ON PHRAGMÉN–LINDELÖF PRINCIPLES OF TYPE L^p

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ABSTRACT. Integrability conditions of type L^p along straight lines within a strip in $\mathbb{C}$ are derived for holomorphic functions which are integrable over the area of the strip.

The Phragmén–Lindelöf principle is a substitute for the maximum-modulus principle when the domain under consideration is unbounded. One of its innumerable applications is Hadamard's three-lines theorem [1, Section 5.65], which roughly states that the growth order of a holomorphic function in a strip in $\mathbb{C}$ is determined by the growth order on the boundary of the strip, and that the suprema over straight lines within the strip are logarithmically convex. This convexity result was generalized by Littlewood [2, 3] from suprema to L^p-norms, which in turn has numerous applications, most notably in harmonic analysis and to Hardy spaces [4].

Not much is known, however, about sufficient conditions for the L^p-integrability of a holomorphic function along straight lines in a strip. Here we remark on some small steps in that direction. For our discussion, fix the strip

$$Y \stackrel{\text{def}}{=} \mathbb{R} + \mathrm{i}[y_1, y_2], \quad \text{with } -\infty < y_1 < y_2 < \infty.$$

Denote by $\mathcal{O}^c(Y)$ the set of functions which are holomorphic in Y° and continuous on the boundary ∂Y. Let $\mathrm{d}(\lambda)$ be the Lebesgue measure on $\mathbb{C}$. The **L^p-means** of a function $f \in \mathcal{O}^c(Y)$ are defined as

$$M_p(y, f) \stackrel{\text{def}}{=} \Bigg(\int\limits_{\operatorname{Im} z = y} |f(z)|^p |\mathrm{d}z| \Bigg)^{1/p}, \quad \text{for } y_1 \le y \le y_2,$$

for $0 < p < \infty$, whenever they exist.

One basic convexity result concerning L^p-means goes back to Hardy, Ingham and Pólya. We cite it following Narasimhan.

Proposition 1 ([5, Proposition 4, p. 244]). *Let $f \in \mathcal{O}^c(Y)$ be such that $M_p(y, f)$ is locally uniformly bounded for $y \in (y_1, y_2)$. Then $\log M_p(y, f)$ is convex in y.*

Our basic result gives a sufficient condition for the assumption of local uniform boundedness in the above theorem in a specal case.

Proposition 2. *Let $f \in \mathcal{O}^c(Y) \cap L^1(Y, \mathrm{d}(\lambda))$. Then $M_1(y, f)$ is finite and locally uniformly bounded in $y \in (y_1, y_2)$, and $\log M_1(y, f)$ is convex there.*

Proof. Let I be a compact subset of (y_1, y_2) and $\Gamma(y) = \{z \mid \operatorname{Im} z = y\}$, $y \in I$. Divide $\Gamma(y)$ in unit intervals

$$L_j = \big\{x + iy \in \Gamma(y) \bigm| x \in [j, j+1]\big\}, \quad \text{for } j \in \mathbb{Z}.$$

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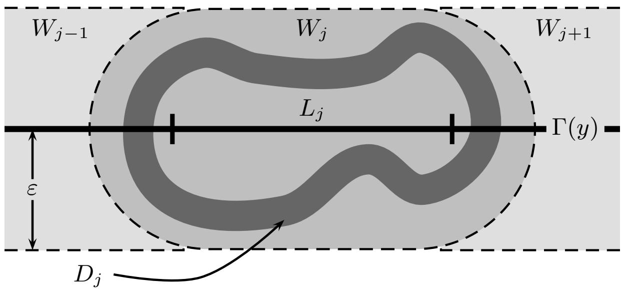
Choose neighbourhoods

$$W_j = \{ z \in \mathbb{C} \mid \operatorname{dist}(z, L_j) < \varepsilon \}$$

whose boundary has distance ε from L_j, where $0 < \varepsilon < 1/2$ is chosen such that $W_j \subset \mathbb{R} + \mathrm{i}I$. Choose a smooth function $\phi_0 \in C_c^\infty(W_0)$ such that $\phi_0 \equiv 1$ in a neighbourhood of L_0. Let $\phi_j(z) = \phi_0(z - j) \in C_c^\infty(W_j)$ be the translates of ϕ_0 for $j \in \mathbb{Z}$, and set

$$D_j = \{ z \in \mathbb{C} \mid 0 < \phi_j(z) < 1 \}, \quad \widetilde{D}_j = \{ z \in \mathbb{C} \mid 0 < \phi_j(z) \}.$$

Then, $D_j = D_0 + j$ is relatively compact in $W_j \setminus L_j$, and $\delta = \operatorname{dist}(D_j, L_j) > 0$ is independent of j. This situation is sketched below.



The integral formula of Cauchy–Stokes [6, Theorem 1.2.1] yields

$$f(z) = (\phi_j f)(z) = \frac{1}{2\pi\mathrm{i}} \left(\int_{\partial \widetilde{D}_j} \frac{(\phi_j f)(w)}{w - z} \mathrm{d}w + \int_{\widetilde{D}_j} \frac{\partial_{\overline{w}}(\phi_j f)(w)}{w - z} \mathrm{d}w \wedge \mathrm{d}\overline{w} \right),$$

for all $z \in L_j$. Now $\phi_j = 0$ on $\partial \widetilde{D}_j$, and thus the first term vanishes. For the second term, we use the holomporphy of f in $\widetilde{D}_j$ to conclude $\partial_{\overline{w}}(\phi_j f) = \partial_{\overline{w}} \phi_j \cdot f$, and since $\partial_{\overline{w}} \phi_j = 0$ on $\widetilde{D}_j \setminus D_j$ we can restrict the integral to D_j. The result is

$$f(z) = \frac{-1}{\pi} \int_{D_j} \frac{\partial \phi_j}{\partial \overline{w}} \frac{f(w)}{w - z} \mathrm{d}\lambda(w), \quad \text{for } z \in L_j,$$

taking $-2\mathrm{i}\mathrm{d}\lambda(w) = \mathrm{d}w \wedge \mathrm{d}\overline{w}$ into account. Therefore we can estimate

$$\sup_{z \in L_j} |f(z)| \le K \int_{D_j} |f(w)| \mathrm{d}\lambda(w), \quad \text{with } K = \frac{1}{\pi\delta} \cdot \sup_{w \in D_j} \left| \frac{\mathrm{d}\phi_j}{\mathrm{d}\overline{w}} \right|.$$

The constant K is independent of j, allowing us to estimate

$$\begin{aligned} M_1(y, f) &= \int_{\Gamma(y)} |f(z)| |\mathrm{d}z| \\ &\le \sum_{j=-\infty}^{+\infty} \sup_{z \in L_j} |f(z)| \\ &\le K \sum_{j=-\infty}^{+\infty} \int_{D_j} |f(w)| \mathrm{d}\lambda(w) \\ &\le 2K \|f\|_{L^1(I)} \\ &\le 2K \|f\|_{L^1(Y)} < \infty. \end{aligned}$$

Here, a factor 2 appears since we have chosen ε such that at most two neighbouring W_j have nonempty intersection. Since $I \Subset (y_1, y_2)$ was arbitrary, $M_1(y, f)$ exists in (y_1, y_2) and is locally uniformly bounded there, with local bound $2K\|f\|_{L^1(I)}$ on I. Proposition 1 then shows the claimed convexity. $\square$

Our first generalization is that to L^p, $0 < p < \infty$. Here we use subharmonic functions, which we avoided in the elementary proof of the L^1-case above.

Proposition 3. *Let $f \in \mathcal{O}^c(Y) \cap L^p(Y, \mathrm{d}(\lambda))$, for $0 < p < \infty$. Then $M_p(y, f)$ is finite and locally uniformly bounded in $y \in (y_1, y_2)$, and $\log M_p(y, f)$ is convex.*

Proof. We use the notation of the proof of Proposition 2. Let $D_\varepsilon(z)$ be the open disc of radius ε around $z \in L_j$. Now, the function $g(z) = |f(z)|^p$ is subharmonic on Y°, and it has the sub-mean-value property [4, Theorem 1.4], which can be written in the form

$$rg(z) \leq \frac{1}{2\pi} \int_0^{2\pi} g(z + r\mathrm{e}^{\mathrm{i}\theta}) r \mathrm{d}\theta,$$

for $z \in L_j$, and all $0 < r < \varepsilon$, using that the right hand side is nondecreasing in r [4, Theorem 1.6]. This inequality can be integrated with respect to r over the range $(0, \varepsilon)$ to yield

$$g(z) \leq \frac{1}{\pi\varepsilon^2} \int_{D_\varepsilon(z)} g(w) \mathrm{d}\lambda(w).$$

Letting z vary in L_j, the discs $D_\varepsilon(z)$ sweep out W_j, and thus

$$\sup_{z \in L_j} g(z) \leq \int_{W_j} g(z) \mathrm{d}\lambda(z).$$

The same summation argument as in the proof of Proposition 2 can be applied to conclude the proof, where the local constant of uniform boundedness K has to be replaced by $K' = 1/(\pi\varepsilon^2)$. □

Both proofs above require $\Gamma(y)$ to have an arbitrarily small, but finite, distance from ∂Y. To obtain a condition for the existence of M_p in the strip that depends only on the boundary as in the original Phragmén-Lindelöf theorem, we have to pose additional conditions. One problem is here, that even if f is L^p on the boundary, it still can behave rather badly there (it need not even vanish at infinity). We give a sufficient condition for f to behave good enough on ∂Y, valid if f is exact for the Cauchy–Riemann operator $\overline{\partial}$.

Corollary 4. *Assume there exists $F \in C^1(Y^\circ)$ such that $|f|^p = \overline{\partial} F$ on Y°, and*

$$\left| \int_{\partial Y} F(z) \mathrm{d}z \right| < \infty, \quad \text{and } I = \sup_{x \in \mathbb{R}} \left| \int_{y_1}^{y_2} F(x + \mathrm{i}y) \mathrm{d}y \right| < \infty.$$

Then $M_p(y, f)$ is finite and locally uniformly bounded in $y \in (y_1, y_2)$, and $\log M_p(y, f)$ is convex.

Proof. Set $R_j = [-j, j] + \mathrm{i}[y_1, y_2]$. Then Stokes' theorem implies

$$\int_{R_j} |f|^p \mathrm{d}\lambda(z) = \int_{R_j} \overline{\partial} F(z) \mathrm{d}\lambda(z) = \frac{1}{2\mathrm{i}} \int_{\partial R_j} F(z) \mathrm{d}z,$$

where we chose ∂R_j to be positively oriented. Thus

$$\begin{aligned} \int_Y |f|^p \mathrm{d}\lambda(z) &= \lim_{j \to \infty} \frac{1}{2\mathrm{i}} \int_{\partial R_j} F(z) \mathrm{d}z \\ &\leq \frac{1}{2} \left| \int_{\partial Y} F(z) \mathrm{d}z \right| + I < \infty \end{aligned}$$

by assumption. The claim now follows from Proposition 3. □

One natural generalization of Proposition 3 concerns integrability with respect to a weight function on the strip. We consider the case of polynomial weights to exhibit the method. This has been used in [7] to construct the duality theory of hyperfunctions with polynomial growth order in one dimension.

Corollary 5. *Let* $\gamma \in \mathbb{R}$ *and* $f \in \mathcal{O}^c(Y) \cap L^p\big(Y, (1+|\operatorname{Re} z|)^\gamma \mathrm{d}\lambda(z)\big)$. *Then*

$$M_{p,\gamma}(y,f) \stackrel{\text{def}}{=} \Bigg(\int\limits_{\operatorname{Im} z = y} |f(z)|^p (1+|\operatorname{Re} z|)^\gamma |\mathrm{d}z| \Bigg)^{1/p} < \infty,$$

for all $y_1 \leq y \leq y_2$, *and* $M_{p,\gamma}(y,f)$ *is convex in* (y_1, y_2).

Proof. The function

$$j_\gamma(z) = \big((y_2 - y_1)^2 + (z - \mathrm{i}y_1)^2\big)^{(\gamma-p)/2}$$

is holomorphic in a neighbourhood of Y and does not vanish there. Furthermore $j_\gamma = O(|\operatorname{Re} z|^{\gamma - p})$, and thus the function $j_\gamma f$ satisfies the assumption of Proposition 3 if and only if f satisfies the one of this corollary. □

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