Testing a non-perturbative mechanism for elementary fermion mass generation: lattice setup

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Abstract. In this contribution we lay down a lattice setup that allows for the non-perturbative study of a field theoretical model where a SU(2) fermion doublet, subjected to non-Abelian gauge interactions, is also coupled to a complex scalar field doublet via a Yukawa and an "irrelevant" Wilson-like term. Using naive fermions in quenched approximation and based on the renormalized Ward identities induced by purely fermionic chiral transformations, lattice observables are discussed that enable: a) in the Wigner phase, the determinations of the critical Yukawa coupling value where the purely fermionic chiral transformation become a symmetry up to lattice artifacts; b) in the Nambu-Goldstone phase of the resulting critical theory, a stringent test of the actual generation of a fermion mass term of non-perturbative origin. A soft twisted fermion mass term is introduced to circumvent the problem of exceptional configurations, and observables are then calculated in the limit of vanishing twisted mass.

1 Introduction

In [1] a new non-perturbative (NP) mechanism for elementary particle mass generation was conjectured. Existence and main properties of this phenomenon can be tested in the toy model described by

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the Lagrangian

$$\mathcal{L}_{\rm toy}(\Psi, A, \Phi) = \mathcal{L}_{kin}(\Psi, A, \Phi) + \mathcal{V}(\Phi) + \mathcal{L}_{Wil}(\Psi, A, \Phi) + \mathcal{L}_{Yuk}(\Psi, \Phi)\,, \tag{1}$$

$$\mathcal{L}_{kin}(\Psi, A, \Phi) = \frac{1}{4}(F \cdot F) + \bar{\Psi}_L \not{\mathcal{D}} \Psi_L + \bar{\Psi}_R \not{\mathcal{D}} \Psi_R + \frac{1}{2}\,{\rm tr}\,[\partial_\mu \Phi^\dagger \partial_\mu \Phi] \tag{2}$$

$$\mathcal{V}(\Phi) = \frac{\mu_0^2}{2}\,{\rm tr}\,[\Phi^\dagger\Phi] + \frac{\lambda_0}{4}\big({\rm tr}\,[\Phi^\dagger\Phi]\big)^2 \tag{3}$$

$$\mathcal{L}_{Wil}(\Psi, A, \Phi) = \frac{b^2}{2}\rho\,(\bar{\Psi}_L \overleftarrow{\mathcal{D}}_\mu \Phi \mathcal{D}_\mu \Psi_R + \bar{\Psi}_R \overleftarrow{\mathcal{D}}_\mu \Phi^\dagger \mathcal{D}_\mu \Psi_L) \tag{4}$$

$$\mathcal{L}_{Yuk}(\Psi, \Phi) = \ \eta\,(\bar{\Psi}_L \Phi \Psi_R + \bar{\Psi}_R \Phi^\dagger \Psi_L)\,, \tag{5}$$

where $b^{-1} = \Lambda_{UV}$ is the UV-cutoff. The Lagrangian (1) describes a SU(2) fermion doublet subjected to non-Abelian gauge interaction and coupled to a complex scalar field via Wilson-like (eq. (4)) and Yukawa (eq. (5)) terms. For short we use a compact SU(2)-like notation where $\Psi_L = (u_L\ d_L)^T$ and $\Psi_R = (u_R\ d_R)^T$ are fermion iso-doublets and Φ is a 2×2 matrix with $\Phi = (\phi, -i\tau^2\phi^*)$ and ϕ an iso-doublet of complex scalar fields. The term $\mathcal{V}(\Phi)$ in eq. (3) is the standard quartic scalar potential where the (bare) parameters λ_0 and μ_0^2 control the self-interaction and the mass of the scalar field. In the equations above we have introduced the covariant derivatives

$$\mathcal{D}_\mu = \partial_\mu - i g_s \lambda^a A^a_\mu\,, \qquad \overleftarrow{\mathcal{D}}_\mu = \overleftarrow{\partial}_\mu + i g_s \lambda^a A^a_\mu\,, \tag{6}$$

where A^a_μ is the gluon field ($a = 1, 2, \dots, N_c^2 - 1$) with field strengt $F^a_{\mu\nu}$. . The model (1) is power-counting renormalizable (as LQCD is) with counter-terms constrained by the exact symmetries of the Lagrangian. Besides Lorentz, gauge and C, P, T, CPF_2 symmetries (see Appendix B of [1]), $\mathcal{L}_{\rm toy}$ is invariant under the following (global) transformations χ_L and χ_R

$$\bullet\,\chi_L: \quad \tilde{\chi}_L \otimes (\Phi \to \Omega_L \Phi) \qquad \bullet\ \chi_R: \quad \tilde{\chi}_R \otimes (\Phi \to \Phi\Omega_R^\dagger) \tag{7}$$

$$\tilde{\chi}_{L/R}: \begin{cases} \Psi_{L/R} \to \Omega_{L/R}\Psi_{L/R} \\ \bar{\Psi}_{L/R} \to \bar{\Psi}_{L/R}\Omega^\dagger_{L/R} \end{cases} \qquad \Omega_{L/R} \in {\rm SU}(2)_{L/R}\quad, \tag{8}$$

which forbid power divergent fermion mass terms. The $d = 4$ Yukawa term $\mathcal{L}_{Yuk}$ and the Wilson-like $d = 6$ operator $\mathcal{L}_{Wil}$, which for dimensional reasons enters in the Lagrangian multiplied by b^2, break explicitly chiral transformations $\tilde{\chi}_L$ and $\tilde{\chi}_R$. To study possible enhancement of $\tilde{\chi}_L$ symmetry (by parity the same will hold also for $\tilde{\chi}_R$) we consider the bare Schwinger Dyson Equation (SDE)

$$\partial_\mu \langle \tilde{J}^{L\,i}_\mu(x)\,\hat{O}(0)\rangle = \langle \tilde{\Delta}^i_L \hat{O}(0)\rangle\delta(x) - \eta\,\langle (\bar{\Psi}_L \frac{\tau^i}{2}\Phi\Psi_R - \bar{\Psi}_R\Phi^\dagger\frac{\tau^i}{2}\Psi_L)(x)\,\hat{O}(0)\rangle + \\ - \frac{b^2}{2}\rho\,\langle\Big(\bar{\Psi}_L \overleftarrow{\mathcal{D}}_\mu \frac{\tau^i}{2}\Phi\mathcal{D}_\mu\Psi_R - \bar{\Psi}_R\overleftarrow{\mathcal{D}}_\mu\Phi^\dagger\frac{\tau^i}{2}\mathcal{D}_\mu\Psi_L\Big)(x)\,\hat{O}(0)\rangle\,, \tag{9}$$

where $\tilde{\Delta}^i_L \hat{O}(0)$ is the variation of $\hat{O}(0)$ under $\tilde{\chi}_L$ and the associated non-conserved currents are

$$\tilde{J}^{L\,i}_\mu = \bar{\Psi}_L\gamma_\mu\frac{\tau^i}{2}\Psi_L - \frac{b^2}{2}\rho\Big(\bar{\Psi}_L\frac{\tau^i}{2}\Phi\mathcal{D}_\mu\Psi_R - \bar{\Psi}_R\overleftarrow{\mathcal{D}}_\mu\Phi^\dagger\frac{\tau^i}{2}\Psi_L\Big)\,. \tag{10}$$

Under renormalization the $d = 6$ operator $O_6^{L\,i} = \frac{1}{2}\rho\Big[\bar{\Psi}_L\overleftarrow{\mathcal{D}}_\mu\frac{\tau^i}{2}\Phi\mathcal{D}_\mu\Psi_R - \text{h.c.}\Big]$ mixes with two $d = 4$ operators, plus a set of six-dimensional ones that we globally denote by $[O_6^{L\,i}]_{sub}$ [1], viz.

$$O_6^{L\,i} = \Big[O_6^{L\,i}\Big]_{sub} + \frac{Z_{\partial\tilde{J}} - 1}{b^2}\partial_\mu\tilde{J}_\mu^{L\,i} - \frac{\bar{\eta}}{b^2}\Big[\bar{\Psi}_L\frac{\tau^i}{2}\Phi\Psi_R - \text{h.c.}\Big] + \ldots \tag{11}$$

where $Z_{\partial\tilde{J}}$ and $\bar{\eta}$ are functions of the dimensionless bare parameters entering (1) and hence depend on the subtracted scalar squared mass $\mu_{sub}^2 = \mu_0^2 - b^2\tau$ through the combination $b^2\mu_{sub}^2$ that is a negligible $O(b^2)$ quantity [1]. Thus we write $Z_{\partial\tilde{J}} = Z_{\partial\tilde{J}}(\eta; g_s^2, \rho, \lambda_0)$ and $\bar{\eta} = \bar{\eta}(\eta; g_s^2, \rho, \lambda_0)$. Ellipses in the r.h.s. of eq. (11) denote possible NP contributions to operator mixing, the possible occurrence of which is a key point that will be discussed below. Plugging (11) into (10) we get

$$\partial_\mu\langle Z_{\partial\tilde{J}}\tilde{J}_\mu^{L,i}(x)\,\hat{O}(0)\rangle = \langle\tilde{\Delta}_L^i\hat{O}(0)\rangle\delta(x) - (\eta - \overline{\eta})\,\langle O_{Yuk}^{L,i}(x)\,\hat{O}(0)\rangle + \ldots + \text{O}(b^2).$$

We define $\eta_{cr}(g_s^2, \rho, \lambda_0) - \bar{\eta}(\eta_{cr}; g_s^2, \rho, \lambda_0) = 0$. Setting $\eta = \eta_{cr}(g_s^2, \rho, \lambda_0)$ the SDE takes the form of a WTI

$$\partial_\mu\langle Z_{\partial\tilde{J}}\tilde{J}_\mu^{L,i}(x)\,\hat{O}(0)\rangle = \langle\tilde{\Delta}_L^i\hat{O}(0)\rangle\delta(x) + \ldots + \text{O}(b^2)\,, \tag{12}$$

implying restoration of the fermionic $\tilde{\chi}_L \otimes \tilde{\chi}_R$ symmetries up to $\text{O}(b^2)$ UV cutoff effects.

1.1 Mass generation mechanism in the critical model (Nambu-Goldstone phase)

The physics of the model (1) at the critical value η_{cr} crucially depends on whether the parameter μ_0^2 is such that $\mathcal{V}(\Phi)$ has a unique minimum (Wigner phase of the χ_L symmetry, $\mu_{sub}^2 > 0$) or whether $\mathcal{V}(\Phi)$ develops the typical "mexican hat" shape (Nambu-Goldstone phase $\mu_{sub}^2 < 0$). Here $\mu_{sub}^2 = \mu_0^2 - \mu_{cr}$, with μ_{cr}^2 being the phase transition point. In the Wigner phase no NP terms (i.e. ellipses) are expected to occur in the mixing pattern of eq. (11) and the transformations $\tilde{\chi}_L$ leads to eq. (12) without the ellipses [1].

In the Nambu-Goldstone phase a non-perturbative term is expected/conjectured [1] to appear in the mixing pattern of eqs. (11) leading to a WTI of the form

$$\partial_\mu\langle Z_{\partial\tilde{J}}\tilde{J}_\mu^{L,i}(x)\,\hat{O}(0)\rangle_{\eta_{cr}} = \langle\tilde{\Delta}_L^i\hat{O}(0)\rangle_{\eta_{cr}}\delta(x) + \langle C_1\Lambda_s[\overline{\Psi}_L\frac{\tau^i}{2}\mathcal{U}\Psi_R + \text{h.c.}]\hat{O}(0)\rangle + \text{O}(b^2) \tag{13}$$

where

$$\mathcal{U} = \frac{\Phi}{\sqrt{\Phi^\dagger\Phi}} = \frac{v + \sigma + i\vec{\tau}\vec{\pi}}{\sqrt{(v + \sigma)^2 + \vec{\pi}\vec{\pi}}}\,. \tag{14}$$

$\mathcal{U}$ is a dimensionless non-analytic function of Φ that has the same transformation properties as the latter under $\chi_L \times \chi_R$ and is well defined only if $\langle\Phi\rangle = v \neq 0$. Occurrence of the $c_1\Lambda_s$ term in the (13) implies the presence of $c_1\Lambda_s\bar{\Psi}\Psi$ term in Γ_{NG}^{loc}, the local effective action in the NG phase. This term describe NP breaking of $\tilde{\chi}_L\otimes\tilde{\chi}_R$ and in particular gives fermions a mass $c_1\Lambda_s$. It does not stem from the Yukawa term and, interestingly, can give a natural (in the sense of 't Hooft [2]) understanding of the fermion mass hierarchy problem (see discussion in [1]). An idea of how the mechanism works can be obtained from a perturbative expansion where Feynman diagrams are evaluated with the Lagrangian (1) augmented by few extra terms representing the expected $O(b^2)$ NP effective vertices [1], as those shown in fig. 1. These vertices can be inserted together with $O(b^2)$ vertices coming from the term (4)

[1] We do not need to resolve the mixing among the different $d = 6$ operators, as they only yield negligible $\text{O}(b^2)$ effects. To simplify the mixing pattern (11) we used $\partial_\mu J_\mu^{L,i} = 0$, where $J_\mu^{L,i}$ is the Noether current associated with the exact symmetry χ_L (7).

in diagrams like the ones depicted in fig. 2, giving rise to finite self-energy contributions.
It is worth noticing that if the mechanism we have conjectured really exists it will generate a NP mass term for the fermions even in the quenched approximation where the vertices (b) and (c) of fig. 1, and thus the two rightmost diagrams of fig. 2, are still present.

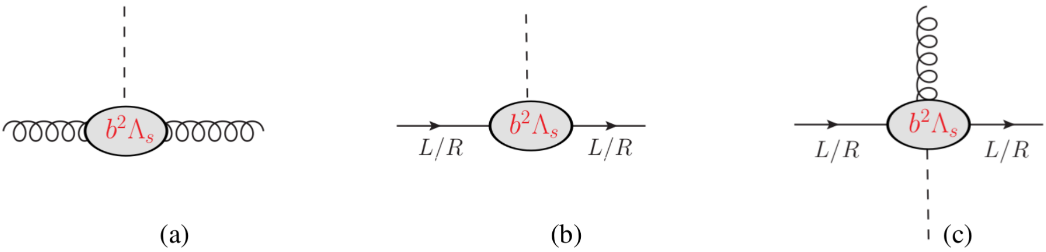

Figure 1. Some of the NP $O(b^2\Lambda_s\alpha_s^2)$ effective vertices that are conjectured to arise [1] in the Nambu-Goldstone phase of the model.

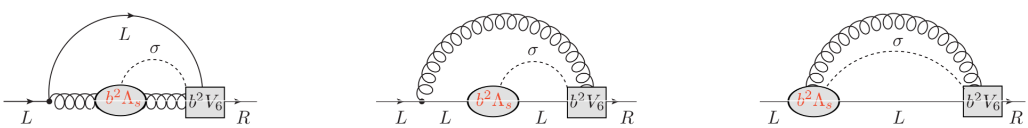

Figure 2. Typical lowest order self-energy "diagrams" giving rise to dynamically generated quark mass terms. The grey box represents the insertion of the Wilson-like vertex stemming from $\mathcal{L}_{Wil}$. The dotted line represents the propagation of a scalar particle. The b^{-4} loop divergency is cancelled by the two vertices $O(b^2)$ giving rise to a finite result.

2 Lattice quenched study of $\mathcal{L}_{toy}$: regularization and renormalization

Numerical simulations of lattice models with gauge, fermions and scalars are not common and technically challenging[2]. In this first numerical study of the model (1) we can limit ourselves to a *quenched-fermion* simulation of the lattice model specified below. In fact in quenched approximation the gauge and the scalar fields can be updated independently of each other. The lattice regularized action [3] we consider reads

$$S_{lat} = b^4 \sum_x \left\{ \mathcal{L}_{kin}^{YM}[U] + \mathcal{L}_{kin}^{sca}(\Phi) + \mathcal{V}(\Phi) + \overline{\Psi} D_{lat}[U,\Phi]\Psi \right\} \tag{15}$$

$$\mathcal{L}_{kin}^{YM}[U] \,:\, \text{SU(3) plaquette action} \tag{16}$$

$$\mathcal{L}_{kin}^{sca}(\Phi) + \mathcal{V}(\Phi) = \frac{1}{2}\,\mathrm{tr}\,[\Phi^\dagger(-\partial_\mu^*\partial_\mu)\Phi] + \frac{\mu_0^2}{2}\,\mathrm{tr}\,[\Phi^\dagger\Phi] + \frac{\lambda_0}{4}(\,\mathrm{tr}\,[\Phi^\dagger\Phi])^2\,, \tag{17}$$

[2]To our knowledge what we presented here is the first numerical study of a model with fermions, scalars and non-Abelian gauge fields in the strong interaction regime.
[3]For a presentation of preliminary numerical results see [3]

where $\Phi = \varphi_0 1\!\!1 + i\varphi_j\tau^j$ is a matrix-valued field and $\mathrm{tr}\,[\Phi^\dagger\Phi] = \varphi_0^2 + \varphi_1^2 + \varphi_2^2 + \varphi_3^2$

$$(D_{lat}[U,\Phi] \quad \Psi)(x) = \gamma_\mu\widetilde{\nabla}_\mu\Psi(x) + \eta F(x)\Psi(x) - b^2\rho\frac{1}{2}F(x)\widetilde{\nabla}_\mu\widetilde{\nabla}_\mu\Psi(x) \tag{18}$$

$$-b^2\rho\frac{1}{4}\Big[(\partial_\mu F)(x)U_\mu(x)\widetilde{\nabla}_\mu\Psi(x+\hat{\mu}) + (\partial^*_\mu F)(x)U^\dagger_\mu(x-\hat{\mu})\widetilde{\nabla}_\mu\Psi(x-\hat{\mu})\Big], \tag{19}$$

with $F(x) \equiv [\varphi_0 1\!\!1 + i\gamma_5\tau^j\varphi_j](x)$, the fermionic SU(2) doublet $\Psi^T = (u,d)$ and the lattice derivatives defined as

$$\nabla_\mu f(x) \equiv \frac{1}{b}(U_\mu(x)f(x+\hat{\mu}) - f(x)) \qquad \nabla^*_\mu f(x) \equiv \frac{1}{b}(f(x) - U^\dagger_\mu(x-\hat{\mu})f(x-\hat{\mu})) \tag{20}$$

$$\widetilde{\nabla}_\mu f(x) \equiv \frac{1}{2}(\nabla_\mu + \nabla^*_\mu)F(x). \tag{21}$$

The Wilson-like term does not remove the doublers because it involves the scalar field Φ and it has dimension six. This makes no harm in this quenched study aimed at testing whether the mass generation mechanism occurs at all. The analysis done in [4], [5] and [6] for staggered fermions can be used to analyze the fermions in the Lagrangian (15): first we rewrite the action in terms of the field $\chi(x) = \mathcal{A}_x^{-1}\Psi(x)$ with $\mathcal{A}_x = \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}\gamma_4^{x_4}$, then we perform a second change of variables

$$q^B_{\alpha,a}(y) = \frac{1}{8}\sum_\xi \overline{U}(2y, 2y+\xi)[\Gamma_\xi]_{\alpha,a}(1 - b\sum_\mu \xi_\mu\widetilde{\nabla}_\mu)\chi^B(2y+\xi), \tag{22}$$

where y runs over the coarse lattice $x_\mu = 2y_\mu + \xi_\mu$, $\xi_\mu = 0, 1$ and $\overline{U}(2y, 2y+\xi)$ is the average of link products along the shortest paths from y to $y+\xi$. With these changes of variables the action becomes

$$S^{fer}_{lat} = \sum_{y,B}\bar{q}^B(y)\left\{\sum_\mu(\gamma_\mu\otimes 1\!\!1)D_\mu + (\eta-\bar{\eta})\mathcal{F}(y)\right\}q^B(y) + O(b^2) \tag{23}$$

where $\mathcal{F}(y) = \varphi_0(2y)(1\!\!1\otimes 1\!\!1) + s_B i\tau^i\varphi_i(2y)(\gamma_5\otimes t_5)$, $s_B = \pm 1$ and $t_\mu = \gamma^*_\mu$ are the taste matrices. The action (23) is diagonal in taste and replicas $B = 1,2,3,4$ indices up to $O(b^2)$; it describes 32 fermions species namely 4 replicas of 4 tastes of the SU(2) doublet $q^T = (u,d)$.
The quark bilinears in the Ψ basis have well defined quantum numbers in the classical continuum limit once expressed in the q^B basis. For example the point-split vector current

$$\widetilde{J}^{V^i}_\mu(x) = \overline{\Psi}(x-\hat{\mu})\gamma_\mu\frac{\tau^i}{2}U_\mu(x-\hat{\mu})\Psi(x) + \overline{\Psi}(x)\gamma_\mu\frac{\tau^i}{2}U^\dagger_\mu(x-\hat{\mu})\Psi(x-\hat{\mu}), \quad x_\mu = 2y_\mu + \xi_\mu, \tag{24}$$

once summed over the hypercube coordinate ξ and expressed in the q^B basis becomes

$$\sum_\xi \widetilde{J}^{V^i}_\mu(2y+\xi) = \sum_{B=1}^4 \overline{q}^B(y)(\gamma_\mu\otimes 1\!\!1)\frac{\tau^i}{2}q^B(y) + O(b^2). \tag{25}$$

One can prove that loop effects do not generate $d \leq 4$ operators besides $F_{\mu\nu}F_{\mu\nu}$, $\partial_\mu\Phi^\dagger\partial_\mu\Phi$, $q^B(\gamma_\mu\otimes 1\!\!1)\tilde{\nabla}q^B$, $\Phi^\dagger\Phi$, $(\Phi^\dagger\Phi)^2$ and $\eta\overline{q}^B(y)\mathcal{F}^B(y)q^B(y)$ which are all present in the action (15). A way of seeing this is based on "spectrum doubling symmetry" [7]

$$\Psi(x) \to \Psi'(x) = e^{-ix\cdot\pi_H}M_H\Psi(x) \qquad \overline{\Psi}(x) \to \overline{\Psi}'(x) = \overline{\Psi}(x)M^\dagger_H e^{ix\cdot\pi_H} \tag{26}$$

where H is an ordered set of four-vectors indices $H \equiv \{\mu_1, ..., \mu_h\}$, $(\mu_1 < \mu_2 < ... < \mu_h)$. For $0 \leq h \leq 4$ there are 16 four-vectors π_H with $\pi_{H,\mu} = \pi$ if $\mu \in H$ or $\pi_{H,\mu} = 0$ otherwise and 16 matrices $M_H \equiv (i\gamma_5\gamma_{\mu_1})...(i\gamma_5\gamma_{\mu_h})$. This is an exact symmetry of S_{lat}, thus also of the effective action $\Gamma_{lat}[U, \Phi, \Psi]$. Now in order to respect the spectrum doubling symmetry Γ_{lat} can only have terms with symmetric covariant derivatives $\widetilde{\nabla}_\mu$ acting on Ψ. Close to the continuum limit among the local terms of Γ_{lat} only the fermion kinetic term $\bar{\Psi}\widetilde{\nabla}\Psi$ and Yukawa term $\eta\bar{\Psi}\Phi\Psi$ are relevant.

As a consequence we find that η_{cr}, the critical value of η, is well defined (even in the presence of fermion doubling), unique and independent of the subtracted scalar squared mass μ^2_{sub} (thus equal for the Wigner phase and the Nambu-Goldstone phase).

Since we are doing a quenched study of the model (15) exceptional configurations of the gauges fileds and the scalars with small eigenvalues of D_{lat} can occur in the Monte Carlo sampling leading to small eigenvalues of D_{lat}. In order to get control over exceptional configurations we add a twisted mass term in the action

$$S^{toy+tm}_{lat} = S_{lat} + i\mu b^4 \sum_x \overline{\Psi}\gamma_5\tau_3\Psi \tag{27}$$

at the price of introducing a soft (hence harmless) breaking of $\chi_{L,R}$ (and $\widetilde{\chi}_{L,R}$ when restored).

3 Strategy of numerical study

To study whether the NP mechanism occurs we consider the renormlize axial $\widetilde{\chi}$ SDE (see eq. 13)

$$Z_{\partial\tilde{A}}\partial_\mu \widetilde{J}^{A^\pm}_\mu = 2(\eta - \eta_{cr})\widetilde{D}^{P\pm} + \delta_{ph,NG} C_1 \Lambda_s \mathcal{P}^\pm + O(b^2) \tag{28}$$

with $\delta_{ph,NG} = 0, 1$ for the NG and Wigner phase respectively, the current

$$\widetilde{J}^{A^i}_\mu(x) = \overline{\Psi}(x-\hat{\mu})\gamma_\mu\gamma_5\frac{\tau^i}{2}U_\mu(x-\hat{\mu})\Psi(x) + \overline{\Psi}(x)\gamma_\mu\gamma_5\frac{\tau^i}{2}U^\dagger_\mu(x-\hat{\mu})\Psi(x-\hat{\mu}) \tag{29}$$

and the densities

$$\widetilde{D}^{P\pm} = \overline{\Psi}_L\left\{\Phi, \frac{\tau^\pm}{2}\right\}\Psi_R - \overline{\Psi}_R\left\{\frac{\tau^\pm}{2}, \Phi^\dagger\right\}\Psi_L\,, \qquad \mathcal{P}^\pm = \overline{\Psi}_L\left\{\mathcal{U}, \frac{\tau^\pm}{2}\right\}\Psi_R - \overline{\Psi}_R\left\{\frac{\tau^\pm}{2}, \mathcal{U}\right\}\Psi_L\,. \tag{30}$$

In the Wigner phase ($\delta_{ph,NG} = 0$) one can determine η_{cr} by studying the SDE (28) for various η values. In NG phase the SDE (28) at $\eta = \eta_{cr}$ takes the form of a $\tilde{\chi}$ WTI with NP breaking term up to $O(b^2)$ that we shall neglect from now on

$$Z_{\partial\tilde{A}}\langle 0|\partial_\mu \widetilde{J}^{A^\pm}_\mu|M_{PS\pm}\rangle = C_1\Lambda_s\langle 0|\mathcal{P}^\pm|M_{PS\pm}\rangle. \tag{31}$$

Expanding $\mathcal{U}$ around the vacuum, $\mathcal{U} = 1\!\!1 + i\frac{\vec{\tau}\cdot\vec{\varphi}}{v} + O(\frac{\sigma^2}{v^2}, \frac{\pi^2}{v^2})$, we get the corresponding expansion for $\mathcal{P}^\pm$

$$\mathcal{P}^\pm = \overline{\Psi}_L\left\{\mathcal{U}, \frac{\tau^\pm}{2}\right\}\Psi_R - h.c. = \overline{\Psi}_L\left\{1\!\!1 + i\frac{\vec{\tau}\cdot\vec{\varphi}}{v} + ..., \frac{\tau^\pm}{2}\right\}\Psi_R - h.c. = P^\pm + ... \tag{32}$$

χ invariance implies that $\mathcal{P}^\pm$ has the same renormalization constant as $P^\pm = \overline{\Psi}\gamma_5\tau^\pm\Psi$ which we call Z_P. Thus a renormalized measure of the effective NP $\widetilde{\chi}$ breaking is given by the dimensionful quantity

$$2m^{ren}_{AWI} \equiv \frac{Z_{\partial\tilde{A}}}{Z_P}\frac{\langle 0|\partial_\mu \widetilde{J}^{A^\pm}_\mu|M_{PS\pm}\rangle}{\langle 0|P^\pm|M_{PS\pm}\rangle} = C_{1,ren}\Lambda_s(1 + ...) \qquad C_{1,ren} = \frac{C_1}{Z_P} \tag{33}$$

In spite of its name m^{ren}_{AWI} is not the renormalized counterpart of any parameter in the lattice action. Since at $\eta = \eta_{cr}$ in Wigner phase the current $\tilde{J}^{A^\pm}_\mu$ is conserved up to $O(b^2)$, $\tilde{\chi}$-current algebra implies vanishing anomalous dimension for $\tilde{J}^{A^\pm}_\mu$ and $\tilde{J}^{V^3}_\mu$, hence at $\eta = \eta_{cr}$ the NP term on the r.h.s. of the (31) is RG-invariant.

In principle the study of SDE (28) would involve evaluation of disconnected diagrams due to isospin changing mediated by the field Φ. However in the quenched approximation one can prove by duplicating the fermion content $\Psi_\ell = (u, d)$ and $\Psi_h = (c, s)$, and considering the SDE involving a generation off diagonal $\tilde{J}^{A\pm}_\mu$ current, that eg. (28) holds for fermionic disconnected and connected diagrams separately. Hence both the determination of η_{cr} in Wigner phase and the evaluation of m_{AWI} and related quantities in the NG phase can be carried out in practice without computing fermionic disconnected diagrams.

4 Renormalisation procedure

In a quenched lattice study the renormalization condition on the action parameters can be chosen such that the tuning of η to its critical value, the renormalization of the gauge coupling and the renormalization of the scalar squared mass and quartic coupling can be carried out separately from each other. For instance: the relation between $g_0^2 \equiv 6/\beta$ and the lattice spacing is determined by keeping fixed the Sommer length scale r_0 in physical units [8, 9]; the bare scalar mass m_0^2 and quartic coupling λ_0 are determined by keeping fixed (as $b \to 0$ together with $g_0 \to 0$) both $M_\sigma r_0$, where M_σ is the mass of the non Goldstone boson scalar particle in the NG-phase, and a suitable non-perturbative definition of the renormalized quartic scalar coupling, $\lambda_{NP} = M_\sigma^2/(2v_R^2)$. In the NG phase the scalar vev, $v_R = Z_\phi^{1/2}\langle\Phi^0\rangle$, with Z_ϕ the renormalization constant of ϕ, is thus also fixed in physical units. The ϕ-field renormalization constant is computed enforcing

$$Z_\phi^{1/2} = [M_\sigma\langle\phi^0_{p=0}\phi^0_{p=0}\rangle] - V\langle\Phi^0\rangle^2]^{-1}\,, \tag{34}$$

the mass M_σ is extracted from exponential decay in time correlator $\langle\Phi^0_{x_0}\Phi^0_0\rangle$, then λ_R is computed. In order to eliminate the dependence from Z_P in the (33) we define the following quantity

$$z^{ren}_{AWI} = 2m_{AWI}r_0 Z_{\partial\tilde{V}} G^{Wigner}_{PS} r_0^2 \tag{35}$$

where $G^{Wigner}_{PS} \equiv \langle 0|P^\pm|\text{PS-meson}\rangle$ is the matrix element of $P^\pm$ between vacuum and the pseudo-scalar meson in the Wigner phase of the theory at $\eta = \eta_{cr}$. Note that $z^{ren}_{AWI} = 0$ if and only if $m^{ren}_{AWI} = 0$. The $\chi_L \otimes \chi_R$ symmetry implies that the renormalization constant $Z_{\partial\tilde{A}}$ is equal to $Z_{\partial\tilde{V}}$, the renormalization factor of the vector current (25). At $\eta = \eta_{cr}$ we have that the vector χ-SDE in the NG phase reads

$$Z_{\partial\tilde{V}}\langle\tilde{J}^{V,i}_\mu(x)\hat{O}(0)\rangle = \langle\Delta\hat{O}\rangle\delta(x) + C_1\Lambda_S\langle\mathcal{S}^1(x)\hat{O}(0)\rangle + 2\mu\langle P^1(x)\hat{O}(0)\rangle \tag{36}$$

where $\mathcal{S}^1 = \overline{\Psi}_L\left[\mathcal{U}, \frac{\tau^i}{2}\right]\Psi_R - h.c.$. Taking $\hat{O} = P^1$ and exploring parity we have

$$\sum_{\vec{x}} Z_{\partial\tilde{V}}\langle\tilde{J}^{V,i}_\mu(x)P^1(0)\rangle = \sum_{\vec{x}} 2\mu\langle P^1(x)P^1(0)\rangle\,, \tag{37}$$

since $\sum_{\vec{x}}\langle\mathcal{S}^1(x)P^1(0)\rangle = 0$. The latter equation can be used to determine $Z_{\partial\tilde{V}}$ which is equal to $Z_{\partial\tilde{A}}$.

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References

[1] R. Frezzotti, G.C. Rossi, Phys. Rev. **D92**, 054505 (2015), 1402.0389

[2] G. 't Hooft, NATO Sci. Ser. B **59**, 135 (1980)

[3] S. Capitani et al., *Testing a non-perturbative mechanism for elementary fermion mass generation: numerical results*, in *Proceedings, 35th International Symposium on Lattice Field Theory (Lattice2017): Granada, Spain*, to appear in EPJ Web Conf.

[4] H. Kluberg-Stern, A. Morel, O. Napoly, B. Petersson, Nucl. Phys. **B220**, 447 (1983)

[5] A. Patel, S.R. Sharpe, Nucl. Phys. **B395**, 701 (1993), hep-lat/9210039

[6] Y.b. Luo, Phys. Rev. **D55**, 353 (1997), hep-lat/9604025

[7] I. Montvay, G. Münster, *Quantum Fields on a Lattice*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1994)

[8] M. Guagnelli, R. Sommer, H. Wittig (ALPHA), Nucl. Phys. **B535**, 389 (1998), hep-lat/9806005

[9] S. Necco, R. Sommer, Nucl. Phys. **B622**, 328 (2002), hep-lat/0108008