on $\Omega$, what we mean by the empty product measure $\bigotimes_{\gamma=1}^0 \pi_\gamma$ is just the measure $\pi_0$ on $\Omega^0$.

Further, for a real $\lambda \geq 0$ and an integer $y \geq 1$ we let

$$p_\lambda(y) = \lambda^y \exp(-\lambda)/y!.$$

Moreover, for $i \in [k]$ we let $\Gamma_i$ be the set of all non-negative integer vectors $\boldsymbol{\gamma} = (\gamma_j)_{j \in [k]\setminus\{i\}}$ and for $\boldsymbol{\gamma} \in \Gamma_i$ we set

$$p_i(\boldsymbol{\gamma}) = \prod_{h \in [k]\setminus\{i\}} p_{\frac{d}{k-1}}(\gamma_h).$$

We also let $\Omega^{\boldsymbol{\gamma}} = \prod_{h \in [k]\setminus\{i\}} \prod_{j \in [\gamma_h]} \Omega$ for $\boldsymbol{\gamma} \in \Gamma_i$. The elements of $\Omega^{\boldsymbol{\gamma}}$ are denoted by $\mu_{\boldsymbol{\gamma}} = (\mu_{h,j})_{h \in [k]\setminus\{i\}, j \in [\gamma_h]}$. Moreover, let

$$\pi_{i,\boldsymbol{\gamma}} = \bigotimes_{h \in [k]\setminus\{i\}} \bigotimes_{j \in [\gamma_h]} \pi_h.$$

Thus, with the convention from the previous paragraph, in the case $\boldsymbol{\gamma} = 0$ the set $\Omega^{\boldsymbol{\gamma}} = \{\emptyset\}$ contains only one element, namely $\mu_0 = \emptyset$. Moreover, $\pi_{i,\boldsymbol{\gamma}}$ is the probability measure on $\Omega^0$ that gives mass one to the point $\emptyset$. We recall the map $\mathcal{B} : \bigcup_{\gamma \geq 1} \Omega^\gamma \to \Omega$ from (4.1.1) and extend this map to $\Omega^0$ by letting $\mathcal{B}(\emptyset) = \frac{1}{k}\mathbf{1}$ be the uniform distribution on $\Omega$. We start the proof of Lemma A.3.1 by establishing the following identity.

**Lemma A.3.3.** *If $\pi$ is fixed point of $\mathcal{F}_{d,k}$, then for any $i \in [k]$ we have*

$$\pi_i = \sum_{\boldsymbol{\gamma} \in \Gamma_i} \int_{\Omega^{\boldsymbol{\gamma}}} \delta_{\mathcal{B}[\mu_{\boldsymbol{\gamma}}]} p_i(\boldsymbol{\gamma}) \mathrm{d}\pi_{i,\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}).$$

To establish Lemma A.3.3 we need to calculate the normalising quantities $Z_\gamma(\pi)$.

**Lemma A.3.4.** *If $\pi$ is fixed point of $\mathcal{F}_{d,k}$, then $Z_\gamma(\pi) = (k-1)^\gamma / k^{\gamma-1}$.*

*Proof.* Assume that $\pi$ is fixed point of $\mathcal{F}_{d,k}$. We claim that

$$\int_\Omega \mu(h) \mathrm{d}\pi(\mu) = 1/k \qquad \text{for all } h \in [k]. \tag{A.3.1}$$

Indeed, set $\nu(h) = \int_\Omega \mu(h)\mathrm{d}\pi(\mu)$. Then $\nu$ is a probability distribution on $[k]$. Since $\pi$ is a fixed point of $\mathcal{F}_{d,k}$, we find

$$\nu(h)=\int_\Omega \mu(h)\mathrm{d}\mathcal{F}_{d,k}[\pi](\mu)=\sum_{\gamma=0}^{\infty}\frac{p_d(\gamma)}{Z_\gamma(\pi)}\int_{\Omega^\gamma}\left[\sum_{h=1}^{k}\prod_{j=1}^{\gamma}1-\mu_j(h)\right]\mathcal{B}[\mu_1,\dots,\mu_\gamma](h)\bigotimes_{j=1}^{\gamma}\mathrm{d}\pi(\mu_j)$$

$$=\sum_{\gamma=0}^{\infty}\frac{p_d(\gamma)}{Z_\gamma(\pi)}\int_{\Omega^\gamma}\prod_{j=1}^{\gamma}1-\mu_j(h)\bigotimes_{j=1}^{\gamma}\mathrm{d}\pi(\mu_j)\qquad\text{[plugging in (4.1.1)]}$$

$$=\sum_{\gamma=0}^{\infty}\frac{p_d(\gamma)}{Z_\gamma(\pi)}\left[\int_\Omega 1-\mu(h)\mathrm{d}\pi(\mu)\right]^\gamma=\sum_{\gamma\geq 0}\frac{(1-\nu(h))^\gamma\,p_d(\gamma)}{\sum_{h'\in[k]}(1-\nu(h'))^\gamma}\ \text{[due to (4.1.1)].}\qquad\text{(A.3.2)}$$

Now, assume that $h_1,h_2\in[k]$ are such that $\nu(h_1)\leq\nu(h_2)$. Then (A.3.2) yields

$$\nu(h_2)\;=\;\sum_{\gamma\geq 0}\frac{(1-\nu(h_1))^\gamma\,p_d(\gamma)}{\sum_{h'\in[k]}(1-\nu(h'))^\gamma}\leq\sum_{\gamma\geq 0}\frac{(1-\nu(h_2))^\gamma\,p_d(\gamma)}{\sum_{h'\in[k]}(1-\nu(h'))^\gamma}=\nu(h_1).$$

Hence, $\nu(h_1)=\nu(h_2)$ for all $h_1,h_2\in[k]$, which implies (A.3.1). Finally, the assertion follows from (A.3.1) and the definition (4.1.1) of $Z_\gamma(\pi)$. □

*Proof of Lemma A.3.3.* If $\pi$ is a fixed point of $\mathcal{F}_{d,k}$, then by Lemma A.3.4 and the definition (4.1.1) of the map $\mathcal{B}$ we have

$$\pi_i=\int_\Omega k\mu(i)\delta_\mu\mathrm{d}\pi(\mu)=\int_\Omega k\mu(i)\delta_\mu\mathrm{d}\mathcal{F}_{d,k}[\pi](\mu)$$

$$=\sum_{\gamma=0}^{\infty}\frac{p_d(\gamma)}{Z_\gamma(\pi)}\int_{\Omega^\gamma}\left[\sum_{h=1}^{k}\prod_{j=1}^{\gamma}1-\mu_j(h)\right]k\mathcal{B}[\mu_1,\dots,\mu_\gamma](i)\delta_{\mathcal{B}[\mu_1,\dots,\mu_\gamma]}\bigotimes_{j=1}^{\gamma}\mathrm{d}\pi(\mu_j)$$

$$=\sum_{\gamma=0}^{\infty}\frac{k^\gamma p_d(\gamma)}{(k-1)^\gamma}\int_{\Omega^\gamma}\left[\prod_{j=1}^{\gamma}1-\mu_j(i)\right]\cdot\delta_{\mathcal{B}[\mu_1,\dots,\mu_\gamma]}\bigotimes_{j=1}^{\gamma}\mathrm{d}\pi(\mu_j).$$

Further, for any $\mu\in\Omega$ we have $1-\mu(i)=\sum_{i'\neq i}\mu(i')$. Hence,

$$\pi_i=\sum_{\gamma=0}^{\infty}\frac{k^\gamma p_d(\gamma)}{(k-1)^\gamma}\sum_{i_1,\dots,i_\gamma\in[k]\setminus\{i\}}\int_{\Omega^\gamma}\left[\prod_{j=1}^{\gamma}\mu_j(i_j)\right]\cdot\delta_{\mathcal{B}[\mu_1,\dots,\mu_\gamma]}\bigotimes_{j=1}^{\gamma}\mathrm{d}\pi(\mu_j)$$

$$=\sum_{\gamma=0}^{\infty}\frac{p_d(\gamma)}{(k-1)^\gamma}\sum_{i_1,\dots,i_\gamma\in[k]\setminus\{i\}}\int_{\Omega^\gamma}\delta_{\mathcal{B}[\mu_1,\dots,\mu_\gamma]}\bigotimes_{j=1}^{\gamma}\mathrm{d}\pi_{i_j}(\mu_j).\qquad\text{(A.3.3)}$$

In the last expression, we can think of generating the sequence $i_1,\dots,i_\gamma$ as follows: first, choose $\gamma$ from the Poisson distribution $\mathrm{Po}(d)$. Then, choose the sequence $i_1,\dots,i_\gamma$ by independently choosing $i_j$ from the set $[k]\setminus\{i\}$ uniformly at random. Thus, in the overall experiment the number of times that each colour $h$ occurs has distribution $\mathrm{Po}(d/(k-1))$, independently for all $h \in [k] \setminus \{i\}$, whence (A.3.3) implies the assertion. □

**Corollary A.3.5.** *If $\pi$ is fixed point of $\mathcal{F}_{d,k}$, then $(\rho_i(\pi))_{i\in[k]}$ is a fixed point of the function $F_{d,k}$ from Lemma 5.2.1.*

*Proof.* Invoking Lemma A.3.3, we obtain for any $i \in [k]$

$$\rho_i(\pi) = \pi(\{\delta_i\}) = \frac{\pi_i(\{\delta_i\})}{k} = \frac{1}{k}\sum_{\gamma\in\Gamma_i}\int_{\Omega^{\gamma}} \mathbf{1}_{\delta_i=\mathcal{B}[\mu_\gamma]} p_i(\gamma)\mathrm{d}\pi_{i,\gamma}(\mu_\gamma). \tag{A.3.4}$$

A glimpse at the definition (4.1.1) of $\mathcal{B}$ reveals that $\delta_i = \mathcal{B}[\mu_\gamma]$ if and only if for each $h \in [k] \setminus \{i\}$ there is $j \in [\gamma_h]$ such that $\mu_{h,j} = \delta_h$. Further, in (A.3.4) the $\mu_{h,j}$ are chosen independently from the distribution $\pi_h$, and $\pi_h(\{\delta_h\}) = k\rho_h(\pi)$. In effect, the r.h.s. of (A.3.4) is simply the probability that if we choose numbers $\gamma_h$ independently from the Poisson distribution with mean $d/(k-1)$ for $h \neq i$ and then perform $\gamma_h$ independent Bernoulli experiments with success probability $k\rho_h(\pi)$, then there occurs at least one success for each $h \neq i$. Of course, this is nothing but the probability that $k-1$ independent Poisson variables $(\mathrm{Po}(\rho_h(\pi)dk/(k-1)))_{h\neq i}$ are all strictly positive. Hence,

$$\begin{aligned}\rho_i(\pi) &= \frac{1}{k}\prod_{h\in[k]\setminus\{i\}} \mathbb{P}[\mathrm{Po}(\rho_h(\pi)dk/(k-1)) > 0]\\ &= \frac{1}{k}\prod_{h\in[k]\setminus\{i\}} 1-\exp(-\rho_h(\pi)d') \quad \text{for any } i \in [k]\,.\end{aligned}$$

Consequently, $(\rho_i(\pi))_{i\in[k]} = F_{d,k}((\rho_i(\pi))_{i\in[k]})$. □

*Proof of Lemma A.3.1.* Assume that $\pi \in \mathcal{P}$ is a frozen fixed point of $\mathcal{F}_{d,k}$. Then $\rho_i(\pi) \geq \frac{2}{3k}$ for all $i \in [k]$. Hence, Corollary A.3.5 yields $(\rho_1(\pi),\ldots,\rho_k(\pi)) \in [\frac{2}{3k},\frac{1}{k}]^k$ is a fixed point of $F_{d,k}$. Therefore, Lemma 5.2.1 implies that $\rho_i(\pi) = q^*/k$ for all $i \in [k]$.

To prove the second assertion, let $(i,\ell) \in \mathcal{T}$. Then Lemma A.3.3 yields

$$\rho_{i,\ell}(\pi) = \sum_{\gamma\in\Gamma_i}\int_{\Omega^{\gamma}} \mathbf{1}_{\mathcal{B}[\mu_\gamma]\in\Omega_\ell} p_i(\gamma)\mathrm{d}\pi_{i,\gamma}(\mu_\gamma). \tag{A.3.5}$$

Now, the definition (4.1.1) is such that $\mathcal{B}[\mu_\gamma] \in \Omega_\ell$ if and only if

1. for each $h \in [k] \setminus \ell$ there is $j \in [\gamma_h]$ such that $\mu_{h,j} = \delta_h$, and
2. for each $h \in \ell \setminus \{i\}$ and any $j \in [\gamma_h]$ we have $\mu_{h,j} \neq \delta_h$.

Given $\gamma$, the distributions $\mu_{h,j}$ are chosen independently from $\pi_h$ for all $h \neq i$, $j \in [\gamma_h]$. Hence, for a given $\gamma$ the probability that (1) and (2) occur is precisely

$$\eta(\gamma) = \prod_{h\in\ell\setminus\{i\}} (1-\pi_h(\{\delta_h\}))^{\gamma_h} \cdot \prod_{h\in[k]\setminus\ell} 1-(1-\pi_h(\{\delta_h\}))^{\gamma_h}$$
$$= \prod_{h\in\ell\setminus\{i\}} (1-k\rho_h(\pi))^{\gamma_h} \cdot \prod_{h\in[k]\setminus\ell} 1-(1-k\rho_h(\pi))^{\gamma_h}. \tag{A.3.6}$$

Thus, combining (A.3.5) and (A.3.6), we see that

$$\rho_{i,\ell}(\pi) = \sum_{\gamma\in\Gamma_i} \eta(\gamma)p_i(\gamma)$$
$$= \prod_{h\in\ell\setminus\{i\}} \left[\sum_{\gamma_h\geq 0}(1-k\rho_h(\pi))^{\gamma_h} p_{\frac{d}{k-1}}(\gamma_h)\right] \prod_{h\in[k]\setminus\ell} \left[\sum_{\gamma_h\geq 0}(1-(1-k\rho_h(\pi))^{\gamma_h})p_{\frac{d}{k-1}}(\gamma_h)\right]$$
$$= \prod_{h\in\ell\setminus\{i\}} \mathbb{P}\left[\mathrm{Po}(dk\rho_h(\pi)/(k-1)=0)\right] \prod_{h\in[k]\setminus\ell} \mathbb{P}\left[\mathrm{Po}(dk\rho_h(\pi)/(k-1)>0)\right]$$
$$= \prod_{h\in\ell\setminus\{i\}} \exp(-d'\rho_h(\pi)) \prod_{h\in[k]\setminus\ell} 1-\exp(-d'\rho_h(\pi)). \tag{A.3.7}$$

Finally, as we already know from the first paragraph that $\rho_h(\pi) = q^*/k$, (A.3.7) implies that $\rho_{i,\ell}(\pi) = kq^*_{i,\ell}$. □

### A.3.2. The fixed point

The objective in this section is to establish

**Lemma A.3.6.** *Suppose that $d \geq (2k-1)\ln k - 2$. Then $\pi_{d,k,\boldsymbol{q}^*}$ is the unique frozen fixed point of $\mathcal{F}_{d,k}$.*

To prove Lemma A.3.6, let $\mathcal{P}_\ell$ be the set of all probability measures $\pi \in \mathcal{P}$ whose support is contained in $\Omega_\ell$ (i.e. $\pi(\Omega_\ell) = 1$). For each $\pi \in \mathcal{P}$ and any $(i,\ell) \in \mathcal{T}$, we define a measure $\pi_{i,\ell}$ by letting

$$\mathrm{d}\pi_{i,\ell}(\mu) = \frac{\mathbf{1}_{\mu\in\Omega_\ell}}{kq^*_{i,\ell}}\mathrm{d}\pi_i(\mu) = \frac{\mu(i)}{q^*_{i,\ell}}\mathbf{1}_{\mu\in\Omega_\ell}\mathrm{d}\pi(\mu).$$

In addition, let $\widetilde{\mathcal{P}} = \prod_{(i,\ell)\in\mathcal{T}} \mathcal{P}_\ell$ be the set of all families $(\pi_{i,\ell})_{i,\ell\in\mathcal{T}}$ such that $\pi_{i,\ell} \in \mathcal{P}_\ell$ for all $(i,\ell)$.

**Lemma A.3.7.** *If $\pi$ if a frozen fixed point of $\mathcal{F}_{d,k}$, then $\widetilde{\pi} = (\pi_{i,\ell})_{(i,\ell)\in\mathcal{T}} \in \widetilde{\mathcal{P}}$.*

*Proof.* Let $(i,\ell) \in \mathcal{T}$. By construction, the support of $\pi_{i,\ell}$ is contained in $\Omega_\ell$. Furthermore, Lemma A.3.1 implies that

$$\pi_{i,\ell}(\Omega_\ell) = \frac{1}{kq^*_{i,\ell}} \int_\Omega \mathbf{1}_{\mu \in \Omega_\ell} \mathrm{d}\pi_i(\mu) = \frac{\pi_i(\Omega_\ell)}{kq^*_{i,\ell}} = \frac{\rho_{i,\ell}(\pi)}{kq^*_{i,\ell}} = 1.$$

Thus, $\pi_{i,\ell}$ is a probability measure. □

Let $\Gamma_{i,\ell}$ be the set of all non-negative integer vectors $\widehat{\gamma} = (\widehat{\gamma}_{i',\ell'})_{(i',\ell') \in \mathcal{T}_{i,\ell}}$. For $\widehat{\gamma} \in \Gamma_{i,\ell}$, we let

$$p_{i,\ell}(\widehat{\gamma}) = \prod_{(i',\ell') \in \mathcal{T}_{i,\ell}} p_{d' q^*_{i',\ell'}}(\widehat{\gamma}_{i',\ell'}).$$

Moreover, we let $\Omega^{\widehat{\gamma}} = \prod_{(i',\ell') \in \mathcal{T}_{i,\ell}} \prod_{j \in [\widehat{\gamma}_{i',\ell'}]} \Omega$ and by $\mu_{\widehat{\gamma}} = (\mu_{i',\ell',j})_{(i',\ell') \in \mathcal{T}_{i,\ell}, j \in [\widehat{\gamma}_{i',\ell'}]}$ we denote its points. In addition, if $\pi$ is a probability measure on $\Omega$ and $\widehat{\gamma} \in \Gamma_{i,\ell}$, we set

$$\pi_{i,\ell,\widehat{\gamma}} = \bigotimes_{(i',\ell') \in \mathcal{T}_{i,\ell}} \bigotimes_{j=1}^{\widehat{\gamma}_{i',\ell'}} \pi_{i',\ell'}.$$

Further, we define for any non-empty set $\ell \subset [k]$ a map

$$\mathcal{B}_\ell : \bigcup_{\gamma=1}^{\infty} \Omega^\gamma \to \Omega, \quad (\mu_1, \ldots, \mu_\gamma) \mapsto \mathcal{B}_\ell[\mu_1, \ldots, \mu_\gamma], \text{ where} \tag{A.3.8}$$

$$\mathcal{B}_\ell[\mu_1, \ldots, \mu_\gamma](h) = \begin{cases} \frac{\mathbf{1}_{h \in \ell}}{|\ell|} & \text{if } \sum_{h' \in \ell} \prod_{j=1}^{\gamma} 1 - \mu_j(h') = 0, \\ \frac{\mathbf{1}_{h \in \ell} \cdot \prod_{j=1}^{\gamma} 1 - \mu_j(h)}{\sum_{h' \in \ell} \prod_{j=1}^{\gamma} 1 - \mu_j(h')} & \text{if } \sum_{h' \in \ell} \prod_{j=1}^{\gamma} 1 - \mu_j(h') > 0. \end{cases}$$

Additionally, to cover the case $\gamma = 0$ we define $\mathcal{B}_\ell[\emptyset](h) = \frac{\mathbf{1}_{h \in \ell}}{|\ell|}$. Thus, $\mathcal{B}_\ell[\emptyset]$ is the uniform distribution on $\ell$.

**Lemma A.3.8.** *Let $\mathcal{X}$ be the set of all frozen fixed points of $\mathcal{F}_{d,k}$. Moreover, let $\widetilde{\mathcal{X}}$ be the set of all fixed points of*

$$\widetilde{\mathcal{F}}_{d,k} : \widetilde{\mathcal{P}} \to \widetilde{\mathcal{P}}, \qquad (\pi_{i,\ell})_{(i,\ell) \in \mathcal{T}} \mapsto \left( \sum_{\widehat{\gamma} \in \Gamma_{i,\ell}} \int_{\Omega^{\widehat{\gamma}}} \delta_{\mathcal{B}_\ell[\mu_{\widehat{\gamma}}]} p_{i,\ell}(\widehat{\gamma}) \mathrm{d}\pi_{i,\ell,\widehat{\gamma}}(\mu_{\widehat{\gamma}}) \right)_{(i,\ell) \in \mathcal{T}}.$$

*Then the map $\pi \in \mathcal{X} \mapsto \widetilde{\pi} = (\pi_{i,\ell})_{(i,\ell) \in \mathcal{T}}$ induces a bijection between $\mathcal{X}$ and $\widetilde{\mathcal{X}}$.*

*Proof.* Suppose that $\pi \in \mathcal{X}$. Let $(i,\ell) \in \mathcal{T}$. Then Lemma A.3.3 yields

$$\pi_{i,\ell} = \int_{\Omega_\ell} \frac{\delta_\mu}{kq^*_{i,\ell}} \mathrm{d}\pi_i(\mu) = \sum_{\boldsymbol{\gamma} \in \Gamma_i} \int_{\Omega^{\boldsymbol{\gamma}}} \frac{\mathbf{1}_{\mathcal{B}[\mu_{\boldsymbol{\gamma}}] \in \Omega_\ell} \delta_{\mathcal{B}[\mu_{\boldsymbol{\gamma}}]}}{kq^*_{i,\ell}} p_i(\boldsymbol{\gamma}) \mathrm{d}\pi_{i,\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}). \tag{A.3.9}$$

Now let us fix a pair $(i, \ell) \in \mathcal{T}$ and $(\gamma, \mu_\gamma)$. We denote, for $h \neq i$, by $\widehat{\gamma}_h = \widehat{\gamma}_h(\mu_\gamma)$ the number of occurrences of $\delta_h$ in the tuple $\mu_\gamma$. The event $\mathcal{B}[\mu_\gamma] \in \Omega_\ell$ occurs if and only if

1. for each $h \in [k] \setminus \ell$ there is $j \in [\gamma_h]$ such that $\mu_{h,j} = \delta_h$, i.e. $\widehat{\gamma}_h > 0$,
2. for each $h \in \ell \setminus \{i\}$ and all $j \in [\gamma_h]$ we have $\mu_{h,j} = \delta_h$, i.e. $\widehat{\gamma}_h = 0$,

Thus, Lemma A.3.1 implies that

$$\sum_{\gamma \in \Gamma_i} \int_{\Omega^\gamma} \frac{\mathbf{1}_{\mathcal{B}[\mu_\gamma] \in \Omega_\ell}}{k q^*_{i,\ell}} p_i(\gamma) \mathrm{d}\pi_{i,\gamma}(\mu_\gamma) = \frac{1}{k q^*_{i,\ell}} \prod_{h \in [k] \setminus \ell} \mathbb{P}\left[\mathrm{Po}(q^*_h d') > 0\right] \prod_{h \in \ell \setminus \{i\}} \mathbb{P}\left[\mathrm{Po}(q^*_h d') = 0\right] = 1. \tag{A.3.10}$$

Furthermore, given that the event $\mathcal{B}[\mu_\gamma] \in \Omega_\ell$ occurs, the measure $\mathcal{B}[\mu_\gamma]$ is determined by those components $\mu_{i',\ell',j}$ with $(i', \ell') \in \mathcal{T}_{i,\ell}$ only. Thus, defining $\widehat{\gamma} = (\widehat{\gamma}_{i',\ell'})_{(i',\ell') \in \mathcal{T}_{i,\ell}}$ and $\mu_{\widehat{\gamma}} = (\mu_{i',\ell',j})_{(i',\ell') \in \mathcal{T}_{i,\ell}, j \in [\widehat{\gamma}_{i',\ell'}]}$ we obtain from (A.3.9) and (A.3.10)

$$\pi_{i,\ell} = \sum_{\widehat{\gamma} \in \Gamma_{i,\ell}} \int_{\Omega^{\widehat{\gamma}}} \delta_{\mathcal{B}_\ell[\mu_{\widehat{\gamma}}]} p_{i,\ell}(\widehat{\gamma}) \mathrm{d}\pi_{i,\ell,\widehat{\gamma}}(\mu_{\widehat{\gamma}}).$$

Thus, if $\pi$ is a frozen fixed point of $\mathcal{F}_{d,k}$, then $\widetilde{\pi}$ is a fixed point of $\widetilde{\mathcal{F}}_{d,k}$.

Conversely, if $\widetilde{\pi} = (\pi_{i,\ell})$ is a fixed point of $\widetilde{\mathcal{F}}_{d,k}$, then the measure $\pi$ defined by

$$\mathrm{d}\pi(\mu) = \sum_{\ell \subset [k]} \frac{1}{|\ell|} \sum_{i \in \ell} \frac{q^*_{i,\ell}}{\mu(i)} \mathrm{d}\pi_{i,\ell}(\mu)$$

is easily verified to be a fixed point of $\mathcal{F}_{d,k}$. Moreover, for $i \in [k]$, $\rho_i(\pi) = q^*_{i,\{i\}} = q^*/k \geq 2/(3k)$ and $\pi$ is thus a frozen fixed point of $\mathcal{F}_{d,k}$. □

**Corollary A.3.9.** *The distribution $\pi_{d,k,\boldsymbol{q}^*}$ is a fixed point of $\mathcal{F}_{d,k}$.*

*Proof.* To unclutter the notation we write $\pi = \pi_{d,k,q^*}$. Moreover, we let $\boldsymbol{T} = \boldsymbol{T}_{d,k,\boldsymbol{q}^*}$; by Lemma 5.2.1 we may always assume that $\boldsymbol{T}$ is a finite tree. Recall that $\pi$ is the distribution of $\mu_{\boldsymbol{T}}$, which is the distribution of the colour of the root under a random legal colouring of $\boldsymbol{T}$. In light of Lemma A.3.8 it suffices to show that $\widetilde{\pi} = (\pi_{i,\ell})$ is a fixed point of $\widetilde{\mathcal{F}}_{d,k}$. Thus, we need to show that for all $(i, \ell) \in \mathcal{T}$,

$$\pi_{i,\ell} = \sum_{\widehat{\gamma} \in \Gamma_{i,\ell}} \int_{\Omega^{\widehat{\gamma}}} \delta_{\mathcal{B}_\ell[(\mu_{i',\ell',j})]} \prod_{(i',\ell') \in \mathcal{T}_{i,\ell}} p_{d' q^*_{i',\ell'}}(\widehat{\gamma}_{i',\ell'}) \bigotimes_{j=1}^{\widehat{\gamma}_{i',\ell'}} \mathrm{d}\pi_{i',\ell'}(\mu_{i',\ell',j}). \tag{A.3.11}$$

Let us denote by $\boldsymbol{T}_{i,\ell}$ the random tree $\boldsymbol{T}$ given that the root has type $(i, \ell)$. We claim that $\pi_{i,\ell}$ is the distribution of $\mu_{\boldsymbol{T}_{i,\ell}}$. Indeed, let $\ell \subset [k]$. If the root $v_0$ of $\boldsymbol{T}$ has type $(i,\ell)$ for some $i \in \ell$, then the support of the measure $\mu_{\boldsymbol{T}}$ is contained in $\ell$ (because under any legal colouring, $v_0$ receives a colour from $\ell$). Moreover, all children of $v_0$ have types in $\mathcal{T}_{i,\ell}$, and if $(i',\ell') \in \mathcal{T}_{i,\ell}$, then $|\ell'| \geq 2$. Hence, inductively we see that if $v_0$ has type $(i,\ell)$, then for any colour $h \in \ell$ there is a legal colouring under which $v_0$ receives colour $h$. Consequently, the support of $\mu_{\boldsymbol{T}}$ is precisely $\ell$. Furthermore, the distribution $\mu_{\boldsymbol{T}}$ is invariant under the following operation: obtain a random tree $\boldsymbol{T}'$ by choosing a legal colour $\boldsymbol{\tau}$ of $\boldsymbol{T}$ randomly and then changing the types $\vartheta(v) = (i_v, \ell_v)$ of the vertices to $\vartheta'(v) = (\boldsymbol{\tau}(i_v), \ell_v)$; this is because the trees $\boldsymbol{T}$ and $\boldsymbol{T}'$ have the same set of legal colourings. These observations imply that for any measurable set $A$ we have

$$\begin{aligned}\mathbb{P}\left[\mu_{\boldsymbol{T}} \in A | \vartheta(v_0) = (i,\ell)\right] &= \frac{\mathbb{P}\left[\mu_{\boldsymbol{T}} \in A, \vartheta(v_0) = (i,\ell)\right]}{\mathbb{P}\left[\vartheta(v_0) = (i,\ell)\right]} \\ &= \frac{\mathbb{P}\left[\mu_{\boldsymbol{T}} \in A \cap \Omega_\ell, \vartheta(v_0) = (i,\ell)\right]}{q^*_{i,\ell}} \\ &= \frac{1}{q^*_{i,\ell}} \int_A \mu(i) \mathbf{1}_{\mu \in \Omega_\ell} \mathrm{d}\pi(\mu) = \pi_{i,\ell}(A).\end{aligned}$$

To prove that $\widetilde{\pi}$ is a fixed point of $\widetilde{\mathcal{F}}_{d,k}$, we observe that the random tree $\boldsymbol{T}_{i,\ell}$ can be described by the following recurrence. There is a root $v_0$ of type $(i,\ell)$. For each $(i',\ell')$, $v_0$ has a random number $\gamma_{i',\ell'} = \mathrm{Po}(d' q^*_{i,\ell})$ of children $(v_{i',\ell',j})_{j=1,\ldots,\gamma_{i',\ell'}}$ of type $(i',\ell')$. Moreover, each $v_{i',\ell',j}$ is the root of a random tree $\boldsymbol{T}_{i',\ell',j}$. Of course, the random variables $(\gamma_{i',\ell'})_{(i',\ell') \in \mathcal{T}_{i,\ell}}$ and the random trees $\boldsymbol{T}_{i',\ell',j}$ are chosen independently.

This recursive description of the random tree $\boldsymbol{T}_{i,\ell}$ leads to a recurrence for the distribution $\pi_{i,\ell}$. Indeed, given the numbers $(\gamma_{i',\ell'})_{i',\ell'}$, the distribution $\mu_{\boldsymbol{T}_{i',\ell',j}}$ of the colour of the root of the random tree $\boldsymbol{T}_{i',\ell',j}$ is an $\Omega_{\ell'}$-valued random variable with distribution $\pi_{i',\ell'}$ for each $j = 1, \ldots, \gamma_{i',\ell'}$. Moreover, the random variables $(\mu_{\boldsymbol{T}_{i',\ell',j}})_{i',\ell',j}$ are mutually independent. In addition, we claim that given the distributions $(\mu_{\boldsymbol{T}_{i',\ell',j}})_{i',\ell',j}$, the colour of the root $v_0$ of the entire tree $\boldsymbol{T}_{i,\ell}$ has distribution

$$\mu_{\boldsymbol{T}_{i,\ell}} = \mathcal{B}_\ell[(\mu_{\boldsymbol{T}_{i',\ell',j}})_{i',\ell',j}]. \tag{A.3.12}$$

Indeed, given that $v_0$ has type $(i,\ell)$, $v_0$ receives a colour from $\ell$ under any legal colouring. Further, for any $h \in \ell$ the probability that $v_0$ takes colour $h$ under a random colouring of $\boldsymbol{T}_{i,\ell}$ is proportional to the probability that none of its children $v_{i',\ell',j}$ takes colour $h$ in a random colouring of the tree $\boldsymbol{T}_{i',\ell',j}$ whose root $v_{i',\ell',j}$ is.

Finally, we recall that $\pi_{i,\ell}$ is the distribution of $\mu_{\boldsymbol{T}_{i,\ell}}$. Hence, (A.3.12) implies together with the fact that the $\gamma_{i',\ell',j}$ are independent Poisson variables that $\pi_{i,\ell}$ satisfies (A.3.11). □

**Lemma A.3.10.** *The map $\widetilde{\mathcal{F}}_{d,k}$ has at most one fixed point.*

*Proof.* As before, we let $\boldsymbol{T}$ denote the random tree $\boldsymbol{T}_{d,k,\boldsymbol{q}^*}$. Moreover, $\boldsymbol{T}_{i,\ell}$ is the random tree $\boldsymbol{T}$ given that the root has type $(i, \ell)$.

Let $t \geq 0$ be an integer and let $\widetilde{\pi} = (\pi_{i,\ell}) \in \widetilde{\mathcal{P}}$. We define a distribution $\widetilde{\pi}_t = (\pi_{i,\ell,t}) \in \widetilde{\mathcal{P}}$ by means of the following experiment. Let $(i, \ell) \in \mathcal{T}$. Let $v_0$ denote the root of $\boldsymbol{T}_{i,\ell}$ and let $\vartheta(v)$ signify the type of each vertex $v$.

**TR1** Let $\boldsymbol{T}_{i,\ell,t}$ be the tree obtained from $\boldsymbol{T}_{i,\ell}$ by deleting all vertices at distance greater than $t$ from $v_0$.

**TR2** Let $V_t$ be the set of all vertices at distance exactly $t$ from $v_0$. For each $v \in V_t$ independently, choose $\mu_v \in \Omega$ from the distribution $\pi_{\vartheta(v)}$.

**TR3** Let $\mu_{i,\ell,t}$ be the distribution of the colour of $v_0$ under a random colouring $\boldsymbol{\tau}$ chosen as follows.
- Independently for each vertex $v \in V_t$ choose a colour $\boldsymbol{\tau}_t(v)$ from the distribution $\mu_v$.
- Let $\boldsymbol{\tau}$ be a uniformly random legal colouring of $\boldsymbol{T}_{i,\ell,t}$ such that $\boldsymbol{\tau}(v) = \boldsymbol{\tau}_t(v)$ for all $v \in V_t$; if there is no such colouring, discard the experiment.

Step **TR3** of the above experiment yields a *distribution* $\mu_{i,\ell,t} \in \Omega$. Clearly $\mu_{i,\ell,t}$ is determined by the random choices in steps **TR1**–**TR2**. Thus, let we let $\pi_{i,\ell,t}$ be the distribution of $\mu_{i,\ell,t}$ with respect to **TR1**–**TR2**.

We now claim that for any integer $t \geq 0$ the following is true.

$$\text{If } \widetilde{\pi} \text{ is a fixed point of } \widetilde{\mathcal{F}}_{d,k}, \text{ then } \widetilde{\pi} = \widetilde{\pi}_t. \tag{A.3.13}$$

The proof of (A.3.13) is by induction on $t$. It is immediate from the construction that $\pi_{i,\ell,0} = \pi_{i,\ell}$ for all $(i, \ell) \in \mathcal{T}$. Thus, assume that $t \geq 1$. By induction, it suffices to show that $\widetilde{\pi}_t = \widetilde{\pi}_{t-1}$. To this end, let us condition on the random tree $\boldsymbol{T}_{i,\ell,t-1}$. Consider a vertex $v \in V_{t-1}$ of type $\vartheta(v) = (i_v, \ell_v)$. We obtain the random tree $\boldsymbol{T}_{i,\ell,t}$ from $\boldsymbol{T}_{i,\ell,t-1}$ by attaching to each such $v \in V_{t-1}$ a random number $\gamma_{i',\ell',v} = \mathrm{Po}(d' q^*_{i',\ell'})$ of children of each type $(i', \ell') \in \mathcal{T}_{i_v,\ell_v}$ where, of course, the random variables $\gamma_{i',\ell',v}$ are mutually independent. Further, in step **TR2** of the above experiment we choose $\mu_{i',\ell',v,j} \in \Omega_{\ell'}$ independently from $\pi_{i',\ell'}$ for each $v \in V_{t-1}$, $(i', \ell') \in \mathcal{T}_{i,\ell}$ and $j = 1, \ldots, \gamma_{i',\ell',v}$.

Given the distributions $\mu_{i',\ell',v,j}$, suppose that we choose a legal colouring $\boldsymbol{\tau}_v$ of the sub-tree consisting of $v \in V_{t-1}$ and its children only from the following distribution.

- Independently choose the colours $\boldsymbol{\tau}_v(u_{i',\ell',j})$ of the children $u_{i',\ell',j}$ of $v$ of type $(i', \ell')$ from $\mu_{i',\ell',v,j}$.
- Choose a colour $\boldsymbol{\tau}_v(v)$ for $v$ uniformly from the set of all colours $h \in \ell$ that are not already assigned to a child of $v$ if possible.

Let $\mu_v$ denote the distribution of the colour $\boldsymbol{\tau}_v(v)$. Then by construction,

$$\mu_v = \mathcal{B}_\ell[(\mu_{i',\ell',v,j})_{(i',\ell')\in\mathcal{T}_{i,\ell},j\in[\gamma_{i',\ell',v}]}].$$

Hence, the distribution of $\mu_v$ with respect to the choice of the numbers $\gamma_{i',\ell',v}$ and the distributions $\mu_{i',\ell',v,j}$ is given by

$$\sum_{\gamma\in\Gamma_{i,\ell}} \int_{\Omega^\gamma} \delta_{\mathcal{B}_\ell[(\mu_{i',\ell',v,j})]} \prod_{(i',\ell')\in\mathcal{T}_{i,\ell}} p_{d'q^*_{i',\ell'}}(\gamma_{i',\ell',v}) \bigotimes_{j=1}^{\gamma_{i',\ell',v}} \mathrm{d}\pi_{i',\ell'}(\mu_{i',\ell',v,j}) = \pi_{i,\ell},$$

because $\widetilde{\pi}$ is a fixed point of $\widetilde{\mathcal{F}}_{d,k}$. Therefore, the experiment of first choosing $\boldsymbol{T}_{i,\ell,t}$, then choosing distributions $\mu_u$ independently from $\pi_{\vartheta(u)}$ for the vertices at distance $t$, and then choosing a random legal colouring $\boldsymbol{\tau}$ as in **TR3** is equivalent to performing the same experiment with $t-1$ instead. Hence, $\widetilde{\pi}_t = \widetilde{\pi}_{t-1}$.

To complete the proof, assume that $\widetilde{\pi}, \widetilde{\pi}'$ are fixed points of $\widetilde{\mathcal{F}}_{d,k}$. Then for any integer $t \geq 0$ we have $\widetilde{\pi} = \widetilde{\pi}_t$, $\widetilde{\pi}' = \widetilde{\pi}'_t$. Furthermore, as $\widetilde{\pi}_t$, $\widetilde{\pi}'_t$ result from the experiment **TR1–TR3**, whose first step **TR1** can be coupled, we see that for any $(i,\ell) \in \mathcal{T}$,

$$\left\|\pi_{i,\ell} - \pi'_{i,\ell}\right\|_{\mathrm{TV}} = \left\|\pi_{i,\ell,t} - \pi'_{i,\ell,t}\right\|_{\mathrm{TV}} \leq 2\,\mathbb{P}\left[|\boldsymbol{T}_{i,\ell}| \geq t\right]. \tag{A.3.14}$$

Because Lemma 5.2.1 shows that $\boldsymbol{T}$ results from a sub-critical branching process, we have

$$\lim_{t\to\infty} \mathbb{P}\left[|\boldsymbol{T}_{i,\ell}| \geq t\right] = 0$$

for any $(i,\ell) \in \mathcal{T}$. Consequently, (A.3.14) shows that $\widetilde{\pi} = \widetilde{\pi}'$. □

Finally, Lemma A.3.6 follows directly from Lemma A.3.8, Corollary A.3.9 and Lemma A.3.10.

### A.3.3. The number of legal colourings

The final step of the proof of Proposition 5.1.4 is to relate $\phi_{d,k}(\pi_{d,k,\boldsymbol{q}^*})$ to the number of legal colourings of $\boldsymbol{T}_{d,k,\boldsymbol{q}^*}$. The starting point for this is a formula for the (logarithm of the) number of legal colourings of a decorated tree $T, \vartheta$. To write this formula down, we recall the map $\mathcal{B}_\ell$ from (A.3.8). Moreover, suppose that $\ell \subset [k]$ and $\mu_1, \ldots, \mu_\gamma \in \Omega$ are such that:

$$\exists h \in \ell\, \forall j \in [\gamma] : \mu_j(h) < 1. \tag{A.3.15}$$

Then we let

$$\begin{aligned}\phi_\ell(\mu_1,\ldots,\mu_\gamma) &= \phi^v_\ell(\mu_1,\ldots,\mu_\gamma) - \frac{1}{2}\phi^e_\ell(\mu_1,\ldots,\mu_\gamma), \quad \text{where}\\ \phi^v_\ell(\mu_1,\ldots,\mu_\gamma) &= \ln\sum_{h\in\ell}\prod_{j=1}^{\gamma} 1-\mu_j(h),\\ \phi^e_\ell(\mu_1,\ldots,\mu_\gamma) &= \sum_{j=1}^{\gamma}\ln\left[1-\sum_{h\in\ell}\mu_j(h)\mathcal{B}_\ell[\mu_1,\ldots,\mu_{j-1},\mu_{j+1},\ldots,\mu_\gamma](h)\right];\end{aligned}$$

the condition (A.3.15) ensures that these quantities are well-defined (i.e. the argument of the logarithm is positive in both instances). Additionally, to cover the case $\gamma = 0$ we set $\phi_\ell(\emptyset) = \ln|\ell|$.

Further, suppose that $T, \vartheta, v$ is a rooted decorated tree that has at least one legal colouring $\sigma$. Let $v_1, \ldots, v_\gamma$ be the neighbours of the root vertex $v$ and suppose that $\vartheta(v) = (i, \ell)$ and $\vartheta(v_j) = (i_j, \ell_j)$ for $j = 1, \ldots, \gamma$. If we remove the root $v$ from $T$, then each of the vertices $v_1, \ldots, v_\gamma$ lies in a connected component $T_i$ of the resulting forest. By considering the restrictions $\vartheta_i$ of $\vartheta$ to the vertex set of $T_i$, we obtain decorated trees $T_i, \vartheta_i$. Recall that $\mu_{T_j,\vartheta_j,v_j}$ denotes the distribution of the colour of the root in a random legal colouring of $T_j, \vartheta_j, v_j$. Since $\sigma$ is a legal colouring, for $h = \sigma(v)$ for all $j \in [\gamma]$ we have $\mu_{T_j,\vartheta_j,v_j}(h) < 1$. Thus, we can define

$$\phi(T, \vartheta, v) = \phi_\ell(\mu_{T_1,\vartheta_1,v_1}, \ldots, \mu_{T_\gamma,\vartheta_\gamma,v_\gamma}).$$

**Fact A.3.11.** *Let $T, \vartheta$ be a decorated tree such that $\mathcal{Z}(T, \vartheta) \geq 1$. Then we have $\ln\mathcal{Z}(T, \vartheta) = \sum_{v\in V(T)}\phi(T, \vartheta, v)$.*

*Proof.* This follows from [DM10, Proposition 3.7]. More specifically, let $(i_v, \ell_v) = \vartheta(v)$ be the type of vertex $v$. In the terminology of [DM10] (and of the physicists "cavity method"), $\phi(T, \vartheta, v)$ is the *Bethe free entropy* of the Boltzmann distribution

$$\nu : [k]^{V(T)} \to [0, 1], \qquad \nu(\tau) = \frac{1}{\mathcal{Z}(T, \vartheta)}\prod_{v\in V(T)}\mathbf{1}_{\tau(v)\in\ell_v}\cdot\prod_{e=\{u,w\}\in E(T)}\mathbf{1}_{\tau(u)\neq\tau(w)}.$$

Thus, $\nu$ is simply the uniform distribution over legal $k$-colourings of $T, \vartheta$, and $\mathcal{Z}(T, \vartheta)$ is its partition function. □

Let $\boldsymbol{T}$ denote the random rooted decorated tree $\boldsymbol{T}_{d,k,\boldsymbol{q}^*}$. Moreover, for $(i, \ell) \in \mathcal{T}$ we let $\boldsymbol{T}_{i,\ell}$ denote the random tree $\boldsymbol{T}$ given that the root has type $(i, \ell)$. The starting point of the proof is the following key observation. Furthermore, if $(T, \vartheta, v)$ is a rooted decorated tree, then we let $(T, \vartheta, v)^\star$ signify the isomorphism class of the random rooted decorated tree $(T, \vartheta, u)$ obtained from $(T, \vartheta, v)$ by choosing a vertex $u$ of $T$ uniformly at random and rooting the tree at $u$. In other words, $(T, \vartheta, v)^\star$ is obtained by re-rooting $(T, \vartheta, v)$ at a random vertex.

**Lemma A.3.12.** *Let $\boldsymbol{T}^\star$ be the random rooted decorated tree obtained by re-rooting $\boldsymbol{T}$ at a random vertex. Then the distribution of $\boldsymbol{T}^\star$ coincides with the distribution of $\boldsymbol{T}$.*

*Proof.* This follows from the general fact that Galton-Watson trees are unimodular in the sense of [BC15]. □

**Corollary A.3.13.** *We have $\mathbb{E}\left[\frac{\ln \mathcal{Z}(\boldsymbol{T})}{|\boldsymbol{T}|}\right] = \mathbb{E}[\phi(\boldsymbol{T})]$.*

*Proof.* Letting $(T, \vartheta, v)$ range over rooted decorated trees, we find

$$\begin{aligned}
\mathbb{E}\left[\frac{\ln \mathcal{Z}(\boldsymbol{T})}{|\boldsymbol{T}|}\right] &= \sum_{(T,\vartheta,v)} \mathbb{P}\left[\boldsymbol{T} \cong (T, \vartheta, v)\right] \cdot \frac{\ln \mathcal{Z}(T, \vartheta, v)}{|V(T)|} \\
&= \sum_{(T,\vartheta,v)} \sum_{u \in V(T)} \frac{\mathbb{P}\left[\boldsymbol{T} \cong (T, \vartheta, v)\right] \phi(T, \vartheta, u)}{|V(T)|} && \text{[by Fact A.3.11]} \\
&= \sum_{(T,\vartheta,v)} \sum_{u \in V(T)} \frac{\mathbb{P}\left[\boldsymbol{T} \cong (T, \vartheta, u)\right] \phi(T, \vartheta, u)}{|V(T)|} && \text{[by Lemma A.3.12]} \\
&= \sum_{(T,\vartheta,v)} \mathbb{P}\left[\boldsymbol{T} \cong (T, \vartheta, v)\right] \phi(T, \vartheta, v) = \mathbb{E}[\phi(\boldsymbol{T})],
\end{aligned}$$

as claimed. □

**Lemma A.3.14.** *We have*

$$\begin{aligned}
\mathbb{E}[\phi(\boldsymbol{T}_{i,\ell})] = &\sum_{\boldsymbol{\gamma} \in \Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \int_{\Omega^{\boldsymbol{\gamma}}} \phi^v_\ell(\mu_{\boldsymbol{\gamma}}) \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}) \\
&- \sum_{(\hat{i},\hat{\ell}) \in \mathcal{T}_{i,\ell}} \frac{q_{\hat{i},\hat{\ell}} d'}{2} \int_{\Omega^2} \ln\left[1 - \sum_{h=1}^k \hat{\mu}(h)\mu(h)\right] \mathrm{d}\pi_{i,\ell}(\mu) \otimes \pi_{\hat{i},\hat{\ell}}(\hat{\mu}).
\end{aligned}$$

*Proof.* Writing $\pi = \pi_{d,k,\boldsymbol{q}^*}$ for the distribution of $\mu_{\boldsymbol{T}}$, we know from Corollary A.3.9 that $\pi_{i,\ell}$ is the distribution of $\mu_{\boldsymbol{T}_{i,\ell}}$ for any type $(i, \ell)$. Furthermore, the distribution of $\boldsymbol{T}_{i,\ell}$ can be described by the following recurrence: there is a root $v_0$ of type $(i, \ell)$, to which we attach for each $(i', \ell') \in \mathcal{T}_{i,\ell}$ independently a number $\gamma_{i',\ell'} = \mathrm{Po}(d' q^*_{i',\ell'})$ of trees $(T_{i',\ell',j})_{j=1,\ldots,\gamma_{i',\ell'}}$ that are chosen independently from the distribution $\boldsymbol{T}_{i',\ell'}$. By independence, the distribution of the colour of the root of each $T_{i',\ell',j}$

is just an independent sample from the distribution $\pi_{i',\ell'}$. Therefore, we obtain the expansion

$$\mathbb{E}[\phi(\boldsymbol{T}_{i,\ell})] = \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} \int_{\Omega^{\boldsymbol{\gamma}}} \phi_\ell(\mu_{\boldsymbol{\gamma}}) p_{i,\ell}(\boldsymbol{\gamma}) \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}).$$

Substituting in the definition of $\phi_\ell$, we obtain

$$\mathbb{E}[\phi(\boldsymbol{T}_{i,\ell})] = I_{i,\ell} - \frac{1}{2} J_{i,\ell},$$

where $I_{i,\ell} = \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \int_{\Omega^{\boldsymbol{\gamma}}} \phi_\ell^v(\mu_{\boldsymbol{\gamma}}) \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}})$ and $J_{i,\ell} = \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \int_{\Omega^{\boldsymbol{\gamma}}} \phi_\ell^e(\mu_{\boldsymbol{\gamma}}) \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}})$. Further, by the definition of $\phi_\ell^e$ we have

$$\begin{aligned}
J_{i,\ell} &= \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \sum_{(\hat{\imath},\hat{\ell})\in\mathcal{T}_{i,\ell}} \sum_{\hat{\jmath}=1}^{\gamma_{\hat{\imath},\hat{\ell}}} \int_{\Omega^{\boldsymbol{\gamma}}} \ln\left[1 - \sum_{h\in\ell} \mu_{\hat{\imath},\hat{\ell},\hat{\jmath}}(h) \mathcal{B}[(\mu_{i',\ell',j})_{(i',\ell',j)\neq(\hat{\imath},\hat{\ell},\hat{\jmath})}](h)\right] \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}) \\
&= \sum_{(\hat{\imath},\hat{\ell})\in\mathcal{T}_{i,\ell}} \sum_{g\geq 1} p_{q^*_{\hat{\imath},\hat{\ell}}}(g) \sum_{\hat{\jmath}=1}^{g} \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \mathbf{1}_{\gamma_{\hat{\imath},\hat{\ell}}=g} \\
&\qquad \cdot \int_{\Omega\times\Omega^{\boldsymbol{\gamma}}} \ln\left[1 - \sum_{h\in\ell} \mu(h) \mathcal{B}[(\mu_{i',\ell',j})_{(i',\ell',j)\neq(\hat{\imath},\hat{\ell},1)}](h)\right] \mathrm{d}\pi_{\hat{\imath},\hat{\ell}}(\mu) \otimes \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}) \\
&= \sum_{(\hat{\imath},\hat{\ell})\in\mathcal{T}_{i,\ell}} \sum_{g\geq 1} p_{q^*_{\hat{\imath},\hat{\ell}} d'}(g) \sum_{\hat{\jmath}=1}^{g} \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \mathbf{1}_{\gamma_{\hat{\imath},\hat{\ell}}=g-1} \\
&\qquad \cdot \int_{\Omega\times\Omega^{\boldsymbol{\gamma}}} \ln\left[1 - \sum_{h\in\ell} \mu(h) \mathcal{B}[\mu_{\boldsymbol{\gamma}}](h)\right] \mathrm{d}\pi_{\hat{\imath},\hat{\ell}}(\mu) \otimes \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}).
\end{aligned}$$

To simplify this, we use the following elementary relation: if $X : \mathbf{Z} \to \mathbb{R}_{\geq 0}$ is a function and $g$ is a Poisson random variable, then $\mathbb{E}[\mathbf{1}_{g\geq 1} g X(g-1)] = \mathbb{E}[g]\mathbb{E}[X(g)]$. Applying this observation to

$$X(g) = \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \mathbf{1}_{\gamma_{i,\ell}=g-1} \int_{\Omega\times\Omega^{\boldsymbol{\gamma}}} \ln\left[1 - \sum_{h\in\ell} \mu(h) \mathcal{B}[\mu_{\boldsymbol{\gamma}}](h)\right] \mathrm{d}\pi_{\hat{\imath},\hat{\ell}}(\mu) \otimes \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}),$$

we obtain

$$\begin{aligned}
J_{i,\ell} &= \sum_{(\hat{\imath},\hat{\ell})\in\mathcal{T}_{i,\ell}} q_{\hat{\imath},\hat{\ell}} d' \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \\
&\qquad \cdot \int_{\Omega\times\Omega^{\boldsymbol{\gamma}}} \ln\left[1 - \sum_{h\in\ell} \mu_{\hat{\imath},\hat{\ell}}(h) \mathcal{B}[\mu_{\boldsymbol{\gamma}}](h)\right] \mathrm{d}\pi_{\hat{\imath},\hat{\ell}}(\mu) \otimes \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}).
\end{aligned}$$

Now, since $\pi$ is a fixed point of $\mathcal{F}_{d,k}$, the distribution of the measure $\mathcal{B}[\mu_{\boldsymbol{\gamma}}]$ is just $\pi_{i,\ell}$. Hence,

$$J_{i,\ell} = \sum_{(\hat{i},\hat{\ell})\in\mathcal{T}_{i,\ell}} q_{\hat{i},\hat{\ell}} d' \int_{\Omega^2} \ln\left[1 - \sum_{h\in\ell} \hat{\mu}(h)\mu(h)\right] \mathrm{d}\pi_{i,\ell}(\mu) \otimes \mathrm{d}\pi_{\hat{i},\hat{\ell}}(\hat{\mu}).$$

Thus, we obtain the assertion. □

**Lemma A.3.15.** *We have* $\mathbb{E}\left[\phi(\boldsymbol{T}_{d,k,\boldsymbol{q}^*})\right] = \phi_{d,k}(\pi_{d,k,q^*})$.

*Proof.* Summing over all $(i,\ell)\in\mathcal{T}$, we obtain from Lemma A.3.14 that

$$\mathbb{E}[\phi(\boldsymbol{T})] = I - \frac{1}{2}J, \qquad \text{where}$$

$$I = \sum_{(i,\ell)\in\mathcal{T}} q^*_{i,\ell} \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} p_{i,\ell}(\boldsymbol{\gamma}) \int_{\Omega^{\boldsymbol{\gamma}}} \phi^v_\ell(\mu_{\boldsymbol{\gamma}}) \mathrm{d}\pi_{\boldsymbol{\gamma}}(\mu_{\boldsymbol{\gamma}}),$$

$$J = d' \sum_{(i,\ell)\in\mathcal{T}} \sum_{(\hat{i},\hat{\ell})\in\mathcal{T}_{i,\ell}} q^*_{i,\ell} q^*_{\hat{i},\hat{\ell}} \int_{\Omega^2} \ln\left[1 - \sum_{h=1}^k \hat{\mu}(h)\mu(h)\right] \mathrm{d}\pi_{i,\ell}(\mu)\otimes \pi_{\hat{i},\hat{\ell}}(\hat{\mu}).$$

Recalling that $\mathrm{d}\pi_{i,\ell}(\mu) = \frac{\mathbf{1}_{\mu\in\Omega_\ell}}{kq^*_{i,\ell}} \mathrm{d}\pi_i(\mu)$ and $\mathrm{d}\pi_{\hat{i},\hat{\ell}}(\hat{\mu}) = \frac{\mathbf{1}_{\hat{\mu}\in\Omega_{\hat{\ell}}}}{kq^*_{\hat{i},\hat{\ell}}} \mathrm{d}\pi_{\hat{i}}(\hat{\mu})$, we get

$$\begin{aligned}
J &= \frac{d'}{k^2} \sum_{(i,\ell)\in\mathcal{T}} \sum_{(\hat{i},\hat{\ell})\in\mathcal{T}_{i,\ell}} \int_{\Omega^2} \ln\left[1 - \sum_{h=1}^k \hat{\mu}(h)\mu(h)\right] \mathbf{1}_{\mu\in\Omega_\ell} \mathbf{1}_{\hat{\mu}\in\Omega_{\hat{\ell}}} \mathrm{d}\pi_i(\mu)\otimes\pi_{\hat{i}}(\hat{\mu}) \\
&= \frac{d}{k(k-1)} \sum_{i,\hat{i}\in[k]:i\neq\hat{i}} \int_{\Omega^2} \sum_{\ell:(i,\ell)\in\mathcal{T}} \sum_{\hat{\ell}:(\hat{i},\hat{\ell})\in\mathcal{T}} \ln\left[1 - \sum_{h=1}^k \hat{\mu}(h)\mu(h)\right] \\
&\qquad \cdot \mathbf{1}_{\mu\in\Omega_\ell} \mathbf{1}_{\hat{\mu}\in\Omega_{\hat{\ell}}} \mathrm{d}\pi_i(\mu)\otimes\pi_{\hat{i}}(\hat{\mu}) \\
&= \frac{d}{k(k-1)} \sum_{i,\hat{i}\in[k]:i\neq\hat{i}} \int_{\Omega^2} \ln\left[1 - \sum_{h=1}^k \hat{\mu}(h)\mu(h)\right] \mathrm{d}\pi_i(\mu)\otimes\pi_{\hat{i}}(\hat{\mu}) = \phi^e_{d,k}(\pi).
\end{aligned}$$

It finally remains to simplify the expression for $I$. To this aim, we introduce $\mathcal{T}_i = \{(i',\ell')\in\mathcal{T}, i'\neq i\}$ and let $\overline{\Gamma}_i$ be the set of non-negative vectors $\overline{\boldsymbol{\gamma}} = (\overline{\gamma}_{i',\ell'})_{(i',\ell')\in\mathcal{T}_i}$. Moreover, we define $\Omega^{\overline{\boldsymbol{\gamma}}} = \prod_{(i',\ell')\in\mathcal{T}_i}\prod_{j\in[\overline{\gamma}_{i',\ell'}]}\Omega$ and denote its points by $\mu_{\overline{\boldsymbol{\gamma}}} = (\mu_{i',\ell',j})_{(i',\ell')\in\mathcal{T}_i, j\in[\overline{\gamma}_{i',\ell'}]}$. We note that if $\boldsymbol{\gamma}\in\Gamma_{i,\ell}$ and $\overline{\boldsymbol{\gamma}}\in\overline{\Gamma}_i$ are such that:

(a) $\forall i'\in\ell\setminus\{i\}, \overline{\gamma}_{i',\{i'\}} = 0$,
(b) $\forall i'\in[k]\setminus\ell, \overline{\gamma}_{i',\{i'\}} > 0$

(c) $\forall (i',\ell') \in \mathcal{T}_{i,\ell}, \gamma_{i',\ell'} = \overline{\gamma}_{i',\ell'}$,

and that $\mu_{\boldsymbol{\gamma}}, \overline{\mu}_{\overline{\boldsymbol{\gamma}}}$ satisfy

(d) $\forall (i',\ell') \in \mathcal{T}_{i,\ell}, \forall j \in [\gamma_{i',\ell'}], \mu_{i',\ell',j} = \overline{\mu}_{i',\ell',j}$,
(e) $\forall (i',\ell') \in \mathcal{T}_{i}, \forall j \in [\overline{\gamma}_{i',\ell'}], \overline{\mu}_{i',\ell',j} \in \Omega_{\ell'}$,

then

$$\prod_{(i',\ell')\in\mathcal{T}_i} \prod_{j\in[\overline{\gamma}_{i',\ell'}]} 1 - \overline{\mu}_{i',\ell',j}(h) = \begin{cases} 0 \text{ if } h \notin \ell, \\ \prod_{(i',\ell')\in\mathcal{T}_{i,\ell}} \prod_{j\in[\overline{\gamma}_{i',\ell'}]} 1 - \mu_{i',\ell',j}(h) \text{ if } h \in \ell. \end{cases}$$

Consequently

$$\phi^v_\ell(\mu_{\boldsymbol{\gamma}}) = \ln\left[\sum_{h\in[k]} \prod_{(i',\ell')\in\mathcal{T}_i} \prod_{j\in[\overline{\gamma}_{i',\ell'}]} 1 - \overline{\mu}_{i',\ell',j}(h)\right]. \tag{A.3.16}$$

Moreover, choosing the $\overline{\gamma}_{i',\ell'}$ from a Poisson distribution of parameter $q^*_{i',\ell'}d'$, the event "(a) and (b)" happens with probability exactly $kq^*_{i,\ell}$. This allows us to write:

$$\begin{aligned}
I &= \sum_{(i,\ell)\in\mathcal{T}} q^*_{i,\ell} \sum_{\boldsymbol{\gamma}\in\Gamma_{i,\ell}} \prod_{(i',\ell')\in\mathcal{T}_{i,\ell}} p_{q^*_{i',\ell'}d'}(\gamma_{i',\ell'}) \cdot \int_{\Omega^{\boldsymbol{\gamma}}} \phi^v_\ell(\mu_{\boldsymbol{\gamma}}) \bigotimes_{(i,'\ell')\in\mathcal{T}_{i,\ell}} \bigotimes_{j\in[\overline{\gamma}_{i',\ell'}]} \mathrm{d}\pi_{i',\ell'}(\mu_{i',\ell',j}) \\
&= \frac{1}{k} \sum_{(i,\ell)\in\mathcal{T}} \sum_{\overline{\boldsymbol{\gamma}}\in\overline{\Gamma}_i} \prod_{(i',\ell')\in\mathcal{T}_i} p_{q^*_{i',\ell'}d'}(\overline{\gamma}_{i',\ell'}) \prod_{i'\in\ell\setminus\{i\}} \mathbf{1}_{\overline{\gamma}_{i',\{i'\}}=0} \prod_{i'\in[k]\setminus\ell} \mathbf{1}_{\overline{\gamma}_{i',\{i'\}}>0} \\
&\qquad \cdot \int_{\Omega^{\overline{\boldsymbol{\gamma}}}} \ln\left[\sum_{h\in[k]} \prod_{(i',\ell')\in\mathcal{T}_i} \prod_{j\in[\overline{\gamma}_{i',\ell'}]} 1 - \overline{\mu}_{i',\ell',j}(h)\right] \\
&\qquad \cdot \prod_{(i',\ell')\in\mathcal{T}_i} \prod_{j\in[\overline{\gamma}_{i',\ell'}]} \frac{\mathbf{1}_{\overline{\mu}_{i',\ell',j}\in\Omega_{\ell'}}}{kq^*_{i',\ell'}} \bigotimes_{(i,'\ell')\in\mathcal{T}_i} \bigotimes_{j\in[\overline{\gamma}_{i',\ell'}]} \mathrm{d}\pi_{i'}(\overline{\mu}_{i',\ell',j}) \\
&= \frac{1}{k} \sum_{i\in[k]} \sum_{\overline{\boldsymbol{\gamma}}\in\overline{\Gamma}_i} \prod_{(i',\ell')\in\mathcal{T}_i} p_{q^*_{i',\ell'}d'}(\overline{\gamma}_{i',\ell'}) \\
&\qquad \cdot \int_{\Omega^{\overline{\boldsymbol{\gamma}}}} \ln\left[\sum_{h\in[k]} \prod_{(i',\ell')\in\mathcal{T}_i} \prod_{j\in[\overline{\gamma}_{i',\ell'}]} 1 - \overline{\mu}_{i',\ell',j}(h)\right] \\
&\qquad \cdot \prod_{(i',\ell')\in\mathcal{T}_i} \prod_{j\in[\overline{\gamma}_{i',\ell'}]} \frac{\mathbf{1}_{\overline{\mu}_{i',\ell',j'}\in\Omega_{\ell'}}}{kq^*_{i',\ell'}} \bigotimes_{(i,'\ell')\in\mathcal{T}_i} \bigotimes_{j\in[\overline{\gamma}_{i',\ell'}]} \mathrm{d}\pi_{i'}(\overline{\mu}_{i',\ell',j}).
\end{aligned}$$

We used (A.3.16) to go from the first to the second line, and summed over $\ell \ni i$ to go from the second to the third. Re-indexing the vector $\overline{\mu}_{\overline{\boldsymbol{\gamma}}}$ in a vector $\mu_{\boldsymbol{\gamma}}$, $\boldsymbol{\gamma} \in \Gamma_i$ (with $\gamma_{i'} = \sum_{\ell':(i',\ell')\in\mathcal{T}} \overline{\gamma}_{i',\ell'}$), we obtain with Lemma A.3.1:

$$I = \frac{1}{k} \sum_{i \in [k]} \sum_{\boldsymbol{\gamma} \in \Gamma_i} \prod_{i' \neq i} p_{\frac{d}{k-1}}(\gamma_{i'})$$

$$\cdot \int_{\Omega^{\gamma}} \ln \left[ \sum_{h \in [k]} \prod_{i' \neq i} \prod_{j \in [\gamma_{i'}]} 1 - \mu_{i',j}(h) \right] \bigotimes_{i' \neq i} \bigotimes_{j \in [\gamma_{i'}]} \mathrm{d}\pi_{i'}(\mu_{i',j})$$

$$= \frac{1}{k} \sum_{i \in [k]} \sum_{\gamma_1, \ldots, \gamma_h = 0}^{\infty} \prod_{i' \in [k]} p_{\frac{d}{k-1}}(\gamma_{i'})$$

$$\cdot \int_{\Omega^{\gamma_1 + \cdots + \gamma_h}} \ln \left[ \sum_{h \in [k]} \prod_{i' \neq i} \prod_{j \in [\gamma_{i'}]} 1 - \mu_{i',j}(h) \right] \bigotimes_{i' \in [k]} \bigotimes_{j \in [\gamma_{i'}]} \mathrm{d}\pi_{i'}(\mu_{i',j}).$$

□

*Proof of Proposition 5.1.4.* The first assertion is immediate from Lemma 5.2.1, while the second assertion follows from Lemma A.3.6. The third claim follows by combining Corollary A.3.13 with Lemma A.3.15. With respect to the last assertion, we observe that for $d = (2k-1)\ln k - 2\ln 2 + o_k(1)$ we have

$$\ln k + \frac{d}{2} \ln(1 - 1/k) = \frac{\ln 2 + o_k(1)}{k}.$$

Moreover, as $q^* = 1 - 1/k + o_k(1/k)$ by Lemma 5.2.1, one checks easily that

$$\mathbb{E}\left[\frac{\ln \mathcal{Z}(\boldsymbol{T}_{d,k,\boldsymbol{q}^*})}{|\boldsymbol{T}_{d,k,\boldsymbol{q}^*}|}\right] = \frac{\ln 2 + o_k(1)}{k}. \tag{A.3.17}$$

Further, by Lemma 5.2.1

$$\frac{\partial}{\partial d}\mathbb{E}\left[\frac{\ln \mathcal{Z}(\boldsymbol{T}_{d,k}(\boldsymbol{q}^*))}{|\boldsymbol{T}_{d,k}(\boldsymbol{q}^*)|}\right] = \tilde{O}_k(k^{-2}) \quad \text{while} \quad \frac{\partial}{\partial d} \ln k + \frac{d}{2}\ln(1 - 1/k) = \Omega_k(1/k). \tag{A.3.18}$$

Combining (A.3.17) and (A.3.18) and using the third part of Proposition 5.1.4, we conclude that $\Sigma_k$ has a unique zero $d_{\mathrm{cond}}$, as claimed. □

# B Complementary proofs: Number of solutions in random graph $k$-colouring

This chapter presents the remaining parts of the proofs of statements in Chapter 8. It is a verbatim copy of parts of the paper *On the number of solutions in random graph $k$-colouring* [Ras16b+] submitted to *Combinatorics, Probability and Computing*.

*Proof of Corollary 8.1.6.* We fix $s \in S_{k,\omega,\nu}$ and let $\mathcal{E}$ denote the event $\{\forall 2 \le l \le L : C_{l,n} = c_l\}$. Let $\mathcal{Z}_n = \widetilde{Z}^s_{k,\omega,\nu}(\mathcal{G}(n,m))$ for the sake of brevity. Since $\mathcal{Z}_n \le Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))$, equation (8.1.5) yields the upper bound

$$\frac{\mathbb{E}\left[\mathcal{Z}_n|\mathcal{E}\right]}{\mathbb{E}\left[\mathcal{Z}_n\right]} \le \frac{\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{E}\right]}{(1+o(1))\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]} \sim \prod_{l=2}^{L}\left[1+\delta_l\right]^{c_l}\exp\left[-\delta_l\lambda_l\right]. \tag{B.0.1}$$

We show the following matching lower bound:

$$\mathbb{E}\left[\mathcal{Z}_n|\mathcal{E}\right] \ge (1-o(1))\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{E}\right]. \tag{B.0.2}$$

Indeed, assume for contradiction that (B.0.2) is false. Then we can find an $n$-independent $\varepsilon > 0$ such that for infinitely many $n$,

$$\mathbb{E}\left[\mathcal{Z}_n|\mathcal{E}\right] < (1-\varepsilon)\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{E}\right]. \tag{B.0.3}$$

By Fact 8.1.2 there exists an $n$-independent $\xi = \xi(c_2, \ldots, c_L) > 0$ such that $\mathbb{P}\left[\mathcal{E}\right] \ge \xi$. Hence, (B.0.3) and Bayes' formula imply that

$$\begin{aligned}
\mathbb{E}\left[\mathcal{Z}_n\right] &= \mathbb{E}\left[\mathcal{Z}_n|\mathcal{E}\right]\mathbb{P}\left[\mathcal{E}\right] + \mathbb{E}\left[\mathcal{Z}_n|\neg\mathcal{E}\right]\mathbb{P}\left[\neg\mathcal{E}\right] \\
&\le (1-\varepsilon)\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{E}\right]\mathbb{P}\left[\mathcal{E}\right] + \mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\neg\mathcal{E}\right]\mathbb{P}\left[\neg\mathcal{E}\right] \\
&\le \mathbb{E}[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))] - \varepsilon\xi \cdot \mathbb{E}[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{E}] \\
&= \mathbb{E}[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))] \cdot \left(1 + o(1) - \varepsilon\xi\prod_{l=2}^{L}[1+\delta_l]^{c_l}\exp\left[-\delta_l\lambda_l\right]\right) \\
&= (1-\Omega(1))\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right],
\end{aligned} \tag{B.0.4}$$

where the last equality holds since $\delta_l$, $\lambda_l$ and $c_l$ remain fixed as $n \to \infty$. As (B.0.4) contradicts (8.1.5), we have established (B.0.2). Finally, combining (B.0.2) with (8.1.4) and (8.1.5), we get

$$\frac{\mathbb{E}\left[\mathcal{Z}_n|S\right]}{\mathbb{E}\left[\mathcal{Z}_n\right]} \geq \frac{(1-o(1))\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|S\right]}{(1+o(1))\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right]} \sim \prod_{l=2}^{L}\left[1+\delta_l\right]^{c_l}\exp\left[-\delta_l\lambda_l\right], \tag{B.0.5}$$

and the assertion follows from (B.0.1) and (B.0.5). □

## B.1. Calculating the first moment

The following proofs are very close to analogous proofs in [BCOE14+].

*Proof of Lemma 8.2.1.* As the edges in $\mathcal{G}(n,m)$ are independent by construction, the expected number of $k$-colourings with colour density $\rho$ is given by

$$\mathbb{E}\left[Z_{k,\rho}(\mathcal{G}(n,m))\right] = \binom{n}{\rho_1 n,\ldots,\rho_k n}\left(1-\frac{1}{N}\sum_{i=1}^{k}\binom{\rho_i n}{2}\right)^m, \quad \text{where } N=\binom{n}{2}. \tag{B.1.1}$$

Further, the number of forbidden edges is given by

$$\sum_{i=1}^{k}\binom{\rho_i n}{2} = N\left(\sum_{i=1}^{k}\rho_i^2\right)+\frac{n}{2}\left(\sum_{i=1}^{k}\rho_i^2-1\right)+O(1)$$

and thus

$$\begin{aligned} m\ln\left(1-\frac{1}{N}\sum_{i=1}^{k}\binom{\rho_i n}{2}\right) &= m\ln\left[\left(1+\frac{n}{2N}\right)\left(1-\sum_{i=1}^{k}\rho_i^2\right)\right]+o(1) \\ &= n\frac{d}{2}\ln\left(1-\sum_{i=1}^{k}\rho_i^2\right)+\frac{d}{2}+o(1). \end{aligned} \tag{B.1.2}$$

Equation (8.2.1) follows from (B.1.1), (B.1.2) and Stirling's formula. Moreover, (8.2.2) follows from (B.1.1) and (B.1.2) because $\|\rho-\rho^\star\|_2 = o(1)$ implies that $\sum_{i=1}^{k}\rho_i^2 \sim 1/k$ and

$$\binom{n}{\rho_1 n,\ldots,\rho_k n} \sim (2\pi n)^{\frac{1-k}{2}}\, k^{k/2}\exp\left[n\mathcal{H}(\rho)\right].$$

□

*Proof of Corollary 8.2.2.* The functions $\rho \in \mathcal{A}_k \mapsto \mathcal{H}(\rho)$ and $\rho \in \mathcal{A}_k \mapsto \frac{d}{2}\ln(1-\sum_{i=1}^{k}\rho_i^2)$ are both concave and attain their maximum at $\rho=\rho^\star$. Consequently, setting $B(d,k) = k(1+\frac{d}{k-1})$ and expanding around $\rho = \rho^\star$, we obtain

$$f_1(\rho^\star) - \frac{B(d,k)}{2}\|\rho-\rho^\star\|_2^2 - O\left(\|\rho-\rho^\star\|_2^3\right) \le f_1(\rho) \le f_1(\rho^\star) - \frac{B(d,k)}{2}\|\rho-\rho^\star\|_2^2. \quad \text{(B.1.3)}$$

Plugging the upper bound from (B.1.3) into (8.2.1) and observing that $|\mathcal{A}_k(n)| \le n^k = \exp[o(n)]$, we find

$$S_1 = \sum_{\substack{\rho\in\mathcal{A}_k(n)\\ \|\rho-\rho^\star\|_2 > n^{-3/8}}} \mathbb{E}\left[Z_{k,\rho}(\mathcal{G}(n,m))\right] \le C_2 \exp\left[f_1(\rho^\star)\right]\exp\left[-\frac{B(d,k)}{2}n^{1/6}\right]. \quad \text{(B.1.4)}$$

On the other hand, (8.2.2) implies that

$$S_2 = \sum_{\substack{\rho\in\mathcal{A}_k(n)\\ \|\rho-\rho^\star\|_2 \le n^{-3/8}}} \mathbb{E}\left[Z_{k,\rho}(\mathcal{G}(n,m))\right] \sim \sum_{\substack{\rho\in\mathcal{A}_k(n)\\ \|\rho-\rho^\star\|_2 \le n^{-3/8}}} (2\pi n)^{\frac{1-k}{2}} k^{k/2}\exp[d/2]\exp[nf_1(\rho)]$$

$$\sim (2\pi n)^{\frac{1-k}{2}} k^{k/2}\exp\left[d/2 + nf_1(\rho^\star)\right]\sum_{\rho\in\mathcal{A}_k(n)}\exp\left[-n\frac{B(d,k)}{2}\|\rho-\rho^\star\|_2^2\right]. \quad \text{(B.1.5)}$$

The last sum is nearly in the standard form of a Gaussian summation, just that the vectors $\rho \in \mathcal{A}_k(n)$ that we sum over are subject to the linear constraint $\rho_1 + \cdots + \rho_k = 1$. We rid ourselves of this constraint by substituting $\rho_k = 1 - \rho_1 - \cdots - \rho_{k-1}$. Formally, let $J$ be the $(k-1)\times(k-1)$-matrix with diagonal entries equal to 2 and remaining entries equal to 1. We observe that $\det J = k$. Then

$$\sum_{\rho\in\mathcal{A}_k(n)}\exp\left[-n\frac{B(d,k)}{2}\|\rho-\rho^\star\|_2^2\right] \sim \sum_{y\in\frac{1}{n}\mathbb{Z}^k}\exp\left[-n\frac{B(d,k)}{2}\langle Jy,y\rangle\right]$$

$$\sim (2\pi n)^{\frac{k-1}{2}} k^{-\frac{k}{2}}\left(1+\frac{d}{k-1}\right)^{-\frac{k-1}{2}}. \quad \text{(B.1.6)}$$

Plugging (B.1.6) into (B.1.5), we obtain

$$S_2 \sim (2\pi n)^{\frac{1-k}{2}} k^{k/2}\exp\left[d/2 + nf_1(\rho^\star)\right](2\pi n)^{\frac{k-1}{2}} k^{-\frac{k}{2}}\left(1+\frac{d}{k-1}\right)^{-\frac{k-1}{2}}$$

$$= \exp\left[d/2 + nf_1(\rho^\star)\right]\left(1+\frac{d}{k-1}\right)^{-\frac{k-1}{2}}. \quad \text{(B.1.7)}$$

Finally, comparing (B.1.4) and (B.1.7), we see that $S_1 = o(S_2)$. Thus, $\mathbb{E}[Z_k(\mathcal{G}(n,m))] = S_1 + S_2 \sim S_2$, and the assertion follows from (B.1.7). □

## B.2. Calculating the second moment

The following proof is very close to an analogous proof in [BCOE14+].

*Proof of 8.3.1.* To calculate the expected number of pairs of colourings $\sigma, \tau$ with overlap $\rho \in \mathcal{B}_k(n)$, we first observe that

$$\mathbb{P}\left[\sigma, \tau \text{ are } k\text{-colourings of } \mathcal{G}(n,m)\right] = \left(1 - \frac{\mathcal{F}(\sigma,\tau)}{N}\right)^m,$$

where $\mathcal{F}(\sigma,\tau)$ is the number of "forbidden" edges joining two vertices with the same colour under either $\sigma$ or $\tau$ and $N = \binom{n}{2}$. We have

$$\begin{aligned}\mathcal{F}(\sigma,\tau) &= \sum_{i=1}^{k}\binom{\rho_{i\star}n}{2} + \sum_{j=1}^{k}\binom{\rho_{\star j}n}{2} - \sum_{i,j=1}^{k}\binom{\rho_{ij}n}{2} \\ &= N\left(\sum_{i=1}^{k}\rho_{i\star}^2 + \sum_{j=1}^{k}\rho_{\star j}^2 - \sum_{i,j=1}^{k}\rho_{ij}^2\right) + \frac{n}{2}\left(\sum_{i=1}^{k}\rho_{i\star}^2 + \sum_{j=1}^{k}\rho_{\star j}^2 - \sum_{i,j=1}^{k}\rho_{ij}^2 - 1\right) + O(1)\end{aligned}$$

and thus, the probability that $\sigma$ and $\tau$ are both colourings of $\mathcal{G}(n,m)$ only depends on their overlap $\rho$ and is given by

$$\mathbb{P}\left[\sigma, \tau \text{ are } k\text{-colourings of } \mathcal{G}(n,m)\right] \sim \exp\left[m \ln\left(1 - \sum_{i=1}^{k}\rho_{i\star}^2 - \sum_{j=1}^{k}\rho_{\star j}^2 + \sum_{i,j=1}^{k}\rho_{ij}^2\right) + \frac{d}{2}\right]. \tag{B.2.1}$$

It remains to multiply this by the total number of $\sigma, \tau$ with overlap $\rho \in \mathcal{B}_k(n)$. By Stirling's formula, this number is given by

$$\binom{n}{\rho_{11}n, \ldots, \rho_{kk}n} \sim \sqrt{2\pi}n^{-\frac{k^2-1}{2}}\left(\prod_{i,j}\frac{1}{\sqrt{2\pi\rho_{ij}}}\right)\exp\left[n\mathcal{H}(\rho)\right]. \tag{B.2.2}$$

Equation (8.3.2) is obtained by combining (B.2.1) and (B.2.2). To prove (8.3.3), we observe that if $\|\rho - \bar{\rho}\|_2^2 = o(1)$, we have

$$\frac{\sqrt{2\pi}n^{\frac{1-k^2}{2}}}{\prod_{i,j=1}^{k}\sqrt{2\pi\rho_{ij}}} \sim k^{k^2}\left(2\pi n\right)^{\frac{1-k^2}{2}}$$

and the statement follows. □

## B.3. Counting short cycles

In this section we count the number of cycles of a short fixed length in order to prove Proposition 8.1.3. The results in this section were already obtained in [BCOE14+] and the proofs are a very close adaption of the ones in [BCOE14+]. We recall that for $l = 2, \dots, L$ we denoted by $C_{l,n}$ the number of cycles of length exactly $l$ in $\mathcal{G}(n, m)$. We let $c_2, \dots, c_L$ be a sequence of non-negative integers and $\mathcal{E}$ the event that $C_{l,n} = c_l$ for $l = 2, \dots, L$. We recall $\lambda_l, \delta_l$ from (8.1.3). For a map $\sigma : [n] \mapsto [k]$, we define $\mathcal{V}(\sigma)$ as the event that $\sigma$ is a $k$-colouring of the random graph $\mathcal{G}(n, m)$. Our starting point is the following lemma concerning the distribution of the random variables $C_{l,n}$ given $\mathcal{V}(\sigma)$.

**Lemma B.3.1.** *Let* $\mu_l = \frac{d^l}{2l}\left[1 + \frac{(-1)^l}{(k-1)^{l-1}}\right]$. *Then* $\mathbb{P}[\mathcal{E}|\mathcal{V}(\sigma)] \sim \prod_{l=2}^{L} \frac{\exp[-\mu_l]}{c_l!}\mu_l^{c_l}$ *for any* $\sigma$ *with* $\rho(\sigma) \in \mathcal{A}_{k,\omega}(n)$.

*Proof.* All we have to show is that for any fixed sequence of integers $m_2, \dots, m_L \geq 0$, the joint factorial moments satisfy

$$\mathbb{E}\left[(C_{2,n})_{m_2} \cdots (C_{L,n})_{m_L} | \mathcal{V}(\sigma)\right] \sim \prod_{l=2}^{L} \mu_l^{m_l}. \tag{B.3.1}$$

Then Lemma B.3.1 follows from [Bol01, Theorem 1.23].

To establish (B.3.1), we interpret $(C_{2,n})_{m_2} \cdots (C_{L,n})_{m_L}$ as the number of sequences of $m_2 + \cdots + m_L$ distinct cycles such that $m_2$ is the number of cycles of length 2, and so on. We let $Y$ be the number of those sequences of cycles such that any two cycles are vertex-disjoint and $Y'$ be the number of sequences having intersecting cycles. Obviously, we have

$$\mathbb{E}\left[(C_{2,n})_{m_2} \cdots (C_{L,n})_{m_L} | \mathcal{V}(\sigma)\right] = \mathbb{E}\left[Y | \mathcal{V}(\sigma)\right] + \mathbb{E}\left[Y' | \mathcal{V}(\sigma)\right]. \tag{B.3.2}$$

For $\mathbb{E}\left[Y' | \mathcal{V}(\sigma)\right]$, we use the following claim that we prove at the end of this section.

**Claim B.3.2.** *It holds that* $\mathbb{E}\left[Y' | \mathcal{V}(\sigma)\right] = O(n^{-1})$.

Thus, it remains to count the number of vertex disjoint cycles conditioned on $\mathcal{V}(\sigma)$. The line of arguments we use is similar to [KPGW10, Section 2]. To simplify the calculations, we define $D_{l,n}$ as the number of rooted, directed cycles of length $l$ in $\mathcal{G}(n, m)$, implying that $D_{l,n} = 2lC_{l,n}$.

For a rooted directed cycle $(v_1, \dots, v_l)$ of length $l$, we call $(\sigma(v_1), \dots, \sigma(v_l))$ the *type* of the cycle under $\sigma$. Let $D_{l,n}^t$ denote the number of rooted, directed cycles of length $l$ and type $t = (t_1, ..., t_l)$. We claim that

$$\mathbb{E}\left[D_{l,n}^t|\mathcal{V}(\sigma)\right] \sim \left(\frac{n}{k}\right)^l \frac{(m)_l}{(N-\mathcal{F}(\sigma))^l} \sim \left(\frac{d}{k-1}\right)^l \quad \text{with } N = \binom{n}{2}. \tag{B.3.3}$$

Indeed, as $\sigma$ is $(\omega, n)$-balanced, the number of ways of choosing $l$ vertices $(v_1, \ldots, v_l)$ such that $\sigma(v_i) = t_i$ for all $i$ is $(1+o(1))(n/k)^l$ and each edge $\{v_i, v_{i+1}\}$ of the cycle is present in the graph with a probability asymptotically equal to $m/(N - \mathcal{F}(\sigma))$. This explains the first asymptotic equality in (B.3.3). The second one follows because $m = dn/2$ and $\mathcal{F}(\sigma) \sim N/k$.

In particular, the r.h.s. of (B.3.3) is independent of the type $t$. For a given $l$, let $T_l$ signify the number of all possible types of cycles of length $l$. Thus, $T_l$ is the set of all sequences $(t_1, \ldots, t_l)$ such that $t_{i+1} \neq t_i$ for all $1 \leq i < l$ and $t_l \neq t_1$. Let $T_1 = 0$. Then $T_l$ satisfies the recurrence

$$T_l + T_{l-1} = k(k-1)^{l-1}. \tag{B.3.4}$$

To see this, observe that $k(k-1)^{l-1}$ is the number of all sequences $(t_1, \ldots, t_l)$ such that $t_{i+1} \neq t_i$ for all $1 \leq i < l$. Any such sequence either satisfies $t_l \neq t_1$, which is accounted for by $T_l$, or $t_l = t_1$ and $t_{l-1} \neq t_1$, in which case it is contained in $T_{l-1}$.

Hence, iterating (B.3.4) gives $T_l = (k-1)^l + (-1)^l(k-1)$. Combining this formula with (B.3.3), we obtain

$$\mathbb{E}\left[D_{l,n}|\mathcal{V}(\sigma)\right] \sim T_l \cdot \mathbb{E}\left(D_{l,n}^t|\mathcal{V}(\sigma)\right) \sim d^l \left(1 + \frac{(-1)^l}{(k-1)^{l-1}}\right).$$

Recalling that $C_{l,n} = D_{l,n}/(2l)$, we get

$$\mathbb{E}\left[C_{l,n}|\mathcal{V}(\sigma)\right] \sim \frac{d^l}{2l}\left(1 + \frac{(-1)^l}{(k-1)^{l-1}}\right). \tag{B.3.5}$$

Since $Y$ considers only vertex disjoint cycles and $l, m_2, \ldots, m_L$ remain fixed as $n \to \infty$, equation (B.3.5) yields

$$\mathbb{E}\left[Y|\mathcal{V}(\sigma)\right] \sim \prod_{l=2}^{L}\left(\frac{d^l}{2l}\left(1 + \frac{(-1)^l}{(k-1)^{l-1}}\right)\right)^{m_l}.$$

Plugging the above relation and Claim B.3.2 into (B.3.2), we get (B.3.1) and the assertion follows. □

*Proof of Proposition 8.1.3:* Let $s \in S_{k,\omega,\nu}$. By Bayes' rule and Lemma B.3.1 we have

$$\begin{aligned}\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))|\mathcal{E}\right] &= \frac{1}{\mathbb{P}[\mathcal{E}]}\sum_{\tau\in\mathcal{A}^s_{k,\omega,\nu}(n)}\mathbb{P}[\mathcal{V}(\tau)]\,\mathbb{P}\left[\mathcal{E}|\mathcal{V}(\tau)\right]\\ &\sim \frac{\prod_{l=2}^{L}\frac{\exp[-\mu_l]}{c_l!}\mu_l^{c_l}}{\mathbb{P}[\mathcal{E}]}\sum_{\tau\in\mathcal{A}^s_{k,\omega,\nu}(n)}\mathbb{P}\left[\mathcal{V}(\tau)\right]\\ &\sim \frac{\prod_{l=2}^{L}\frac{\exp[-\mu_l]}{c_l!}\mu_l^{c_l}}{\mathbb{P}[\mathcal{E}]}\mathbb{E}\left[Z^s_{k,\omega,\nu}(\mathcal{G}(n,m))\right].\end{aligned}$$

From Lemma B.3.1 and Fact 8.1.2 we get that

$$\frac{\prod_{l=2}^{L}\frac{\exp[-\mu_l]}{c_l!}\mu_l^{c_l}}{\mathbb{P}[\mathcal{E}]} \sim \prod_{l=2}^{L}\left[1+\delta_l\right]^{c_l}\exp\left[-\delta_l\lambda_l\right],$$

whence Proposition 8.1.3 follows. □

*Proof of Claim B.3.2:* For every subset $R$ of $l \le L$ vertices, let $\mathbb{I}_R$ be equal to 1 if the number of edges with both ends in $R$ is at least $|R|+1$. Let $H_L$ be the event that $\{\sum_{R:|R|\le L}\mathbb{I}_R > 0\}$. By definition, if $Y' > 0$ then the event $H_L$ occurs. This implies that

$$\mathbb{P}\left[Y' > 0|\mathcal{V}(\sigma)\right] \le \mathbb{P}\left[H_L|\mathcal{V}(\sigma)\right].$$

Thus, it suffices to appropriately bound $\mathbb{P}[H_L|\mathcal{V}(\sigma)]$. Markov's inequality yields

$$\mathbb{P}\left[H_L|\mathcal{V}(\sigma)\right] \le \mathbb{E}\left[\sum_{R:|R|\le L}\mathbb{I}_R|\mathcal{V}(\sigma)\right] = \sum_{l=2}^{L}\sum_{R:|R|=l}\mathbb{E}\left[\mathbb{I}_R|\mathcal{V}(\sigma)\right].$$

For any set $R$ such that $|R| = l$, we can put $l+1$ edges inside the set in at most $\binom{\binom{l}{2}}{l+1}$ ways. Clearly conditioning on $\mathcal{V}(\sigma)$ can only reduce the number of different placings of the edges. For a fixed set $R$ of cardinality $l$, we get, using inclusion/exclusion and the Binomial theorem as well as the fact that $\mathcal{F}(\sigma) \sim N/k$:

$$\begin{aligned}\mathbb{E}\left[\mathbb{I}_R|\mathcal{V}(\sigma)\right] &\le \binom{\binom{l}{2}}{l+1}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i\left(1-\frac{i}{N-\mathcal{F}(\sigma)}\right)^m\\ &\le \binom{\binom{l}{2}}{l+1}\left(\frac{m}{N-\mathcal{F}(\sigma)}\right)^{l+1} \sim \binom{\binom{l}{2}}{l+1}\left(\frac{d}{n(1-1/k)}\right)^{l+1}.\end{aligned}$$

As $\binom{i}{j} \leq (ie/j)^j$, it follows that

$$\begin{aligned}
\mathbb{P}\left[H_L|\mathcal{V}(\sigma)\right] &\leq (1+o(1))\sum_{l=2}^{L}\binom{n}{l}\binom{\binom{l}{2}}{l+1}\left(\frac{d}{n(1-1/k)}\right)^{l+1} \\
&\leq (1+o(1))\sum_{l=2}^{L}\left(\frac{ne}{l}\right)^{l}\left(\frac{le}{2}\right)^{l+1}\left(\frac{d}{n(1-1/k)}\right)^{l+1} \\
&\leq \frac{1+o(1)}{n}\sum_{l=2}^{L}\frac{led}{2(1-1/k)}\left(\frac{e^2d}{2(1-1/k)}\right)^{l} = O(n^{-1}),
\end{aligned}$$

where the last equality holds since $L$ is a fixed number. The proves the claim. □

# Deutsche Zusammenfassung

Die vorliegende Doktorarbeit beschäftigt sich mit zwei Fragestellungen im Bereich der Erforschung von zufälligen Graph- und Hypergraphstrukturen.
Zum einen geht es um den Beweis der Existenz und die Bestimmung der Lage des sogenannten Kondensations-Phasenübergangs. Dieser wird für große Werte von $k$ sowohl im Problem der $k$-Färbbarkeit von zufälligen Graphen als auch im Problem der 2-Färbbarkeit von zufälligen $k$-uniformen Hypergraphen untersucht, wobei in letzterem ein erweitertes Modell mit sogenannter *endlicher Temperatur* betrachtet wird.
Zum anderen beschäftigt sich die Arbeit mit der asymptotischen Bestimmung der Verteilung der Anzahl der Lösungen in ebendiesen Strukturen in Dichtebereichen unterhalb des Kondensations-Phasenübergangs.
Die präsentierten Ergebnisse resultieren aus vier Artikeln, die eingereicht und teilweise bereits veröffentlicht sind.

Zunächst folgt nun ein kurzer historischer Überblick über die Entwicklung der Erforschung von Phasenübergängen in zufälligen Bedingungserfüllungsproblemen. Anschließend werden die verwendeten Modelle kurz vorgestellt und danach die Hauptresultate präsentiert und eingeordnet. Es folgt ein weiterer Abschnitt über die verwendeten Methoden, bevor am Ende ein kurzer Ausblick zukünftige Forschungsfragen erläutert.

## Historischer Überblick

Die Untersuchung von zufälligen Graphen geht zurück auf die einflussreiche Arbeit von Erdős und Rényi aus dem Jahr 1960 [ER60]. Seit diesem Zeitpunkt ist die Erforschung von zufälligen diskreten Strukturen, insbesondere von Bedingungserfüllungsproblemen, ein aktives und breites Forschungsgebiet. In den 1990ern entwickelten sich, gestützt durch Computersimulationen, mehrere Hypothesen zum Verhalten dieser zufälligen Probleme bei wachsender Kantendichte[21].
Eine wesentliche Hypothese besagte, dass bei vielen zufälligen Bedingungserfüllungsproblemen die Wahrscheinlichkeit, eine Lösung zu besitzen, rapide von 1 auf 0 abfällt, sobald die Kantendichte einen gewissen 'kritischen Punkt' passiert, das Problem also einen 'scharfen Erfüllbarkeits-Phasenübergang' aufweist.
Viele Jahre versuchte man, diese und andere Hypothesen zu verifizieren, scheiterte aber weitgehend,

[21] Die Kantendichte bezeichnet das Verhältnis von Kanten zu Knoten oder allgemeiner von Bedingungen zu Variablen.

und konnte zumeist weder die Existenz des Phasenübergangs beweisen, noch seine genaue Lage bestimmen. Seit Anfang der 2000er jedoch ermöglichten von Physikern aus der statistischen Mechanik entwickelte Methoden, insbesondere die 'cavity method' [KMRTSZ07], die kombinatorische Struktur besagter Probleme besser zu verstehen. Mit Hilfe dieser Methoden ließen sich Vorhersagen zum Auftreten des Erfüllbarkeits-Phasenübergangs machen und zusätzlich entfaltete sich ein differenziertes Bild über die Entwicklung der Struktur des Raums der Lösungen. Zu diesem Bild gehört unter anderem das Auftreten eines weiteren Phasenübergangs, *Kondensations-Phasenübergang* genannt, an dem sich die Geometrie des Raums der Lösungen grundlegend ändert und den man verantwortlich macht für die Schwierigkeiten, die sich bei der Untersuchung der Probleme ergeben hatten. Da die physikalischen Methoden allerdings mathematisch nicht rigoros sind, öffnete sich für Mathematiker ein neues Betätigungsfeld. Die Resultate der vorliegenden Arbeit tragen dazu bei, mathematisch exakte Grundlagen für diese Methoden zu entwickeln.

## Verwendete Modelle

Als zufällige Graphenmodelle betrachten wir die Erdős-Rényi Graphen $G(n,p)$ und $G(n,m)$ mit Knotenmenge $[n]$ und Kantenmenge $E$. Eine *$k$-Färbung* dieser Graphen ist eine Abbildung $\sigma : [n] \to [k]$ mit $\sigma(i) \neq \sigma(j)$ für alle $\{i,j\} \in E$. Die *Kantendichte* ist definiert als $d = pn$ bzw. $d = 2m/n$ und bestimmt die Schwierigkeit des Problems.
Analog dazu untersuchen wir die $k$-uniformen Hypergraphen $H_k(n,p)$ und $H_k(n,m)$ mit Knotenmenge $[n]$ und Kantenmenge $E$ und die 2-Färbungen $\sigma : [n] \to \{\pm 1\}$ mit $|\sigma(e)| = 2$ für alle $e \in E$ (d.h. Färbungen der Knoten, so dass keine monochromatischen Kanten entstehen). Hier ist die Kantendichte definiert als $d = p\binom{n-1}{k-1}$ bzw. $d = km/n$.
Zumeist sind wir an asymptotischen Resultaten in $n$ interessiert, setzen also stillschweigend voraus, dass $n$ beliebig groß wird. Wir betrachten dünn besetzte Graphen und Hypergraphen, also solche, bei denen die Kantendichte beschränkt bleibt, während $n$ ins Unendliche wächst. In allen vorgestellten Problemen bezeichnen wir mit $Z$ die Anzahl der Färbungen.

## Ergebnisse

Das erste Ergebnis der Doktorarbeit (aus *The condensation phase transition in random graph coloring* [BCOHRV16] zusammen mit Bapst, Coja-Oghlan, Hetterich und Vilenchik, veröffentlicht in *Communications in Mathematical Physics 341 (2016)*) beinhaltet den Beweis der Existenz sowie eine exakte, mathematisch rigorose Lokalisierung des Kondensations-Phasenübergangs im Graph-$k$-Färbbarkeitsproblem für große $k$. Der Phasenübergang wird nicht explizit, sondern als Lösung eines Fixpunktproblems angegeben, was der sehr komplizierten kombinatorischen Struktur des Problems geschuldet ist. Es ist das erste Resultat dieser Art für eine breite Klasse von Problemen und es stimmt mit der Vorhersage der 'cavity method' überein.

Zunächst ein paar grundlegende Definitionen: Wir betrachten die $n$-te Wurzel der Anzahl der Lösungen des $k$-Färbbarkeitsproblems im Grenzwert für großes $n$:

$$\Phi_k(d) = \lim_{n\to\infty} \mathbb{E}\left[Z^{1/n}\right]$$

Die Skalierung ist sinnvoll, da $Z$ üblicherweise von exponentieller Größenordnung ist. Im Allgemeinen ist nicht bewiesen, dass der Grenzwert $\Phi_k(d)$ für alle Werte von $d$ und $k$ existiert. Für gegebenes $k$ nennen wir $d_0 \in (0, \infty)$ daher *glatt*, falls es ein $\varepsilon > 0$ gibt, so dass

- für jedes $d \in (d_0 - \varepsilon, d_0 + \varepsilon)$ der Grenzwert $\Phi_k(d)$ existiert und
- die Abbildung $d \in (d_0 - \varepsilon, d_0 + \varepsilon) \mapsto \Phi_k(d)$ eine Entwicklung als absolut konvergente Potenzreihe um $d_0$ hat.

Falls $d_0$ nicht glatt ist, sagen wir, dass ein *Phasenübergang* bei $d_0$ auftritt.

Im folgenden Theorem bezeichnet die Funktion $\mathcal{F}_{d,k} : \mathcal{P} \to \mathcal{P}$ die Verteilungsversion eines Operators, der in der Physik als 'Belief Propagation'-Operator bekannt ist und vom Raum $\mathcal{P}$ aller Verteilungen auf einem $k$-Simplex in sich selbst abbildet. Im Allgemeinen hat diese Abbildung mehrere Fixpunkte, also Punkte $\pi^* \in \mathcal{P}$, so dass $\mathcal{F}_{d,k}[\pi^*] = \pi^*$. Wir nennen einen solchen Fixpunkt *gefroren*, falls die Masse auf den $k$ Ecken des Simplex zusammen mindestens $2/3$ beträgt.
Das Funktional $\phi_{d,k}$ ist eine Darstellung einer generischen Formel, der sogenannten 'Bethe free entropy'. Die 'Bethe free entropy' liefert eine gute Approximation der freien Entropie des Systems, falls wir als Argumente Verteilungen verwenden, deren Marginale 'nah' an den Marginalen der korrekten Verteilung über die Färbungen der Knoten liegen.
All diese Konzepte wurden systematisch mit Hilfe der 'cavity method' hergeleitet [MM09]. Sie werden in Abschnitt 4.1 ausführlich dargestellt.

**Theorem.** *Es existiert eine Konstante $k_0 \geq 3$, so dass für jedes $k \geq k_0$ folgendes gilt: Falls $d \geq (2k-1)\ln k - 2$, so hat $\mathcal{F}_{d,k}$ genau einen gefrorenen Fixpunkt $\pi^*_{d,k}$. Weiterhin hat die Funktion*

$$\Sigma_k : d \mapsto \ln k + \frac{d}{2}\ln(1 - 1/k) - \phi_{d,k}(\pi^*_{d,k})$$

*eine eindeutige Nullstelle $d_{\text{cond}}$ im Intervall $[(2k-1)\ln k - 2, (2k-1)\ln k - 1]$. Für diese Zahl $d_{\text{cond}}$ gelten die folgenden drei Aussagen:*

*(i) Jedes $0 < d < d_{\text{cond}}$ ist glatt und $\Phi_k(d) = k(1 - 1/k)^{d/2}$.*
*(ii) Es gibt einen Phasenübergang bei $d_{\text{cond}}$.*
*(iii) Falls $d > d_{\text{cond}}$, so gilt*

$$\limsup_{n\to\infty} \mathbb{E}[Z_k(G(n, d/n))^{1/n}] < k(1 - 1/k)^{d/2}.$$

*Das bedeutet, falls $d$ glatt ist, gilt* $\Phi_k(d) < k(1-1/k)^{d/2}$.

Im zweiten Resultat (aus *A positive temperature phase transition in random hypergraph 2-coloring* [BCOR16], zusammen mit Bapst und Coja-Oghlan, veröffentlicht in *Annals of Applied Probability 26 (2016)*) wird für große $k$ die Existenz des Kondensations-Phasenübergangs im 2-Färbbarkeitsproblem für $k$-uniforme Hypergraphen mit endlicher Temperatur bewiesen und die Lage des Phasenübergangs wird asymptotisch exakt in $k$ bestimmt.
Die Erweiterung des klassischen Modells auf endliche Temperatur wird in der physikalischen Literatur oft betrachtet und bedeutet im Wesentlichen, dass man sich nicht nur für gültige Färbungen des Problems interessiert, sondern alle möglichen Zuweisungen von Farben zu Knoten betrachtet und diese proportional zur Anzahl der erzeugten monochromatischen Kanten gewichtet. Man definiert die sogenannte Boltzmann-Verteilung für einen Hypergraphen $H$ und Parameter $\beta$ als

$$\pi_{H,\beta}[\sigma] = \frac{\exp\left[-\beta E_H(\sigma)\right]}{Z_\beta(H)}, \qquad \text{mit } Z_\beta(H) = \sum_{\tau:[n]\to\{\pm 1\}} \exp\left[-\beta E_H(\tau)\right],$$

wobei $E_H(\sigma)$ die Anzahl der monochromatischen Kanten in $H$ unter der Farbzuweisung $\sigma$ bezeichnet.

Wir definieren dann

$$\Phi_{d,k}(\beta) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\ln Z_\beta(H)\right].$$

Die formale Definition eines Phasenübergangs in diesem Szenario ist wie folgt: Wir nennen $\beta_0 > 0$ *glatt*, falls es ein $\varepsilon > 0$ gibt, so dass die Funktion $\beta \in (\beta_0 - \varepsilon, \beta_0 + \varepsilon) \mapsto \Phi_{d,k}(\beta)$ eine Entwicklung als absolut konvergente Potenzreihe um $\beta_0$ hat. Ansonsten sagen wir, dass ein *Phasenübergang* bei $\beta_0$ eintritt.

**Theorem.** *Für jedes feste $C > 0$ existiert eine Folge $\varepsilon_k > 0$ mit $\lim_{k\to\infty} \varepsilon_k = 0$, so dass folgendes gilt: Sei*

$$\Sigma_{k,d}(\beta) = (\beta+1)\exp\left[-\beta + k\ln 2\right]\ln 2 - 2\left(\frac{d}{k} - 2^{k-1}\ln 2 + \ln 2\right).$$

1. *Falls $d/k < 2^{k-1}\ln 2 - \ln 2 - \varepsilon_k$, ist jedes $\beta > 0$ glatt und*

$$\Phi_{d,k}(\beta) = \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right). \tag{B.3.6}$$

2. *Falls $2^{k-1}\ln 2 - \ln 2 + \varepsilon_k < d/k < 2^{k-1}\ln 2 + C$, hat $\Sigma_{k,d}(\beta)$ eine eindeutige Nullstelle $\beta_{\text{cond}}(d,k) \geq k\ln 2$ und*
   - *jedes $\beta \in (0, \beta_{\text{cond}}(d,k) + \varepsilon_k)$ ist glatt und $\Phi_{d,k}(\beta)$ ist gegeben durch* (B.3.6),
   - *es gibt einen Phasenübergang bei $\beta_{\text{cond}}(d,k) + \varepsilon_k$*
   - *für $\beta > \beta_{\text{cond}}(d,k) + \varepsilon_k$ gilt*

$$\Phi_{d,k}(\beta) < \ln 2 + \frac{d}{k}\ln\left(1 - 2^{1-k}\left(1 - \exp\left[-\beta\right]\right)\right).$$

Dieses Resultat ist das erste, das den Kondensations-Phasenübergang in Bezug auf $\beta$ mit solcher Genauigkeit bestimmt. Bis auf den Fehler $\varepsilon_k$ bestätigt es die Vorhersage der nicht-rigorosen 'cavity method'.

Die letzten beiden Hauptresultate beschäftigen sich mit der asymptotischen Verteilung der Anzahl der Lösungen in zwei verschiedenen Färbbarkeitsproblemen. In der Doktorarbeit werden die folgenden Resultate aus *On the number of solutions in random hypergraph 2-colouring* [Ras16a+], eingereicht bei *The Electronic Journal of Combinatorics* und *On the number of solutions in random graph $k$-colouring* [Ras16b+], eingereicht bei *Combinatorics, Probability and Computing* präsentiert: Ist $Z$ die Anzahl der Lösungen im zufälligen Hypergraph-2-Färbbarkeitsproblem oder im zufälligen Graph-$k$-Färbbarkeitsproblem, so zeigen wir, dass $\ln Z - \ln \mathbb{E}[Z]$ in Verteilung gegen eine Zufallsvariable konvergiert, die wir explizit angeben können.
Für das 2-Färbbarkeitsproblem von $k$-uniformen Hypergraphen lautet das Resultat:

**Theorem.** *Sei $k \geq 3$ und $d'$ eine feste Zahl, so dass $m = \lceil d'n/k \rceil$ und $d'/k \leq 2^{k-1} \ln 2 - 2$ sowie*

$$\lambda_l = \frac{[d(k-1)]^l}{2l} \quad \textit{und} \quad \delta_l = \frac{(-1)^l}{(2^{k-1}-1)^l}.$$

*Ist dann $(X_l)_l$ eine Familie von unabhängigen Zufallsvariablen mit $\mathbb{E}[X_l] = \lambda_l$, alle auf dem gleichen Wahrscheinlichkeitsraum definiert, so gilt für die Zufallsvariable*

$$W = \sum_l X_l \ln(1+\delta_l) - \lambda_l \delta_l,$$

*dass $\mathbb{E}|W| < \infty$ und $\ln Z - \ln \mathbb{E}[Z]$ in Verteilung gegen $W$ konvergiert.*

Aus diesem Resultat folgt, dass die Fluktuation des Logarithmus der Anzahl der Lösungen in $n$ divergiert, allerdings beliebig langsam. Zusätzlich zeigen wir eine Aussage über das qualitative Verhalten des 'planted model', einer Wahrscheinlichkeitsverteilung über Paare von Graph und Färbung, die oft alternativ zur 'natürlich auftretenden' Verteilung untersucht wird, da sie leichter zu handhaben ist.

Für das Graph-$k$-Färbbarkeitsproblem lautet das Resultat:

**Theorem.** *Es gibt eine Konstante $k_0 > 3$, so dass folgendes gilt: Sei $d'$ eine feste Zahl, so dass $m = \lceil d'n/2 \rceil$ und sei entweder $k \geq 3$ sowie $d' \leq 2(k-1)\ln(k-1)$ oder $k \geq k_0$ sowie $d' < d_{\mathrm{cond}}$. Sei weiterhin*

$$\lambda_l = \frac{d^l}{2l} \quad \textit{und} \quad \delta_l = \frac{(-1)^l}{(k-1)^{l-1}}.$$

*Ist dann $(X_l)_l$ eine Familie von unabhängigen Zufallsvariablen mit $\mathbb{E}[X_l] = \lambda_l$, alle auf dem gleichen*

*Wahrscheinlichkeitsraum definiert, so gilt für die Zufallsvariable*

$$W = \sum_{l \geq 3} X_l \ln(1 + \delta_l) - \lambda_l \delta_l - d^2/(4(k-1)),$$

*dass* $\mathbb{E}|W| < \infty$ *und* $\ln Z - \ln \mathbb{E}[Z]$ *in Verteilung gegen* $W$ *konvergiert.*

## Methoden

Übliche Werkzeuge bei der Untersuchung von Phasenübergängen in zufälligen Bedingungserfüllungsproblemen sind die erste und zweite Moment-Methode, die obere und untere Schranken an den Erfüllbarkeits-Phasenübergang liefern und die auch bei allen hier präsentierten Resultaten verwendet werden. Insbesondere die zweite Moment-Methode kann entweder in ihrer klassischen Form Anwendung finden, oder in einer von den physikalischen Methoden inspirierten erweiterten Form (wie in [COP16]).

Des Weiteren benutzen wir das 'planted model', das uns erlaubt, die Struktur des Lösungsraums von zufälligen Bedingungserfüllungsproblemen unterhalb des Kondensations-Phasenübergangs zu untersuchen. Wir verwenden Konzentrationsargumente sowie Aussagen über die Eigenschaften des 'core', einer dicht-verlinkten Menge von 'gefrorenen' Knoten, die aufgrund der geometrischen Strukturen im Wesentlichen auf eine Farbe fixiert sind.

Für den Beweis des ersten Resultats finden wir eine explizite Verbindung zwischen der kombinatorischen Struktur des Graph-Färbbarkeitsproblems und dem Verteilungs-Fixpunktproblem aus [ZK07]. Wir benutzen dazu den in der physikalischen Literaur eingeführten 'message-passing'-Prozess *Warning Propagation* (vgl. [MM09]) und zeigen, dass wir mit seiner Hilfe einen detaillierten Einblick in die Geometrie der Komponenten des Lösungsraums erhalten.

Im Falle endlicher Temperatur entwickeln wir eine rigorose Version der vorhergesagten Zerlegung des Raums der Lösungen in sogenannte 'cluster', wie es sie für das klassische Modell (ohne Temperatur-Parameter) schon gab [ACO08, COZ12] und beschäftigen uns mit der Bestimmung der Größe dieser 'cluster'.

Die Resultate zur asymptotischen Verteilung der Anzahl der Lösungen erhalten wir mit einer Variante der Methode 'small subgraph conditioning', die von Robinson und Wormald [RW94] eingeführt wurde und von Janson [Jan95] weiterentwickelt. Der Beweis beruht auf der Beobachtung, dass die Fluktuationen in der Anzahl der Lösungen zurückgeführt werden können auf die Fluktuationen in der Anzahl der kurzen Kreise in den zugrundeliegenden Graph- und Hypergraphstrukturen. Die Verwendung dieser Methode erfordert eine sehr exakte Berechnung des zweiten Moments und eine Analyse der Varianz nach dem Vorbild von [RW94].

# Ausblick

Die in dieser Arbeit entwickelten und verwendeten Methoden lassen sich vermutlich auf eine ganze Reihe weiterer Probleme und Fragestellungen anwenden.
Da sich die Vorhersagen der Physiker in vielen Bedingungserfüllungsproblemen ähneln, ist es plausibel, dass sich z.B. die Beweistechniken zur Lokalisierung des Kondensations-Phasenübergangs im $k$-Färbbarkeitsproblem auf andere Probleme übertragen lassen. Ebenso ist zu erwarten, dass auch Modelle mit endlicher Temperatur analog zur Hypergraph 2-Färbbarkeit untersucht werden können. Selbstverständlich ist neben der Frage zum Kondensations-Phasenübergang auch die Frage nach der Existenz und Lage des Erfüllbarkeits-Phasenübergangs (zumindest in Problemen ohne den Temperaturparameter) wegweisend. Bis jetzt existieren nur wenige rigorose Ergebnisse zur Bestimmung dieses Übergangs [DSS15, DSS16, DSS16+, COP16]. Tatsächlich ist in vielen Problemen nicht einmal die Vorhersage der 'cavity method' bewiesen, dass sich der Kondensations-Phasenübergang vom Erfüllbarkeits-Phasenübergang unterscheidet. In jedem Fall wird die vollständige mathematische Präzisierung der 'cavity method' auf absehbare Zeit eine spannende Herausforderung bleiben.
Was die Verteilung der Anzahl der Lösungen betrifft, so liegt die Annahme nahe, dass eine Kombination der zweiten Moment-Methode und 'small subgraph conditioning' in vielen anderen Problemen zur Bestimmung der asymptotischen Verteilung der Anzahl der Lösungen genutzt werden kann. Probleme, für die das vorstellbar ist, sind z.B. zufälliges NAE-$k$-SAT, zufälliges $k$-XORSAT, zufällige Hypergraph $k$-Färbbarkeit oder Probleme auf zufälligen regulären Strukturen. Für asymmetrische Probleme wie das bekannte zufällige $k$-SAT erwarten wir jedoch, dass die Anzahl der Lösungen stärker fluktuiert und bezweifeln daher, dass ein ähnliches Resultat erzielt werden kann.
Tatsächlich wäre es sehr interessant, eine komplette Klassifizierung aller Probleme zu erstellen, für die eine solche Grenzverteilung gefunden werden kann. Es ist denkbar, dass dies alle Modelle betrifft, bei denen die Verteilungsfunktion auf einem Baum mit $n$ Knoten konstant ist. In diesem Fall wäre eine Verallgemeinerung der Beweistechnik lohnenswert, sodass alle betrachteten Modelle abgedeckt werden.

Zudem ist auch die Frage nach dem effizienten Auffinden von Lösungen aus algorithmischer Sicht noch weitgehend unbeantwortet. Insbesondere die präzise Analyse von 'message-passing' Algorithmen ist ein aktives Forschungsgebiet. Obwohl es einige experimentelle Ergebnisse gibt, steckt die mathematisch rigorose Analyse noch in den Kinderschuhen.

Zusammenfassend und abschließend lässt sich sagen, dass in den letzten Jahrzehnten ein großer Schritt getan wurde, die Eigenschaften und Besonderheiten von Bedingungserfüllungsproblemen zu verstehen. Es gibt auf diesem Gebiet jedoch noch sehr viel zu erforschen und man ist noch weit davon entfernt, alle auftretenden Phänomene grundlegend verstanden zu haben.

# Felicia Raßmann

*Curriculum Vitae*


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*Nationality: German*
*Date of birth: 26/01/1988*
*Place of birth: Frankfurt (Main)*


## Education

| | |
|---|---|
| 10/2012 – today | **PhD in Mathematics**, Goethe University, Frankfurt (Main) |
| Advisor | Prof. Dr. Amin Coja-Oghlan, Institute for Mathematics: Discrete Mathematics |
| 10/2010 – 09/2012 | **Master of Science in Mathematics**, Goethe University, Frankfurt (Main)<br>*Minor subject:* Informatics<br>*Final Grade:* "with distinction" (1.1) |
| Master Thesis | *Concentration and Negative Association in Urn Models* |
| Advisor | Prof. Dr. Ralph Neininger, Institute for Mathematics: Stochastics |
| 10/2007 – 06/2010 | **Bachelor of Science in Mathematics**, Goethe University, Frankfurt (Main)<br>*Minor subject:* Informatics<br>*Final Grade:* "with distinction" (1.2) |
| Bachelor Thesis | *Quadratic Reciprocity in Finite Groups* |
| Advisor | Dr. Amir Džambić, Institute for Mathematics: Algebra and Number Theory |
| 09/2005 – 07/2007 | Woehler Gymnasium, Frankfurt (Main)<br>*Degree*: Abitur (secondary school leaving examination)<br>*Final Grade:* 1.0 |
| 08/1998 – 06/2005 | Ziehen Gymnasium, Frankfurt (Main)<br>Bilingual education with French as first foreign language |

## Scholarships

| | |
|---|---|
| 10/2007 – 09/2012 | Scholarship of the "Studienstiftung des deutschen Volkes" (German National Academic Foundation) |

## Work Experience and Teaching

| | |
|---|---|
| 10/2012 – today | **Goethe University**, Frankfurt (Main)<br>Scientific research assistant for Prof. Dr. Amin Coja-Oghlan |
| 10/2011 – 09/2012 | **Goethe University**, Frankfurt (Main)<br>Teaching assistant at the learning center of the Institute for Mathematics |
| 04/2010 – 09/2011 | **Goethe University**, Frankfurt (Main)<br>Teaching assistant for the lectures "Analysis 2", "Geometry and Topology" and "Linear Algebra" |

## Conferences and workshops

21/08 – 26/08/2016 **Workshop: Meeting of scientific research groups**, Kleinwalsertal (Austria)

25/07 – 29/07/2016 **Workshop: Phase Transitions in Discrete Structures**, *Member of the organizing team*, Goethe University Frankfurt (Germany)

02/05 – 06/05/2016 **Workshop: Random Instances and Phase Transitions**, Simons Institute, Berkeley (USA)

05/08 – 07/08/2015 **11. Doktorandentreffen Stochastik**, Berlin (Germany)

16/03 – 20/03/2015 **IMA Workshop: The Power of Randomness in Computation**, Georgia Institute of Technology, Atlanta (USA)

27/06 – 31/06/2015 **Random Structures and Algorithms** - 17th International Conference, Pittsburgh (USA)

26/08 – 04/09/2014 **Spin glasses: An old tool for new problems**, Institut d'Etudes Scientifiques de Cargese (France)

05/05 – 09/05/2014 **EPSRC Symposium on Statistical Mechanics** - Phase transitions in discrete structures and computational problems, Warwick (UK)

11/11 – 13/11/2013 **Workshop on Statistical Issues in Compressive Sensing**, Göttingen (Germany)

30/09 – 11/10/2013 **Statistical physics, Optimization, Inference and Message-Passing algorithms**, Autumn School, Ecole de physique des Houches (France)

19/08 – 30/08/2013 **School on Mathematical Statistical Physics** - Organized by the Center for Theoretical Study of the Charles University of Prague, Prague (Czech Republic)

05/08 – 09/08/2013 **Random Structures and Algorithms** - 16th International Conference, Poznan (Poland)

24/02 – 11/03/2013 **Georgia Institute of Technology** - two-week research period, Atlanta (USA)

## Talks

23/08/2016 *On the number of solutions in random graph $k$-colouring* (Workshop: Meeting of scientific research groups, Kleinwalsertal (Austria))

27/07/2016 *On the number of solutions in random hypergraph 2-colouring* (Workshop Phase Transitions in Discrete Structures, Frankfurt (Germany))

05/08/2015 *Chasing phase tranistions in random graphs* (11. Doktorandentreffen Stochastik, Berlin (Germany))

28/07/2015 *A positive temperature phase transition in random hypergraph 2-coloring* (Random Structures and Algorithms, Pittsburgh (USA))

04/09/2014 *The condensation phase transition in random graph coloring* (Institut d'Etudes Scientifiques de Cargese (France))

## Publications

**A positive temperature phase transition in random hypergraph 2-coloring**
joint with Victor Bapst and Amin Coja-Oghlan
Annals of Applied Probability **26(3)** (2016), pp. 1362–1406.

**The condensation phase transition in random graph coloring**
joint with Victor Bapst, Amin Coja-Oghlan, Samuel Hetterich and Dan Vilenchik
Communications in Mathematical Physics **341** (2016), pp. 543–606.

## Preprints

**On the number of solutions in random graph $k$-colouring**
http://arxiv.org/abs/1609.04191.

**On the number of solutions in random hypergraph 2-colouring**
http://arxiv.org/abs/1603.07523.