

Wigner Function for Spin-1/2 Fermions in Electromagnetic Fields

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Xin-li Sheng

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Dekan:

Gutachter : Prof. Dr. Dirk H. Rischke

Prof. Dr. Qun Wang

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Abstract

We study the Wigner function for massive spin-1/2 fermions in electromagnetic fields. The covariant Wigner function is a four by four matrix function in 8-dimensional phase space $\{x^\mu, p^\mu\}$, whose components give various physical quantities such as the particle distribution, the current density, and the spin distribution, etc.. The kinetic equations for the Wigner function are obtained from the Dirac equation. We derive the Dirac form equations with first order differential operators, as well as the Klein-Gordon form equations with second order differential operators, both are matrix equations in Dirac space. We prove that some component equations are automatically satisfied if the rest are fulfilled, which means both the Dirac form and the Klein-Gordon form equations have redundancy. In this thesis two methods are proposed for calculating the Wigner function, which are proved to be equivalent. In addition to the covariant Wigner function, the equal-time Wigner function will also be introduced. The equal-time one is a function of time and 6-dimensional phase space variables $\{\mathbf{x}, \mathbf{p}\}$, which can be derived from the covariant one by taking an integration over energy p^0 . The equal-time Wigner function is not Lorentz-covariant but it is a powerful tool to deal with dynamical problems. In this thesis, it is used to study the Schwinger pair-production in the presence of an electric field.

The Wigner function can be analytically calculated following the standard second-quantization procedure. We consider three cases: free fermions with or without chiral imbalance, and fermions in constant magnetic field with chiral imbalance. The computations are achieved via firstly deriving a set of orthonormal single-particle wavefunctions from the Dirac equation, then constructing the quantized field operator, and finally inserting the field operator into the Wigner function and determine the expectation values of operators under the wave-packet description. The Wigner functions are computed to leading order in spatial gradients. In Strong electric field the vacuum can decay into a pair of particle and anti-particle. The pair-production process is studied using the equal-time Wigner function. General solutions are obtained for pure constant electric fields and for constant parallel electromagnetic fields. We also solve the case of a Sauter-type electric field numerically.

For an arbitrary space-time dependent electromagnetic field, the Dirac equation does not have an analytical solution and neither has the Wigner function. A semi-classical expansion with respect to the reduced Planck's constant \hbar are performed for the Wigner function as well as the kinetic equations. We calculate the Wigner function (and all of its components equivalently) to leading order in \hbar , in which order the spin component start playing a role. Up to this order, the Wigner function contains four independent degrees of freedoms, three of which describe the polarization density and the remaining one describes the net particle number density. A generalized Bargmann-Michel-Telegdi (BMT) equation and a generalized Boltzmann equation are obtained for these undetermined parts, which can be used to construct spin-hydrodynamics in the future.

Using analytical results and semi-classical solutions, we compute physical quantities in thermal equilibrium. In semi-classical expansion, we introduce the chiral chemical potential μ_5 in the thermal distribution. This naive treatment is straightforward extension of the massless case but provides a good estimate of physical quantities when μ_5 is comparable or smaller than the typical energy scale, i.e., the temperature in a

thermal system. Meanwhile, by making comparison of the results of the semi-classical expansion and the ones in a constant magnetic field, we find that the semi-classical method works well for the chiral effects, including the Chiral Magnetic Effect, the Chiral Separation Effect, as well as the energy flux along the direction of the magnetic field. But when the mass and chemical potentials are much larger than the temperature, the semi-classical results over estimate these chiral effects. The magnetic field strength dependence of physical quantities is discussed. If we fix the thermodynamical variables, the net fermion number density, energy density, and the longitudinal pressure are proportional to the field strength, while the axial-charge density and the transverse pressure are inversely proportional to it.

Schwinger pair-production rates in a thermal background are computed for a Sauter-type electric field and a constant parallel electromagnetic field, respectively. For the Sauter-type field, the total number of newly generated pairs is proportional to the field strength and the life time of the field. On the other hand, a parallel magnetic field will enhance the pair-production rate. Due to Pauli's exclusion principle, the creation of pairs is forbidden for particles already existing in the same quantum state. Thus in both cases, the pair-production rate is proved to be inversely proportional to the chemical potential and temperature.

Keywords: Wigner function, electromagnetic fields, chiral effect, pair-production.

Zusammenfassung

In dieser Arbeit haben wir uns mit dem Wignerfunktionsansatz für Spin-1/2-Teilchen beschäftigt und diese Herangehensweise verwendet, um die chiralen Effekte und die Paarbildung in Gegenwart eines elektromagnetischen Feldes zu untersuchen. Die Wignerfunktion ist als Quasiverteilungsfunktion im Phasenraum definiert. Die Wignerfunktion ist eine komplexe 4×4 -Matrix, die in die Generatoren der Clifford-Algebra Γ_i zerlegt werden kann. Die Zerlegungskoeffizienten werden gemäß ihrer Transformationseigenschaften unter Lorentz-Transformationen und Paritätsinversion jeweils als Skalar, Pseudoskalar, Vektor, Axialvektor und Tensor identifiziert. Sie können nach Integration über den Impuls mit verschiedenen Arten von makroskopischen physikalischen Größen wie dem Fermionstrom, der Spinpolarisation und dem magnetischen Dipolmoment in Beziehung gesetzt werden. Sie werden also als die Dichten im Phasenraum interpretiert.

Da die Wignerfunktion mit Hilfe des Dirac-Feldes konstruiert wird, haben wir die kinetischen Gleichungen für die Wignerfunktion aus der Dirac-Gleichung erhalten. In dieser Arbeit haben wir die Dirac-Form-Gleichung abgeleitet, die im Differentialoperator linear ist. Daneben haben wir auch die Klein-Gordon-Form-Gleichung erhalten, die die Operatoren zur zweiten Ordnung enthält. Diese Gleichungen werden dann wie auch die Wignerfunktion selbst in Γ_i zerlegt, sodass sie ein System mehrerer partieller Differentialgleichungen (PDG) liefern. Glücklicherweise sind die zerlegten Gleichungen nicht unabhängig voneinander. Durch Eliminieren der redundanten Gleichungen erhielten wir zwei Möglichkeiten, die Lösung für die Wignerfunktion im massiven Fall zu bestimmen. Diese Redundanz beruht auf der Tatsache, dass die Vektor- und Axialvektorkomponenten \mathcal{V}^μ , \mathcal{A}^μ der Wignerfunktion in Form der Skalar-, Pseudoskalar- und Tensorkomponenten \mathcal{F} , \mathcal{P} , $\mathcal{S}^{\mu\nu}$ ausgedrückt werden können oder umgekehrt. Ein Ansatz zur Lösung des PDG-Systems besteht daher darin, \mathcal{V}^μ , \mathcal{A}^μ als Basisfunktionen zu verwenden und sich auf ihre Massenschalenbedingungen zu konzentrieren. Daneben besteht der andere Ansatz darin, \mathcal{F} , \mathcal{P} , $\mathcal{S}^{\mu\nu}$ als Basisfunktionen zu verwenden. Durch eine Entwicklung in \hbar , die als semiklassische Entwicklung bezeichnet wird, haben wir die allgemeine Lösung der Wignerfunktion bis zur ersten Ordnung in \hbar erhalten. Es hat sich gezeigt, dass die beiden oben genannten Ansätze zu dem gleichen Ergebnis führen und somit äquivalent sind. Die endgültige Lösung hat nur vier unabhängige Freiheitsgrade, was durch eine Eigenwertanalyse bewiesen wird. Zur Ordnung \hbar wird die übliche Massenschale $p^2 - m^2 = 0$ durch die spinmagnetische Kopplung verschoben.

Wir haben weiterhin die Wignerfunktion für den masselosen Fall durch eine semiklassische Entwicklung reproduziert. Im masselosen Fall können die Fermionen nach ihrer Chiralität in zwei

Gruppen eingeteilt werden. Unter Verwendung von \mathcal{V}^μ und \mathcal{A}^μ konstruierten wir die linkshändigen und rechtshändigen Ströme, die zur Ordnung \hbar bestimmt werden. Die übrigen Komponenten \mathcal{F} , \mathcal{P} , $\mathcal{S}^{\mu\nu}$ sind proportional zur Teilchenmasse und verschwinden somit im masselosen Limes. In dieser Arbeit haben wir eine direkte Beziehung zwischen den masselosen und massiven Strömen \mathcal{V}^μ , \mathcal{A}^μ gefunden. Dies könnte darauf hinweisen, dass unsere massiven Ergebnisse allgemeiner sind als die chirale kinetische Theorie.

In dieser Arbeit haben wir mehrere analytisch lösbare Fälle diskutiert. In den folgenden drei Fällen konnten aus der Dirac-Gleichung analytisch Einteilchenwellenfunktionen bestimmt werden, aus denen die Wignerfunktion abgeleitet wird. Wir haben nur den Beitrag zur führenden Ordnung in räumlichen Gradienten der Wignerfunktionen aufgelistet, aber auch einen potentiellen Ansatz gefunden, um Beiträge höherer Ordnung abzuleiten.

1. Quantisierung der ebenen Wellen: In diesem Fall enthält die Dirac-Gleichung keine äußere Wechselwirkung und wird somit durch freie ebene Wellen gelöst. Die Ergebnisse dieses Ansatzes bilden den Grundstein für die Methode der semiklassischen Entwicklung: Sie dienen als Lösungen nullter Ordnung in \hbar , während solche höherer Ordnung automatisch Ordnung für Ordnung erscheinen.

2. Chirale Quantisierung: In diesem Fall haben wir μ und μ_5 als konstante Variablen für die Selbstenergie eingeführt. Diese Variablen tragen in Form von $\mu\hat{N} + \mu_5\hat{N}_5$ zum gesamten Hamiltonian bei, wobei \hat{N} und \hat{N}_5 Operatoren für Teilchenzahl und Axialladungszahl sind. Im masselosen Limes konnten wir μ als das chemische Vektorpotential und μ_5 als das chirale chemische Potential identifizieren. Wir betonen, dass das chirale chemische Potenzial im massiven Fall nicht wohldefiniert ist, da die konjugierte Größe, die Axialladung, nicht erhalten ist. Im massiven Fall ist μ_5 also nur eine Variable, die das Spin-Ungleichgewicht beschreibt. Der modifizierte Hamilton-Operator führt zu einer neuen Dirac-Gleichung, die gelöst werden könnte, wenn wir annehmen, dass μ und μ_5 Konstanten sind. Die Wignerfunktion wird dann durch Einteilchenwellenfunktionen konstruiert. Da jedoch das Vorhandensein von μ und μ_5 die Dirac-Gleichung ändert, müssen auch die kinetischen Gleichungen für die Wignerfunktion weiter modifiziert werden. Darüber hinaus können wir die Einteilchenlösung für allgemeine raum- / zeitabhängige μ und μ_5 nicht erhalten. Die Methode der chiralen Quantisierung dient somit nur als Gegenprobe für die Methode der semiklassischen Entwicklung. Hier sind die elektromagnetischen Felder noch nicht enthalten.

3. Landau-Quantisierung: Basierend auf dem Fall 2 führen wir weiterhin ein konstantes Magnetfeld ein. Die Energieniveaus werden dann durch die Landau-Niveaus mit Modifikation von den chemischen Potentialen μ und μ_5 beschrieben. In diesem Fall können wir die Phänomene im

Magnetfeld wie CME, CSE und anomalen Energiefluss explizit untersuchen. Da das Feld das Energiespektrum ändert, haben wir festgestellt, dass die Gesamtfermionzahldichte, die Energiedichte und der Druck von der Stärke des Magnetfelds abhängen.

Darüber hinaus haben wir basierend auf der Quantisierung der ebenen Wellen eine semiklassische Entwicklung in der reduzierten Plancksche Konstante \hbar durchgeführt. Die Wignerfunktion wird dann bis zur Ordnung \hbar gelöst. Man beachte, dass die Methode der semiklassischen Entwicklung für ein elektromagnetisches Feld mit beliebiger Raum-/Zeitabhängigkeit verwendet werden kann. Bei dieser Methode setzen wir μ und μ_5 in die thermischen Gleichgewichtsverteilungen anstatt in die Hamilton-Verteilung ein und machen Gebrauch von der spezifischen Annahme, dass alle Fermionen in longitudinaler Richtung polarisiert sind. Dieses Verfahren stellt eine naive Erweiterung des Verfahrens für den masselosen Fall dar. Numerische Berechnungen zeigen, dass die auf diese Weise erhaltene Gesamtfermionzahldichte und Axialladungsdichte mit denjenigen aus der chiralen Quantisierung übereinstimmen, wenn μ_5 und Masse m vergleichbar mit der oder kleiner als die Temperatur sind. Gleichzeitig zeigen die Energiedichten und Drücke bei diesen beiden Methoden ebenfalls Übereinstimmungen.

Neben den obigen drei analytisch lösbaren Fällen haben wir auch die Wignerfunktion im elektrischen Feld diskutiert. Basierend auf den Ergebnissen der Quantisierung der ebenen Wellen und der Landauquantisierung erhalten wir durch dynamische Betrachtungen jeweils Wignerfunktionen in Gegenwart eines konstanten elektrischen Feldes. Anschließend werden Paarproduktionen berechnet, die, wie bewiesen wird, durch ein paralleles Magnetfeld verstärkt, durch Temperatur und chemisches Potential dagegen unterdrückt werden. Die Unterdrückung der Paarproduktion im thermischen System wird auf das Pauli'sch Ausschlussprinzip zurückgeführt.

Die Methode der semiklassischen Entwicklung bietet eine allgemeine Möglichkeit, die Spinkorrektur zu berechnen. Zu nullter Ordnung in \hbar haben wir die klassische spinlose Boltzmann-Gleichung reproduziert. Zur Ordnung \hbar treten automatisch Spinkorrekturen wie die Energieverschiebung durch spin-magnetische Kopplung auf. In dieser Arbeit haben wir eine allgemeine Boltzmann-Gleichung und eine allgemeine BMT-Gleichung erhalten, die jeweils die Entwicklungen der Teilchenverteilung und der Spinpolarisationsdichte bestimmen. Kollisionen zwischen Teilchen sind jedoch noch nicht enthalten. Nach der Methode der Momente könnten wir die semiklassischen Ergebnisse zu einer hydrodynamischen Beschreibung ausweiten, was unsere zukünftige Arbeit wäre. Allerdings erscheinen bei der Methode der semiklassischen Entwicklung die elektromagnetischen Felder zur Ordnung \hbar , was für den Limes schwacher Feldstärke gilt. Im Anfangsstadium von Schwerionenkollisionen ist die Magnetfeldstärke jedoch vergleichbar mit m_π^2 . Auch in späteren Stadien

können die Schwankungen der elektromagnetischen Felder m_π^2 erreichen. In der starken Laserphysik sind die elektromagnetischen Felder von erheblicher Stärke, aber es gibt fast keine Teilchen. Ob die semiklassische Entwicklung in diesen Fällen eingesetzt werden kann oder nicht, bedarf einer genaueren Diskussion. Die Untersuchungen des konstanten Magnetfeldes in dieser Arbeit können als Ausgangspunkt der kinetischen Theorie in einem starken Hintergrundfeld dienen.

Eine weitere mögliche Erweiterung dieser Arbeit ist die Axialladungserzeugung. Bei Vorhandensein paralleler elektrischer und magnetischer Felder erzeugt das elektrische Feld Teilchenpaare aus dem Vakuum und die neu erzeugten Paare werden durch das Magnetfeld polarisiert. Infolgedessen trägt die Paarproduktion auf dem niedrigsten Landau-Niveau zur axialen Ladungsdichte bei. Die Realzeitaxialladungserzeugung von massiven Teilchen im thermischen Hintergrund wurde noch nicht gelöst. Und der Wignerfunktionsansatz in dieser Arbeit könnte einen möglichen Zugang zu diesem Ziel liefern.

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I. INTRODUCTION

A. Chiral effects and pair-production

The quark-gluon plasma (QGP) is a new state of matter created in relativistic heavy-ion collisions. It is the hot and dense matter of strong interaction governed by quantum chromodynamics (QCD). The universe is in the QGP phase in its early stage. Thus creating and studying the QGP helps us to better understand both the properties of QCD and the evolution of the universe. There are two big collider experiments for heavy-ion collisions that are running in the world: the Large Hadron Collider (LHC) at CERN [1] and the Relativistic Heavy Ion Collider (RHIC) at BNL [2]. There are also other colliders under construction: the Facility for Antiproton and Ion Research (FAIR) at GSI [3], Nuclotron-based Ion Collider fAcility (NICA) at Dubna, etc..

The evolution of the QGP is dominated by the strong interaction. The interaction rate is sufficiently large such that the plasma reaches hydrodynamization rapidly after its generation [4]. Here, hydrodynamization means the QGP can be accurately described by relativistic hydrodynamics. In non-central collisions, a strong magnetic field is generated by the fast-moving protons. The field strength depends on the type of colliding nuclei and the center-of-mass collision energy. For example, in Au+Au collisions at RHIC with collision energy $\sqrt{s} = 200\text{GeV}$ per nucleon, the magnetic field can reach several $m_\pi^2 \sim 10^{18}\text{G}$ [5–8], which is the strongest field that humans have ever made. The magnetic field decays quickly because it is mainly generated by the spectators which move far away from the interaction region soon after the collision moment. Most simulations use the Lienard-Wiechert potential, developed by A. M Lienard in 1898 and E. Wiechert in 1900, to describe the electromagnetic field of a moving point charge in vacuum. However, the QGP is a conducting medium, with the conductivity having been calculated via lattice QCD [9, 10] and holographic models [11]. A non-vanishing electrical conductivity will significantly extend the life-time of the magnetic field [12, 13]. Recently, an analytical formula has been derived for the electromagnetic field generated by a moving point charge in a medium with constant electrical conductivity σ and chiral magnetic conductivity σ_χ [14, 15], which can serve for numerical simulations in the future.

For massless particles, the chirality operator is commutable with the Hamiltonian. One can then separate the massless particles into the right-handed (RH) ones and left-handed (LH) ones according to their chirality. In the QGP, an imbalance between RH and LH particles can be generated by topological fluctuations of the gluonic sector, fluctuations of the quark sector, or glasma flux tubes [16–20]. The corresponding thermodynamic states can be specified by a chiral chemical potential μ_5 ,

which is defined as the parameter conjugate to the topological charge. A nonzero topological charge breaks the charge-parity symmetry locally and induces a charge current along the direction of the magnetic field, i.e. Chiral Magnetic Effect (CME) [19, 21, 22]. The CME can also be understood through Landau quantization: In the presence of a magnetic field, charged particles will occupy energy states with specific spin and orbital angular momentum, which are called Landau levels. The ground state, i.e., the lowest Landau level is occupied by positively charged particles whose spins are parallel to the magnetic field, or negatively charged particles whose spins are anti-parallel to the magnetic field. Such a spin configuration is required by the principle of minimum energy. Because of the nonzero topological charge, the momentum of positively (or negatively) charged particle has a preference direction with respect to its spin and thus generates a collective current. The CME is proportional to the magnetic field and the chiral chemical potential μ_5 [22, 23],

$$\mathbf{J} = \frac{\mu_5}{2\pi^2} q \mathbf{B}, \quad (1.1)$$

where q is the electric charge of particles. For systems with multiple species of particles, we need to take a sum over all species.

In heavy-ion collision experiments, the chiral imbalance can be spontaneously generated in the initial stage of the collision [16–20, 24]. Thus, the CME is expected to be observed in non-central collisions through the azimuthal distribution of charge [25],

$$\frac{dN}{d\phi} \propto 1 + 2 \sum_n \{v_n \cos[n(\phi - \Psi_{\text{RP}})] + a_n \sin[n(\phi - \Psi_{\text{RP}})]\}, \quad (1.2)$$

where v_n and a_n denote the parity-even and parity-odd Fourier coefficients and $\phi - \Psi_{\text{RP}}$ is the azimuthal angle with respect to the reaction plane. In experiments, the reaction plane cannot be detected directly, thus in practice we use the event plane Ψ_{EP} as an approximation. Here the event plane is determined by the beam direction and the direction of maximal energy density. The CME was first expected to be observed through the charge correlation [25, 26],

$$\gamma_{\alpha\beta} = \langle \cos(\phi_\alpha + \phi_\beta - 2\Psi_{\text{EP}}) \rangle, \quad (1.3)$$

where α, β denote particles with the same or opposite charge sign and $\langle \dots \rangle$ means average over all the particles. Moreover, the determination of the event plane is not necessary: Ψ_{EP} can be replaced by a third particle, which gives the three-particle correlation [25, 27],

$$\gamma_{\alpha\beta} = \frac{1}{v_{2,c}} \langle \cos(\phi_\alpha + \phi_\beta - 2\phi_c) \rangle. \quad (1.4)$$

In the CME, the correlation for the same charge sign is observed to be positive while that for the opposite sign is negative, at RHIC [26–28] and at LHC [29–31]. However, these correlation

functions have significant background contributions from the cluster particle correlations [32] and the coupling between local charge conservation and v_2 of the QGP [33]. Meanwhile, the difference between same-charge-sign correlations and opposite-charge-sign correlations, $\Delta\gamma \equiv \gamma_{SS} - \gamma_{OS}$, are of the same magnitude in Pb+Pb and p+Pb collisions at LHC [30]. This is a challenge for the CME interpretation of the charge correlation because the magnetic field in p+Pb collisions is expected to be much smaller than in Pb+Pb collisions. On the other hand, the direction of magnetic field is random with respect to the reaction plane in p+Pb collisions according to the Glauber Monte Carlo simulation [30, 34], thus the event-by-event average of the CME contribution is expected to be small, which indicates that the large part of observables measured in Pb+Pb collisions may come from the background instead of the CME. Various new methods are proposed to isolate the CME from the background [35–38]. Isobaric collisions have been proposed at RHIC for this purpose [35]. The isobars are chosen to be ${}^{96}_{44}\text{Ru}+{}^{96}_{44}\text{Ru}$ and ${}^{96}_{40}\text{Zr}+{}^{96}_{40}\text{Zr}$ since they have the same nucleon number but different proton number. Due to different proton numbers in collisions of two isobars, the magnetic field would be 10% different and so is the CME signal, while the backgrounds are expected to be of the same magnitude because they are dominated by the strong interaction. Thus, the isobaric collisions would provide controlled experiment for the CME [35].

In non-central collisions, the colliding nuclei carry large orbital angular momentum. For example, the total angular momentum is about $10^6\hbar$ for Au+Au collisions at $\sqrt{s_{NN}} = 200$ GeV and $b = 10$ fm [39, 40]. Most of the total orbital momentum will be taken away by spectators while about 10% is left in the QGP [40]. The rotation of QGP can be described by a kinematic vorticity $\boldsymbol{\omega} = \frac{1}{2}\boldsymbol{\nabla} \times \mathbf{v}$, where \mathbf{v} is the fluid velocity. Analogous to the CME, the vorticity can also induce an electrical current along its direction because of the spin-orbit coupling, which is known as the Chiral Vortical Effect (CVE) [41, 42]. In the massless case, 2/3 of the total CVE is attributed to the magnetization current and the remaining 1/3 is attributed to the modified particle distribution because of the spin-vorticity coupling [43–45]. Since the CVE is blind to the charge, it can induce a separation of baryons. Thus, the CVE is expected to be detected through baryon-baryon correlations [46, 47].

Because of the spin-magnetic-field and spin-vorticity couplings, both the magnetic field and vorticity can polarize particles, known as the Chiral Separation Effect (CSE) [48, 49] and the Axial Chiral Vortical Effect (ACVE) [41, 42, 50]. Note that these two effects exist even if the chiral imbalance vanishes, i.e. $\mu_5 = 0$. The ACVE can induce a global polarization of hyperons, which has been observed through the polarization of Λ hyperon at STAR [51, 52]. Here the Λ decays into proton and π^- through weak interaction, which breaks the parity symmetry. The spin of the Λ can then be detected by the azimuthal distribution of daughter protons.

Since the CME depends on the axial-charge imbalance and induces an electrical current, while the CSE depends on the electrical charge imbalance and induces an axial current, the interplay between the CME and the CSE will generate a propagating wave along the direction of the magnetic field, the so-called Chiral Magnetic Wave [53]. Analogously, the interplay between the CVE and the ACVE excites collective flow along the vorticity called the Chiral Vortical Wave [54].

In the past few years, a lot of progress has been made in the chiral effects in heavy-ion collisions, see e.g. Ref. [55, 56] for a recent review. Note that the QGP is a complicated many-particle system, in which the chiral conductivities receive many corrections [57, 58]. Thus a self-consistent kinetic theory is needed for numerical simulations. Recent works along this line include the kinetic theory with Berry curvature [59–63], the Chiral Kinetic Theory [64–69], and Anomalous Hydrodynamics [24, 42, 67, 70–73] but most of these works are for massless particles. Even through the u, d quarks are almost massless compared with the typical temperature of the QGP, the s quarks are quite massive. The Wigner-function method in this thesis provides a possible way to develop the kinetic theory for massive spin-1/2 particles [74–77].

In addition to the physics related to the magnetic field and vorticity, we also study the effects of the strong electric field in the QGP. This electric field is induced by the fast decreasing magnetic field according to the Maxwell's equation. It is of the same magnitude as the magnetic field and both of them are sufficiently large in the initial stage of heavy-ion collisions, see Refs. [8, 15] for some numerical simulations. In a strong electric field, the QED vacuum becomes unstable due to fermion/anti-fermion pair-production, the so-called Schwinger process [78]. The pair-production rate was first derived by Julian Schwinger in 1951 via quantum field theory [78] and then reproduced through various kinds of methods, such as the WKB method [79–81], instanton method [82–84], holographic method [85–87], and the Wigner-function method [88–92]. The pair-production process can be analytically solved for a constant electric field $\mathbf{E}(t) = \mathbf{E}_0$ or a Sauter-type field $\mathbf{E}(t) = \mathbf{E}_0 \text{sech}^2(t/\tau)$ using the quantum kinetic theory [93, 94]. In heavy-ion collisions, the electric field strongly depends on the space and time. One may estimate the pair-production rate via firstly dividing the whole space into small cells, and then applying the pair-production rate for constant electric field in each cell. However, in the instanton method [95], one can show that spatial inhomogeneities tend to suppress the pair-production while the temporal ones tend to enhance it. Thus it is difficult to judge whether the constant-field approximation overestimates or underestimates the total pair-production rate. Some methods such as the Wigner-function method [89–92, 96] can deal with space- and time-dependent electric fields, but one has to solve a system of non-linear partial differential equations. Thanks to the development of computing power, the pair-production

for more general field configurations becomes numerically solvable [97–100].

Moreover, the thermal background and strong magnetic field makes the calculation of pair-production more challenging in a medium than in the vacuum. In a thermal system, the existing particles prohibit the creation of new particles with the same quantum number because of the Pauli exclusion principle. Thus the pair-production will be suppressed in thermal systems [92, 101–104]. Meanwhile, the magnetic field can increase the pair-production rate if it is parallel to the electric field. This enhancement has been analytically shown many years ago through the proper-time method [80, 105, 106] and recently been reproduced in string theory [107, 108], holographic theory [109], and the Wigner function approach [92]. On the other hand, due to the fact that particles in the lowest Landau level behave like chiral fermions, the pair-production in parallel electromagnetic fields is related to the production of axial charge [20, 109, 110] and to the pseudoscalar condensation [111, 112].

Non-central heavy-ion collisions always lead to strong electromagnetic fields. The study of fermions (quarks) in an electromagnetic field will help us to better understand the early-stage evolution of the QGP. In central collisions, the event-by-event average of the field is zero but its fluctuation is sufficiently large [8]. Although the chiral effects and the pair-production, have been extensively studied for many years, they have not yet been fully understood in terms of experimental observables. For example, how to extract the very weak signal out of large backgrounds is still unsolved [35–38]. As another example, the Λ polarization in the longitudinal and the transverse directions: the results of hydrodynamical calculation [113–115] have opposite sign with respect to the experiment data [52, 116]. Thus deeper and more extensive studies about fermions in electromagnetic and vorticity fields are necessary in the frontier of high-energy physics.

B. Wigner-function method

In classical statistical theory, a multi-particle system is described by a classical particle distribution $f(t, \mathbf{x}, \mathbf{p})$, as a function of the time t , the spatial coordinates \mathbf{x} , and the 3-momentum \mathbf{p} of the particle. This description is valid because the spatial position and 3-momentum of a classical particle can be determined simultaneously to arbitrary precision. However, in quantum mechanics, the classical distribution $f(t, \mathbf{x}, \mathbf{p})$ is not well-defined because Heisenberg's uncertainty principle states that the more precisely the momentum of one particle is determined, the more uncertainty is its position, and vice versa. This principle was firstly proposed by Werner Heisenberg [117] in

1927 and then mathematically derived by Earle Kennard [118] and Hermann Weyl [119],

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}, \quad (1.5)$$

where $\sigma_{x(p)}$ is the standard deviation when measuring the position x or the momentum p , and \hbar is the reduced Planck's constant. In 1932, Eugene Wigner introduced a quasi-probability distribution to study quantum statistical mechanics [120], which is now called the Wigner function (or Wigner quasi-probability distribution). The Wigner function is derived from a two-point correlation function by taking the Fourier transform with respect to the distance between the two points, so the Wigner function is a function in phase space $\{x^\mu, p^\mu\}$. The spatial densities of physical observables can be derived from the Wigner function by integrating over the 4-momentum p^μ [121]. A more detailed discussion of the Wigner function will be presented in Sec. II A. For spin-1/2 particles, the Wigner function is defined using the Dirac field operator, thus the kinetic equations for the Wigner function can be derived from the Dirac equation without loss of generality [121].

The Wigner function can be analytically computed only in very limited cases, such as in constant electromagnetic fields [89, 92, 122, 123]. Meanwhile, the numerical calculation is challenging because the parameter space is 8-dimensional, which is too large for finite-difference methods. A general way to deal with the Wigner function is to treat the space-time derivative and electromagnetic field as small quantities and expand all the Wigner function as well as all the operators in terms of \hbar . This method is known as the semi-classical expansion [121, 124]. Since \hbar is the unit of the angular momentum, the expansion in \hbar is also an expansion in spin. Up to $\mathcal{O}(\hbar)$, general solutions for the Wigner function have been obtained, in both the massless [62, 66, 69, 125] and the massive [74–77] case. In the massless case, the Chiral Kinetic Theory can be obtained from the Wigner-function approach [66–69]. The chiral effects are successfully reproduced at the order $\mathcal{O}(\hbar)$ [62, 66, 68, 69, 125]. In the massive case, kinetic equations are obtained which agree with the relativistic Boltzmann-Vlasov equation and the Bargmann-Michel-Telegdi (BMT) equation in the classical limit [74–77] and recover the Chiral Kinetic Theory in the massless limit [75–77].

On the other hand, the equal-time Wigner function can be derived from the covariant one by integrating out the energy p^0 [88, 122, 126]. The equal-time formula only depends on $\{t, \mathbf{x}, \mathbf{p}\}$, thus is suitable for time-dependent problems, such as out-of-equilibrium physics [127] and pair-production [89–92, 96].

C. System of units, notations and conventions

In this subsection, we declare the system of units, notations and conventions we will use throughout this thesis. In the International System of Units (SI), the reduced Planck's constant, the speed of light, and the electron volt are

$$[\hbar] = [kg \cdot m^2 \cdot s^{-1}], \quad [c] = [m \cdot s^{-1}], \quad [eV] = [kg \cdot m^2 \cdot s^{-2}]. \quad (1.6)$$

Here the square bracket $[\dots]$ represents the unit or the dimension, and m , kg , s are meter, kilogram and second, the unit of mass, length, and time, respectively. On the other hand, from Eq. (1.6), we can express the units of mass, length and time in terms of \hbar , c and eV,

$$[kg] = [c^{-2} \cdot eV], \quad [m] = [\hbar \cdot c \cdot eV^{-1}], \quad [s] = [\hbar^2 \cdot c \cdot eV^{-2}]. \quad (1.7)$$

Natural units are convenient to us, in which the units of physical quantities are selected as physical constants. For example, the speed of light is the natural unit of speed. In natural units, the values of the reduced Planck's constant \hbar and the speed of light c are set to 1, while the unit of energy is set to eV. In SI units, the unit of any physical quantity can be expressed as $m^a \cdot s^b \cdot kg^c$, with some rational numbers a , b , c . Then it can be rewritten as eV^{c-a-2b} in natural units, where \hbar and c are hidden because they are set to 1. If we consider, for example, a charged particle in a static electric field, the unit of electric force is given by

$$[qE] = [kg \cdot m \cdot s^{-2}] = [\hbar^{-1} \cdot c^{-1} \cdot eV^2]. \quad (1.8)$$

Thus in natural units, qE has the unit of energy squared, eV^2 . Here q denotes the charge of the particles considered. As a convention, q always comes in front of the gauge potential \mathbb{A}^μ , electric and magnetic fields E^μ , B^μ and the field strength tensor $F^{\mu\nu}$. In the thesis, we only consider charged fermions of one species, where the charge of fermions is $+1$ (and -1 for anti-fermions). Thus we can absorb the charge q in the definition of the electromagnetic field tensor $F^{\mu\nu}$ and the gauge potential \mathbb{A}^μ .

Since the spin has the unit of \hbar , we will use \hbar as a parameter to label its quantum nature. In the calculation of the Wigner function, we will recover \hbar in Sec. II and treat \hbar as an expansion parameter in Sec. IV. This method is already known as the semi-classical expansion [121, 124], which at leading order in \hbar can reproduce the classical results.

Throughout this thesis, we assume the mass m of a particle is constant. We use natural units $\hbar = c = k_B = 1$ but show \hbar explicitly in Sec. IV since \hbar is used as a parameter for power-counting

in that section. We work in Minkowski space with the metric tensor $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and the Levi-Civita symbol $\epsilon^{0123} = -\epsilon_{0123} = 1$. We use bold symbols such as \mathbf{p} to represent 3-dimensional vectors. The electromagnetic potential is denoted by \mathbb{A}^μ in order to distinguish it from the axial-vector component of the Wigner function. The electric charge q is set to $+1$ for fermions and -1 for anti-fermions and thus q will be hidden in this thesis. The operators in the Dirac theory, such as the Dirac field operator, the Hamiltonian operator, etc., are denoted with hat. Meanwhile, operators constructed by the space-time derivative ∂_x^μ and the momentum derivative ∂_p^μ are denoted without hat. The components of a Lorentz vector are labeled by $\{0, x, y, z\}$, for example, the 4-momentum is denoted as $p^\mu = (p^0, p^x, p^y, p^z)^T$. Sometime we use the transpose operation to change a 4-momentum in the line vector into a column vector. In this thesis we also used M^{-1} to denote the inverse of the matrix M . The unit matrix is denoted by \mathbb{I}_n , with n being the dimension of the matrix.

D. Outline

In this thesis we will first give an overview of the Wigner-function method in Sec. II. This section includes the definition of the covariant Wigner function and its kinetic equations. Two kinds of kinetic equations are obtained, one of which is the analog of the Dirac equation and the other one is the analog of the Klein-Gordon equation. These equations are differential equations of first order and second order in time, respectively. The Wigner function, as well as its kinetic equations, are then decomposed in terms of the generators of the Clifford algebra, i.e., the gamma matrices $\{\mathbb{I}_4, i\gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \frac{1}{2}\sigma^{\mu\nu}\}$. The equal-time Wigner function is also introduced in Sec. II. In Sec. III we focus on several analytically solvable cases:

1. Free fermions, without electromagnetic field and without chiral imbalance.
2. Without electromagnetic field, but a chiral imbalance is introduced in the Dirac equation as a self-energy correction.
3. A constant magnetic field, otherwise as in case 2.
4. A constant electric field, otherwise as in case 1.
5. Constant electromagnetic fields added to case 1, where the electric and magnetic fields are assumed to be parallel to each other.

In the cases 1, 2, and 3, the Dirac equation has analytical single-particle solutions, which are then used to derive the Wigner functions. Note that the magnetic field and the additional self-energy term break the Lorentz symmetry. For example, if we take a Lorentz boost along the direction perpendicular to the magnetic field, we find that an electric field automatically appears in the new frame. That is, the magnetic field itself is not Lorentz covariant. Meanwhile, chemical potentials also breaks the Lorentz covariance. So for cases 2 and 3, we are working in specific frames in which the chemical potential μ is the conjugate parameter for the net particle number and μ_5 is the one for the axial charge. In the presence of an electric field, i.e., cases 4 and 5, the existing particles will be accelerated and new fermion/anti-fermion pairs will be excited from the vacuum. Thus systems in an electric field are evolving over time and the equal-time Wigner function is used for these systems. In Sec. III we analytically solve the Wigner function for the case of a constant electric field. Meanwhile in subsection III D 3 we numerically calculate the solution for a Sauter-type electric field $E(t) = E_0 \text{sech}^2(t/\tau)$. A more general field configuration is considered in Sec. IV using the method of semi-classical expansion. Solutions are obtained up to order \hbar for both massless and massive particles. In Sec. V we relate the Wigner function with several physical quantities such as the net fermion current, spin polarization, energy-momentum tensor, etc.. The analytical results from Sec. III are used under a thermal-equilibrium assumption. The physical quantities show a dependence with respect to the thermodynamical variables and the magnetic field. Different methods are used and compared with each other, which show both coincidences and differences. pair-production is also discussed in Sec. V. The results show that the magnetic field enhances the pair-production rate, while the thermal background suppresses it. A summary and outlook of this thesis are given in Sec. VI. In App. A we listed the gamma matrices and their properties. Other useful auxiliary functions are discussed in App. B, which appear when dealing with the Wigner function in constant electromagnetic fields. In App. C we present the standard wave-packet description for a quantum particle, which will be used when solving the Wigner function. The relation between the pair-production rate and the Wigner function is derived from a quantum field description in App. D.

II. OVERVIEW OF WIGNER FUNCTION

A. Definition of Wigner function

In quantum mechanics, the space-time position x^μ and the 4-momentum p^μ cannot be specified simultaneously for a single particle, which is a straightforward consequence of the Heisenberg's uncertainty principle [117–119]. Thus, the classical particle distribution function $f(t, \mathbf{x}, \mathbf{p})$ is not well-defined in the quantum case. In order to find a proper way to describe quantum kinetics, we first consider a system of two particles, whose space-time coordinate operators are \hat{x}_1^μ and \hat{x}_2^μ and 4-momentum operators are \hat{p}_1^μ and \hat{p}_2^μ , respectively. The uncertainty principle gives the following relations

$$[\hat{x}_a^\mu, \hat{p}_b^\nu] = -i\hbar g^{\mu\nu} \delta_{ab}, \quad [\hat{x}_a^\mu, \hat{x}_b^\nu] = 0, \quad [\hat{p}_a^\mu, \hat{p}_b^\nu] = 0. \quad (2.1)$$

where $a, b = 1, 2$. We now define the center position and the relative momentum as

$$\hat{x}^\mu \equiv \frac{1}{2}(\hat{x}_1^\mu + \hat{x}_2^\mu), \quad \hat{p}^\mu \equiv \frac{1}{2}(\hat{p}_1^\mu - \hat{p}_2^\mu). \quad (2.2)$$

Then using the commutators in Eq. (2.1) we can check that these two quantities are commutable with each other

$$[\hat{x}^\mu, \hat{p}^\nu] = 0, \quad [\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{p}^\mu, \hat{p}^\nu] = 0. \quad (2.3)$$

Thus according to the uncertainty principle, even if we can not determine the position and momentum simultaneously for each particle, the center position and relative momentum can specified simultaneously. Meanwhile, the relative position and the total momentum are defined as

$$\hat{y}^\mu \equiv \hat{x}_1^\mu - \hat{x}_2^\mu, \quad \hat{q}^\mu \equiv \hat{p}_1^\mu + \hat{p}_2^\mu. \quad (2.4)$$

These operators also commute with each other

$$[\hat{y}^\mu, \hat{q}^\nu] = 0, \quad [\hat{y}^\mu, \hat{y}^\nu] = 0, \quad [\hat{q}^\mu, \hat{q}^\nu] = 0, \quad (2.5)$$

and thus they are not constrained by the uncertainty principle. On the other hand, we have the following commutators

$$[\hat{y}^\mu, \hat{p}^\nu] = [\hat{x}^\mu, \hat{q}^\nu] = -i\hbar g^{\mu\nu} \delta_{ab}, \quad (2.6)$$

which indicates that the relative momentum is the conjugate variable of the relative position, while the total momentum is the conjugate variable of the center position.

The Wigner operator for a free Dirac field is defined from the two-point correlation function [120],

$$\hat{\mathcal{W}}_{\text{free}}(x, p) \equiv \int \frac{d^4 y}{(2\pi)^4} \exp(-iy^\mu p_\mu) \hat{\psi}\left(x + \frac{y}{2}\right) \otimes \hat{\psi}\left(x - \frac{y}{2}\right), \quad (2.7)$$

where the operator \otimes represents the tensor product and $\hat{\psi}$ is the Dirac field operator. In this definition, the two field operators are defined at two different space-time points, $x^\mu \pm \frac{y^\mu}{2}$, where x^μ the center position and y^μ the relative position. A Fourier transform is taken with respect to y^μ , whose conjugate momentum p_μ can be identified as the relative momentum of two fields in classical mechanics. According to discussion in previous paragraph, p^μ and x^μ can be determined simultaneously. Thus the Wigner operator is a well-defined quasi-distribution in phase-space $\{x^\mu, p^\mu\}$. We will show in Sec. III that the Wigner function is related to the classical distribution $f(t, \mathbf{x}, \mathbf{p})$ at leading order in spatial gradients.

Note that the Wigner operator defined in Eq. (2.7) is not gauge-invariant. Under a local gauge transformation $\theta(x)$, the field operators transform as follows,

$$\hat{\psi}\left(x - \frac{y}{2}\right) \rightarrow e^{i\theta(x-y/2)} \hat{\psi}\left(x - \frac{y}{2}\right), \quad \hat{\psi}\left(x + \frac{y}{2}\right) \rightarrow e^{-i\theta(x+y/2)} \hat{\psi}\left(x + \frac{y}{2}\right), \quad (2.8)$$

and the Wigner operator transforms as

$$\hat{\mathcal{W}}_{\text{free}}(x, p) \rightarrow \exp[i\theta(x-y/2) - i\theta(x+y/2)] \hat{\mathcal{W}}_{\text{free}}(x, p). \quad (2.9)$$

The exponential factor is not 1 for general local transformation with $\theta(x-y/2) \neq \theta(x+y/2)$. Thus the Wigner operator in Eq. (2.7) is not gauge-invariant. In order to define a gauge-invariant quantity, we first express the Dirac field at the position $x - \frac{y}{2}$ in a Taylor expansion as follows,

$$\hat{\psi}\left(x - \frac{y}{2}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}y_\mu \partial_x^\mu\right)^n \hat{\psi}(x) = \exp\left(-\frac{1}{2}y_\mu \partial_x^\mu\right) \hat{\psi}(x). \quad (2.10)$$

In the presence of an electromagnetic field, we replace the ordinary derivative ∂_x^μ by the covariant one $D_x^\mu = \partial_x^\mu + i\mathbb{A}^\mu$ so that the gauge invariance is automatically ensured. Here \mathbb{A}^μ is the four-vector potential of the electromagnetic field. Inserting the field operator into Eq. (2.7), we define the following Wigner operator,

$$\hat{\mathcal{W}}(x, p) \equiv \int \frac{d^4 x}{(2\pi)^4} \exp(-iy^\mu p_\mu) \left[\exp\left(\frac{1}{2}y_\mu D_x^\mu\right) \hat{\psi}(x) \right]^\dagger \gamma^0 \otimes \left[\exp\left(-\frac{1}{2}y_\mu D_x^\mu\right) \hat{\psi}(x) \right], \quad (2.11)$$

which is covariant and gauge-invariant.

Now we define the gauge link between two space-time points

$$U(x_2, x_1) = \exp\left[-i \int_{x_1}^{x_2} dx^\mu \mathbb{A}_\mu(x)\right], \quad (2.12)$$

where the path of integration is taken as a straight line between two points. Then we obtain the gauge link between $x^\mu - \frac{1}{2}y^\mu$ and $x^\mu + \frac{1}{2}y^\mu$,

$$U\left(x + \frac{y}{2}, x - \frac{y}{2}\right) = \exp\left[-iy^\mu \int_{-1/2}^{1/2} ds \mathbb{A}_\mu(x + sy)\right], \quad (2.13)$$

We can prove

$$e^{\frac{1}{2}y_\mu D_x^\mu} = U\left(x, x + \frac{y}{2}\right) e^{\frac{1}{2}y_\mu \partial_x^\mu}, \quad (2.14)$$

with the help of two auxiliary functions

$$\begin{aligned} f(s) &\equiv e^{sy_\mu D_x^\mu}, \\ g(s) &\equiv U\left(x, x + sy\right) e^{sy_\mu \partial_x^\mu}. \end{aligned} \quad (2.15)$$

Their derivatives with respect to the parameter s are

$$\begin{aligned} \frac{d}{ds}f(s) &= f(s)y_\nu D_x^\nu, \\ \frac{d}{ds}g(s) &= U\left(x, x + sy\right) e^{sy_\mu \partial_x^\mu} y_\nu \partial_x^\nu + \left[\frac{d}{ds}U\left(x, x + sy\right)\right] e^{sy_\mu \partial_x^\mu} \\ &= U\left(x, x + sy\right) \left\{ e^{sy_\mu \partial_x^\mu} y_\nu \partial_x^\nu + [iy^\nu \mathbb{A}_\nu(x + sy)] e^{sy_\mu \partial_x^\mu} \right\} \\ &= g(s)y_\nu D_x^\nu, \end{aligned} \quad (2.16)$$

which means the two functions defined in Eq. (2.15) satisfy the same differential equation. Furthermore they also share the same value at point $s = 0$, we conclude that they are equivalent for arbitrary s . Equation (2.14) is then proved by taking $s = \frac{1}{2}$. Substituting Eq. (2.14) into Eq. (2.11), we obtain another form of the Wigner operator

$$\hat{\mathcal{W}}(x, p) = \int \frac{d^4y}{(2\pi)^4} \exp(-iy^\mu p_\mu) U\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \hat{\psi}\left(x + \frac{y}{2}\right) \otimes \hat{\psi}\left(x - \frac{y}{2}\right). \quad (2.17)$$

The Wigner function is then derived by taking the expectation value of the Wigner operator on the physical state of the system $|\Omega\rangle$

$$W(x, p) = \int \frac{d^4y}{(2\pi)^4} \exp(-iy^\mu p_\mu) U\left(x + \frac{y}{2}, x - \frac{y}{2}\right) \langle \Omega | \hat{\psi}\left(x + \frac{y}{2}\right) \otimes \hat{\psi}\left(x - \frac{y}{2}\right) | \Omega \rangle. \quad (2.18)$$

Note that here we have taken the gauge link out of the expectation value. We treat the gauge field (i.e., the electromagnetic field in this thesis) as a classical C-number field, while the fermionic field as a quantum field. This is known as the Hartree approximation, which is valid when higher-loop corrections are negligible or the field is large enough.

The Wigner function in Eq. (2.18) is not Hermitian but it transforms as

$$W^\dagger = \gamma^0 W \gamma^0, \quad (2.19)$$

which is the same as the property (A6) of the generators of the Clifford algebra $\Gamma_i = \{\mathbb{I}_4, i\gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \frac{1}{2}\sigma^{\mu\nu}\}$, where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Thus the Wigner function can be expanded in terms of Γ_i ,

$$W(x, p) = \frac{1}{4} \left(\mathbb{I}_4 \mathcal{F} + i\gamma^5 \mathcal{P} + \gamma^\mu \mathcal{V}_\mu + \gamma^5 \gamma^\mu \mathcal{A}_\mu + \frac{1}{2} \sigma^{\mu\nu} \mathcal{S}_{\mu\nu} \right). \quad (2.20)$$

The Wigner function has 16 independent components because it is a complex 4×4 matrix and Eq. (2.19) provides 16 constraints, which correspond to Γ_i . Inserting the decomposition (2.20) into Eq. (2.19) and using the property (A6) of Γ_i , one can prove that all the coefficients in Eq. (2.20) are real functions. The expansion coefficients can be derived from the Wigner function by multiplying the corresponding generators and then taking the trace,

$$\begin{aligned} \mathcal{F} &= \text{Tr}(W), \\ \mathcal{P} &= -\text{Tr}(i\gamma^5 W), \\ \mathcal{V}^\mu &= \text{Tr}(\gamma^\mu W), \\ \mathcal{A}^\mu &= \text{Tr}(\gamma^\mu \gamma^5 W), \\ \mathcal{S}^{\mu\nu} &= \text{Tr}(\sigma^{\mu\nu} W). \end{aligned} \quad (2.21)$$

The tensor component is anti-symmetric and has 6 independent members. We can equivalently introduce two vector functions

$$\mathbf{T} = \frac{1}{2} \mathbf{e}_i (\mathcal{S}^{i0} - \mathcal{S}^{0i}), \quad \mathbf{S} = \frac{1}{2} \epsilon_{ijk} \mathbf{e}_i \mathcal{S}_{jk}. \quad (2.22)$$

By Lorentz and parity transformations, \mathcal{F} , \mathcal{P} , \mathcal{V}^μ , \mathcal{A}^μ , $\mathcal{S}^{\mu\nu}$ are the scalar, pseudo-scalar, vector, axial-vector, and tensor, respectively. The properties under charge conjugation, parity and time reversal are shown in Tab. I. It has been shown in Ref. [121] that some components in Eq. (2.20) have obvious physical meaning. For example, the vector component is the fermion number current density and the axial-vector component is the polarization density. The physical meanings are listed in Tab. II. We will have a more detailed discussion in Sec. (V).

B. Equations for the Wigner function

In this subsection we will derive kinetic equations for the Wigner function. The Wigner function is defined in Eq. (2.18) for spin-1/2 fermions, whose kinetic equation will be derived from the Dirac

	$\mathcal{F}(t, \mathbf{x})$	$\mathcal{P}(t, \mathbf{x})$	$\mathcal{V}_\mu(t, \mathbf{x})$	$\mathcal{A}_\mu(t, \mathbf{x})$	$\mathcal{S}_{\mu\nu}(t, \mathbf{x})$
C	$\mathcal{F}(t, \mathbf{x})$	$\mathcal{P}(t, \mathbf{x})$	$-\mathcal{V}_\mu(t, \mathbf{x})$	$\mathcal{A}_\mu(t, \mathbf{x})$	$-\mathcal{S}_{\mu\nu}(t, \mathbf{x})$
P	$\mathcal{F}(t, -\mathbf{x})$	$-\mathcal{P}(t, -\mathbf{x})$	$\mathcal{V}_\mu(t, -\mathbf{x})$	$-\mathcal{A}_\mu(t, -\mathbf{x})$	$\mathcal{S}_{\mu\nu}(t, -\mathbf{x})$
T	$\mathcal{F}(-t, \mathbf{x})$	$-\mathcal{P}(-t, \mathbf{x})$	$\mathcal{V}_\mu(-t, \mathbf{x})$	$\mathcal{A}_\mu(-t, \mathbf{x})$	$-\mathcal{S}_{\mu\nu}(-t, \mathbf{x})$
CPT	$\mathcal{F}(-t, -\mathbf{x})$	$\mathcal{P}(-t, -\mathbf{x})$	$-\mathcal{V}_\mu(-t, -\mathbf{x})$	$-\mathcal{A}_\mu(-t, -\mathbf{x})$	$\mathcal{S}_{\mu\nu}(-t, -\mathbf{x})$

Table I: Transformation properties of the components of the Wigner function under charge conjugation (C), parity (P), and time reversal (T). The dependence on the momentum p^μ is suppressed here.

Component	Physical meaning (distribution in phase space)
\mathcal{F}	Mass
\mathcal{P}	Pseudoscalar condensate
\mathcal{V}^μ	Net fermion current
\mathcal{A}^μ	Polarization (or spin current, or axial-charge current)
\mathcal{T}	Electric dipole-moment
\mathcal{S}	Magnetic dipole-moment

Table II: Physical meaning of the components of the Wigner function.

equation and its conjugate,

$$\begin{aligned}
[i\gamma^\mu(\vec{\partial}_{x\mu} + i\mathbb{A}_\mu) - m\mathbb{I}_4]\psi &= 0, \\
\bar{\psi}[i\gamma^\mu(\overleftarrow{\partial}_{x\mu} - i\mathbb{A}_\mu) + m\mathbb{I}_4] &= 0.
\end{aligned} \tag{2.23}$$

Note that we have adopted the Hartree approximation, in which the electromagnetic field is assumed to be a classical field instead of a quantum one. In these equations, ψ and $\bar{\psi}$ represent either the field or the field operators after second quantization.

1. Dirac form

In order to derive a Dirac form kinetic equation with the first order in the time derivative, we first act with $i\gamma^\sigma \partial_{x\sigma}$ on Eq. (2.18),

$$\begin{aligned}
& i\gamma^\sigma \partial_{x\sigma} W(x, p) \\
&= i\gamma^\sigma \int \frac{d^4 y}{(2\pi)^4} \exp(-iy^\mu p_\mu) \left[\partial_{x\sigma} U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \right] \left\langle \Omega \left| \hat{\psi} \left(x + \frac{y}{2} \right) \otimes \hat{\psi} \left(x - \frac{y}{2} \right) \right| \Omega \right\rangle \\
&+ \int \frac{d^4 y}{(2\pi)^4} \exp(-iy^\mu p_\mu) U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \left\langle \Omega \left| \left[\partial_{x\sigma} \hat{\psi} \left(x + \frac{y}{2} \right) \right] \otimes i\gamma^\sigma \hat{\psi} \left(x - \frac{y}{2} \right) \right| \Omega \right\rangle \\
&+ \int \frac{d^4 y}{(2\pi)^4} \exp(-iy^\mu p_\mu) U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \left\langle \Omega \left| \hat{\psi} \left(x + \frac{y}{2} \right) \otimes \left[i\gamma^\sigma \partial_{x\sigma} \hat{\psi} \left(x - \frac{y}{2} \right) \right] \right| \Omega \right\rangle.
\end{aligned} \tag{2.24}$$

In the second and third lines we have used the following property of the tensor product to put the gamma matrix into the expectation value,

$$A(B \otimes C) = B \otimes (AC). \tag{2.25}$$

With this property, the operator $i\gamma^\sigma \partial_{x\sigma}$ in the last line of Eq. (2.24) directly acts on $\psi \left(x - \frac{y}{2} \right)$ and thus the Dirac equation can be used for further simplification. On the other hand, the conjugate of the Dirac equation in Eq. (2.23) cannot be directly used to simplify the second line of Eq. (2.24) because γ^σ does not come with $\partial_{x\sigma} \bar{\psi} \left(x + \frac{y}{2} \right)$. However, since the field operator $\hat{\psi} \left(x + \frac{y}{2} \right)$ depends on $x^\mu + \frac{1}{2}y^\mu$, we can replace the derivative with respect to x^μ by that with respect to y^μ ,

$$\partial_{x\sigma} \hat{\psi} \left(x + \frac{y}{2} \right) = 2\partial_{y\sigma} \hat{\psi} \left(x + \frac{y}{2} \right). \tag{2.26}$$

Inserting this into Eq. (2.24) and integrating by parts, we obtain

$$\begin{aligned}
i\gamma^\sigma \partial_{x\sigma} W(x, p) &= -2\gamma^\sigma p_\sigma W(x, p) \\
&+ i\gamma^\sigma \int \frac{d^4 y}{(2\pi)^4} \exp(-iy^\mu p_\mu) \left[(\partial_{x\sigma} - 2\partial_{y\sigma}) U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \right] \\
&\quad \times \left\langle \Omega \left| \hat{\psi} \left(x + \frac{y}{2} \right) \otimes \hat{\psi} \left(x - \frac{y}{2} \right) \right| \Omega \right\rangle \\
&+ 2 \int \frac{d^4 y}{(2\pi)^4} \exp(-iy^\mu p_\mu) U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \\
&\quad \times \left\langle \Omega \left| \hat{\psi} \left(x + \frac{y}{2} \right) \otimes \left[i\gamma^\sigma \partial_{x\sigma} \hat{\psi} \left(x - \frac{y}{2} \right) \right] \right| \Omega \right\rangle.
\end{aligned} \tag{2.27}$$

Here we have dropped the boundary term which is assumed to vanish if we take an infinitely large volume. Now the Dirac equation (2.23) can be used to further simplify the last term in Eq. (2.27),

$$\begin{aligned}
i\gamma^\sigma \partial_{x\sigma} W(x, p) &= -2\gamma^\sigma p_\sigma W(x, p) + 2m\mathbb{I}_4 W(x, p) \\
&+ \gamma^\sigma \int \frac{d^4 y}{(2\pi)^4} \exp(-iy^\mu p_\mu) \left\langle \Omega \left| \hat{\psi} \left(x + \frac{y}{2} \right) \otimes \hat{\psi} \left(x - \frac{y}{2} \right) \right| \Omega \right\rangle \\
&\times \left[i(\partial_{x\sigma} - 2\partial_{y\sigma}) + 2\mathbb{A}_\sigma \left(x - \frac{y}{2} \right) \right] U \left(x + \frac{y}{2}, x - \frac{y}{2} \right). \tag{2.28}
\end{aligned}$$

Using the definition of the gauge link in Eq. (2.13), we can explicitly calculate the derivative of the gauge link,

$$\begin{aligned}
&\left[i(\partial_{x\sigma} - 2\partial_{y\sigma}) + 2\mathbb{A}_\sigma \left(x - \frac{y}{2} \right) \right] U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \\
&= U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \left[y^\mu \int_{-1/2}^{1/2} ds (1-2s) F_{\sigma\mu}(x+sy) \right]. \tag{2.29}
\end{aligned}$$

Inserting Eq. (2.29) into Eq. (2.28) we can obtain

$$\begin{aligned}
i\gamma^\sigma \partial_{x\sigma} W(x, p) &= -2(\gamma^\sigma p_\sigma - m\mathbb{I}_4) W(x, p) \\
&+ \gamma^\sigma \int \frac{d^4 y}{(2\pi)^4} \exp(-iy^\mu p_\mu) y^\rho \int_{-1/2}^{1/2} ds (1-2s) F_{\sigma\rho}(x+sy) \\
&\times U \left(x + \frac{y}{2}, x - \frac{y}{2} \right) \left\langle \Omega \left| \hat{\psi} \left(x + \frac{y}{2} \right) \otimes \hat{\psi} \left(x - \frac{y}{2} \right) \right| \Omega \right\rangle. \tag{2.30}
\end{aligned}$$

Due to the phase factor $\exp(-iy^\mu p_\mu)$, the relative coordinate y^ρ can be replaced by the momentum derivative $i\partial_p^\rho$. Furthermore, the integral over the field tensor can be calculated using a Taylor expansion

$$\begin{aligned}
\int_{-1/2}^{1/2} ds (1-2s) F_{\sigma\rho}(x+sy) &= \sum_{n=0}^{\infty} \frac{1}{n!} (y^\nu \partial_{x\nu})^n F_{\sigma\rho}(x) \int_{-1/2}^{1/2} ds (1-2s) s^n \\
&= \sum_{n=0}^{\infty} \frac{1 + (-1)^n (3+2n)}{(n+2)! 2^{n+1}} (y^\nu \partial_{x\nu})^n F_{\sigma\rho}(x) \\
&= \sum_{n=0}^{\infty} \frac{1 + (-1)^n (3+2n)}{(n+2)! 2^{n+1}} (i\partial_p^\nu \partial_{x\nu})^n F_{\sigma\rho}(x). \tag{2.31}
\end{aligned}$$

We now separate the even and odd terms in the series expansion, so the above formula can be written in a more concise form,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1 + (-1)^n (3+2n)}{(n+2)! 2^{n+1}} (i\partial_p^\nu \partial_{x\nu})^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! 2^{2n}} (\partial_p^\nu \partial_{x\nu})^{2n} - i \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)(2n+1)! 2^{2n+1}} (\partial_p^\nu \partial_{x\nu})^{2n+1} \\
&= j_0 \left(\frac{1}{2} \partial_p^\nu \partial_{x\nu} \right) - i j_1 \left(\frac{1}{2} \partial_p^\nu \partial_{x\nu} \right), \tag{2.32}
\end{aligned}$$

where the following spherical Bessel functions were used

$$j_0(x) = \frac{\sin x}{x}, \quad j_1(x) = \frac{\sin x - x \cos x}{x^2}. \quad (2.33)$$

Thus we finally obtain the following kinetic equation

$$\begin{aligned} i\gamma^\sigma \partial_{x\sigma} W(x, p) &= -2(\gamma^\sigma p_\sigma - m\mathbb{I}_4)W(x, p) \\ &+ i\gamma^\sigma \left[j_0 \left(\frac{1}{2} \partial_p^\nu \partial_{x\nu} \right) F_{\sigma\rho}(x) - i j_1 \left(\frac{1}{2} \partial_p^\nu \partial_{x\nu} \right) F_{\sigma\rho}(x) \right] \partial_p^\rho W(x, p). \end{aligned} \quad (2.34)$$

Defining the following operators

$$\begin{aligned} K^\mu &\equiv \Pi^\mu + \frac{i}{2} \nabla^\mu, \\ \Pi^\mu &\equiv p^\mu - \frac{1}{2} j_1(\Delta) F^{\mu\nu}(x) \partial_{p\nu}, \\ \nabla^\mu &\equiv \partial_x^\mu - j_0(\Delta) F^{\mu\nu}(x) \partial_{p\nu}, \end{aligned} \quad (2.35)$$

with $\Delta \equiv \frac{1}{2} \partial_p^\nu \partial_{x\nu}$, the kinetic equation for the Wigner function can be written in a compact form,

$$(\gamma^\mu K_\mu - m\mathbb{I}_4)W(x, p) = 0, \quad (2.36)$$

We note that the derivative $\partial_{x\nu}$ in the operator Δ only acts on $F^{\mu\nu}(x)$ but not on the Wigner function. The operators Π^μ and ∇^μ are generalized 4-momentum and spatial-derivative operators, respectively, which can be reduced to the ordinary ones without the electromagnetic field. In the limit of vanishing electromagnetic field, the equation for the Wigner function in Eq. (2.36) takes the same form as the Dirac equation. Note that Eq. (2.36) is first order in space-time derivatives. Thus, in the remainder part of the thesis, we call Eq. (2.36) the Dirac-form kinetic equation for the Wigner function.

Note that in Sec. IV we will expand the Wigner function in powers of the Planck's constant. To this end, we will show \hbar explicitly. Recalling the discussions about natural units in Sec. IC, the product of p^μ and x_μ has the unit of \hbar . The field strength $F^{\mu\nu}$ has the unit $[kg \cdot m \cdot s^{-2}]$ in the SI units, thus $F^{\mu\nu}(x) \partial_{p\nu}$ has the unit $[s^{-1}]$. In order to make sure K^μ has the unit of momentum and Δ is unit-less, we recover \hbar as follows,

$$\begin{aligned} K^\mu &\equiv \Pi^\mu + \frac{i\hbar}{2} \nabla^\mu, \\ \Pi^\mu &\equiv p^\mu - \frac{\hbar}{2} j_1(\Delta) F^{\mu\nu}(x) \partial_{p\nu}, \\ \nabla^\mu &\equiv \partial_x^\mu - j_0(\Delta) F^{\mu\nu}(x) \partial_{p\nu}, \end{aligned} \quad (2.37)$$

with the operator $\Delta \equiv \frac{\hbar}{2} \partial_p^\nu \partial_{x\nu}$.

2. Klein-Gordon form

In the previous part of this subsection, we have derived the Dirac-form kinetic equation (2.36), which is of first order in space-time derivatives. A second-order equation can be obtained by multiplying the Dirac-form equation (2.36) with $\gamma^\mu K_\mu + m$ and using the following relation,

$$\gamma^\mu \gamma^\nu = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} + \frac{1}{2} [\gamma^\mu, \gamma^\nu] = g^{\mu\nu} - i\sigma^{\mu\nu}. \quad (2.38)$$

Then the new kinetic equation, which is called the Klein-Gordon-form kinetic equation, reads,

$$\left(K^\mu K_\mu - \frac{i}{2} \sigma^{\mu\nu} [K_\mu, K_\nu] - m^2 \right) W(x, p) = 0, \quad (2.39)$$

where K^μ is defined in Eq. (2.35). Since both the Wigner function and the operators are complex matrices, taking the Hermitian conjugate of Eq. (2.39) and using the property (2.19) of the Wigner function, we obtain the conjugate equation

$$[(K^\mu K_\mu)^* - m^2] W(x, p) + \frac{i}{2} [K_\mu, K_\nu]^* W(x, p) \sigma^{\mu\nu} = 0. \quad (2.40)$$

Since K^μ is complex, one can separate the real and imaginary parts of $K^\mu K_\mu$ and $[K_\mu, K_\nu]$ as $K^\mu K_\mu = \Re K^2 + i\Im K^2$ and $[K_\mu, K_\nu] = \Re K_{\mu\nu} + i\Im K_{\mu\nu}$. We will give the explicit expressions for these operators later. The Klein-Gordon-form equation (2.39) and its conjugate (2.40) now become

$$\begin{aligned} (\Re K^2 - m^2) W + i\Im K^2 W - \frac{i}{2} \Re K_{\mu\nu} \sigma^{\mu\nu} W + \frac{1}{2} \Im K_{\mu\nu} \sigma^{\mu\nu} W &= 0, \\ (\Re K^2 - m^2) W - i\Im K^2 W + \frac{i}{2} \Re K_{\mu\nu} W \sigma^{\mu\nu} + \frac{1}{2} \Im K_{\mu\nu} W \sigma^{\mu\nu} &= 0. \end{aligned} \quad (2.41)$$

Note that the above two equations should be satisfied simultaneously. Thus we can form linear combinations by taking the sum and the difference,

$$\begin{aligned} (\Re K^2 - m^2) W - \frac{i}{4} \Re K_{\mu\nu} [\sigma^{\mu\nu}, W] + \frac{1}{4} \Im K_{\mu\nu} \{\sigma^{\mu\nu}, W\} &= 0, \\ \Im K^2 W - \frac{1}{4} \Re K_{\mu\nu} \{\sigma^{\mu\nu}, W\} - \frac{i}{4} \Im K_{\mu\nu} [\sigma^{\mu\nu}, W] &= 0. \end{aligned} \quad (2.42)$$

The first equation obviously depends on the mass while the second one does not. These equations are the generalized on-shell condition and the Vlasov equation, respectively.

The operator K^μ is the linear combination of the generalized momentum operator Π^μ and the generalized space-time derivative operator ∇^μ , as defined in Eq. (2.37). Using the operators Π^μ and ∇^μ , we obtain

$$\begin{aligned} K^\mu K_\mu &= \Pi^\mu \Pi_\mu - \frac{\hbar^2}{4} \nabla^\mu \nabla_\mu + \frac{i\hbar}{2} \{\nabla^\mu, \Pi_\mu\}, \\ [K_\mu, K_\nu] &= -\frac{\hbar^2}{4} [\nabla_\mu, \nabla_\nu] + [\Pi_\mu, \Pi_\nu] + \frac{i\hbar}{2} ([\Pi_\mu, \nabla_\nu] - [\Pi_\nu, \nabla_\mu]). \end{aligned} \quad (2.43)$$

Since both Π^μ and ∇^μ are real-defined operators, one can read off the real and imaginary parts

$$\begin{aligned}
\Re K^2 &\equiv \Pi^\mu \Pi_\mu - \frac{\hbar^2}{4} \nabla^\mu \nabla_\mu, \\
\Im K^2 &\equiv \frac{\hbar}{2} \{ \nabla^\mu, \Pi_\mu \}, \\
\Re K_{\mu\nu} &\equiv -\frac{\hbar^2}{4} [\nabla_\mu, \nabla_\nu] + [\Pi_\mu, \Pi_\nu], \\
\Im K_{\mu\nu} &\equiv \frac{\hbar}{2} ([\Pi_\mu, \nabla_\nu] - [\Pi_\nu, \nabla_\mu]).
\end{aligned} \tag{2.44}$$

More detailed calculations give

$$\begin{aligned}
\Re K^2 &= p_\mu p^\mu - \frac{\hbar^2}{4} \partial_{x\mu} \partial_x^\mu - \hbar p^\mu [j_1(\Delta) F_{\mu\nu}] \partial_p^\nu \\
&\quad + \frac{\hbar^2}{2} [j_0(\Delta) F_{\mu\nu}] \partial_p^\nu \left\{ \partial_x^\mu - \frac{1}{2} [j_0(\Delta) F^{\mu\alpha}] \partial_{p,\alpha} \right\} \\
&\quad - \frac{\hbar^2}{4} \{ [j_1'(\Delta) - j_0(\Delta)] \partial_x^\mu F_{\mu\nu} \} \partial_p^\nu \\
&\quad + \frac{\hbar^2}{4} [j_1(\Delta) F_{\mu\nu}] [j_1(\Delta) F^{\mu\alpha}] \partial_p^\nu \partial_{p,\alpha}, \\
\Im K^2 &= \hbar p_\mu \{ \partial_x^\mu - [j_0(\Delta) F^{\mu\alpha}] \partial_{p,\alpha} \} \\
&\quad - \frac{\hbar^2}{2} \{ \partial_x^\mu - [j_0(\Delta) F^{\mu\alpha}] \partial_{p,\alpha} \} [j_1(\Delta) F_{\mu\nu}] \partial_p^\nu, \\
\Re K_{\mu\nu} &= -\hbar \Delta j_0(\Delta) F_{\mu\nu}(x), \\
\Im K_{\mu\nu} &= -\hbar j_0(\Delta) F_{\mu\nu} + \hbar \Delta j_1(\Delta) F_{\mu\nu}.
\end{aligned} \tag{2.45}$$

These expressions seem to be complicated but if we truncate $\mathcal{O}(\hbar^2)$ and higher order terms, these operators become

$$\begin{aligned}
\Re K^2 &= p^2 + \mathcal{O}(\hbar^2), \\
\Im K^2 &= \hbar p_\mu (\partial_x^\mu - F^{\mu\alpha} \partial_{p,\alpha}) + \mathcal{O}(\hbar^2), \\
\Re K_{\mu\nu} &= \mathcal{O}(\hbar^2), \\
\Im K_{\mu\nu} &= -\hbar F_{\mu\nu} + \mathcal{O}(\hbar^2),
\end{aligned} \tag{2.46}$$

which are quite concise and will be useful in semi-classical expansion. Inserting the truncated operators into Eq. (2.42), we obtain

$$\begin{aligned}
(p^2 - m^2) W - \frac{\hbar}{4} F_{\mu\nu} \{ \sigma^{\mu\nu}, W \} &= 0, \\
p_\mu (\partial_x^\mu - F^{\mu\alpha} \partial_{p,\alpha}) W + \frac{i}{4} F_{\mu\nu} [\sigma^{\mu\nu}, W] &= 0.
\end{aligned} \tag{2.47}$$

The first equation coincides with the on-shell condition, because if the electromagnetic field vanishes, the non-trivial solution of $W(x, p)$ should ensure $p^2 = m^2$. Here the term $-\frac{\hbar}{4} F_{\mu\nu} \{ \sigma^{\mu\nu}, W \}$ in the

first equation plays the role of a coupling between the electromagnetic field and the dipole-moment. The second equation is the Vlasov equation. Note that the first equation does not contain any information on the dynamical evolution. Up to order \hbar , the evolution of the Wigner function is determined by the second equation (2.47) while the first equation just provides a constraint.

C. Component equations

In the previous subsection we have derived the Dirac form and Klein-Gordon form of the kinetic equation. In this subsection, we decompose the Wigner function into the scalar, pseudoscalar, vector, axial-vector, and tensor parts, as shown in Eq. (2.20) and derive the equations for all 16 independent components. Inserting the decomposition (2.20) into the Dirac-form equation (2.36) and extracting the coefficients of different matrices Γ_i , we find following complex-valued equations

$$\begin{aligned}
K^\mu \mathcal{V}_\mu - m\mathcal{F} &= 0, \\
K^\mu \mathcal{A}_\mu + im\mathcal{P} &= 0, \\
K_\mu \mathcal{F} + iK^\nu \mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu &= 0, \\
iK_\mu \mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} K^\nu \mathcal{S}^{\alpha\beta} + m\mathcal{A}_\mu &= 0, \\
-iK_{[\mu} \mathcal{V}_{\nu]} - \epsilon_{\mu\nu\alpha\beta} K^\alpha \mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} &= 0,
\end{aligned} \tag{2.48}$$

where $A_{[\mu} B_{\nu]} \equiv A_\mu B_\nu - A_\nu B_\mu$. Since all components $\{\mathcal{F}, \mathcal{P}, \mathcal{V}^\mu, \mathcal{A}^\mu, \mathcal{S}^{\mu\nu}\}$ are real functions and the operator K^μ is given by Eq. (2.37), the real and imaginary parts of the above equations can be easily separated and the real parts read

$$\begin{aligned}
\Pi^\mu \mathcal{V}_\mu - m\mathcal{F} &= 0, \\
\frac{\hbar}{2} \nabla^\mu \mathcal{A}_\mu + m\mathcal{P} &= 0, \\
\Pi_\mu \mathcal{F} - \frac{\hbar}{2} \nabla^\nu \mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu &= 0, \\
-\frac{\hbar}{2} \nabla_\mu \mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} \Pi^\nu \mathcal{S}^{\alpha\beta} + m\mathcal{A}_\mu &= 0, \\
\frac{\hbar}{2} \nabla_{[\mu} \mathcal{V}_{\nu]} - \epsilon_{\mu\nu\alpha\beta} \Pi^\alpha \mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} &= 0,
\end{aligned} \tag{2.49}$$

while the imaginary parts are

$$\begin{aligned}
\hbar\nabla^\mu\mathcal{V}_\mu &= 0, \\
\Pi^\mu\mathcal{A}_\mu &= 0, \\
\frac{\hbar}{2}\nabla_\mu\mathcal{F} + \Pi^\nu\mathcal{S}_{\nu\mu} &= 0, \\
\Pi_\mu\mathcal{P} + \frac{\hbar}{4}\epsilon_{\mu\nu\alpha\beta}\nabla^\nu\mathcal{S}^{\alpha\beta} &= 0, \\
\Pi_{[\mu}\mathcal{V}_{\nu]} + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\mathcal{A}^\beta &= 0.
\end{aligned} \tag{2.50}$$

Note that the real parts of these equations explicitly depend on the particle mass while the imaginary parts do not. Equations (2.49) and (2.50) contains 32 component equations in total, but they can be simplified in massless case. If the mass is zero, then the terms proportional to the mass in Eq. (2.49) vanish, while the imaginary parts (2.50) do not change. Then the vector and axial-vector components decouple from the other components, the corresponding equations read,

$$\begin{aligned}
\hbar\nabla^\mu\mathcal{V}_\mu &= 0, & \hbar\nabla^\mu\mathcal{A}_\mu &= 0, \\
\Pi^\mu\mathcal{V}_\mu &= 0, & \Pi^\mu\mathcal{A}_\mu &= 0, \\
\Pi_{[\mu}\mathcal{V}_{\nu]} + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\mathcal{A}^\beta &= 0, & \Pi_{[\mu}\mathcal{A}_{\nu]} + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\nabla^\alpha\mathcal{V}^\beta &= 0.
\end{aligned} \tag{2.51}$$

We observe that these equations are symmetric with respect to $\mathcal{V}^\mu \rightleftharpoons \mathcal{A}^\mu$. This can be understood from another point of view: in the massless limit the chiral symmetry is restored, thus the net fermion number current \mathcal{V}^μ and the axial current \mathcal{A}^μ are related by chiral symmetry. The remaining equations are for the scalar, pseudoscalar and tensor parts

$$\begin{aligned}
\Pi_\mu\mathcal{F} - \frac{\hbar}{2}\nabla^\nu\mathcal{S}_{\nu\mu} &= 0, & \frac{\hbar}{2}\nabla_\mu\mathcal{F} + \Pi^\nu\mathcal{S}_{\nu\mu} &= 0, \\
\Pi_\mu\mathcal{P} + \frac{\hbar}{4}\epsilon_{\mu\nu\alpha\beta}\nabla^\nu\mathcal{S}^{\alpha\beta} &= 0, & -\frac{\hbar}{2}\nabla_\mu\mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\Pi^\nu\mathcal{S}^{\alpha\beta} &= 0.
\end{aligned} \tag{2.52}$$

On the other hand, we can derive the on-shell conditions and Vlasov equations from Eq. (2.42). Combining commutators and anti-commutators between gamma matrices and $\sigma^{\mu\nu}$, which is listed in Eq. (A7), with the decomposition (2.20), we have

$$\begin{aligned}
[\sigma^{\mu\nu}, W(x, p)] &= \frac{i}{2} \left\{ \gamma^{[\mu}\mathcal{V}^{\nu]} + \gamma^5\gamma^{[\mu}\mathcal{A}^{\nu]} + \frac{1}{2} \left(g^{\rho[\mu}\sigma^{\nu]\sigma} - g^{\sigma[\mu}\sigma^{\nu]\rho} \right) \mathcal{S}_{\sigma\rho} \right\}, \\
\{\sigma^{\mu\nu}, W(x, p)\} &= \frac{1}{2} \left\{ \sigma^{\mu\nu}\mathcal{F} - \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}\mathcal{P} + \epsilon^{\mu\nu\alpha\beta}\gamma^5\gamma_\beta\mathcal{V}_\alpha + \epsilon^{\mu\nu\alpha\beta}\gamma_\beta\mathcal{A}_\alpha \right. \\
&\quad \left. + \frac{1}{2} \left(g^{\mu[\sigma}g^{\rho]\nu} + i\epsilon^{\mu\nu\sigma\rho}\gamma^5 \right) \mathcal{S}_{\sigma\rho} \right\}.
\end{aligned} \tag{2.53}$$

Inserting this into Eq. (2.42) and separating different coefficients of the matrices, we obtain the on-shell conditions

$$\begin{aligned}
(\mathfrak{R}K^2 - m^2) \mathcal{F} + \frac{1}{2} \mathfrak{S}K_{\mu\nu} \mathcal{S}^{\mu\nu} &= 0, \\
(\mathfrak{R}K^2 - m^2) \mathcal{P} + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} \mathfrak{S}K^{\mu\nu} \mathcal{S}^{\alpha\beta} &= 0, \\
(\mathfrak{R}K^2 - m^2) \mathcal{V}_\mu + \mathfrak{R}K_{\mu\nu} \mathcal{V}^\nu - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathfrak{S}K^{\alpha\beta} \mathcal{A}^\nu &= 0, \\
(\mathfrak{R}K^2 - m^2) \mathcal{A}_\mu + \mathfrak{R}K_{\mu\nu} \mathcal{A}^\nu - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathfrak{S}K^{\alpha\beta} \mathcal{V}^\nu &= 0, \\
(\mathfrak{R}K^2 - m^2) \mathcal{S}_{\mu\nu} + \mathfrak{R}K^\alpha_{[\mu} \mathcal{S}_{\nu]\alpha} + \mathfrak{S}K_{\mu\nu} \mathcal{F} - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathfrak{S}K^{\alpha\beta} \mathcal{P} &= 0,
\end{aligned} \tag{2.54}$$

and the Vlasov equations

$$\begin{aligned}
\mathfrak{S}K^2 \mathcal{F} - \frac{1}{2} \mathfrak{R}K_{\mu\nu} \mathcal{S}^{\mu\nu} &= 0, \\
\mathfrak{S}K^2 \mathcal{P} - \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} \mathfrak{R}K^{\mu\nu} \mathcal{S}^{\alpha\beta} &= 0, \\
\mathfrak{S}K^2 \mathcal{V}_\mu + \mathfrak{S}K_{\mu\nu} \mathcal{V}^\nu + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathfrak{R}K^{\alpha\beta} \mathcal{A}^\nu &= 0, \\
\mathfrak{S}K^2 \mathcal{A}_\mu + \mathfrak{S}K_{\mu\nu} \mathcal{A}^\nu + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathfrak{R}K^{\alpha\beta} \mathcal{V}^\nu &= 0, \\
\mathfrak{S}K^2 \mathcal{S}_{\mu\nu} + \mathfrak{S}K^\alpha_{[\mu} \mathcal{S}_{\nu]\alpha} - \mathfrak{R}K_{\mu\nu} \mathcal{F} + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \mathfrak{R}K^{\alpha\beta} \mathcal{P} &= 0.
\end{aligned} \tag{2.55}$$

Here the operators $\mathfrak{R}K^2$, $\mathfrak{S}K^2$, $\mathfrak{R}K_{\mu\nu}$, $\mathfrak{S}K_{\mu\nu}$ are given in Eq. (2.44). We observe that in these equations, the vector and axial-vector components \mathcal{V}_μ , \mathcal{A}_μ decouple from all the other components \mathcal{F} , \mathcal{P} , $\mathcal{S}_{\mu\nu}$, in both the massive and the massless cases. We will show in the next subsection that, the scalar, pseudoscalar, and tensor components \mathcal{F} , \mathcal{P} , $\mathcal{S}_{\mu\nu}$ can be derived from the other components \mathcal{V}_μ , \mathcal{A}_μ , or vice versa. Thus the on-shell conditions and Vlasov equations for \mathcal{F} , \mathcal{P} , $\mathcal{S}_{\mu\nu}$ in Eqs. (2.54), (2.55) can be obtained using the equations for components \mathcal{V}_μ , \mathcal{A}_μ (and vice versa). This means that equations (2.49), (2.50), (2.54), and (2.55) are reducible: when solving for the Wigner function, it is not necessary to check that all these equations are fulfilled. More detailed arguments will be given in the next subsection.

Both the decomposed equations (2.49), (2.50) and the above on-shell conditions (2.54) and Vlasov equations (2.55) are derived from the Dirac-form equation for the Wigner function (2.36). The difference is that Eqs. (2.49), (2.50) are first-order equations with respect to the operators Π^μ , ∇^μ while Eqs. (2.54), (2.55) are second-order ones. Analogous to the fact that the Klein-Gordon equation for fermions can be derived from the Dirac equation, one can reproduce Eqs. (2.54), (2.55) from Eqs. (2.49), (2.50) by multiplying with the appropriate operators and then taking linear combinations. Taking the on-shell and Vlasov equations for the scalar component \mathcal{F} as an

example, from the third line of Eq. (2.49) we express \mathcal{V}_μ in terms of \mathcal{F} and $\mathcal{S}_{\nu\mu}$

$$\mathcal{V}_\mu = \frac{1}{m}\Pi_\mu\mathcal{F} - \frac{\hbar}{2m}\nabla^\nu\mathcal{S}_{\nu\mu}. \quad (2.56)$$

Multiplying the first line of Eq. (2.49) by the mass m and using the above relation, an on-shell condition for the scalar component \mathcal{F} is obtained,

$$(\Pi^\mu\Pi_\mu - m^2)\mathcal{F} - \frac{\hbar}{2}\Pi^\mu\nabla^\nu\mathcal{S}_{\nu\mu} = 0. \quad (2.57)$$

Using the anti-symmetric property of $\mathcal{S}_{\mu\nu}$, the second term can be simplified as

$$\begin{aligned} -\frac{\hbar}{2}\Pi^\mu\nabla^\nu\mathcal{S}_{\nu\mu} &= \frac{\hbar}{2}[\Pi^\mu, \nabla^\nu]\mathcal{S}_{\mu\nu} + \frac{\hbar}{2}\nabla^\nu\Pi^\mu\mathcal{S}_{\mu\nu} \\ &= \frac{\hbar}{4}([\Pi^\mu, \nabla^\nu] - [\Pi^\nu, \nabla^\mu])\mathcal{S}_{\mu\nu} + \frac{\hbar^2}{4}\nabla^\nu\nabla_\nu\mathcal{F}, \end{aligned} \quad (2.58)$$

where we have used the third line of Eq. (2.50) in the last step. Comparing with the operators listed in Eq. (2.44) we directly reproduce the first line in Eq. (2.54). On the other hand, inserting the relation (2.56) into the first line of Eq. (2.50) we obtain

$$\frac{\hbar}{2}\nabla^\mu\Pi_\mu\mathcal{F} - \frac{\hbar^2}{4}\nabla^\mu\nabla^\nu\mathcal{S}_{\nu\mu} = 0. \quad (2.59)$$

Then multiplying the third line of Eq. (2.50) with operator Π^μ gives

$$\frac{\hbar}{2}\Pi^\mu\nabla_\mu\mathcal{F} + \Pi^\mu\Pi^\nu\mathcal{S}_{\nu\mu} = 0. \quad (2.60)$$

Taking the sum of Eqs. (2.59) and (2.60) and comparing with the operators in Eq. (2.44), we reproduce the Vlasov equation for the scalar component \mathcal{F} , which is the first line of Eq. (2.55). Similar procedures can be performed for all the other components of the Wigner function, which proves that the on-shell and Vlasov equations in (2.54), (2.55) can be derived from Eqs. (2.49), (2.50) without any additional assumptions.

D. Redundancy of equations

In this section we will prove that Eqs. (2.49), (2.50) are reducible: the third and fourth lines of Eq. (2.50) can be derived from the others and \mathcal{F} , \mathcal{P} , and $\mathcal{S}_{\mu\nu}$ can be expressed by \mathcal{V}_μ and \mathcal{A}_μ , or vice versa. Thus the kinetic equations can be further simplified. Two approaches are proposed for solving the Wigner function: one approach is based on \mathcal{V}_μ and \mathcal{A}_μ , and the other approach is based on \mathcal{F} , \mathcal{P} , $\mathcal{S}_{\mu\nu}$. In Sec. IV both of these two methods are used and obtain the same results.

Now we prove that Eqs. (2.49), (2.50) are not independent from each other for the massive case.

We make the following combination using the first and last lines in Eq. (2.49) and Eq. (2.50)

$$\begin{aligned}
0 &= \frac{\hbar}{2m} \nabla_\mu (\Pi^\nu \mathcal{V}_\nu - m\mathcal{F}) - \frac{1}{2m} \Pi_\mu (\hbar \nabla^\nu \mathcal{V}_\nu) \\
&\quad - \frac{1}{m} \Pi^\nu \left(\frac{\hbar}{2} \nabla_{[\mu} \mathcal{V}_{\nu]} - \epsilon_{\mu\nu\alpha\beta} \Pi^\alpha \mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} \right) \\
&\quad + \frac{\hbar}{2m} \nabla^\nu \left(\Pi_{[\mu} \mathcal{V}_{\nu]} + \frac{\hbar}{2} \epsilon_{\mu\nu\alpha\beta} \nabla^\alpha \mathcal{A}^\beta \right). \tag{2.61}
\end{aligned}$$

The right-hand-side vanishes because all the terms inside parentheses vanish. After some calculations we obtain

$$\begin{aligned}
\frac{\hbar}{2} \nabla_\mu \mathcal{F} + \Pi^\nu \mathcal{S}_{\nu\mu} &= -\frac{\hbar}{2m} ([\Pi_\mu, \nabla_\nu] + [\Pi_\nu, \nabla_\mu]) \mathcal{V}^\nu + \frac{\hbar}{2m} [\Pi_\nu, \nabla^\nu] \mathcal{V}_\mu \\
&\quad + \frac{1}{2m} \epsilon_{\mu\nu\alpha\beta} \left([\Pi^\nu, \Pi^\alpha] + \frac{\hbar^2}{4} [\nabla^\nu, \nabla^\alpha] \right) \mathcal{A}^\beta. \tag{2.62}
\end{aligned}$$

Using Eq. (2.37) one can calculate the commutators

$$\begin{aligned}
[\Pi_\mu, \Pi_\nu] &= -\hbar \left[j_1(\Delta) + \frac{1}{2} \Delta j_1'(\Delta) \right] F_{\mu\nu}, \\
[\Pi_\mu, \nabla_\nu] &= [\Delta j_1(\Delta) - j_0(\Delta)] F_{\mu\nu}, \\
\hbar^2 [\nabla_\mu, \nabla_\nu] &= 2\hbar \Delta j_0(\Delta) F_{\mu\nu}, \tag{2.63}
\end{aligned}$$

where $j_1'(x) \equiv \frac{d}{dx} j_1(x)$. Thus we can find the following relations

$$\begin{aligned}
[\Pi_\mu, \nabla_\nu] + [\Pi_\nu, \nabla_\mu] &= [\Delta j_1(\Delta) - j_0(\Delta)] (F_{\mu\nu} + F_{\nu\mu}) = 0, \\
[\Pi_\nu, \nabla^\nu] &= [\Delta j_1(\Delta) - j_0(\Delta)] F_\nu{}^\nu = 0, \\
[\Pi_\mu, \Pi_\nu] + \frac{\hbar^2}{4} [\nabla_\mu, \nabla_\nu] &= \frac{\hbar}{2} [\Delta j_0(\Delta) - 2j_1(\Delta) - \Delta j_1'(\Delta)] F_{\mu\nu} = 0, \tag{2.64}
\end{aligned}$$

where we have used the anti-symmetry of $F_{\mu\nu}$ and the following relation for spherical Bessel functions

$$xj_0(x) - 2j_1(x) - xj_1'(x) = 0. \tag{2.65}$$

Inserting the commutators in Eq. (2.64) into Eq. (2.62), we confirm that the right-hand side vanishes and we obtain the third line of Eq. (2.50). Analogously, we construct the following equation from the second and the last lines of Eqs. (2.49), (2.50),

$$\begin{aligned}
0 &= \frac{1}{m} \Pi_\mu \left(\frac{\hbar}{2} \nabla^\nu \mathcal{A}_\nu + m\mathcal{P} \right) - \frac{\hbar}{2m} \nabla_\mu (\Pi^\nu \mathcal{A}_\nu) \\
&\quad - \frac{\hbar}{4m} \epsilon_{\mu\nu\alpha\beta} \nabla^\nu \left(\frac{\hbar}{2} \nabla^{[\alpha} \mathcal{V}^{\beta]} - \epsilon^{\alpha\beta\rho\sigma} \Pi_\rho \mathcal{A}_\sigma - m\mathcal{S}^{\alpha\beta} \right) \\
&\quad - \frac{1}{2m} \epsilon_{\mu\nu\alpha\beta} \Pi^\nu \left(\Pi^{[\alpha} \mathcal{V}^{\beta]} + \frac{\hbar}{2} \epsilon^{\alpha\beta\rho\sigma} \nabla_\rho \mathcal{A}_\sigma \right). \tag{2.66}
\end{aligned}$$

After some calculations we obtain

$$\begin{aligned} \Pi_\mu \mathcal{P} + \frac{\hbar}{4} \epsilon_{\mu\nu\alpha\beta} \nabla^\nu \mathcal{S}^{\alpha\beta} &= -\frac{\hbar}{2m} ([\Pi_\mu, \nabla_\nu] + [\Pi_\nu, \nabla_\mu]) \mathcal{A}^\nu + \frac{\hbar}{2m} [\Pi_\nu, \nabla^\nu] \mathcal{A}_\mu \\ &+ \frac{1}{4m} \epsilon_{\mu\nu\alpha\beta} \left([\Pi^\nu, \Pi^\alpha] + \frac{\hbar^2}{4} [\nabla^\nu, \nabla^\alpha] \right) \mathcal{V}^\beta. \end{aligned} \quad (2.67)$$

where the right-hand side vanishes according to (2.64). In this way we recover the fourth line in Eq. (2.50). Thus according to the above discussions, the third and fourth lines in Eq. (2.50) can be obtained from the other lines in Eqs. (2.49) and (2.50).

Now we will construct a proper way for computing the Wigner function. As discussed in the previous subsection, the dynamical evolution and constraints of the Wigner function are determined by Eqs. (2.49) and (2.50), or equivalently by Eqs. (2.54) and (2.55). However, we note that according to the first, second, and last lines of Eq. (2.49), we can express the scalar, pseudo-scalar, and tensor components in terms of \mathcal{V}^μ , \mathcal{A}^μ ,

$$\begin{aligned} \mathcal{F} &= \frac{1}{m} \Pi^\mu \mathcal{V}_\mu, \\ \mathcal{P} &= -\frac{\hbar}{2m} \nabla^\mu \mathcal{A}_\mu, \\ \mathcal{S}_{\mu\nu} &= \frac{\hbar}{2m} \nabla_{[\mu} \mathcal{V}_{\nu]} - \frac{1}{m} \epsilon_{\mu\nu\alpha\beta} \Pi^\alpha \mathcal{A}^\beta, \end{aligned} \quad (2.68)$$

Substituting \mathcal{F} , \mathcal{P} , $\mathcal{S}_{\mu\nu}$ into the third and fourth lines of Eq. (2.49) by Eq. (2.68), one obtains

$$\begin{aligned} (\Re K^2 - m^2) \mathcal{V}_\mu + \Re K_{\mu\nu} \mathcal{V}^\nu - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \Im K^{\alpha\beta} \mathcal{A}^\nu &= 0, \\ (\Re K^2 - m^2) \mathcal{A}_\mu + \Re K_{\mu\nu} \mathcal{A}^\nu - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \Im K^{\alpha\beta} \mathcal{V}^\nu &= 0, \end{aligned} \quad (2.69)$$

where the operators $\Re K^2$, $\Im K^2$, $\Re K_{\mu\nu}$, $\Im K_{\mu\nu}$ are defined in Eq. (2.44). These equations are nothing new but the vector and axial-vector components of the on-shell conditions in Eq. (2.54). The functions \mathcal{V}^μ , \mathcal{A}^μ should satisfy the equations listed in Eq. (2.50),

$$\begin{aligned} \hbar \nabla^\mu \mathcal{V}_\mu &= 0, \\ \Pi^\mu \mathcal{A}_\mu &= 0, \\ \Pi_{[\mu} \mathcal{V}_{\nu]} + \frac{\hbar}{2} \epsilon_{\mu\nu\alpha\beta} \nabla^\alpha \mathcal{A}^\beta &= 0, \end{aligned} \quad (2.70)$$

while the remaining two equations, i.e., the third and fourth lines of Eq. (2.49), are satisfied automatically according to the previous discussion. In the massless limit, we have chiral fermion whose spin is quantized along its momentum. We define spin-up and spin-down currents as

$$\mathcal{J}_\chi^\mu \equiv \frac{1}{2} (\mathcal{V}^\mu + \chi \mathcal{A}^\mu), \quad (2.71)$$

where $\chi = \pm$ labels chirality. Analogous to the massless case, we adopt the same definition (2.71) in the massive case. The corresponding on-shell conditions for \mathcal{J}_χ^μ are derived from Eq. (2.69),

$$(\Re K^2 - m^2) \mathcal{J}_\chi^\mu + \Re K^{\mu\nu} \mathcal{J}_{\chi\nu} - \frac{\chi}{2} \epsilon^{\mu\nu\alpha\beta} \Im K_{\alpha\beta} \mathcal{J}_{\chi\nu} = 0. \quad (2.72)$$

We conclude that one method for computing the Wigner function is firstly solving \mathcal{V}_μ , \mathcal{A}_μ from the on-shell equations (2.72) together with Eq. (2.70). Then the remaining components \mathcal{F} , \mathcal{P} , and $\mathcal{S}_{\mu\nu}$ are given by Eq. (2.68).

On the other hand, according to Eq. (2.49), we can prove that \mathcal{V}^μ and \mathcal{A}^μ can also be expressed by \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$. This can be done by

$$\begin{aligned} \mathcal{V}_\mu &= \frac{1}{m} \Pi_\mu \mathcal{F} + \frac{\hbar}{2m} \nabla^\nu \mathcal{S}_{\mu\nu}, \\ \mathcal{A}_\mu &= \frac{\hbar}{2m} \nabla_\mu \mathcal{P} - \frac{1}{2m} \epsilon_{\mu\nu\alpha\beta} \Pi^\nu \mathcal{S}^{\alpha\beta}. \end{aligned} \quad (2.73)$$

The functions \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$ satisfy Eq. (2.50), which gives the following constraints

$$\begin{aligned} \frac{\hbar}{2} \nabla_\mu \mathcal{F} + \Pi^\nu \mathcal{S}_{\nu\mu} &= 0, \\ \Pi_\mu \mathcal{P} + \frac{\hbar}{4} \epsilon_{\mu\nu\alpha\beta} \nabla^\nu \mathcal{S}^{\alpha\beta} &= 0. \end{aligned} \quad (2.74)$$

The other equations, i.e., the first, second, and last lines of Eq. (2.50) are automatically fulfilled. In order to prove this, we form the combinations,

$$\begin{aligned} 0 &= \frac{1}{m} \Pi^\mu \left(-\frac{\hbar}{2} \nabla_\mu \mathcal{P} + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \Pi^\nu \mathcal{S}^{\alpha\beta} + m \mathcal{A}_\mu \right) \\ &\quad + \frac{\hbar}{2m} \nabla^\mu \left(\Pi_\mu \mathcal{P} + \frac{\hbar}{4} \epsilon_{\mu\nu\alpha\beta} \nabla^\nu \mathcal{S}^{\alpha\beta} \right), \end{aligned} \quad (2.75)$$

and

$$\begin{aligned} 0 &= -\frac{\hbar}{m} \nabla^\mu \left(\Pi_\mu \mathcal{F} - \frac{\hbar}{2} \nabla^\nu \mathcal{S}_{\nu\mu} - m \mathcal{V}_\mu \right) \\ &\quad + \frac{2}{m} \Pi^\mu \left(\frac{\hbar}{2} \nabla_\mu \mathcal{F} + \Pi^\nu \mathcal{S}_{\nu\mu} \right), \end{aligned} \quad (2.76)$$

together with

$$\begin{aligned} 0 &= -\frac{1}{m} \Pi_{[\mu} \left(\Pi_{\nu]} \mathcal{F} + \frac{\hbar}{2} \nabla^\alpha \mathcal{S}_{\nu\alpha} - m \mathcal{V}_{\nu]} \right) \\ &\quad - \frac{\hbar}{2m} \nabla_{[\mu} \left(\frac{\hbar}{2} \nabla_{\nu]} \mathcal{F} - \Pi^\alpha \mathcal{S}_{\nu\alpha} \right) \\ &\quad - \frac{1}{m} \epsilon_{\mu\nu\alpha\beta} \Pi^\alpha \left(\Pi^\beta \mathcal{P} + \frac{\hbar}{4} \epsilon^{\beta\gamma\rho\sigma} \nabla_\gamma \mathcal{S}_{\rho\sigma} \right) \\ &\quad + \frac{\hbar}{2m} \epsilon_{\mu\nu\alpha\beta} \nabla^\alpha \left(-\frac{\hbar}{2} \nabla^\beta \mathcal{P} + \frac{1}{2} \epsilon^{\beta\gamma\rho\sigma} \Pi_\gamma \mathcal{S}_{\rho\sigma} + m \mathcal{A}^\beta \right). \end{aligned} \quad (2.77)$$

These equations are satisfied because the terms inside the parentheses are zero according to Eqs. (2.73), (2.74). After complicated but straightforward calculations and with the help of Eq. (2.64), we reproduce the first, second, and last lines of Eq. (2.50). Meanwhile, substituting Eq. (2.73) into Eq. (2.49), one obtains the following on-shell conditions

$$\begin{aligned}
(\Re K^2 - m^2) \mathcal{F} + \frac{1}{2} \Im K_{\mu\nu} \mathcal{S}^{\mu\nu} &= 0, \\
(\Re K^2 - m^2) \mathcal{P} + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} \Im K^{\mu\nu} \mathcal{S}^{\alpha\beta} &= 0, \\
(\Re K^2 - m^2) \mathcal{S}_{\mu\nu} + \Re K^\alpha_{[\mu} \mathcal{S}_{\nu]\alpha} + \Im K_{\mu\nu} \mathcal{F} - \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \Im K^{\alpha\beta} \mathcal{P} &= 0.
\end{aligned} \tag{2.78}$$

So we conclude that another approach for computing the Wigner function is: first obtain a solution for \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$ which satisfies Eqs. (2.74) and (2.78) and then derive \mathcal{V}^μ and \mathcal{A}^μ using Eq. (2.73).

Note that in the massless case, the above discussion seem to be useless because the mass appears in the denominators in Eqs. (2.68), (2.73) and $1/m$ will be divergent when $m \rightarrow 0$. However, detailed calculations in the next section show that the numerators are also proportional to the mass, which leads to a finite quotient. In this way, the results in the massive case are expected to smoothly converge to the results in the massless case. For massless particles, the vector and axial-vector components \mathcal{V}^μ and \mathcal{A}^μ decouple from the other components, as given in (2.51). Adopting the definition (2.71) of spin-up/spin-down currents, the equations can be rewritten in a compact form,

$$\begin{aligned}
\hbar \nabla_\mu \mathcal{J}_\chi^\mu &= 0, \quad \Pi_\mu \mathcal{J}_\chi^\mu = 0, \\
\Pi_{[\mu} \mathcal{J}_{\nu]}^\chi + \frac{\chi \hbar}{2} \epsilon_{\mu\nu\alpha\beta} \nabla^\alpha \mathcal{J}_\chi^\beta &= 0.
\end{aligned} \tag{2.79}$$

Properly taking combinations of the above equations, one can derive the on-shell equations, which agree with Eq. (2.72) by putting $m = 0$.

As a brief summary of this subsection, we list once more the approaches for computing the Wigner function:

1. For the massless case, the vector and axial-vector components \mathcal{V}^μ and \mathcal{A}^μ can be written in terms of the spin-up and spin-down currents as shown in Eq. (2.71). These currents should satisfy Eq. (2.79).
2. For massive particles, one can take the vector and axial-vector components \mathcal{V}^μ and \mathcal{A}^μ as basic quantities, which satisfy the on-shell conditions (2.69) and Eq. (2.70). The other components, \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$, are then derived from \mathcal{V}^μ , \mathcal{A}^μ using relation (2.68).

3. For massive particles, another possible method is to take the scalar, pseudoscalar, and tensor components \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$ as basic quantities, which satisfy Eq. (2.74) and the on-shell conditions (2.78). Equation (2.73) shows how to derive the other components \mathcal{V}^μ and \mathcal{A}^μ from \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$.

The more detailed semi-classical calculations in Sec. IV show that the approach 2 is equivalent with 3.

E. Equal-time Wigner function

In some dynamical problems, it appears to be more convenient to use the equal-time Wigner function, which was first proposed in Refs. [88, 126]. In this thesis, we define the equal-time Wigner function as follows

$$W(t, \mathbf{x}, \mathbf{p}) = \int dp^0 W(x, p), \quad (2.80)$$

which is derived from the covariant Wigner function by integrating over energy p^0 . Obviously the equal-time Wigner function is not Lorentz covariant because the observer's frame has been fixed. From Eq. (2.18), we can finish the integration over energy p^0 and obtain

$$W(t, \mathbf{x}, \mathbf{p}) \equiv \int \frac{d^3\mathbf{y}}{(2\pi)^3} \exp(i\mathbf{x} \cdot \mathbf{p}) U\left(t, \mathbf{x} + \frac{\mathbf{y}}{2}, \mathbf{x} - \frac{\mathbf{y}}{2}\right) \left\langle \Omega \left| \hat{\psi}\left(t, \mathbf{x} + \frac{\mathbf{y}}{2}\right) \otimes \hat{\psi}\left(t, \mathbf{x} - \frac{\mathbf{y}}{2}\right) \right| \Omega \right\rangle. \quad (2.81)$$

Here the two field operators are defined at the same time t but at different spatial points. A 3-dimensional Fourier transform is made with respect to the relative coordinate \mathbf{y} , which gives the dependence on the kinetic 3-momentum \mathbf{p} . Similar to the covariant form, the gauge field (i.e., the electromagnetic field) is assumed to be a classical C -number and thus the gauge link is taken out of the quantum expectation value. Meanwhile, the covariant Wigner function can be described by its energy moments $\int dp^0 (p^0)^n W(x, p)$ and the equal-time Wigner function is just the zeroth order moment. Thus, from the covariant Wigner function one can derive the equal-time Wigner function, but from the equal-time one we cannot reproduce the covariant one because the higher-order energy moments, $\int dp^0 (p^0)^n W(x, p)$ for $n > 0$, are unknown. But if particles are on the usual mass-shell $p^2 = m^2$, the covariant Wigner function and the equal-time one are equivalent to each other.

The equation of motion for the equal-time Wigner function can be obtained from the Dirac equation, or equivalently from the equation of motion for the covariant Wigner function via taking an energy integral. From Eq. (2.36), we can obtain the Dirac-form equation for the equal-time

Wigner function by integrating over p^0 and dropping boundary terms such as $\int dp^0 \partial_{p^0} W(x, p)$,

$$\gamma^0 \int dp^0 p^0 W(x, p) + \gamma^0 \Pi^0 W(x, p) + \left(\frac{i\hbar}{2} \gamma^0 D_t - \boldsymbol{\gamma} \cdot \mathbf{K} - m \right) W(t, \mathbf{x}, \mathbf{p}) = 0, \quad (2.82)$$

where the operator is defined as

$$\mathbf{K} \equiv \boldsymbol{\Pi} - \frac{i\hbar}{2} \mathbf{D}_{\mathbf{x}}. \quad (2.83)$$

Here the generalized time derivative operator D_t , the spatial derivative operator $\mathbf{D}_{\mathbf{x}}$, the energy shift Π^0 , and the momentum operator $\boldsymbol{\Pi}$ are given by

$$D_t \equiv \partial_t + j_0(\Delta) \mathbf{E}(x) \cdot \nabla_{\mathbf{p}},$$

$$\mathbf{D}_{\mathbf{x}} \equiv \nabla_{\mathbf{x}} + j_0(\Delta) \mathbf{B}(x) \times \nabla_{\mathbf{p}}, \quad (2.84)$$

$$\Pi^0 = \frac{\hbar}{2} j_1(\Delta) \mathbf{E}(x) \cdot \nabla_{\mathbf{p}},$$

$$\boldsymbol{\Pi} \equiv \mathbf{p} - \frac{\hbar}{2} j_1(\Delta) \mathbf{B}(x) \times \nabla_{\mathbf{p}}, \quad (2.85)$$

with $\Delta \equiv -\frac{\hbar}{2} \nabla_{\mathbf{p}} \cdot \nabla_{\mathbf{x}}$ where $\nabla_{\mathbf{x}}$ only acts on the electromagnetic fields. These generalized operators D_t , $\mathbf{D}_{\mathbf{x}}$, and $\boldsymbol{\Pi}$ are reduced to the normal time derivative, spatial derivative, and 3-momentum when the electromagnetic fields vanish. They are real operators, thus the Hermitian conjugate of Eq. (2.82) reads,

$$\gamma^0 \int dp^0 p^0 W(x, p) + \gamma^0 \Pi^0 W(x, p) + \gamma^0 W(t, \mathbf{x}, \mathbf{p}) \left[-\frac{i\hbar}{2} (\gamma^0 D_t + \boldsymbol{\gamma} \cdot \mathbf{D}_{\mathbf{x}}) - \boldsymbol{\gamma} \cdot \boldsymbol{\Pi} - m \right] \gamma^0 = 0, \quad (2.86)$$

where we have used the property $W^\dagger = \gamma^0 W \gamma^0$. Multiplying Eqs. (2.82) and (2.86) with γ^0 from the left, we obtain

$$\int dp^0 p^0 W(x, p) + \Pi^0 W(x, p) + \left[\frac{i\hbar}{2} (D_t + \gamma^0 \boldsymbol{\gamma} \cdot \mathbf{D}_{\mathbf{x}}) - \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\Pi} - m \gamma^0 \right] W(t, \mathbf{x}, \mathbf{p}) = 0,$$

$$\int dp^0 p^0 W(x, p) + \Pi^0 W(x, p) + W(t, \mathbf{x}, \mathbf{p}) \left[-\frac{i\hbar}{2} (D_t - \gamma^0 \boldsymbol{\gamma} \cdot \mathbf{D}_{\mathbf{x}}) + \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\Pi} - m \gamma^0 \right] = 0. \quad (2.87)$$

Taking the difference of these two equations we obtain the equation of motion for the equal-time Wigner function

$$i\hbar D_t W(t, \mathbf{x}, \mathbf{p}) + \frac{i\hbar}{2} \mathbf{D}_{\mathbf{x}} \cdot [\gamma^0 \boldsymbol{\gamma}, W(t, \mathbf{x}, \mathbf{p})] - \boldsymbol{\Pi} \cdot \{\gamma^0 \boldsymbol{\gamma}, W(t, \mathbf{x}, \mathbf{p})\} - m [\gamma^0, W(t, \mathbf{x}, \mathbf{p})] = 0, \quad (2.88)$$

while the sum gives

$$\int dp^0 p^0 W(x, p)$$

$$= -\frac{i\hbar}{2} \mathbf{D}_{\mathbf{x}} \cdot \{\gamma^0 \boldsymbol{\gamma}, W(t, \mathbf{x}, \mathbf{p})\} - \Pi^0 W(x, p) + \boldsymbol{\Pi} \cdot [\gamma^0 \boldsymbol{\gamma}, W(t, \mathbf{x}, \mathbf{p})] + m \{\gamma^0, W(t, \mathbf{x}, \mathbf{p})\}. \quad (2.89)$$

We note that the time-evolution of the equal-time Wigner function is determined by Eq. (2.88) while Eq. (2.89) provides the relation between the first-order energy moment $\int dp^0 p^0 W(x, p)$ and the equal-time Wigner function.

Analogously to the covariant Wigner function, the equal-time Wigner function can be decomposed in 16 independent generators of the Clifford algebra, $\Gamma_i = \{1, i\gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \frac{1}{2}\sigma^{\mu\nu}\}$, as shown in Eq. (2.20). Here the coefficients are now functions of $\{t, \mathbf{x}, \mathbf{p}\}$. Inserting the decomposed Wigner function into Eq. (2.88) and taking the trace over Γ_i we obtain the following equations of motion,

$$\begin{aligned}
\hbar D_t \mathcal{F} &= 2\mathbf{\Pi} \cdot \mathcal{T}, \\
\hbar D_t \mathcal{P} &= -2\mathbf{\Pi} \cdot \mathcal{S} + 2m\mathcal{A}^0, \\
\hbar D_t \mathcal{V}^0 &= -\hbar \mathbf{D}_x \cdot \mathcal{V}, \\
\hbar D_t \mathcal{V} &= -\hbar \mathbf{D}_x \mathcal{V}^0 + 2\mathbf{\Pi} \times \mathcal{A} - 2m\mathcal{T}, \\
\hbar D_t \mathcal{A}^0 &= -\hbar \mathbf{D}_x \cdot \mathcal{A} - 2m\mathcal{P}, \\
\hbar D_t \mathcal{A} &= -\hbar \mathbf{D}_x \mathcal{A}^0 + 2\mathbf{\Pi} \times \mathcal{V}, \\
\hbar D_t \mathcal{T} &= -\hbar \mathbf{D}_x \times \mathcal{S} - 2\mathbf{\Pi} \mathcal{F} + 2m\mathcal{V}, \\
\hbar D_t \mathcal{S} &= \hbar \mathbf{D}_x \times \mathcal{T} + 2\mathbf{\Pi} \mathcal{P},
\end{aligned} \tag{2.90}$$

where we have suppressed the dependence on $\{t, \mathbf{x}, \mathbf{p}\}$ for all component functions. These equations describe how these component functions evolve with time. On the other hand, decomposing Eq. (2.89) we derive the first-order energy moments,

$$\begin{aligned}
\int dp^0 p^0 \mathcal{F}(x, p) &= \frac{\hbar}{2} \mathbf{D}_x \cdot \mathcal{T} + m\mathcal{V}^0 - \mathbf{\Pi}^0 \mathcal{F}, \\
\int dp^0 p^0 \mathcal{P}(x, p) &= -\frac{\hbar}{2} \mathbf{D}_x \cdot \mathcal{S} - \mathbf{\Pi}^0 \mathcal{P}, \\
\int dp^0 p^0 \mathcal{V}^0(x, p) &= \mathbf{\Pi} \cdot \mathcal{V} + m\mathcal{F} - \mathbf{\Pi}^0 \mathcal{V}^0, \\
\int dp^0 p^0 \mathcal{V}(x, p) &= \frac{\hbar}{2} \mathbf{D}_x \times \mathcal{A} + \mathbf{\Pi} \mathcal{V}^0 - \mathbf{\Pi}^0 \mathcal{V}, \\
\int dp^0 p^0 \mathcal{A}^0(x, p) &= \mathbf{\Pi} \cdot \mathcal{A} - \mathbf{\Pi}^0 \mathcal{A}^0, \\
\int dp^0 p^0 \mathcal{A}(x, p) &= \frac{\hbar}{2} \mathbf{D}_x \times \mathcal{V} + \mathbf{\Pi} \mathcal{A}^0 + m\mathcal{S} - \mathbf{\Pi}^0 \mathcal{A}, \\
\int dp^0 p^0 \mathcal{T}(x, p) &= -\frac{\hbar}{2} \mathbf{D}_x \mathcal{F} + \mathbf{\Pi} \times \mathcal{S} - \mathbf{\Pi}^0 \mathcal{T}, \\
\int dp^0 p^0 \mathcal{S}(x, p) &= \frac{\hbar}{2} \mathbf{D}_x \mathcal{P} - \mathbf{\Pi} \times \mathcal{T} + m\mathcal{A} - \mathbf{\Pi}^0 \mathcal{S},
\end{aligned} \tag{2.91}$$

where the functions on the right-hand side are equal-time ones, while the functions on the left-hand side are covariant ones.

Now we divide the 16 functions into four groups, each group having four functions,

$$\begin{aligned}\mathcal{G}_1 &= \begin{pmatrix} \mathcal{F} \\ \mathcal{S} \end{pmatrix}, & \mathcal{G}_2 &= \begin{pmatrix} \mathcal{V}^0 \\ \mathcal{A} \end{pmatrix}, \\ \mathcal{G}_3 &= \begin{pmatrix} \mathcal{A}^0 \\ \mathcal{V} \end{pmatrix}, & \mathcal{G}_4 &= \begin{pmatrix} \mathcal{P} \\ \mathcal{T} \end{pmatrix}.\end{aligned}\quad (2.92)$$

The introduction of these four groups proves to be useful for dealing with the Wigner function when the observer's frame is fixed [92, 123]. This form will be used in Sec. III for the case of constant electromagnetic fields. Using Eq. (2.92), Eq. (2.90) takes a matrix form,

$$\hbar D_t \begin{pmatrix} \mathcal{G}_1\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_2\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_3\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_4\{t, \mathbf{x}, \mathbf{p}\} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & M_1 \\ 0 & 0 & -M_2 & 0 \\ 0 & -M_2 & 0 & -2m\mathbb{I}_4 \\ M_1 & 0 & 2m\mathbb{I}_4 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{G}_1\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_2\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_3\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_4\{t, \mathbf{x}, \mathbf{p}\} \end{pmatrix}, \quad (2.93)$$

while the constraint equation reads

$$\int dp^0 p^0 \begin{pmatrix} \mathcal{G}_1(x, p) \\ \mathcal{G}_2(x, p) \\ \mathcal{G}_3(x, p) \\ \mathcal{G}_4(x, p) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 2m\mathbb{I}_4 & 0 & M_2 \\ 2m\mathbb{I}_4 & 0 & M_1 & 0 \\ 0 & M_1 & 0 & 0 \\ -M_2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{G}_1\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_2\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_3\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_4\{t, \mathbf{x}, \mathbf{p}\} \end{pmatrix} - \Pi^0 \begin{pmatrix} \mathcal{G}_1\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_2\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_3\{t, \mathbf{x}, \mathbf{p}\} \\ \mathcal{G}_4\{t, \mathbf{x}, \mathbf{p}\} \end{pmatrix}. \quad (2.94)$$

Here we define two matrices which are constructed from \mathbf{D}_x , and $\mathbf{\Pi}$,

$$M_1 = \begin{pmatrix} 0 & 2\mathbf{\Pi}^T \\ 2\mathbf{\Pi} & \hbar\mathbf{D}_x^\times \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & \hbar\mathbf{D}_x^T \\ \hbar\mathbf{D}_x & -2\mathbf{\Pi}^\times \end{pmatrix}. \quad (2.95)$$

For any 3-dimensional column vector, for example, the momentum operator $\mathbf{\Pi}$, we use $\mathbf{\Pi}^T$ for its transpose, a row vector. In Eq. (2.95), $\mathbf{\Pi}^\times$ represents an anti-symmetric matrix whose elements are $(\mathbf{\Pi}^\times)^{ij} = -\epsilon^{ijk}\Pi^k$,

$$\mathbf{\Pi}^\times = \begin{pmatrix} 0 & -\Pi^z & \Pi^y \\ \Pi^z & 0 & -\Pi^x \\ -\Pi^y & \Pi^x & 0 \end{pmatrix}. \quad (2.96)$$

When acting with the matrix $\mathbf{\Pi}^\times$ onto another column vector \mathbf{V} , we obtain the cross product of two vectors,

$$\mathbf{\Pi}^\times \mathbf{V} = \mathbf{\Pi} \times \mathbf{V}. \quad (2.97)$$

The operators defined in Eq. (2.85) coincide with the ones used in Refs. [77, 88, 126] because we have the following relations

$$\begin{aligned}
j_0(\Delta)\mathbf{E}(x) &= \int_{-1/2}^{1/2} ds \mathbf{E}(\mathbf{x} + is\hbar\nabla_{\mathbf{p}}), \\
-\frac{i}{2}j_1(\Delta)\mathbf{E}(x) &= \int_{-1/2}^{1/2} dss \mathbf{E}(\mathbf{x} + is\hbar\nabla_{\mathbf{p}}).
\end{aligned} \tag{2.98}$$

With the help of these relations, the operators in Eq. (2.85) can be written in another form,

$$\begin{aligned}
D_t &= \partial_t + \int_{-1/2}^{1/2} ds \mathbf{E}(\mathbf{x} + is\hbar\nabla_{\mathbf{p}}) \cdot \nabla_{\mathbf{p}}, \\
\mathbf{D}_{\mathbf{x}} &= \nabla_{\mathbf{x}} + \int_{-1/2}^{1/2} ds \mathbf{B}(\mathbf{x} + is\hbar\nabla_{\mathbf{p}}) \times \nabla_{\mathbf{p}}, \\
\Pi^0 &= i\hbar \int_{-1/2}^{1/2} dss \mathbf{E}(\mathbf{x} + is\hbar\nabla_{\mathbf{p}}) \cdot \nabla_{\mathbf{p}}, \\
\Pi &= \mathbf{p} - i\hbar \int_{-1/2}^{1/2} dss \mathbf{B}(\mathbf{x} + is\hbar\nabla_{\mathbf{p}}) \times \nabla_{\mathbf{p}},
\end{aligned} \tag{2.99}$$

which are used in Refs. [77, 88, 126].

III. ANALYTICAL SOLUTIONS

In the previous section we have introduced the definition of the covariant Wigner function in Eq. (2.18) and its equal-time formula in Eq. (2.81). Kinetic equations are also derived but we still need the initial conditions for numerically solving the equations. In this section we will give several analytically solvable cases. The results of this section can serve as initial conditions for numerical calculations. In the following three cases, the Dirac equation has analytical solutions

1. A system consisting of fermions without any interaction.
2. Fermions with chiral imbalance. The chemical potential μ and the chiral chemical potential μ_5 are included in the Dirac equation but still without the electromagnetic field.
3. Fermions in a constant magnetic field. As in case 2, μ and μ_5 are included in the Dirac equation.

In all three cases, the Dirac equation can be analytically solved and we derive the eigenenergies and corresponding eigenwavefunctions. Then the field operator is derived following the standard procedure of second quantization. The Wigner function is then solved up to zeroth order in the spatial derivative. The chiral chemical potential μ_5 is included for the further study of chiral effects. In this section, two dynamical problems will also be considered,

1. Fermions in an electric field. The existence of the electric field leads to the decay of the vacuum into fermion/anti-fermion pairs. At the same time, charged particles in the system will be accelerated.
2. Fermions in constant electromagnetic fields. The magnetic field is assumed to be parallel to the electric field.

We use the equal-time Wigner function for these dynamical problems. These discussions show that the Wigner function approach can also be used for the study of pair-production. Furthermore, in parallel electric and magnetic field, the existence of the magnetic field will enhance the pair-production rate since it changes the structure of energy levels. Meanwhile, at the lowest Landau level, spins of positive charged particles are locked to the direction of the magnetic field, and the newly generated positively charged particles move along the electric field. Thus these particles have RH chirality. Similar arguments show that the negatively charged particles (anti-fermions) have RH chirality, too. This gives rise to interesting effects, such as axial-charge production and

axial-current production [110]. In this section we display the analytical procedure for deriving the Wigner function in the above five cases, while in Sec. V we will numerically calculate physical quantities. Throughout this section we will suppress \hbar but it can be recovered by carefully counting the units.

A. Free fermions

1. Plane-wave solutions

In this subsection we will focus on free fermions with spin- $\frac{1}{2}$ in the absence of electromagnetic fields. Interactions among particles are also neglected. In this case, fermions satisfy the free Dirac equation (2.23) with vanishing gauge potential $\mathbb{A}_\mu = 0$,

$$(i\gamma^\mu \partial_{x\mu} - m\mathbb{I}_4) \psi(x) = 0. \quad (3.1)$$

The Dirac equation can be rewritten in the form of a Schrödinger equation

$$i\frac{\partial}{\partial t}\psi = (-i\gamma^0\boldsymbol{\gamma} \cdot \boldsymbol{\partial}_{\mathbf{x}} + m\gamma^0)\psi. \quad (3.2)$$

Note that the spatial derivative operator $\boldsymbol{\partial}_{\mathbf{x}}$ commutes with the Hamilton operator $\hat{H} = -i\gamma^0\boldsymbol{\gamma} \cdot \boldsymbol{\partial}_{\mathbf{x}} + m\gamma^0$, so we can introduce a kinetic 3-momentum \mathbf{p} by making a Fourier expansion for the field $\psi(x)$,

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip^\mu x_\mu} \psi(p). \quad (3.3)$$

Applying this into the Dirac equation we obtain

$$p^0\psi(p) = (\gamma^0\boldsymbol{\gamma} \cdot \mathbf{p} + m\gamma^0)\psi(p). \quad (3.4)$$

The on-shell condition can be obtained by acting with $(\gamma^0\boldsymbol{\gamma} \cdot \mathbf{p} + m\gamma^0)$ onto Eq. (3.4),

$$(p^0)^2\psi(p) = (m^2 + \mathbf{p}^2)\psi(p). \quad (3.5)$$

Solving the on-shell condition, we obtain positive-energy states with $p^0 > 0$, and negative-energy states with $p^0 < 0$.

$$p^0 = \pm E_{\mathbf{p}} = \pm\sqrt{m^2 + \mathbf{p}^2}. \quad (3.6)$$

The positive-energy states will be identified as fermions, while the negative-energy states are identified as anti-fermions.

In order to obtain the corresponding eigenwavefunctions, we perform a Lorentz transformation and work in the particle's rest frame. We parameterize the Lorentz transformation using $\omega_{\mu\nu}$, which is anti-symmetric with respect to $\mu \leftrightarrow \nu$. The transformation matrix for a Lorentz vector is

$$\Lambda^\mu{}_\nu = \exp \left[-\frac{i}{2} \omega_{\alpha\beta} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu \right], \quad (3.7)$$

where $(\mathcal{J}^{\alpha\beta})^\mu{}_\nu$ is the generator of the Lorentz algebra. In the coordinate representation, this generator is given by

$$(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(\delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta). \quad (3.8)$$

Inserting $(\mathcal{J}^{\alpha\beta})_{\mu\nu}$ into the transformation matrix $\Lambda^\mu{}_\nu$, we obtain

$$\Lambda^\mu{}_\nu = \exp(\omega^\mu{}_\nu). \quad (3.9)$$

Any vector, for example the 4-momentum, transforms as

$$p^\mu \rightarrow \Lambda^\mu{}_\nu p^\nu. \quad (3.10)$$

Meanwhile, the Dirac-spinor field $\psi(x)$ transforms as

$$\psi(x) \rightarrow \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1}x), \quad (3.11)$$

where the spinor representation of the Lorentz transformation is given by

$$\Lambda_{\frac{1}{2}} = \exp \left(-\frac{i}{4} \omega_{\mu\nu} \sigma^{\mu\nu} \right), \quad (3.12)$$

with $\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu]$. Now we consider the transformation from the particle's rest frame to the lab frame. For one particle which has 4-momentum $(E_{\mathbf{p}}, \mathbf{p})$ in the lab frame, its 3-velocity is

$$\boldsymbol{\beta} = \frac{\mathbf{p}}{E_{\mathbf{p}}}, \quad (3.13)$$

and we define the rapidity vector as

$$\boldsymbol{\zeta} = \frac{\boldsymbol{\beta}}{\beta} \tanh^{-1} \beta, \quad (3.14)$$

with $\beta \equiv |\boldsymbol{\beta}|$. Then we define the parameters for the Lorentz transformation from the particle's rest frame to the lab frame

$$\omega^{0i} = -\zeta^i, \quad \omega^{ij} = 0, \quad (3.15)$$

which leads to the following transformation matrix

$$\Lambda^\mu{}_\nu = \exp \left[\left(\begin{array}{cccc} 0 & \beta^x & \beta^y & \beta^z \\ \beta^x & 0 & 0 & 0 \\ \beta^y & 0 & 0 & 0 \\ \beta^z & 0 & 0 & 0 \end{array} \right) \frac{1}{\beta} \tanh^{-1} \beta \right]. \quad (3.16)$$

This matrix can be calculated using the Taylor expansion,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\beta^x & \gamma\beta^y & \gamma\beta^z \\ \gamma\beta^x & 1 + (\gamma - 1)(\hat{\beta}^x)^2 & (\gamma - 1)\hat{\beta}^x\hat{\beta}^y & (\gamma - 1)\hat{\beta}^x\hat{\beta}^z \\ \gamma\beta^y & (\gamma - 1)\hat{\beta}^x\hat{\beta}^y & 1 + (\gamma - 1)(\hat{\beta}^y)^2 & (\gamma - 1)\hat{\beta}^y\hat{\beta}^z \\ \gamma\beta^z & (\gamma - 1)\hat{\beta}^x\hat{\beta}^z & (\gamma - 1)\hat{\beta}^y\hat{\beta}^z & 1 + (\gamma - 1)(\hat{\beta}^z)^2 \end{pmatrix}, \quad (3.17)$$

where $\beta^{x,y,z}$ are the three components of the 3-velocity $\boldsymbol{\beta}$, $\hat{\beta}^{x,y,z}$ are the components of the velocity direction $\boldsymbol{\beta}/\beta$, and the Lorentz factor $\gamma = 1/\sqrt{1 - \beta^2} = E_{\mathbf{p}}/m$. The spinor representation of this transformation is

$$\Lambda_{\frac{1}{2}} = \exp \left(\frac{1}{2} \gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\zeta} \right). \quad (3.18)$$

Thus the Dirac field $\psi(p)$ can be written in terms of the field in the particle's rest frame

$$\psi(p) = \Lambda_{\frac{1}{2}} \psi_{\text{rf}}. \quad (3.19)$$

In the particle's rest frame, where the 3-momentum vanishes $\mathbf{p} = 0$ and $E_{\mathbf{p}} = m$, the Dirac equation (3.4) reads

$$\pm m\psi_{\text{rf}} = m\gamma^0\psi_{\text{rf}}. \quad (3.20)$$

Here we adopt the Weyl basis for the gamma matrices in Eq. (A2). Then the wavefunctions for positive- and negative-energy states are given by

$$\psi_{\text{rf},s}^{(+)} = \sqrt{m} \begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}, \quad \psi_{\text{rf},s}^{(-)} = \sqrt{m} \begin{pmatrix} \xi_s \\ -\xi_s \end{pmatrix}, \quad (3.21)$$

where ξ_s are two-component spinors which satisfy the orthonormality relation $\xi_r^\dagger \xi_s = \delta_{rs}$. We have introduced a factor \sqrt{m} in these solutions for convenience. The spinor ξ_s define the spin direction in the rest frame. For example, $\xi = (1, 0)^T$ corresponds to a spin-up state in the z-direction and $\xi = \frac{1}{\sqrt{2}}(1, 1)^T$ corresponds to a spin-up state in the x-direction. Note that $\xi = \frac{1}{\sqrt{2}}(1, 1)^T$ is a superposition of $(1, 0)^T$ and $(0, 1)^T$, which respectively represent the spin-up and spin-down states

in the z-direction. Thus we can choose $\xi_+ = (1, 0)^T$ and $\xi_- = (0, 1)^T$ without loss of generality and all possible spin configuration can be written as a superposition of ξ_{\pm} . Generally, the spinors ξ_s can be chosen as the eigenvectors of an arbitrary linear combination of Pauli matrices. If we choose ξ_s as the eigenvectors of the 2×2 matrix $\mathbf{n} \cdot \boldsymbol{\sigma}$, then $\psi_{\text{rf},s}^{(+)}$ represent fermions with spin parallel ($s = +$) or anti-parallel ($s = -$) to the vector \mathbf{n} in their rest frame, while $\psi_{\text{rf},s}^{(-)}$ represent anti-fermions with spin parallel ($s = -$) or anti-parallel ($s = +$) to \mathbf{n} .

Then we boost from the particle's rest frame to the lab frame. Inserting the gamma matrices (A2) and the rapidity vector (3.14) into the definition of $\Lambda_{\frac{1}{2}}$ in Eq. (3.18), we obtain

$$\Lambda_{\frac{1}{2}} = \exp \left[-\frac{1}{2|\mathbf{p}|} \begin{pmatrix} \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \\ 0 & -\boldsymbol{\sigma} \cdot \mathbf{p} \end{pmatrix} \tanh^{-1} \frac{|\mathbf{p}|}{E_{\mathbf{p}}} \right], \quad (3.22)$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. Note that an exponential of a matrix is defined as the Taylor expansion

$$\Lambda_{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2|\mathbf{p}|} \tanh^{-1} \frac{|\mathbf{p}|}{E_{\mathbf{p}}} \right)^n \begin{pmatrix} (\boldsymbol{\sigma} \cdot \mathbf{p})^n & 0 \\ 0 & (-\boldsymbol{\sigma} \cdot \mathbf{p})^n \end{pmatrix}. \quad (3.23)$$

In order to calculate $\Lambda_{\frac{1}{2}}$, we first focus on the 2-dimensional matrix $\boldsymbol{\sigma} \cdot \mathbf{p}$. Note that $\boldsymbol{\sigma} \cdot \mathbf{p}$ is Hermitian, which means that it can be diagonalized. The normalized eigenstates of $\boldsymbol{\sigma} \cdot \mathbf{p}$ are given by

$$\frac{1}{\sqrt{2|\mathbf{p}|(p^z + |\mathbf{p}|)}} \begin{pmatrix} p^z + |\mathbf{p}| \\ p^x + ip^y \end{pmatrix}, \quad \frac{1}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^z)}} \begin{pmatrix} p^z - |\mathbf{p}| \\ p^x + ip^y \end{pmatrix}, \quad (3.24)$$

which correspond to the eigenvalues $\pm |\mathbf{p}|$, respectively. With these eigenvectors, one can define the following transformation matrix $S_{\mathbf{p}}$ and its Hermitian conjugate,

$$S_{\mathbf{p}} = \begin{pmatrix} \frac{p^z + |\mathbf{p}|}{\sqrt{2|\mathbf{p}|(p^z + |\mathbf{p}|)}} & \frac{p^z - |\mathbf{p}|}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^z)}} \\ \frac{p^x + ip^y}{\sqrt{2|\mathbf{p}|(p^z + |\mathbf{p}|)}} & \frac{p^x + ip^y}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^z)}} \end{pmatrix}, \quad S_{\mathbf{p}}^{\dagger} = \begin{pmatrix} \frac{p^z + |\mathbf{p}|}{\sqrt{2|\mathbf{p}|(p^z + |\mathbf{p}|)}} & \frac{p^x - ip^y}{\sqrt{2|\mathbf{p}|(p^z + |\mathbf{p}|)}} \\ \frac{p^z - |\mathbf{p}|}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^z)}} & \frac{p^x - ip^y}{\sqrt{2|\mathbf{p}|(|\mathbf{p}| - p^z)}} \end{pmatrix}. \quad (3.25)$$

The matrix $S_{\mathbf{p}}$ is a unitary matrix, i.e., its inverse is equivalent to its Hermitian conjugate, $S_{\mathbf{p}}^{\dagger} S_{\mathbf{p}} = S_{\mathbf{p}} S_{\mathbf{p}}^{\dagger} = \mathbb{I}_2$. The matrix $\boldsymbol{\sigma} \cdot \mathbf{p}$ is then diagonalized as

$$\boldsymbol{\sigma} \cdot \mathbf{p} = |\mathbf{p}| S_{\mathbf{p}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S_{\mathbf{p}}^{\dagger}. \quad (3.26)$$

With the help of Eq. (3.26), we can calculate the n -th power of $\boldsymbol{\sigma} \cdot \mathbf{p}$,

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^n = |\mathbf{p}|^n S_{\mathbf{p}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^n S_{\mathbf{p}}^{\dagger} = |\mathbf{p}|^n S_{\mathbf{p}} \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n \end{pmatrix} S_{\mathbf{p}}^{\dagger}. \quad (3.27)$$

We can calculate the terms in Eq. (3.23)

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2|\mathbf{p}|} \tanh^{-1} \frac{|\mathbf{p}|}{E_{\mathbf{p}}} \right)^n (\boldsymbol{\sigma} \cdot \mathbf{p})^n &= \frac{1}{\sqrt{m}} \sqrt{p_{\mu} \sigma^{\mu}}, \\ \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2|\mathbf{p}|} \tanh^{-1} \frac{|\mathbf{p}|}{E_{\mathbf{p}}} \right)^n (-\boldsymbol{\sigma} \cdot \mathbf{p})^n &= \frac{1}{\sqrt{m}} \sqrt{p_{\mu} \bar{\sigma}^{\mu}},\end{aligned}\quad (3.28)$$

where we have used

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\pm \frac{1}{2} \tanh^{-1} \frac{|\mathbf{p}|}{E_{\mathbf{p}}} \right)^n = \exp \left[\pm \frac{1}{2} \tanh^{-1} \frac{|\mathbf{p}|}{E_{\mathbf{p}}} \right] = \frac{1}{\sqrt{m}} \sqrt{E_{\mathbf{p}} \pm |\mathbf{p}|}, \quad (3.29)$$

and introduced the following short-hand notations

$$\sqrt{p_{\mu} \sigma^{\mu}} = S_{\mathbf{p}} \begin{pmatrix} \sqrt{E_{\mathbf{p}} - |\mathbf{p}|} & 0 \\ 0 & \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \end{pmatrix} S_{\mathbf{p}}^{\dagger}, \quad \sqrt{p_{\mu} \bar{\sigma}^{\mu}} = S_{\mathbf{p}} \begin{pmatrix} \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} & 0 \\ 0 & \sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \end{pmatrix} S_{\mathbf{p}}^{\dagger}. \quad (3.30)$$

These matrices are real,

$$(\sqrt{p_{\mu} \sigma^{\mu}})^{\dagger} = \sqrt{p_{\mu} \sigma^{\mu}}, \quad (\sqrt{p_{\mu} \bar{\sigma}^{\mu}})^{\dagger} = \sqrt{p_{\mu} \bar{\sigma}^{\mu}}, \quad (3.31)$$

and satisfy following relations,

$$\begin{aligned}(\sqrt{p_{\mu} \sigma^{\mu}})^2 &= E_{\mathbf{p}} - \boldsymbol{\sigma} \cdot \mathbf{p}, \quad (\sqrt{p_{\mu} \bar{\sigma}^{\mu}})^2 = E_{\mathbf{p}} + \boldsymbol{\sigma} \cdot \mathbf{p}, \\ \sqrt{p_{\mu} \sigma^{\mu}} \sqrt{p_{\nu} \bar{\sigma}^{\nu}} &= \sqrt{p_{\nu} \bar{\sigma}^{\nu}} \sqrt{p_{\mu} \sigma^{\mu}} = m,\end{aligned}\quad (3.32)$$

which will be useful in checking the normalization relation of the wavefunctions and calculating the Wigner function. Substituting the Taylor series in Eq. (3.23) into Eq. (3.28), the transformation matrix $\Lambda_{\frac{1}{2}}$ has the form

$$\Lambda_{\frac{1}{2}} = \frac{1}{\sqrt{m}} \begin{pmatrix} \sqrt{p_{\mu} \sigma^{\mu}} & 0 \\ 0 & \sqrt{p_{\mu} \bar{\sigma}^{\mu}} \end{pmatrix}, \quad (3.33)$$

where $\sqrt{p_{\mu} \sigma^{\mu}}$ and $\sqrt{p_{\mu} \bar{\sigma}^{\mu}}$ are defined in Eq. (3.30). Now we act with the transformation matrix $\Lambda_{\frac{1}{2}}$ in Eq. (3.33) onto the wavefunctions in the particle's rest frame to obtain the wavefunctions in the lab frame

$$\psi_s^{(+)}(\mathbf{p}) = \begin{pmatrix} \sqrt{p_{\mu} \sigma^{\mu}} \xi_s \\ \sqrt{p_{\mu} \bar{\sigma}^{\mu}} \xi_s \end{pmatrix}, \quad \psi_s^{(-)}(\mathbf{p}) = \begin{pmatrix} \sqrt{p_{\mu} \sigma^{\mu}} \xi_s \\ -\sqrt{p_{\mu} \bar{\sigma}^{\mu}} \xi_s \end{pmatrix}. \quad (3.34)$$

They are properly normalized,

$$\begin{aligned}\psi_s^{(+)\dagger}(\mathbf{p}) \psi_{s'}^{(+)}(\mathbf{p}) &= 2E_{\mathbf{p}} \delta_{ss'}, \\ \psi_s^{(-)\dagger}(\mathbf{p}) \psi_{s'}^{(-)}(\mathbf{p}) &= 2E_{\mathbf{p}} \delta_{ss'},\end{aligned}\quad (3.35)$$

and the positive-energy states are orthogonal to the negative-energy ones,

$$\psi_s^{(+)\dagger}(\mathbf{p})\psi_{s'}^{(-)}(-\mathbf{p}) = \psi_s^{(-)\dagger}(\mathbf{p})\psi_{s'}^{(+)}(-\mathbf{p}) = 0. \quad (3.36)$$

Although these solutions have already been obtained in many textbooks, we have repeated the details in this thesis because we want to clarify how to calculate the square root of a matrix, i.e., the terms $\sqrt{p_\mu\sigma^\mu}$ and $\sqrt{p_\mu\bar{\sigma}^\mu}$ in the solutions (3.34). These details will help us in calculating the Wigner function in the latter part of this subsection.

2. Plane-wave quantization

Using the single-particle wavefunction in Eq. (3.34), the Dirac-field operator can be quantized as

$$\hat{\psi}(x) = \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \left[e^{-iE_{\mathbf{p}}t+i\mathbf{p}\cdot\mathbf{x}}\psi_s^{(+)}(\mathbf{p})\hat{a}_{\mathbf{p},s} + e^{iE_{\mathbf{p}}t-i\mathbf{p}\cdot\mathbf{x}}\psi_s^{(-)}(\mathbf{p})\hat{b}_{\mathbf{p},s}^\dagger \right], \quad (3.37)$$

where $\hat{a}_{\mathbf{p},s}$ represents the annihilation operator for a fermion with momentum \mathbf{p} and spin s , and $\hat{b}_{\mathbf{p},s}^\dagger$ is the creation operator for an anti-fermion with the same quantum numbers $\{\mathbf{p}, s\}$. Here the particle energy is on the mass-shell $p^2 = m^2$, thus we can rewrite the integration over the 3-momentum \mathbf{p} as a 4-dimensional covariant integration over the 4-momentum p^μ ,

$$\hat{\psi}(x) = \sum_s \int \frac{d^4p}{(2\pi)^3} e^{-ip^\mu x_\mu} \delta(p^2 - m^2) \sqrt{2E_{\mathbf{p}}} \left[\theta(p^0)\psi_s^{(+)}(\mathbf{p})\hat{a}_{\mathbf{p},s} + \theta(-p^0)\psi_s^{(-)}(-\mathbf{p})\hat{b}_{-\mathbf{p},s}^\dagger \right], \quad (3.38)$$

where we have used the following property of the delta-function,

$$\theta(\pm p^0)\delta(p^2 - m^2) = \frac{1}{2E_{\mathbf{p}}} \delta(p^0 \mp E_{\mathbf{p}}). \quad (3.39)$$

We demand that the creation and annihilation operators satisfy the following anti-commutation relations

$$\left\{ \hat{a}_{\mathbf{p},s}, \hat{a}_{\mathbf{p}',s'}^\dagger \right\} = \left\{ \hat{b}_{\mathbf{p},s}, \hat{b}_{\mathbf{p}',s'}^\dagger \right\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{ss'}, \quad (3.40)$$

with all other anti-commutators vanishing,

$$\left\{ \hat{a}_{\mathbf{p},s}, \hat{a}_{\mathbf{p}',s'} \right\} = \left\{ \hat{b}_{\mathbf{p},s}, \hat{b}_{\mathbf{p}',s'} \right\} = \left\{ \hat{a}_{\mathbf{p},s}, \hat{b}_{\mathbf{p}',s'}^\dagger \right\} = \left\{ \hat{b}_{\mathbf{p},s}, \hat{a}_{\mathbf{p}',s'}^\dagger \right\} = 0. \quad (3.41)$$

Then it is easy to verify the equal-time anti-commutation relation for the field operator

$$\left\{ \hat{\psi}_a(t, \mathbf{x}), \hat{\psi}_b^\dagger(t, \mathbf{x}') \right\} = \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \left[u_{s,a}(\mathbf{p})u_{s,b}^\dagger(\mathbf{p}) + v_{s,a}(\mathbf{p})v_{s,b}^\dagger(\mathbf{p}) \right], \quad (3.42)$$

where $a, b = 1, 2, 3, 4$ label components of $\hat{\psi}$ or $\hat{\psi}^\dagger$. Inserting the explicit expressions for $\psi_s^{(+)}(\mathbf{p})$ and $\psi_s^{(-)}(\mathbf{p})$ in Eq. (3.34) into the above equation, one obtains

$$\left\{ \hat{\psi}_a(t, \mathbf{x}), \hat{\psi}_b^\dagger(t, \mathbf{x}') \right\} = \delta^{(3)}(\mathbf{x} - \mathbf{x}') \delta_{ab}, \quad (3.43)$$

while the other anti-commutators are zero,

$$\left\{ \hat{\psi}_a(t, \mathbf{x}), \hat{\psi}_b(t, \mathbf{x}') \right\} = \left\{ \hat{\psi}_a^\dagger(t, \mathbf{x}), \hat{\psi}_b^\dagger(t, \mathbf{x}') \right\} = 0. \quad (3.44)$$

By checking these equal-time anti-commutators, we confirm that the quantized field operator in Eq. (3.37) has the correct property. A further calculation gives the Hamiltonian operator,

$$\hat{H} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \sum_{s=\pm} \left(\hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} + \hat{b}_{-\mathbf{p},s}^\dagger \hat{b}_{-\mathbf{p},s} - 1 \right), \quad (3.45)$$

while the momentum operator is given by

$$\hat{\mathbf{P}} = \int d^3\mathbf{x} \hat{\psi}^\dagger(-i\nabla_{\mathbf{x}})\hat{\psi} = \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p} \left(\hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} - \hat{b}_{-\mathbf{p},s}^\dagger \hat{b}_{-\mathbf{p},s} \right). \quad (3.46)$$

In deriving the Hamiltonian and momentum operators, we have used the orthonormality relations in Eqs. (3.35) and (3.36). In quantum electrodynamics, the Dirac spinor ψ is used to describe spin-1/2 particles, such as electrons and positrons. Here the spin is quantized along a given direction, which is determined by the choice of Pauli spinors ξ_s in (3.34). If we adopt the quantization procedure in this subsection, the operator $\hat{a}_{\mathbf{p},s}^\dagger$ creates an electron with momentum \mathbf{p} and spin parallel ($s = +$) or anti-parallel ($s = -$) to the spin quantization direction. On the other hand, the operator $\hat{b}_{\mathbf{p},s}^\dagger$ creates a positron with momentum \mathbf{p} and spin parallel ($s = -$) or anti-parallel ($s = +$) to the spin quantization direction. The interpretation of spin can be obtained via computing the spin angular momentum operator.

3. Wigner function

In the previous parts of this subsection we have derived the plane-wave solutions and quantized the Dirac-field in Eq. (3.38). Inserting the field operator into the definition of the Wigner function (2.18), one obtains

$$\begin{aligned} W(x, p) &= \int \frac{d^4q d^4q'}{(2\pi)^6} \sum_{ss'} \exp [i(q^\mu - q'^\mu)x_\mu] \sqrt{2E_{\mathbf{q}}} \sqrt{2E_{\mathbf{q}'}} \\ &\times \delta^{(4)} \left(p_\mu - \frac{q_\mu + q'_\mu}{2} \right) \delta(q^\mu q_\mu - m^2) \delta(q'^\mu q'_\mu - m^2) \\ &\times \left\langle \Omega \left| \left[\theta(q^0) \bar{\psi}_s^{(+)}(\mathbf{q}) \hat{a}_{\mathbf{q},s}^\dagger + \theta(-q^0) \bar{\psi}_s^{(-)}(-\mathbf{q}) \hat{b}_{-\mathbf{q},s} \right] \right. \right. \\ &\otimes \left. \left. \left[\theta(q'^0) \psi_{s'}^{(+)}(\mathbf{q}') \hat{a}_{\mathbf{q}',s'} + \theta(-q'^0) \psi_{s'}^{(-)}(-\mathbf{q}') \hat{b}_{-\mathbf{q}',s'}^\dagger \right] \right| \Omega \right\rangle. \end{aligned} \quad (3.47)$$

In the Wigner function two field operators are defined at different space-time points, thus here after the Fourier transformations, we have two momentum variables q^μ and q'^μ . Then we define the average and relative momentum as follows

$$k^\mu = \frac{1}{2}(q^\mu + q'^\mu), \quad u^\mu = q^\mu - q'^\mu, \quad (3.48)$$

in terms of which can we express q^μ and q'^μ ,

$$q^\mu = k^\mu + \frac{1}{2}u^\mu, \quad q'^\mu = k^\mu - \frac{1}{2}u^\mu. \quad (3.49)$$

Since the Jacobian for this substitution equals 1, we have

$$d^4q d^4q' = d^4k d^4u. \quad (3.50)$$

Using the new variables k^μ and u^μ , the delta functions in Eq. (3.47) can be simplified as

$$\begin{aligned} \delta(q^\mu q_\mu - m^2) \delta(q'^\mu q'_\mu - m^2) &= \delta\left(k^\mu k_\mu + \frac{1}{4}u^\mu u_\mu - m^2 + k^\mu u_\mu\right) \delta\left(k^\mu k_\mu + \frac{1}{4}u^\mu u_\mu - m^2 - k^\mu u_\mu\right) \\ &= \frac{1}{2} \delta\left(k^\mu k_\mu + \frac{1}{4}u^\mu u_\mu - m^2\right) \delta(k^\mu u_\mu), \end{aligned} \quad (3.51)$$

On the other hand, we have to deal with the step functions in the Wigner function (3.47). The product of two step functions can be rewritten as

$$\theta(x)\theta(y) = \theta(x+y)\theta(x+y-|x-y|), \quad (3.52)$$

So we obtain

$$\begin{aligned} \theta(q^0)\theta(q'^0) &= \theta(k^0)\theta\left(k^0 - \left|\frac{1}{2}u^0\right|\right), \\ \theta(-q^0)\theta(-q'^0) &= \theta(-k^0)\theta\left(-k^0 - \left|\frac{1}{2}u^0\right|\right), \\ \theta(q^0)\theta(-q'^0) &= \theta(u^0)\theta\left(\frac{1}{2}u^0 - |k^0|\right), \\ \theta(-q^0)\theta(q'^0) &= \theta(-u^0)\theta\left(-\frac{1}{2}u^0 - |k^0|\right). \end{aligned} \quad (3.53)$$

Since q^μ and q'^μ are fixed on the mass-shell, their zeroth component is $q^0 = \pm E_{\mathbf{q}}$ and $q'^0 = \pm E_{\mathbf{q}'}$. Thus we can check the following relations between the absolute values of $|k^0|$ and $|u^0|$

$$\begin{cases} \frac{1}{2}|u^0| < |k^0|, & \text{sgn}(q^0)\text{sgn}(q'^0) = 1, \\ \frac{1}{2}|u^0| > |k^0|, & \text{sgn}(q^0)\text{sgn}(q'^0) = -1, \end{cases} \quad (3.54)$$

Using these relations we find that the products of two step functions in Eq. (3.53) can be simplified as

$$\begin{aligned}
\theta(q^0)\theta(q'^0) &= \theta(k^0), \\
\theta(-q^0)\theta(-q'^0) &= \theta(-k^0), \\
\theta(q^0)\theta(-q'^0) &= \theta(u^0), \\
\theta(-q^0)\theta(q'^0) &= \theta(-u^0).
\end{aligned} \tag{3.55}$$

Using the new variables k^μ , u^μ , the Wigner function can be put into the form

$$\begin{aligned}
W(x, p) &= \int \frac{d^4 u}{(2\pi)^6} \sum_{ss'} \exp(iu^\mu x_\mu) \delta\left(p^\mu p_\mu + \frac{1}{4}u^\mu u_\mu - m^2\right) \delta(p^\mu u_\mu) \sqrt{\left|p^0 + \frac{1}{2}u^0\right| \left|p^0 - \frac{1}{2}u^0\right|} \\
&\times \left[\theta(p^0) \bar{\psi}_s^{(+)}\left(\mathbf{p} + \frac{1}{2}\mathbf{u}\right) \otimes \psi_{s'}^{(+)}\left(\mathbf{p} - \frac{1}{2}\mathbf{u}\right) \left\langle \Omega \left| \hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{u},s}^\dagger \hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{u},s'} \right| \Omega \right\rangle \right. \\
&\quad + \theta(-p^0) \bar{\psi}_s^{(-)}\left(-\mathbf{p} - \frac{1}{2}\mathbf{u}\right) \otimes \psi_{s'}^{(-)}\left(-\mathbf{p} + \frac{1}{2}\mathbf{u}\right) \left\langle \Omega \left| \hat{b}_{-\mathbf{p}-\frac{1}{2}\mathbf{u},s} \hat{b}_{-\mathbf{p}+\frac{1}{2}\mathbf{u},s'}^\dagger \right| \Omega \right\rangle \\
&\quad + \theta(u^0) \bar{\psi}_s^{(+)}\left(\mathbf{p} + \frac{1}{2}\mathbf{u}\right) \otimes \psi_{s'}^{(-)}\left(-\mathbf{p} + \frac{1}{2}\mathbf{u}\right) \left\langle \Omega \left| \hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{u},s}^\dagger \hat{b}_{-\mathbf{p}+\frac{1}{2}\mathbf{u},s'}^\dagger \right| \Omega \right\rangle \\
&\quad \left. + \theta(-u^0) \bar{\psi}_s^{(-)}\left(-\mathbf{p} - \frac{1}{2}\mathbf{u}\right) \otimes \psi_{s'}^{(+)}\left(\mathbf{p} - \frac{1}{2}\mathbf{u}\right) \left\langle \Omega \left| \hat{b}_{-\mathbf{p}-\frac{1}{2}\mathbf{u},s} \hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{u},s'} \right| \Omega \right\rangle \right].
\end{aligned} \tag{3.56}$$

Note that the last two lines contribute if and only if there is mixture between the fermion state and the anti-fermion state. Since we choose to neglect collisions between particles, processes such as pair-production or pair-annihilation are not included yet. Then the last two lines in Eq. (3.56) will be dropped in future discussions.

We consider a fermionic system where eigenstates of fermions with different momenta are not mixed together. Then the expectation values in Eq. (3.56) contain a delta function of \mathbf{u} ,

$$\begin{aligned}
\left\langle \Omega \left| \hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{u},s}^\dagger \hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{u},s'} \right| \Omega \right\rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{u}) f_{ss'}^{(+)}(\mathbf{p}), \\
\left\langle \Omega \left| \hat{b}_{-\mathbf{p}-\frac{1}{2}\mathbf{u},s} \hat{b}_{-\mathbf{p}+\frac{1}{2}\mathbf{u},s'}^\dagger \right| \Omega \right\rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{u}) \left[1 - f_{s's}^{(-)}(-\mathbf{p})\right],
\end{aligned} \tag{3.57}$$

where $f_{ss'}^{(+)}(\mathbf{p})$ and $f_{s's}^{(-)}(-\mathbf{p})$ are distribution functions of fermions and anti-fermions, respectively. Since we do not consider any spin interaction, the energy states are degenerate with respect to the spin direction. If the polarization of the system is not parallel to the spin quantization direction, these distribution functions are then not diagonal with respect to ss' . The delta function $\delta^{(3)}(\mathbf{u})$, together with $\delta(p^\mu u_\mu)$ in Eq. (3.56), gives a four-dimensional delta function,

$$\delta^{(3)}(\mathbf{u}) \delta(p^\mu u_\mu) = \delta^{(3)}(\mathbf{u}) \delta(p^0 u^0) = \frac{1}{|p^0|} \delta^{(4)}(u). \tag{3.58}$$

Thus we can carry out the integration over d^4u and obtain the following Wigner function

$$\begin{aligned}
W(x, p) &= \frac{1}{(2\pi)^3} \sum_{ss'} \delta(p^\mu p_\mu - m^2) \\
&\times \left\{ \theta(p^0) \bar{\psi}_s^{(+)}(\mathbf{p}) \otimes \psi_{s'}^{(+)}(\mathbf{p}) f_{ss'}^{(+)}(\mathbf{p}) + \theta(-p^0) \bar{\psi}_s^{(-)}(-\mathbf{p}) \otimes \psi_{s'}^{(-)}(-\mathbf{p}) \left[1 - f_{s's}^{(-)}(-\mathbf{p}) \right] \right\}.
\end{aligned} \tag{3.59}$$

In this formula, the Wigner function is independent of the space-time coordinates x^μ . This is because from the beginning we have assumed that the fermions are described by plane waves, which are homogeneous with respect to x^μ .

As we discuss in Appendix C, the plane wave cannot describe a quantum particle which is located at a given spatial point. According to the uncertainty principle, the momentum uncertainty for the plane wave is zero, $\sigma_p = 0$, thus its conjugate variable, the uncertainty of spatial position is infinity, $\sigma_x = \infty$. In order to introduce the x -dependence into distribution functions, we adopt the wave-packet description, as shown in Eq. (C2). This wave packet describes quantum particles at given center positions and average momenta. The expectation value of $a_{\mathbf{p}+\frac{1}{2}\mathbf{u},s}^\dagger a_{\mathbf{p}-\frac{1}{2}\mathbf{u},s}$ in a wave-packet state is

$$\begin{aligned}
&\left\langle \mathbf{p}', s, + \left| \hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{u},s}^\dagger \hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{u},s'} \right| \mathbf{p}', s, + \right\rangle \\
&= \frac{1}{N^2} \int \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(2\pi)^6} \exp \left[-\frac{(\mathbf{p}' - \mathbf{p}_1)^2 + (\mathbf{p}' - \mathbf{p}_2)^2}{4\sigma_p^2} \right] \left\langle 0 \left| \hat{a}_{\mathbf{p}_2,s} \hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{u},s}^\dagger \hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{u},s} \hat{a}_{\mathbf{p}_1,s}^\dagger \right| 0 \right\rangle \\
&= \frac{1}{N^2} \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 \exp \left[-\frac{(\mathbf{p}' - \mathbf{p}_1)^2 + (\mathbf{p}' - \mathbf{p}_2)^2}{4\sigma_p^2} \right] \delta^{(3)}(\mathbf{p} + \frac{1}{2}\mathbf{u} - \mathbf{p}_2) \delta^{(3)}(\mathbf{p} - \frac{1}{2}\mathbf{u} - \mathbf{p}_1) \\
&= \frac{1}{N^2} \exp \left[-\frac{(\mathbf{p}' - \mathbf{p})^2 + \frac{1}{4}\mathbf{u}^2}{2\sigma_p^2} \right].
\end{aligned} \tag{3.60}$$

In general, since the whole system is made of many wave packets, it is reasonable to expect that the expectation values in Eq. (3.56) are given by a distribution functions which depends on the parameters \mathbf{p} , \mathbf{u} , s , and s' ,

$$\begin{aligned}
\left\langle \Omega \left| \hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{u},s}^\dagger \hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{u},s'} \right| \Omega \right\rangle &= f_{ss'}^{(+)}(\mathbf{p}, \mathbf{u}), \\
\left\langle \Omega \left| \hat{b}_{-\mathbf{p}-\frac{1}{2}\mathbf{u},s} \hat{b}_{-\mathbf{p}+\frac{1}{2}\mathbf{u},s'}^\dagger \right| \Omega \right\rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{u}) - f_{s's}^{(-)}(-\mathbf{p}, \mathbf{u}),
\end{aligned} \tag{3.61}$$

where the first term in the second line, e.g. $(2\pi)^3 \delta^{(3)}(\mathbf{u})$, comes from the anti-commutator of $\hat{b}_{-\mathbf{p}-\frac{1}{2}\mathbf{u},s}$ and $\hat{b}_{-\mathbf{p}+\frac{1}{2}\mathbf{u},s'}^\dagger$. Here $f_{ss'}^{(+)}(\mathbf{p}, \mathbf{u})$ and $f_{s's}^{(-)}(-\mathbf{p}, \mathbf{u})$ are functions determined by the state of

the system $|\Omega\rangle$. Inserting these expectation values back into the Wigner function (3.56) we obtain

$$\begin{aligned}
W(x, p) &= \int \frac{d^4 u}{(2\pi)^6} \sum_{ss'} \exp[iu^\mu x_\mu] \delta\left(p^\mu p_\mu + \frac{1}{4}u^\mu u_\mu - m^2\right) \delta(p^\mu u_\mu) \sqrt{(p^0)^2 - \frac{1}{4}(u^0)^2} \\
&\times \left[\theta(p^0) \bar{\psi}_s^{(+)}\left(\mathbf{p} + \frac{1}{2}\mathbf{u}\right) \otimes \psi_{s'}^{(+)}\left(\mathbf{p} - \frac{1}{2}\mathbf{u}\right) f_{ss'}^{(+)}(\mathbf{p}, \mathbf{u}) \right. \\
&\quad \left. - \theta(-p^0) \bar{\psi}_s^{(-)}\left(-\mathbf{p} - \frac{1}{2}\mathbf{u}\right) \otimes \psi_{s'}^{(-)}\left(-\mathbf{p} + \frac{1}{2}\mathbf{u}\right) f_{s's}^{(-)}(-\mathbf{p}, \mathbf{u}) \right] \\
&+ \frac{1}{(2\pi)^3} \sum_{ss'} \delta(p^\mu p_\mu - m^2) \theta(-p^0) \bar{\psi}_s^{(-)}(-\mathbf{p}) \otimes \psi_{s'}^{(-)}(-\mathbf{p}). \tag{3.62}
\end{aligned}$$

Note that in general the uncertainty in momentum is small, which means the spread of the wave packet in momentum space is not large. So we can expect that the functions $f_{ss'}^{(+)}(\mathbf{p}, \mathbf{u})$ and $f_{s's}^{(-)}(-\mathbf{p}, \mathbf{u})$ are narrow with respect to \mathbf{u} . The Wigner function can then be expanded in terms of the small variable \mathbf{u} and higher-order terms can be dropped. The wavefunction part is expanded as follows

$$\begin{aligned}
\bar{\psi}_s^{(+)}\left(\mathbf{p} + \frac{1}{2}\mathbf{u}\right) \otimes \psi_{s'}^{(+)}\left(\mathbf{p} - \frac{1}{2}\mathbf{u}\right) &\simeq \bar{\psi}_s^{(+)}(\mathbf{p}) \otimes \psi_{s'}^{(+)}(\mathbf{p}) \\
&+ \frac{1}{2}\mathbf{u} \cdot \left\{ \left[\nabla_{\mathbf{p}} \bar{\psi}_s^{(+)}(\mathbf{p}) \right] \otimes \psi_{s'}^{(+)}(\mathbf{p}) - \bar{\psi}_s^{(+)}(\mathbf{p}) \otimes \nabla_{\mathbf{p}} \psi_{s'}^{(+)}(\mathbf{p}) \right\} + \mathcal{O}(\mathbf{u}^2). \tag{3.63}
\end{aligned}$$

Inserting (3.63) into the Wigner function, the leading-order term is

$$\begin{aligned}
W^{(0)}(x, p) &= \frac{1}{(2\pi)^3} \delta(p^\mu p_\mu - m^2) \sum_{ss'} \left\{ \theta(p^0) \bar{\psi}_s^{(+)}(\mathbf{p}) \otimes \psi_{s'}^{(+)}(\mathbf{p}) f_{ss'}^{(+)}(x, \mathbf{p}) \right. \\
&\quad \left. + \theta(-p^0) \bar{\psi}_s^{(-)}(-\mathbf{p}) \otimes \psi_{s'}^{(-)}(-\mathbf{p}) \left[1 - f_{s's}^{(-)}(x, -\mathbf{p}) \right] \right\}. \tag{3.64}
\end{aligned}$$

On the other hand, the first-order correction in the expansion (3.63) contributes to the Wigner function as

$$\begin{aligned}
W^{(1)}(x, p) &= \frac{1}{2} \delta(p^\mu p_\mu - m^2) \theta(p^0) \sum_{ss'} \\
&\quad \times \left\{ \left[\nabla_{\mathbf{p}} \bar{\psi}_s^{(+)}(\mathbf{p}) \right] \otimes \psi_{s'}^{(+)}(\mathbf{p}) - \bar{\psi}_s^{(+)}(\mathbf{p}) \otimes \nabla_{\mathbf{p}} \psi_{s'}^{(+)}(\mathbf{p}) \right\} \cdot i \nabla_{\mathbf{x}} f_{ss'}^{(+)}(x, \mathbf{p}) \\
&+ \frac{1}{2} \delta(p^\mu p_\mu - m^2) \theta(-p^0) \sum_{ss'} \\
&\quad \times \left\{ \left[\nabla_{\mathbf{p}} \bar{\psi}_s^{(-)}(-\mathbf{p}) \right] \otimes \psi_{s'}^{(-)}(-\mathbf{p}) - \bar{\psi}_s^{(-)}(-\mathbf{p}) \otimes \nabla_{\mathbf{p}} \psi_{s'}^{(-)}(-\mathbf{p}) \right\} \cdot i \nabla_{\mathbf{x}} f_{s's}^{(-)}(x, -\mathbf{p}). \tag{3.65}
\end{aligned}$$

Here we have defined the semi-distribution functions,

$$f_{ss'}^{(\pm)}(x, \mathbf{p}) = \int \frac{d^4 u}{(2\pi)^3} \delta\left(u^0 - \frac{\mathbf{p} \cdot \mathbf{u}}{p^0}\right) f_{ss'}^{(\pm)}(\mathbf{p}, \mathbf{u}) \exp(iu^\mu x_\mu). \tag{3.66}$$

In the first-order part, we have replaced $\mathbf{u}f_{ss'}^{(\pm)}(x, \mathbf{p})$ by the spatial derivative $i\nabla_{\mathbf{x}}f_{ss'}^{(\pm)}(x, \mathbf{p})$, thus $W^{(1)}(x, p)$ is of first order in the spatial gradients of the distributions $f_{ss'}^{(\pm)}(x, \mathbf{p})$. If we consider classical particles, the semi-distributions $f_{ss'}^{(\pm)}(x, \mathbf{p})$ can be interpreted as the classical distributions of fermions or anti-fermions at the phase space point $\{t, \mathbf{x}, \mathbf{p}\}$. Making a comparison with the results from the semi-classical expansion, which will be done in Sec. IV, we identify $W^{(0)}(x, p)$ as of zeroth order in \hbar and $W^{(1)}(x, p)$ as of first order in \hbar . The first order contribution $W^{(1)}(x, p)$ can be calculated using Eq. (3.65), since the wavefunctions in this equation have already been derived. But actually the calculation is too complicated, so that, in the following part of this subsection, we will only compute the leading order contribution $W^{(0)}(x, p)$.

4. Components of the Wigner function

The Wigner function can be calculated via inserting Eq. (3.34) into Eq. (3.64). In the following we will decompose the Wigner function as shown in Eq. (2.20) and compute the different components.

Before we do so, we first discuss the transformation matrix $S_{\mathbf{p}}$ in Eq. (3.25), which is useful because the matrices $\sqrt{p_{\mu}\sigma^{\mu}}$ and $\sqrt{p_{\mu}\bar{\sigma}^{\mu}}$ in the wavefunction are defined with $S_{\mathbf{p}}$ as shown in Eq. (3.30). Under the transformation $S_{\mathbf{p}}$, the Pauli matrices transform as

$$\begin{aligned} S_{\mathbf{p}}^{\dagger}\sigma^x S_{\mathbf{p}} &= \frac{1}{|\mathbf{p}|} \left[\frac{p^x p^z}{\sqrt{(p^x)^2 + (p^y)^2}} \sigma^x - \frac{p^y |\mathbf{p}|}{\sqrt{(p^x)^2 + (p^y)^2}} \sigma^y + p^x \sigma^z \right], \\ S_{\mathbf{p}}^{\dagger}\sigma^y S_{\mathbf{p}} &= \frac{1}{|\mathbf{p}|} \left[\frac{p^y p^z}{\sqrt{(p^x)^2 + (p^y)^2}} \sigma^x + \frac{p^x |\mathbf{p}|}{\sqrt{(p^x)^2 + (p^y)^2}} \sigma^y + p^y \sigma^z \right], \\ S_{\mathbf{p}}^{\dagger}\sigma^z S_{\mathbf{p}} &= \frac{1}{|\mathbf{p}|} \left[-\sqrt{(p^x)^2 + (p^y)^2} \sigma^x + p^z \sigma^z \right]. \end{aligned} \quad (3.67)$$

Multiplying each equation by $S_{\mathbf{p}}$ on the left and $S_{\mathbf{p}}^{\dagger}$ on the right, and using $S_{\mathbf{p}}^{\dagger}S_{\mathbf{p}} = S_{\mathbf{p}}S_{\mathbf{p}}^{\dagger} = \mathbb{I}_2$, one obtains

$$\begin{aligned} \sigma^x &= \frac{1}{|\mathbf{p}|} \left[\frac{p^x p^z}{\sqrt{(p^x)^2 + (p^y)^2}} S_{\mathbf{p}} \sigma^x S_{\mathbf{p}}^{\dagger} - \frac{p^y |\mathbf{p}|}{\sqrt{(p^x)^2 + (p^y)^2}} S_{\mathbf{p}} \sigma^y S_{\mathbf{p}}^{\dagger} + p^x S_{\mathbf{p}} \sigma^z S_{\mathbf{p}}^{\dagger} \right], \\ \sigma^y &= \frac{1}{|\mathbf{p}|} \left[\frac{p^y p^z}{\sqrt{(p^x)^2 + (p^y)^2}} S_{\mathbf{p}} \sigma^x S_{\mathbf{p}}^{\dagger} + \frac{p^x |\mathbf{p}|}{\sqrt{(p^x)^2 + (p^y)^2}} S_{\mathbf{p}} \sigma^y S_{\mathbf{p}}^{\dagger} + p^y S_{\mathbf{p}} \sigma^z S_{\mathbf{p}}^{\dagger} \right], \\ \sigma^z &= \frac{1}{|\mathbf{p}|} \left[-\sqrt{(p^x)^2 + (p^y)^2} S_{\mathbf{p}} \sigma^x S_{\mathbf{p}}^{\dagger} + p^z S_{\mathbf{p}} \sigma^z S_{\mathbf{p}}^{\dagger} \right]. \end{aligned} \quad (3.68)$$

From this we can obtain that

$$\begin{aligned}
S_{\mathbf{p}}\sigma^x S_{\mathbf{p}}^\dagger &= p^z \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{|\mathbf{p}| \sqrt{(p^x)^2 + (p^y)^2}} - \frac{|\mathbf{p}|}{\sqrt{(p^x)^2 + (p^y)^2}} \sigma^z, \\
S_{\mathbf{p}}\sigma^y S_{\mathbf{p}}^\dagger &= -\frac{\sigma^x p^y - \sigma^y p^x}{\sqrt{(p^x)^2 + (p^y)^2}}, \\
S_{\mathbf{p}}\sigma^z S_{\mathbf{p}}^\dagger &= \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{|\mathbf{p}|}.
\end{aligned} \tag{3.69}$$

These properties for the Pauli matrices will help us when we compute the axial-vector and tensor components of the Wigner function.

The Wigner function in Eq. (3.64) will now be decomposed in terms of the generators of the Clifford algebra $\Gamma_i = \{1, -i\gamma^5, \gamma^\mu, \gamma^\mu \gamma^5, \sigma^{\mu\nu}\}$ as in Eq. (2.20). The expansion coefficients are calculated via Eq. (2.21), and the traces in Eq. (2.21) are given by

$$\begin{aligned}
\text{Tr} \left[\Gamma_i W^{(0)}(x, p) \right] &= \frac{1}{(2\pi)^3} \delta(p^\mu p_\mu - m^2) \sum_{ss'} \left\{ \theta(p^0) \bar{\psi}_s^{(+)}(\mathbf{p}) \Gamma_i \psi_{s'}^{(+)}(\mathbf{p}) f_{ss'}^{(+)}(x, \mathbf{p}) \right. \\
&\quad \left. + \theta(-p^0) \bar{\psi}_s^{(-)}(-\mathbf{p}) \Gamma_i \psi_{s'}^{(-)}(-\mathbf{p}) \left[1 - f_{s's}^{(-)}(x, -\mathbf{p}) \right] \right\}.
\end{aligned} \tag{3.70}$$

We observe that Eq. (3.70) consists of a fermion part and an anti-fermion part. We first focus on the fermion part and then the anti-fermion part can be derived in the same way. The key point is to calculate

$$\bar{\psi}_s^{(+)}(\mathbf{p}) \Gamma_i \psi_{s'}^{(+)}(\mathbf{p}), \tag{3.71}$$

where $\psi_s^{(+)}(\mathbf{p})$ is the single particle wavefunction in momentum space, which is given in Eq. (3.34). The scalar and pseudoscalar parts can be derived directly by inserting Eq. (3.34) into Eq. (3.71) and using the relations (3.32),

$$\begin{aligned}
\bar{\psi}_s^{(+)}(\mathbf{p}) \psi_{s'}^{(+)}(\mathbf{p}) &= 2m \delta_{ss'}, \\
-i \bar{\psi}_s^{(+)}(\mathbf{p}) \gamma^5 \psi_{s'}^{(+)}(\mathbf{p}) &= 0.
\end{aligned} \tag{3.72}$$

Now we focus on the vector part, the calculation of the zeroth component is straightforward

$$\bar{\psi}_s^{(+)}(\mathbf{p}) \gamma^0 \psi_{s'}^{(+)}(\mathbf{p}) = 2E_{\mathbf{p}} \delta_{ss'}, \tag{3.73}$$

while the spatial components read

$$\begin{aligned}
\bar{\psi}_s^{(+)}(\mathbf{p}) \boldsymbol{\gamma} \psi_{s'}^{(+)}(\mathbf{p}) &= \xi_s^\dagger \left(\sqrt{p_\mu \bar{\sigma}^\mu} \boldsymbol{\sigma} \sqrt{p_\mu \bar{\sigma}^\mu} - \sqrt{p_\mu \sigma^\mu} \boldsymbol{\sigma} \sqrt{p_\mu \sigma^\mu} \right) \xi_{s'} \\
&= 2\xi_s^\dagger S_{\mathbf{p}} \begin{pmatrix} |\mathbf{p}| (S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}})_{11} & 0 \\ 0 & -|\mathbf{p}| (S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}})_{22} \end{pmatrix} S_{\mathbf{p}}^\dagger \xi_{s'}.
\end{aligned} \tag{3.74}$$

In the last step of Eq. (3.74) we have used the definitions in (3.30) for $\sqrt{p_\mu \bar{\sigma}^\mu}$ and $\sqrt{p_\mu \sigma^\mu}$. Here $S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}}$ is a 2×2 matrix and the subscript labels different elements of this matrix. From the transformation properties of the Pauli matrices in Eq. (3.67) we obtain

$$(S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}})_{11} = -(S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}})_{22} = \frac{\mathbf{p}}{|\mathbf{p}|}. \quad (3.75)$$

Inserting this into Eq. (3.74), we have

$$\bar{\psi}_s^{(+)}(\mathbf{p}) \boldsymbol{\gamma} \psi_{s'}^{(+)}(\mathbf{p}) = 2\mathbf{p} \delta_{ss'}. \quad (3.76)$$

On the other hand, the axial-vector components of Eq. (3.71) are given by

$$\begin{aligned} \bar{\psi}_s^{(+)}(\mathbf{p}) \gamma^0 \boldsymbol{\gamma}^5 \psi_{s'}^{(+)}(\mathbf{p}) &= 2\xi_s^\dagger \mathbf{p} \cdot \boldsymbol{\sigma} \xi_{s'}, \\ \bar{\psi}_s^{(+)}(\mathbf{p}) \boldsymbol{\gamma} \boldsymbol{\gamma}^5 \psi_{s'}^{(+)}(\mathbf{p}) &= 2\xi_s^\dagger S_{\mathbf{p}} \begin{pmatrix} E_{\mathbf{p}}(S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}})_{11} & m(S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}})_{12} \\ m(S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}})_{21} & E_{\mathbf{p}}(S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}})_{22} \end{pmatrix} S_{\mathbf{p}}^\dagger \xi_{s'}. \end{aligned} \quad (3.77)$$

Since we do not have a universal formula for $S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}}$, we have to calculate different components one by one. The computations are straightforward using the explicit expressions of $S_{\mathbf{p}}^\dagger \boldsymbol{\sigma} S_{\mathbf{p}}$ and $S_{\mathbf{p}} \boldsymbol{\sigma} S_{\mathbf{p}}^\dagger$ in Eqs. (3.67) and (3.69). The final result reads,

$$\bar{\psi}_s^{(+)}(\mathbf{p}) \boldsymbol{\gamma} \boldsymbol{\gamma}^5 \psi_{s'}^{(+)}(\mathbf{p}) = 2\xi_s^\dagger \left(m\boldsymbol{\sigma} + \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E_{\mathbf{p}} + m} \mathbf{p} \right) \xi_{s'}. \quad (3.78)$$

The tensor component of Eq. (3.71) is given by

$$\begin{aligned} \bar{\psi}_s^{(+)}(\mathbf{p}) \sigma^{0i} \psi_{s'}^{(+)}(\mathbf{p}) &= 2i\xi_s^\dagger S_{\mathbf{p}} \begin{pmatrix} 0 & -|\mathbf{p}|(S_{\mathbf{p}}^\dagger \sigma^i S_{\mathbf{p}})_{12} \\ |\mathbf{p}|(S_{\mathbf{p}}^\dagger \sigma^i S_{\mathbf{p}})_{21} & 0 \end{pmatrix} S_{\mathbf{p}}^\dagger \xi_{s'}, \\ \bar{\psi}_s^{(+)}(\mathbf{p}) \sigma^{ij} \psi_{s'}^{(+)}(\mathbf{p}) &= 2\epsilon^{ijk} \xi_s^\dagger S_{\mathbf{p}} \begin{pmatrix} m(S_{\mathbf{p}}^\dagger \sigma^k S_{\mathbf{p}})_{11} & E_{\mathbf{p}}(S_{\mathbf{p}}^\dagger \sigma^k S_{\mathbf{p}})_{12} \\ E_{\mathbf{p}}(S_{\mathbf{p}}^\dagger \sigma^k S_{\mathbf{p}})_{21} & m(S_{\mathbf{p}}^\dagger \sigma^k S_{\mathbf{p}})_{22} \end{pmatrix} S_{\mathbf{p}}^\dagger \xi_{s'}, \end{aligned} \quad (3.79)$$

where $i, j, k = 1, 2, 3$. Again these terms are calculated using the properties in Eqs. (3.67) and (3.69) and the results are

$$\bar{\psi}_s^{(+)}(\mathbf{p}) \sigma^{0i} \psi_{s'}^{(+)}(\mathbf{p}) = -2\epsilon^{ijk} p^j \xi_s^\dagger \sigma^k \xi_{s'}, \quad (3.80)$$

and

$$\bar{\psi}_s^{(+)}(\mathbf{p}) \sigma^{ij} \psi_{s'}^{(+)}(\mathbf{p}) = 2\epsilon^{ijk} \left(E_{\mathbf{p}} \xi_s^\dagger \sigma^k \xi_s - \frac{p^k}{E_{\mathbf{p}} + m} \xi_s^\dagger \mathbf{p} \cdot \boldsymbol{\sigma} \xi_s \right). \quad (3.81)$$

As a conclusion, we now collect all results from the above calculations, where we have written the vector, axial-vector, and tensor components in a covariant form,

$$\begin{aligned}
\bar{\psi}_s^{(+)}(\mathbf{p})\psi_{s'}^{(+)}(\mathbf{p}) &= 2m\delta_{ss'}, \\
-i\bar{\psi}_s^{(+)}(\mathbf{p})\gamma^5\psi_{s'}^{(+)}(\mathbf{p}) &= 0, \\
\bar{\psi}_s^{(-)}(\mathbf{p})\gamma^\mu\psi_{s'}^{(-)}(\mathbf{p}) &= 2p^\mu\delta_{ss'}, \\
\bar{\psi}_s^{(+)}(\mathbf{p})\gamma^\mu\gamma^5\psi_{s'}^{(+)}(\mathbf{p}) &= 2\xi_s^\dagger n^\mu(\mathbf{p})\xi_{s'}, \\
\bar{\psi}_s^{(+)}(\mathbf{p})\sigma^{\mu\nu}\psi_{s'}^{(+)}(\mathbf{p}) &= -\frac{2}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha\xi_s^\dagger n_\beta(\mathbf{p})\xi_{s'},
\end{aligned} \tag{3.82}$$

where $p^0 = E_{\mathbf{p}}$ is the on-shell energy and we have defined a vector for the spin polarization

$$n^\mu(\mathbf{p}) \equiv \left(\mathbf{p} \cdot \boldsymbol{\sigma}, m\boldsymbol{\sigma} + \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E_{\mathbf{p}} + m} \mathbf{p} \right)^T. \tag{3.83}$$

In spin space, the scalar, pseudoscalar and vector parts are diagonalized, while the axial-vector and tensor parts depend on $\xi_s^\dagger n^\mu(\mathbf{p})\xi_{s'}$, which is in general not diagonal. This is because generally the spin quantization direction is different from the spin polarization direction. In the last part of this section we will discuss the effect of different choices for the spin quantization direction.

The antiparticle contributions can be computed repeating above calculations. An easier way is to use the relation between the particle and antiparticle wavefunctions

$$\psi_s^{(-)}(\mathbf{p}) = -\gamma^5\psi_s^{(+)}(\mathbf{p}), \tag{3.84}$$

so we have

$$\bar{\psi}_s^{(-)}(-\mathbf{p})\Gamma_i\psi_{s'}^{(-)}(-\mathbf{p}) = -\bar{\psi}_s^{(+)}(-\mathbf{p})\gamma^5\Gamma_i\gamma^5\psi_{s'}^{(+)}(-\mathbf{p}), \tag{3.85}$$

Substitute Γ_i with different matrices $\{1, -i\gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\}$, we obtain

$$\begin{aligned}
\bar{\psi}_s^{(-)}(-\mathbf{p})\psi_{s'}^{(-)}(-\mathbf{p}) &= -2m\delta_{ss'}, \\
-i\bar{\psi}_s^{(-)}(-\mathbf{p})\gamma^5\psi_{s'}^{(-)}(-\mathbf{p}) &= 0, \\
\bar{\psi}_s^{(-)}(-\mathbf{p})\gamma^\mu\psi_{s'}^{(-)}(-\mathbf{p}) &= -2p^\mu\delta_{ss'}, \\
\bar{\psi}_s^{(-)}(-\mathbf{p})\gamma^\mu\gamma^5\psi_{s'}^{(-)}(-\mathbf{p}) &= 2\xi_s^\dagger \hat{n}^\mu(-\mathbf{p})\xi_{s'}, \\
\bar{\psi}_s^{(-)}(-\mathbf{p})\sigma^{\mu\nu}\psi_{s'}^{(-)}(-\mathbf{p}) &= -\frac{2}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha\xi_s^\dagger \hat{n}_\beta(-\mathbf{p})\xi_{s'},
\end{aligned} \tag{3.86}$$

where $p^0 = -E_{\mathbf{p}}$ is the on-shell energy for the anti-fermions.

Inserting the fermion contributions in (3.82) and the anti-fermion contributions in (3.86) into Eq. (3.70), we derive different components of the Wigner function,

$$\begin{aligned}
\mathcal{F}^{(0)} &= m \frac{2\delta(p^2 - m^2)}{(2\pi)^3} \sum_s \left\{ \theta(p^0) f_{ss}^{(+)}(x, \mathbf{p}) - \theta(-p^0) \left[1 - f_{ss}^{(-)}(x, -\mathbf{p}) \right] \right\}, \\
\mathcal{P}^{(0)} &= 0, \\
\mathcal{V}_\mu^{(0)} &= p_\mu \frac{2\delta(p^2 - m^2)}{(2\pi)^3} \sum_s \left\{ \theta(p^0) f_{ss}^{(+)}(x, \mathbf{p}) - \theta(-p^0) \left[1 - f_{ss}^{(-)}(x, -\mathbf{p}) \right] \right\}, \\
\mathcal{A}_\mu^{(0)} &= \frac{2\delta(p^2 - m^2)}{(2\pi)^3} \sum_{ss'} \left\{ \theta(p^0) \xi_s^\dagger n^\mu(\mathbf{p}) \xi_{s'} f_{ss'}^{(+)}(x, \mathbf{p}) + \theta(-p^0) \xi_s^\dagger n^\mu(-\mathbf{p}) \xi_{s'} \left[1 - f_{s's}^{(-)}(x, -\mathbf{p}) \right] \right\}, \\
\mathcal{S}_{\mu\nu}^{(0)} &= -\frac{2\delta(p^2 - m^2)}{(2\pi)^3} \frac{1}{m} \epsilon_{\mu\nu\alpha\beta} p^\alpha \\
&\quad \times \sum_{ss'} \left\{ \theta(p^0) \xi_s^\dagger n^\beta(\mathbf{p}) \xi_{s'} f_{ss'}^{(+)}(x, \mathbf{p}) + \theta(-p^0) \xi_s^\dagger n^\beta(-\mathbf{p}) \xi_{s'} \left[1 - f_{s's}^{(-)}(x, -\mathbf{p}) \right] \right\}. \tag{3.87}
\end{aligned}$$

The Wigner function is then recovered by Eq. (2.20). Note that the above results are of zeroth order in spatial gradients of the distribution function. Higher-order terms in spatial gradients are calculated via Eq. (3.65) but will not be done here, because it is too complicated. Now we define functions which can be interpreted as the net fermion density and polarization, respectively,

$$\begin{aligned}
V^{(0)}(x, p) &\equiv \frac{2}{(2\pi)^3} \sum_s \left\{ \theta(p^0) f_{ss}^{(+)}(x, \mathbf{p}) - \theta(-p^0) \left[1 - f_{ss}^{(-)}(x, -\mathbf{p}) \right] \right\}, \\
n^{(0)\mu}(x, p) &\equiv \frac{2}{(2\pi)^3} \sum_{ss'} \left\{ \theta(p^0) \xi_s^\dagger n^\mu(\mathbf{p}) \xi_{s'} f_{ss'}^{(+)}(x, \mathbf{p}) + \theta(-p^0) \xi_s^\dagger n^\mu(-\mathbf{p}) \xi_{s'} \left[1 - f_{s's}^{(-)}(x, -\mathbf{p}) \right] \right\}. \tag{3.88}
\end{aligned}$$

Then the components of the Wigner function at the zeroth order in spatial gradients read

$$\begin{aligned}
\mathcal{F}^{(0)} &= \delta(p^2 - m^2) m V^{(0)}(x, p), \\
\mathcal{P}^{(0)} &= 0, \\
\mathcal{V}_\mu^{(0)} &= \delta(p^2 - m^2) p_\mu V^{(0)}(x, p), \\
\mathcal{A}_\mu^{(0)} &= \delta(p^2 - m^2) n_\mu^{(0)}(x, p), \\
\mathcal{S}_{\mu\nu}^{(0)} &= -\delta(p^2 - m^2) \frac{1}{m} \epsilon_{\mu\nu\alpha\beta} p^\alpha n^{(0)\beta}(x, p). \tag{3.89}
\end{aligned}$$

These results agree with the ones from the semi-classical expansion [75, 112, 121]. Note that the results are independent of the choice of the spin quantization direction. Different spin quantizations are related by rotations in spin space. Both $f_{ss'}^{(\pm)}(x, \mathbf{p})$ and $\xi_s^\dagger n^\mu(\mathbf{p}) \xi_{s'}$ depend on the quantization direction but $V^{(0)}$ and $n^{(0)\mu}$ only depend on the trace in spin space, which are invariant under spin-rotations [128]. The solutions in Eq. (3.89) are all on the normal mass shell $p^2 = m^2$ because

we have not considered any electromagnetic field. In the semi-classical expansion discussed in Sec. IV we will clearly show that the normal mass shell is shifted by the spin-electromagnetic coupling.

5. Diagonalization of distributions

According to Eq. (3.61), we can find the relation between the function $f_{ss'}^{(+)}(\mathbf{p}, \mathbf{u})$ and its complex conjugate,

$$\left[f_{ss'}^{(\pm)}(\mathbf{p}, \mathbf{u}) \right]^* = f_{s's}^{(\pm)}(\mathbf{p}, -\mathbf{u}). \quad (3.90)$$

Then a relation between the distribution functions $f_{ss'}^{(\pm)}(x, \mathbf{p})$ in Eq. (3.66) and their complex conjugates can be derived,

$$\left[f_{ss'}^{(\pm)}(x, \mathbf{p}) \right]^* = \int \frac{d^4 u}{(2\pi)^3} \delta \left(u^0 - \frac{\mathbf{p} \cdot \mathbf{u}}{p^0} \right) f_{s's}^{(\pm)}(\mathbf{p}, -\mathbf{u}) \exp(-iu^\mu x_\mu) = f_{s's}^{(\pm)}(x, \mathbf{p}), \quad (3.91)$$

Here in the second step we have made a replacement $u^\mu \rightarrow -u^\mu$. The distribution $f_{ss'}^{(\pm)}(x, \mathbf{p})$ is actually the ss' element of a 2×2 matrix distribution $f^{(\pm)}(x, \mathbf{p})$ in spin space. So the relation (3.91) indicates that $f^{(\pm)}(x, \mathbf{p})$ is a Hermitian matrix, which can be diagonalized by a unitary transformation. The unitary transformation can be interpreted as a rotation of the spin quantization direction [128].

We take the fermion part $f^{(+)}(x, \mathbf{p})$ as an example to show the procedure of diagonalizing the distribution functions $f^{(\pm)}(x, \mathbf{p})$. Note that any 2-dimensional Hermitian matrix can be parameterized using the Pauli matrices $\boldsymbol{\sigma}$ together with the unit matrix,

$$f^{(+)}(x, \mathbf{p}) = a \mathbb{I}_2 + \mathbf{b} \cdot \boldsymbol{\sigma}. \quad (3.92)$$

Here a and \mathbf{b} are real functions of x^μ and \mathbf{p} . The matrix $\mathbf{b} \cdot \boldsymbol{\sigma}$ has eigenvalues $\pm |\mathbf{b}|$ with $|\mathbf{b}| \equiv \sqrt{\mathbf{b}^2}$ is the length of the 3-vector \mathbf{b} . Assuming that the corresponding eigenvectors are \vec{d}_\pm , which are 2-dimensional column vectors and satisfy

$$(\mathbf{b} \cdot \boldsymbol{\sigma}) \vec{d}_\pm = \pm |\mathbf{b}| \vec{d}_\pm. \quad (3.93)$$

Then the distribution function can be diagonalized as

$$\tilde{f}_s^{(+)}(x, \mathbf{p}) \delta_{rs} = \sum_{r's'} (D^\dagger)_{rr'} f_{r's'}^{(+)}(x, \mathbf{p}) D_{s's}, \quad (3.94)$$

where the transformation matrix D is a 2×2 matrix in spin space and constructed from the eigenvectors \vec{d}_\pm ,

$$D \equiv \begin{pmatrix} \vec{d}_+ & \vec{d}_- \end{pmatrix}. \quad (3.95)$$

The new distribution functions are then given by

$$\tilde{f}_{\pm}^{(+)}(x, \mathbf{p}) = a \pm |\mathbf{b}|. \quad (3.96)$$

In general, due to the fact that a and \mathbf{b} are defined locally, the transformation matrix D should be a function of $\{x^\mu, \mathbf{p}\}$. We also rotate the wavefunctions and define the following local ones,

$$\tilde{\psi}_s^{(+)}(x, \mathbf{p}) \equiv \sum_{s'} (D^\dagger)_{ss'} \psi_{s'}^{(+)}(\mathbf{p}). \quad (3.97)$$

Note that these new basis functions are still normalized because the transformation is unitary. The plane-wavefunctions $\psi_{s'}^{(+)}(\mathbf{p})$ are given in Eq. (3.34), from which the new wavefunctions are obtained,

$$\tilde{\psi}_s^{(+)}(x, \mathbf{p}) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu} \tilde{\xi}_s \\ \sqrt{p_\mu \bar{\sigma}^\mu} \tilde{\xi}_s \end{pmatrix}, \quad (3.98)$$

where

$$\tilde{\xi}_s \equiv \sum_{s'} (D^\dagger)_{ss'} \xi_{s'}. \quad (3.99)$$

Since s and s' label the spin state parallel or anti-parallel to a given quantization direction, the transformation matrix D is then interpreted as the $SU(2)$ representation of a rotation of the quantization direction. Analogously, we can take similar procedure for anti-particles, and finally the Wigner function (3.64) can be put into the form

$$\begin{aligned} W^{(0)}(x, p) = & \frac{1}{(2\pi)^3} \delta(p^\mu p_\mu - m^2) \sum_s \left\{ \theta(p^0) \tilde{\psi}_s^{(+)}(x, \mathbf{p}) \otimes \tilde{\psi}_s^{(+)}(x, \mathbf{p}) \tilde{f}_s^{(+)}(x, \mathbf{p}) \right. \\ & \left. + \theta(-p^0) \tilde{\psi}_s^{(-)}(x, -\mathbf{p}) \otimes \tilde{\psi}_s^{(-)}(x, -\mathbf{p}) \left[1 - \tilde{f}_s^{(-)}(x, -\mathbf{p}) \right] \right\}, \end{aligned} \quad (3.100)$$

where the anti-particle parts are diagonalized as

$$\begin{aligned} \tilde{f}_s^{(-)}(x, -\mathbf{p}) \delta_{rs} &= \sum_{r's'} (\bar{D}^\dagger)_{rr'} f_{r's'}^{(-)}(x, -\mathbf{p}) \bar{D}_{s's}, \\ \tilde{\psi}_s^{(-)}(x, -\mathbf{p}) &= \sum_{s'} (\bar{D}^\dagger)_{ss'} \psi_{s'}^{(-)}(-\mathbf{p}), \end{aligned} \quad (3.101)$$

with the transformation matrix \bar{D} is a function of $\{x^\mu, \mathbf{p}\}$.

The redefinition of $\tilde{\xi}_s$ in Eq. (3.99) corresponds to a new spin quantization direction. If we assume before the transformation $\xi_+ = (1, 0)^T$, $\xi_- = (0, 1)^T$, the new spinors then read

$$\tilde{\xi}_+ = \vec{d}_+, \quad \tilde{\xi}_- = \vec{d}_-, \quad (3.102)$$

which are eigenvectors of $\mathbf{b} \cdot \boldsymbol{\sigma}$ with eigenvalues $\pm |\mathbf{b}|$. This indicates that the new quantization direction is the direction of \mathbf{b} . The components of the Wigner function are computed from Eq. (3.47), where $V^{(0)}(x, p)$ and $n^{(0)\mu}(x, p)$ are given by Eq. (3.88). The rotation of the spin quantization direction does not change the trace of the matrix distribution $f^{(+)}(x, \mathbf{p})$, i.e., the following relation holds in any case,

$$\sum_s f_{ss}^{(+)}(x, \mathbf{p}) = \sum_s \tilde{f}_s^{(+)}(x, \mathbf{p}). \quad (3.103)$$

Thus the function $V^{(0)}(x, p)$ can be expressed in terms of the new distribution functions

$$V^{(0)}(x, p) = \frac{2}{(2\pi)^3} \sum_s \left\{ \theta(p^0) \tilde{f}_s^{(+)}(x, \mathbf{p}) - \theta(-p^0) \left[1 - \tilde{f}_s^{(-)}(x, -\mathbf{p}) \right] \right\}. \quad (3.104)$$

Meanwhile, the polarization part reads

$$\sum_{ss'} \xi_s^\dagger n^\mu(\mathbf{p}) \xi_{s'} f_{ss'}^{(+)}(x, \mathbf{p}) = \sum_s \tilde{\xi}_s^\dagger n^\mu(\mathbf{p}) \tilde{\xi}_s \tilde{f}_s^{(+)}(x, \mathbf{p}), \quad (3.105)$$

where $n^\mu(\mathbf{p})$ is given by Eq. (3.83). Note that $\tilde{\xi}_s^\dagger \boldsymbol{\sigma} \tilde{\xi}_s = s \frac{\mathbf{b}}{|\mathbf{b}|}$ because the new spinors $\tilde{\xi}_\pm$ are now eigenvectors of $\mathbf{b} \cdot \boldsymbol{\sigma}$. Thus the right-hand-side of the above equation can be computed and we obtain

$$\sum_{ss'} \xi_s^\dagger n^\mu(\mathbf{p}) \xi_{s'} f_{ss'}^{(+)}(x, \mathbf{p}) = \frac{1}{|\mathbf{b}|} \left(\mathbf{p} \cdot \mathbf{b}, m\mathbf{b} + \frac{\mathbf{p} \cdot \mathbf{b}}{E_{\mathbf{p}} + m} \mathbf{p} \right)^T \sum_s s \tilde{f}_s^{(+)}(x, \mathbf{p}). \quad (3.106)$$

Similar results can be done for anti-fermions. Finally the function $n^{(0)\mu}(x, p)$, defined in Eq. (3.88), becomes

$$\begin{aligned} n^{(0)\mu}(x, p) &= \left[\theta(p^0) n_0^\mu(\mathbf{p}, \mathbf{b}^{(+)}) - \theta(p^0) n_0^\mu(-\mathbf{p}, \mathbf{b}^{(-)}) \right] \\ &\times \frac{2}{(2\pi)^3} \sum_s s \left\{ \theta(p^0) \tilde{f}_s^{(+)}(x, \mathbf{p}) - \theta(-p^0) \left[1 - \tilde{f}_s^{(-)}(x, -\mathbf{p}) \right] \right\}, \end{aligned} \quad (3.107)$$

where $\mathbf{b}^{(+)}$ represents the diagonalization parameter for fermions while $\mathbf{b}^{(-)}$ is that for anti-fermions. Here we defined

$$n_0^\mu(\mathbf{p}, \mathbf{b}) \equiv \frac{1}{|\mathbf{b}|} \left(\mathbf{p} \cdot \mathbf{b}, m\mathbf{b} + \frac{\mathbf{p} \cdot \mathbf{b}}{E_{\mathbf{p}} + m} \mathbf{p} \right)^T. \quad (3.108)$$

If we boost to the rest frame of the particles with momentum \mathbf{p} , the above polarization direction is $n_0^\mu \propto (0, \mathbf{b})^T$. Thus $\mathbf{b}^{(\pm)}$ can be identified as the spin polarization direction in the rest frame of fermions and anti-fermions respectively. In general the polarization of fermions can be different from that of anti-fermions, i.e., $\mathbf{b}^{(+)}$ can be different from $\mathbf{b}^{(-)}$.

B. Free fermions with chiral imbalance

In this subsection we will study a system of free fermions with a non-vanishing chiral chemical potential. Since for massive fermions the helicity is not a conserved quantity, the chiral chemical potential μ_5 is no longer a well-defined conjugate variable of the axial charge. However, on a time scale which is much smaller than the one for varying axial charges, one can still use μ_5 to describe a thermal equilibrium system. We work with an effective theory where the chemical potentials μ and μ_5 are introduced in the Dirac equation as self-energy corrections. The effective Lagrangian reads,

$$\mathcal{L} = \bar{\psi} (i\gamma_\mu \partial_x^\mu - m\mathbb{I}_4) \psi + \mu \psi^\dagger \psi + \mu_5 \psi^\dagger \gamma^5 \psi. \quad (3.109)$$

We can find the similar treatment in the Nambu-Jona-Lasinio model [129] or other QCD effective theories with topological charge [18, 22]. The Dirac equation is then given by

$$(i\gamma_\mu \partial_x^\mu - m\mathbb{I}_4 + \mu\gamma^0 + \mu_5\gamma^0\gamma^5) \psi = 0. \quad (3.110)$$

In general the mass m , the chemical potential μ , and the chiral chemical potential μ_5 are dynamical quantities which depend on the space-time coordinates, but here we assume all these variables are constants. Under this assumption, ∂_x^μ commute with the Hamiltonian, so we can define a conserved 4-momentum p^μ .

1. Plane-wave solutions with chiral imbalance

Analogous to the case in the previous subsection, we first take a Fourier transformation of the Dirac equation. Then in momentum space, the Dirac field satisfies the following equation

$$(p^0 + \mu)\psi(p) = [\gamma^0 \boldsymbol{\gamma} \cdot \mathbf{p} + m\gamma^0 - \mu_5\gamma^5]\psi(p). \quad (3.111)$$

Here the chemical potential μ shifts the energy levels. We now define

$$\tilde{\psi}(\mathbf{p}) = (\mathbb{I}_2 \otimes S_{\mathbf{p}}^\dagger) \psi(\mathbf{p}), \quad (3.112)$$

where the matrix $S_{\mathbf{p}}$ is the transformation matrix in Eq. (3.25) which diagonalizes $\mathbf{p} \cdot \boldsymbol{\sigma}$ as shown in Eq. (3.26). With this definition, Eq. (3.111) is put into the following form,

$$(p^0 + \mu)\tilde{\psi}(\mathbf{p}) = \begin{pmatrix} -|\mathbf{p}| + \mu_5 & 0 & m & 0 \\ 0 & |\mathbf{p}| + \mu_5 & 0 & m \\ m & 0 & |\mathbf{p}| - \mu_5 & 0 \\ 0 & m & 0 & -|\mathbf{p}| - \mu_5 \end{pmatrix} \tilde{\psi}(p). \quad (3.113)$$

This can be treated as an eigenvalue problem, with $p^0 + \mu$ being the eigenvalue of the coefficient matrix on the right-hand-side while $\tilde{\psi}(p)$ is the corresponding eigenstate. Direct forward calculations give the eigenvalues

$$p^0 = -\mu \pm E_{\mathbf{p},s}, \quad (3.114)$$

with $E_{\mathbf{p},s} = \sqrt{m^2 + (|\mathbf{p}| - s\mu_5)^2}$ and $s = \pm$. The eigenstates corresponding to positive energies $p^0 + \mu = E_{\mathbf{p},s}$ are given by

$$\tilde{\psi}_+^{(+)}(\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p},+} - (|\mathbf{p}| - \mu_5)} \\ 0 \\ \sqrt{E_{\mathbf{p},+} + (|\mathbf{p}| - \mu_5)} \\ 0 \end{pmatrix}, \quad \tilde{\psi}_-^{(+)}(\mathbf{p}) = \begin{pmatrix} 0 \\ \sqrt{E_{\mathbf{p},-} + (|\mathbf{p}| + \mu_5)} \\ 0 \\ \sqrt{E_{\mathbf{p},-} - (|\mathbf{p}| + \mu_5)} \end{pmatrix}, \quad (3.115)$$

while the eigenstates for negative energies $p^0 + \mu = -E_{\mathbf{p},s}$ read

$$\tilde{\psi}_+^{(-)}(\mathbf{p}) = \begin{pmatrix} -\sqrt{E_{\mathbf{p},+} + |\mathbf{p}| - \mu_5} \\ 0 \\ \sqrt{E_{\mathbf{p},+} - (|\mathbf{p}| - \mu_5)} \\ 0 \end{pmatrix}, \quad \tilde{\psi}_-^{(-)}(\mathbf{p}) = \begin{pmatrix} 0 \\ -\sqrt{E_{\mathbf{p},-} - (|\mathbf{p}| + \mu_5)} \\ 0 \\ \sqrt{E_{\mathbf{p},-} + (|\mathbf{p}| + \mu_5)} \end{pmatrix}. \quad (3.116)$$

The wavefunctions in Eqs. (3.115) and (3.116) are normalized

$$\tilde{\psi}_{s_2}^{(s_1)\dagger}(\mathbf{p})\tilde{\psi}_{s_2'}^{(s_1')}(\mathbf{p}) = 2E_{\mathbf{p},s_2}\delta_{s_1s_1'}\delta_{s_2s_2'}. \quad (3.117)$$

Here they are normalized to the corresponding eigenenergies in order to smoothly reproduce the normalization relations without chiral chemical potential in Eq. (3.35). The wavefunctions in coordinate space are then computed by adding a Fourier factor

$$\psi_{s_2}^{(s_1)}(x, \mathbf{p}) = e^{i\mu t} \exp(-i s_1 E_{\mathbf{p},s_2} t + i\mathbf{p} \cdot \mathbf{x}) (\mathbb{I}_2 \otimes \mathbf{S}_{\mathbf{p}}) \tilde{\psi}_{s_2}^{(s_1)}(\mathbf{p}), \quad (3.118)$$

In the solution (3.118), $s_1 = \pm$ labels fermions (+) or anti-fermions (-). The states with $s_1 = +$ are interpreted as fermions with the kinetic momentum \mathbf{p} while $s_1 = -$ as anti-fermions with the kinetic momentum $-\mathbf{p}$. Meanwhile, $s_2 = \pm$ does not have an explicit meaning in the massive case. But in the massless limit it parameterizes the chirality.

The wavefunctions in Eqs. (3.115), (3.116) are superpositions of the LH states and the RH ones. In the massless limit, the eigenenergies are given by $E_{\mathbf{p},s} = (|\mathbf{p}| - s\mu_5)\text{sgn}(|\mathbf{p}| - s\mu_5)$, where sgn is the sign function. Meanwhile, the wavefunction $\tilde{\psi}_+^{(+)}(\mathbf{p})$ in Eq. (3.115) reduces to the following

expression,

$$\tilde{\psi}_+^{(+)}(\mathbf{p}) = \|\mathbf{p} - \mu_5\| \begin{pmatrix} \sqrt{1 - \text{sgn}(|\mathbf{p} - \mu_5|)} \\ 0 \\ \sqrt{1 + \text{sgn}(|\mathbf{p} - \mu_5|)} \\ 0 \end{pmatrix}, \quad (3.119)$$

which represents a RH wavefunction if $|\mathbf{p}| - \mu_5$ is positive, and a LH wavefunction if $|\mathbf{p}| - \mu_5$ is negative. Similar discussion can be done for other functions in Eqs. (3.115), (3.116). We conclude that if $|\mathbf{p}| - \mu_5 > 0$, $\tilde{\psi}_+^{(+)}(\mathbf{p})$ and $\tilde{\psi}_-^{(-)}(\mathbf{p})$ are RH while $\tilde{\psi}_-^{(+)}(\mathbf{p})$, and $\tilde{\psi}_+^{(-)}(\mathbf{p})$ are LH. On the other hand, if $|\mathbf{p}| - \mu_5 < 0$, $\tilde{\psi}_+^{(+)}(\mathbf{p})$ and $\tilde{\psi}_-^{(-)}(\mathbf{p})$ are LH while $\tilde{\psi}_-^{(+)}(\mathbf{p})$ and $\tilde{\psi}_+^{(-)}(\mathbf{p})$ are RH.

On the other hand, we can consider the limit $\mu_5 \rightarrow 0$, which corresponds to a state where the chiral symmetry is restored. We find that the eigenenergies are now independent of s , which read $E_{\mathbf{p},s} = E_{\mathbf{p}} = \sqrt{m^2 + |\mathbf{p}|^2}$. The states $\tilde{\psi}_{\pm}^{(+)}(\mathbf{p})$ then have the same eigenenergy $p^0 = -\mu + E_{\mathbf{p}}$ while $\tilde{\psi}_{\pm}^{(-)}(\mathbf{p})$ have eigenenergy $p^0 = -\mu - E_{\mathbf{p}}$. The wavefunctions in Eqs. (3.115) and (3.116) reduce to the following forms in this limit,

$$\begin{aligned} \tilde{\psi}_+^{(+)}(\mathbf{p}) &= \begin{pmatrix} \sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \\ 0 \\ \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \\ 0 \end{pmatrix}, & \tilde{\psi}_-^{(+)}(\mathbf{p}) &= \begin{pmatrix} 0 \\ \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \\ 0 \\ \sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \end{pmatrix}, \\ \tilde{\psi}_-^{(-)}(\mathbf{p}) &= \begin{pmatrix} 0 \\ -\sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \\ 0 \\ \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \end{pmatrix}, & \tilde{\psi}_+^{(-)}(\mathbf{p}) &= \begin{pmatrix} -\sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \\ 0 \\ \sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \\ 0 \end{pmatrix}, \end{aligned} \quad (3.120)$$

where all these functions are normalized to $2E_{\mathbf{p}}$ respectively. For a nonzero mass, all these states are superposition of RH and LH spinors. But in the massless case, the energy $E_{\mathbf{p}} = |\mathbf{p}|$ and the subscript labels the helicity. In order to show the coincidence with the plane-wave solution in Eq. (3.34), we form a linear combination between the states with the same eigenenergies,

$$\begin{aligned} \tilde{\xi}_1 \tilde{\psi}_+^{(+)}(\mathbf{p}) \pm \tilde{\xi}_2 \tilde{\psi}_-^{(+)}(\mathbf{p}) &\rightarrow \tilde{\psi}_{\pm}^{(+)}(\mathbf{p}), \\ -\left[\tilde{\xi}_1 \tilde{\psi}_+^{(-)}(\mathbf{p}) \pm \tilde{\xi}_2 \tilde{\psi}_-^{(-)}(\mathbf{p}) \right] &\rightarrow \tilde{\psi}_{\pm}^{(-)}(-\mathbf{p}). \end{aligned}$$

These wavefunctions read,

$$\tilde{\psi}_{\pm}^{(+)}(\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \tilde{\xi}_1 \\ \pm \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \tilde{\xi}_2 \\ \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \tilde{\xi}_1 \\ \pm \sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \tilde{\xi}_2 \end{pmatrix}, \quad \tilde{\psi}_{\pm}^{(-)}(-\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \tilde{\xi}_1 \\ \pm \sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \tilde{\xi}_2 \\ -\sqrt{E_{\mathbf{p}} - |\mathbf{p}|} \tilde{\xi}_1 \\ \mp \sqrt{E_{\mathbf{p}} + |\mathbf{p}|} \tilde{\xi}_2 \end{pmatrix}. \quad (3.121)$$

We further demand that

$$\xi_s \equiv S_{\mathbf{p}} \begin{pmatrix} \tilde{\xi}_1 \\ \pm \tilde{\xi}_2 \end{pmatrix}, \quad (3.122)$$

and using Eq. (3.30) and Eq. (3.112) we finally obtain

$$\psi_s^{(+)}(\mathbf{p}) = \begin{pmatrix} \sqrt{p_{\mu} \sigma^{\mu} \xi_s} \\ \sqrt{p_{\mu} \bar{\sigma}^{\mu} \xi_s} \end{pmatrix}, \quad \psi_s^{(-)}(-\mathbf{p}) = \begin{pmatrix} \sqrt{p_{\mu} \bar{\sigma}^{\mu} \xi_s} \\ -\sqrt{p_{\mu} \sigma^{\mu} \xi_s} \end{pmatrix}, \quad (3.123)$$

which agree with the previous results (3.34). Thus we conclude that in the presence of a constant chiral chemical potential μ_5 , the single-particle wavefunctions in (3.118) when setting $\mu_5 = 0$, up to a linear combination, coincide with the solutions without μ_5 in the previous subsection.

2. Chiral quantization

Analogous to the case without chiral imbalance, the fermionic field can be quantized using the single-particle wavefunctions with finite μ_5 , which are given in Eq. (3.118),

$$\hat{\psi}(x) = e^{i\mu t} \sum_s \int \frac{d^3 \mathbf{p}}{(2\pi)^3 \sqrt{2E_{\mathbf{p},s}}} e^{i\mathbf{p} \cdot \mathbf{x}} (\mathbb{I}_2 \otimes S_{\mathbf{p}}) \left[e^{-iE_{\mathbf{p},s}t} \tilde{\psi}_s^{(+)}(\mathbf{p}) \hat{a}_{\mathbf{p},s} + e^{iE_{\mathbf{p},s}t} \tilde{\psi}_s^{(-)}(\mathbf{p}) \hat{b}_{-\mathbf{p},s}^{\dagger} \right], \quad (3.124)$$

where the creation and annihilation operators satisfy the following canonical anti-commutation relations

$$\left\{ \hat{a}_{\mathbf{p},s}, \hat{a}_{\mathbf{p}',s'}^{\dagger} \right\} = \left\{ \hat{b}_{\mathbf{p},s}, \hat{b}_{\mathbf{p}',s'}^{\dagger} \right\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}') \delta_{ss'}, \quad (3.125)$$

while all other anti-commutators vanish. In order to check whether the fermionic field is correctly quantized, we calculate the anti-commutator for the field operator $\hat{\psi}$ and its Hermitian conjugate $\hat{\psi}^{\dagger}$,

$$\left\{ \hat{\psi}_{\alpha}(t, \mathbf{x}), \hat{\psi}_{\beta}^{\dagger}(t, \mathbf{x}') \right\} = \delta_{\alpha\beta} \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \quad (3.126)$$

where α, β label components of the Dirac field. Furthermore, other anti-commutators, such as $\{\psi_\alpha(t, \mathbf{x}), \psi_\beta(t, \mathbf{x}')\}$ and $\{\psi_\alpha^\dagger(t, \mathbf{x}), \psi_\beta^\dagger(t, \mathbf{x}')\}$, vanish because they do not contain any nonzero anti-commutator (3.125). Note that the field operator in Eq. (3.124) recovers the one in Eq. (3.37) if we take $\mu_5 = 0$.

The Hamiltonian operator \hat{H} is now given by

$$\hat{H} = \int d^3\mathbf{x} \hat{\psi}^\dagger (-i\gamma^0 \boldsymbol{\gamma} \cdot \nabla_{\mathbf{x}} - m - \mu - \mu_5 \gamma^5) \hat{\psi} = \hat{H}_0 - \mu \hat{N} - \mu_5 \hat{N}_5, \quad (3.127)$$

where $\hat{H}_0 = \psi^\dagger (-i\gamma^0 \boldsymbol{\gamma} \cdot \nabla_{\mathbf{x}} - m) \psi$ is the free fermion Hamiltonian, $\hat{N} = \psi^\dagger \psi$ is the net particle number operator and $\hat{N}_5 = \psi^\dagger \gamma^5 \psi$ is the axial-charge operator. Inserting the quantized field operator (3.124) into the Hamiltonian, we obtain

$$\hat{H} = \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[(E_{\mathbf{p},s} - \mu) \hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} + (E_{\mathbf{p},s} + \mu) \left(\hat{b}_{-\mathbf{p},s}^\dagger \hat{b}_{-\mathbf{p},s} - 1 \right) \right]. \quad (3.128)$$

From the above expression we observe that the lowest energy state is no longer empty. In the lowest energy state, all the states with $E_{\mathbf{p},s} < \mu$ are occupied, which agrees with our expectation. The chemical potential means the system has non-vanishing net fermion number. And the lowest energy state is reached when the thermal temperature is zero and the fermions occupy all states below the Fermi surface. On the other hand, the momentum operator is

$$\hat{\mathbf{P}} = \int d^3\mathbf{x} \hat{\psi}^\dagger (-i\nabla_{\mathbf{x}}) \hat{\psi} = \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p} \left(\hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} + \hat{b}_{\mathbf{p},s}^\dagger \hat{b}_{\mathbf{p},s} \right). \quad (3.129)$$

The Hamiltonian and momentum operators indicate that $\hat{a}_{\mathbf{p},s}^\dagger$ plays as the creation operator of a fermion with the momentum \mathbf{p} and the energy $E_{\mathbf{p},s} - \mu$ while $\hat{b}_{\mathbf{p},s}^\dagger$ creates an anti-fermion with the same momentum \mathbf{p} and the energy $E_{\mathbf{p},s} + \mu$.

3. Wigner function

Inserting the field operator (3.124) into the definition (2.18) of the Wigner function, one obtains

$$\begin{aligned} W(x, p) = & \sum_{ss'} \int \frac{dt' d^3\mathbf{x}'}{(2\pi)^4} \int \frac{d^3\mathbf{q} d^3\mathbf{q}'}{(2\pi)^6 \sqrt{2E_{\mathbf{q},s}} \sqrt{2E_{\mathbf{q}',s'}}} \exp \left[i\mathbf{x}' \cdot \left(\mathbf{p} - \frac{\mathbf{q} + \mathbf{q}'}{2} \right) + i\mathbf{x} \cdot (\mathbf{q} - \mathbf{q}') \right] \\ & \times \left\{ \exp \left[-it' \left(p^0 + \mu - \frac{E_{\mathbf{q}',s'} + E_{\mathbf{q},s}}{2} \right) + it (E_{\mathbf{q}',s'} - E_{\mathbf{q},s}) \right] \right. \\ & \quad \times \tilde{\psi}_{s'}^{(+)\dagger}(\mathbf{q}') \left(\mathbb{I}_2 \otimes S_{\mathbf{q}'}^\dagger \right) \gamma^0 \otimes \left[(\mathbb{I}_2 \otimes S_{\mathbf{q}}) \tilde{\psi}_s^{(+)}(\mathbf{q}) \right] \left\langle \Omega \left| \hat{a}_{\mathbf{q}',s'}^\dagger \hat{a}_{\mathbf{q},s} \right| \Omega \right\rangle \\ & + \exp \left[-it' \left(p^0 + \mu + \frac{E_{\mathbf{q}',s'} + E_{\mathbf{q},s}}{2} \right) - it (E_{\mathbf{q}',s'} - E_{\mathbf{q},s}) \right] \\ & \quad \times \tilde{\psi}_{s'}^{(-)\dagger}(\mathbf{q}') \left(\mathbb{I}_2 \otimes S_{\mathbf{q}'}^\dagger \right) \gamma^0 \otimes \left[(\mathbb{I}_2 \otimes S_{\mathbf{q}}) \tilde{\psi}_s^{(-)}(\mathbf{q}) \right] \left\langle \Omega \left| \hat{b}_{-\mathbf{q}',s'}^\dagger \hat{b}_{-\mathbf{q},s} \right| \Omega \right\rangle \left. \right\}, \quad (3.130) \end{aligned}$$

where we have dropped terms of $a_{\mathbf{q}',s'}^\dagger b_{-\mathbf{q},s}^\dagger$ or $b_{-\mathbf{q}',s'} a_{\mathbf{q},s}$. These terms represent the mixtures between fermion states and anti-fermion states, which are not considered in this thesis. Analogous to the discussion in subsection III A, we introduce the average and relative momenta,

$$\mathbf{k} = \frac{\mathbf{q} + \mathbf{q}'}{2}, \quad \mathbf{u} = \mathbf{q} - \mathbf{q}'. \quad (3.131)$$

The integration measure is then invariant under the momentum redefinition,

$$d^3\mathbf{q}d^3\mathbf{q}' = d^3\mathbf{k}d^3\mathbf{u}. \quad (3.132)$$

In the Wigner function, the integration over $d^3\mathbf{x}'$ gives a 3-dimensional delta-function for the momentum, while the integration over dt' gives a delta-function for the energy. After a straightforward calculation, we obtain

$$\begin{aligned} W(x, p) = & \sum_{ss'} \int \frac{d^3\mathbf{u}}{(2\pi)^6 \sqrt{2E_{\mathbf{p}+\mathbf{u}/2,s}} \sqrt{2E_{\mathbf{p}-\mathbf{u}/2,s'}}} e^{i\mathbf{u}\cdot\mathbf{x}} \\ & \times \left\{ \delta \left(p^0 + \mu - \frac{E_{\mathbf{p}-\mathbf{u}/2,s'} + E_{\mathbf{p}+\mathbf{u}/2,s}}{2} \right) \exp \left[it \left(E_{\mathbf{p}-\mathbf{u}/2,s'} - E_{\mathbf{p}+\mathbf{u}/2,s} \right) \right] \right. \\ & \quad \times \tilde{\psi}_{s'}^{(+)\dagger} \left(\mathbf{p} - \frac{\mathbf{u}}{2} \right) \left(\mathbb{I}_2 \otimes S_{\mathbf{p}-\frac{\mathbf{u}}{2}}^\dagger \right) \gamma^0 \otimes \left[\left(\mathbb{I}_2 \otimes S_{\mathbf{p}+\frac{\mathbf{u}}{2}} \right) \tilde{\psi}_s^{(+)} \left(\mathbf{p} + \frac{\mathbf{u}}{2} \right) \right] \\ & \quad \times \left\langle \Omega \left| a_{\mathbf{p}-\frac{\mathbf{u}}{2},s'}^\dagger a_{\mathbf{p}+\frac{\mathbf{u}}{2},s} \right| \Omega \right\rangle \\ & + \delta \left(p^0 + \mu + \frac{E_{\mathbf{p}-\mathbf{u}/2,s'} + E_{\mathbf{p}+\mathbf{u}/2,s}}{2} \right) \exp \left[-it \left(E_{\mathbf{p}-\mathbf{u}/2,s'} - E_{\mathbf{p}+\mathbf{u}/2,s} \right) \right] \\ & \quad \times \tilde{\psi}_{s'}^{(-)\dagger} \left(\mathbf{p} - \frac{\mathbf{u}}{2} \right) \left(\mathbb{I}_2 \otimes S_{\mathbf{p}-\frac{\mathbf{u}}{2}}^\dagger \right) \gamma^0 \otimes \left[\left(\mathbb{I}_2 \otimes S_{\mathbf{p}+\frac{\mathbf{u}}{2}} \right) \tilde{\psi}_s^{(-)} \left(\mathbf{p} + \frac{\mathbf{u}}{2} \right) \right] \\ & \quad \times \left\langle \Omega \left| b_{-\mathbf{p}+\frac{\mathbf{u}}{2},s'} b_{-\mathbf{p}-\frac{\mathbf{u}}{2},s}^\dagger \right| \Omega \right\rangle \left. \right\}. \quad (3.133) \end{aligned}$$

Again we adopt the wave-packet prescription and assume that the expectation value is given by some distribution function,

$$\begin{aligned} \left\langle \Omega \left| a_{\mathbf{p}-\frac{\mathbf{u}}{2},s'}^\dagger a_{\mathbf{p}+\frac{\mathbf{u}}{2},s} \right| \Omega \right\rangle &= f_s^{(+)}(\mathbf{p}, \mathbf{u}) \delta_{ss'}, \\ \left\langle \Omega \left| b_{-\mathbf{p}+\frac{\mathbf{u}}{2},s'} b_{-\mathbf{p}-\frac{\mathbf{u}}{2},s}^\dagger \right| \Omega \right\rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{u}) \delta_{ss'} - f_s^{(-)}(-\mathbf{p}, \mathbf{u}) \delta_{ss'}. \quad (3.134) \end{aligned}$$

Here the presence of $\delta_{ss'}$ takes into account that states with different s have different energy shells. We further define the distribution function in phase space,

$$f_s^{(\pm)}(x, \mathbf{p}) \equiv \int \frac{d^3\mathbf{u}}{(2\pi)^3} e^{i\mathbf{u}\cdot\mathbf{x}} \exp \left[\pm it \left(E_{\mathbf{p}-\mathbf{u}/2,s} - E_{\mathbf{p}+\mathbf{u}/2,s} \right) \right] f_s^{(\pm)}(\mathbf{p}, \mathbf{u}). \quad (3.135)$$

Analogous to what we did for the free fermion case in subsection III A, we expand the Wigner function in terms of \mathbf{u} . Since we adopt the wave-packet prescription, the relative momentum \mathbf{u}

contributes if it is much smaller than the width of the wave packet. Thus \mathbf{u} is treated as a small variable and the Wigner function at leading order in spatial gradient reads,

$$W^{(0)}(x, p) = \sum_s \frac{1}{(2\pi)^3} \left\{ \delta(p^0 + \mu - E_{\mathbf{p},s}) W_s^{(+)}(\mathbf{p}) f_s^{(+)}(x, \mathbf{p}) + \delta(p^0 + \mu + E_{\mathbf{p},s}) W_s^{(-)}(\mathbf{p}) \left[1 - f_s^{(-)}(x, -\mathbf{p}) \right] \right\}, \quad (3.136)$$

where we have defined the following terms for contributions from fermions or anti-fermions,

$$W_s^{(\pm)}(\mathbf{p}) \equiv \frac{1}{2E_{\mathbf{p},s}} \tilde{\psi}_s^{(\pm)\dagger}(\mathbf{p}) \left(\mathbb{I}_2 \otimes S_{\mathbf{p}}^\dagger \right) \gamma^0 \otimes \left[(\mathbb{I}_2 \otimes S_{\mathbf{p}}) \tilde{\psi}_s^{(\pm)}(\mathbf{p}) \right]. \quad (3.137)$$

Here the single-particle wavefunctions $\tilde{\psi}_s^{(r)}(\mathbf{p})$ are listed in Eqs. (3.115) and (3.116), meanwhile the transformation matrices $S_{\mathbf{p}}$ and $S_{\mathbf{p}}^\dagger$ are given in Eq. (3.25). The delta-function for the energy can be written in a covariant form

$$\frac{1}{2E_{\mathbf{p},s}} \delta(p^0 + \mu - rE_{\mathbf{p},s}) = \theta[r(p^0 + \mu)] \delta[(p^0 + \mu)^2 - (|\mathbf{p}| - s\mu_5)^2 - m^2], \quad (3.138)$$

which recovers with the normal on-shell condition when setting $\mu = \mu_5 = 0$.

Then the next step is to insert the single-particle wavefunctions (3.115) and (3.116) into Eq. (3.136) and calculate 16 components of the Wigner function. First we focus on the fermion part, $\tilde{\psi}_s^{(+)}(\mathbf{p})$, which can be written in terms of the Kronecker product of two column vectors,

$$\tilde{\psi}_s^{(+)}(\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p},+} - (|\mathbf{p}| - \mu_5)} \\ \sqrt{E_{\mathbf{p},+} + (|\mathbf{p}| - \mu_5)} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\psi}_s^{(+)}(\mathbf{p}) = \begin{pmatrix} \sqrt{E_{\mathbf{p},-} + (|\mathbf{p}| + \mu_5)} \\ \sqrt{E_{\mathbf{p},-} - (|\mathbf{p}| + \mu_5)} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.139)$$

Then using the property of the Kronecker product in Eq. (A10), we obtain

$$W_s^{(+)}(\mathbf{p}) = \frac{1}{2E_{\mathbf{p},s}} \left[\begin{pmatrix} m & E_{\mathbf{p},s} - s(|\mathbf{p}| - s\mu_5) \\ E_{\mathbf{p},s} + s(|\mathbf{p}| - s\mu_5) & m \end{pmatrix} \right] \otimes \left[\begin{pmatrix} \delta_{s+} & \delta_{s-} \\ \delta_{s+} & \delta_{s-} \end{pmatrix} S_{\mathbf{p}}^\dagger \otimes S_{\mathbf{p}} \begin{pmatrix} \delta_{s+} \\ \delta_{s-} \end{pmatrix} \right]. \quad (3.140)$$

The explicit forms of $S_{\mathbf{p}}$ and $S_{\mathbf{p}}^\dagger$ are given in Eq. (3.25). Then after some complicated but straightforward calculation we obtain

$$\begin{pmatrix} \delta_{s+} & \delta_{s-} \\ \delta_{s+} & \delta_{s-} \end{pmatrix} S_{\mathbf{p}}^\dagger \otimes S_{\mathbf{p}} \begin{pmatrix} \delta_{s+} \\ \delta_{s-} \end{pmatrix} = \frac{1}{2} \mathbb{I}_2 + s \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2|\mathbf{p}|}. \quad (3.141)$$

With the help of the Kronecker-product of the gamma matrices in Eq. (A8), we obtain

$$W_s^{(+)}(\mathbf{p}) = \frac{1}{4E_{\mathbf{p},s}} \left[m \left(\mathbb{I}_4 + s \frac{1}{2|\mathbf{p}|} \epsilon^{ijk} \sigma^{ij} \mathbf{p}^k \right) + E_{\mathbf{p},s} \left(\gamma^0 - s \frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \boldsymbol{\gamma} \right) + (|\mathbf{p}| - s\mu_5) \left(s\gamma^5 \gamma^0 - \frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \boldsymbol{\gamma} \right) \right]. \quad (3.142)$$

An analogous calculation can be done for anti-fermions, which gives

$$W_s^{(-)}(\mathbf{p}) = \frac{1}{2E_{\mathbf{p},s}} \left[\begin{pmatrix} -m & E_{\mathbf{p},s} + s(|\mathbf{p}| - s\mu_5) \\ E_{\mathbf{p},s} - s(|\mathbf{p}| - s\mu_5) & -m \end{pmatrix} \right] \otimes \left[\begin{pmatrix} \delta_{s+} & \\ & \delta_{s-} \end{pmatrix} S_{\mathbf{p}}^\dagger \otimes S_{\mathbf{p}} \begin{pmatrix} \delta_{s+} \\ \delta_{s-} \end{pmatrix} \right]. \quad (3.143)$$

Then using Eqs. (3.141) and (A8) we can express $W_s^{(-)}(\mathbf{p})$ in gamma matrices,

$$W_s^{(-)}(\mathbf{p}) = \frac{1}{4E_{\mathbf{p},s}} \left[-m \left(\mathbb{I}_4 + s \frac{1}{2|\mathbf{p}|} \epsilon^{ijk} \sigma^{ij} \mathbf{p}^k \right) + E_{\mathbf{p},s} \left(\gamma^0 - s \frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \boldsymbol{\gamma}^5 \boldsymbol{\gamma} \right) - (|\mathbf{p}| - s\mu_5) \left(s\gamma^5 \gamma^0 - \frac{1}{|\mathbf{p}|} \mathbf{p} \cdot \boldsymbol{\gamma} \right) \right]. \quad (3.144)$$

Inserting these back into the Wigner function in Eq. (3.136) and taking the trace after multiplying with $\Gamma_i = \{\mathbb{I}_4, i\gamma^5, \gamma^\mu, \gamma^5 \gamma^\mu, \frac{1}{2} \sigma^{\mu\nu}\}$, we can extract different components of the Wigner function,

$$\begin{aligned} \mathcal{F} &= mV(x, p), \\ \mathcal{P} &= 0, \\ \mathcal{V}^0 &= (p^0 + \mu)V(x, p), \\ \boldsymbol{\mathcal{V}} &= \mathbf{p} \left[V(x, \mathbf{p}) - \frac{\mu_5}{|\mathbf{p}|} A(x, p) \right], \\ \mathcal{A}^0 &= |\mathbf{p}| \left[A(x, p) - \frac{\mu_5}{|\mathbf{p}|} V(x, p) \right], \\ \boldsymbol{\mathcal{A}} &= \frac{\mathbf{p}}{|\mathbf{p}|} (p^0 + \mu) A(x, p), \\ \mathcal{S}^{0i} &= 0, \\ \mathcal{S}^{ij} &= \frac{m}{|\mathbf{p}|} \epsilon^{ijk} \mathbf{p}^k A(x, p), \end{aligned} \quad (3.145)$$

where we have defined

$$\begin{aligned} V(x, p) &\equiv \frac{2}{(2\pi)^3} \sum_s \delta \left[(p^0 + \mu)^2 - (|\mathbf{p}| - s\mu_5)^2 - m^2 \right] \\ &\quad \times \left\{ \theta(p^0 + \mu) f_s^{(+)}(x, \mathbf{p}) - \theta[-(p^0 + \mu)] \left[1 - f_s^{(-)}(x, -\mathbf{p}) \right] \right\}, \\ A(x, p) &\equiv \frac{2}{(2\pi)^3} \sum_s \delta \left[(p^0 + \mu)^2 - (|\mathbf{p}| - s\mu_5)^2 - m^2 \right] \\ &\quad \times s \left\{ \theta(p^0 + \mu) f_s^{(+)}(x, \mathbf{p}) - \theta[-(p^0 + \mu)] \left[1 - f_s^{(-)}(x, -\mathbf{p}) \right] \right\}. \end{aligned} \quad (3.146)$$

Note that the presence of μ and μ_5 breaks the Lorentz covariance of the Wigner function. That is why in (3.145), we separately listed \mathcal{V}^0 and $\boldsymbol{\mathcal{V}}$ instead of a four-vector \mathcal{V}^μ . For the same reason, the

axial-vector \mathcal{A}^μ is separated into \mathcal{A}^0 and \mathbf{A} , while the tensor $\mathcal{S}^{\mu\nu}$ is separated into the electric-like components \mathcal{S}^{0i} and the magnetic-like components \mathcal{S}^{ij} .

In the case without chiral imbalance, we have four undetermined functions, $V^{(0)}(x, p)$ and $n^{(0)\mu}(x, p)$, which are defined in (3.88). (Here $p_\mu n^{(0)\mu} = 0$ thus $n^{(0)\mu}$ has only three independent components.) However, in (3.145), we only have two functions, $V(x, p)$ and $A(x, p)$. This loss of degrees of freedom is attributed to the spin degeneracy. In the case without μ_5 , the energy states are degenerate with respect to spin. In the particle's rest frame, its spin can take arbitrary spatial direction. However, a finite chiral chemical potential μ_5 breaks the spin degeneracy, so the eigenstates, given by Eqs. (3.115) and (3.116), have fixed spin directions, i.e. along the direction of \mathbf{p} . Hence in (3.145) the polarization density \mathbf{A} is parallel to \mathbf{p} . This is different to the case $\mu_5 = 0$, where the polarization density \mathbf{A} can point in any direction. The reason for this difference is because we forbid the mixture between different energy levels. In Eq. (3.134) we assume that the expectation values of $a_{\mathbf{p}-\frac{\mathbf{u}}{2}, s'}^\dagger a_{\mathbf{p}+\frac{\mathbf{u}}{2}, s}$ and $b_{-\mathbf{p}+\frac{\mathbf{u}}{2}, s'}^\dagger b_{-\mathbf{p}-\frac{\mathbf{u}}{2}, s}$ are proportional to $\delta_{ss'}$, because states with different s are supported on different energy levels. If we allow the mixture between different spin states, the final result would have more degrees of freedom and thus is expected to coincide with the result in subsection III A in the limit $\mu_5 \rightarrow 0$.

C. Fermions in a constant magnetic field

1. Dirac equation

In a constant magnetic field, the transverse momentum of a particle is discrete while the momentum along the direction of the magnetic field stays continuous. The eigenenergies are given by the well-known Landau energy levels

$$E_{p^z}^{(n)} = \sqrt{m^2 + (p^z)^2 + 2nB_0}, \quad (3.147)$$

where B_0 is the strength of the magnetic field and $n = 0, 1, 2, \dots$ label the Landau levels. Note that the electric charge has been absorbed into the field. We can rewrite the quantum number n as $n = n' + \frac{1}{2} + \frac{1}{2}s$, with $n' = 0, 1, 2, \dots$ denotes the orbital quantum number and $s = \pm$ represents the spin direction. Then the lowest Landau level $n = 0$ corresponds to $n' = 0$ and $s = -$, which means the particles in the lowest Landau level $n = 0$ have a definite spin direction. According to the principle of minimum energy, the spins of fermions with positive charges are parallel, while those of negatively charged anti-fermions are anti-parallel, to the direction of the magnetic field.

Meanwhile, higher Landau levels $n > 0$ are degenerate for $n' = n, s = -$ and $n' = n - 1, s = +$, which means these levels are 2-fold degenerate.

In this section we will consider fermions in a constant magnetic field. Since the magnetic field is not Lorentz-covariant itself, we should choose a specific frame. The choice of the frame will break the covariance. We also consider finite μ and μ_5 . The Dirac equation reads in this case

$$i\frac{\partial}{\partial t}\psi = [\gamma^0\boldsymbol{\gamma} \cdot (-i\nabla - \mathbb{A}) + m\gamma^0 - \mu - \mu_5\gamma^5]\psi. \quad (3.148)$$

Here μ and μ_5 are assumed to be constant. We can read off the Hamilton operator from the above equation

$$\hat{H} = \gamma^0\boldsymbol{\gamma} \cdot (-i\nabla - \mathbb{A}) + m\gamma^0 - \mu - \mu_5\gamma^5. \quad (3.149)$$

Without loss of generality, the magnetic field is taken in the z -direction. The magnetic field and gauge potential are

$$\begin{aligned} \mathbf{B} &= B_0\mathbf{e}_z, \\ \mathbb{A} &= -B_0y\mathbf{e}_x, \end{aligned} \quad (3.150)$$

where the field strength being a positive constant $B_0 > 0$. This choice of the gauge potential is known as the Landau gauge. Another widely used gauge is the symmetric gauge with $\mathbb{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$ but here we adopt the Landau gauge because under this gauge the wavefunctions take the simplest form. The Wigner function will only depend on the magnetic field and then be independent from the choice of gauge.

Since the gauge potential \mathbb{A} only depends on the y -coordinate in the Landau gauge, while the mass m , the chemical potential μ , and the chiral chemical potential μ_5 are assumed to be constant, one can check that the spatial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$ commute with the Hamiltonian \hat{H} in Eq. (3.149). This indicates that we can introduce the momenta p^x and p^z as conserved variables. The solution of the Dirac equation (3.148) can be cast into the Fourier mode

$$\psi(t, \mathbf{x}) = \int \frac{dp^x dp^z}{(2\pi)^2} e^{-iEt + ip^x x + ip^z z} \psi(p^x, p^z, y). \quad (3.151)$$

Here we adopt the Weyl representation, i.e., the Dirac spinor can be decomposed into the LH and RH Pauli spinors,

$$\psi(p^x, p^z, y) = \begin{pmatrix} \chi_L(p^x, p^z, y) \\ \chi_R(p^x, p^z, y) \end{pmatrix}. \quad (3.152)$$

Inserting Eq. (3.151) into the Dirac equation (3.148), one obtains the equations for the LH and RH spinors,

$$\begin{aligned} \left[E + \sigma^1(p^x + B_0 y) + \sigma^2(-i\frac{\partial}{\partial y}) + \sigma^3 p^z + \mu - \mu_5 \right] \chi_L(p^x, p^z, y) &= m \chi_R(p^x, p^z, y), \\ \left[E - \sigma^1(p^x + B_0 y) - \sigma^2(-i\frac{\partial}{\partial y}) - \sigma^3 p^z + \mu + \mu_5 \right] \chi_R(p^x, p^z, y) &= m \chi_L(p^x, p^z, y). \end{aligned} \quad (3.153)$$

For massless fermions $m = 0$, the LH spinor is decoupled from the RH one, which indicates that particles are either LH or RH. But in the massive case there is a mixture between χ_L and χ_R , so the chirality is no longer a good quantum number. Inserting the explicit formula for the Pauli matrices into the equations of $\chi_{R,L}$, we obtain the matrix form,

$$\begin{aligned} \begin{pmatrix} E + \mu - \mu_5 + p^z & -\frac{\partial}{\partial y} + (p^x + B_0 y) \\ \frac{\partial}{\partial y} + (p^x + B_0 y) & E + \mu - \mu_5 - p^z \end{pmatrix} \chi_L(p^x, p^z, y) &= m \chi_R(p^x, p^z, y), \\ \begin{pmatrix} E + \mu + \mu_5 - p^z & \frac{\partial}{\partial y} - (p^x + B_0 y) \\ -\frac{\partial}{\partial y} - (p^x + B_0 y) & E + \mu + \mu_5 + p^z \end{pmatrix} \chi_R(p^x, p^z, y) &= m \chi_L(p^x, p^z, y). \end{aligned} \quad (3.154)$$

In order to make the formula simpler, we introduce the creation and annihilation operators

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2B_0}} \left[\frac{\partial}{\partial y} + (p^x + B_0 y) \right], \\ \hat{a}^\dagger &= \frac{1}{\sqrt{2B_0}} \left[-\frac{\partial}{\partial y} + (p^x + B_0 y) \right]. \end{aligned} \quad (3.155)$$

They are, respectively, the creation and annihilation operators of a harmonic oscillator around an equilibrium point p^x/B_0 with the frequency $\omega = \sqrt{B_0}$. We can check that they satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. But we should note that here \hat{a}^\dagger is not the Hermitian transpose of \hat{a} . Using these operators, the equations for the Pauli spinors read

$$\begin{aligned} \begin{pmatrix} E + \mu - \mu_5 + p^z & \sqrt{2B_0} \hat{a}^\dagger \\ \sqrt{2B_0} \hat{a} & E + \mu - \mu_5 - p^z \end{pmatrix} \chi_L(p^x, p^z, y) &= m \chi_R(p^x, p^z, y), \\ \begin{pmatrix} E + \mu + \mu_5 - p^z & -\sqrt{2B_0} \hat{a}^\dagger \\ -\sqrt{2B_0} \hat{a} & E + \mu + \mu_5 + p^z \end{pmatrix} \chi_R(p^x, p^z, y) &= m \chi_L(p^x, p^z, y). \end{aligned} \quad (3.156)$$

Inserting the second line into the first line or vice versa, one derives the equation for RH or LH spinors by eliminating χ_L or χ_R

$$\begin{pmatrix} (E + \mu)^2 - \Lambda^- & 2\mu_5 \sqrt{2B_0} \hat{a}^\dagger \\ 2\mu_5 \sqrt{2B_0} \hat{a} & (E + \mu)^2 - \Lambda^+ \end{pmatrix} \chi_{R,L}(p^x, p^z, y) = 0, \quad (3.157)$$

where

$$\Lambda^\pm \equiv m^2 + (p^z \pm \mu_5)^2 + 2B_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \pm B_0 \quad (3.158)$$

is the energy squared of particles with spin parallel or anti-parallel to the magnetic field. We see that the RH and LH spinors satisfy the same differential equation, thus we can solve one of them and derive the other through the relation (3.156). Note that if the chiral chemical potential vanishes, $\mu_5 = 0$, the off-diagonal terms in Eq. (3.157) also vanish and a straightforward calculation gives the eigenenergy

$$E_{s_1 s_2} = -\mu + s_1 \sqrt{m^2 + (p^z)^2 + 2B_0 \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right)} + s_2 B_0. \quad (3.159)$$

The term $2B_0 (\hat{a}^\dagger \hat{a} + \frac{1}{2})$ is the transverse energy squared, which comes from the coupling between the magnetic field and orbital angular momentum. The last term in the square root, $\pm B_0$, is the spin-magnetic coupling. This energy level agrees with the well-known Landau levels in Eq. (3.147).

In the case of finite μ_5 , the off-diagonal terms in Eq. (3.157) take non-vanishing values. In order to solve Eq. (3.157), we choose the basis of the harmonic oscillator, i.e., eigenstates of $\hat{a}^\dagger \hat{a}$,

$$\phi_n(p^x, y) = \left(\frac{B_0}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp \left[-\frac{B_0}{2} \left(y + \frac{p^x}{B_0} \right)^2 \right] H_n \left[\sqrt{B_0} \left(y + \frac{p^x}{B_0} \right) \right]. \quad (3.160)$$

Here H_n are the Hermite polynomials. One can check the completeness and orthonormality of ϕ_n as

$$\begin{aligned} \sum_{n=0}^{\infty} \phi_n(p^x, y) \phi_n(p^{x'}, y') &= \delta \left(y + \frac{p^x}{B_0} - y' - \frac{p^{x'}}{B_0} \right), \\ \int dy \phi_n(p^x, y) \phi_{n'}(p^x, y) &= \delta_{nn'}. \end{aligned} \quad (3.161)$$

When acting on the basis functions $\phi_n(p^x, y)$, the operators \hat{a}^\dagger and \hat{a} raise or decrease the quantum number n ,

$$\begin{aligned} \hat{a} \phi_n(p^x, y) &= \sqrt{n} \phi_{n-1}(p^x, y), \\ \hat{a}^\dagger \phi_n(p^x, y) &= \sqrt{n+1} \phi_{n+1}(p^x, y). \end{aligned} \quad (3.162)$$

Due to the completeness of ϕ_n , the spinors can be expanded as

$$\chi_{R/L}(p^x, p^z, y) = \sum_{n=0}^{\infty} \begin{pmatrix} c_n(p^x, p^z) \\ d_n(p^x, p^z) \end{pmatrix} \phi_n(p^x, y), \quad (3.163)$$

where all the y dependence was put into ϕ_n . Then, from Eq. (3.157) we derive

$$\sum_{n=0}^{\infty} \begin{pmatrix} [(E + \mu)^2 - \lambda_n^-] \phi_n(p_x, y) & 2\mu_5 \sqrt{2(n+1)B_0} \phi_{n+1}(p_x, y) \\ 2\mu_5 \sqrt{2nB_0} \phi_{n-1}(p_x, y) & [(E + \mu)^2 - \lambda_{n+1}^+] \phi_n(p_x, y) \end{pmatrix} \begin{pmatrix} c_n \\ d_n \end{pmatrix} = 0, \quad (3.164)$$

where

$$\lambda_n^\pm \equiv m^2 + (p^z \pm \mu_5)^2 + 2nB_0. \quad (3.165)$$

Using the orthonormality conditions in Eq. (3.161) we can derive the equations for the coefficients c_n and d_n

$$\begin{aligned} [(E + \mu)^2 - \lambda_0^-] c_0 &= 0, \\ 2\mu_5 \sqrt{2nB_0} c_n + [(E + \mu)^2 - \lambda_n^+] d_{n-1} &= 0, \quad n > 0, \\ [(E + \mu)^2 - \lambda_n^-] c_n + 2\mu_5 \sqrt{2nB_0} d_{n-1} &= 0, \quad n > 0. \end{aligned} \quad (3.166)$$

Here the coefficient c_0 decouples from all others, while c_n always couples with d_{n-1} for any $n \geq 1$.

2. Lowest Landau level

The lowest Landau level is given by demanding a non-vanishing c_0 . From the first line in Eq. (3.166), we obtain the eigenenergies $E = -\mu \pm E_{p^z}^{(0)}$, there the upper/lower sign labels fermions/anti-fermions. The energy of the lowest Landau level is

$$E_{p^z}^{(0)} = \sqrt{m^2 + (p^z - \mu_5)^2}. \quad (3.167)$$

Inserting the eigenenergy $E = -\mu \pm E_{p^z}^{(0)}$ into the second and third lines of Eq. (3.166), we obtain $c_n = d_{n-1} = 0$ for any $n > 0$. Then from the decomposition (3.163), one can construct the unnormalized eigenspinor for the lowest Landau level

$$\chi^{(0)}(p^x, p^z, y) = \begin{pmatrix} c_0(p^x, p^z) \\ 0 \end{pmatrix} \phi_0(p^x, y). \quad (3.168)$$

Since the functions $\phi_0(p^x, y)$ satisfies the orthonormality relation in Eq. (3.161), we demand that the spinor satisfies the normalization condition $\int dy \chi^{(0)\dagger} \chi^{(0)} = 1$. With the help of Eq. (3.161), we find that $c_0(p^x, p^z) = 1$, so the normalized eigenspinor is independent of the longitudinal momentum p^z , which agrees with the case without chemical potentials. The normalized eigenspinor for the lowest Landau level is

$$\chi^{(0)}(p^x, y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \phi_0(p^x, y). \quad (3.169)$$

The lower component is zero, so this state is occupied by a particle with the spin along the positive z-direction or an anti-particle with the spin along the negative z-direction. Recalling that the z-direction is the direction of the magnetic field, we see that the spin configuration in the lowest Landau level ensures the lowest spin-magnetic coupling. Since the LH and RH spinors satisfy the same equation (3.157), we take $\chi_L^{(0)}(p^x, y) = \chi^{(0)}(p^x, y)$ without loss of generality. The RH spinor is then derived from Eq. (3.156),

$$\chi_R^{(0)}(p^x, p^z, y) = \frac{E + \mu - \mu_5 + p^z}{m} \chi^{(0)}(p^x, y). \quad (3.170)$$

Here the energy takes the eigenvalue for the lowest Landau level $E = -\mu \pm E_{p^z}^{(0)}$. Then the wavefunction in Eq. (3.152) becomes

$$\psi_r^{(0)}(p^x, p^z, y) = \frac{1}{N_r} \begin{pmatrix} 1 \\ \frac{E_r + \mu - \mu_5 + p^z}{m} \end{pmatrix} \otimes \chi^{(0)}(p^x, y), \quad (3.171)$$

where $E_r = rE_{p^z}^{(0)} - \mu$ with $r = \pm$, and the symbol \otimes represents the tensor product of two matrices. The normalization factor N_r is determined by the normalization condition for the wavefunction,

$$\int dy \psi_r^{(0)\dagger}(p^x, p^z, y) \psi_{r'}^{(0)}(p^x, p^z, y) = \delta_{rr'}, \quad (3.172)$$

which gives

$$N_r = \frac{1}{\sqrt{E_{p^z}^{(0)} - r(p^z - \mu_5)}}. \quad (3.173)$$

Inserting N_r into the solution we obtain the normalized wavefunction

$$\psi_r^{(0)}(p^x, p^z, y) = \frac{1}{\sqrt{2E_{p^z}^{(0)}}} \begin{pmatrix} \sqrt{E_{p^z}^{(0)} - r(p^z - \mu_5)} \\ r\sqrt{E_{p^z}^{(0)} + r(p^z - \mu_5)} \end{pmatrix} \otimes \chi^{(0)}(p^x, y). \quad (3.174)$$

The terms in the square root in the solution are always positive because

$$E_{p^z}^{(0)} = \sqrt{m^2 + (p^z - \mu_5)^2} \geq |p^z - \mu_5|. \quad (3.175)$$

In the massless limit we obtain $E_{p^z}^{(0)} = |p^z - \mu_5|$, and the wavefunction reduces to the chiral one,

$$\psi_r(p^x, p^z, y) = \begin{cases} \begin{pmatrix} 0 \\ \chi^{(0)}(p^x, y) \end{pmatrix}, & r \operatorname{sgn}(p^z - \mu_5) > 0, \\ \begin{pmatrix} \chi^{(0)}(p^x, y) \\ 0 \end{pmatrix}, & r \operatorname{sgn}(p^z - \mu_5) < 0. \end{cases} \quad (3.176)$$

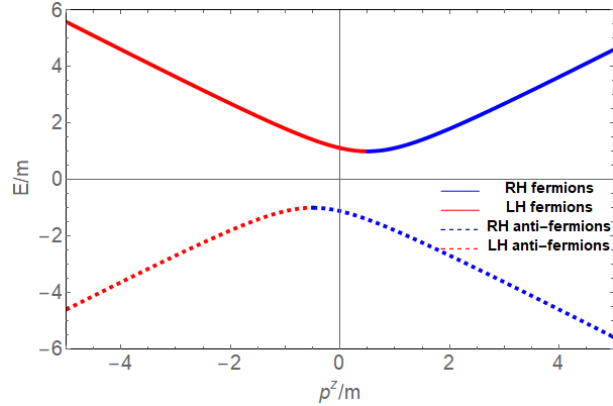


Figure 1: Energy spectrum for fermions (solid line with $E > 0$) and anti-fermions (dashed line with $E < 0$) in the lowest Landau level. We take the mass as the energy unit. The chiral chemical potential is taken to be $\mu_5/m = 0.5$. RH particles are shown in red, while LH in blue.

Here $r = \pm$ represent fermions and anti-fermions respectively. The fermion states with $p^z > \mu_5$ are RH, while states with $p^z < \mu_5$ are LH. On the other hand, for anti-fermion states, $p^z > \mu_5$ corresponds to LH, while $p^z < \mu_5$ corresponds to RH. We plot the energy spectrum $\pm E_{p^z}^{(0)}$ as a function of p^z in Fig. 1. In the Figure, the x-axis represents the dimensionless longitudinal momentum p^z/m and the y-axis represents the dimensionless energy E/m . The branch with the positive energy is for fermions while the one with the negative energy is for anti-fermions. We use the blue color for fermions/anti-fermions with RH chirality while the red color for LH chirality. We observe an energy gap $2m$ between fermions and anti-fermions induced by the mass. There is also a gap $2\mu_5$ in the x -direction is induced by the chiral chemical potential.

The lowest Landau level is related to the CME. The fermions fill in the positive-energy states from the lowest one. Hence fermions are more likely to have positive p^z because the energy spectrum is not symmetric. On the hand, anti-fermions are more likely to have negative longitudinal momentum p^z , which can be observed from the dashed line in Fig. 1. Therefore there will be a net fermion current along the positive z -direction, i.e., the direction of the magnetic field. Later on we will show that the higher Landau levels do not contribute to the CME because they are symmetric in p^z .

3. Higher Landau levels

Similar to the lowest Landau level, the higher Landau levels are obtained by demanding a non-vanishing c_n with $n > 0$. According to Eq. (3.166), in the presence of a non-vanishing μ_5 , the

coefficient c_n is always coupled to the coefficient d_{n-1} . Eliminating d_{n-1} we obtain an equation for c_n ,

$$\{[(E + \mu)^2 - \lambda_n^+] [(E + \mu)^2 - \lambda_n^-] - 8nB_0\mu_5^2\} c_n = 0. \quad (3.177)$$

In order to have a non-trivial c_n , the coefficient must vanish, which gives the eigenenergies are $E = -\mu + s_1 E_{p^z s_2}^{(n)}$, where

$$E_{p^z s}^{(n)} = \sqrt{m^2 + \left[\sqrt{(p^z)^2 + 2nB_0} - s\mu_5 \right]^2} \quad (3.178)$$

is the energy of the n -th Landau level, with $n > 0$ and $s = \pm$. The coefficient c_m with $m \neq n$ must vanish because c_m and c_n correspond to different eigenenergies.

In the massless limit and assuming $\sqrt{(p^z)^2 + 2nB_0} \gg |\mu_5|$ (this is possible because μ_5 labels the chiral imbalance which in general should be a small variable compared to the momentum), we have the eigenenergies

$$E = r\sqrt{(p^z)^2 + 2nB_0} - (\mu + rs\mu_5). \quad (3.179)$$

Note that $r = \pm$ label states with the positive (+) and negative (-) energy. In the massless case, $\mu + \mu_5$ is the chemical potential for RH particles while $\mu - \mu_5$ is the one for LH ones, so the product rs denotes the chirality. The parameter s labels the helicity because fermions ($r = +$) with the RH helicity ($s = +$) and anti-fermions ($r = -$) with the LH helicity ($s = -$) have the RH chirality ($rs = +$), or vice versa. In the case $\mu_5 = 0$, the eigenenergies in Eqs. (3.167) and (3.178) reproduce the well-known Landau energy levels in (3.147). The higher Landau levels $E_{p^z s}^{(n)}$ in Eq. (3.178) are degenerate for $s = \pm$ and any $n > 0$.

In Fig. 2 we plot the energy spectrum for the first Landau level $n = 1$. In this figure we use the blue color to label the branches which in the massless limit reproduce the states of the RH chirality and use the red color for the LH chirality. We observe a gap between LH and RH branches, which is attributed to a finite μ_5 . Note that the energy spectrum is symmetric for flipping the sign of the longitudinal momentum $p^z \leftrightarrow -p^z$. Thus, if the distribution only depends on the energy spectrum, the number of particles moving in the positive z -direction equals to that moving in the negative z -direction. The corresponding currents cancel with each other and there is no macroscopic current for the Landau levels with $n > 0$.

Now we derive the wavefunction of the n -th Landau level. Inserting the eigenenergy into Eq. (3.166) we obtain a relation between c_n and d_{n-1} . All other coefficients c_m , d_{m-1} with $m \neq n$ have

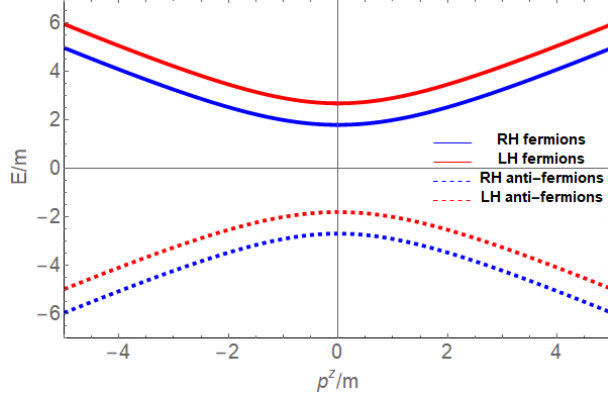


Figure 2: Energy spectrum for fermions (solid lines with $E > 0$) and anti-fermions (dashed lines with $E < 0$) in the Landau level $n = 1$. The mass m is taken to be the unit of the energy and momentum. The magnetic field strength is chosen to be $B_0/m^2 = 2$ and the chiral chemical potential $\mu_5/m = 0.5$. We use the blue color for particles with the RH chirality and the red color for those with the LH chirality. The curves are even functions of p^z .

to vanish. According to the expansion in Eq. (3.163), we obtain the unnormalized Pauli spinors

$$\chi_s^{(n)}(p^x, p^z, y) = c_n(p^x, p^z) \begin{pmatrix} \sqrt{2nB_0}\phi_n(p^x, y) \\ (s\sqrt{(p^z)^2 + 2nB_0} - p^z)\phi_{n-1}(p^x, y) \end{pmatrix}. \quad (3.180)$$

Again we demand the orthonormality condition $\int dy \chi_s^{(n)\dagger} \chi_{s'}^{(n)} = \delta_{ss'}$ to determine the coefficient c_n . The normalized eigenspinors read

$$\chi_s^{(n)}(p^x, p^z, y) = \frac{1}{\sqrt{2\sqrt{(p^z)^2 + 2nB_0}}} \begin{pmatrix} \sqrt{\sqrt{(p^z)^2 + 2nB_0} + sp^z} \phi_n(p^x, y) \\ s\sqrt{\sqrt{(p^z)^2 + 2nB_0} - sp^z} \phi_{n-1}(p^x, y) \end{pmatrix}. \quad (3.181)$$

Note that the spinors $\chi_s^{(n)}(p^x, p^z, y)$ are real functions because a) $\phi_n(p^x, y)$ are real; b) the magnetic field strength B_0 is positive; c) $\sqrt{(p^z)^2 + 2nB_0} > |p^z|$. If the chiral chemical potential vanish, $\mu_5 = 0$, according to Eq. (3.166), the state with $c_n \neq 0$ and the one with $d_{n-1} \neq 0$ have the same energy $E = -\mu \pm \sqrt{\lambda^\pm}$ with $\lambda^\pm = m^2 + (p^z)^2 + 2nB_0$. Since $c_n \neq 0$ corresponds to a spin-up state and $d_{n-1} \neq 0$ corresponds to a spin-down state, we conclude that the higher Landau levels are two-fold degenerate with respect to spin if $\mu_5 = 0$. But for a finite μ_5 , the eigenenergy $E = -\mu + s_1 E_{p^z s_2}^{(n)}$ depends on s_1 and s_2 , while the eigenspinors in Eq. (3.181) are mixture of spin-up and spin-down states. Using the orthonormality condition for ϕ_n in Eq. (3.161) we can check that the spinors for higher Landau levels satisfy the orthonormality condition, and they are also orthogonal to the one

for the lowest Landau level,

$$\begin{aligned} \int dy \chi_s^{(n)\dagger}(p^x, p^z, y) \chi^{(0)}(p^x, y) &= 0, \\ \int dy \chi_{s'}^{(n')\dagger}(p^x, p^z, y) \chi_s^{(n)}(p^x, p^z, y) &= \delta_{ss'} \delta_{nn'}. \end{aligned} \quad (3.182)$$

The wavefunction in momentum space can be obtained following the procedure for the lowest Landau level. First the LH spinor is assumed to take the form given in Eq. (3.181), $\chi_{L,s}^{(n)}(p^x, p^z, y) = \chi_s^{(n)}(p^x, p^z, y)$. Then the RH spinor can be derived from Eq. (3.156),

$$\chi_{R,s}^{(n)}(p^x, p^z, y) = \frac{1}{m} \begin{pmatrix} E + \mu - \mu_5 + p^z & \sqrt{2B_0} \hat{a}^\dagger \\ \sqrt{2B_0} \hat{a} & E + \mu - \mu_5 - p^z \end{pmatrix} \chi_s^{(n)}(p^x, p^z, y), \quad (3.183)$$

where $E = rE_{p^z s}^{(n)} - \mu$ which depends on s and n . Here $r = \pm$ label the fermion (+) and anti-fermion (-). Inserting the solution (3.181) into the above equation, and using Eq. (3.162) to deal with \hat{a}^\dagger and \hat{a} , we obtain the RH spinor and then the unnormalized wavefunction

$$\psi_{rs}^{(n)}(p^x, p^z, y) = \frac{1}{mN_{rs}^{(n)}} \begin{pmatrix} 1 \\ rE_{p^z s}^{(n)} + s\sqrt{(p^z)^2 + 2nB_0} - \mu_5 \end{pmatrix} \otimes \chi_s^{(n)}(p^x, p^z, y). \quad (3.184)$$

After proper normalization, we obtain

$$\psi_{rs}^{(n)}(p^x, p^z, y) = \frac{1}{\sqrt{2E_{p^z s}^{(n)}}} \begin{pmatrix} r\sqrt{E_{p^z s}^{(n)} + r\mu_5 - rs\sqrt{p_z^2 + 2nB_0}} \\ \sqrt{E_{p^z s}^{(n)} - r\mu_5 + rs\sqrt{p_z^2 + 2nB_0}} \end{pmatrix} \otimes \chi_s^{(n)}(p^x, p^z, y). \quad (3.185)$$

Again, due to the non-zero mass, the terms in square roots are always positive, and the wavefunction is real. With the help of Eq. (3.161), we can check that the wavefunctions satisfy the orthonormality conditions

$$\begin{aligned} \int dy \psi_{r's}^{(n)\dagger}(p^x, p^z, y) \psi_r^{(0)}(p^x, y) &= 0, \\ \int dy \psi_{r's'}^{(n')\dagger}(p^x, p^z, y) \psi_{rs}^{(n)}(p^x, p^z, y) &= \delta_{rr'} \delta_{ss'} \delta_{nn'}. \end{aligned} \quad (3.186)$$

Note that p^x in the wavefunctions (3.174) and (3.185) is the momentum in the x direction, but it also determines the center position in the y direction. The obtained wavefunctions are plane waves in the x and z directions but have a finite extent in the y direction because of the harmonic oscillator eigenfunctions $\phi_n(p^x, y)$. One can have a more realistic description of a quantum particle with given position and momentum by constructing wave packets by superposition of the single particle wavefunctions with different p^x and p^z . In this thesis, the wave-packet description will be used in computing the Wigner function.

Now we briefly discuss the density of state. We consider a finite volume $l_x \times l_y \times l_z$ with periodic boundary conditions. Then p^x takes discrete values,

$$p^x = \frac{2\pi n_x}{l_x}, \quad n_x = \dots, -1, 0, 1, 2, \dots \quad (3.187)$$

All the wavefunctions are constructed from the harmonic oscillator wavefunction $\phi_n(p^x, y)$, and the center position of the harmonic oscillator is $y = -p^x/B_0$. In order to make sure this center position is located inside the area considered, we demand

$$0 \leq -\frac{p^x}{B_0} \leq l_y, \quad (3.188)$$

from which we obtain

$$-\frac{B_0 l_x l_y}{2\pi} \leq n^x \leq 0. \quad (3.189)$$

So the density of state is $B_0 l_x l_y / (2\pi)$ for given p^z and n, r, s . This result is consistent with our knowledge about Landau levels.

4. Landau quantization

We have given in (3.174) and (3.185) the wavefunctions for the lowest Landau level and higher Landau levels respectively. Using these wavefunctions, the Dirac operator can be quantized as

$$\begin{aligned} \hat{\psi}(t, \mathbf{x}) = e^{i\mu t} \sum_{n,s} \int \frac{dp^x dp^z}{(2\pi)^2} e^{ip^x x + ip^z z} \left[\exp\left(-iE_{p^z s}^{(n)} t\right) \psi_{+,s}^{(n)}(p^x, p^z, y) \hat{a}_s^{(n)}(p^x, p^z) \right. \\ \left. + \exp\left(iE_{p^z s}^{(n)} t\right) \psi_{-,s}^{(n)}(p^x, p^z, y) \hat{b}_s^{(n)\dagger}(-p^x, -p^z) \right], \end{aligned} \quad (3.190)$$

where we have defined

$$\sum_{n,s} f_s^{(n)} \equiv f^{(0)} + \sum_{s=\pm} \sum_{n>0} f_s^{(n)}, \quad (3.191)$$

for any function $f_s^{(n)}$ which depends on the helicity index s and Landau level n . The particles in the lowest Landau level always have the fixed spin. Here $\hat{a}_s^{(n)}(p^x, p^z)$ is the annihilation operator for a fermion in the n -th Landau level with p^x, p^z , and s . Similarly, $\hat{b}_s^{(n)\dagger}(p^x, p^z)$ is the creation operator for an anti-fermion in the n -th Landau level with the same p^x, p^z , and s . We observe that the contribution from the chemical potential μ to the field operator is an overall factor $e^{i\mu t}$, while the chiral chemical potential μ_5 enters the energy levels $E_{p^z s}^{(n)}$. We further assume that the creation and annihilation operators satisfy the following anti-commutation relations

$$\begin{aligned} \left\{ \hat{a}_{p^x p^z s}^{(n)}, \hat{a}_{q^x q^z s'}^{(n')\dagger} \right\} &= (2\pi)^2 \delta(p^x - q^x) \delta(p^z - q^z) \delta_{nn'} \delta_{ss'}, \\ \left\{ \hat{b}_{p^x p^z s}^{(n)}, \hat{b}_{q^x q^z s'}^{(n')\dagger} \right\} &= (2\pi)^2 \delta(p^x - q^x) \delta(p^z - q^z) \delta_{nn'} \delta_{ss'}, \end{aligned} \quad (3.192)$$

with all other anti-commutators vanishing. These relations are straightforward extensions of the free case, but it is reasonable because we can derive the following equal-time anti-commutation relations for the field operators

$$\begin{aligned}\left\{\hat{\psi}_\alpha(t, \mathbf{x}), \hat{\psi}_\beta^\dagger(t, \mathbf{x}')\right\} &= \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{x}'), \\ \left\{\hat{\psi}_\alpha(t, \mathbf{x}), \hat{\psi}_\beta(t, \mathbf{x}')\right\} &= \left\{\hat{\psi}_\alpha^\dagger(t, \mathbf{x}), \hat{\psi}_\beta^\dagger(t, \mathbf{x}')\right\} = 0,\end{aligned}\quad (3.193)$$

where α and β are indices of the Dirac spinors.

Since we have computed the eigenstates of the Hamilton operator and used them to quantize the Dirac field in Eq. (3.190), it is straightforward to rewrite the Hamiltonian using the creation and annihilation operators,

$$\begin{aligned}\hat{H} &= \sum_{n,s} \int \frac{dp^x dp^z}{(2\pi)^2} \left[\left(E_{p^z s}^{(n)} - \mu \right) a_s^{(n)\dagger}(p^x, p^z) a_s^{(n)}(p^x, p^z) \right. \\ &\quad \left. - \left(E_{p^z s}^{(n)} + \mu \right) b_s^{(n)}(-p^x, -p^z) b_s^{(n)\dagger}(-p^x, -p^z) \right].\end{aligned}\quad (3.194)$$

The momentum operators are given by

$$\begin{aligned}\hat{P}_x &= \sum_{n,s} \int \frac{dp^x dp^z}{(2\pi)^2} p^x \left[a_s^{(n)\dagger}(p^x, p^z) a_s^{(n)}(p^x, p^z) - b_s^{(n)}(p^x, p^z) b_s^{(n)\dagger}(p^x, p^z) \right], \\ \hat{P}_z &= \sum_{n,s} \int \frac{dp^x dp^z}{(2\pi)^2} p^z \left[a_s^{(n)\dagger}(p^x, p^z) a_s^{(n)}(p^x, p^z) - b_s^{(n)}(p^x, p^z) b_s^{(n)\dagger}(p^x, p^z) \right].\end{aligned}\quad (3.195)$$

In these calculations we have used the orthonormality conditions in Eqs. (3.172) and (3.186).

5. Wigner function

We have derived the wavefunctions in Eqs. (3.174) and (3.185) and the field operator is quantized in Eq. (3.190). Inserting the field operator into the definition of the Wigner function (2.18) we

obtain

$$\begin{aligned}
W(x, p) = & \sum_{n,s} \sum_{n',s'} \int \frac{dy'}{(2\pi)} \int \frac{dq_1^x dq_1^z dq_2^x dq_2^z}{(2\pi)^4} \delta \left(p^x - B_0 y - \frac{q_1^x + q_2^x}{2} \right) \delta \left(p^z - \frac{q_1^z + q_2^z}{2} \right) \\
& \times \exp \left[i(q_1^x - q_2^x)x + i(q_1^z - q_2^z)z + ip^y y' \right] \\
& \times \left\{ \left\langle a_{s'}^{(n')\dagger}(q_2^x, q_2^z) a_s^{(n)}(q_1^x, q_1^z) \right\rangle \bar{\psi}_{+,s'}^{(n')} \left(q_2^x, q_2^z, y + \frac{y'}{2} \right) \otimes \psi_{+,s}^{(n)} \left(q_1^x, q_1^z, y - \frac{y'}{2} \right) \right. \\
& \quad \times \exp \left[i \left(E_{q_2^z s'}^{(n')} - E_{q_1^z s}^{(n)} \right) t \right] \delta \left(p^0 + \mu - \frac{E_{q_2^z s'}^{(n')} + E_{q_1^z s}^{(n)}}{2} \right) \\
& \quad + \left\langle b_{s'}^{(n')\dagger}(-q_2^x, -q_2^z) b_s^{(n)\dagger}(-q_1^x, -q_1^z) \right\rangle \bar{\psi}_{-,s'}^{(n')} \left(q_2^x, q_2^z, y + \frac{y'}{2} \right) \otimes \psi_{-,s}^{(n)} \left(q_1^x, q_1^z, y - \frac{y'}{2} \right) \\
& \quad \left. \times \exp \left[-i \left(E_{q_2^z s'}^{(n')} - E_{q_1^z s}^{(n)} \right) t \right] \delta \left(p^0 + \mu + \frac{E_{q_2^z s'}^{(n')} + E_{q_1^z s}^{(n)}}{2} \right) \right\}, \tag{3.196}
\end{aligned}$$

where y is the y -component of the four-vector x^μ , and y' is an integration variable. Here we have dropped mixing terms of fermions and anti-fermions, i.e., $a_{q_2^x q_2^z s'}^{(n')\dagger} b_{-q_1^x, -q_1^z, s}^{(n)\dagger}$ and $b_{-q_2^x, -q_2^z, s'}^{(n')} a_{q_1^x q_1^z s}^{(n)}$. These terms only contribute if there is a mixture between the fermion and anti-fermion state. Analogous to what we did in the case of free fermions in subsection III A, we define the average and relative momenta,

$$k^x = \frac{q_1^x + q_2^x}{2}, \quad k^z = \frac{q_1^z + q_2^z}{2}, \quad u^x = q_1^x - q_2^x, \quad u^z = q_1^z - q_2^z. \tag{3.197}$$

Using these new variables, the integration measure stays unchanged

$$dq_1^x dq_1^z dq_2^x dq_2^z = dk^x dk^z du^x du^z. \tag{3.198}$$

Then the Wigner function reads

$$\begin{aligned}
W(x, p) = & \sum_{n,s} \sum_{n',s'} \int \frac{dy'}{(2\pi)} \int \frac{dk^x dk^z du^x du^z}{(2\pi)^4} \delta(p^x - B_0 y - k^x) \delta(p^z - k^z) \exp(iu^x x + iu^z z + ip^y y') \\
& \times \left\{ \left\langle a_{s'}^{(n')\dagger} \left(k^x - \frac{u^x}{2}, k^z - \frac{u^z}{2} \right) a_s^{(n)} \left(k^x + \frac{u^x}{2}, k^z + \frac{u^z}{2} \right) \right\rangle \right. \\
& \quad \times \bar{\psi}_{+,s'}^{(n')} \left(k^x - \frac{u^x}{2}, k^z - \frac{u^z}{2}, y + \frac{y'}{2} \right) \otimes \psi_{+,s}^{(n)} \left(k^x + \frac{u^x}{2}, k^z + \frac{u^z}{2}, y - \frac{y'}{2} \right) \\
& \quad \times \exp \left[i \left(E_{k^z - \frac{1}{2}u^z, s'}^{(n')} - E_{k^z + \frac{1}{2}u^z, s}^{(n)} \right) t \right] \delta \left(p^0 + \mu - \frac{E_{k^z - \frac{1}{2}u^z, s'}^{(n')} + E_{k^z + \frac{1}{2}u^z, s}^{(n)}}{2} \right) \\
& + \left\langle b_{s'}^{(n')} \left(-k^x + \frac{u^x}{2}, -k^z + \frac{u^z}{2} \right) b_s^{(n)\dagger} \left(-k^x - \frac{u^x}{2}, -k^z - \frac{u^z}{2} \right) \right\rangle \\
& \quad \times \bar{\psi}_{-,s'}^{(n')} \left(k^x - \frac{u^x}{2}, k^z - \frac{u^z}{2}, y + \frac{y'}{2} \right) \otimes \psi_{-,s}^{(n)} \left(k^x + \frac{u^x}{2}, k^z + \frac{u^z}{2}, y - \frac{y'}{2} \right) \\
& \quad \times \exp \left[-i \left(E_{k^z - \frac{1}{2}u^z, s'}^{(n')} - E_{k^z + \frac{1}{2}u^z, s}^{(n)} \right) t \right] \delta \left(p^0 + \mu + \frac{E_{k^z - \frac{1}{2}u^z, s'}^{(n')} + E_{k^z + \frac{1}{2}u^z, s}^{(n)}}{2} \right) \left. \right\}. \tag{3.199}
\end{aligned}$$

The Wigner function now is a two-point correlation function in momentum space. We adopt the wave-packet description and assume the expectation values are given by some distribution functions that will be determined later,

$$\begin{aligned}
\left\langle a_{s'}^{(n')\dagger} \left(k^x - \frac{u^x}{2}, k^z - \frac{u^z}{2} \right) a_s^{(n)} \left(k^x + \frac{u^x}{2}, k^z + \frac{u^z}{2} \right) \right\rangle &= \delta_{ss'} \delta_{nn'} f_s^{(+)(n)}(k^x, k^z, u^x, u^z), \\
\left\langle b_{s'}^{(n')} \left(-k^x + \frac{u^x}{2}, -k^z + \frac{u^z}{2} \right) b_s^{(n)\dagger} \left(-k^x - \frac{u^x}{2}, -k^z - \frac{u^z}{2} \right) \right\rangle &= \delta_{ss'} \delta_{nn'} \left[(2\pi)^2 \delta(u^x) \delta(u^z) \right. \\
& \quad \left. - f_s^{(-)(n)}(-k^x, -k^z, u^x, u^z) \right]. \tag{3.200}
\end{aligned}$$

The expectation values are proportional to the Kronecker-deltas, $\delta_{ss'} \delta_{nn'}$ because we assume the wave packets are constructed by states at the same Landau level n with the same helicity s . In the free fermion case, energies are two-fold degenerated for the spin, but for non-zero μ_5 , this spin degeneracy disappears and all the quantum states are not degenerate any more. Inserting these

expectation values back to the Wigner function we obtain

$$\begin{aligned}
W(x, p) &= \sum_{n,s} \int \frac{dy'}{(2\pi)} \int \frac{du^x du^z}{(2\pi)^4} \exp(iu^x x + iu^z z + ip^y y') \\
&\times \left\{ f_s^{(+)(n)}(p^x - B_0 y, p^z, u^x, u^z) \right. \\
&\quad \times \exp \left[i \left(E_{p^z - \frac{1}{2}u^z, s}^{(n)} - E_{p^z + \frac{1}{2}u^z, s}^{(n)} \right) t \right] \delta \left(p^0 + \mu - \frac{E_{p^z - \frac{1}{2}u^z, s}^{(n)} + E_{p^z + \frac{1}{2}u^z, s}^{(n)}}{2} \right) \\
&\quad \times \bar{\psi}_{+,s}^{(n)} \left(p^x - B_0 y - \frac{u^x}{2}, p^z - \frac{u^z}{2}, y + \frac{y'}{2} \right) \\
&\quad \otimes \psi_{+,s}^{(n)} \left(p^x - B_0 y + \frac{u^x}{2}, p^z + \frac{u^z}{2}, y - \frac{y'}{2} \right) \\
&\quad + \left[(2\pi)^2 \delta(u^x) \delta(u^z) - f_s^{(-)(n)}(-p^x + B_0 y, -p^z, u^x, u^z) \right] \\
&\quad \times \exp \left[-i \left(E_{p^z - \frac{1}{2}u^z, s}^{(n)} - E_{p^z + \frac{1}{2}u^z, s}^{(n)} \right) t \right] \delta \left(p^0 + \mu + \frac{E_{p^z - \frac{1}{2}u^z, s}^{(n)} + E_{p^z + \frac{1}{2}u^z, s}^{(n)}}{2} \right) \\
&\quad \times \bar{\psi}_{-,s}^{(n)} \left(p^x - B_0 y - \frac{u^x}{2}, p^z - \frac{u^z}{2}, y + \frac{y'}{2} \right) \\
&\quad \left. \otimes \psi_{-,s}^{(n)} \left(p^x - B_0 y + \frac{u^x}{2}, p^z + \frac{u^z}{2}, y - \frac{y'}{2} \right) \right\}. \tag{3.201}
\end{aligned}$$

Assuming that the relative momenta u^x and u^z are small we can expand $E_{p^z \pm \frac{1}{2}u^z, s}^{(n)}$ as well as the wavefunctions in u^x and u^z

$$E_{p^z + \frac{1}{2}u^z, s}^{(n)} \equiv E_{p^z, s}^{(n)} + \mathcal{O}(u). \tag{3.202}$$

The local distributions are defined as

$$\begin{aligned}
f_s^{(+)(n)}(p^x, p^z, \mathbf{x}) &\equiv \int \frac{du^x du^z}{(2\pi)^2} \exp(iu^x x + iu^z z) f_s^{(+)(n)}(p^x - B_0 y, p^z, u^x, u^z), \\
f_s^{(-)(n)}(-p^x, -p^z, \mathbf{x}) &\equiv \int \frac{du^x du^z}{(2\pi)^2} \exp(iu^x x + iu^z z) f_s^{(-)(n)}(-p^x + B_0 y, p^z, u^x, u^z). \tag{3.203}
\end{aligned}$$

Then the Wigner function at the leading order in the spatial gradient reads

$$\begin{aligned}
W(x, p) &= \frac{1}{(2\pi)^3} \sum_{n,s} f_s^{(+)(n)}(p^x, p^z, \mathbf{x}) W_{+,s}^{(n)}(\mathbf{p}) \delta \left(p^0 + \mu - E_{p^z, s}^{(n)} \right) \\
&\quad + \frac{1}{(2\pi)^3} \sum_{n,s} \left[1 - f_s^{(-)(n)}(-p^x, -p^z, \mathbf{x}) \right] W_{-,s}^{(n)}(\mathbf{p}) \delta \left(p^0 + \mu + E_{p^z, s}^{(n)} \right), \tag{3.204}
\end{aligned}$$

where the contribution from the n -th Landau level is

$$W_{rs}^{(n)}(\mathbf{p}) = \frac{1}{(2\pi)^3} \int dy' \exp(ip_y y') \bar{\psi}_{rs}^{(n)} \left(p_x, p_z, \frac{y'}{2} \right) \otimes \psi_{rs}^{(n)} \left(p_x, p_z, -\frac{y'}{2} \right). \tag{3.205}$$

Here we have used the property $\phi_n(p_x - eBy, y - \frac{y'}{2}) = \phi_n(p_x, -\frac{y'}{2})$ and the fact that the dependence of $\psi_{rs}^{(n)}$ on p_x and y only appears in the eigenfunctions ϕ_n . The contribution from fermions is separated from that of anti-fermions. The distributions $f_s^{(\pm)(n)}$ turn out to be locally defined, which also depend on n , s , p^x , and p^z . The Dirac-delta functions in Eq. (3.204) can be written by an on-shell condition multiplied with a step function,

$$\delta(p_0 + \mu - rE_{p^z, s}^{(n)}) = 2E_{p^z, s}^{(n)} \delta\left\{(p_0 + \mu)^2 - [E_{p^z, s}^{(n)}]^2\right\} \theta[r(p_0 + \mu)]. \quad (3.206)$$

The lowest Landau level $n = 0$ does not depend on s . The wavefunctions are given in Eq. (3.174). Inserting the wavefunctions into Eq. (3.205), we obtain

$$W_r^{(0)}(\mathbf{p}) = \frac{1}{(2\pi)^3} \frac{1}{4E_{p^z}^{(0)}} \left[rm\mathbb{I}_2 + E_{p^z}^{(0)}\sigma^1 - r(p_z - \mu_5)i\sigma^2 \right] \otimes (\mathbb{I}_2 + \sigma^3) I_{00}(p^x, p^y), \quad (3.207)$$

where $I_{ij}(p^x, p^y)$ is defined in Eq. (B1). The tensor product of two Pauli matrices can be written in terms of gamma matrices, as shown in Eq. (A8). Thus we obtain

$$W_r^{(0)}(\mathbf{p}) = \frac{r}{(2\pi)^3 4E_{p^z}^{(0)}} I_{00}(p^x, p^y) \left[m(\mathbb{I}_4 + \sigma^{12}) + rE_{p^z}^{(0)}(\gamma^0 - \gamma^5\gamma^3) - (p^z - \mu_5)(\gamma^3 - \gamma^5\gamma^0) \right]. \quad (3.208)$$

Here $I_{00}(p^x, p^y)$ is calculated in Eq. (B10). Similarly, using the wavefunctions for higher Landau levels in Eq. (3.185), we can compute the contributions of the higher Landau levels

$$W_{rs}^{(n)}(\mathbf{p}) = \frac{1}{(2\pi)^3} \frac{1}{2E_{p^z, s}^{(n)}} \left[rm\mathbb{I}_2 + E_{p^z, s}^{(n)}\sigma^1 + (i\sigma^2)r(\mu_5 - s\sqrt{(p^z)^2 + 2nB_0}) \right] \\ \otimes \frac{1}{2\sqrt{(p^z)^2 + 2nB_0}} \begin{pmatrix} (\sqrt{(p^z)^2 + 2nB_0} + sp^z)I_{nn} & s\sqrt{2nB_0}I_{n-1, n} \\ s\sqrt{2nB_0}I_{n, n-1} & (\sqrt{(p^z)^2 + 2nB_0} - sp^z)I_{n-1, n-1} \end{pmatrix}. \quad (3.209)$$

The matrix in the second line of Eq. (3.209) can be decomposed in terms of the unit matrix and Pauli matrices,

$$\begin{pmatrix} (\sqrt{(p^z)^2 + 2nB_0} + sp^z)I_{nn} & s\sqrt{2nB_0}I_{n-1, n} \\ s\sqrt{2nB_0}I_{n, n-1} & (\sqrt{(p^z)^2 + 2nB_0} - sp^z)I_{n-1, n-1} \end{pmatrix} \\ = \left(\frac{\sqrt{(p^z)^2 + 2nB_0} I_{nn} + I_{n-1, n-1}}{2} + sp^z \frac{I_{nn} - I_{n-1, n-1}}{2} \right) \mathbb{I}_2 \quad (3.210)$$

$$+ s\sqrt{2nB_0} \frac{I_{n-1, n} + I_{n, n-1}}{2} \sigma^1 + s\sqrt{2nB_0} \frac{I_{n-1, n} - I_{n, n-1}}{2} i\sigma^2 \\ + \left(\frac{\sqrt{(p^z)^2 + 2nB_0} I_{nn} - I_{n-1, n-1}}{2} + sp^z \frac{I_{nn} + I_{n-1, n-1}}{2} \right) \sigma^3. \quad (3.211)$$

Note that these functions are independent of the choice of gauge. We start from the Landau gauge where p^x is a well-defined momentum while p^y is not. But Eq. (B8) only depends on p_T , where p^x

and p^y have the same importance. The functions $I_{ij}(p^x, p^y)$ are computed in Eq. (B5). Using the results (B9), the Wigner function for the higher Landau levels $n > 0$ can be written as

$$\begin{aligned}
W_{rs}^{(n)}(\mathbf{p}) = & \frac{r}{(2\pi)^3 4E_{p^z s}^{(n)}} \left\{ \left[m\mathbb{I}_4 + rE_{p^z s}^{(n)}\gamma^0 + (s\sqrt{(p^z)^2 + 2nB_0} - \mu_5)\gamma^5\gamma^0 \right] \right. \\
& \times \left(\Lambda_+^{(n)}(p_T) + s\frac{p^z}{\sqrt{(p^z)^2 + 2nB_0}}\Lambda_-^{(n)}(p_T) \right) \\
& + \left[m\sigma^{12} - rE_{p^z s}^{(n)}\gamma^5\gamma^3 + (\mu_5 - s\sqrt{(p^z)^2 + 2nB_0})\gamma^3 \right] \\
& \times \left(\Lambda_-^{(n)}(p_T) + s\frac{p^z}{\sqrt{(p^z)^2 + 2nB_0}}\Lambda_+^{(n)}(p_T) \right) \\
& + \left[m(\sigma^{23}p^x + \sigma^{31}p^y) - rE_{p^z s}^{(n)}(\gamma^5\gamma^1p^x + \gamma^5\gamma^2p^y) \right. \\
& \left. + (\mu_5 - s\sqrt{(p^z)^2 + 2nB_0})(\gamma^1p^x + \gamma^2p^y) \right] \times s\frac{2nB_0}{p_T^2\sqrt{(p^z)^2 + 2nB_0}}\Lambda_+^{(n)}(p_T) \left. \right\}, \tag{3.212}
\end{aligned}$$

where we have defined a new function for $n > 0$. Different components of the Wigner function can be extracted using the trace properties in Eq. (2.21),

$$\begin{aligned}
\begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{pmatrix} &= \left[\sum_{n=0} V_n e_1^{(n)} + \sum_{n>0} A_n \frac{1}{\sqrt{(p^z)^2 + 2nB_0}} (p^z e_2^{(n)} + \sqrt{2nB_0} e_3^{(n)}) \right] \begin{pmatrix} m \\ p_0 + \mu \end{pmatrix}, \\
\mathcal{G}_3 &= (p^z - \mu_5)V_0 e_1^{(0)} + \sum_{n>0} \left[\sqrt{(p^z)^2 + 2nB_0} A_n - \mu_5 V_n \right] e_1^{(n)} \\
&+ \sum_{n>0} \left[V_n - \frac{\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} A_n \right] (p^z e_2^{(n)} + \sqrt{2nB_0} e_3^{(n)}), \\
\mathcal{G}_4 &= 0, \tag{3.213}
\end{aligned}$$

where the basis vectors $e_1^{(n)}$, $e_2^{(n)}$, and $e_3^{(n)}$ are defined in Eq. (B15) and we have defined two functions for $n > 0$

$$\begin{aligned}
V_n &\equiv \frac{2}{(2\pi)^3} \sum_s \delta \left\{ (p_0 + \mu)^2 - [E_{p^z s}^{(n)}]^2 \right\} \\
&\times \left\{ f_s^{(+)(n)}(p^x, p^z, \mathbf{x})\theta(p_0 + \mu) + \left[f_s^{(-)(n)}(-p^x, -p^z, \mathbf{x}) - 1 \right] \theta(-p_0 - \mu) \right\}, \\
A_n &\equiv \frac{2}{(2\pi)^3} \sum_s s \delta \left\{ (p_0 + \mu)^2 - [E_{p^z s}^{(n)}]^2 \right\} \\
&\times \left\{ f_s^{(+)(n)}(p^x, p^z, \mathbf{x})\theta(p_0 + \mu) + \left[f_s^{(-)(n)}(-p^x, -p^z, \mathbf{x}) - 1 \right] \theta(-p_0 - \mu) \right\}, \tag{3.214}
\end{aligned}$$

and

$$\begin{aligned}
V_0 &= \frac{2}{(2\pi)^3} \delta \left\{ (p_0 + \mu)^2 - [E_{p^z}^{(0)}]^2 \right\} \\
&\times \left\{ f^{(+)(0)}(p^x, p^z, \mathbf{x})\theta(p_0 + \mu) + \left[f^{(-)(0)}(-p^x, -p^z, \mathbf{x}) - 1 \right] \theta(-p_0 - \mu) \right\}. \tag{3.215}
\end{aligned}$$

In Eq. (3.213) we have separated the 16 components of the Wigner function into four groups \mathcal{G}_i with $i = 1, 2, 3, 4$ as shown in Eq. (2.92). From the solutions (3.213) we observe that the pseudoscalar condensate \mathcal{P} and the electric dipole-moment \mathcal{T} vanish and the remaining components can be decomposed into different Landau levels while there is no mixture among different levels.

The Wigner function computed in this subsection is useful for studying the physics in strong magnetic fields. In the Landau-level description, we find the eigenstates of the Dirac equation, thus thermal equilibrium can be well defined. This allows us to calculate various physical quantities under the assumption of thermal equilibrium. We will show in Sec. V that the CME and CSE can be correctly obtained from the Wigner function (3.213). In the limit of sufficiently strong magnetic fields, all particles will stay in the lowest Landau level. The system then reaches a fully polarized state, which means that the average polarization of spin-1/2 particles can reach its maximum value 1/2. In Sec. V we will also calculate the average polarization and the results agree with those expected.

D. Fermions in an electric field

In this subsection we will focus on a system in a pure electric field. If the electric field is large enough, fermion-antifermion pairs can be excited from the vacuum, which is known as Schwinger pair production [78]. Since this process is a time-evolution problem, in this subsection we will use the equal-time Wigner function for convenience. The Schwinger process can be analytically solved in the case of a constant electric field $\mathbf{E}(t, \mathbf{x}) = E_0 \mathbf{e}^z$ and in the case of a Sauter-type field $\mathbf{E}(t, \mathbf{x}) = E_0 \cosh^{-2}(t/\tau) \mathbf{e}^z$, where \mathbf{e}^z is the unit vector in the z direction. Both cases are homogeneous in space, while the Sauter-type field depends on the time t . Analytically solving the Sauter-type field requires knowledge about special functions, thus in this subsection we will solve this case numerically instead of analytically. The numerical calculation for a Sauter-type field also provides an universal approach for solving the Wigner function in a time-dependent but spatially homogeneous electric field. In this subsection we analytically solve the Wigner function in a constant electric field.

1. Asymptotic condition

In order to solve a time-evolution problem, we need the equations of motions and an asymptotic condition. The equations of motions for the equal-time Wigner function have been derived in

subsection II E while the asymptotic condition can be chosen as the Wigner function for vanishing electric field.

Since the Schwinger pair-production process does not depend on the spin, we choose to neglect the spin of the particles. According to the calculations for the free-particle case in subsection III A, where the Wigner function is given in Eq. (3.47), we obtain

$$\begin{pmatrix} \mathcal{F}(x, p) \\ \mathcal{V}_\mu(x, p) \end{pmatrix} = \begin{pmatrix} m \\ p_\mu \end{pmatrix} \delta(p^2 - m^2) V(x, p), \quad (3.216)$$

with

$$V(x, p) \equiv \frac{2d_s}{(2\pi)^3} \left\{ \theta(p^0) f^{(+)}(x, \mathbf{p}) - \theta(-p^0) \left[1 - f^{(-)}(x, -\mathbf{p}) \right] \right\}. \quad (3.217)$$

Here d_s is the spin degeneracy, for spin-1/2 fermions we have $d_s = 2$. Other components \mathcal{P} , \mathcal{A}_μ , and $\mathcal{S}_{\mu\nu}$ vanish as desired because we have neglected spin effects. In general the distribution functions $f^{(\pm)}$ should be space-time dependent. But if a spatial inhomogeneity is taken into account, the Wigner function cannot be analytically computed. In this subsection we assume that the electric field as well as the distributions $f^{(\pm)}$ are independent of spatial coordinates, which indicates that the whole system has translationally invariance in space.

First we derive the equal-time Wigner function from the covariant one by integrating over energy p^0 . The mass-shell delta-function in Eq. (3.216) can be integrated out and we obtain

$$\begin{pmatrix} \mathcal{F}(\mathbf{p}) \\ \mathcal{V}(\mathbf{p}) \end{pmatrix} = \frac{1}{E_{\mathbf{p}}} \begin{pmatrix} m \\ \mathbf{p} \end{pmatrix} C_1(\mathbf{p}), \quad (3.218)$$

and

$$\mathcal{V}^0(\mathbf{p}) \equiv C_2(\mathbf{p}), \quad (3.219)$$

where we have defined $C_1(\mathbf{p}) \equiv \int dp^0 E_{\mathbf{p}} V(p) \delta(p^2 - m^2)$ and $C_2(\mathbf{p}) \equiv \int dp^0 p^0 V(p) \delta(p^2 - m^2)$, respectively. More explicitly, the integration can be performed and yields

$$\begin{aligned} C_1(\mathbf{p}) &= \frac{2}{(2\pi)^3} \left[f^{(+)}(\mathbf{p}) + f^{(-)}(-\mathbf{p}) - 1 \right], \\ C_2(\mathbf{p}) &= \frac{2}{(2\pi)^3} \left[f^{(+)}(\mathbf{p}) - f^{(-)}(-\mathbf{p}) + 1 \right]. \end{aligned} \quad (3.220)$$

The last term ± 1 in Eq. (3.220) represents the contribution from the vacuum, which arises because the operators in the Wigner function are not normal-ordered. If the Wigner function is defined with a normal ordering, the vacuum contribution vanishes and the function $C_1(\mathbf{p})$ is the sum of fermion and anti-fermion distributions. Meanwhile, $C_2(\mathbf{p})$ is the net fermion number density.

2. Equations of motion

In a time-dependent but spatially homogeneous electric field, the equations of motions (2.90) take a simple form. The operators used in these equations, which are defined in Eq. (2.85), have the following expressions, where the electric field is assumed to be in the z -direction.

$$\begin{aligned} D_t &= \partial_t + E(t)\partial_{p^z}, \\ \mathbf{D}_x &= 0, \\ \mathbf{\Pi} &= \mathbf{p}. \end{aligned} \tag{3.221}$$

Here we have dropped spatial derivatives because the whole system is translationally invariant. Due to the translation invariance, we find that the 16 components of the Wigner function can be divided into several subgroups. The members in each group are coupled together according to Eq. (2.90). It is a good feature that the component $\mathcal{V}^0(t, \mathbf{p})$ decouples from all other components, which satisfy the following equation

$$D_t \mathcal{V}^0(t, \mathbf{p}) = 0. \tag{3.222}$$

Since the net charge density in coordinate space can be derived from $\mathcal{V}^0(t, \mathbf{p})$ by integrating over $d^3\mathbf{p}$, the above equation is nothing but the conservation law of the net charge density. Taking the solution in Eq. (3.219) at time t_0 , the solution for $\mathcal{V}^0(t, \mathbf{p})$ reads

$$\mathcal{V}^0(t, \mathbf{p}) = C_2 \left(\mathbf{p} - \int_{t_0}^t dt' E(t') \mathbf{e}^z \right), \tag{3.223}$$

where \mathbf{e}^z is the unit vector along the electric field direction. This solution reflects the overall acceleration of fermions in an electric field, with $-\int_{t_0}^t dt' E(t') \mathbf{e}^z$ is the momentum shift due to the electric field.

Meanwhile, the equations for \mathcal{P} , \mathcal{A}^0 , and \mathcal{S} decouple from others and thus form a closed subsystem. Since these components are all zero when the electric field vanishes, they will remain zeros even after the electric field is turned on. The rest ten components, \mathcal{F} , \mathcal{V} , \mathcal{A} , \mathcal{T} , form another subsystem, which satisfy the equations of motion in a matrix form

$$D_t \mathbf{w}(t, \mathbf{p}) = M(\mathbf{p}) \mathbf{w}(t, \mathbf{p}), \tag{3.224}$$

where the column vector $\mathbf{w}(t, \mathbf{p}) \equiv (\mathcal{F}, \mathcal{V}, \mathcal{A}, \mathcal{T})^T$ has ten elements and $M(\mathbf{p})$ is a 10×10 coefficient

matrix

$$M(\mathbf{p}) = 2 \begin{pmatrix} 0 & 0 & 0 & \mathbf{p}^T \\ 0 & 0 & \mathbf{p}^\times & -m\mathbb{I}_3 \\ 0 & \mathbf{p}^\times & 0 & 0 \\ -\mathbf{p} & m\mathbb{I}_3 & 0 & 0 \end{pmatrix}. \quad (3.225)$$

The initial condition at a given time t_0 for the equations of motion in (3.224) is taken to be the solution (3.218) without the electric field. Based on the fact that all fermions will be accelerated in the electric field, we make the following ansatz for $\mathbf{w}(t, \mathbf{p})$,

$$\mathbf{w}(t, \mathbf{p}) = C_1 \left(\mathbf{p} - \int_{t_0}^t dt' E(t') \mathbf{e}^z \right) \sum_{i=1}^{10} \chi_i(t, \mathbf{p}) \mathbf{e}_i(\mathbf{p}). \quad (3.226)$$

Here the overall factor $C_1 \left(\mathbf{p} - \int_{t_0}^t dt' E(t') \mathbf{e}^z \right)$ is constructed from the distribution function for fermions with the momentum $\mathbf{p} - \int_{t_0}^t dt' E(t') \mathbf{e}^z$ and that for anti-fermions with the opposite momentum $-\mathbf{p} + \int_{t_0}^t dt' E(t') \mathbf{e}^z$. Thus we observe that particles are accelerated along the direction of electric field, while antiparticles are accelerated along the opposite direction. Meanwhile, we take ten basis vectors $\mathbf{e}_i(\mathbf{p})$ because $\mathbf{w}(t, \mathbf{p})$ has ten components. The basis vectors are assumed to be time-independent, while the expanding coefficients $\chi_i(t, \mathbf{p})$ are time-dependent. The first three basis vectors read

$$\mathbf{e}_1 = \begin{pmatrix} 0 \\ \mathbf{e}_z \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{e}_2(\mathbf{p}_T) = \frac{1}{m_T} \begin{pmatrix} m \\ \mathbf{p}_T \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{e}_3(\mathbf{p}_T) = \frac{1}{m_T} \begin{pmatrix} 0 \\ \mathbf{0} \\ \mathbf{e}^z \times \mathbf{p}_T \\ -m\mathbf{e}^z \end{pmatrix}, \quad (3.227)$$

where $m_T \equiv \sqrt{m^2 + \mathbf{p}_T^2}$ is the transverse mass, which ensures that these vectors are properly normalized and orthogonal to each other $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Since they are independent of t and p^z , we have $D_t \mathbf{e}_i = 0$ for $i = 1, 2, 3$. We can check that these basis vectors form a closed sub-Hilbert space under the operator $M(\mathbf{p})$,

$$M(\mathbf{p}) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & -m_T \\ 0 & 0 & p^z \\ m_T & -p^z & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}. \quad (3.228)$$

Note that the initial condition, i.e., the Wigner function when the electric field vanishes, stays in such a subspace, the system will be in this subspace at later time of the evolution. The other basis vectors $\mathbf{e}_i(\mathbf{p})$, $i = 4, 5, \dots, 10$, in Eq. (3.226), are not necessary because the first three are sufficient

to describe the time evolution. The evolution of the coefficients $\chi_i(t, \mathbf{p})$, $i = 1, 2, 3$ are then derived from Eqs. (3.224), (3.226), and (3.228),

$$D_t \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} (t, \mathbf{p}) = 2 \begin{pmatrix} 0 & 0 & m_T \\ 0 & 0 & -p^z \\ -m_T & p^z & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} (t, \mathbf{p}). \quad (3.229)$$

This system of partial differential equations is equivalent to the well-known Vlasov equation for pair production in quantum kinetic theory [89]. Once the functions χ_i are solved from Eq. (3.229), the Wigner function can be reproduced by inserting χ_i and Eq. (3.227) into Eq. (3.226).

3. Solutions for a Sauter-type field

In order to solve Eq. (3.229), we first need an initial condition. One naive choice is, assuming that the electric field does not exist before time t_0 ,

$$\chi_1(t_0, \mathbf{p}) = \frac{p^z}{E_{\mathbf{p}}}, \quad \chi_2(t_0, \mathbf{p}) = \frac{m_T}{E_{\mathbf{p}}}, \quad \chi_3(t_0, \mathbf{p}) = 0. \quad (3.230)$$

This initial condition corresponds to a field which is suddenly switched on at t_0 , i.e., a time-dependent electric field as

$$E(t) = \theta(t - t_0)E(t). \quad (3.231)$$

Such an initial condition is useful when dealing with a field which vanishes when $t \rightarrow t_0$. For example, for a Sauter-type field $E(t) = E_0 \cosh^{-2}(t/\tau)$, we can specify the solution (3.230) for $t_0 \rightarrow -\infty$ and the system evolves with time according to Eq. (3.229).

We now take the Sauter-type electric field $E(t) = E_0 \cosh^{-2}(t/\tau)$ as an example. In Fig. 3 we plot the time dependence of the field strength. The Sauter-type field can be used to describe a pulse, which converges to zero in the limit $t \rightarrow \pm\infty$. We define the canonical momentum q^z as

$$q^z = p^z - E_0\tau [\tanh(t/\tau) + 1], \quad (3.232)$$

which ensures that $q^z = p^z$ in the limit $t \rightarrow -\infty$. Then we substitute the kinetic momentum p^z in the operator D_t by the canonical one q^z and obtain

$$[\partial_t + E_0 \cosh^{-2}(t/\tau) \partial_{p^z}] \chi_i(t, \mathbf{p}) = \frac{d}{dt} \chi_i(t, \mathbf{p}_T, q^z). \quad (3.233)$$

The equations of motions (3.229) now transform into ordinary differential equations,

$$\frac{d}{dt} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & m_T \\ 0 & 0 & -q^z - E_0\tau [\tanh(t/\tau) + 1] \\ -m_T & q^z + E_0\tau [\tanh(t/\tau) + 1] & 0 \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}, \quad (3.234)$$

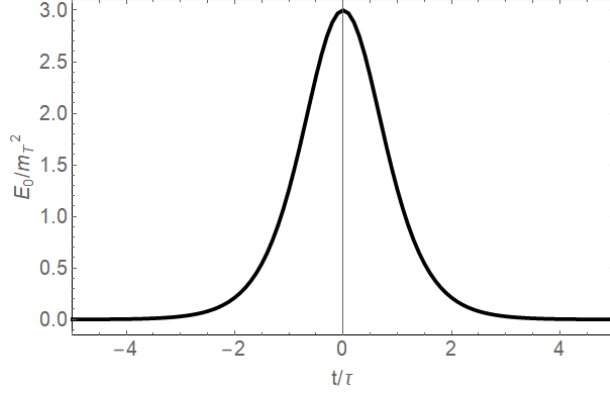


Figure 3: The time dependence of a Sauter-type electric field $E(t) = E_0 \cosh^{-2}(t/\tau)$. Here we take the transverse mass m_T as the energy unit and the peak value of the electric field is taken to be $3m_T^2$.

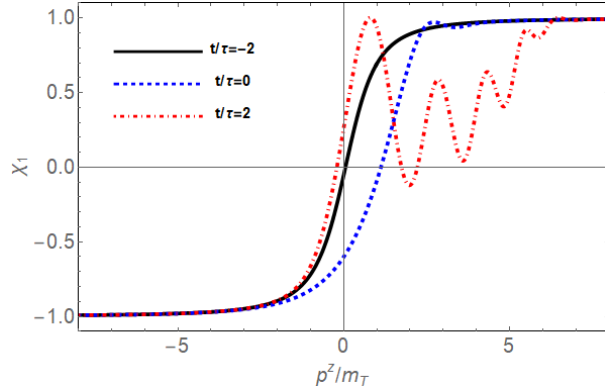


Figure 4: The p^z -dependence of χ_1 at times $t = -2\tau$ (solid line), $t = 0$ (dashed line), and $t = 2\tau$ (dot-dashed line).

with initial conditions

$$\lim_{t_0 \rightarrow -\infty} \chi_1(t_0, \mathbf{p}_T, q^z) = \frac{q^z}{E_{\mathbf{p}}}, \quad \lim_{t_0 \rightarrow -\infty} \chi_2(t_0, \mathbf{p}_T, q^z) = \frac{m_T}{E_{\mathbf{p}}}, \quad \lim_{t_0 \rightarrow -\infty} \chi_3(t_0, \mathbf{p}_T, q^z) = 0. \quad (3.235)$$

This system of ordinary differential equations can be easily solved using the finite-difference method. One can also see Ref. [89] for the analytical solution from quantum kinetic theory.

As an example, we take the transverse mass $m_T = \sqrt{m^2 + \mathbf{p}_T^2}$ as the energy unit and $\tau = 1/m_T$ as the time unit. The peak value of the Sauter-type electric field is chosen to be $E_0/m_T^2 = 3$. In Figs. 4, 5, and 6 we plot the p^z -dependence of χ_1 , χ_2 , and χ_3 , respectively, at several times, $t = -2\tau$, 0 , and 2τ . We emphasize that in these figures the x -axis is the kinetic momentum p^z . According to our calculation, even though the electric field strength turns to zero in the limit $t \rightarrow +\infty$, the functions χ_1 , χ_2 , and χ_3 cannot reach stationary states. Instead, these functions will oscillate and the oscillations become more and more pronounced at later times. In Sec. V E we will clearly see

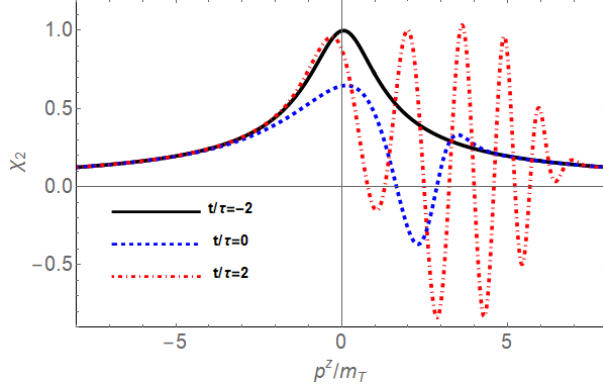


Figure 5: The p^z -dependence of χ_2 at times $t = -2\tau$ (solid line), $t = 0$ (dashed line), and $t = 2\tau$ (dot-dashed line).

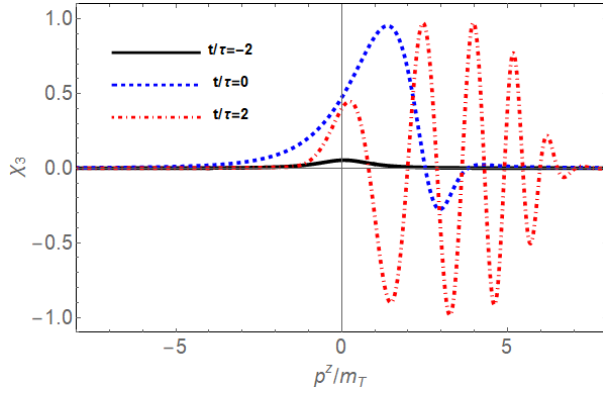


Figure 6: The p^z -dependence of χ_3 at times $t = -2\tau$ (solid line), $t = 0$ (dashed line), and $t = 2\tau$ (dot-dashed line).

that the oscillation does not contribute to the pair-production rate, and the pair spectrum finally reaches a stationary state.

4. Solutions in a constant electric field.

However, the initial conditions in Eq. (3.230) do not work for a constant electric field $E(t) = E_0$. Since a constant field is not integrable, the momentum shift $\int_{t_0}^t dt' E(t') \mathbf{e}_z$ will be infinitely large if we take the limit $t_0 \rightarrow -\infty$. From the physical point of view, the fermions can collide with each other and the kinetic energy will be converted to the thermal energy through collisions. The collision processes will retard the movement of particles and the system would finally reach a new balance state in the electric field. If the system has a boundary, the particles would accumulate near the boundary and the chemical potential μ then becomes spatially-dependent. Finally the force

from the Pauli exclusion principle, i.e., the effect from the gradient of μ , will cancel with that from the electric field. If the system is infinitely large, the system would reach a state with a collective charge current, and more fermions (assumed to have positive charge) moving in the direction of the electric field. In this case, the current can be independent of the spatial coordinate and so does the distribution.

Here we assume that the system is described by spatial independent distribution functions at time t_0 and we focus on a short period after this moment. The system w deviates from the initial state during this period because of pair production. Then our goal is to find a solution which coincides with Eq. (3.218) when the electric field vanishes,

$$\left. \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} (t, \mathbf{p}) \right|_{E_0 \rightarrow 0} = \frac{1}{E_{\mathbf{p}}} \begin{pmatrix} p^z \\ m_T \\ 0 \end{pmatrix}. \quad (3.236)$$

The Wigner function in a constant electric field is then given by Eqs. (3.226) and (3.227), where the coefficients χ_i with $i = 1, 2, 3$ are solved from Eq. (3.229) with the condition (3.236), while other χ_i with $i = 4, 5, \dots, 10$ are zeros. From quantum kinetic theory one can obtain the following solution [89],

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} (\mathbf{p}) = \begin{pmatrix} d_1 \left(\eta, \sqrt{\frac{2}{E_0}} p^z \right) \\ \frac{m_T}{\sqrt{2E_0}} d_2 \left(\eta, \sqrt{\frac{2}{E_0}} p^z \right) \\ \frac{m_T}{\sqrt{2E_0}} d_3 \left(\eta, \sqrt{\frac{2}{E_0}} p^z \right) \end{pmatrix}, \quad (3.237)$$

where $\eta \equiv m_T^2/E_0$ is the dimensionless transverse mass square. One can check that this solution satisfies Eq. (3.229) and the constraint (3.236). Here the auxiliary functions d_1 , d_2 , and d_3 are

defined in Eq. (B19). Then the Wigner function can be reproduced using Eq. (3.226),

$$\begin{aligned}
\mathcal{F} &= \frac{m}{\sqrt{2E_0}} d_2 \left(\eta, \sqrt{\frac{2}{E_0}} p^z \right) C_1(\mathbf{p} - E_0 t \mathbf{e}_z), \\
\mathcal{P} &= 0, \\
\mathcal{V}^0 &= C_2(\mathbf{p} - E_0 t \mathbf{e}_z), \\
\mathcal{V}_T &= \frac{\mathbf{p}_T}{\sqrt{2E_0}} d_2 \left(\eta, \sqrt{\frac{2}{E_0}} p^z \right) C_1(\mathbf{p} - E_0 t \mathbf{e}_z), \\
\mathcal{V}^z &= d_1 \left(\eta, \sqrt{\frac{2}{E_0}} p^z \right) C_1(\mathbf{p} - E_0 t \mathbf{e}_z), \\
\mathcal{A}^0 &= 0, \\
\mathcal{A} &= \frac{\mathbf{e}_z \times \mathbf{p}_T}{\sqrt{2E_0}} d_3 \left(\eta, \sqrt{\frac{2}{E_0}} p^z \right) C_1(\mathbf{p} - E_0 t \mathbf{e}_z), \\
\mathcal{T} &= -\frac{m \mathbf{e}_z}{\sqrt{2E_0}} d_3 \left(\eta, \sqrt{\frac{2}{E_0}} p^z \right) C_1(\mathbf{p} - E_0 t \mathbf{e}_z), \\
\mathcal{S} &= 0.
\end{aligned} \tag{3.238}$$

When taking the limit $E_0 \rightarrow 0$, the Wigner function recovers the results in Eqs. (3.218) and (3.219). At the moment $t = 0$, the existing particles are assumed to produce distributions which are determined by $C_1(\mathbf{p})$ and $C_2(\mathbf{p})$. Note that due to the lack of collisions, the solutions in Eq. (3.238) can only be used to describe a short period after $t = 0$, i.e., for times smaller than the mean free time.

E. Fermions in constant parallel electromagnetic fields

1. Asymptotic condition

In subsection III C, we have derived the Wigner function in a constant magnetic field. In this subsection we will add an electric field, which is assumed to be parallel to the magnetic field. Both the electric field and the magnetic field are chosen to be constant so that the problem can be analytically solved. Similar to the case of a pure electric field, the case in this subsection is a time-evolution problem. Particles in constant magnetic field are described by the Landau levels, which is used as the initial condition for the time evolution. Taking the chiral chemical potential $\mu_5 = 0$, and integrating over energy p^0 , we obtain the following equal-time Wigner function from

Eq. (3.213),

$$\begin{aligned}
\mathcal{G}_1 &= \sum_{n=0} \frac{m}{E_{p^z}^{(n)}} C_1^{(n)}(p^z) e_1^{(n)}(p_T), \\
\mathcal{G}_2 &= \sum_{n=0} C_2^{(n)}(p^z) e_1^{(n)}(p_T), \\
\mathcal{G}_3 &= \frac{p^z}{E_{p^z}^{(0)}} C_1^{(0)}(p^z) e_1^{(0)}(p_T) + \sum_{n>0} \frac{1}{E_{p^z}^{(n)}} C_1^{(n)}(p^z) \left[p^z e_2^{(n)}(p_T) + \sqrt{2nB_0} e_3^{(n)}(p_T) \right], \\
\mathcal{G}_4 &= 0.
\end{aligned} \tag{3.239}$$

Here \mathcal{G}_i are constructed from the Wigner function as shown in Eq. (2.92), $C_1^{(n)} \equiv \int dp^0 E_{p^z}^{(n)} V^{(n)}$ and $C_2^{(n)} \equiv \int dp^0 (p^0 + \mu) V^{(n)}$. The basis vectors $e_{1,2,3}^{(0)}(\mathbf{p}_T)$ are defined in Appendix B. The function $V^{(n)}$ is defined in (3.214) and (3.215), from which we obtain an explicit relation between $C_1^{(n)}$, $C_2^{(n)}$ and $f^{(\pm)(n)}$,

$$\begin{aligned}
C_1^{(n)}(p^z) &= \frac{2 - \delta_{n0}}{(2\pi)^3} \left[f^{(+)(n)}(p^z) + f^{(-)(n)}(p^z) - 1 \right], \\
C_2^{(n)}(p^z) &= \frac{2 - \delta_{n0}}{(2\pi)^3} \left[f^{(+)(n)}(p^z) - f^{(-)(n)}(p^z) + 1 \right].
\end{aligned} \tag{3.240}$$

Up to a vacuum contribution, the auxiliary function $C_1^{(n)}(p^z)$ is the sum of the fermion and anti-fermion distribution in the n -th Landau level, while $C_2^{(n)}(p^z)$ is the difference. In general the distributions depend on p^x , p^z , and \mathbf{x} , where $p^x - B_0 y$ plays a role as the center position in the y -direction. Here we choose to neglect the spatial dependence of the distributions. Thus, there is no dependence on \mathbf{x} and p^x in Eq. (3.240). The pre-factor $2 - \delta_{n0}$ is the spin degeneracy for the n -th Landau level. We find that the equal-time Wigner function, whose components are given in Eq. (3.239), is a sum over different Landau levels. Later on we will show that in the presence of a constant electric field, different Landau levels evolve independently.

2. Equations of motion

In the presence of constant electromagnetic fields, we assume that the whole system is spatially homogeneous so that the spatial derivative $\nabla_{\mathbf{x}}$ can be dropped. The operators in Eq. (2.85) are now given by

$$\begin{aligned}
D_t &= \partial_t + E_0 \partial_{p^z}, \\
\mathbf{D}_{\mathbf{x}} &= B_0 \mathbf{e}_z \times \nabla_{\mathbf{p}}, \\
\mathbf{\Pi} &= \mathbf{p},
\end{aligned} \tag{3.241}$$

where E_0 and B_0 are strengths of electric and magnetic field, respectively. Then the matrix operators M_1 and M_2 , defined in Eq. (2.95), take the following forms,

$$M_1 = \begin{pmatrix} 0 & 2p^x & 2p^y & 2p^z \\ 2p^x & 0 & 0 & \hbar B_0 \partial_{p^x} \\ 2p^y & 0 & 0 & \hbar B_0 \partial_{p^y} \\ 2p^z & -\hbar B_0 \partial_{p^x} & -\hbar B_0 \partial_{p^y} & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & -\hbar B_0 \partial_{p^y} & \hbar B_0 \partial_{p^x} & 0 \\ -\hbar B_0 \partial_{p^y} & 0 & -2p^z & 2p^y \\ \hbar B_0 \partial_{p^x} & 2p^z & 0 & -2p^x \\ 0 & -2p^y & 2p^x & 0 \end{pmatrix}. \quad (3.242)$$

For the lowest Landau level, only the basis vector $e_1^{(0)}(p_T)$ defined in (B15), contributes to the solution in (3.239). Furthermore, we can check that $e_1^{(0)}(p_T)$ is an eigenvector of the operators D_t , M_1 , and M_2 ,

$$D_t e_1^{(0)}(p_T) = 0, \quad M_1 e_1^{(0)}(p_T) = 2p^z e_1^{(0)}(p_T), \quad M_2 e_1^{(0)}(p_T) = 0. \quad (3.243)$$

Thus we only need $e_1^{(0)}(p_T)$ to describe the dynamics of the lowest Landau level.

For higher Landau levels, the initial Wigner function in (3.239) contains all three basis vectors $e_i^{(n)}(\mathbf{p}_T)$, $i = 1, 2, 3$ and $n > 0$, which are defined in Eq. (B15). But we can check that these basis vectors are not closed under the operator M_2 . In order to construct a closed Hilbert space under the operators D_t , M_1 , and M_2 , we need another basis vector, i.e., $e_4^{(n)}(\mathbf{p}_T)$ defined in Eq. (B15). Acting the matrix operators M_1 and M_2 on these basis vectors, we obtain

$$M_1 e_i^{(n)}(\mathbf{p}_T) = \sum_{j=1}^4 (c_1^{(n)})_{ij}^T e_j^{(n)}(\mathbf{p}_T),$$

$$M_2 e_i^{(n)}(\mathbf{p}_T) = \sum_{j=1}^4 (c_2^{(n)})_{ij}^T e_j^{(n)}(\mathbf{p}_T), \quad (3.244)$$

where the coefficients are

$$c_1^{(n)} = 2 \begin{pmatrix} 0 & p^z & \sqrt{2nB_0} & 0 \\ p^z & 0 & 0 & 0 \\ \sqrt{2nB_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c_2^{(n)} = -2 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2nB_0} \\ 0 & 0 & 0 & -p^z \\ 0 & -\sqrt{2nB_0} & p^z & 0 \end{pmatrix}. \quad (3.245)$$

Note that in Eq. (3.244), we have used the transposes of $c_1^{(n)}$ and $c_2^{(n)}$ for convenience of further calculations. Due to the fact that the basis vectors $e_i^{(n)}(\mathbf{p}_T)$, $i = 1, 2, 3, 4$, are independent of t and p^z , we find

$$D_t e_i^{(n)}(\mathbf{p}_T) = 0. \quad (3.246)$$

When the electric field vanishes, the equal-time Wigner function in Eq. (3.239) can be expressed in terms of the basis vectors $e_1^{(0)}(p_T)$ and $e_i^{(n)}(\mathbf{p}_T)$. Analogous to the case in the previous subsection, we take the Wigner function in a constant magnetic field as an asymptotic condition when $E_0 \rightarrow 0$. Then it is straightforward to conclude that the Wigner function will stay in the Hilbert space formed by $e_1^{(0)}(p_T)$ and $e_i^{(n)}(\mathbf{p}_T)$ because this space is closed for all operators D_t , M_1 and M_2 in the equations of motion (2.93). We thus decompose the Wigner function as

$$\mathcal{G}_i(t, \mathbf{p}) = f_i^{(0)}(t, p^z) e_1^{(0)}(p_T) + \sum_{n>0} \sum_{j=1}^4 f_{ij}^{(n)}(t, p^z) e_j^{(n)}(\mathbf{p}_T), \quad (3.247)$$

where $i = 1, 2, 3, 4$. Note that since we focus on a constant magnetic field, the basis vectors are independent to time and all the time-dependence is put into the coefficients $f_i^{(0)}$ and $f_{ij}^{(n)}$. We also find that the transverse momentum is separated in Eq. (3.247). Inserting the decomposition (3.247) into the equations of motion (2.93), and then using the orthonormality conditions in Eqs. (B16), (B17), and (B18) to separate the coefficients of different basis vectors, we derive the equations of motions for $f_i^{(0)}(t, p^z)$ and $f_{ij}^{(n)}(t, p^z)$. For the lowest Landau level, the equations of motions read

$$D_t \begin{pmatrix} f_1^{(0)} \\ f_2^{(0)} \\ f_3^{(0)} \\ f_4^{(0)} \end{pmatrix} (t, p^z) = 2 \begin{pmatrix} 0 & 0 & 0 & p^z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m \\ -p^z & 0 & m & 0 \end{pmatrix} \begin{pmatrix} f_1^{(0)} \\ f_2^{(0)} \\ f_3^{(0)} \\ f_4^{(0)} \end{pmatrix} (t, p^z), \quad (3.248)$$

while for the higher Landau levels we obtain

$$D_t \begin{pmatrix} \mathbf{f}_1^{(n)} \\ \mathbf{f}_2^{(n)} \\ \mathbf{f}_3^{(n)} \\ \mathbf{f}_4^{(n)} \end{pmatrix} (t, p^z) = \begin{pmatrix} 0 & 0 & 0 & c_1^{(n)} \\ 0 & 0 & -c_2^{(n)} & 0 \\ 0 & -c_2^{(n)} & 0 & -2m\mathbb{I}_4 \\ -c_1^{(n)} & 0 & 2m\mathbb{I}_4 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{f}_1^{(n)} \\ \mathbf{f}_2^{(n)} \\ \mathbf{f}_3^{(n)} \\ \mathbf{f}_4^{(n)} \end{pmatrix} (t, p^z), \quad (3.249)$$

where $\mathbf{f}_i^{(n)} \equiv (f_{i1}^{(n)}, f_{i2}^{(n)}, f_{i3}^{(n)}, f_{i4}^{(n)})^T$ is a four-dimensional column vector. From these equations we observe that different Landau levels decouple from each other and thus evolve separately.

3. Lowest Landau level

In the lowest Landau level, the spin of a positive charged particle is parallel to the magnetic field. Meanwhile, the higher Landau levels are 2-fold degenerate with respect to spin. Thus the lowest Landau level is special and needs a careful treatment. The equations of motion for the lowest

Landau level (3.248) are obviously distinct from those for the higher Landau levels (3.249). We note that in Eq. (3.248), the equation for $f_2^{(0)}$ decouples from the others, which gives

$$D_t f_2^{(0)}(t, p^z) = 0. \quad (3.250)$$

Since the net fermion number at the lowest Landau level is given by

$$n^{(0)} = \int d^3 \mathbf{p} f_2^{(0)}(t, p^z) \Lambda_+^{(0)}(p_T), \quad (3.251)$$

we find that the equation for $f_2^{(0)}$ correspond to the conservation of $n^{(0)}$, i.e.,

$$\partial_t n^{(0)} = \int d^3 \mathbf{p} \left[D_t f_2^{(0)}(t, p^z) \right] \Lambda_+^{(0)}(p_T) = 0, \quad (3.252)$$

Here we have integrated the p^z -derivative by parts and dropped the boundary term. Equation (3.250), together with the asymptotic condition $f_2^{(0)}(t, p^z) \Big|_{E_0 \rightarrow 0} = C_2^{(0)}(p^z)$, give the following specific solution

$$f_2^{(0)}(t, p^z) = C_2^{(0)}(p^z - E_0 t). \quad (3.253)$$

It describes the overall acceleration of particles along the direction of the electric field. Note that due to the absence of collisions, the particle distribution will be far from the initial one after a long time period. But in reality, collisions prevent the acceleration and the system will stay near the thermal equilibrium and the specific solution is only suitable to describe the physics for $t < t_{\text{relax}}$, where t_{relax} is the relaxation time of the system.

The other three functions, $f_1^{(0)}$, $f_3^{(0)}$ and $f_4^{(4)}$ can be parametrized as

$$\left\{ f_1^{(0)}, f_3^{(0)}, f_4^{(4)} \right\} = \left\{ \chi_1^{(0)}, \chi_2^{(0)}, \chi_3^{(0)} \right\} C_1^{(0)}(p^z - E_0 t), \quad (3.254)$$

where $C_1^{(0)}$ is defined in Eq. (3.240). Here the canonical momentum $p^z - E_0 t$ again reflects the acceleration of particles. Comparing with Eq. (3.239), we obtain the asymptotic condition when the electric field vanishes,

$$\left(\begin{array}{c} \chi_1^{(0)} \\ \chi_2^{(0)} \\ \chi_3^{(0)} \end{array} \right) (t, p^z) \Big|_{E_0 \rightarrow 0} = \frac{1}{E_{p^z}^{(0)}} \left(\begin{array}{c} m \\ p^z \\ 0 \end{array} \right). \quad (3.255)$$

The equations of motion for $\chi_1^{(0)}, \chi_2^{(0)}, \chi_3^{(0)}$ are derived from Eq. (3.248) by using the fact that $D_t C_1^{(0)}(p^z - E_0 t) = 0$,

$$D_t \left(\begin{array}{c} \chi_1^{(0)} \\ \chi_2^{(0)} \\ \chi_3^{(0)} \end{array} \right) (t, p^z) = 2 \left(\begin{array}{ccc} 0 & 0 & p^z \\ 0 & 0 & -m \\ -p^z & m & 0 \end{array} \right) \left(\begin{array}{c} \chi_1^{(0)} \\ \chi_2^{(0)} \\ \chi_3^{(0)} \end{array} \right) (t, p^z). \quad (3.256)$$

Comparing Eq. (3.256) and the asymptotic condition in Eq. (3.255) with Eqs. (3.229) and (3.236), we find that they are exactly the same if we substitute $\chi_1 \rightarrow \chi_2^{(0)}, \chi_2 \rightarrow \chi_1^{(0)}, \chi_3 \rightarrow -\chi_3^{(0)}$, and $m_T \rightarrow m$ in Eqs. (3.229) and (3.236). This indicates that the pair production in the lowest Landau level and in a pure electric field are controlled by the same system of partial differential equations. The solution for Eq. (3.256) in a constant electric field is straightforward,

$$\begin{pmatrix} \chi_1^{(0)} \\ \chi_2^{(0)} \\ \chi_3^{(0)} \end{pmatrix} (p^z) = \begin{pmatrix} \frac{m}{\sqrt{2E_0}} d_2(\eta^{(0)}, \sqrt{\frac{2}{E_0}} p^z) \\ d_1(\eta^{(0)}, \sqrt{\frac{2}{E_0}} p^z) \\ -\frac{m}{\sqrt{2E_0}} d_3(\eta^{(0)}, \sqrt{\frac{2}{E_0}} p^z) \end{pmatrix}, \quad (3.257)$$

where $\eta^{(0)} \equiv m^2/E_0$ and d_i are defined in Eq. (B19). The functions $f_1^{(0)}, f_3^{(0)}$ and $f_4^{(4)}$ can be reproduced using Eq. (3.254). To summarize, we list all the functions for the Lowest Landau level,

$$\begin{pmatrix} f_1^{(0)} \\ f_2^{(0)} \\ f_3^{(0)} \\ f_4^{(0)} \end{pmatrix} (t, p^z) = \begin{pmatrix} \frac{m}{\sqrt{2E_0}} d_2(\eta^{(0)}, \sqrt{\frac{2}{E_0}} p^z) C_1^{(0)}(p^z - E_0 t) \\ C_2^{(0)}(p^z - E_0 t) \\ d_1(\eta^{(0)}, \sqrt{\frac{2}{E_0}} p^z) C_1^{(0)}(p^z - E_0 t) \\ -\frac{m}{\sqrt{2E_0}} d_3(\eta^{(0)}, \sqrt{\frac{2}{E_0}} p^z) C_1^{(0)}(p^z - E_0 t) \end{pmatrix}. \quad (3.258)$$

By inserting them into Eq. (3.247) one can obtain the contribution of the lowest Landau level to the Wigner function, which will be given later.

4. Higher Landau levels

For all the higher Landau levels, the equations of motion in Eq. (3.249) take the same form for different n . Note that these equations contain 16 functions, $f_{ij}^{(n)}$ with $i, j = 1, 2, 3, 4$, for each n . Solving such a system seems to be very difficult, thus we first analyze the relation between $f_{ij}^{(n)}$, which is listed in Table III. Using this table we can divide the 16 functions into several subgroups. For example, we start from $f_{11}^{(n)}$, which directly couples with $f_{42}^{(n)}$ and $f_{43}^{(n)}$. Then $f_{42}^{(n)}$ couples with $f_{32}^{(n)}$, while $f_{43}^{(n)}$ couples with $f_{33}^{(n)}$. Furthermore, $f_{32}^{(n)}$ and $f_{33}^{(n)}$ couple with $f_{24}^{(n)}$. Thus, these six functions, $\{f_{11}^{(n)}, f_{24}^{(n)}, f_{32}^{(n)}, f_{33}^{(n)}, f_{42}^{(n)}, f_{43}^{(n)}\}$, form one subgroup because every member only couples with other members in this group. Analogously, we can find other subgroups from Table III: $\{f_{12}^{(n)}, f_{13}^{(n)}, f_{31}^{(n)}, f_{41}^{(n)}\}$, $\{f_{14}^{(n)}\}$, $\{f_{21}^{(n)}\}$, $\{f_{22}^{(n)}, f_{23}^{(n)}, f_{34}^{(n)}, f_{44}^{(n)}\}$. Note that not all of these subgroups contribute to the Wigner function, which can be understood as follows: according to the solution in Eq. (3.239), when the electric field vanishes $E_0 \rightarrow 0$, the non-vanishing functions are $f_{11}^{(n)}, f_{21}^{(n)}, f_{32}^{(n)}$, and $f_{33}^{(n)}$. During the time evolution, only the terms coupled with them, i.e.,

	$f_{11}^{(n)}$	$f_{12}^{(n)}$	$f_{13}^{(n)}$	$f_{14}^{(n)}$	$f_{21}^{(n)}$	$f_{22}^{(n)}$	$f_{23}^{(n)}$	$f_{24}^{(n)}$	$f_{31}^{(n)}$	$f_{32}^{(n)}$	$f_{33}^{(n)}$	$f_{34}^{(n)}$	$f_{41}^{(n)}$	$f_{42}^{(n)}$	$f_{43}^{(n)}$	$f_{44}^{(n)}$
$f_{11}^{(n)}$														✓	✓	
$f_{12}^{(n)}$													✓			
$f_{13}^{(n)}$													✓			
$f_{14}^{(n)}$																
$f_{21}^{(n)}$																
$f_{22}^{(n)}$												✓				
$f_{23}^{(n)}$												✓				
$f_{24}^{(n)}$										✓	✓					
$f_{31}^{(n)}$													✓			
$f_{32}^{(n)}$								✓						✓		
$f_{33}^{(n)}$								✓							✓	
$f_{34}^{(n)}$						✓	✓									✓
$f_{41}^{(n)}$		✓	✓						✓							
$f_{42}^{(n)}$	✓									✓						
$f_{43}^{(n)}$	✓										✓					
$f_{44}^{(n)}$												✓				

Table III: Coupling relations among $f_{ij}^{(n)}$, $i, j = 1, 2, 3, 4$. The table describes the coupling for each pair of functions. Here ✓ means that we can find one equation in Eq. (3.249) which contains both of these two functions. On the other hand, blank means we cannot find such an equation in Eq. (3.249).

$\{f_{21}^{(n)}\}$ and $\{f_{11}^{(n)}, f_{24}^{(n)}, f_{32}^{(n)}, f_{33}^{(n)}, f_{42}^{(n)}, f_{43}^{(n)}\}$, can have non-trivial solutions, while other terms will stay zero.

First we focus on the function $f_{21}^{(n)}$. It decouples from all the other functions and the corresponding equation reads $D_t f_{21}^{(n)}(t, p^z) = 0$. Analogous to the case in the lowest Landau level, this equation gives nothing but the conservation of the net fermion number in each Landau levels. Its specific solution is

$$f_{21}^{(n)}(t, p^z) = C_2^{(n)}(p^z - E_0 t), \quad (3.259)$$

where $C_2^{(n)}$ is defined in Eq. (3.240). Again this solution describes the overall acceleration of charged particles, and at $t = 0$ the system is described by $C_1^{(n)}(p^z)$ and $C_2^{(n)}(p^z)$.

As mentioned above, $\{f_{11}^{(n)}, f_{24}^{(n)}, f_{32}^{(n)}, f_{33}^{(n)}, f_{42}^{(n)}, f_{43}^{(n)}\}$ form one subgroup for the equations of

motion. They can be further decoupled by introducing a linear recombination,

$$\begin{pmatrix} g_1^{(n)} & g_3^{(n)} \\ g_4^{(n)} & g_2^{(n)} \end{pmatrix} = \frac{1}{m^{(n)}} \begin{pmatrix} m & \sqrt{2nB_0} \\ \sqrt{2nB_0} & -m \end{pmatrix} \begin{pmatrix} f_{11}^{(n)} & f_{24}^{(n)} \\ f_{33}^{(n)} & f_{42}^{(n)} \end{pmatrix}, \quad (3.260)$$

where we define $m^{(n)} \equiv \sqrt{m^2 + 2nB_0}$ as the effective mass in the n -th Landau level. The transformation matrix is unitary, so the inverse transformation reads

$$\begin{pmatrix} f_{11}^{(n)} & f_{24}^{(n)} \\ f_{33}^{(n)} & f_{42}^{(n)} \end{pmatrix} = \frac{1}{m^{(n)}} \begin{pmatrix} m & \sqrt{2nB_0} \\ \sqrt{2nB_0} & -m \end{pmatrix} \begin{pmatrix} g_1^{(n)} & g_3^{(n)} \\ g_4^{(n)} & g_2^{(n)} \end{pmatrix}. \quad (3.261)$$

Then from the equations of motion (3.249) we obtain the following two groups of equations,

$$D_t \begin{pmatrix} g_1^{(n)} \\ g_2^{(n)} \\ f_{32}^{(n)} \end{pmatrix} (t, p^z) = 2 \begin{pmatrix} 0 & -p^z & 0 \\ p^z & 0 & -m^{(n)} \\ 0 & m^{(n)} & 0 \end{pmatrix} \begin{pmatrix} g_1^{(n)} \\ g_2^{(n)} \\ f_{32}^{(n)} \end{pmatrix} (t, p^z), \quad (3.262)$$

and

$$D_t \begin{pmatrix} g_3^{(n)} \\ g_4^{(n)} \\ f_{43}^{(n)} \end{pmatrix} (t, p^z) = 2 \begin{pmatrix} 0 & -p^z & 0 \\ p^z & 0 & m^{(n)} \\ 0 & -m^{(n)} & 0 \end{pmatrix} \begin{pmatrix} g_3^{(n)} \\ g_4^{(n)} \\ f_{43}^{(n)} \end{pmatrix} (t, p^z). \quad (3.263)$$

When the electric field vanishes, we can calculate $g_i^{(n)}$ and the result reads,

$$\left. \begin{pmatrix} g_1^{(n)} \\ g_2^{(n)} \\ f_{32}^{(n)} \end{pmatrix} \right|_{E_0 \rightarrow 0} = \frac{1}{E_{p^z}^{(n)}} \begin{pmatrix} m^{(n)} \\ 0 \\ p^z \end{pmatrix} C_1^{(n)}(p^z), \quad \left. \begin{pmatrix} g_3^{(n)} \\ g_4^{(n)} \\ f_{43}^{(n)} \end{pmatrix} \right|_{E_0 \rightarrow 0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.264)$$

The equations for $g_3^{(n)}$, $g_4^{(n)}$, and $f_{43}^{(n)}$ will have trivial solutions, i.e., all of three stay zero even after the electric field is turned on. Therefore, for the higher Landau levels we only need to focus on $g_1^{(n)}$, $g_2^{(n)}$, and $f_{32}^{(n)}$. We take the overall acceleration ansatz and parameterize them as

$$\{g_1^{(n)}, g_2^{(n)}, f_{32}^{(n)}\} = \{\chi_1^{(n)}, \chi_2^{(n)}, \chi_3^{(n)}\} C_1^{(n)}(p^z - E_0 t). \quad (3.265)$$

Since $D_t C_1^{(n)}(p^z - E_0 t) = 0$, the equations of motion for $\chi_1^{(n)}$, $\chi_2^{(n)}$, and $\chi_3^{(n)}$ can be derived from Eq. (3.262)

$$D_t \begin{pmatrix} \chi_1^{(n)} \\ \chi_2^{(n)} \\ \chi_3^{(n)} \end{pmatrix} (t, p^z) = 2 \begin{pmatrix} 0 & -p^z & 0 \\ p^z & 0 & -m^{(n)} \\ 0 & m^{(n)} & 0 \end{pmatrix} \begin{pmatrix} \chi_1^{(n)} \\ \chi_2^{(n)} \\ \chi_3^{(n)} \end{pmatrix} (t, p^z), \quad (3.266)$$

with the asymptotic condition

$$\left. \begin{pmatrix} \chi_1^{(n)} \\ \chi_2^{(n)} \\ \chi_3^{(n)} \end{pmatrix} (t, p^z) \right|_{E_0 \rightarrow 0} = \frac{1}{E p^z} \begin{pmatrix} m^{(n)} \\ 0 \\ p^z \end{pmatrix}. \quad (3.267)$$

The equations and asymptotic conditions coincide with Eqs. (3.229) and (3.236) if we make the replacements $\chi_1 \rightarrow \chi_3^{(n)}$, $\chi_2 \rightarrow \chi_1^{(n)}$, $\chi_3 \rightarrow \chi_2^{(n)}$, and $m_T \rightarrow m^{(n)}$. The solution for the case of a constant electric field is then straightforward to obtain,

$$\begin{pmatrix} \chi_1^{(n)} \\ \chi_2^{(n)} \\ \chi_3^{(n)} \end{pmatrix} (t, p^z) = \begin{pmatrix} \frac{m^{(n)}}{\sqrt{2E_0}} d_2 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) \\ \frac{m^{(n)}}{\sqrt{2E_0}} d_3 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) \\ d_1 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) \end{pmatrix}, \quad (3.268)$$

where $\eta^{(n)} \equiv (m^2 + 2nB_0)/E_0$ is the dimensionless effective mass squared. Inserting the solutions into Eq. (3.265) and then using the inverse transformation in Eq. (3.261), one obtains the non-vanishing functions,

$$\begin{aligned} \begin{pmatrix} f_{11}^{(n)} \\ f_{33}^{(n)} \end{pmatrix} &= \begin{pmatrix} m \\ \sqrt{2nB_0} \end{pmatrix} \frac{1}{\sqrt{2E_0}} d_2 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) C_1^{(n)}(p^z - E_0 t), \\ \begin{pmatrix} f_{24}^{(n)} \\ f_{42}^{(n)} \end{pmatrix} &= \begin{pmatrix} \sqrt{2nB_0} \\ -m \end{pmatrix} \frac{1}{\sqrt{2E_0}} d_3 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) C_1^{(n)}(p^z - E_0 t), \\ f_{32}^{(n)} &= d_1 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) C_1^{(n)}(p^z - E_0 t), \end{aligned} \quad (3.269)$$

together with $f_{21}^{(n)}$ listed in Eq. (3.259). The remaining ten of $f_{ij}^{(n)}$ are zero.

5. Wigner function

In the above parts of this subsection, we have solved the Wigner function in parallel electromagnetic fields by properly choosing basis functions. Finally we obtained a system of partial differential equations, which is the same as the one in a pure constant electric field. The only difference is, in a electric field, the equations depends on the magnitude of the transverse momentum p_T , while in parallel electromagnetic fields we have to replace p_T by the quantized momentum $p_T \rightarrow \sqrt{2nB_0}$. For convenience of future works, we list all the components of Wigner function in the following. These components are obtained by inserting the solutions (3.258), (3.259), and (3.269) into Eq.

(3.247). The four groups \mathcal{G}_i , $i = 1, 2, 3, 4$, defined in Eq. (2.92), are given by

$$\begin{aligned}
\begin{pmatrix} \mathcal{F} \\ \mathcal{S} \end{pmatrix} &= \frac{m}{\sqrt{2E_0}} \sum_{n=0} d_2 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) C_1^{(n)}(p^z - E_0 t) \begin{pmatrix} \Lambda_+^{(n)}(p_T) \\ 0 \\ 0 \\ \Lambda_-^{(n)}(p_T) \end{pmatrix}, \\
\begin{pmatrix} \mathcal{P} \\ \mathcal{T} \end{pmatrix} &= -\frac{m}{\sqrt{2E_0}} \sum_{n=0} d_3 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) C_1^{(n)}(p^z - E_0 t) \begin{pmatrix} \Lambda_-^{(n)}(p_T) \\ 0 \\ 0 \\ \Lambda_+^{(n)}(p_T) \end{pmatrix}, \\
\begin{pmatrix} \mathcal{V}^0 \\ \mathcal{A} \end{pmatrix} &= \sum_{n=0} C_2^{(n)}(p^z - E_0 t) \begin{pmatrix} \Lambda_+^{(n)}(p_T) \\ 0 \\ 0 \\ \Lambda_-^{(n)}(p_T) \end{pmatrix} \\
&\quad + \frac{1}{\sqrt{2E_0}} \sum_{n>0} d_3 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) C_1^{(n)}(p^z - E_0 t) \frac{2nB_0}{p_T^2} \Lambda_+^{(n)}(p_T) \begin{pmatrix} 0 \\ -p^y \\ p^x \\ 0 \end{pmatrix}, \\
\begin{pmatrix} \mathcal{A}^0 \\ \mathcal{V} \end{pmatrix} &= \sum_{n=0} d_1 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) C_1^{(n)}(p^z - E_0 t) \begin{pmatrix} \Lambda_-^{(n)}(p_T) \\ 0 \\ 0 \\ \Lambda_+^{(n)}(p_T) \end{pmatrix} \\
&\quad + \frac{1}{\sqrt{2E_0}} \sum_{n>0} d_2 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) C_1^{(n)}(p^z - E_0 t) \frac{2nB_0}{p_T^2} \Lambda_+^{(n)}(p_T) \begin{pmatrix} 0 \\ p^x \\ p^y \\ 0 \end{pmatrix},
\end{aligned} \tag{3.270}$$

where $\eta^{(n)} \equiv (m^2 + 2nB_0)/E_0$ is the dimensionless effective mass squared. The functions $C_1^{(n)}$, $C_2^{(n)}$, d_i with $i = 1, 2, 3$, and $\Lambda_{\pm}^{(n)}(p_T)$ are defined in (3.240), (B19), and (B8), respectively. If the magnetic field is sufficiently small, the sum over all Landau levels can be done using Eqs. (B12) and (B13). The discrete Landau levels are then replaced by the continuous transverse momentum squared p_T^2 and (3.270) reproduce the results (3.238) in a constant electric field.

IV. SEMI-CLASSICAL EXPANSION

A. Introduction to the \hbar expansion

In Sec. III we have shown several cases in which the Wigner function has an analytical solution. The semi-classical expansion is a more general approach which can be used for a general space-time dependent field. In the semi-classical expansion approach, we make Taylor expansions for the Wigner function, all the operators, as well as all the equations in terms of the reduced Planck's constant \hbar , and then solve the equations order by order. The Wigner function, taking its scalar component as an example, is expanded as follows,

$$\mathcal{F} = \sum_{n=0}^{\infty} \hbar^n \mathcal{F}^{(n)}. \quad (4.1)$$

Here we use the superscript (n) to label different orders in \hbar . The operators in Eq. (2.37) are expanded as

$$\begin{aligned} \Pi^\mu &= \sum_{n=0}^{\infty} \hbar^{2n} \Pi^{(2n)\mu} = p^\mu - \frac{\hbar}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+3)(2n+1)!} \Delta^{2n+1} F^{\mu\nu}(x) \partial_{p\nu}, \\ \nabla^\mu &= \sum_{n=0}^{\infty} \hbar^{2n} \nabla^{(2n)\mu} = \partial_x^\mu - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \Delta^{2n} F^{\mu\nu}(x) \partial_{p\nu}, \end{aligned} \quad (4.2)$$

where the spatial-derivative $\partial_{x\alpha}$ in the product $\Delta \equiv \frac{\hbar}{2} \partial_p^\alpha \partial_{x\alpha}$ acts only on the electromagnetic field tensor $F^{\mu\nu}(x)$. The other operators, $\Re K^2$, $\Im K^2$, $\Re K_{\mu\nu}$, and $\Im K_{\mu\nu}$ can be written in terms of Π^μ and ∇^μ as shown in (2.44). The leading-order contributions to these operators are

$$\begin{aligned} \Pi^\mu &= p^\mu + \mathcal{O}(\hbar^2), \\ \nabla^\mu &= \partial_x^\mu - F^{\mu\nu} \partial_{p\nu} + \mathcal{O}(\hbar^2), \\ \Re K^2 &= p^2 + \mathcal{O}(\hbar^2), \\ \Im K^2 &= \hbar p_\mu (\partial_x^\mu - F^{\mu\nu} \partial_{p,\nu}) + \mathcal{O}(\hbar^2), \\ \Re K_{\mu\nu} &= -\frac{\hbar^2}{2} (\partial_x^\alpha F_{\mu\nu}) \partial_{p\alpha} + \mathcal{O}(\hbar^3), \\ \Im K_{\mu\nu} &= -\hbar F_{\mu\nu} + \mathcal{O}(\hbar^2), \end{aligned} \quad (4.3)$$

The reduced Planck's constant \hbar labels the strength of the spin-electromagnetic coupling. For example, the quantum of the spin-angular momentum along a given direction is $\pm\hbar/2$ for a spin-1/2 particle. Thus the method of the semi-classical expansion, in some sense, is the Taylor expansion of the spin effect. In the zeroth order of \hbar , particles can be treated as spinless classical ones. The first order in \hbar gives the leading-order correction from spin. In this section we truncate at the order \hbar

because the equations will be more and more complicated and hard to solve at higher order. In this section we will perform a semi-classical expansion and then in Sec. V we will compare the physical quantities calculated using the semi-classical expansion with those from analytical calculations.

The semi-classical expansion works well if and only if high orders in \hbar are much smaller than the leading order one. This requirement is ensured by the following inequality, which is derived from Eq. (4.2)

$$\left(\frac{\hbar}{2}\partial_p^\alpha\partial_{x\alpha}\right)^{2n+2}F^{\mu\nu}(x)W(x,p)\ll\left(\frac{\hbar}{2}\partial_p^\alpha\partial_{x\alpha}\right)^{2n}F^{\mu\nu}(x)W(x,p). \quad (4.4)$$

Assuming that the fluctuations of the electromagnetic field are significant over a typical spatial scale ΔR , while the Wigner function fluctuates over a typical momentum scale ΔP , we demand that

$$\Delta R\Delta P\gg\hbar. \quad (4.5)$$

In the unit of $\text{MeV}\cdot\text{fm}$, the value of reduced Planck's constant is $\hbar = 197 \text{ MeV}\cdot\text{fm}$. If we focus on cosmic systems such as a neutron star, then the typical spatial scale is large enough to make sure that the semi-classical expansion is valid. If we consider heavy-ion collisions such as Au+Au collisions at 200 GeV/A at RHIC, the typical momentum would be several GeV while the typical spatial scale would be several fm, which means Eq. (4.5) can also be satisfied. Thus the method discussed in this section could be useful for both cosmic and microscopic systems. On the other hand, the zeroth-order part of the Dirac-form equation (2.36) for the Wigner function is $(\gamma^\mu p_\mu - m)W$, while the first-order part is $\frac{i\hbar}{2}\gamma^\mu\partial_{x\mu}W$. Thus we demand

$$|\hbar\gamma^\mu\partial_{x\mu}W|\ll m|W|. \quad (4.6)$$

to make sure the \hbar -order correction is much smaller than the zeroth-order contribution. Note that \hbar/m is the Compton wave length, thus the above condition means the wave length of macroscopic fluctuations should be much larger than the Compton wave length.

The semi-classical expansion is widely used in previous papers [66, 68, 69, 74–77, 121]. In this section we will solve the Wigner function up to order \hbar and derive the corresponding kinetic equation at the same order. Higher orders can be solved employing a similar procedure but the results would be too complicated for further analysis and thus are not listed in this thesis.

B. Massless case

As we discussed in Sec. II, in the massless case the system of partial differential equations is much simpler because the vector and axial-vector components \mathcal{V}^μ and \mathcal{A}^μ decouple from the others. Meanwhile, \mathcal{V}^μ is equivalent to \mathcal{A}^μ because of the chiral symmetry of massless fermions. In this section we will solve \mathcal{V}^μ and \mathcal{A}^μ up to order \hbar . First we form a linear combination and define the LH and RH currents \mathcal{J}_\pm^μ as in Eq. (2.71). Inserting the operators in Eq. (4.3) into the on-shell equation (2.72) and the constraint equations (2.79), at zeroth order in \hbar we have

$$\begin{aligned} p_\mu \mathcal{J}_\chi^{(0)\mu} &= 0, \\ p_\mu \mathcal{J}_{\chi,\nu}^{(0)} - p_\nu \mathcal{J}_{\chi,\mu}^{(0)} &= 0, \\ p^2 \mathcal{J}_\chi^{(0)\mu} &= 0. \end{aligned} \quad (4.7)$$

Here $\mathcal{J}_\chi^{(0)\mu}$ represents the zeroth-order part of \mathcal{J}_χ^μ , where $\chi = \pm$ labels the chirality. The last line ensures that $\mathcal{J}_\chi^{(0)\mu}$ should be on the mass-shell $p^2 = 0$ otherwise we will obtain the trivial solution $\mathcal{J}_\chi^{(0)\mu} = 0$. The general nontrivial zeroth-order solution reads

$$\mathcal{J}_\chi^{(0)\mu} = p^\mu f_\chi^{(0)} \delta(p^2). \quad (4.8)$$

Here the distribution $f_\chi^{(0)}$ is still undetermined, but it should not have a singularity on the mass-shell $p^2 = 0$. The zeroth-order currents are parallel to p^μ , which agrees with our expectation: the spins of chiral fermions are always parallel or anti-parallel to their momenta, so the number current and axial-charge current of massless particles are both proportional to p^μ , and is their linear combination.

At first order in \hbar , Eqs. (2.79), (2.72) can be separated into two groups, one of which only depends on the zeroth-order function $\mathcal{J}_\chi^{(0)\mu}$,

$$\nabla_\mu^{(0)} \mathcal{J}_\chi^{(0)\mu} = 0, \quad (4.9)$$

and the other group depends on the first-order function $\mathcal{J}_\chi^{(1)\mu}$,

$$\begin{aligned} p_\mu \mathcal{J}_\chi^{(1)\mu} &= 0, \\ p_\mu \mathcal{J}_{\chi,\nu}^{(1)} - p_\nu \mathcal{J}_{\chi,\mu}^{(1)} + \frac{\chi}{2} \epsilon_{\mu\nu\alpha\beta} \nabla^{(0)\alpha} \mathcal{J}_\chi^{(0)\beta} &= 0, \\ p^2 \mathcal{J}_\chi^{(1)\mu} + \frac{\chi}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \mathcal{J}_{\chi\nu}^{(0)} &= 0. \end{aligned} \quad (4.10)$$

Inserting the zeroth-order solution (4.8) into Eq. (4.9) we obtain the kinetic equations for $f_\chi^{(0)}$,

$$\delta(p^2) p^\mu \nabla_\mu^{(0)} f_\chi^{(0)} = 0, \quad (4.11)$$

which agrees with the collisionless Boltzmann-Vlasov equation [128]. The first-order current $\mathcal{J}_\chi^{(1)\mu}$ can be assumed to have a solution of the form

$$\mathcal{J}_\chi^{(1)\mu} = j_\chi^\mu \delta(p^2) + \chi \tilde{F}^{\mu\nu} p_\nu f_\chi^{(0)} \delta'(p^2), \quad (4.12)$$

where the derivative of the Dirac-delta function is $\delta'(x) = -\delta(x)/x$, which can be proved using the method of integrating by parts. Using $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ we can check that the solution (4.12) automatically satisfies the third line of Eq. (4.10) for an arbitrary j_χ^μ . Inserting the solution (4.12) into the first and second lines of Eq. (4.10) we obtain the following relations

$$\begin{aligned} 0 &= \delta(p^2) p_\mu j_\chi^\mu, \\ 0 &= \delta(p^2) \left[p_\mu j_{\chi,\nu} - p_\nu j_{\chi,\mu} + \frac{\chi}{2} \epsilon_{\mu\nu\alpha\beta} \nabla^{(0)\alpha} \left(p^\beta f_\chi^{(0)} \right) \right] \\ &\quad + \chi \delta'(p^2) \left(p_\mu \tilde{F}_{\nu\alpha} p^\alpha - p_\nu \tilde{F}_{\mu\alpha} p^\alpha - \epsilon_{\mu\nu\alpha\beta} p^\beta F^{\alpha\gamma} p_\gamma \right) f_\chi^{(0)}. \end{aligned} \quad (4.13)$$

Here the last line can be simplified using the Schouten identity ,

$$p_\mu \epsilon_{\nu\alpha\beta\gamma} + p_\nu \epsilon_{\alpha\beta\gamma\mu} + p_\alpha \epsilon_{\beta\gamma\mu\nu} + p_\beta \epsilon_{\gamma\mu\nu\alpha} + p_\gamma \epsilon_{\mu\nu\alpha\beta} = 0. \quad (4.14)$$

This identity holds because in 4-dimensional Minkowski space, the indices can take the values 1–4, thus at least two of the indices $\mu\nu\alpha\beta\gamma$ are identical. Considering without loss of generality the case $\mu = \nu$, the Levi-Civita symbols $\epsilon_{\beta\gamma\mu\nu}$, $\epsilon_{\gamma\mu\nu\alpha}$, and $\epsilon_{\mu\nu\alpha\beta}$ vanish, and the remaining two terms cancel with each other due to the anti-symmetric property of the Levi-Civita symbol. In the case that three or more indices are equal to each other, all Levi-Civita symbols give zero values. With the help of the Schouten identity, the term which multiplies $\delta'(p^2)$ can be simplified as

$$\begin{aligned} & p_\mu \tilde{F}_{\nu\alpha} p^\alpha - p_\nu \tilde{F}_{\mu\alpha} p^\alpha - \epsilon_{\mu\nu\alpha\beta} p^\beta F^{\alpha\gamma} p_\gamma \\ &= \frac{1}{2} (p_\mu \epsilon_{\nu\alpha\beta\gamma} + p_\nu \epsilon_{\alpha\beta\gamma\mu}) p^\alpha F^{\beta\gamma} - \epsilon_{\mu\nu\alpha\beta} p^\beta F^{\alpha\gamma} p_\gamma \\ &= -\frac{1}{2} (p_\alpha \epsilon_{\beta\gamma\mu\nu} + p_\beta \epsilon_{\gamma\mu\nu\alpha} + p_\gamma \epsilon_{\mu\nu\alpha\beta}) p^\alpha F^{\beta\gamma} - \epsilon_{\mu\nu\alpha\beta} p^\beta F^{\alpha\gamma} p_\gamma \\ &= -p^2 \tilde{F}_{\mu\nu}, \end{aligned} \quad (4.15)$$

Inserting back into Eq. (4.13) one obtains two constraints

$$\begin{aligned} 0 &= \delta(p^2) p_\mu j_\chi^\mu, \\ 0 &= \delta(p^2) \left[p_\mu j_{\chi,\nu} - p_\nu j_{\chi,\mu} + \frac{\chi}{2} \epsilon_{\mu\nu\alpha\beta} p^\beta \nabla^{(0)\alpha} f_\chi^{(0)} \right]. \end{aligned} \quad (4.16)$$

On account of Eq. (4.14) and using Eq. (4.11), the general solution reads

$$j_{\chi,\mu} = p_\mu f_\chi^{(1)} + \frac{\chi}{2(p \cdot u)} \epsilon_{\mu\nu\alpha\beta} p^\nu u^\alpha \nabla^{(0)\beta} f_\chi^{(0)}, \quad (4.17)$$

where u^μ is an arbitrary reference vector with $p \cdot u \neq 0$. Substituting j_χ^μ into Eq. (4.12), we obtain

$$\mathcal{J}_\chi^{(1)\mu} = \left[p^\mu f_\chi^{(1)} + \frac{\chi}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_\nu u_\alpha \nabla_\beta^{(0)} f_\chi^{(0)} \right] \delta(p^2) + \chi \tilde{F}^{\mu\nu} p_\nu f_\chi^{(0)} \delta'(p^2). \quad (4.18)$$

Here we again demand that the undetermined distribution $f_\chi^{(1)}$ is non-singular at $p^2 = 0$. The first-order solution derived here agrees with previous results [66, 68, 69].

At order \hbar^2 , Eqs. (2.72) and (2.79) contain the second-order currents $\mathcal{J}_\chi^{(2)\mu}$. Since we only focus on the leading two orders of the solution, the equations for $\mathcal{J}_\chi^{(2)\mu}$ will be neglected. Only one equation is independent of $\mathcal{J}_\chi^{(2)\mu}$ at order \hbar^2 ,

$$\nabla_\mu^{(0)} \mathcal{J}_\chi^{(1)\mu} = 0. \quad (4.19)$$

The kinetic equation for $f_\chi^{(1)}$ is then derived by substituting $\mathcal{J}_\chi^{(1)\mu}$ in Eq. (4.18) into the above equation,

$$\begin{aligned} 0 = \delta(p^2) & \left\{ p^\mu \nabla_\mu^{(0)} f_\chi^{(1)} + \frac{\chi}{2} \left(\nabla_\mu^{(0)} \frac{u_\alpha}{p \cdot u} \right) \epsilon^{\mu\nu\alpha\beta} p_\nu \nabla_\beta^{(0)} f_\chi^{(0)} \right. \\ & \left. - \frac{\chi}{2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} \nabla_\beta^{(0)} f_\chi^{(0)} + \frac{\chi}{4(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_\nu u_\alpha \left[\nabla_\mu^{(0)}, \nabla_\beta^{(0)} \right] f_\chi^{(0)} \right\} \\ & + \chi \delta'(p^2) \left[\nabla_\mu^{(0)} \left(\tilde{F}^{\mu\nu} p_\nu f_\chi^{(0)} \right) - \frac{1}{p \cdot u} \epsilon^{\mu\nu\alpha\beta} F_{\mu\gamma} p^\gamma p_\nu u_\alpha \nabla_\beta^{(0)} f_\chi^{(0)} \right] \\ & - 2\chi \tilde{F}^{\mu\nu} p_\nu F_{\mu\alpha} p^\alpha f_\chi^{(0)} \delta''(p^2). \end{aligned} \quad (4.20)$$

The Schouten identity (4.14) are then used for further simplification. We also use the following relation

$$\tilde{F}^{\mu\nu} p_\nu F_{\mu\alpha} p^\alpha = \frac{1}{4} p^2 F_{\beta\gamma} \tilde{F}^{\beta\gamma}, \quad (4.21)$$

and the properties of the Dirac-delta function

$$\begin{aligned} p^2 \delta'(p^2) &= -\delta(p^2), \\ p^2 \delta''(p^2) &= -2\delta'(p^2). \end{aligned} \quad (4.22)$$

After simplification we finally obtain

$$\begin{aligned} 0 = \delta(p^2) & \left\{ p^\mu \nabla_\mu^{(0)} f_\chi^{(1)} + \frac{\chi}{2} \left(\nabla_\mu^{(0)} \frac{u_\alpha}{p \cdot u} \right) \epsilon^{\mu\nu\alpha\beta} p_\nu \nabla_\beta^{(0)} f_\chi^{(0)} + \frac{\chi}{2(p \cdot u)} p_\nu u_\alpha (\partial_{x^\gamma} \tilde{F}^{\nu\alpha}) \partial_p^\gamma f_\chi^{(0)} \right\} \\ & + \chi \delta'(p^2) \left[(\partial_{x^\mu} \tilde{F}^{\mu\nu}) p_\nu f_\chi^{(0)} - \frac{1}{p \cdot u} \tilde{F}^{\mu\nu} p_\mu u_\nu p^\alpha \nabla_\alpha^{(0)} f_\chi^{(0)} \right], \end{aligned} \quad (4.23)$$

which is the kinetic equation for the first-order distribution function $f_\chi^{(1)}$.

Collecting the zeroth- and first-order solutions, we find that the LH and RH currents in the massless case are given by

$$\mathcal{J}_\chi^\mu = \left[p^\mu f_\chi + \frac{\hbar\chi}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_\nu u_\alpha \nabla_\beta^{(0)} f_\chi \right] \delta(p^2) + \chi \hbar \tilde{F}^{\mu\nu} p_\nu f_\chi \delta'(p^2) + \mathcal{O}(\hbar^2), \quad (4.24)$$

where

$$f_\chi \equiv f_\chi^{(0)} + \hbar f_\chi^{(1)} + \mathcal{O}(\hbar^2), \quad (4.25)$$

is the full distribution function for RH ($\chi = +$) or LH ($\chi = -$) particles, which depends on the phase-space position $\{x^\mu, p^\mu\}$. Note that the full distribution f_χ contains contributions of any order in \hbar , but higher order terms should be much smaller than the leading two orders, which ensures the validity of the semi-classical expansion. The kinetic equation for f_χ can be derived from the zeroth order one in (4.11) and the first order one in (4.23),

$$0 = \delta(p^2) \left\{ p^\mu \nabla_\mu^{(0)} f_\chi + \frac{\chi\hbar}{2} \left(\nabla_\mu^{(0)} \frac{u_\alpha}{p \cdot u} \right) \epsilon^{\mu\nu\alpha\beta} p_\nu \nabla_\beta^{(0)} f_\chi + \frac{\chi\hbar}{2(p \cdot u)} p_\nu u_\alpha (\partial_{x^\gamma} \tilde{F}^{\nu\alpha}) \partial_p^\gamma f_\chi \right\} \\ + \chi \hbar \delta'(p^2) \left[(\partial_{x^\mu} \tilde{F}^{\mu\nu}) p_\nu f_\chi - \frac{1}{p \cdot u} \tilde{F}^{\mu\nu} p_\mu u_\nu p^\alpha \nabla_\alpha^{(0)} f_\chi \right] + \mathcal{O}(\hbar^2). \quad (4.26)$$

The kinetic equation agrees with the result of Refs. [66, 68, 69]. In the classical limit $\hbar \rightarrow 0$, only the first term survives and the equation reduces to the well-known Boltzmann-Vlasov equation [128]. The vector and axial-vector currents can be recovered from the LH and RH currents as

$$\mathcal{V}^\mu = \frac{1}{2} \sum_{\chi=\pm} \mathcal{J}_\chi^\mu, \quad \mathcal{A}^\mu = \frac{1}{2} \sum_{\chi=\pm} \chi \mathcal{J}_\chi^\mu. \quad (4.27)$$

We define the scalar distribution V and the axial distribution A as

$$V \equiv \frac{1}{2} \sum_{\chi=\pm} f_\chi, \quad A \equiv \frac{1}{2} \sum_{\chi=\pm} \chi f_\chi. \quad (4.28)$$

Then we obtain

$$\mathcal{V}^\mu = \left[p^\mu V + \frac{\hbar}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_\nu u_\alpha \nabla_\beta^{(0)} A \right] \delta(p^2) + \hbar \tilde{F}^{\mu\nu} p_\nu A \delta'(p^2) + \mathcal{O}(\hbar^2), \\ \mathcal{A}^\mu = \left[p^\mu A + \frac{\hbar}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_\nu u_\alpha \nabla_\beta^{(0)} V \right] \delta(p^2) + \hbar \tilde{F}^{\mu\nu} p_\nu V \delta'(p^2) + \mathcal{O}(\hbar^2), \quad (4.29)$$

where the distributions V , A in general depend on the coordinates $\{x^\mu, p^\mu\}$ in the phase space. Here the vector u^μ plays the role of the reference frame, and V , A are identified as the net fermion and axial-charge distributions in the frame u^μ , respectively [45, 66, 69]. Note that the functions V and A should depend on the choice of u^μ so that the whole currents \mathcal{V}^μ , \mathcal{A}^μ are independent of u^μ . In the massless case, the dependence on the reference frame is the result of the side-jump effect

[44, 66]. In the next subsections we will prove that in the massive case the reference frame can be chosen as the rest frame of the particle and then the solution will not depend on the auxiliary vector u^μ .

C. Massive case 1: taking vector and axial-vector components as basis

In this subsection we will focus on the massive case $m \neq 0$. As discussed in Sec. II, we can take either the vector and axial-vector components \mathcal{V}^μ and \mathcal{A}^μ or the rest ones \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$ as basis functions. In this subsection, \mathcal{V}^μ and \mathcal{A}^μ are taken as basis and \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$ are derived from \mathcal{V}^μ and \mathcal{A}^μ as shown in Eq. (2.68).

In the massive case, \mathcal{J}_χ^μ defined in (2.71) no longer have definite chirality but we still use \mathcal{J}_χ^μ to represent the linear combination of \mathcal{V}^μ and \mathcal{A}^μ . We first insert the expanded operators in Eq. (4.3) into the on-shell condition (2.72). The zeroth-order part and the first-order part read

$$\begin{aligned} (p^2 - m^2) \mathcal{J}_\chi^{(0)\mu} &= 0, \\ (p^2 - m^2) \mathcal{J}_\chi^{(1)\mu} + \chi \tilde{F}_{\mu\nu} \mathcal{J}_\chi^{(0)\nu} &= 0. \end{aligned} \quad (4.30)$$

The general nontrivial solution of these on-shell conditions is

$$\begin{aligned} \mathcal{J}_\chi^{(0)\mu} &= f_\chi^{(0)\mu} \delta(p^2 - m^2), \\ \mathcal{J}_\chi^{(1)\mu} &= f_\chi^{(1)\mu} \delta(p^2 - m^2) + \chi \tilde{F}^{\mu\nu} f_{\chi\nu}^{(0)} \delta'(p^2 - m^2), \end{aligned} \quad (4.31)$$

where $f_\pm^{(0)\mu}$ and $f_\pm^{(1)\mu}$ are arbitrary functions which are non-singular at $p^2 - m^2 = 0$. We find that the zeroth-order solutions are on the mass-shell $p^2 = m^2$, while the first-order ones have off-shell contributions. From the definition of \mathcal{J}_χ^μ in (2.71) we can recover the vector and axial-vector currents. Up to order \hbar , we obtain

$$\begin{aligned} \mathcal{V}^\mu &= \delta(p^2 - m^2) \sum_{\chi=\pm} \left(f_\chi^{(0)\mu} + \hbar f_\chi^{(1)\mu} \right) + \hbar \tilde{F}^{\mu\nu} \delta'(p^2 - m^2) \sum_{\chi=\pm} \chi f_{\chi\nu}^{(0)} + \mathcal{O}(\hbar^2), \\ \mathcal{A}^\mu &= \delta(p^2 - m^2) \sum_{\chi=\pm} \chi \left(f_\chi^{(0)\mu} + \hbar f_\chi^{(1)\mu} \right) + \hbar \tilde{F}^{\mu\nu} \delta'(p^2 - m^2) \sum_{\chi=\pm} f_{\chi\nu}^{(0)} + \mathcal{O}(\hbar^2). \end{aligned} \quad (4.32)$$

The solutions should satisfy Eq. (2.70), which gives several constraints on the functions $f_\chi^{(0)\mu}$, $f_\chi^{(1)\mu}$. Up to first order in \hbar , the last line of Eq. (2.70) reads

$$p_{[\mu} \left(\mathcal{V}_{\nu]}^{(0)} + \hbar \mathcal{V}_{\nu]}^{(1)} \right) + \frac{\hbar}{2} \epsilon_{\mu\nu\alpha\beta} \nabla^{(0)\alpha} \mathcal{A}^{(0)\beta} = \mathcal{O}(\hbar^2). \quad (4.33)$$

Substituting the vector and axial-vector currents by the ones in (4.32), we obtain

$$\begin{aligned} \delta(p^2 - m^2)p_{[\mu} \sum_{\chi=\pm} \left(f_{\chi\nu}^{(0)} + \hbar f_{\chi\nu}^{(1)} \right) + \hbar p_{[\mu} \tilde{F}_{\nu]\alpha} \delta'(p^2 - m^2) \sum_{\chi=\pm} \chi f_{\chi}^{(0)\alpha} \\ + \frac{\hbar}{2} \epsilon_{\mu\nu\alpha\beta} \nabla^{(0)\alpha} \left[\delta(p^2 - m^2) \sum_{\chi=\pm} \chi f_{\chi}^{(0)\beta} \right] = 0. \end{aligned} \quad (4.34)$$

Contracting this equation with p^μ and taking out different orders in \hbar gives

$$\begin{aligned} 0 &= \delta(p^2 - m^2) \left[m^2 \sum_{\chi=\pm} f_{\chi\nu}^{(0)} - p_\nu p^\mu \sum_{\chi=\pm} f_{\chi\mu}^{(0)} \right], \\ 0 &= \delta(p^2 - m^2) \left[m^2 \sum_{\chi=\pm} f_{\chi\nu}^{(1)} - p_\nu p^\mu \sum_{\chi=\pm} f_{\chi\mu}^{(1)} + \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} p^\mu \nabla^{(0)\alpha} \sum_{\chi=\pm} \chi f_{\chi}^{(0)\beta} \right] \\ &\quad + \delta'(p^2 - m^2) \left[p^2 \tilde{F}_{\nu\alpha} \sum_{\chi=\pm} \chi f_{\chi}^{(0)\alpha} - p_\nu p^\mu \tilde{F}_{\mu\alpha} \sum_{\chi=\pm} \chi f_{\chi}^{(0)\alpha} - p^\mu \epsilon_{\mu\nu\alpha\beta} F^{\alpha\gamma} p_\gamma \sum_{\chi=\pm} \chi f_{\chi}^{(0)\beta} \right]. \end{aligned} \quad (4.35)$$

Now we define the distribution functions and polarization vectors using $f_{\chi}^{(0)\mu}$ and $f_{\chi}^{(1)\mu}$,

$$\begin{aligned} V^{(0)} &\equiv \frac{1}{m^2} p_\mu \sum_{\chi=\pm} f_{\chi}^{(0)\mu}, & V^{(1)} &\equiv \frac{1}{m^2} p_\mu \sum_{\chi=\pm} f_{\chi}^{(1)\mu}, \\ n^{(0)\mu} &\equiv \frac{1}{m} \sum_{\chi=\pm} \chi f_{\chi}^{(0)\mu}, & n^{(1)\mu} &\equiv \frac{1}{m} \sum_{\chi=\pm} \chi f_{\chi}^{(1)\mu}. \end{aligned} \quad (4.36)$$

Using these new functions, the solutions of Eq. (4.35) can be expressed as

$$\begin{aligned} \sum_{\chi=\pm} f_{\chi}^{(0)\mu} &= p^\mu V^{(0)} + (p^2 - m^2) g^{(0)\mu}, \\ \sum_{\chi=\pm} f_{\chi}^{(1)\mu} &= p^\mu V^{(1)} + \frac{1}{2m} \epsilon^{\mu\nu\alpha\beta} p_\nu \nabla_\alpha^{(0)} n_\beta^{(0)} + \frac{p^2}{m(p^2 - m^2)} \tilde{F}^{\mu\nu} n_\nu^{(0)} \\ &\quad - \frac{1}{m(p^2 - m^2)} p^\mu \tilde{F}_{\alpha\beta} p^\alpha n^{(0)\beta} + \frac{1}{m(p^2 - m^2)} \epsilon^{\mu\nu\alpha\beta} p_\nu F_{\alpha\gamma} p^\gamma n_\beta^{(0)} + (p^2 - m^2) g^{(1)\mu}, \end{aligned} \quad (4.37)$$

where we have used $\delta'(x) = -\delta(x)/x$. Here $g_\mu^{(0)}$ and $g_\mu^{(1)}$ are some undetermined functions which are non-singular on the mass-shell $p^2 - m^2$. Since they are multiplied with $p^2 - m^2$, they will not contribute when calculating the vector component. In terms of $V^{(0)}$, $V^{(1)}$, $n^{(0)\mu}$, and $n^{(1)\mu}$, the solutions for \mathcal{V}^μ and \mathcal{A}^μ in Eq. (4.32) are given by

$$\begin{aligned} \mathcal{V}^\mu &= \delta(p^2 - m^2) \left[p^\mu \left(V^{(0)} + \hbar V^{(1)} \right) + \frac{\hbar}{2m} \epsilon^{\mu\nu\alpha\beta} \nabla_\alpha^{(0)} (p_\nu n_\beta^{(0)}) \right] \\ &\quad + \hbar \delta'(p^2 - m^2) \left(\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta^{(0)} F_{\nu\gamma} p^\gamma + \frac{1}{2m} p^\mu \epsilon^{\nu\alpha\beta\gamma} p_\nu n_\alpha^{(0)} F_{\beta\gamma} \right), \\ \mathcal{A}^\mu &= \delta(p^2 - m^2) \left(m n^{(0)\mu} + \hbar m n^{(1)\mu} - \hbar \tilde{F}^{\mu\nu} g_\nu^{(0)} \right) + \hbar \tilde{F}^{\mu\nu} p_\nu V^{(0)} \delta'(p^2 - m^2). \end{aligned} \quad (4.38)$$

Now we define the resummed distribution V and the resummed polarization n^μ ,

$$\begin{aligned} V &\equiv V^{(0)} + \hbar V^{(1)} + \mathcal{O}(\hbar^2), \\ n^\mu &\equiv n^{(0)\mu} + \hbar n^{(1)\mu} - \frac{\hbar}{m} \tilde{F}^{\mu\nu} g_\nu^{(0)} + \mathcal{O}(\hbar^2), \end{aligned} \quad (4.39)$$

and for simplicity we also define the following dipole-moment tensor,

$$\Sigma^{\mu\nu} \equiv -\frac{1}{m} \epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta. \quad (4.40)$$

Then the solutions for \mathcal{V}^μ and \mathcal{A}^μ are given by

$$\begin{aligned} \mathcal{V}^\mu &= \delta(p^2 - m^2) \left[p^\mu V + \frac{\hbar}{2} \nabla_\nu^{(0)} \Sigma^{\mu\nu} \right] - \hbar \delta'(p^2 - m^2) \left(\Sigma^{\mu\alpha} F_{\alpha\beta} p^\beta + \frac{1}{2} p^\mu \Sigma^{\alpha\beta} F_{\alpha\beta} \right) + \mathcal{O}(\hbar^2), \\ \mathcal{A}^\mu &= \delta(p^2 - m^2) m n^\mu + \hbar \tilde{F}^{\mu\nu} p_\nu V \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2). \end{aligned} \quad (4.41)$$

Inserting them into Eq. (2.68) we can obtain the scalar, pseudo-scalar and tensor components \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$

$$\begin{aligned} \mathcal{F} &= m \left[\delta(p^2 - m^2) V - \frac{\hbar}{2} \Sigma^{\mu\nu} F_{\mu\nu} \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\ \mathcal{P} &= -\delta(p^2 - m^2) \frac{\hbar}{2} \nabla^\mu n_\mu + \hbar F^{\mu\nu} p_\nu n_\mu \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2), \\ \mathcal{S}_{\mu\nu} &= \delta(p^2 - m^2) \left(m \Sigma_{\mu\nu} - \frac{\hbar}{2m} p_{[\mu} \nabla_{\nu]} V \right) - m \hbar F_{\mu\nu} V \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2). \end{aligned} \quad (4.42)$$

Here the undetermined functions are V and the vector n^μ . In classical limit $\hbar \rightarrow 0$, we have

$$\begin{aligned} \mathcal{V}^\mu &\rightarrow p^\mu V \delta(p^2 - m^2), \\ \mathcal{A}^\mu &\rightarrow m n^\mu \delta(p^2 - m^2). \end{aligned} \quad (4.43)$$

Thus V can be interpreted as the fermion distribution and n^μ as the polarization density. Substituting the solutions (4.41) into the second line of Eq. (2.70), one obtains a constraint for n^μ ,

$$\delta(p^2 - m^2) p_\mu n^\mu = \mathcal{O}(\hbar^2), \quad (4.44)$$

which can be identified as a requirement for the spin: for massive particles, their spins must be perpendicular to their momenta. On the other hand, the kinetic equation for V can be derived from the first line of Eq. (2.70). Up to order \hbar^2 , we obtain

$$\hbar \nabla^{(0)\mu} \left(\mathcal{V}_\mu^{(0)} + \hbar \mathcal{V}_\mu^{(1)} \right) = \mathcal{O}(\hbar^2). \quad (4.45)$$

Replacing the vector components by the solution (4.41), we obtain the following kinetic equation

$$\delta(p^2 - m^2) \left[p^\mu \nabla_\mu^{(0)} V + \frac{\hbar}{4} (\partial_{x\alpha} F_{\mu\nu}) \partial_p^\alpha \Sigma^{\mu\nu} \right] - \frac{\hbar}{2} \delta'(p^2 - m^2) F_{\alpha\beta} p^\mu \nabla_\mu^{(0)} \Sigma^{\alpha\beta} = 0. \quad (4.46)$$

In the classical limit $\hbar \rightarrow 0$, this kinetic equation reduces to the Boltzmann-Vlasov equation [128]. On the other hand, an addition kinetic equation is necessary to determine n^μ , which can be derived from the last line of Eq. (2.70) by contracting with $\epsilon^{\rho\sigma\mu\nu} p_\sigma$. We also have another approach to derive the kinetic equation for n^μ . In Sec. II we have listed all Vlasov equations in Eq. (2.55), where the one for the axial-vector component reads,

$$p_\nu \nabla^{(0)\nu} \mathcal{A}_\mu - F_{\mu\nu} \mathcal{A}^\nu - \frac{\hbar}{2} (\partial_x^\alpha \tilde{F}_{\mu\nu}) \partial_{p\alpha} \mathcal{V}^\nu = \mathcal{O}(\hbar^2). \quad (4.47)$$

Substituting \mathcal{V}^μ and \mathcal{A}^μ by those in Eq. (4.41), Eq. (4.47) gives the following kinetic equation for n^μ ,

$$\delta(p^2 - m^2) \left[p_\nu \nabla^{(0)\nu} n_\mu - F_{\mu\nu} n^\nu - \frac{\hbar}{2m} (\partial_x^\alpha \tilde{F}_{\mu\nu}) p^\nu \partial_{p\alpha} V \right] + \frac{\hbar}{m} \delta'(p^2 - m^2) \tilde{F}_{\mu\alpha} p^\alpha p_\nu \nabla^{(0)\nu} V = 0. \quad (4.48)$$

In the classical limit $\hbar \rightarrow 0$, it reproduces the Bargmann-Michel-Telegdi equation [130] for the classical spin precession in an electromagnetic field.

D. Massive case 2: taking scalar, pseudo-scalar, and tensor components as basis

In this subsection we will take the scalar, pseudo-scalar, and tensor components \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$ as basis functions and solve the Wigner function in the massive case. The remaining components, the vector and axial-vector ones \mathcal{V}^μ , \mathcal{A}^μ are then given by Eq. (2.73). We start from the zeroth order in \hbar . At this order, the on-shell conditions (2.54) read

$$\begin{aligned} (p^2 - m^2) \mathcal{F}^{(0)} &= 0, \\ (p^2 - m^2) \mathcal{P}^{(0)} &= 0, \\ (p^2 - m^2) \mathcal{S}_{\mu\nu}^{(0)} &= 0, \end{aligned} \quad (4.49)$$

which means that the zeroth-order functions are on the normal mass shell $p^2 - m^2 = 0$. The corresponding constraint conditions (2.74) are

$$\begin{aligned} p^\nu \mathcal{S}_{\nu\mu}^{(0)} &= 0, \\ p_\mu \mathcal{P}^{(0)} &= 0. \end{aligned} \quad (4.50)$$

Thus we obtain the following general solutions

$$\begin{aligned}
\mathcal{F}^{(0)} &= mV^{(0)}\delta(p^2 - m^2), \\
\mathcal{P}^{(0)} &= 0, \\
\mathcal{S}_{\mu\nu}^{(0)} &= m\Sigma_{\mu\nu}^{(0)}\delta(p^2 - m^2),
\end{aligned} \tag{4.51}$$

where $V^{(0)}$ and $\Sigma_{\mu\nu}^{(0)}$ are now arbitrary functions of the phase-space position $\{x^\mu, p^\mu\}$. In order to satisfy the on-shell condition, $V^{(0)}$ and $\Sigma_{\mu\nu}^{(0)}$ should not have any singularities for an on-shell momentum $p^2 = m^2$. We also demand that

$$\delta(p^2 - m^2)\Sigma^{(0)\mu\nu}p_\nu = 0, \tag{4.52}$$

in order to satisfy the constraint condition for $\mathcal{S}_{\mu\nu}^{(0)}$ in Eq. (4.50). Since the Wigner function has the dimension of the energy, we find that both $V^{(0)}$ and $\Sigma_{\mu\nu}^{(0)}$ are dimensionless. Recalling that \mathcal{F} is interpreted as the mass density, $V^{(0)}$ is then identified as the zeroth-order fermion distribution. And $\Sigma_{\mu\nu}^{(0)}$ is the dimensionless zeroth-order dipole-moment tensor. Due to the second line of the constraint equation (4.50), the pseudoscalar component vanishes at zeroth order in \hbar .

The first-order on-shell conditions (2.54) read

$$\begin{aligned}
(p^2 - m^2)\mathcal{F}^{(1)} - \frac{1}{2}F_{\mu\nu}\mathcal{S}^{(0)\mu\nu} &= 0, \\
(p^2 - m^2)\mathcal{P}^{(1)} - \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}\mathcal{S}^{(0)\alpha\beta} &= 0, \\
(p^2 - m^2)\mathcal{S}_{\mu\nu}^{(1)} - F_{\mu\nu}\mathcal{F}^{(0)} + \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\mathcal{P}^{(0)} &= 0.
\end{aligned} \tag{4.53}$$

Inserting the zeroth-order solutions (4.51), we obtain the following general solutions at order \hbar ,

$$\begin{aligned}
\mathcal{F}^{(1)} &= m \left[V^{(1)}\delta(p^2 - m^2) - \frac{1}{2}F_{\mu\nu}\Sigma^{(0)\mu\nu}\delta'(p^2 - m^2) \right], \\
\mathcal{P}^{(1)} &= m \left[G^{(1)}\delta(p^2 - m^2) - \frac{1}{2}\tilde{F}_{\mu\nu}\Sigma^{(0)\mu\nu}\delta'(p^2 - m^2) \right], \\
\mathcal{S}_{\mu\nu}^{(1)} &= m \left[\Sigma_{\mu\nu}^{(1)}\delta(p^2 - m^2) - F_{\mu\nu}V^{(0)}\delta'(p^2 - m^2) \right],
\end{aligned} \tag{4.54}$$

where $\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is the dual field tensor. Here $V^{(1)}$, $G^{(1)}$, and $\Sigma_{\mu\nu}^{(1)}$ are still undetermined, which are dimensionless and non-singular on the mass-shell $p^2 = m^2$. The first-order part of constraint equations (2.74) read

$$\begin{aligned}
\frac{1}{2}\nabla_\mu^{(0)}\mathcal{F}^{(0)} + p^\nu\mathcal{S}_{\nu\mu}^{(1)} &= 0, \\
p_\mu\mathcal{P}^{(1)} + \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\nabla^{(0)\nu}\mathcal{S}^{(0)\alpha\beta} &= 0.
\end{aligned} \tag{4.55}$$

Substituting the zeroth-order and first-order functions by the general solutions in (4.51) and (4.54), we obtain

$$\delta(p^2 - m^2) \left[p^\nu \Sigma_{\mu\nu}^{(1)} - \frac{1}{2} \nabla_\mu^{(0)} V^{(0)} \right] = 0, \quad (4.56)$$

and

$$\begin{aligned} 0 = & \delta(p^2 - m^2) \left[p_\mu G^{(1)} + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} \nabla^{(0)\nu} \Sigma^{(0)\alpha\beta} \right] \\ & - \frac{1}{4} [p_\mu \epsilon_{\alpha\beta\nu\gamma} + p_\gamma \epsilon_{\mu\nu\alpha\beta} - p_\nu \epsilon_{\mu\gamma\alpha\beta}] F^{\nu\gamma} \Sigma^{(0)\alpha\beta} \delta'(p^2 - m^2). \end{aligned} \quad (4.57)$$

Then using the Schouten identity in (4.14) to simplify the second line of Eq. (4.57) and we obtain a constraint equation for $G^{(1)}$

$$\delta(p^2 - m^2) \left[p_\mu G^{(1)} + \frac{1}{4} \epsilon_{\mu\nu\alpha\beta} \nabla^{(0)\nu} \Sigma^{(0)\alpha\beta} \right] + \tilde{F}_{\mu\nu} \Sigma^{(0)\nu\alpha} p_\alpha \delta'(p^2 - m^2) = 0, \quad (4.58)$$

which leads to a solution

$$G^{(1)} = -\frac{1}{4m^2} \epsilon_{\mu\nu\alpha\beta} p^\mu \nabla^{(0)\nu} \Sigma^{(0)\alpha\beta} + \frac{1}{m^2(p^2 - m^2)} p^\mu \tilde{F}_{\mu\nu} \Sigma^{(0)\nu\alpha} p_\alpha. \quad (4.59)$$

Since the zeroth-order dipole-moment tensor $\Sigma^{(0)\nu\alpha}$ satisfies Eq. (4.52), we find that $G^{(1)}$ is not singular on the mass-shell $p^2 - m^2 = 0$, which agrees with our requirement. Then we can prove that the pseudo-scalar component $\mathcal{P}^{(1)}$ can be written as

$$\mathcal{P}^{(1)} = \frac{1}{4m} \nabla^{(0)\mu} \left[\epsilon_{\mu\nu\alpha\beta} p^\nu \Sigma^{(0)\alpha\beta} \delta(p^2 - m^2) \right]. \quad (4.60)$$

Thus up to order \hbar , the undetermined functions are $V^{(0)}$, $V^{(1)}$, $\Sigma_{\mu\nu}^{(0)}$, and $\Sigma_{\mu\nu}^{(1)}$.

Now we define the resummed functions

$$\begin{aligned} V & \equiv V^{(0)} + \hbar V^{(1)} + \mathcal{O}(\hbar^2), \\ \Sigma_{\mu\nu} & \equiv \Sigma_{\mu\nu}^{(0)} + \hbar \Sigma_{\mu\nu}^{(1)} + \frac{\hbar}{2m^2} p_{[\mu} \nabla_{\nu]}^{(0)} V + \mathcal{O}(\hbar^2). \end{aligned} \quad (4.61)$$

Here in the definition of the resummed dipole-moment tensor we add an additional term $\frac{\hbar}{2m^2} p_{[\mu} \nabla_{\nu]}^{(0)} V$ so that the final results are comparable with the ones in the previous subsection. In terms of these resummed functions, the up to \hbar order solutions for the scalar, pseudo-scalar, and tensor components are written as

$$\begin{aligned} \mathcal{F} & = m \left[V \delta(p^2 - m^2) - \frac{\hbar}{2} F_{\mu\nu} \Sigma^{\mu\nu} \delta'(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\ \mathcal{P} & = \frac{\hbar}{4m} \nabla^{(0)\mu} \left[\epsilon_{\mu\nu\alpha\beta} p^\nu \Sigma^{\alpha\beta} \delta(p^2 - m^2) \right] + \mathcal{O}(\hbar^2), \\ \mathcal{S}_{\mu\nu} & = \delta(p^2 - m^2) \left(m \Sigma_{\mu\nu} - \frac{\hbar}{2m} p_{[\mu} \nabla_{\nu]}^{(0)} V \right) - m \hbar F_{\mu\nu} V \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2). \end{aligned} \quad (4.62)$$

The resummed dipole-moment tensor $\Sigma_{\mu\nu}$ satisfies the following constraint equation, which is derived from Eqs. (4.52) and (4.56)

$$\delta(p^2 - m^2) \left(\Sigma_{\mu\nu} p^\nu - \frac{\hbar}{2m^2} p_\mu p^\nu \nabla_\nu^{(0)} V \right) = \mathcal{O}(\hbar^2). \quad (4.63)$$

On the other hand, from Eq. (2.55) we obtain the Vlasov equations for \mathcal{F} and $\mathcal{S}_{\mu\nu}$,

$$\begin{aligned} \hbar p_\mu \nabla^{(0)\mu} \mathcal{F} + \frac{\hbar^2}{4} (\partial_x^\alpha F_{\mu\nu}) \partial_{p\alpha} \mathcal{S}^{\mu\nu} &= \mathcal{O}(\hbar^3), \\ \hbar p_\alpha \nabla^{(0)\alpha} \mathcal{S}_{\mu\nu} - \hbar F_{[\mu}^\alpha \mathcal{S}_{\nu]\alpha} + \frac{\hbar^2}{2} (\partial_x^\alpha F_{\mu\nu}) \partial_{p\alpha} \mathcal{F} - \frac{\hbar^2}{4} \epsilon_{\mu\nu\alpha\beta} (\partial_x^\gamma F^{\alpha\beta}) \partial_{p\gamma} \mathcal{P} &= \mathcal{O}(\hbar^3). \end{aligned} \quad (4.64)$$

Here the Vlasov equation for the pseudo-scalar component \mathcal{P} is not listed because it does not contain any new undetermined function. Inserting the solutions (4.62) into the above Vlasov equations, we obtain

$$\begin{aligned} \delta(p^2 - m^2) \left[p_\mu \nabla^{(0)\mu} V + \frac{\hbar}{4} (\partial_x^\alpha F_{\mu\nu}) \partial_{p\alpha} \Sigma^{\mu\nu} \right] - \frac{\hbar}{2} \delta'(p^2 - m^2) F_{\mu\nu} p_\alpha \nabla^{(0)\alpha} \Sigma^{\mu\nu} &= \mathcal{O}(\hbar^2), \\ \delta(p^2 - m^2) \left[p_\alpha \nabla^{(0)\alpha} \bar{\Sigma}_{\mu\nu} - F_{[\mu}^\alpha \bar{\Sigma}_{\nu]\alpha} + \frac{\hbar}{2} (\partial_x^\alpha F_{\mu\nu}) \partial_{p\alpha} V \right] - \delta'(p^2 - m^2) \hbar F_{\mu\nu} p_\alpha \nabla^{(0)\alpha} V &= \mathcal{O}(\hbar^2). \end{aligned} \quad (4.65)$$

where in the second line $\bar{\Sigma}_{\mu\nu} \equiv \Sigma_{\mu\nu} - \frac{\hbar}{2m^2} p_{[\mu} \nabla_{\nu]}^{(0)} V$. This redefinition does not introduce new functions but it makes the Vlasov equations more concise. With the help of the Vlasov equation for V , we find that the constraint equation (4.63) for $\Sigma_{\mu\nu}$ can be further simplified,

$$\delta(p^2 - m^2) \Sigma_{\mu\nu} p^\nu = \mathcal{O}(\hbar^2). \quad (4.66)$$

This means that the dipole-moment tensor is perpendicular to the momentum. The vector and axial-vector components are calculated using Eq. (2.73). Up to order \hbar , we obtain

$$\begin{aligned} \mathcal{V}_\mu &= \delta(p^2 - m^2) \left[p_\mu V + \frac{\hbar}{2} \nabla^{(0)\nu} \Sigma_{\mu\nu} \right] - \hbar \delta'(p^2 - m^2) \left[\frac{1}{2} F_{\alpha\beta} \Sigma^{\alpha\beta} p_\mu + \Sigma_{\mu\nu} F^{\nu\alpha} p_\alpha \right] + \mathcal{O}(\hbar^2), \\ \mathcal{A}_\mu &= -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} p^\nu \Sigma^{\alpha\beta} \delta(p^2 - m^2) + \hbar \tilde{F}_{\mu\nu} p^\nu V \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2). \end{aligned} \quad (4.67)$$

The solutions in (4.62) and (4.67) provide all components of the Wigner function. Defining

$$n_\mu \equiv -\frac{1}{2m} \epsilon_{\mu\nu\alpha\beta} p^\nu \Sigma^{\alpha\beta}, \quad (4.68)$$

the axial-vector component can be written as

$$\mathcal{A}_\mu = m n_\mu \delta(p^2 - m^2) + \hbar \tilde{F}_{\mu\nu} p^\nu V \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2). \quad (4.69)$$

The kinetic equation for n_μ can be derived by acting the operator $p_\alpha \nabla^{(0)\alpha}$ onto the definition of n_μ , and then using the second line of Eq. (4.65). A careful calculation reproduces Eq. (4.48). Hence the results in this subsection, i.e., Eqs. (4.62), (4.67), and (4.65), coincide with the results in the previous subsection IV C, i.e., Eqs. (4.41), (4.42), (4.46), and (4.48).

In the classical limit $\hbar \rightarrow 0$, the solutions (4.62), (4.67) coincide with the results from the first-principle calculations in Sec. III. So the analytical solutions give a constructive suggestion for the undetermined functions. In practice, one assumes that the undetermined functions takes their equilibrium form, i.e., they are solutions of the collisionless Boltzmann-Vlasov equation. However, the equilibrium form of V at order \hbar is still under discussion. In Ref. [75] we proposed a possible equilibrium distribution but it can only be used in very limited cases. This is because we have neglected the momentum dependence of the dipole-moment tensor. But in realistic cases, the dipole-moment tensor should be computed from its kinetic equation and thus in general depends on x^μ and p^μ . A self-consistent treatment for the kinetic equations in Eq. (4.65) has not been done yet.

E. Ambiguity of functions

Comparing the results in subsection IV C with that in subsection IV D, we find that even though in these subsections we start from different points, the final solutions as well as the corresponding kinetic equations and constraint equations are exactly the same. This agrees with our expectation that the Wigner function should have only one solution. Although the Wigner function has 16 components, the solutions up to order \hbar only depends on four functions V and n_μ (note here that n_μ is a 4-vector which is perpendicular to p_μ , so it only has three components). In this section we will first analyze the Wigner function as an eigenvalue problem, which will clearly show why there are only four independent degrees of freedom. Then we will discuss the ambiguity of the undetermined functions, where we find some transformations which change the basis functions without changing the Wigner function. And finally, we will show in this subsection how to smoothly reproduce the massless results from the massive ones.

1. *eigenvalue problem*

We first focus on the Dirac-form equation for the Wigner function in Eq. (2.36). The leading two orders in \hbar read

$$(\gamma^\mu p_\mu - m) \left(W^{(0)} + \hbar W^{(1)} \right) = -\frac{i\hbar}{2} \gamma^\mu \nabla_\mu^{(0)} W^{(0)} + \mathcal{O}(\hbar^2). \quad (4.70)$$

Here we have moved the spatial gradient term to the right-hand-side of the equation. Since the Wigner function has 16 components as shown in Eq. (2.20), we first put these components in a column vector as follows

$$w(x, p) \equiv (\mathcal{F}, \mathcal{P}, \mathcal{V}^0, \mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^3, \mathcal{A}^0, \mathcal{A}^1, \mathcal{A}^2, \mathcal{A}^3, \mathcal{S}^{01}, \mathcal{S}^{02}, \mathcal{S}^{03}, \mathcal{S}^{23}, \mathcal{S}^{31}, \mathcal{S}^{12})^T. \quad (4.71)$$

These component can be derive from the Wigner function by multiplying with $\Gamma_i = \{\mathbb{I}_4, i\gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \frac{1}{2}\sigma^{\mu\nu}\}$ and then taking the trace. Then Eq. (4.70) can be written as

$$(M - m\mathbb{I}_{16}) w(x, p) = \hbar\delta w, \quad (4.72)$$

where \mathbb{I}_{16} is a 16-dimensional unit matrix, δw represents the order- \hbar correction, which can be calculated from the left-hand-side of Eq. (4.70). In this section we only focus on the properties of the solution, thus we will not list the exact formula for δw . The vector $w(x, p)$ contains both the zeroth order and first order in \hbar . The coefficient matrix M is a 16×16 complex matrix given by

$$M = \begin{pmatrix} 0 & 0 & p^0 & -p_x & -p_y & -p_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ip^0 & -ip_x & -ip_y & -ip_z & 0 & 0 & 0 & 0 & 0 & 0 \\ p^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -ip_x & -ip_y & -ip_z & 0 & 0 & 0 \\ p_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -ip^0 & 0 & 0 & 0 & -ip_z & ip_y \\ p_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -ip^0 & 0 & ip_z & 0 & -ip_x \\ p_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -ip^0 & -ip_y & ip_x & 0 \\ 0 & -ip^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_x & p_y & p_z \\ 0 & -ip_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_z & p_y & p^0 & 0 & 0 \\ 0 & -ip_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_x & 0 & -p_x & 0 & p^0 & 0 \\ 0 & -ip_z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -p_y & p_x & 0 & 0 & 0 & p^0 \\ 0 & 0 & -ip_x & ip^0 & 0 & 0 & 0 & 0 & -p_z & p_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ip_y & 0 & ip^0 & 0 & 0 & p_z & 0 & -p_x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -ip_z & 0 & 0 & ip^0 & 0 & -p_y & p_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ip_z & -ip_y & -p_x & p^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -ip_z & 0 & ip_x & -p_y & 0 & p^0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ip_y & -ip_x & 0 & -p_z & 0 & 0 & p^0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.73)$$

The solution of Eq. (4.72) can be decomposed into one specific solution and several general solutions, where the general solutions are solved by taking $\delta w \rightarrow 0$. In the limit $\delta w \rightarrow 0$, Eq. (4.72) is

the characteristic equation for the matrix M , with m the eigenvalue and w the corresponding eigenvector. This characteristic equation has a nontrivial solution if and only if the determinant of its coefficient matrix vanishes

$$\det(M - m\mathbb{I}_{16}) = 0, \quad (4.74)$$

which gives

$$(p^2 - m^2)^8 = 0. \quad (4.75)$$

So the matrix M has eight positive eigenvalues $m = \sqrt{p^2}$ and eight negative ones $m = -\sqrt{p^2}$. In real cases the mass of the particle is positive, thus the negative eigenvalues are non-physical. For the positive eigenvalues, we can solve the following eigenvectors

$$\begin{aligned} v_1 &= (m, 0, p^0, p_x, p_y, p_z, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ v_2 &= (0, im, 0, 0, 0, 0, p^0, p_x, p_y, p_z, 0, 0, 0, 0, 0, 0), \\ v_3 &= (0, -imp_x, -ip_y p_z, 0, -ip^0 p_z, 0, -p^0 p_x, -(p_x^2 + p_z^2), -p_x p_y, 0, 0, mp_z, 0, 0, 0, 0), \\ v_4 &= (0, imp_y, -ip_x p_z, -ip^0 p_z, 0, 0, p^0 p_y, p_x p_y, p_y^2 + p_z^2, 0, mp_z, 0, 0, 0, 0, 0), \\ v_5 &= (0, -imp^0, 0, ip_y p_z, -ip_x p_z, 0, -(p^0)^2 + p_z^2, -p^0 p_x, -p^0 p_y, 0, 0, 0, 0, 0, 0, mp_z), \\ v_6 &= (imp^0, 0, i(p^0)^2 - ip_z^2, ip^0 p_x, ip^0 p_y, 0, 0, p_y p_z, -p_x p_z, 0, 0, 0, mp_z, 0, 0, 0), \\ v_7 &= (-imp_x, 0, -ip^0 p_x, -i(p_x^2 + p_z^2), -ip_x p_y, 0, p_y p_z, 0, p^0 p_z, 0, 0, 0, 0, 0, 0, mp_z), \\ v_8 &= (imp_y, 0, ip^0 p_y, ip_x p_y, i(p_y^2 + p_z^2), 0, p_x p_z, p^0 p_z, 0, 0, 0, 0, 0, 0, 0, mp_z). \end{aligned} \quad (4.76)$$

Note that these vectors are neither properly normalized nor orthogonal to each other. The general solution for w can be written in terms of the above eigenvectors,

$$w = \sum_{i=1}^8 c_i v_i. \quad (4.77)$$

The property of the Wigner function tell us that all components \mathcal{F} , \mathcal{P} , \mathcal{V}^μ , \mathcal{A}^μ , and $\mathcal{S}^{\mu\nu}$ are real functions, but the eigenvectors v_2, \dots, v_8 are complex vectors. In order to make sure all the components of w are real, we can obtain the following constraints for coefficients c_i ,

$$\begin{aligned} p^0 c_6 - p_x c_7 - p_y c_8 &= 0, \\ c_2 - p_x c_3 + p_y c_4 - p^0 c_5 &= 0, \\ -p_y p_z c_3 - p_x p_z c_4 + [(p^0)^2 - p_z^2] c_6 - p^0 p_x c_7 + p^0 p_y c_8 &= 0, \\ -p^0 p_z c_4 + p_y p_z c_5 + p^0 p_x c_6 - (p_x^2 + p_z^2) c_7 + p_x p_y c_8 &= 0, \\ -p^0 p_z c_3 - p_x p_z c_5 + p^0 p_y c_6 - p_x p_y c_7 + (p_y^2 + p_z^2) c_8 &= 0. \end{aligned} \quad (4.78)$$

Using these constraints, the coefficients c_5 , c_6 , c_7 , and c_8 can be expressed in terms of c_2 , c_3 , and c_4 ,

$$\begin{aligned} c_5 &= \frac{c_2 - p_x c_3 + p_y c_4}{p^0}, & c_6 &= -\frac{p_y c_3 + p_z c_4}{p_z}, \\ c_7 &= \frac{p_y c_2 - p_x p_y c_3 - (m^2 + p_x^2 + p_z^2) c_4}{p^0 p_z}, & c_8 &= \frac{p_x c_2 + (m^2 + p_y^2 + p_z^2) c_3 + p_x p_y c_4}{p^0 p_z}. \end{aligned} \quad (4.79)$$

If the coefficients do not satisfy these relations, then the vector $\sum_{i=1}^8 c_i v_i$ may have an imaginary part, which cannot be a correct solution for the Wigner function. So in order to construct the general solution of $w(x, p)$, we only need four parameters c_i with $i = 1, 2, 3, 4$. This means the general order- \hbar solution has only four independent degrees of freedom, which agrees with the conclusion of the previous subsections IV C and IV D.

2. Shift of mass-shell

In this section we will show how the energies of the particles are shifted by the coupling between the electromagnetic field and the dipole-moment. First we find that the solutions (4.62) and (4.67) are invariant under transformations

$$\begin{aligned} \widehat{\Sigma}_{\mu\nu} &= \Sigma_{\mu\nu} + (p^2 - m^2) \delta \Sigma_{\mu\nu}, \\ \widehat{V} &= V - \frac{\hbar}{2} F^{\mu\nu} \delta \Sigma_{\mu\nu}, \end{aligned} \quad (4.80)$$

and

$$\begin{aligned} \widehat{V} &= V + (p^2 - m^2) \delta V, \\ \widehat{\Sigma}_{\mu\nu} &= \Sigma_{\mu\nu} - \hbar F_{\mu\nu} \delta V - \frac{\hbar}{m^2} p_{[\mu} F_{\nu]\alpha} p^\alpha \delta V. \end{aligned} \quad (4.81)$$

Here $\delta \Sigma_{\mu\nu}$ and δV are arbitrary functions which should be non-singular on the mass-shell $p^2 = m^2$. The invariance can be easily proven by inserting the transformations into Eqs. (4.62) and (4.67), and using the property of the Dirac delta-function $-x \delta'(x) = \delta(x)$. Note that the transformation (4.80) does not affect the on-shell value of $\Sigma_{\mu\nu}$ because the factor $p^2 - m^2$ in front of the additional term vanishes on the mass-shell. But the transformation (4.80) changes the on-shell value of V by a term $-\frac{\hbar}{2} F^{\mu\nu} \delta \Sigma_{\mu\nu}$. Similarly, the transformation (4.81) does not change the on-shell value of V but changes the on-shell value of $\Sigma_{\mu\nu}$.

Since $p^2 \equiv (p^0)^2 - \mathbf{p}^2$, we have the following relation

$$p^0 = \pm \sqrt{(p^2 - m^2) + E_{\mathbf{p}}^2}, \quad (4.82)$$

where $E_{\mathbf{p}} \equiv \sqrt{m^2 + \mathbf{p}^2}$ is the on-shell energy. The sign of p^0 labels fermions ($p^0 > 0$) or anti-fermions ($p^0 < 0$). Now we define $\delta m^2 \equiv p^2 - m^2$ as a new parameter, which describes the distance between the mass-shell and a given p^μ . Using the chain rule for computing the derivative we obtain, for an arbitrary function $f(p^0, \mathbf{p})$, the Taylor expansion in δm^2

$$f(p^0, \mathbf{p}) = f(p^0, \mathbf{p})|_{p^0 \rightarrow \pm E_{\mathbf{p}}} + \frac{1}{2}(p^2 - m^2) \frac{\partial}{\partial p^0} f(p^0, \mathbf{p}) \Big|_{p^0 \rightarrow \pm E_{\mathbf{p}}} + \mathcal{O}[(\delta m^2)^2], \quad (4.83)$$

where the first term is the on-shell value of $f(p^0, \mathbf{p})$ while the second term is related to the on-shell value of $\frac{\partial}{\partial p^0} f(p^0, \mathbf{p})$. Here we make the replacement $p^0 \rightarrow E_{\mathbf{p}}$ if we focus on fermions and $p^0 \rightarrow -E_{\mathbf{p}}$ if we focus on anti-fermions. Comparing the expansion (4.83) with the transformations (4.80) and (4.81) we immediately find that the transformations δV , $\delta \Sigma_{\mu\nu}$ change the on-shell values of $\frac{\partial}{\partial p^0} f(p^0, \mathbf{p})$. If we take a specific choice as

$$\delta \Sigma_{\mu\nu} = - \frac{\partial}{\partial p^0} \Sigma_{\mu\nu} \Big|_{p^0 \rightarrow \pm E_{\mathbf{p}}}, \quad \delta V = - \frac{\partial}{\partial p^0} V \Big|_{p^0 \rightarrow \pm E_{\mathbf{p}}}, \quad (4.84)$$

then after the transformations (4.80) and (4.81) we have

$$\begin{aligned} \widehat{\Sigma}_{\mu\nu} &= \Sigma_{\mu\nu}|_{p^0 \rightarrow \pm E_{\mathbf{p}}} + \mathcal{O}[(p^2 - m^2)^2], \\ \widehat{V} &= V|_{p^0 \rightarrow \pm E_{\mathbf{p}}} + \mathcal{O}[(p^2 - m^2)^2]. \end{aligned} \quad (4.85)$$

All these undetermined functions take their on-shell values plus high order corrections in $p^2 - m^2$.

If we take the energy integration for the covariant Wigner function (this is how the equal-time formula is obtained), we find that the equal-time formula will depend on the following terms,

$$V|_{p^0 \rightarrow \pm E_{\mathbf{p}}}, \quad \bar{\Sigma}^{\mu\nu}|_{p^0 \rightarrow \pm E_{\mathbf{p}}}, \quad \frac{\partial}{\partial p^0} V \Big|_{p^0 \rightarrow \pm E_{\mathbf{p}}}, \quad \frac{\partial}{\partial p^0} \bar{\Sigma}^{\mu\nu} \Big|_{p^0 \rightarrow \pm E_{\mathbf{p}}}. \quad (4.86)$$

The transformations (4.80) and (4.81) indicate that the above four terms are not independent from each other. For example, the transformation (4.80) changes $V|_{p^0 \rightarrow \pm E_{\mathbf{p}}}$ and $\frac{\partial}{\partial p^0} \bar{\Sigma}^{\mu\nu} \Big|_{p^0 \rightarrow \pm E_{\mathbf{p}}}$ at the same time. Since the covariant Wigner function is invariant under transformations (4.80) and (4.81), the equal-time Wigner function should only depend on the following invariant combinations,

$$\begin{aligned} V - \frac{\hbar}{2} F^{\mu\nu} \frac{\partial}{\partial p^0} \bar{\Sigma}^{\mu\nu} \Big|_{p^0 \rightarrow \pm E_{\mathbf{p}}}, \\ \bar{\Sigma}^{\mu\nu} - \hbar \left(F_{\mu\nu} + \frac{1}{m^2} p_{[\mu} F_{\nu]\alpha} p^\alpha \right) \frac{\partial}{\partial p^0} V \Big|_{p^0 \rightarrow \pm E_{\mathbf{p}}}. \end{aligned} \quad (4.87)$$

They are identified respectively as the net fermion distribution and dipole-moment tensor, which appear in the semi-classical solution of the equal-time Wigner function in Ref. [77].

The δ' terms in the kinetic equations (4.65) can be dropped if we properly choice a transformation. We take the first equation in (4.65) as an example. For one on-shell p^μ , we obtain $p \cdot \nabla^{(0)} V = \mathcal{O}(\hbar)$. However, analogous to Eq. (??), if there exists a transformation which satisfies

$$p \cdot \nabla^{(0)} \delta V = - \frac{p \cdot \nabla^{(0)} V - p \cdot \nabla^{(0)} V|_{p^0 \rightarrow \pm E_{\mathbf{p}}}}{p^2 - m^2}. \quad (4.88)$$

Then we obtain that $p \cdot \nabla^{(0)} \widehat{V} = \mathcal{O}(\hbar)$ holds for any p^μ , either on-shell or off-shell. Similarly, due to the ambiguity of $\delta \Sigma_{\alpha\beta}$ we can also find a transformation which ensures $p \cdot \nabla^{(0)} \widehat{\Sigma}_{\mu\nu} - F_{[\mu}^\alpha \widehat{\Sigma}_{\nu]\alpha} = \mathcal{O}(\hbar)$ hold for any p^μ . Note that after the transformations, the terms which are proportional to $\delta'(p^2 - m^2)$ in Eq. (4.65) are $\mathcal{O}(\hbar^2)$ and we obtain

$$\begin{aligned} 0 &= p \cdot \nabla^{(0)} \widehat{V} + \frac{\hbar}{4} (\partial_x^\alpha F^{\mu\nu}) \partial_{p\alpha} \widehat{\Sigma}_{\mu\nu} + \mathcal{O}(\hbar^2), \\ 0 &= p \cdot \nabla^{(0)} \widehat{\Sigma}_{\mu\nu} - F_{[\mu}^\alpha \widehat{\Sigma}_{\nu]\alpha} + \frac{\hbar}{2} (\partial_{x\alpha} F_{\mu\nu}) \partial_p^\alpha \widehat{V} + \mathcal{O}(\hbar^2). \end{aligned} \quad (4.89)$$

These kinetic equations were used for deriving an thermal equilibrium distribution in the presence of vorticity in Ref. [75].

In practice, the transformations (4.80) and (4.81) can be interpreted as a shift of mass-shell. We take the scalar component in Eq. (4.62) as an example. In the solution of the scalar component \mathcal{F} , there is one term which is proportional to $\delta'(p^2 - m^2)$ which contribute the off-shell effect [69, 74, 75, 125]. We now focus on the fermions and neglect the anti-fermions. Assuming that the average dipole-moment per particle is $\widetilde{\Sigma}^{\mu\nu}$, then we have the following relation

$$\Sigma^{\mu\nu} = \widetilde{\Sigma}^{\mu\nu} V, \quad (4.90)$$

because V is the fermion distribution. Then the scalar component can be written in terms of a modified on-shell condition

$$\mathcal{F} = m\theta(p^0) \delta \left(p^2 - m^2 - \frac{\hbar}{2} F_{\mu\nu} \widetilde{\Sigma}^{\mu\nu} \right) V(x, p) + \mathcal{O}(\hbar^2). \quad (4.91)$$

The modified on-shell delta-function is defined via a Taylor expansion,

$$\delta \left(p^2 - m^2 - \frac{\hbar}{2} F_{\mu\nu} \widetilde{\Sigma}^{\mu\nu} \right) = \delta(p^2 - m^2) - \frac{\hbar}{2} F_{\mu\nu} \widetilde{\Sigma}^{\mu\nu} \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2). \quad (4.92)$$

We find that the normal mass-shell is changed by a spin-magnetic coupling term. This term can be expanded near $p^0 = E_{\mathbf{p}}$, and the term in the delta function can be written as

$$\begin{aligned} p^2 - m^2 - \frac{\hbar}{2} F_{\mu\nu} \widetilde{\Sigma}^{\mu\nu} &= (p^0)^2 - E_{\mathbf{p}}^2 - \frac{\hbar}{2} F_{\mu\nu} \widetilde{\Sigma}^{\mu\nu} \Big|_{p^0 \rightarrow E_{\mathbf{p}}} - (p^0 - E_{\mathbf{p}}) \frac{\hbar}{2} F_{\mu\nu} \frac{\partial}{\partial p^0} \widetilde{\Sigma}^{\mu\nu} \Big|_{p^0 \rightarrow E_{\mathbf{p}}} \\ &= (p^0 + \delta p^0)^2 - (E_{\mathbf{p}} + \delta E_{\mathbf{p}})^2, \end{aligned} \quad (4.93)$$

where we have dropped the order \hbar^2 terms. The shift of p^0 and the shift of $E_{\mathbf{p}}$ are defined as

$$\begin{aligned}\delta p^0 &= -\frac{\hbar}{4}F_{\mu\nu}\frac{\partial}{\partial p^0}\tilde{\Sigma}^{\mu\nu}\Big|_{p^0\rightarrow E_{\mathbf{p}}}, \\ \delta E_{\mathbf{p}} &= \frac{\hbar}{4E_{\mathbf{p}}}F_{\mu\nu}\tilde{\Sigma}^{\mu\nu}\Big|_{p^0\rightarrow E_{\mathbf{p}}} - \frac{\hbar}{4}F_{\mu\nu}\frac{\partial}{\partial p^0}\tilde{\Sigma}^{\mu\nu}\Big|_{p^0\rightarrow E_{\mathbf{p}}}.\end{aligned}\quad (4.94)$$

Thus we obtain

$$\theta(p^0)\delta\left(p^2 - m^2 - \frac{\hbar}{2}F_{\mu\nu}\tilde{\Sigma}^{\mu\nu}\right) = \frac{1}{2(p^0 + \delta p^0)}\theta(p^0 - E_{\mathbf{p}} + \delta p^0 - \delta E_{\mathbf{p}}). \quad (4.95)$$

When integrating the scalar component \mathcal{F} over p^0 we have

$$\begin{aligned}\int dp^0 \mathcal{F} &= m \frac{1}{2(p^0 + \delta p^0)}V(x, p)\Big|_{p^0\rightarrow E_{\mathbf{p}}+\delta E_{\mathbf{p}}-\delta p^0} \\ &= m \frac{1}{2p^0}V(x, p)\Big|_{p^0\rightarrow E_{\mathbf{p}}} + m \frac{\hbar}{4E_{\mathbf{p}}}F_{\mu\nu}\frac{\partial}{\partial p^0}\left[\frac{1}{2p^0}\tilde{\Sigma}^{\mu\nu}V\right]\Big|_{p^0\rightarrow E_{\mathbf{p}}}.\end{aligned}\quad (4.96)$$

Here in the last line, p^0 takes its on-shell value $E_{\mathbf{p}}$, thus we can introduce a normal mass-shell delta function

$$\int dp^0 \mathcal{F} = m \int dp^0 \theta(p^0)\delta(p^2 - m^2)\left[V + \frac{\hbar}{2}F_{\mu\nu}\frac{\partial}{\partial p^0}\left(\frac{1}{2p^0}\Sigma^{\mu\nu}\right)\right]. \quad (4.97)$$

Note that this result can also be derived by taking the p^0 -integration for the function \mathcal{F} in Eq. (4.62), and integrating by parts for the $\delta'(p^2 - m^2)$ term. Making a comparison with transformation (4.80), we find that if we take

$$\delta\Sigma^{\mu\nu} = -\frac{\partial}{\partial p^0}\left(\frac{1}{2p^0}\Sigma^{\mu\nu}\right), \quad (4.98)$$

then Eq. (4.97) can be written in terms of the new distribution \hat{V}

$$\int dp^0 \mathcal{F} = m \int dp^0 \theta(p^0)\delta(p^2 - m^2)\hat{V}. \quad (4.99)$$

In Eq. (4.91), the distribution V is on the modified mass-shell while in the above equation the new distribution \hat{V} is on the normal mass-shell. Thus we conclude that the transformations (4.80) and (4.81) change the mass-shell. We can always find some specific transformation, after which we can put the coupling between the electromagnetic field and the dipole moment into the distribution instead of into the mass-shell delta function.

3. Reference-frame dependence

In the solutions of the massless Wigner function (4.29), we have used a reference frame vector, which determines how to separate the currents into a distribution part and a gradient part. Here

we will focus on the massive case and show how the reference vector can be introduced. We will also show how to reproduce the massless results from the massive ones.

First we focus on the tensor component in the solution (4.62). Since any anti-symmetric tensor can be decomposed into an electric-like part and a magnetic-like part, we have the following decomposition,

$$\mathcal{S}_{\mu\nu} = \mathcal{D}_\mu u_\nu - \mathcal{D}_\nu u_\mu - \epsilon_{\mu\nu\alpha\beta} u^\alpha \mathcal{M}^\beta, \quad (4.100)$$

where the electric dipole moment and the magnetic dipole moment are respectively given by,

$$\begin{aligned} \mathcal{D}_\mu &= \mathcal{S}_{\mu\nu} u^\nu = m \mathbb{P}_\mu \delta(p^2 - m^2) - m \hbar E_\mu V \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2), \\ \mathcal{M}_\mu &= -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} u^\nu \mathcal{S}^{\alpha\beta} = m \mathbb{M}_\mu \delta(p^2 - m^2) - m \hbar B_\mu V \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2). \end{aligned} \quad (4.101)$$

Here $E_\mu \equiv F_{\mu\nu} u^\nu$ is the electric-field vector and $B_\mu \equiv \tilde{F}_{\mu\nu} u^\nu$ is the magnetic-field vector. We have defined the following terms for the on-shell parts \mathbb{P}_μ and \mathbb{M}_μ ,

$$\begin{aligned} \mathbb{P}_\mu &= \Sigma_{\mu\nu} u^\nu - \frac{\hbar}{2m} p_\mu u^\nu \nabla_\nu^{(0)} V + \frac{\hbar}{2m^2} (p \cdot u) \nabla_\mu^{(0)} V, \\ \mathbb{M}_\mu &= -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} u^\nu \Sigma^{\alpha\beta} + \frac{\hbar}{2m^2} \epsilon_{\mu\nu\alpha\beta} u^\nu p^\alpha \nabla^{(0)\beta} V, \end{aligned} \quad (4.102)$$

The on-shell dipole-moment tensor $\Sigma_{\mu\nu}$ can be reproduced via

$$\Sigma_{\mu\nu} = \mathbb{P}_\mu u_\nu - \mathbb{P}_\nu u_\mu + \epsilon_{\mu\nu\alpha\beta} u^\alpha \mathbb{M}^\beta + \frac{\hbar}{2m^2} p_{[\mu} \nabla_{\nu]}^{(0)} V. \quad (4.103)$$

Due to the constraint equation (4.66), one obtains the relation between \mathbb{P}_μ and \mathbb{M}_μ ,

$$\delta(p^2 - m^2) \left[(p \cdot u) \mathbb{P}_\mu - \epsilon_{\mu\nu\alpha\beta} p^\nu u^\alpha \mathbb{M}^\beta - \frac{\hbar}{2} (g_{\mu\nu} - u_\mu u_\nu) \nabla^{(0)\nu} V \right] = \mathcal{O}(\hbar^2), \quad (4.104)$$

where we have used the dynamical equation $\delta(p^2 - m^2) p^\nu \nabla_\nu^{(0)} V = \mathcal{O}(\hbar)$. From the above constraint, one can express \mathbb{P}_μ in terms of \mathbb{M}_μ ,

$$\delta(p^2 - m^2) \mathbb{P}_\mu = \delta(p^2 - m^2) \left[\frac{1}{p \cdot u} \epsilon_{\mu\nu\alpha\beta} p^\nu u^\alpha \mathbb{M}^\beta + \frac{\hbar}{2(p \cdot u)} (g_{\mu\nu} - u_\mu u_\nu) \nabla^{(0)\nu} V \right]. \quad (4.105)$$

Inserting back into the decomposition (4.100), we obtain

$$\mathcal{S}_{\mu\nu} = m \delta(p^2 - m^2) \left[-\frac{1}{p \cdot u} \epsilon_{\mu\nu\alpha\beta} p^\alpha \mathbb{M}^\beta - \frac{\hbar}{2(p \cdot u)} u_{[\mu} \nabla_{\nu]}^{(0)} V \right] - m \hbar F_{\mu\nu} V \delta'(p^2 - m^2). \quad (4.106)$$

This indicates that $\mathcal{S}_{\mu\nu}$ can be written in terms of the magnetic dipole-moment vector \mathbb{M}_μ , which is defined in frame u^μ . Meanwhile, we can express the axial-vector component of Wigner function,

$$\mathcal{A}^\mu = \frac{1}{p \cdot u} \left[m^2 \mathbb{M}^\mu - (p \cdot \mathbb{M}) p^\mu \right] \delta(p^2 - m^2) + \frac{\hbar}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_\nu u_\alpha \nabla_\beta^{(0)} V + \hbar \tilde{F}_{\mu\nu} p^\nu V \delta'(p^2 - m^2). \quad (4.107)$$

In the massless limit, we define

$$A \equiv - \lim_{m \rightarrow 0} \frac{p \cdot \mathbb{M}}{p \cdot u}. \quad (4.108)$$

Then \mathcal{A}^μ in Eq. (4.107) correctly reproduce the massless one in Eq. (4.29).

On the other hand, the polarization vector n^μ can be projected into the direction of p^μ and an arbitrary frame vector u^μ . Since p^μ has the unit of energy, we write down the following formula

$$m n^\mu = c_{\parallel} \left(p^\mu - \frac{m^2}{p \cdot u} u^\mu \right) + c_{\perp} m n_{\perp}^\mu. \quad (4.109)$$

Here the coefficient of u^μ is proportional to the coefficient of p^μ in order to satisfy the constraint equation $p \cdot n \delta(p^2 - m^2) = 0$. The vector n_{\perp}^μ is assumed to be a normalized space-like vector $n_{\perp}^\mu n_{\perp\mu} = -1$, which is perpendicular to both u^μ and p^μ . If we observe in the frame u^μ , the first term in Eq. (4.109) would be parallel to the momentum while the second term is perpendicular. Thus the polarization vector is decomposed into one longitudinally polarized part and one transversely polarized part. Here longitudinal/transverse refers to the relation between the particle's momentum direction and its spin direction. We now consider a special case where

$$c_{\parallel} = A \neq 0, \quad c_{\perp} m n_{\perp}^\mu = \frac{\hbar}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_{\alpha} u_{\beta} \nabla_{\nu}^{(0)} V, \quad (4.110)$$

Inserting back into the Wigner function in (4.41), we obtain the axial-vector current

$$\mathcal{A}^\mu = \delta(p^2 - m^2) \left[\left(p^\mu - \frac{m^2}{p \cdot u} u^\mu \right) A + \frac{\hbar}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_{\alpha} u_{\beta} \nabla_{\nu}^{(0)} V \right] + \hbar \tilde{F}^{\mu\nu} p_{\nu} V \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2), \quad (4.111)$$

When taking the massless limit $m \rightarrow 0$, \mathcal{A}^μ in Eq. (4.111) smoothly reproduce the massless one in Eq. (4.29). Here we can identify the transverse part $c_{\perp} m n_{\perp}^\mu$ as the contribution from the side-jump effect [65, 66, 69].

The dipole-moment tensor $\Sigma^{\mu\nu}$ can be expressed in terms of n^μ , as Eq. (4.40) shows. Inserting n^μ and $\Sigma^{\mu\nu}$ into the vector \mathcal{V}^μ in Eq. (4.41), we obtain

$$\begin{aligned} \mathcal{V}^\mu &= \delta(p^2 - m^2) \left(p^\mu V + \frac{\hbar}{2m} \epsilon^{\mu\nu\alpha\beta} p_{\nu} \nabla_{\alpha}^{(0)} n_{\beta} \right) \\ &\quad + \hbar \left(m \tilde{F}_{\mu\nu} n^{\nu} - \frac{p \cdot n}{m} \tilde{F}_{\mu\nu} p^{\nu} \right) \delta'(p^2 - m^2) + \mathcal{O}(\hbar^2), \end{aligned} \quad (4.112)$$

Note that here $p \cdot n \delta'(p^2 - m^2)$ does not have to vanish. In this formula we want to introduce a reference frame for the second term. With the help of the Schouten identity (4.14) and the kinetic

equation for n^μ in Eq. (4.48) we can prove

$$\begin{aligned}
\delta(p^2 - m^2) \frac{\hbar}{2m} \epsilon_{\mu\nu\alpha\beta} p^\nu \nabla^{(0)\alpha} n^\beta &= -\delta(p^2 - m^2) \frac{\hbar}{2m(p \cdot u)} p_\mu \epsilon_{\nu\alpha\beta\gamma} u^\gamma p^\nu \nabla^{(0)\alpha} n^\beta \\
&+ \delta(p^2 - m^2) \frac{\hbar}{2(p \cdot u)} \epsilon_{\mu\nu\alpha\beta} u^\nu \nabla^{(0)\alpha} m n^\beta \\
&+ \delta(p^2 - m^2) \frac{\hbar}{2m(p \cdot u)} \epsilon_{\mu\nu\alpha\beta} u^\nu p^\alpha \nabla^{(0)\beta} (p \cdot n). \quad (4.113)
\end{aligned}$$

We furthermore replace n^μ by Eqs. (4.109) and (4.110). Up to leading order in \hbar we have $p \cdot n = \frac{(p^2 - m^2)}{m} A + \mathcal{O}(\hbar)$. Substituting n^μ into the above equations, we obtain

$$\begin{aligned}
\mathcal{V}^\mu &= \delta(p^2 - m^2) p^\mu \left[V + \frac{\hbar}{2(p \cdot u)} \epsilon_{\nu\alpha\beta\gamma} p^\nu u^\alpha \nabla^{(0)\beta} \left(\frac{u^\gamma}{u \cdot p} A \right) \right] \\
&+ \delta(p^2 - m^2) \frac{\hbar}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} p_\nu u_\alpha \nabla_\beta^{(0)} A + \hbar \tilde{F}^{\mu\nu} p_\nu A \delta'(p^2 - m^2) \\
&- m^2 \left[\delta(p^2 - m^2) \frac{\hbar}{2(p \cdot u)} \epsilon^{\mu\nu\alpha\beta} u_\nu \nabla_\alpha^{(0)} \left(\frac{u_\beta}{p \cdot u} A \right) + \delta'(p^2 - m^2) \hbar \tilde{F}^{\mu\nu} \frac{u_\nu}{p \cdot u} A \right] + \mathcal{O}(\hbar^2), \quad (4.114)
\end{aligned}$$

Taking the massless limit $m \rightarrow 0$, and redefining the distribution function V , the above formula agrees with the massless results in (4.29).

In this subsection, we have found the relation between massive solutions and massless ones. We emphasize that the polarization of massive particles has 3 degrees of freedom, that is because in their rest frames, their spins can point to any spatial directions. Thus in massive case we have four functions to describe the system, three of which describe the polarization and the remaining one is the particle distribution. However, in the massless case, particles are either LH or RH and their spins are either parallel or anti-parallel to their momenta. Thus for the massless case the net fermion number distribution and the axial-charge distribution are sufficient to describe the system. In this section we decompose the polarization into a longitudinal part and a transverse part. Since massless particles cannot be transversely polarized, we presume that the transverse part only comes from the side-jump effect. Under this assumption, we find that the massive solutions smoothly reproduce the massless ones. In heavy ion collisions, the u , d quarks can be treated as massless, because their masses are much smaller than the typical temperature of the QGP. However, the mass of the s quark is about 95 MeV, which is comparable with the chemical freeze-out temperature (~ 160 MeV). Thus the massless chiral kinetic theory is not sufficient for simulations. The kinetic theory for massive particles with spin has been developed many years ago [131] and was here reproduced via the Wigner function approach. However, up to now there is no systematical tool for the simulation of the spin-evolution for massive fermions in heavy-ion collisions. Our study in this section would be a starting

point for future works on the dynamical spin evolution. Comparing with the classical Boltzmann equation and the BMT equation, we have obtained \hbar -order corrections, which are attributed to the coupling between the spin and the electromagnetic fields. In the future, we will use the method of moments to deal with the kinetic equations and derive spin-hydrodynamics.

V. PHYSICAL QUANTITIES

In this section we will consider systems in thermal equilibrium. The Wigner function for free fermions without chiral imbalance is computed at the leading order in \hbar in subsection III A. Higher-order terms in \hbar have been obtained by the semi-classical expansion in Eqs. (4.62) and (4.67) in Sec. IV. On the other hand, the case of fermions with chiral imbalance is analytically solved in subsection III B, which is called the chiral quantization in this thesis. The chiral chemical potential is not well-defined in the massive case, thus in the chiral quantization, μ_5 is treated as a variable for an addition self-energy correction term. However, the chiral quantization can only deal with constant μ_5 . In this section, through a comparison between semi-classical results and chiral-quantization results, we will give a reasonable estimate where the semi-classical results are applicable. In a constant magnetic field there also exist thermal equilibrium, which is then compared with the free-fermion case in this section. Pair production in the presence of an electric field is a dynamical problem, which was analytically solved in subsection III D. In this section we will numerically show the pair-production rates and display the thermal suppression to the production rates.

A. Physical quantities from quantum field theory

Throughout this thesis, we focus on spin-1/2 particles in electromagnetic fields. Since the electromagnetic interaction is a long-range interaction, short-range interactions such as the strong and weak interactions will be neglected. In this subsection we will start from the QED Lagrangian and derive some basic physical quantities. Then we will find a straightforward relation between the Wigner function and these quantities. In subsections VC and VD we will calculate their values using the Wigner function in thermal equilibrium.

Mathematically, QED is an Abelian $U(1)$ gauge theory. The Lagrangian for a Dirac spinor field in an electromagnetic field is given by

$$\hat{\mathcal{L}} = \frac{1}{2} \left[\hat{\psi} \gamma^\mu i \partial_{x\mu} \hat{\psi} - \left(i \partial_{x\mu} \hat{\psi} \right) \gamma^\mu \hat{\psi} \right] - \hat{\psi} (m + \gamma \cdot \mathbb{A}) \hat{\psi} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \quad (5.1)$$

where $\hat{\psi}$ is the spinor field operator, \mathbb{A}^μ is the gauge potential and $F^{\mu\nu} = \partial_x^\mu \mathbb{A}^\nu - \partial_x^\nu \mathbb{A}^\mu$ is the field-strength tensor, whose $0i$ components are the electric fields and ij components are the magnetic fields. The Lagrangian is invariant under the following local gauge transformation

$$\begin{aligned} \mathbb{A}^\mu(x) &\rightarrow \mathbb{A}^\mu(x) - \partial_x^\mu \theta(x), \\ \psi(x) &\rightarrow e^{i\theta(x)} \psi(x), \end{aligned} \quad (5.2)$$

where the gauge potential has an additional derivative term while the Dirac field has a phase rotation. The gauge invariance indicates an ambiguity of the gauge potential. In practice we can fix the gauge by e.g. taking the Lorenz-gauge condition $\partial_x^\mu \mathbb{A}_\mu = 0$, or the temporal gauge $\mathbb{A}^0 = 0$, etc.. From the Lagrangian we obtain the Euler-Lagrangian equations for $\hat{\psi}$, $\hat{\bar{\psi}}$ and \mathbb{A}^μ . They are the Dirac equation, the conjugate of the Dirac equation and the inhomogeneous Maxwell equation, respectively. The Dirac equation and its conjugate are listed in Eq. (2.23) and the Maxwell equation reads

$$\partial_{x\mu} F^{\mu\nu} = \hat{\bar{\psi}} \gamma^\nu \hat{\psi}. \quad (5.3)$$

According to the Noether's theorem, each continuous symmetry of the Lagrangian corresponds to a conserved current. The gauge symmetry is associated with an electric current,

$$\hat{\mathbb{N}}^\mu = \hat{\bar{\psi}} \gamma^\mu \hat{\psi}. \quad (5.4)$$

Here we use hat to distinguish the operator from physical quantities. The conserved currents corresponding to translations and Lorentz transformations are the energy-momentum tensor and the angular-momentum tensor, respectively.

$$\begin{aligned} \hat{\mathbb{T}}^{\mu\nu} &= \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_{x\mu} \hat{\psi})} \partial_x^\nu \hat{\psi} + \partial_x^\nu \hat{\bar{\psi}} \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_{x\mu} \hat{\bar{\psi}})} + \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_{x\mu} \mathbb{A}_\rho)} \partial_x^\nu \mathbb{A}_\rho - g^{\mu\nu} \hat{\mathcal{L}}, \\ \hat{\mathbb{M}}^{\rho\mu\nu} &= x^\mu \hat{\mathbb{T}}^{\rho\nu} - x^\nu \hat{\mathbb{T}}^{\rho\mu} - i \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_{x\rho} \hat{\psi})} \mathcal{S}^{\mu\nu} \hat{\psi} + i \hat{\bar{\psi}} \mathcal{S}^{\mu\nu} \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_{x\rho} \hat{\bar{\psi}})} - i \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_{x\rho} \mathbb{A}_\alpha)} (\mathcal{J}^{\mu\nu})_{\alpha\beta} \mathbb{A}^\beta, \end{aligned} \quad (5.5)$$

where the generators of spin are $\mathcal{S}^{\mu\nu} \equiv \frac{i}{4} [\gamma^\mu, \gamma^\nu] = \frac{1}{2} \sigma^{\mu\nu}$ and $(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\beta^\mu \delta_\alpha^\nu)$. Inserting the Lagrangian into the definition of these currents we obtain

$$\begin{aligned} \hat{\mathbb{T}}^{\mu\nu} &= \hat{\mathbb{T}}_{\text{mat}}^{\mu\nu} + \mathbb{A}^\nu \hat{\mathbb{N}}^\mu + \hat{\mathbb{T}}_{\text{field}}^{\mu\nu}, \\ \hat{\mathbb{M}}^{\rho,\mu\nu} &= \hat{\mathbb{L}}^{\rho,\mu\nu} + \hat{\mathbb{S}}_{\text{mat}}^{\rho,\mu\nu} + \hat{\mathbb{S}}_{\text{field}}^{\rho,\mu\nu}. \end{aligned} \quad (5.6)$$

Here we have separated the total energy-momentum tensor and the total angular-momentum tensor into several parts. The total orbital angular-momentum tensor is given by

$$\hat{\mathbb{L}}^{\rho,\mu\nu} \equiv x^\mu \hat{\mathbb{T}}^{\rho\nu} - x^\nu \hat{\mathbb{T}}^{\rho\mu}. \quad (5.7)$$

The matter parts of the energy-momentum tensor and the spin-angular-momentum tensor are

$$\begin{aligned} \hat{\mathbb{T}}_{\text{mat}}^{\mu\nu} &= \frac{1}{2} \left[i \hat{\bar{\psi}} \gamma^\mu (\overrightarrow{\partial}_x^\nu + i \mathbb{A}^\nu) \hat{\psi} - i \hat{\bar{\psi}} \gamma^\mu (\overleftarrow{\partial}_x^\nu - i \mathbb{A}^\nu) \hat{\psi} \right], \\ \hat{\mathbb{S}}_{\text{mat}}^{\rho,\mu\nu} &= \frac{1}{4} \hat{\bar{\psi}} \{ \gamma^\rho, \sigma^{\mu\nu} \} \hat{\psi}, \end{aligned} \quad (5.8)$$

while the field parts are

$$\begin{aligned}\hat{\mathbb{T}}_{\text{field}}^{\mu\nu} &= \frac{1}{4}g^{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} - F^{\mu\rho}\partial^\nu\mathbb{A}_\rho, \\ \hat{\mathbb{S}}_{\text{field}}^{\rho,\mu\nu} &= -(F^{\rho\mu}\mathbb{A}^\nu - F^{\rho\nu}\mathbb{A}^\mu).\end{aligned}\tag{5.9}$$

Note that after such a decomposition, the matter parts are gauge-invariant while the remaining parts are not. The gauge dependence comes from the definitions (5.5), where the derivative operators are ordinary ones instead of covariant ones. Taking expectation values of the above operators on one specific system $|\Omega\rangle$, we can obtain the net fermion current \mathbb{N}^μ , the matter part of the energy-momentum tensor $\mathbb{T}_{\text{mat}}^{\mu\nu}$, and the spin angular-momentum tensor $\mathbb{S}_{\text{mat}}^{\rho,\mu\nu}$,

$$\mathbb{N}^\mu = \langle\Omega|\hat{\mathbb{N}}^\mu|\Omega\rangle, \quad \mathbb{T}_{\text{mat}}^{\mu\nu} = \langle\Omega|\hat{\mathbb{T}}_{\text{mat}}^{\mu\nu}|\Omega\rangle, \quad \mathbb{S}_{\text{mat}}^{\rho,\mu\nu} = \langle\Omega|\hat{\mathbb{S}}_{\text{mat}}^{\rho,\mu\nu}|\Omega\rangle.\tag{5.10}$$

The Noether currents are conserved, thus the fluid-dynamical quantities satisfy the following conservation laws automatically,

$$\partial_{x\mu}\mathbb{N}^\mu = 0, \quad \partial_{x\mu}\mathbb{T}^{\mu\nu} = 0, \quad \partial_{x\rho}\mathbb{M}^{\rho,\mu\nu} = 0.\tag{5.11}$$

But note that $\mathbb{T}^{\mu\nu}$ and $\mathbb{M}^{\rho,\mu\nu}$ are separated into several parts as in Eq. (5.6). The matter parts of $\mathbb{T}_{\text{mat}}^{\mu\nu}$ and $\mathbb{S}_{\text{mat}}^{\rho,\mu\nu}$ are not conserved themselves,

$$\begin{aligned}\partial_{x\mu}\mathbb{T}_{\text{mat}}^{\mu\nu} &= F^{\nu\alpha}\mathbb{N}_\alpha, \\ \partial_{x\rho}\mathbb{S}_{\text{mat}}^{\rho,\mu\nu} &= -\mathbb{T}_{\text{mat}}^{\mu\nu} + \mathbb{T}_{\text{mat}}^{\nu\mu}.\end{aligned}\tag{5.12}$$

From the second line we observe that the anti-symmetric part of $\mathbb{T}_{\text{mat}}^{\mu\nu}$ is related to the derivative of the spin tensor. For a classical particle, where the spin degrees of freedom are ignored, the spin tensor vanishes and thus $\mathbb{T}_{\text{mat}}^{\mu\nu}$ is symmetric. But in general the canonical energy-momentum tensor is not symmetric for a system with spin.

The tensors above are not uniquely defined. The Lagrangian in Eq. (5.1) can have an additional term, like $\partial_{x\mu}\delta\mathcal{L}^\mu$ with an arbitrary $\delta\mathcal{L}^\mu$. When taking integration over the whole space, this additional term will contribute a boundary term, which in general is neglected. However, changing the definition of the Lagrangian leads to a different energy-momentum tensor and a different spin-angular-momentum tensor. All these different definitions are exactly equivalent since they are related to the canonical formula by so-called pseudo-gauge transformations [132, 133]

$$\begin{aligned}\mathbb{T}'_{\text{mat}}{}^{\mu\nu} &= \mathbb{T}_{\text{mat}}^{\mu\nu} + \frac{1}{2}\partial_\rho(\mathbb{F}^{\rho,\mu\nu} + \mathbb{F}^{\mu,\nu\rho} + \mathbb{F}^{\nu,\mu\rho}), \\ \mathbb{S}'_{\text{mat}}{}^{\rho,\mu\nu} &= \mathbb{S}_{\text{mat}}^{\rho,\mu\nu} - \mathbb{F}^{\rho,\mu\nu},\end{aligned}\tag{5.13}$$

where $\mathbb{F}^{\rho,\mu\nu}$ is an arbitrary tensor which is anti-symmetric under $\mu \leftrightarrow \nu$. We can check that the newly defined quantities still satisfy the conservation equations (5.12). A specific choice for $\mathbb{F}^{\rho,\mu\nu}$ is $\mathbb{S}_{\text{mat}}^{\rho,\mu\nu}$, which makes $\mathbb{S}'_{\text{mat}}{}^{\rho,\mu\nu}$ vanishes. The new energy-momentum tensor is the Belinfante one,

$$\mathbb{T}_{\text{Bel}}^{\mu\nu} \equiv \mathbb{T}_{\text{mat}}^{\mu\nu} + \frac{1}{2} \partial_\rho (\mathbb{S}_{\text{mat}}^{\rho,\mu\nu} + \mathbb{S}_{\text{mat}}^{\mu,\nu\rho} + \mathbb{S}_{\text{mat}}^{\nu,\mu\rho}). \quad (5.14)$$

It is easy to check that the Belinfante energy-momentum tensor is the symmetric part of the canonical one,

$$\mathbb{T}_{\text{Bel}}^{\mu\nu} = \frac{1}{2} (\mathbb{T}_{\text{mat}}^{\mu\nu} + \mathbb{T}_{\text{mat}}^{\nu\mu}). \quad (5.15)$$

Comparing the definition of the Wigner function in Eq. (2.18) with the above fluid-dynamical quantities, we obtain the following relations

$$\begin{aligned} \mathbb{N}^\mu(x) &= \int d^4p \mathcal{V}^\mu(x, p), \\ \mathbb{T}_{\text{mat}}^{\mu\nu}(x) &= \int d^4p p^\nu \mathcal{V}^\mu(x, p), \\ \mathbb{S}_{\text{mat}}^{\rho,\mu\nu}(x) &= -\frac{1}{2} \epsilon^{\rho\mu\nu\alpha} \int d^4p \mathcal{A}_\alpha(x, p), \end{aligned} \quad (5.16)$$

where $\mathcal{V}^\mu(x, p)$ and $\mathcal{A}^\mu(x, p)$ are the vector and axial-vector components of the Wigner function. On the other hand, the Belinfante energy-momentum tensor is given by

$$\mathbb{T}_{\text{Bel}}^{\mu\nu}(x) = \frac{1}{2} \int d^4p [p^\mu \mathcal{V}^\nu(x, p) + p^\nu \mathcal{V}^\mu(x, p)]. \quad (5.17)$$

In the following part of this section, we will specify the Wigner function and calculate the canonical quantities in Eq. (5.16).

B. Thermal Equilibrium

In Sec. III we have derived the analytical solutions of the Wigner function. Note that in these solutions, the distributions for fermions and anti-fermions are still undetermined. In this subsection we will consider systems in thermal equilibrium and give the equilibrium distributions. First we consider massless particles at a given temperature T , chemical potential μ , and chiral chemical potential μ_5 . The helicity of massless particles is a conserved quantity, thus μ_5 is well-defined. The canonical partition function for such a system is given by

$$\hat{Z} = \exp[-\beta(\hat{H}_0 - \mu\hat{N} - \mu_5\hat{N}_5)], \quad (5.18)$$

where the Hamiltonian operator \hat{H}_0 , the fermion number operator \hat{N} , and the axial-charge number \hat{N}_5 are

$$\begin{aligned}\hat{H}_0 &= \int d^3\mathbf{x} \hat{\psi}^\dagger(t, \mathbf{x}) (-i\gamma^0 \boldsymbol{\gamma} \cdot \boldsymbol{\partial}_{\mathbf{x}} + m\gamma^0) \hat{\psi}(t, \mathbf{x}), \\ \hat{N} &= \int d^3\mathbf{x} \hat{\psi}^\dagger(t, \mathbf{x}) \hat{\psi}(t, \mathbf{x}), \\ \hat{N}_5 &= \int d^3\mathbf{x} \hat{\psi}^\dagger(t, \mathbf{x}) \gamma^5 \hat{\psi}(t, \mathbf{x}).\end{aligned}\tag{5.19}$$

Using the quantized field operator in Eq. (3.37) and the single-particle wavefunctions in Eq. (3.34) and (3.32) for the detailed calculations, we derive

$$\begin{aligned}\hat{H}_0 &= \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} |\mathbf{p}| \left(\hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} - \hat{b}_{\mathbf{p},s} \hat{b}_{\mathbf{p},s}^\dagger \right), \\ \hat{N} &= \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} + \hat{b}_{\mathbf{p},s} \hat{b}_{\mathbf{p},s}^\dagger \right), \\ \hat{N}_5 &= \sum_{ss_1} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\xi_s^\dagger(\boldsymbol{\sigma} \cdot \mathbf{p}) \xi_{s_1}}{|\mathbf{p}|} \left(\hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s_1} + \hat{b}_{\mathbf{p},s} \hat{b}_{\mathbf{p},s_1}^\dagger \right),\end{aligned}\tag{5.20}$$

where we have dropped terms $\hat{a}_{\mathbf{p},s}^\dagger \hat{b}_{-\mathbf{p},s}^\dagger$ and $\hat{b}_{\mathbf{p},s} \hat{a}_{-\mathbf{p},s}$. In general, the term $\xi_s^\dagger(\boldsymbol{\sigma} \cdot \mathbf{p}) \xi_{s_1}$ is not diagonalizable in spin space, which means that the physical states are superposition of states with $s = +$ and states with $s = -$, and thus the thermal expectation value of $a_{\mathbf{p},+}^\dagger a_{\mathbf{p},-}$ will be non-zero. We can use the method in subsection III A to diagonalize the distribution. If we choose the spin quantization direction as the direction of \mathbf{p} , which fulfills,

$$(\boldsymbol{\sigma} \cdot \mathbf{p}) \xi_s = s |\mathbf{p}| \xi_s,\tag{5.21}$$

then the operator \hat{N}_5 is diagonalized in spin space,

$$\hat{N}_5 = \sum_s s \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} + \hat{b}_{\mathbf{p},s} \hat{b}_{\mathbf{p},s}^\dagger \right).\tag{5.22}$$

Inserting the quantized operators (5.20), (5.22) into the canonical partition function, we obtain

$$\hat{Z} = \exp \left\{ -\beta \sum_s \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left[(|\mathbf{p}| - \mu - s\mu_5) \hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s} - (|\mathbf{p}| + \mu + s\mu_5) \hat{b}_{\mathbf{p},s} \hat{b}_{\mathbf{p},s}^\dagger \right] \right\}.\tag{5.23}$$

This result coincides with our knowledge for the massless case: for massless particles, the operators \hat{N} and \hat{N}_5 commute with the Hamiltonian \hat{H} , thus the basis states can be chosen as common eigenstates of the operators \hat{H} , \hat{N} , and \hat{N}_5 . The canonical partition function is then diagonalized. The expectation value of any operator \hat{O} is computed via

$$\langle \hat{O} \rangle = \frac{\text{Tr}(\hat{O} \hat{Z})}{\text{Tr} \hat{Z}}.\tag{5.24}$$

Here Tr runs over all possible quantum states. Taking the expectation values of the fermion number operator $\hat{a}_{\mathbf{p},s}^\dagger \hat{a}_{\mathbf{p},s'}$ and anti-fermion number operator $\hat{b}_{\mathbf{p},s}^\dagger \hat{b}_{\mathbf{p},s'}$ we obtain the following Fermi-Dirac distributions,

$$f_{ss'}^{(\pm)}(x, \mathbf{p}) = \frac{1}{1 + \exp[\beta(|\mathbf{p}| \mp \mu \mp s\mu_5)]} \delta_{ss'}, \quad (5.25)$$

which coincide with the equilibrium distributions for chiral particles. Since the spinors satisfy Eq. (5.21), we can calculate

$$\xi_s^\dagger \hat{n}^\mu(\mathbf{p}) \xi_s = sp^\mu. \quad (5.26)$$

Then the Wigner function in Eq. (3.216) reproduces the massless results of Refs. [66, 69, 125].

For the massive case, one possible choice for the thermal equilibrium distributions is the naive extension from the massless ones in Eq. (5.25) by substituting $|\mathbf{p}| \rightarrow E_{\mathbf{p}}$,

$$f_{ss'}^{(\pm)}(x, \mathbf{p}) = \frac{1}{1 + \exp[\beta(E_{\mathbf{p}} \mp \mu \mp s\mu_5)]} \delta_{ss'}. \quad (5.27)$$

However, note that the axial-charge number \hat{N}_5 is not conserved in the massive case because it does not commute with the Hamiltonian, so μ_5 is not well-defined. The correct way is to include additional self-energy terms $\mu\psi^\dagger\psi + \mu_5\psi^\dagger\gamma^5\psi$ in the Hamiltonian, where μ_5 is the conjugate variable of the axial-charge density. Here μ_5 controls the chiral imbalance and is the counterpart of the chiral chemical potential in the massless case. Meanwhile, μ is interpreted as the vector chemical potential. The single-particle wavefunction, as well as Wigner function have been computed in subsection III B, see Eq. (3.145). The Hamiltonian is quantized as shown in Eq. (3.128). Since the chemical potentials are already included in the Hamiltonian, the canonical partition function is given by [134]

$$\hat{Z} = \exp(-\beta\hat{H}), \quad (5.28)$$

with the total Hamiltonian

$$\hat{H} = \hat{H}_0 - \mu\hat{N} - \mu_5\hat{N}_5, \quad (5.29)$$

where the free Hamiltonian \hat{H}_0 , the net fermion number \hat{N} , and the axial-charge number \hat{N}_5 are defined as Eq. (5.19). The equilibrium distributions for fermions and anti-fermions are derived using this partition function, which agree with the Fermi-Dirac distributions,

$$\begin{aligned} f_s^{(+)}(\mathbf{p}) &= \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} - \mu)]}, \\ f_s^{(-)}(\mathbf{p}) &= \frac{1}{1 + \exp[\beta(E_{-\mathbf{p},s} + \mu)]}, \end{aligned} \quad (5.30)$$

where $f_s^{(+)}(\mathbf{p}) \equiv \langle a_{\mathbf{p},s}^\dagger a_{\mathbf{p},s} \rangle$ and $f_s^{(-)}(\mathbf{p}) \equiv \langle b_{\mathbf{p},s}^\dagger b_{\mathbf{p},s} \rangle$ are expectation values of fermion number and anti-fermion number with given \mathbf{p} and s . Note that here the energy $E_{\mathbf{p},s} = \sqrt{m^2 + (|\mathbf{p}| - s\mu_5)^2}$ reduces to $E_{\mathbf{p},s} = ||\mathbf{p}| - s\mu_5|$ in the massless limit and these distributions agree with previous results in Eq. (5.25) when $|\mathbf{p}| > \mu_5$. In subsections VC and VD we will use both the naive distributions in (5.27) and the explicit ones in (5.30) to compute dynamical quantities. They will show a coincidence with each other in some parameter region.

In a constant magnetic field, the Dirac equation is solved in subsection III C and in this case the Hamiltonian is quantized as shown in Eq. (3.194). The corresponding canonical partition function is again defined by Eq. (5.28). Taking the expectation values of $\hat{a}_s^{(n)\dagger}(p^x, p^z)\hat{a}_s^{(n)}(p^x, p^z)$ and $\hat{b}_s^{(n)\dagger}(p^x, p^z)\hat{b}_s^{(n)}(p^x, p^z)$, we obtain the following distributions respectively,

$$\begin{aligned} f_s^{(+)(n)}(p^z) &= \frac{1}{1 + \exp\left[\beta\left(E_{p^z,s}^{(n)} - \mu\right)\right]}, \\ f_s^{(-)(n)}(p^z) &= \frac{1}{1 + \exp\left[\beta\left(E_{-p^z,s}^{(n)} + \mu\right)\right]}. \end{aligned} \quad (5.31)$$

Note that here the equilibrium distributions are independent of the parameter p^x because the energy states, $E_{p^z}^{(0)} = \sqrt{m^2 + (p^z - \mu_5)^2}$ and $E_{p^z,s}^{(n)} = \sqrt{m^2 + \left[\sqrt{(p^z)^2 + 2nB_0} - s\mu_5\right]^2}$ for $n > 0$, are independent of p^x . This agrees with our knowledge about the Landau levels: the transverse momenta are quantized and described by the quantum number n , and p^x is now a parameter for the center position of the wavefunction in the y direction.

The coupling between spin and the magnetic field is already considered when computing the Wigner function because electromagnetic fields are included in the definition of the Wigner function (2.18). However in the presence of vorticity, additional spin-vorticity coupling terms are necessary, otherwise the vortical effects cannot be derived. The vorticity of charged particles generates an effective magnetic field, which couples with the magnetic dipole-moment of the particles, thus the additional coupling term would be

$$\Delta\hat{H} = \frac{\hbar}{4}\omega^{\mu\nu}\hat{\psi}^\dagger\gamma^0\sigma_{\mu\nu}\hat{\psi}, \quad (5.32)$$

where $\omega^{\mu\nu} \equiv \partial_x^\mu(\beta u^\nu) - \partial_x^\nu(\beta u^\mu)$ is the thermal vorticity and $\frac{\hbar}{2}\hat{\psi}^\dagger\gamma^0\sigma_{\mu\nu}\hat{\psi}$ is the operator of the dipole-moment tensor. In general $\Delta\hat{H}$ is diagonal for the eigenstates of Hamiltonian \hat{H} if and only if they commute $[\hat{H}, \Delta\hat{H}] = 0$. Assuming that the dipole-moment tensor of particles points along the direction $m_{\mu\nu}$, then we have

$$\Delta\hat{H} = \frac{\hbar}{4}\omega^{\mu\nu}m_{\mu\nu} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s s \left(a_{\mathbf{p},s}^\dagger a_{\mathbf{p},s} - b_{\mathbf{p},s} b_{\mathbf{p},s}^\dagger \right). \quad (5.33)$$

This can be achieved in the case of free fermions without chiral imbalance because the spin quantization direction is not specified, as discussed in subsection III A. Then the distribution functions would have an order- \hbar correction,

$$f_{ss'}^{(\pm)}(x, \mathbf{p}) = \frac{1}{1 + \exp [\beta (E_{\mathbf{p}} \mp \mu \pm s \frac{\hbar}{4} \omega^{\mu\nu} m_{\mu\nu})]} \delta_{ss'}, \quad (5.34)$$

which agrees with the suggestion of Ref. [43]. Note that the validity of this distribution needs more careful discussion. That is because in the presence of a magnetic field, there are two specific directions: the direction of the magnetic field and the direction of the vorticity. If they differ from each other, the spin-magnetic coupling term and the spin-vorticity coupling term cannot be diagonal simultaneously. Even if the vorticity and the magnetic field are along the same direction, we find it difficult to introduce the chiral chemical potential because the axial-charge \hat{N}_5 is diagonal only if the spin is quantized along p^μ . More precisely, the vorticity in general depends on the spatial coordinates while the dipole-moment depends on the particle's momentum. If we consider these dependences, $\Delta \hat{H}$ cannot commute with \hat{H}_0 , which indicates that it is impossible to find the common eigenstates for $\Delta \hat{H}$ and \hat{H}_0 . Thus the total Hamiltonian is in general not diagonal and the equilibrium in the presence of vorticity cannot be as simple as shown in Eq. (5.34). The Wigner function contains the vortical effect at order \hbar , but in this thesis we will not discuss the vorticity effect because the correct way to define the thermal equilibrium distributions with spin-vorticity coupling is still under discussion.

C. Net fermion current and polarization

As discussed in subsection V A, the net fermion current can be derived from the vector component of the Wigner function. The chiral imbalance for massive particles should be included in the Dirac equation, as we did in subsection III B. But if the mass of the particles is small enough, they can be treated as massless particles. In this case, the chiral chemical potential μ_5 is included in the thermal equilibrium distribution, which is given in Eq. (5.25). The Wigner function is solved up to order \hbar using the semi-classical expansion of Eqs. (4.62) and (4.67). Note that this method is a straightforward extension of the massless case. Another more exact approach is chiral quantization. In this subsection we will compare these two approaches. Meanwhile, we will discuss the magnetic-field dependence of physical quantities.

1. *Semi-classical results*

If we use the semi-classical results in Eq. (4.67), the net fermion number current is given by

$$\mathbb{N}^\mu = \int d^4p p^\mu \left[V \delta(p^2 - m^2) - \frac{\hbar}{2} F_{\alpha\beta} \Sigma^{\alpha\beta} \delta'(p^2 - m^2) \right] + \frac{\hbar}{2} \partial_{x\nu} \int d^4p \Sigma^{\mu\nu} \delta(p^2 - m^2). \quad (5.35)$$

Since $\int d^4p \Sigma^{\mu\nu} \delta(p^2 - m^2)$ is the leading-order dipole-moment tensor, the second term is identified as the magnetization current. Meanwhile, the axial vector current is given by \mathcal{A}^μ ,

$$\mathbb{N}_5^\mu = -\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \int d^4p p^\nu \Sigma^{\alpha\beta} \delta(p^2 - m^2) + \hbar \tilde{F}_{\mu\nu} \int d^4p p^\nu V \delta'(p^2 - m^2). \quad (5.36)$$

Since we have no idea how to determine the dipole-moment tensor, we presume that all particles are longitudinally polarized. That is, at leading order in \hbar , the spins of the particles are either parallel or anti-parallel to their momenta in the observer's rest frame. The frame dependence of spin polarization is discussed in subsection IVE, see Eqs. (4.109) and (4.110). In this case the vector and axial-vector components of the Wigner function are given by Eqs. (4.111), (4.114). Using the equilibrium distributions in (5.27), we obtain the net fermion distribution V and the axial-charge distribution A ,

$$\begin{aligned} V &= \frac{2}{(2\pi)^3} \sum_s \left\{ \theta(p \cdot u) \frac{1}{1 + \exp[\beta(p \cdot u - \mu - s\mu_5)]} \right. \\ &\quad \left. + \theta(-p \cdot u) \frac{1}{1 + \exp[\beta(-p \cdot u + \mu + s\mu_5)]} - \theta(-p \cdot u) \right\}. \\ A &= \frac{2}{(2\pi)^3} \sum_s s \left\{ \theta(p \cdot u) \frac{1}{1 + \exp[\beta(p \cdot u - \mu - s\mu_5)]} \right. \\ &\quad \left. + \theta(-p \cdot u) \frac{1}{1 + \exp[\beta(-p \cdot u + \mu + s\mu_5)]} - \theta(-p \cdot u) \right\}. \end{aligned} \quad (5.37)$$

Note that in Eq. (5.37) we have terms which represent vacuum contributions. When calculating the net fermion density n and the CME coefficient σ_χ , these vacuum terms should be dropped otherwise the results will be infinite. Here we have replaced p^0 , or the energy $E_{\mathbf{p}}$ by $p \cdot u$, which is the energy in the frame u^μ . From Eq. (4.114) we obtain the net fermion current by integrating over d^4p .

$$\mathbb{N}^\mu = \int d^4p p^\mu \left[V \delta(p^2 - m^2) - \hbar \tilde{F}^{\nu\alpha} \frac{p_\nu u_\alpha}{p \cdot u} A \delta'(p^2 - m^2) \right] + \frac{\hbar}{2} \epsilon^{\mu\nu\alpha\beta} \partial_{x\nu} \int d^4p \frac{p_\alpha u_\beta}{p \cdot u} A \delta(p^2 - m^2). \quad (5.38)$$

We find that the first term agrees with the classical fermion current where the shift of energy is attributed to the spin-magnetic coupling, while the second term gives the analogue of the CVE. Note that if we compare with Maxwell's equation we immediately find that the second term is

nothing but the magnetization current. When computing the integration over d^4p , we need to deal with $\delta'(p^2 - m^2)$, which can be achieved by integrating by parts,

$$\mathbb{N}^\mu = \int d^4p \left[p^\mu V + \frac{\hbar}{2} \tilde{F}^{\mu\nu} u_\nu \frac{1}{p \cdot u} A + \frac{\hbar}{2} u^\mu \tilde{F}^{\nu\alpha} \frac{p_\nu u_\alpha}{p \cdot u} \frac{\partial}{\partial(p \cdot u)} A + \frac{\hbar}{2} u^\mu \tilde{F}^{\nu\alpha} \frac{p_\nu u_\alpha}{(p \cdot u)^2} A \right] \delta(p^2 - m^2). \quad (5.39)$$

Note here that the functions V and A only depends on $(p \cdot u)$. Thus the above current can be parametrized as

$$\mathbb{N}^\mu = u^\mu n + \sigma_\chi \hbar \tilde{F}^{\mu\nu} u_\nu, \quad (5.40)$$

where

$$\begin{aligned} n &= \int d^4p (p \cdot u) V \delta(p^2 - m^2), \\ \sigma_\chi &= \frac{1}{2} \int d^4p \frac{1}{p \cdot u} A \delta(p^2 - m^2) \end{aligned} \quad (5.41)$$

are the net fermion number density and the CME conductivity respectively. Using the distributions shown in Eq. (5.37), these quantities can be numerically calculated.

On the other hand, the axial-vector current, or the spin-polarization density is calculated from the axial-vector current by taking an integration over d^4p

$$\begin{aligned} \mathbb{N}_5^\mu &= \int d^4p \left[\left(p^\mu - \frac{m^2}{p \cdot u} u^\mu \right) A - \hbar \tilde{F}^{\mu\nu} u_\nu \frac{\partial}{\partial(p \cdot u)} V + \frac{\hbar}{2} \tilde{F}^{\mu\nu} p_\nu \frac{1}{p \cdot u} \frac{\partial}{\partial(p \cdot u)} V \right. \\ &\quad \left. + \frac{\hbar}{2} u^\mu \tilde{F}^{\nu\alpha} \frac{p_\nu u_\alpha}{p \cdot u} \frac{\partial}{\partial(p \cdot u)} V \right] \delta(p^2 - m^2), \end{aligned} \quad (5.42)$$

where we have neglected the spatial derivative of u^μ . Analogous to the net fermion number current, the spin polarization can be parametrized as

$$\mathbb{N}_5^\mu = u^\mu n_5 + \sigma_5 \hbar \tilde{F}^{\mu\nu} u_\nu, \quad (5.43)$$

where n_5 is the axial-charge density and σ_5 is the coefficient for the CSE,

$$\begin{aligned} n_5 &= \int d^4p \left(u \cdot p - \frac{m^2}{u \cdot p} \right) A \delta(p^2 - m^2), \\ \sigma_5 &= -\frac{1}{2} \int d^4p \delta(p^2 - m^2) \frac{\partial}{\partial(u \cdot p)} V. \end{aligned} \quad (5.44)$$

Using the equilibrium distributions in Eq. (5.37), n_5 and σ_5 can be numerically calculated.

In the massless case, the fermion number density is given by

$$n_{\text{massless}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_{rs} r \frac{1}{1 + \exp[\beta(|\mathbf{p}| - r\mu - rs\mu_5)]}, \quad (5.45)$$

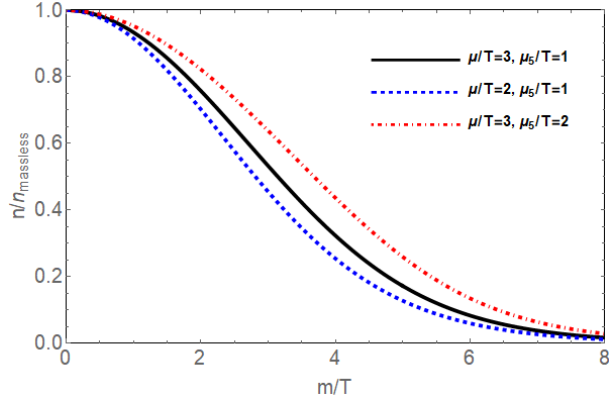


Figure 7: Mass dependence of the net fermion number density n . Here we plot the ratio of the massive one computed using Eq. (5.41) and the massless one in Eq. (5.45). The particles' mass m and chemical potentials μ, μ_5 are normalized by the temperature T .

while the CME conductivity is a constant $\sigma_{\chi, \text{massless}} = \mu_5/(2\pi^2)$ [19, 21–23]. The axial-charge density is

$$n_{5, \text{massless}} = \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \sum_{rs} rs \frac{1}{1 + \exp[\beta(|\mathbf{p}| - r\mu - rs\mu_5)]}, \quad (5.46)$$

while the coefficient for the chiral separation effect is $\sigma_{5, \text{massless}} = \mu/(2\pi)^2$ [23, 48, 49]. In Figs. 7-10 we computed the ratio between the massive results (5.41) and (5.44) and the massless ones (5.45) and (5.46) for the net fermion number density n , the CME conductivity σ_{χ} , the axial-charge density n_5 , and the CSE coefficient σ_5 , respectively. From these figures we observe that the quantities in the massive case smoothly reproduce the massless ones because the ratios become 1 in the massless limit. When the mass grows, both of these quantities decrease. This is because, when the particles' momentum is fixed, heavier ones have larger energy and thus the states are less likely to be occupied.

2. Results from chiral quantization

The coincidence of the massive and massless results is beyond our expectation because the equilibrium distributions are taken to be the ones in Eq. (5.27), which are naive extensions of the massless ones. However, as we discussed in subsection VB, the chiral chemical potential in the massive case should be considered as a self-energy term, which appears in the Dirac equation. The corresponding Wigner function has been given in Eq. (3.145). Here we take \mathcal{V}^0 and \mathcal{A}^0 , which give the net fermion number density and the axial-charge number density respectively. In thermal

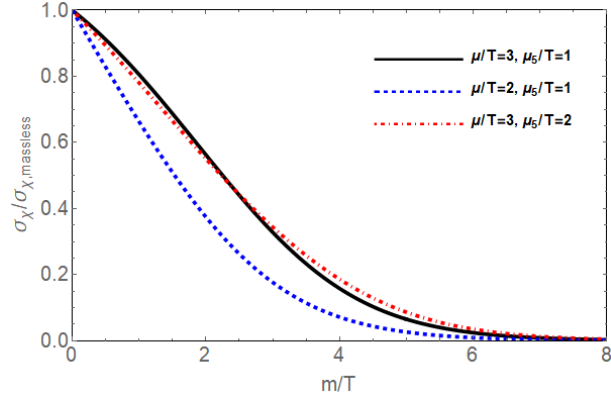


Figure 8: Mass dependence of the CME conductivity σ_χ . Here we plot the ratio of the massive one calculated using Eq. (5.41) and the massless one $\sigma_{\chi, \text{massless}} = \mu_5 / (2\pi^2)$. The particles' mass m and chemical potentials μ, μ_5 are normalized by the temperature T .

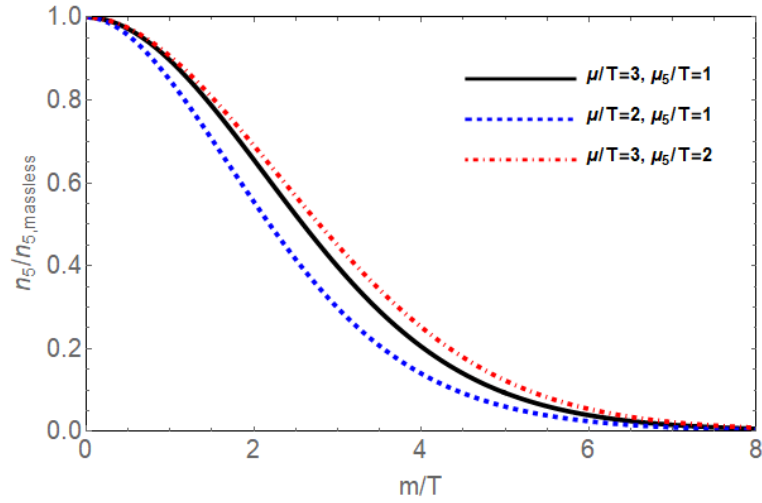


Figure 9: Mass dependence of the axial-charge density n_5 . Here we plot the ratio of the massive one calculated using Eq. (5.44) and the massless one in Eq. (5.46). The particles' mass m and chemical potentials μ, μ_5 are normalized by the temperature T .

equilibrium, these quantities are given by

$$\begin{aligned}
 n &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \left\{ \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} - \mu)]} - \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} + \mu)]} + 1 \right\}, \\
 n_5 &= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \frac{s|\mathbf{p}| - \mu_5}{E_{\mathbf{p},s}} \left\{ \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} - \mu)]} + \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} + \mu)]} - 1 \right\}, \quad (5.47)
 \end{aligned}$$

where $E_{\mathbf{p},s} = \sqrt{m^2 + (|\mathbf{p}| - s\mu_5)^2}$ are the eigenenergies for the Hamiltonian with chiral modification. When computing the net fermion number density, we will simply drop the vacuum contribution, i.e., the last term in the first line of Eq. (5.47). In Fig. 11 we compare n in Eq. (5.47) with

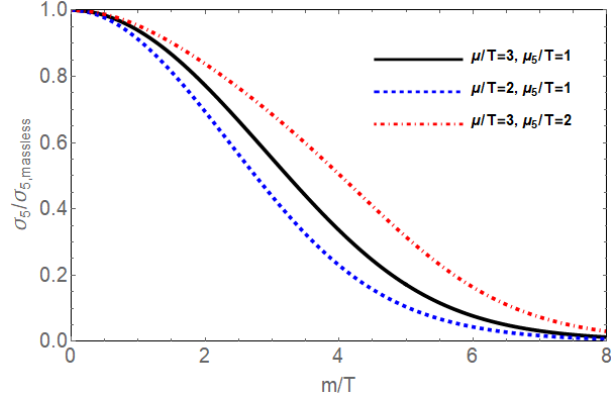


Figure 10: Mass dependence of the CSE coefficient σ_5 . Here we plot the ratio of the one calculated using Eq. (5.44) and the massless one is $\sigma_{5,\text{massless}} = \mu/(2\pi)^2$. The particles' mass m and chemical potentials μ , μ_5 are normalized by the temperature T .

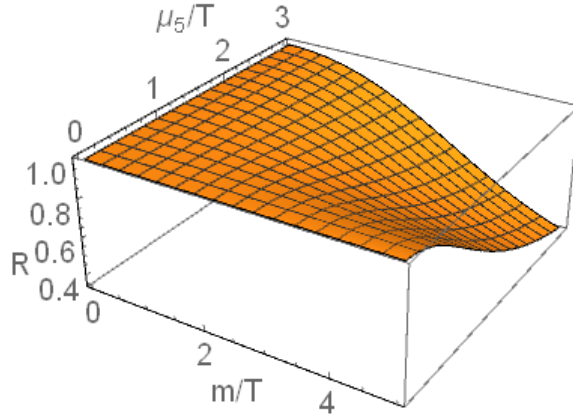


Figure 11: Ratio of the net fermion number density n of the result in (5.47) from chiral quantization and the semi-classical result in (5.41). Here we have fixed the chemical potential $\mu/T = 3$ and plotted the dependence on m and μ_5 .

the semi-classical result in Eq. (5.41). We find that they coincide with each other when the mass m or the chiral chemical potential μ_5 is not too large compared with the temperature. When we have a large m or μ_5 , the semi-classical result overestimates the number density because the ratio is smaller than 1.

Meanwhile, we compare the axial-charge density n_5 calculated using Eq. (5.47) with the one in Eq. (5.44) in Fig. 12. Here in the calculation we have dropped the vacuum contribution. We find that in the massless limit they do not agree with each other. Fortunately, we can attribute the difference to the vacuum contribution. The vacuum part for n_5 in Eq. (5.47) will be divergent for

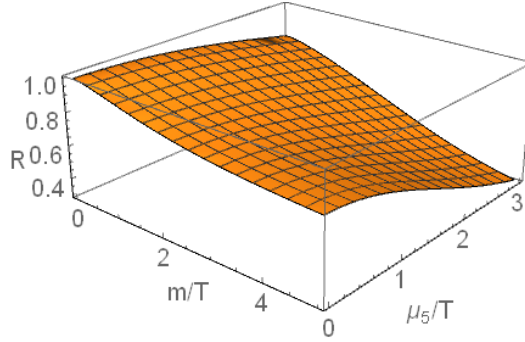


Figure 12: Ratio of the axial-charge density n_5 of the result (5.47) and the semi-classical result in (5.44). Here we have fixed the chemical potential $\mu/T = 3$ and plotted the dependence on m and μ_5 .

non-zero mass, but for the massless case it has finite value

$$\Delta n_{5,vac} = \frac{\mu_5^3}{3\pi^2}. \quad (5.48)$$

Taking this into account, the results in Eq. (5.47) agree with (5.46) in the massless limit. But in general the semi-classical results over-estimate the axial-charge density.

Note that the effects of electromagnetic fields are not included in the chiral-quantization description. In order to derive the CME or CSE, one needs to use the semi-classical method to derive the first order in \hbar , while the results in subsection III B only serve as the zeroth-order solutions. However, note that when the chemical potentials μ and μ_5 appear in the Dirac equation, the kinetic equation (2.36) of the Wigner function will also receive additional terms which are related to μ and μ_5 . So the present semi-classical discussions in Sec. IV need to be repeated for finite μ and μ_5 . In this thesis, the method of chiral quantization will only serve for comparing with the semi-classical method in Sec. IV, and thus the chiral effects will not be discussed using this method.

3. Results in magnetic field

In the presence of a constant magnetic field, fermions are quantized according to the Landau levels. We have derived the Wigner function in subsection III C, where the vector components read

$$\begin{aligned}
\mathcal{V}^0 &= (p^0 + \mu) \sum_{n=0} \Lambda_+^{(n)}(p_T) V_n + (p^0 + \mu) \sum_{n>0} \frac{p^z}{\sqrt{(p^z)^2 + 2nB_0}} \Lambda_-^{(n)}(p_T) A_n, \\
\mathcal{V}^x &= p^x \sum_{n>0} \frac{2nB_0}{p_T^2} \left[V_n - \frac{\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} A_n \right] \Lambda_+^{(n)}(p_T), \\
\mathcal{V}^y &= p^y \sum_{n>0} \frac{2nB_0}{p_T^2} \left[V_n - \frac{\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} A_n \right] \Lambda_+^{(n)}(p_T), \\
\mathcal{V}^z &= (p^z - \mu_5) \Lambda^{(0)}(p_T) V_0 + p^z \sum_{n>0} \left[V_n - \frac{\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} A_n \right] \Lambda_+^{(n)}(p_T) \\
&\quad + \sum_{n>0} \left[\sqrt{(p^z)^2 + 2nB_0} A_n - \mu_5 V_n \right] \Lambda_-^{(n)}(p_T), \tag{5.49}
\end{aligned}$$

The distributions are assumed to take their thermal equilibrium forms,

$$\begin{aligned}
V_n &\equiv \frac{2}{(2\pi)^3} \sum_s \delta \left\{ (p^0 + \mu)^2 - [E_{p^z s}^{(n)}]^2 \right\} \\
&\quad \times \left\{ \theta(p^0 + \mu) \frac{1}{1 + \exp(\beta p^0)} + \theta(-p^0 - \mu) \left[\frac{1}{1 + \exp(-\beta p^0)} - 1 \right] \right\}, \tag{5.50}
\end{aligned}$$

$$A_n \equiv \frac{2}{(2\pi)^3} \sum_s s \delta \left\{ (p^0 + \mu)^2 - [E_{p^z s}^{(n)}]^2 \right\} \tag{5.51}$$

$$\times \left\{ \theta(p^0 + \mu) \frac{1}{1 + \exp(\beta p^0)} + \theta(-p^0 - \mu) \left[\frac{1}{1 + \exp(-\beta p^0)} - 1 \right] \right\}, \tag{5.52}$$

and

$$\begin{aligned}
V_0 &= \frac{2}{(2\pi)^3} \delta \left\{ (p^0 + \mu)^2 - [E_{p^z}^{(0)}]^2 \right\} \\
&\quad \times \left\{ \theta(p^0 + \mu) \frac{1}{1 + \exp(\beta p^0)} + \theta(-p^0 - \mu) \left[\frac{1}{1 + \exp(-\beta p^0)} - 1 \right] \right\}. \tag{5.53}
\end{aligned}$$

Here the eigenenergies are given by $E_{p^z}^{(0)} = \sqrt{m^2 + (p^z)^2}$ for the lowest Landau level and $E_{p^z s}^{(n)} = \sqrt{m^2 + \left[\sqrt{(p^z)^2 + 2nB_0} - s\mu_5 \right]^2}$ for the higher Landau levels with $n > 0$. These eigenenergies are analytically derived from the Dirac equation in subsection III C. Note that these distributions are independent of the transverse momentum p^x and p^y , thus we observe that \mathcal{V}^x is anti-symmetric with respect to p^x . When integrating over d^4p , the component \mathcal{V}_x gives zero, which means there is no current along the x direction. Meanwhile, the current along the y direction vanishes for a similar reason: \mathcal{V}^x is anti-symmetric with respect to p^y . The net fermion number density and current along

the magnetic field are non-vanishing, thus the current can be parametrized as shown in Eq. (5.40), with

$$\begin{aligned}
n &= \sum_{n=0} \int d^4p (p^0 + \mu) \Lambda_+^{(n)}(p_T) V_n, \\
\sigma_\chi &= \frac{1}{B_0} \int d^4p (p^z - \mu_5) \Lambda^{(0)}(p_T) V_0 + \frac{1}{B_0} \sum_{n>0} \int d^4p p^z \left[V_n - \frac{\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} A_n \right] \Lambda_+^{(n)}(p_T).
\end{aligned} \tag{5.54}$$

Here we have dropped the terms of $\Lambda_-^{(n)}(p_T)$ because according to Eq. (B11), these terms vanish when taking an integration over \mathbf{p}_T . With the help of Eq. (B11), $\Lambda_+^{(n)}(p_T)$ can be integrated out, which gives the density of states for the Landau levels. Furthermore, we find that the higher Landau levels $n > 0$ do not contribute to σ_χ because they are anti-symmetric with respect to p^z . Inserting the distributions (5.52) and (5.53) into Eq. (5.54), we finally obtain

$$\begin{aligned}
n &= \frac{B_0}{(2\pi)^2} \int dp^z \sum_{n,s} \left\{ \frac{1}{1 + \exp \left[\beta \left(E_{p^z s}^{(n)} - \mu \right) \right]} - \frac{1}{1 + \exp \left[\beta \left(E_{p^z s}^{(n)} + \mu \right) \right]} + 1 \right\}, \\
\sigma_\chi &= \frac{1}{(2\pi)^2} \int dp^z \frac{p^z - \mu_5}{E_{p^z}^{(0)}} \left\{ \frac{1}{1 + \exp \left[\beta \left(E_{p^z}^{(0)} - \mu \right) \right]} + \frac{1}{1 + \exp \left[\beta \left(E_{p^z}^{(0)} + \mu \right) \right]} - 1 \right\}. \tag{5.55}
\end{aligned}$$

Here σ_χ can be analytically computed,

$$\begin{aligned}
\sigma_\chi &= -\frac{1}{(2\pi)^2 \beta} \left\{ \ln \left[1 + \exp \left(-\beta E_{p^z}^{(0)} + \beta \mu \right) \right] + \ln \left[1 + \exp \left(-\beta E_{p^z}^{(0)} - \beta \mu \right) \right] \right\} \Big|_{-\Lambda}^{\Lambda} \\
&\quad - \frac{1}{(2\pi)^2} \sqrt{m^2 + (p^z - \mu_5)^2} \Big|_{-\Lambda}^{\Lambda} \\
&= \frac{\mu_5}{2\pi^2}, \tag{5.56}
\end{aligned}$$

which agrees with the massless results. The result shows that the CME is independent of mass [22]. In Fig. 13 we plot the dependence of the net fermion number n on magnetic field strength. Here the x -axis is the field strength, where we considered a large region from 0 to $20 T^2$. Note that if we take $T = 100$ MeV, then $20 T^2 \sim 10 m_\pi^2$ is similar with the maximum field strength in Pb+Pb collisions at the LHC energy. In Fig. 13 we compute the ratio between n in Eq. (5.55) and the one in Eq. (5.47). The ratio reaches 1 in the weak-field limit, as desired because the result in Eq. (5.47) is obtained when the magnetic field vanishes. Four parameter configurations are taken into account: 1) $m/T = 1$, $\mu/T = 2$, and $\mu_5/T = 0$, which represents a chirally symmetric system, 2) $m/T = 1$, $\mu/T = 2$, and $\mu_5/T = 1$, which represents a system with chiral imbalance, 3) $m/T = 1$, $\mu/T = 3$, and $\mu_5/T = 1$, which can show the effect of chemical potential by comparing with case

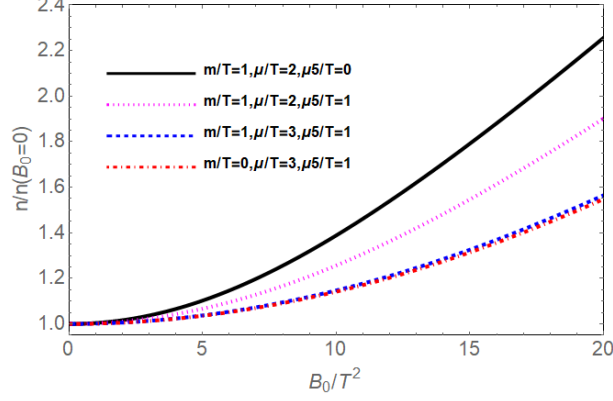


Figure 13: Magnetic-field dependence of the net fermion number density n . Here we compute the ratio of the one in Eq. (5.55), which is derived via Landau quantization, and the one in Eq. (5.47), which is derived via chiral quantization. Several configurations of m , μ , and μ_5 are taken into account.

2), and 4) $m/T = 0$, $\mu/T = 3$, and $\mu_5/T = 1$, which is a system of massless fermions. From Fig. 13 we observe that the chemical potentials distinctly affect the ratio while the mass does not.

Analogous to the vector component, for the axial-vector part, we firstly list all the relevant components of the Wigner function,

$$\begin{aligned}
\mathcal{A}^0 &= (p^z - \mu_5)V_0\Lambda^{(0)}(p_T) + \sum_{n>0} \left[\sqrt{(p^z)^2 + 2nB_0}A_n - \mu_5V_n \right] \Lambda_+^{(n)}(p_T) \\
&\quad + p^z \sum_{n>0} \left[V_n - \frac{\mu_5}{\sqrt{(p^z)^2 + 2nB_0}}A_n \right] \Lambda_-^{(n)}(p_T), \\
\mathcal{A}^z &= (p^0 + \mu)V_0\Lambda^{(0)}(p_T) + (p^0 + \mu) \sum_{n>0} V_n\Lambda_-^{(n)}(p_T) \\
&\quad + (p^0 + \mu)p^z \sum_{n>0} A_n \frac{1}{\sqrt{(p^z)^2 + 2nB_0}} \Lambda_+^{(n)}(p_T).
\end{aligned} \tag{5.57}$$

Here the x - and y -components are not listed because they do not contribute to the axial-vector current for reasons of symmetry. The current \mathbb{N}_5^μ is then parametrized as Eq. (5.43), where the axial-charge density n_5 and coefficient σ_5 for the CSE are given by

$$\begin{aligned}
n_5 &= \int d^4p (p^z - \mu_5)V_0\Lambda^{(0)}(p_T) + \sum_{n>0} \int d^4p \left[\sqrt{(p^z)^2 + 2nB_0}A_n - \mu_5V_n \right] \Lambda_+^{(n)}(p_T), \\
\sigma_5 &= \frac{1}{B_0} \int d^4p (p_0 + \mu)V_0\Lambda^{(0)}(p_T).
\end{aligned} \tag{5.58}$$

Using the distributions in Eqs. (5.52), (5.53) and the property of $\Lambda_+^{(n)}(p_T)$ in Eq. (B11), the

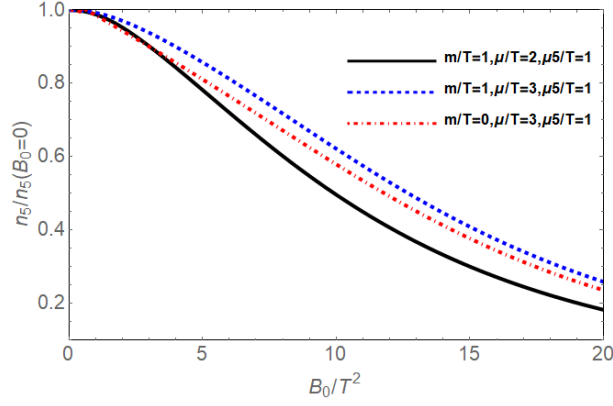


Figure 14: Magnetic-field dependence of n_5 . Here we plot the ratio between the axial-charge density calculated via Eq. (5.59) and the one via Eq. (5.47). Several parameter configurations are considered.

integration over \mathbf{p}_T can be performed and we obtain

$$\begin{aligned}
n_5 &= \frac{B_0}{(2\pi)^2} \int dp^z \frac{p^z - \mu_5}{E_{p^z}^{(0)}} \left\{ \frac{1}{1 + \exp[\beta(E_{p^z}^{(0)} - \mu)]} + \frac{1}{1 + \exp[\beta(E_{p^z}^{(0)} + \mu)]} - 1 \right\} \\
&+ \frac{B_0}{(2\pi)^2} \sum_{n>0} \sum_s \int dp^z \frac{s\sqrt{(p^z)^2 + 2nB_0} - \mu_5}{E_{p^z s}^{(n)}} \\
&\quad \times \left[\frac{1}{1 + \exp[\beta(E_{p^z s}^{(n)} - \mu)]} + \frac{1}{1 + \exp[\beta(E_{p^z s}^{(n)} + \mu)]} - 1 \right], \\
\sigma_5 &= \frac{1}{(2\pi)^2} \int dp^z \left\{ \frac{1}{1 + \exp[\beta(E_{p^z}^{(0)} - \mu)]} - \frac{1}{1 + \exp[\beta(E_{p^z}^{(0)} + \mu)]} + 1 \right\}. \quad (5.59)
\end{aligned}$$

In general these quantities cannot be analytically done. In Fig. 14 we compare the axial-charge density n_5 in a magnetic field, i.e., Eq. (5.59), with the one without magnetic field, i.e., Eq. (5.47). We considered three cases: 1) $m/T = 1$, $\mu/T = 2$, $\mu_5 = 1$, 2) $m/T = 1$, $\mu/T = 3$, $\mu_5 = 1$, and 3) $m/T = 0$, $\mu/T = 3$, $\mu_5 = 1$. Comparing these cases we find that both the mass m and the chemical potential μ affects the ratio. In the weak-field limit, the ratio reaches 1, which indicates that Eq. (5.59) agree with Eq. (5.47) in this limit. But when the magnetic field increases, we find the axial-charge density decreases for fixed m , μ , and μ_5 .

On the other hand, σ_5 is independent of the chiral chemical potential μ_5 , which can be proven by a shift of the integration variable $p^z \rightarrow p^z + \mu_5$. However, in the semi-classical result (5.44), σ_5 depends on μ_5 , which is conflict with the one in Eq. (5.59). In Fig. 15 we computed the ration between σ_5 calculated via Eq. (5.59) with the semi-classical result (5.44). This figure shows that these two agree with each other if μ_5 and m are not too large. The ratio is smaller than 1, which means the semi-classical results overestimate the chiral separation effect. In the limit $m \ll T$, we

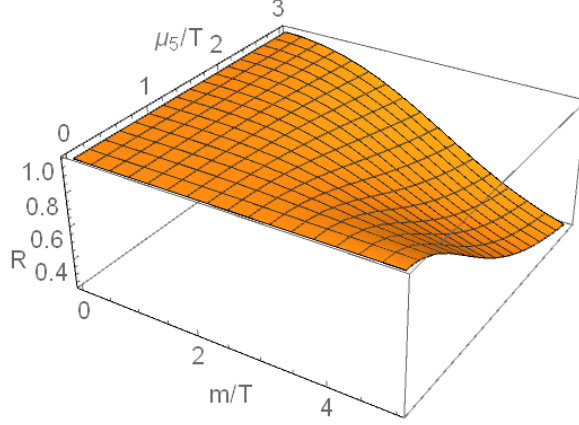


Figure 15: Ratio between the CSE coefficient σ_5 calculated via Eq. (5.59) and the semi-classical result in (5.44). Here the chemical potential $\mu/T = 3$.

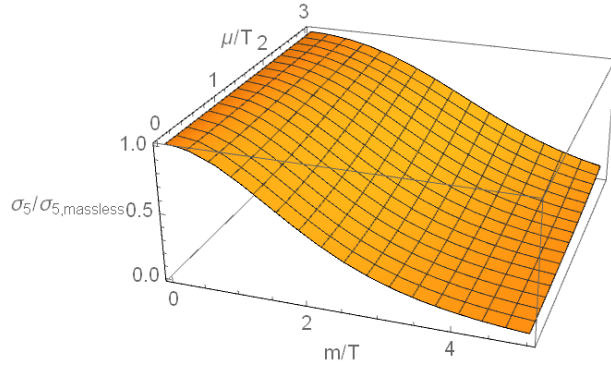


Figure 16: Mass and chemical-potential dependence of σ_5 . Here we calculate the ratio between the one from Landau quantization in Eq. (5.59) and the massless one $\sigma_{5,\text{massless}} = \mu/(2\pi)^2$. Both of these two results are independent of μ_5 .

can expand σ_5 into a series of βm . The leading two terms read

$$\sigma_5 = \frac{\mu}{2\pi^2} - \frac{(\beta m)^2}{(2\pi)^2 \beta} \int_0^\infty dp \frac{e^{\beta(p-\mu)}(e^{2\beta\mu} - 1)(e^{2\beta p} - 1)}{p [1 + e^{\beta(p+\mu)}]^2 [1 + e^{\beta(p-\mu)}]^2} + \mathcal{O}[(\beta m)^4]. \quad (5.60)$$

The leading-order term agrees with the massless result. In Fig. 16 we plotted the mass and chemical-potential dependence of σ_5 . This shows that a finite mass suppresses the chiral separation effect.

Since we have already derived the vector and the axial-vector currents, now we can compute the average spin polarization in a magnetic field. Using the net fermion number density n and the

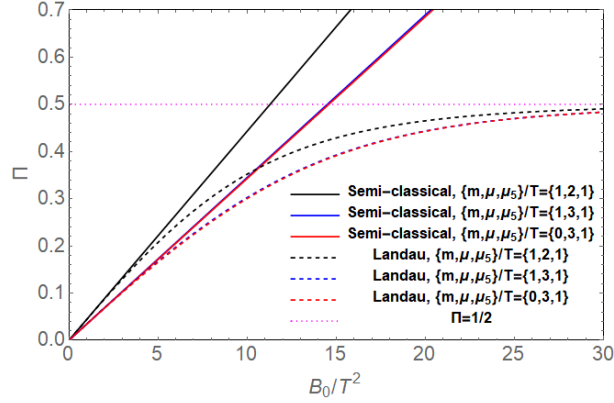


Figure 17: Average spin polarization as function of field strength. Here we have used the solid lines for the results from the semi-classical expansion, where the net fermion number density n and coefficient σ_5 are given in Eqs. (5.41) and (5.44). The dashed lines are results calculated via Landau quantization, where n and σ_5 are given in Eqs. (5.55) and (5.59). The dotted line is the case for fully polarized particles, with the average polarization $\Pi = \frac{1}{2}$.

coefficient σ_5 , the average polarization can be expressed as

$$\Pi = \frac{\hbar \sigma_5 B_0}{2 n}, \quad (5.61)$$

where the factor $\frac{\hbar}{2}$ is the unit of spin and $\frac{\hbar}{2}\sigma_5 B_0$ is the total spin-polarization density in a constant magnetic field. In Fig. 17 we plot the average polarization Π as a function of the magnetic field strength B_0 , where both the semi-classical result and Landau-quantized result are calculated. We find that the average polarization of the semi-classical result grows to infinity when B_0 increases. This is because in the semi-classical result n and σ_5 are independent of B_0 and thus the average polarization is linear in B_0 . Meanwhile, the result from Landau quantization has the upper limit $1/2$. This is because in sufficiently strong magnetic fields, fermions will stay in the lowest Landau level. As we discussed in subsection III C, the spins in the lowest Landau level are fixed. So the system reaches a fully-polarized state if the field strength is large enough and the average spin polarization will reach $\hbar/2$.

D. Energy-momentum tensor and spin tensor

1. Semi-classical results

The canonical energy-momentum tensor and the spin-angular-momentum tensor in the quantum field theory are given in Eq. (5.16). In the semi-classical description, \mathcal{V}^μ and \mathcal{A}^μ are given by Eq.

(4.67). Inserting them into Eq. (5.16) we obtain

$$\begin{aligned}
\mathbb{T}_{\text{mat}}^{\mu\nu} &= \int d^4p p^\mu p^\nu \left[V\delta(p^2 - m^2) - \frac{\hbar}{2} F_{\alpha\beta} \Sigma^{\alpha\beta} \delta'(p^2 - m^2) \right] \\
&\quad + \frac{\hbar}{2} \partial_{x\alpha} \int d^4p p^\nu \Sigma^{\mu\alpha} \delta(p^2 - m^2) + \frac{\hbar}{2} \int d^4p \Sigma^{\mu\alpha} F_\alpha{}^\nu \delta(p^2 - m^2), \\
\mathbb{S}_{\text{mat}}^{\rho,\mu\nu} &= \frac{1}{2} \int d^4p (p^\rho \Sigma^{\mu\nu} + \rho^\mu \Sigma^{\nu\rho} - \rho^\nu \Sigma^{\mu\rho}) \delta(p^2 - m^2) \\
&\quad - \frac{\hbar}{2} \int d^4p (p^\rho F^{\mu\nu} + \rho^\mu F^{\nu\rho} - \rho^\nu F^{\mu\rho}) V \delta'(p^2 - m^2).
\end{aligned} \tag{5.62}$$

Note that in the classical limit $\hbar \rightarrow 0$ the energy-momentum tensor is symmetric with respect to its indices and agrees with the classical formula. But the leading-order term can be non-symmetric. The spin-angular-momentum tensor $\mathbb{S}_{\text{mat}}^{\rho,\mu\nu}$ has a straightforward connection to the axial-vector current $\mathbb{S}_{\text{mat}}^{\rho,\mu\nu} = -\frac{1}{2} \epsilon^{\rho\mu\nu\lambda} \mathbb{N}_{5,\lambda}$. Since the axial-vector current has been discussed in the previous subsection, in this subsection we only focus on the energy-momentum tensor.

Similar to the previous subsection, we have no idea about the equilibrium dipole-moment tensor. Thus we adopt the specific solution in Eqs. (4.111) and (4.114), which smoothly recovers the massless limit. Inserting the dipole-moment tensor into $\mathbb{T}_{\text{mat}}^{\mu\nu}$, we obtain

$$\begin{aligned}
\mathbb{T}_{\text{mat}}^{\mu\nu} &= \int d^4p p^\mu p^\nu \left[V\delta(p^2 - m^2) - \hbar \tilde{F}^{\alpha\beta} \frac{p_\alpha u_\beta}{u \cdot p} A \delta'(p^2 - m^2) \right] \\
&\quad + \frac{\hbar}{2} \epsilon^{\mu\alpha\beta\gamma} \partial_{x\alpha} u_\gamma \int d^4p p^\nu p_\beta \frac{1}{u \cdot p} A \delta(p^2 - m^2) \\
&\quad - \frac{\hbar}{2} \epsilon^{\mu\alpha\beta\gamma} F_\beta{}^\nu u_\gamma \int d^4p \frac{p_\alpha}{u \cdot p} A \delta(p^2 - m^2).
\end{aligned} \tag{5.63}$$

Here we assume that the distributions take their equilibrium form in (5.37), which depends on $u \cdot p$ in the fluid rest frame u^μ . The energy-momentum tensor can be parametrized as

$$\mathbb{T}_{\text{mat}}^{\mu\nu} = u^\mu u^\nu \epsilon - (g^{\mu\nu} - u^\mu u^\nu) P + \hbar \left(u^\mu \tilde{F}^{\nu\beta} u_\beta + u^\nu \tilde{F}^{\mu\beta} u_\beta \right) \xi_B, \tag{5.64}$$

where

$$\begin{aligned}
\epsilon &= \int d^4p (u \cdot p)^2 V \delta(p^2 - m^2), \\
P &= \frac{1}{3} \int d^4p [(u \cdot p)^2 - m^2] V \delta(p^2 - m^2), \\
\xi_B &= \frac{1}{2} \int d^4p A \delta(p^2 - m^2),
\end{aligned} \tag{5.65}$$

are the energy density, the pressure, and the coefficient for energy flux along the magnetic field. In the massless limit, the coefficient ξ_B takes the following form [23]

$$\xi_{B,\text{massless}} = \frac{\mu\mu_5}{2\pi^2}. \tag{5.66}$$

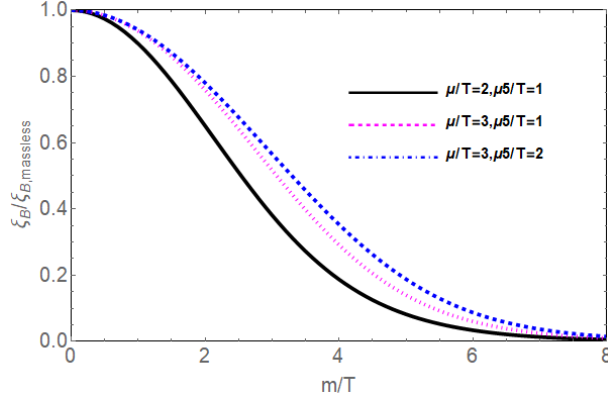


Figure 18: Mass dependence of the ratio between the semi-classical ξ_B in Eq. (5.65) and massless one in Eq. (5.66). We have taken several chemical-potential configurations, 1) $\mu/T = 2$, $\mu_5/T = 1$ (solid line), 2) $\mu/T = 3$, $\mu_5/T = 1$ (solid line).

In Fig. 18 we compare the semi-classical results (5.65) for massive fermions with ξ_B for massless fermions (5.66). We consider a wide region for the value of mass and find that the energy flux decreases for a larger mass. Several configurations of chemical potentials are considered. In the massless limit, the semi-classical results coincide with the massless ones for all the cases considered.

2. Results from chiral quantization

If we adopt the chiral quantization description, the energy-momentum tensor can be calculated at zeroth order in \hbar using the results in Eq. (3.145). Since the formula is not Lorentz-covariant, we take the fluid velocity $u^\mu = (1, 0, 0, 0)^T$. Then the energy density is given by

$$\epsilon = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s E_{\mathbf{p},s} \left\{ \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} - \mu)]} + \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} + \mu)]} \right\}, \quad (5.67)$$

and the pressure is

$$P = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \sum_s \frac{|\mathbf{p}|^2}{3E_{\mathbf{p},s}} \left(1 - s \frac{\mu_5}{|\mathbf{p}|} \right) \left\{ \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} - \mu)]} + \frac{1}{1 + \exp[\beta(E_{\mathbf{p},s} + \mu)]} \right\}, \quad (5.68)$$

while other components of the energy-momentum tensor vanish. In Figs. 19 and 20 we compare the semi-classical results with the ones from the chiral quantization. We find that in the limit $\mu_5 \rightarrow 0$ these results agree with each other, while for large μ_5 , the semi-classical ones would over-estimate both the energy density and the pressure.

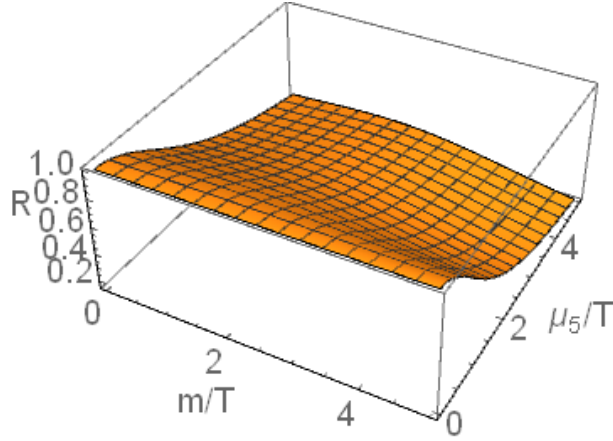


Figure 19: Ratio of energy density calculated using the result in Eq. (5.67) and the semi-classical result in Eq. (5.65), as function of the mass m and the chiral chemical potential μ_5 . Here the vector chemical potential is fixed to $\mu/T = 3$.

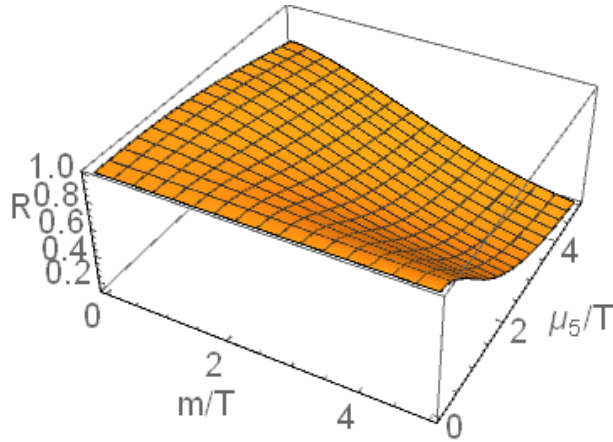


Figure 20: Ratio of pressure calculated using Eq. (5.68) and the semi-classical result in Eq. (5.65), as function of the mass m and the chiral chemical potential μ_5 . Here the vector chemical potential is fixed to $\mu/T = 3$.

3. Results in magnetic field

In the presence of a magnetic field, the Wigner function has been computed in Eq. (3.213). The constant-magnetic field assumption breaks the Lorentz symmetry because an electric field will appear if we perform a Lorentz boost along a direction which is not parallel to the magnetic field. Here we take the observer's frame as $u^\mu = (1, 0, 0, 0)^T$. In this frame, the energy density is $\mathbb{T}_{\text{mat}}^{00}$ and the pressure is $\mathbb{T}_{\text{mat}}^{ii}$, $i = 1, 2, 3$. We first compute the components of $\mathbb{T}_{\text{mat}}^{\mu\nu}$ with the help of Eq.

(5.16) and the solutions in the presence of a magnetic field, which are given by Eq. (3.213),

$$\begin{aligned}
\mathbb{T}_{\text{mat}}^{00} &= \int d^4p (p_0 + \mu)^2 \sum_{n=0} V_n \Lambda_+^{(n)}(p_T), \\
\mathbb{T}_{\text{mat}}^{11} = \mathbb{T}_{\text{mat}}^{22} &= \int d^4p \sum_{n>0} nB_0 \left[V_n - \frac{\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} A_n \right] \Lambda_+^{(n)}(p_T), \\
\mathbb{T}_{\text{mat}}^{33} &= \int d^4p p^z (p^z - \mu_5) V_0 \Lambda^{(0)}(p_T) \\
&\quad + \int d^4p (p^z)^2 \sum_{n>0} \left[V_n - \frac{\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} A_n \right] \Lambda_+^{(n)}(p_T), \\
\mathbb{T}_{\text{mat}}^{03} &= \int d^4p p^z (p_0 + \mu) V_0 \Lambda^{(0)}(p_T),
\end{aligned} \tag{5.69}$$

where we have dropped all terms which vanish when integrating over d^4p . All the unlisted components are zero. Here the distributions V_n and A_n are assumed to take their equilibrium form in Eqs. (5.52) and (5.53). In a Lorentz-covariant form, the energy-momentum tensor can be generalized as follows

$$\mathbb{T}_{\text{mat}}^{\mu\nu} = \epsilon u^\mu u^\nu - P_\perp (g^{\mu\nu} - u^\mu u^\nu + b^\mu b^\nu) + P_\parallel b^\mu b^\nu + \hbar B_0 u^\mu b^\nu \xi_B, \tag{5.70}$$

where b^μ is the direction of the magnetic field. The energy density ϵ , the transverse pressure P_\perp , the longitudinal pressure P_\parallel , and the coefficients ξ_B are respectively given by

$$\begin{aligned}
\epsilon &= \frac{B_0}{(2\pi)^2} \int dp^z \sum_{n,s} E_{p^z s}^{(n)} \left\{ \frac{1}{1 + \exp \left[\beta \left(E_{p^z s}^{(n)} - \mu \right) \right]} + \frac{1}{1 + \exp \left[\beta \left(E_{p^z s}^{(n)} + \mu \right) \right]} - 1 \right\}, \\
P_\perp &= \frac{(B_0)^2}{(2\pi)^2} \int dp^z \sum_{n>0,s} \frac{n}{E_{p^z s}^{(n)}} \left[1 - \frac{s\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} \right] \\
&\quad \times \left\{ \frac{1}{1 + \exp \left[\beta \left(E_{p^z s}^{(n)} - \mu \right) \right]} + \frac{1}{1 + \exp \left[\beta \left(E_{p^z s}^{(n)} + \mu \right) \right]} - 1 \right\}, \\
P_\parallel &= \frac{B_0}{(2\pi)^2} \int dp^z \frac{p^z (p^z - \mu_5)}{E_{p^z}^{(0)}} \left\{ \frac{1}{1 + \exp \left[\beta \left(E_{p^z}^{(0)} - \mu \right) \right]} + \frac{1}{1 + \exp \left[\beta \left(E_{p^z}^{(0)} + \mu \right) \right]} - 1 \right\} \\
&\quad + \frac{B_0}{(2\pi)^2} \int dp^z \sum_{n>0,s} \frac{(p^z)^2}{E_{p^z s}^{(n)}} \left[1 - \frac{s\mu_5}{\sqrt{(p^z)^2 + 2nB_0}} \right] \\
&\quad \times \left\{ \frac{1}{1 + \exp \left[\beta \left(E_{p^z s}^{(n)} - \mu \right) \right]} + \frac{1}{1 + \exp \left[\beta \left(E_{p^z s}^{(n)} + \mu \right) \right]} - 1 \right\}, \\
\xi_B &= \frac{1}{4\pi^2} \int dp^z p^z \left\{ \frac{1}{1 + \exp \left[\beta \left(E_{p^z}^{(0)} - \mu \right) \right]} - \frac{1}{1 + \exp \left[\beta \left(E_{p^z}^{(0)} + \mu \right) \right]} + 1 \right\}.
\end{aligned} \tag{5.71}$$

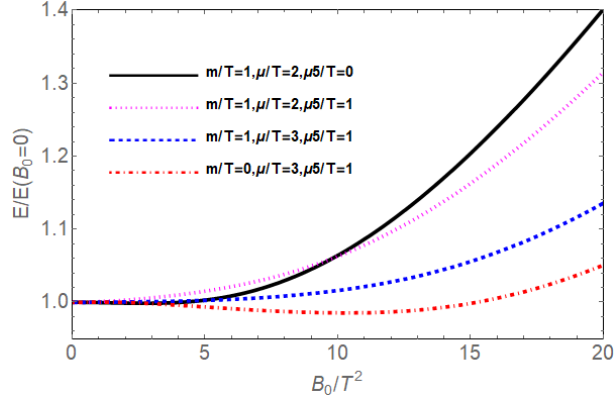


Figure 21: Energy densities as functions of the magnetic field strength.

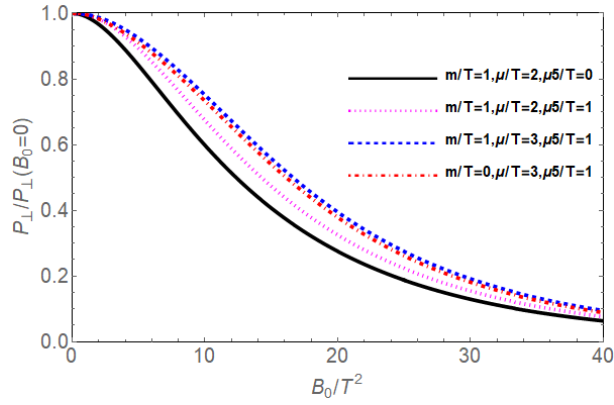


Figure 22: Transverse pressures as functions of the magnetic field strength.

Here in these formula we have kept all vacuum contribution, e.g. the last term “1” in curly braces. But in practice one should neglect the vacuum part otherwise the results will diverge. In Figs. 21, 22, and 23 we compare the energy density and pressure from Landau quantization with those from chiral quantization in Eqs. (5.67) and (5.68). We find that in the weak-field limit, these two approaches coincide with each other. When the field strength increases, the transverse pressure decreases while the longitudinal pressure increases. The decrease of the transverse pressure is attributed to the lowest Landau level: in a strong field, the fermions will more likely to stay in the lowest Landau level, which does not contribute to the transverse pressure. The field-strength dependence of the energy density is a little complicated, for some parameter configuration, the ratio first decreases and then increases when the field strength grows.

On the other hand, the chiral chemical potential also induced an energy flux along the magnetic field direction. Note that if we adopt the Landau quantization, the energy flux $\mathbb{T}_{\text{mat}}^{03}$ exists but the momentum density $\mathbb{T}_{\text{mat}}^{30}$ vanishes, which results in a non-symmetric $\mathbb{T}_{\text{mat}}^{\mu\nu}$. But in the semi-classical

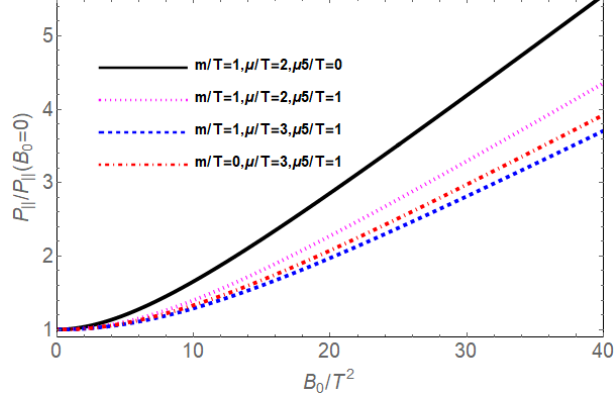


Figure 23: Longitudinal pressures as functions of the magnetic field strength.

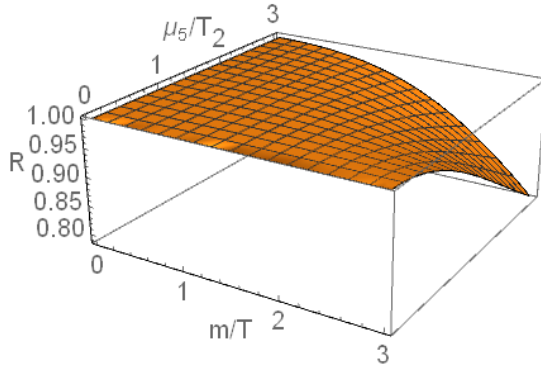


Figure 24: Ratio between the coefficient ξ_B calculated using Eq. (5.71) and the one in Eq. (5.65), as a function of the mass m and μ_5 . The chemical potential is fixed $\mu/T = 3$.

results, $\mathbb{T}_{\text{mat}}^{03}$ and $\mathbb{T}_{\text{mat}}^{30}$ take the same value and $\mathbb{T}_{\text{mat}}^{\mu\nu}$ is symmetric. In Fig. 24 we compute the ratio between the coefficient ξ_B in the Landau-quantization calculation in Eq. (5.71) and the one from the semi-classical method in Eq. (5.65). From this figure we observe that these two results coincide in a wide parameter regime. When m/T and μ_5/T are significantly large, the semi-classical method again overestimates ξ_B .

E. Pair production

1. In Sauter-type field

In this subsection we will focus on pair-production processes in the presence of an electric field. As shown in Eq. (D10), the number of pairs will be expressed in terms of the Wigner function.

First we focus on the Sauter-type field, using Eqs. (3.226), (3.227), and (D10), we obtain

$$n_{\text{pair}}(t, \mathbf{p}) = \frac{m_T \chi_2(t, \mathbf{p}) + p^z \chi_1(t, \mathbf{p})}{2E_{\mathbf{p}}} C_1 \left(\mathbf{p} - \int_{t_0}^t dt' E(t') \mathbf{e}^z \right) + \text{const.}, \quad (5.72)$$

where the transverse mass $m_T = \sqrt{m^2 + \mathbf{p}_T^2}$ and energy $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$. Here we have dropped all the spatial dependence and we further assumed that the distribution function takes the equilibrium form,

$$C_1(\mathbf{p}) = \frac{2}{(2\pi)^3} \left[\frac{1}{1 + \exp[\beta(E_{\mathbf{p}} - \mu)]} + \frac{1}{1 + \exp[\beta(E_{\mathbf{p}} + \mu)]} - 1 \right], \quad (5.73)$$

In the pair spectrum (5.72), the constant term is expected to cancel with the vacuum contribution, thus here we take the constant term to be $\frac{2}{(2\pi)^3}$.

For the Sauter-type field, the coefficients $\chi_2(t, \mathbf{p})$ and $\chi_1(t, \mathbf{p})$ can be numerically computed from Eq. (3.240). The solution is proven to depend on the transverse mass m_T and the longitudinal kinetic momentum p^z . For simplicity we take m_T as the unit of energy and all the quantities are described by dimensionless variables, such as the temperature $\tilde{T} = T/m_T$, the chemical potential $\tilde{\mu} = \mu/m_T$, and the longitudinal kinetic momentum $\tilde{p}^z = p^z/m_T$. As an example, we numerically calculate the evolution of the pair spectrum for a Sauter-type field with peak value $E_0/m_T^2 = 3$ and width $\tau = 2/m_T$ in a thermal system with $T/m_T = 1$ and $\mu/m_T = 2$. We take three typical moments in time $t = -3\tau, 0$, and 3τ and the spectrum of pairs is given in Fig. 25. Since the electric field will be less than 1% for $t < -3\tau$, the initial condition for this system at $t = -\infty$ would be similar to the one for $t = -3\tau$ because before this moment the electric field is not strong enough to generate any effect. After the time $t = 3\tau$, the functions $\chi_1(t, \mathbf{p})$ and $\chi_2(t, \mathbf{p})$ still evolve with time, but the pair spectrum stays unchanged, which means there will be no more pair production after $t = 3\tau$. In Fig. (25), we observe that the spectrums for $t = 0$ and $t = 3\tau$ have two peaks. We identify the peaks on the right-hand-side as contribution from initially existing particles which are accelerated by the electric field. Since the shift of the longitudinal momentum is given by $\int_{-\infty}^t dt' E(t')$, we can obtain that this shift is $E_0\tau$ at moment $t = 0$, and $2E_0\tau$ at moment $t = 3\tau$, which agrees with the locations of peaks. And the peaks on the left-hand-side in the spectra for $t = 0$ and $t = 3\tau$ can be identified as the contribution of the pair-production.

Now we compute the total number of pairs generated in the Sauter-type field. Here we have four variables in this case: the peak value E_0 and the width τ for the Sauter-type field, and the temperature T and the chemical potential μ for the initial thermal equilibrium state. First we fix the thermodynamical quantities $T/m_T = 1$ and $\mu/m_T = 3$ and study the dependence with respect to

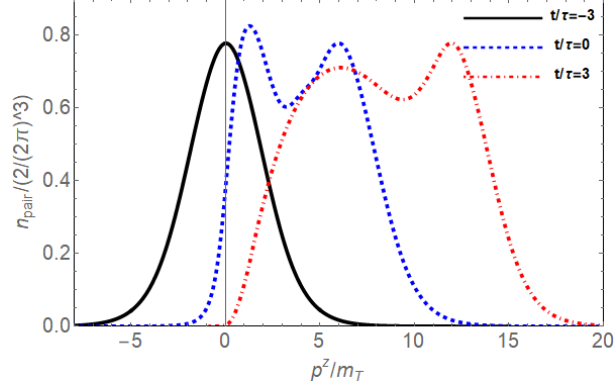


Figure 25: Spectrum of pairs at times $t = -3\tau$ (solid line), $t = 0$ (dashed line), and $t = 3\tau$ (dot-dashed line) for a thermal system with $T/m_T = 1$ and $\mu/m_T = 2$ in a Sauter-type field $E(t) = E_0 \cosh^{-2}(t/\tau)$ with peak value $E_0/m_T^2 = 3$ and width $\tau = 2/m_T$. Here the transverse mass $m_T = \sqrt{m^2 + \mathbf{p}_T^2}$ is taken to be the energy unit. The x -axis is the dimensionless longitudinal kinetic momentum, while the y -axis is the density of pairs in phase space.

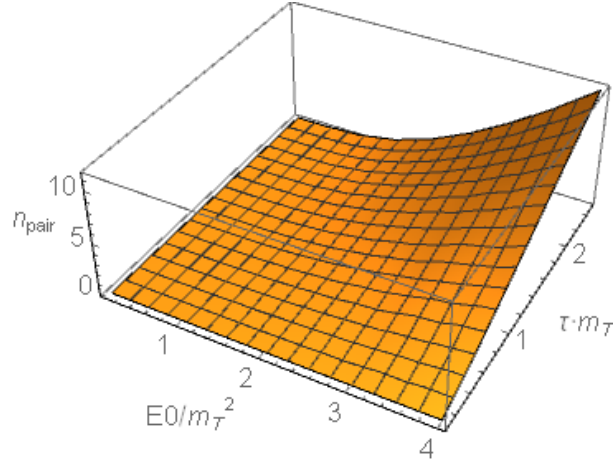


Figure 26: Pair number produced in a Sauter-type field in a thermal system with $T/m_T = 1$ and $\mu/m_T = 3$.

E_0 and τ . In Fig. 26 we plot the total number of produced pairs as function of E_0 and τ . We find that more pairs are generated for a larger peak value E_0 or a longer lifetime τ , as expected.

Then we take $E_0/m_T^2 = 3$ and $\tau = 2/m_T$ and study the dependence with respect to T and μ . The results are shown in Fig. 27. We find that the total number of produced pairs reaches a maximum value at the point $\mu = T = 0$. When the temperature increases, or the chemical potential increases, the pair-production is suppressed. This agrees with our expectation because in high- T or high- μ system the quantum states are more likely to be occupied and the production of new pairs is suppressed due to the Pauli exclusion principle.

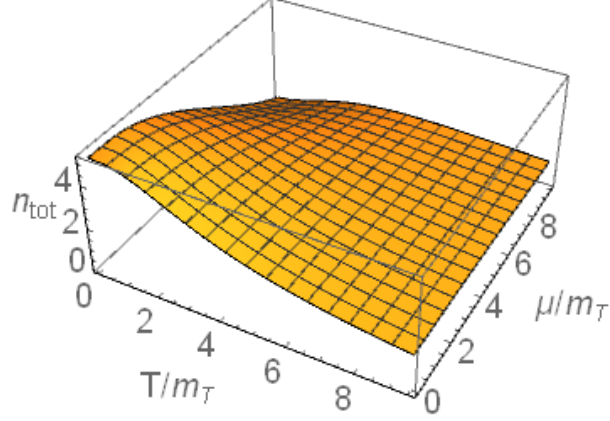


Figure 27: Total pair number generated in a Saute-type field $E(t) = E_0 \cosh^{-2}(t/\tau)$ with peak value $E_0/m_T^2 = 3$ and width $\tau = 2/m_T$ as a function of the thermodynamical quantities T and μ .

2. In parallel electromagnetic fields

In subsection III D we have analytically computed the Wigner function in the case of a constant electric field. Meanwhile we also analytically computed the Wigner function in the case of constant parallel electromagnetic fields in subsection III E. Since the results in parallel electromagnetic fields reduce to the ones in a pure electric field, we will skip the pair production in a constant electric field and directly focus on the process in the presence of both electric and magnetic fields. The spectrum of pairs is related to the Wigner function as shown in Eq. (D10). Analogous to this equation, for a system in a constant magnetic field, the eigenenergies are replaced by the Landau energy levels $E_{p^z}^{(n)} = \sqrt{m^2 + (p^z)^2 + 2nB_0}$, and the number of pairs in the n -th Landau level is

$$n^{(n)}(t, \mathbf{p}) = \frac{m\mathcal{F}^{(n)}(t, \mathbf{p}) + \mathbf{p} \cdot \mathcal{V}^{(n)}(t, \mathbf{p})}{2E_{p^z}^{(n)}} + \text{const..} \quad (5.74)$$

Here $\mathcal{F}^{(n)}$ and $\mathcal{V}^{(n)}$ are components of the Wigner function where the superscript (n) labels contributions from the n -th Landau levels. Then the pair-production rate in the n -th Landau level is calculated via

$$\frac{d}{dt}n^{(n)}(t) = \frac{1}{2} \frac{d}{dt} \int d^3\mathbf{p} \frac{m\mathcal{F}^{(n)}(t, \mathbf{p}) + \mathbf{p} \cdot \mathcal{V}^{(n)}(t, \mathbf{p})}{E_{p^z}^{(n)}}. \quad (5.75)$$

Employing the results in Eq. (3.270) into the pair-production rate, we obtain

$$\begin{aligned} \frac{d}{dt}n^{(n)} &= \frac{1}{2} \frac{d}{dt} \int d^3\mathbf{p} \left[\frac{\eta^{(n)}}{E_{p^z}^{(n)}} \sqrt{\frac{E_0}{2}} d_2 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) + \frac{p^z}{E_{p^z}^{(n)}} d_1 \left(\eta^{(n)}, \sqrt{\frac{2}{E_0}} p^z \right) \right] \\ &\quad \times C_1^{(n)}(p^z - E_0 t) \Lambda_+^{(n)}(p_T), \end{aligned} \quad (5.76)$$

where $C_1^{(n)}$ is given in (3.240). The integration over \mathbf{p}_T can be performed using relation (B11). We also replace the kinetic momentum p^z by the canonical one $q^z = p^z - E_0 t$. Then we obtain the pair-production rate in a multi-particle system,

$$\frac{d}{dt}n^{(n)} = \int dq^z \left[1 - f^{(+)(n)}(q^z) - f^{(-)(n)}(q^z) \right] \frac{d}{dt}n_{\text{vac}}^{(n)}(t, q^z), \quad (5.77)$$

where $\frac{d}{dt}n_{\text{vac}}^{(n)}(t, q^z)$ is the pair-production rate in vacuum for given quantum numbers n and p^z ,

$$\begin{aligned} \frac{d}{dt}n_{\text{vac}}^{(n)}(t, q^z) = & - \left(1 - \frac{\delta_{n0}}{2} \right) \frac{B_0 E_0}{(2\pi)^2} \frac{d}{dq^z} \left\{ \frac{\eta^{(n)}}{E_{q^z+E_0t}^{(n)}} \sqrt{\frac{E_0}{2}} d_2 \left[\eta^{(n)}, \sqrt{\frac{2}{E_0}} (q^z + E_0 t) \right] \right. \\ & \left. + \frac{q^z + E_0 t}{E_{q^z+E_0t}^{(n)}} d_1 \left[\eta^{(n)}, \sqrt{\frac{2}{E_0}} (q^z + E_0 t) \right] \right\}. \end{aligned} \quad (5.78)$$

Summing Eq. (5.77) over all Landau levels yields the total pair-production rate. Here we notice that the pair production in the lowest Landau level is suppressed by the factor $1 - \frac{\delta_{n0}}{2}$. This is because the spin is not degenerate for the lowest Landau level while is two-fold degenerate for the higher Landau levels. In Eq. (5.77), the distribution of fermions and anti-fermions appears in the square bracket, which suppresses the pair-production due to the Pauli exclusion principle. Moreover, if $f^{(+)(n)}(q^z) + f^{(-)(n)}(q^z) > 1$, the pair-production rate will have the opposite sign with the one in the vacuum. This case corresponds to a system where almost all fermion and anti-fermion states are already occupied. Thus pair annihilation is more likely to happen than the pair creation. In a thermal equilibrium system with zero chemical potential and non-zero temperature, the suppression factor is $\tanh(\beta E_{q^z}^{(n)}/2)$, which suppresses the production of pairs with small energies. This factor agrees with Ref. [103]. Later on we will discuss the pair-production in the presence of finite chemical potential and temperature.

First we consider the pair-production in vacuum. The distribution functions $f^{(\pm)(n)}(q^z)$ are set to zero and the pair-production rate in Eq. (5.77) can be calculated using the method of integrating by parts. The asymptotic behavior of functions $d_{1,2}$ are given in Eqs. (B27), (B30), thus the results read

$$\frac{d}{dt}n_{\text{vac}}^{(n)} = \left(1 - \frac{\delta_{n0}}{2} \right) \frac{B_0 E_0}{2\pi^2} \exp \left(-\pi \frac{m^2 + 2nB_0}{E_0} \right), \quad (5.79)$$

and the total pair-production rate is

$$\frac{d}{dt} \sum_{n=0}^{\infty} n_{\text{vac}}^{(n)} = \frac{B_0 E_0}{2\pi^2} \exp \left(-\pi \frac{m^2}{E_0} \right) \coth \left(\pi \frac{B_0}{E_0} \right), \quad (5.80)$$

which agrees with previous results of Refs. [80, 105, 106]. We find that this rate is enhanced for a large magnetic field comparing to the one in a pure electric field.

For more general distribution functions $f^{(\pm)(n)}(q^z)$, the pair-production rate (5.77) requires a numerical calculation. However we find that the function d_2 oscillate strongly for large q^z , which makes the numerical integration slowly converging. In order to solve this problem, we separate the production rate into a vacuum part and a thermal contribution, where the thermal part is given by

$$\frac{d}{dt}n_{\text{thermal}}^{(n)} = - \int dq^z \left[f^{(+)(n)}(q^z) + f^{(-)(n)}(q^z) \right] \frac{d}{dt}n_{\text{vac}}^{(n)}(t, q^z). \quad (5.81)$$

In general the distributions converge quickly at large q^z . So the numerical calculation for this thermal part is easier than directly calculate Eq. (5.77). In order to show the thermal effect we consider a thermal equilibrium system where the distribution functions are given by

$$f^{(\pm)(n)}(q^z) = \frac{1}{1 + \exp \left[\beta \left(E_{q^z}^{(n)} \mp \mu \right) \right]}, \quad (5.82)$$

We introduce a function to describe the ratio between the thermal contribution and the vacuum part. The ratio depends on several dimensionless parameters, the time $\tilde{t} = m^{(n)}t$, the electric field strength $\tilde{E}_0 = E_0 / (m^{(n)})^2$, the temperature $\tilde{T} = T / m^{(n)}$ and the chemical potential $\tilde{\mu} = \mu / T$. And the pair-production rate in the n -th Landau level is given by

$$\frac{d}{dt}n^{(n)} = \left[1 + r \left(\tilde{t}, \tilde{E}_0, \tilde{T}, \tilde{\mu} \right) \right] \frac{d}{dt}n_{\text{vac}}^{(n)}. \quad (5.83)$$

Now we choose the moment $\tilde{t} = 0$ when the kinetic momentum equals the canonical momentum. This reflects pair-production around thermal equilibrium. In Figs. 28 and 29 we plot the dependence with respect to \tilde{E}_0 of $r \left(0, \tilde{E}_0, \tilde{T}, \tilde{\mu} \right)$ for various temperature and chemical-potential configurations. We observe that the ratio lies between $\{-1, 0\}$, which means that the pair production is suppressed in thermal system. We also observe that the suppression, which was described by the absolute value of $r \left(\tilde{t}, \tilde{E}_0, \tilde{T}, \tilde{\mu} \right)$, is larger for a larger chemical potential or a larger temperature. This agrees with our understanding about the Pauli exclusion principle: the states are more likely to be occupied in a system with high T or high μ . And the existing particles will prohibit the generation of pairs with the same quantum number.

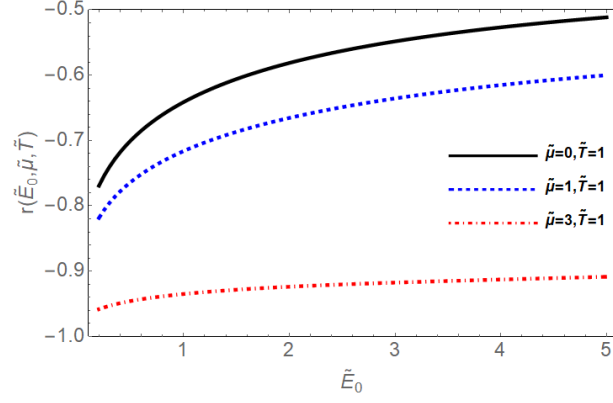


Figure 28: Ratio between the thermal contribution and the vacuum part. Here we fix the temperature $\tilde{T} = 1$ and take three typical values for the chemical potential, 1) $\tilde{\mu} = 0$ (solid line) for a system without net fermion number, 2) $\tilde{\mu} = 1$ (dashed line) for a system with medium chemical potential, and 3) $\tilde{\mu} = 3$ (dot-dashed line) for a system with significantly larger chemical potential.

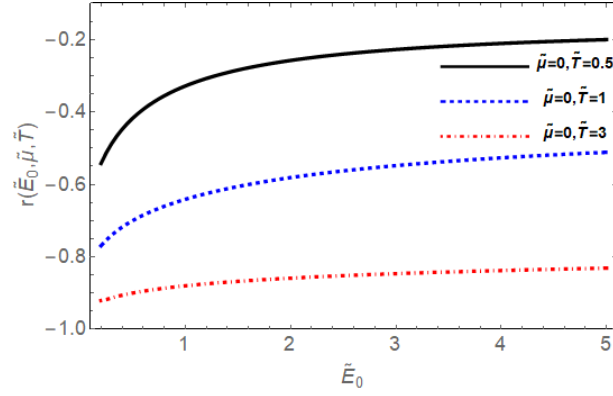


Figure 29: Ratio between the thermal contribution and the vacuum part. Here we considered a system without net fermion number $\tilde{\mu} = 0$. Low temperature $\tilde{T} = 0.5$ (solid line), medium temperature $\tilde{T} = 1$ (dashed line) and high temperature $\tilde{T} = 3$ (dot-dashed line) have been taken into account.

VI. SUMMARY

In this thesis we focused on the Wigner-function approach for spin-1/2 particles and used this approach to study the chiral effects and the pair-production in the presence of an electromagnetic field. The Wigner function is defined as a semi-distribution function in the phase space. The Wigner function is a complex valued 4×4 matrix, which can be expanded in terms of the generators of the Clifford algebra Γ_i . The expansion coefficients are identified to be scalar, pseudoscalar, vector, axial-vector, and tensor according to their transformation properties under Lorentz transformations and parity. After integrating over momentum, they can be related to various kinds of macroscopic physical quantities such as the fermion current, the spin polarization and the magnetic dipole moment. Thus they can be interpreted as densities in the phase space.

Since the Wigner function is constructed from the Dirac field, we can obtain the kinetic equations for the Wigner function from the Dirac equation. In this thesis, we derived the Dirac form equation, which is linear in the derivative operators. Meanwhile, we also obtained the Klein-Gordon form equation, which is second order in the operators. These equations are then decomposed in terms of Γ_i , as we did for the Wigner function itself, and provide several partial differential equations. Fortunately, the decomposed equations are not independent from each other. Eliminating the redundant equations, we obtained two possible ways for computing the Wigner function in massive case. The redundancy is based on the fact that the vector and axial-vector components \mathcal{V}^μ and \mathcal{A}^μ of the Wigner function can be expressed in terms of the scalar, pseudo-scalar, and tensor components \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$, or vice versa. Thus one approach to solve the system is to take \mathcal{V}^μ and \mathcal{A}^μ as basis functions and focus on their on-shell conditions. Meanwhile, the other approach is to take \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$ as basis functions. Carrying out an expansion in \hbar , which is known as the semi-classical expansion, we obtained a general solution of the Wigner function up to first order in \hbar . The two approaches mentioned above are proven to provide the same result and thus are equivalent to each other. The final solution only has four independent degrees of freedom, which is proven through an eigenvalue analysis. At order \hbar , the normal mass-shell $p^2 - m^2 = 0$ is shifted by the spin-magnetic coupling.

We also reproduced the Wigner function for the massless case through the semi-classical expansion. In the massless case, the fermions can be separated into two groups according to their chirality. Using \mathcal{V}^μ and \mathcal{A}^μ , we constructed the LH and RH currents, which are then solved up to order \hbar . The remaining components \mathcal{F} , \mathcal{P} , and $\mathcal{S}^{\mu\nu}$ are proportional to the particle mass and thus vanish in the massless limit. In this thesis, we have found a straightforwardly relation between the

massless Wigner function and the massive one. This indicates that our massive results are more general than the Chiral Kinetic Theory.

In this thesis we have discussed several analytically solvable cases. In the following three cases, analytical single-particle wavefunctions were derived from the Dirac equation, which are then used to compute the Wigner function. We only listed the leading-order contributions in spatial gradients for the Wigner functions, but we have found a viable approach for deriving higher order contributions.

1. Plane-wave quantization: In this case the Dirac equation does not contain any external interaction and thus has free plane-wave solutions. The results obtained in this approach are the cornerstone for the method of the semi-classical expansion: they serve as the solutions to the zeroth order in \hbar , while higher-order ones automatically appear order by order.
2. Chiral quantization: In this case we introduced μ and μ_5 as constant variables. These variables have a contribution $\mu\hat{N} + \mu_5\hat{N}_5$ to the total Hamiltonian, where \hat{N} and \hat{N}_5 are operators for the net fermion number and the axial-charge number. In the massless limit, we identify μ as the vector chemical potential and μ_5 as the chiral chemical potential. We emphasize that the chiral chemical potential is not well-defined in the massive case because its conjugate quantity, the axial-charge, is not conserved. Thus, in the massive case, μ_5 is just a variable which describes the spin imbalance. The modified Hamiltonian leads to a new Dirac equation, which can be solved when μ and μ_5 are constants. The Wigner function is then constructed from the single-particle wavefunctions. However, since the presence of μ and μ_5 changes the Dirac equation, the kinetic equations for the Wigner function also need more modifications. Moreover, we cannot obtain the single-particle wavefunction for a general space-/time-dependent μ or μ_5 . Thus the method of the chiral quantization only serve as a cross-check for the method of semi-classical expansion. Here electromagnetic fields are still not included.
3. Landau quantization: Based on case 2, we further introduced a constant magnetic field. The energy levels are then described by the Landau levels with modifications from μ and μ_5 . This case allows us to explicitly study phenomena in a magnetic field, such as the CME, the CSE, and the anomalous energy flux. Meanwhile, since the field changes the energy spectrum, we find that the net fermion number density, the energy density, and the pressure depend on the strength of the magnetic field.

Furthermore, based on the plane-wave quantization, we carried out a semi-classical expansion in

\hbar . The Wigner function is then solved up to $\mathcal{O}(\hbar)$. Note that the method of the semi-classical expansion can be used for an arbitrary space-time dependent electromagnetic field. In this method, we put μ and μ_5 into the thermal equilibrium distributions instead of into the Hamiltonian and make the specific assumption that all fermions are longitudinally polarized. These treatments are naive extensions of those in the massless case. Numerical calculations show that the net fermion number density and the axial-charge density obtained in this way coincide with the ones from the chiral quantization, if μ_5 and m are comparable or smaller than the temperature. Meanwhile, energy densities and pressures in these two methods also show coincidences.

Besides the above three analytically solvable cases, we also discussed the Wigner function in an electric field. Based on the results from the plane-wave quantization and the ones from the Landau quantization, we obtained the Wigner function in the presence of a constant electric field via a dynamical treatment. Pair production is then computed, which proves to be enhanced by a parallel magnetic field, and suppressed by a temperature and a chemical potential. The suppression of pair production in a thermal system is attributed to the Pauli exclusion principle.

The method of the semi-classical expansion provides a general way to compute spin corrections. At zeroth order in \hbar , we reproduced the classical spinless Boltzmann equation. At order \hbar , spin corrections, such as the energy shift by the spin-magnetic coupling, arise automatically. In this thesis we have obtained a general Boltzmann equation and a general BMT equation, which determine the evolution of the particle distribution and the spin polarization density, respectively. However, collisions between particles are not yet included. Following the method of moments, we can extend the semi-classical results to a hydrodynamical description, which would be topic of future work. Meanwhile, in the method of the semi-classical expansion, electromagnetic fields appear at order \hbar , which work in the weak-field limit. However in the initial stage of heavy-ion collisions, the magnetic field strength is comparable with m_π^2 . Even in later stages, fluctuations of electromagnetic fields can also reach m_π^2 . In strong-laser physics, the electromagnetic fields are significantly strong but there are nearly no particles. Whether the semi-classical expansion can be used in these cases needs more carefully discussions. The study of a constant magnetic field in this thesis may serve as a starting point for the kinetic theory in a strong background field.

Another possible extension of this thesis is axial-charge production. In the presence of parallel electric and magnetic fields, the electric field will excite fermion pairs from vacuum and the newly produced pairs are polarized by the magnetic field. As a result, the pair-production in the lowest Landau level contributes to the axial-charge density. The real-time axial-charge production of massive particles in a thermal background has not yet been computed. And the Wigner function

approach in this thesis may provide a possible approach towards this goal.

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Appendix A: GAMMA MATRICES

In this section we list the gamma matrices used throughout this paper and discuss their properties. The gamma matrices γ^μ should satisfy the following anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}_4, \quad (\text{A1})$$

where $g^{\mu\nu}$ is the Minkowski metric. In principle there are many ways to construct the gamma matrices and the above anti-commutation relation is the only constraint. We can find one 4×4 representation, in which the gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad (\text{A2})$$

with $\sigma^\mu = (\mathbb{I}_2, \sigma^1, \sigma^2, \sigma^3)$ and $\bar{\sigma}^\mu = (\mathbb{I}_2, -\sigma^1, -\sigma^2, -\sigma^3)$. Here \mathbb{I}_2 is a 2×2 unit matrix and $\sigma^{\{1,2,3\}}$ are the Pauli matrices. This representation is known as the Weyl or chiral representation, which is convenient to deal with massless particles. The Weyl representation is used through out this thesis.

The Hermitian conjugate of the gamma matrices γ^μ satisfy the following relation,

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \quad (\text{A3})$$

The anti-commutation relation (A1) indicates that any product of several gamma matrices can be expressed in terms of the anti-symmetric combinations $\gamma^{[\mu_1 \mu_2 \dots \mu_n]}$ with $n = 0, 1, 2, 3, 4$. The maximum n is 4, which equals the number of γ^μ according to the Pigeonhole principle [135]. When taking the Hermitian conjugate, these anti-symmetric combinations satisfy

$$(\gamma^{[\mu_1 \mu_2 \dots \mu_n]})^\dagger = \begin{cases} \gamma^0 \gamma^{[\mu_1 \mu_2 \dots \mu_n]} \gamma^0, & n = 0, 1, 4, \\ -\gamma^0 \gamma^{[\mu_1 \mu_2 \dots \mu_n]} \gamma^0, & n = 2, 3. \end{cases} \quad (\text{A4})$$

Thus we define

$$\begin{aligned} \gamma^5 &\equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \\ \sigma^{\mu\nu} &\equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu], \end{aligned} \quad (\text{A5})$$

and then any combination of γ^μ can be written in terms of $\Gamma_i = \{\mathbb{I}_4, i\gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \frac{1}{2}\sigma^{\mu\nu}\}$, which are 16 matrices in total. These matrices are also known as the independent generators of the Clifford algebra, which automatically satisfy

$$(\Gamma_i)^\dagger = \gamma^0 \Gamma_i \gamma^0, \quad (\text{A6})$$

and will be used to expand our Wigner function.

In Sec. II we find it useful to calculate the commutators and anti-commutators between $\sigma^{\mu\nu}$ and the generators of the Clifford algebra Γ_i . Here we list all results

$$\begin{aligned}
[\sigma^{\mu\nu}, \mathbb{I}_4] &= 0, \\
[\sigma^{\mu\nu}, -i\gamma^5] &= 0, \\
[\sigma^{\mu\nu}, \gamma^\sigma] &= 2i(g^{\nu\sigma}\gamma^\mu - g^{\mu\sigma}\gamma^\nu), \\
[\sigma^{\mu\nu}, \gamma^5\gamma^\sigma] &= 2i(g^{\nu\sigma}\gamma^5\gamma^\mu - g^{\mu\sigma}\gamma^5\gamma^\nu), \\
[\sigma^{\mu\nu}, \sigma^{\sigma\rho}] &= 2i(g^{\mu\rho}\sigma^{\nu\sigma} + g^{\nu\sigma}\sigma^{\mu\rho} - g^{\mu\sigma}\sigma^{\nu\rho} - g^{\nu\rho}\sigma^{\mu\sigma}), \\
\{\sigma^{\mu\nu}, \mathbb{I}_4\} &= 2\sigma^{\mu\nu}, \\
\{\sigma^{\mu\nu}, -i\gamma^5\} &= \epsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}, \\
\{\sigma^{\mu\nu}, \gamma^\alpha\} &= 2\epsilon^{\mu\nu\alpha\beta}\gamma^5\gamma_\beta, \\
\{\sigma^{\mu\nu}, \gamma^5\gamma^\alpha\} &= 2\epsilon^{\mu\nu\alpha\beta}\gamma_\beta, \\
\{\sigma^{\mu\nu}, \sigma^{\sigma\rho}\} &= 2g^{\mu[\sigma}g^{\rho]\nu} + 2i\epsilon^{\mu\nu\sigma\rho}\gamma^5.
\end{aligned} \tag{A7}$$

On the other hand, the matrices Γ_i can be constructed from the Pauli matrices by taking tensor products. The tensor-product form would be useful when calculating the Wigner function from the quantized field operator,

$$\begin{aligned}
\mathbb{I}_4 &= \mathbb{I}_2 \otimes \mathbb{I}_2, \\
\gamma^5 &= -\sigma^3 \otimes \mathbb{I}_2, \\
\gamma^0 &= \sigma^1 \otimes \mathbb{I}_2, \\
\boldsymbol{\gamma} &= i\sigma^2 \otimes \boldsymbol{\sigma}, \\
\gamma^5\gamma^0 &= -i\sigma^2 \otimes \mathbb{I}_2, \\
\gamma^5\boldsymbol{\gamma} &= -\sigma^1 \otimes \boldsymbol{\sigma}, \\
\sigma^{0j} &= i\gamma^0\gamma^j = -i\sigma^3 \otimes \sigma^j, \\
\sigma^{jk} &= i\gamma^j\gamma^k = \epsilon^{jkl}\mathbb{I}_2 \otimes \sigma^l.
\end{aligned} \tag{A8}$$

The tensor product of two matrices, also known as the Kronecker product and denoted by \otimes , is a generalization of the outer product for two vectors. For example, considering two matrices A and

B , the tensor product $A \otimes B$ is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}, \quad (\text{A9})$$

where a_{ij} is the element of A in the i -th row and j -th column. We find it useful to emphasize the mixed-product property,

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD), \quad (\text{A10})$$

where A , B , C , and D are matrices with proper size such that the matrix products make sense. When taking the Hermitian conjugate, we have the following property

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger. \quad (\text{A11})$$

These properties are used when analytically deriving the Wigner function in subsections III A, III B, and III C.

Appendix B: AUXILIARY FUNCTIONS

When calculating the Wigner function in electromagnetic fields, we define some useful auxiliary functions. In this appendix we will list these functions and briefly discuss their properties.

When calculating the Wigner function in a magnetic field, we need to calculate the following integral

$$I_{ij}(p^x, p^y) \equiv \int dy' \exp(ip^y y') \phi_i \left(p^x, \frac{y'}{2} \right) \phi_j \left(p^x, -\frac{y'}{2} \right), \quad (\text{B1})$$

with ϕ_n being the eigenstates of the harmonic oscillator defined in Eq. (3.160). Using the explicit form of ϕ_n one can calculate the integral I_{ij} and obtains

$$\begin{aligned} I_{mn}(p^x, p^y) &= \sqrt{\frac{B_0}{\pi}} \frac{1}{\sqrt{2^{m+n} m! n!}} \exp\left(-\frac{p_T^2}{B_0}\right) \int dy' \exp\left[-B_0 \left(\frac{y'}{2} - \frac{ip^y}{B_0}\right)^2\right] \\ &\times H_m \left[\sqrt{B_0} \left(\frac{1}{2}y' - \frac{ip^y}{B_0}\right) + \frac{p^x + ip^y}{\sqrt{B_0}} \right] H_n \left[-\sqrt{B_0} \left(\frac{1}{2}y' - \frac{ip^y}{B_0}\right) + \frac{p^x - ip^y}{\sqrt{B_0}} \right], \end{aligned} \quad (\text{B2})$$

where $p_T^2 \equiv (p^x)^2 + (p^y)^2$ is the transverse momentum squared. Here $H_n(x)$ are the Hermite polynomials, whose Taylor expansion reads

$$H_n(x+y) = \sum_{i=0}^n \frac{2^i y^i n!}{i!(n-i)!} H_{n-i}(x). \quad (\text{B3})$$

Then $I_{mn}(p^x, p^y)$ can be calculated by firstly expanding the Hermite polynomials around $\pm\sqrt{B_0}\left(\frac{1}{2}y' - \frac{ip^y}{B_0}\right)$, then using the symmetric property $H_n(-x) = (-1)^n H_n(x)$ and the following orthonormality condition

$$\begin{aligned} \sqrt{\frac{B_0}{\pi}} \int dy' \exp\left[-B_0\left(\frac{y'}{2} - \frac{ip^y}{B_0}\right)^2\right] H_{m-i}\left[\sqrt{B_0}\left(\frac{1}{2}y' - \frac{ip^y}{B_0}\right)\right] H_{n-j}\left[\sqrt{B_0}\left(\frac{1}{2}y' - \frac{ip^y}{B_0}\right)\right] \\ = 2^{m-i+1}(m-i)!\delta_{m-i,n-j}. \end{aligned} \quad (\text{B4})$$

Finally we obtain the result

$$\begin{aligned} I_{mn}(p^x, p^y) = \frac{1}{\sqrt{2^{m+n}m!n!}} \exp\left(-\frac{p_T^2}{B_0}\right) \\ \times \sum_{i=0}^m \sum_{j=0}^n \frac{2^{m+1+j}m!n!}{i!j!(n-j)!} \left(\frac{p^x + ip^y}{\sqrt{B_0}}\right)^i \left(\frac{p^x - ip^y}{\sqrt{B_0}}\right)^j (-1)^{m-i} \delta_{m-i,n-j}. \end{aligned} \quad (\text{B5})$$

If we take $m = n$, then it can be written in terms of the Laguerre polynomials

$$I_{nn}(p^x, p^y) = 2(-1)^n \exp\left(-\frac{p_T^2}{B_0}\right) L_n\left(\frac{2p_T^2}{B_0}\right), \quad (\text{B6})$$

where the Laguerre polynomials are defined as

$$L_n(x) = \sum_{i=0}^n \frac{(-1)^i n!}{i!i!(n-i)!} x^i. \quad (\text{B7})$$

For simplicity we define a set of new functions $\Lambda_{\pm}^{(n)}(p_T)$, which depend on the magnitude of transverse momentum p_T ,

$$\Lambda_{\pm}^{(n)}(p_T) \equiv (-1)^n \left[L_n\left(\frac{2p_T^2}{B_0}\right) \mp L_{n-1}\left(\frac{2p_T^2}{B_0}\right) \right] \exp\left(-\frac{p_T^2}{B_0}\right), \quad (\text{B8})$$

where $n > 0$ because $L_{n-1}(x)$ is not well-defined. Then $I_{mn}(p^x, p^y)$ can be related to these $\Lambda_{\pm}^{(n)}(p_T)$,

$$\begin{aligned} \frac{I_{nn} \pm I_{n-1,n-1}}{2} &= \Lambda_{\pm}^{(n)}(p_T), \\ \frac{I_{n,n-1} + I_{n-1,n}}{2} &= \frac{p^x \sqrt{2nB_0}}{p_T^2} \Lambda_+^{(n)}(p_T), \\ \frac{I_{n,n-1} - I_{n-1,n}}{2} &= \frac{ip^y \sqrt{2nB_0}}{p_T^2} \Lambda_+^{(n)}(p_T), \end{aligned} \quad (\text{B9})$$

where the last two lines can be checked by expanding $\Lambda_{\pm}^{(n)}(p_T)$ into a polynomial and comparing with the left-hand-side terms, which are calculated by Eq. (B5). For the case $n = 0$ we specially define

$$I_{00}(p_x, p_y) = \Lambda_{\pm}^{(0)}(p_T) \equiv 2 \exp\left(-\frac{p_T^2}{B_0}\right), \quad (\text{B10})$$

which is independent of the subscript \pm .

In the Wigner function, the functions $\Lambda_{\pm}^{(n)}(p_T)$ in Eqs. (B8), (B10) play roles of distribution with respect to transverse momentum. When integrating over transverse momentum \mathbf{p}_T , $\Lambda_+^{(n)}(p_T)$ give the density of states in the n -th Landau level, while $\Lambda_-^{(n)}(p_T)$ give zero for any $n > 0$,

$$\begin{aligned}\int \frac{d^2\mathbf{p}_T}{(2\pi)^2} \Lambda_+^{(n)}(p_T) &= \frac{B_0}{2\pi}, \\ \int \frac{d^2\mathbf{p}_T}{(2\pi)^2} \Lambda_-^{(n)}(p_T) &= 0, \quad (n \neq 0).\end{aligned}\tag{B11}$$

Both of $\Lambda_{\pm}^{(n)}$ are symmetric with respect to p^x and p^y because they only depend on the magnitude p_T . In Figs. 30 and 31 we plot the first four $\Lambda_{\pm}^{(n)}$. We find that all these functions converge to zero in the limit $p_T \rightarrow \infty$, which is ensured by the exponential term in their definitions (B8) and (B10). In the point $p_T = 0$, the functions $\Lambda_+^{(n)}$ have zero values for all $n > 0$, while $\Lambda_-^{(n)}(0) = 2(-1)^n$ oscillate between $\{-2, 2\}$. The oscillation of $\Lambda_-^{(n)}(0)$ is similar to the Runge's phenomenon, which occurs when using polynomial interpolation. In fact, $\Lambda_+^{(n)}$ plays the role of an interpolation function because numerically we can prove

$$f(p_T^2) = \lim_{B_0 \rightarrow 0} \left[\frac{1}{2} f^{(0)} \Lambda_+^{(0)}(p_T) + \sum_{n>0} f^{(n)} \Lambda_+^{(n)}(p_T) \right],\tag{B12}$$

where $f(x)$ is an arbitrary function and $f^{(n)}$ are the values of the function f at the points $2nB_0$. In the weak-magnetic field limit $B_0 \rightarrow 0$, the interpolation form on the right-hand-side reproduces the function $f(p_T^2)$. On the other hand, when $B_0 \rightarrow 0$ we also have

$$\lim_{B_0 \rightarrow 0} \left[\frac{1}{2} f^{(0)} \Lambda_-^{(0)}(p_T) + \sum_{n>0} f^{(n)} \Lambda_-^{(n)}(p_T) \right] = 0,\tag{B13}$$

which is numerically proven. In Fig. 32 we take an example $f(p_T^2) = 1/[1 + \exp(p_T^2/m^2 - 1)]$, where m is the particles rest mass. Here m plays the role of the energy unit. For convenience we take $B_0 = 0.01 m^2$ and truncate the sum at $n = 150$. We find that the interpolation result formed from $\Lambda_+^{(n)}(p_T)$ coincides with the original function, while the one formed from $\Lambda_-^{(n)}(p_T)$ coincides with zero. There is some disagreement in the large p_T^2 region, which is caused by the truncation at $n = 150$. If the sum is calculated without an upper limit of n , the results would agree with Eq. (B12).

When taking the derivative with respect to p^x , we have the following relations,

$$\begin{aligned}\hbar B_0 \partial_{p^x} \Lambda_+^{(n)}(p_T) &= -2p^x \Lambda_-^{(n)}(p_T), \\ \hbar B_0 \partial_{p^x} \Lambda_-^{(n)}(p_T) &= -2p^x \left(1 - \frac{2nB_0}{p_T^2} \right) \Lambda_+^{(n)}(p_T),\end{aligned}\tag{B14}$$

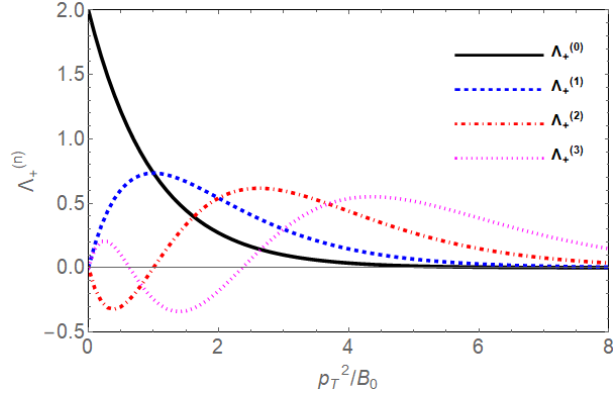


Figure 30: The first four functions of $\Lambda_+^{(n)}$.

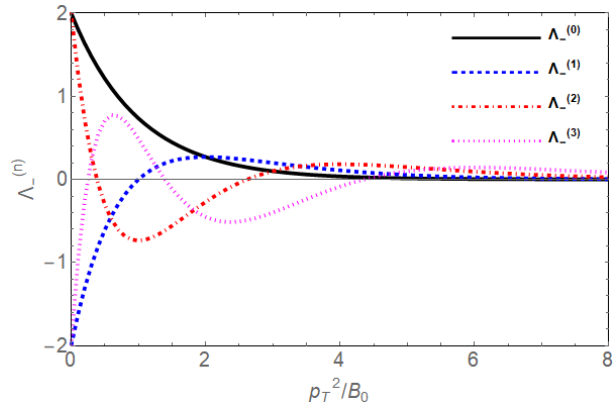


Figure 31: The first four functions of $\Lambda_-^{(n)}$.

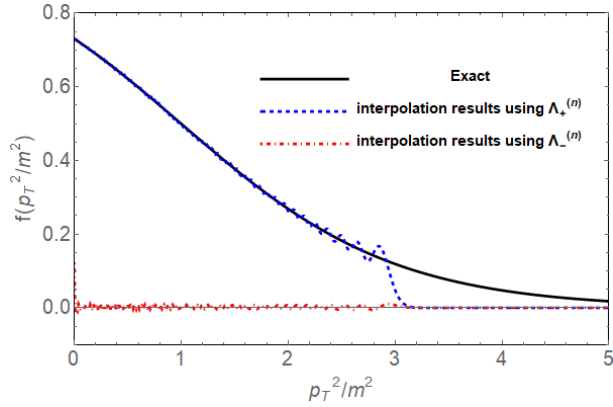


Figure 32: Numerical proof in the weak-field limit. For numerical convenience we take the particles rest mass as natural unit of energy or momentum, and set $B_0 = 0.01 m^2$. The sum over n is truncated at $n = 150$. The test function (solid line) coincides with the interpolation function, which is constructed from $\Lambda_+^{(n)}$ (dashed line), while the one constructed from $\Lambda_-^{(n)}$ (dot-dashed line) agrees with zero.

where the p^y -derivative can be derived by replacing $p^x \leftrightarrow p^y$. These relations help when calculating the Wigner function in parallel electromagnetic fields in subsection III E.

We define four set of basis vectors, which are 4-dimensional column vectors,

$$e_1^{(n)}(p_T) = \begin{pmatrix} \Lambda_+^{(n)}(p_T) \\ 0 \\ 0 \\ \Lambda_-^{(n)}(p_T) \end{pmatrix}, \quad e_2^{(n)}(p_T) = \begin{pmatrix} \Lambda_-^{(n)}(p_T) \\ 0 \\ 0 \\ \Lambda_+^{(n)}(p_T) \end{pmatrix},$$

$$e_3^{(n)}(\mathbf{p}_T) = \begin{pmatrix} 0 \\ p^x \\ p^y \\ 0 \end{pmatrix} \frac{\sqrt{2nB_0}}{p_T^2} \Lambda_+^{(n)}(p_T), \quad e_4^{(n)}(\mathbf{p}_T) \equiv \begin{pmatrix} 0 \\ -p^y \\ p^x \\ 0 \end{pmatrix} \frac{\sqrt{2nB_0}}{p_T^2} \Lambda_+^{(n)}(p_T). \quad (\text{B15})$$

The first two basis vectors only depend on the magnitude p_T of the transverse momentum, while $e_3^{(n)}(\mathbf{p}_T)$ and $e_4^{(n)}(\mathbf{p}_T)$ also depend on the direction of \mathbf{p}_T . Here the functions $\Lambda_{\pm}^{(n)}$ are defined in Eqs. (B8) and (B10). Note that when $n = 0$, the last two, $e_3^{(0)}(\mathbf{p}_T)$ and $e_4^{(0)}(\mathbf{p}_T)$ are not well-defined because they are zero vectors. At the same time, due to the fact that $\Lambda_+^{(0)} = \Lambda_-^{(0)}$, we have $e_1^{(0)}(p_T) = e_2^{(0)}(p_T)$. The basis vector $e_1^{(0)}(p_T)$ is properly normalized with respect to an inner product of transverse momentum \mathbf{p}_T ,

$$\int d^2\mathbf{p}_T e_1^{(0)T}(p_T) e_1^{(0)}(p_T) = 4\pi B_0. \quad (\text{B16})$$

At the same time, the basis vectors with $n > 0$ are orthogonal to $e_1^{(0)}(p_T)$ for the lowest Landau level,

$$\int d^2\mathbf{p}_T e_1^{(0)T}(p_T) e_i^{(n)}(\mathbf{p}_T) = 0, \quad (n > 0, i = 1, 2, 3, 4). \quad (\text{B17})$$

Meanwhile, we have the following orthonormality conditions,

$$\int d^2\mathbf{p}_T e_i^{(m)T}(\mathbf{p}_T) e_j^{(n)}(\mathbf{p}_T) = 2\pi B_0 \delta_{ij} \delta_{mn}, \quad (\text{B18})$$

for any $m, n > 0$ and $i, j = 1, 2, 3, 4$. The basis vectors defined in Eq. (B15) will be used for the Wigner function in the presence of a constant magnetic field in subsections III C and III E.

When a constant electric field exists, the following functions d_1, d_2, d_3 are used in subsections III D and III E,

$$d_1(\eta, u) = -1 + e^{-\frac{\pi\eta}{4}} \eta \left| D_{-1-i\eta/2}(-ue^{i\frac{\pi}{4}}) \right|^2,$$

$$d_2(\eta, u) = e^{-\frac{\pi\eta}{4}} e^{i\frac{\pi}{4}} D_{-1-i\eta/2}(-ue^{i\frac{\pi}{4}}) D_{i\eta/2}(-ue^{-i\frac{\pi}{4}}) + c.c.,$$

$$d_3(\eta, u) = e^{-\frac{\pi\eta}{4}} e^{-i\frac{\pi}{4}} D_{-1-i\eta/2}(-ue^{i\frac{\pi}{4}}) D_{i\eta/2}(-ue^{-i\frac{\pi}{4}}) + c.c., \quad (\text{B19})$$

where $D_\nu(z)$ is the parabolic cylinder function and “c.c.” is short for “complex conjugate”. The complex conjugate of $D_\nu(z)$ is $[D_\nu(z)]^* = D_{\nu^*}(z^*)$. Note that the parabolic cylinder functions satisfy the recurrence relations

$$\begin{aligned} D_{\nu+1}(z) - zD_\nu(z) + \nu D_{\nu-1}(z) &= 0, \\ \frac{\partial}{\partial z} D_\nu(z) + \frac{1}{2}zD_\nu(z) - \nu D_{\nu-1}(z) &= 0. \end{aligned} \quad (\text{B20})$$

Combining them we obtain a relation between $D_\nu(z)$ and $D_{\nu+1}(z)$,

$$\frac{\partial}{\partial z} D_\nu(z) - \frac{1}{2}zD_\nu(z) + D_{\nu+1}(z) = 0. \quad (\text{B21})$$

Using this relation we can obtain the differential equations for d_1, d_2, d_3

$$\begin{aligned} \frac{\partial}{\partial u} d_1(\eta, u) &= \eta d_3(\eta, u), \\ \frac{\partial}{\partial u} d_2(\eta, u) &= -u d_3(\eta, u), \\ \frac{\partial}{\partial u} d_3(\eta, u) &= -2d_1(\eta, u) + u d_2(\eta, u), \end{aligned} \quad (\text{B22})$$

where we have used

$$\left| D_{i\eta/2}(-ue^{-i\frac{\pi}{4}}) \right|^2 = e^{\frac{\pi\eta}{4}} - \frac{1}{2}\eta \left| D_{-1-i\eta/2}(-ue^{i\frac{\pi}{4}}) \right|^2. \quad (\text{B23})$$

In order to prove relation (B23) we first construct another function whose variables are η and u ,

$$d_4(\eta, u) = \left| D_{i\eta/2}(-ue^{-i\frac{\pi}{4}}) \right|^2 + \frac{1}{2}\eta \left| D_{-1-i\eta/2}(-ue^{i\frac{\pi}{4}}) \right|^2. \quad (\text{B24})$$

Then we can prove that this function does not depend on u because $\frac{\partial}{\partial u} d_4(\eta, u) = 0$. Furthermore, the value at $u = 0$ can be calculated using $D_\nu(0) = 2^{\nu/2} \sqrt{\pi} / \Gamma(\frac{1-\nu}{2})$, where $\Gamma(z)$ is the Gamma function.

$$\begin{aligned} d_4(\eta, 0) &= \frac{\pi}{\left| \Gamma\left(\frac{1}{2} - \frac{i\eta}{4}\right) \right|^2} + \frac{\pi\eta}{4 \left| \Gamma\left(1 + \frac{i\eta}{4}\right) \right|^2} \\ &= \cosh\left(-i\frac{\pi\eta}{4}\right) + \sinh\left(i\frac{\pi\eta}{4}\right) \\ &= e^{\frac{\pi\eta}{4}}, \end{aligned} \quad (\text{B25})$$

where we have used the property of the Gamma function $\Gamma(1+z) = z\Gamma(z)$ and a special case of the multiplication theorem,

$$\begin{aligned} |\Gamma(bi)|^2 &= \frac{\pi}{b \sinh(\pi b)}, \\ \left| \Gamma\left(\frac{1}{2} + bi\right) \right|^2 &= \frac{\pi}{\cosh(\pi b)}, \end{aligned} \quad (\text{B26})$$

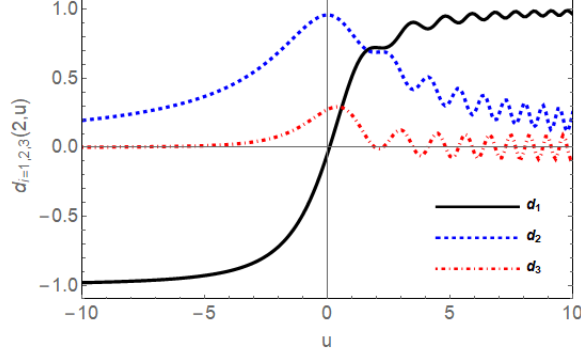


Figure 33: Dependence of the functions $d_i(\eta, u)$ for $i = 1, 2, 3$ with respect to u . Here $\eta = 2$ corresponds to $E_0 = \frac{1}{2}m_T^2$.

where b is a real constant.

In Figs. 33 and 34 we plot the u -dependence of $d_i(\eta, u)$ for two typical values $\eta = 2$ and $\eta = 0.5$. All these functions are convergent when $u \rightarrow -\infty$,

$$\lim_{u \rightarrow -\infty} d_1(\eta, u) = -1, \quad \lim_{u \rightarrow -\infty} d_2(\eta, u) = 0, \quad \lim_{u \rightarrow -\infty} d_3(\eta, u) = 0, \quad (\text{B27})$$

Meanwhile, in the limit $u \rightarrow +\infty$, the functions $d_2(\eta, u)$ and $d_3(\eta, u)$ are highly oscillatory and are not convergent. The function $d_1(\eta, u)$ is also oscillatory but the oscillation amplitude turns to zero when $u \rightarrow +\infty$, thus $d_1(\eta, u)$ converges to a finite value. Explicit analysis of the parabolic cylinder functions give their asymptotic behavior,

$$\lim_{u \rightarrow +\infty} D_{-1-i\eta/2}(-ue^{i\pi/4}) = \frac{\sqrt{2\pi}}{\Gamma(1+i\eta/2)} \exp \left\{ i \left[\frac{u^2}{4} + \frac{\eta}{2} \log(u) \right] - \frac{\pi\eta}{8} \right\}, \quad (\text{B28})$$

where the Gamma function is given by

$$\Gamma(1+i\eta/2) = \int_0^\infty x^{i\eta/2} e^{-x} dx, \quad (\text{B29})$$

Thus we obtain the asymptotic behavior of function $d_1(\eta, u)$,

$$\lim_{u \rightarrow +\infty} d_1(\eta, u) = -1 + e^{-\frac{\pi\eta}{2}} \frac{2\pi\eta}{|\Gamma(1+i\eta/2)|^2} = 1 - 2e^{-\pi\eta}. \quad (\text{B30})$$

Appendix C: WAVE-PACKET DESCRIPTION

Due to the uncertainty principle, in quantum mechanics the momentum and position of one particle cannot be determined at the same time. If we adopt the plane-wave description, then the momentum is fixed, which indicates that the uncertainty of position is infinity. This agrees with

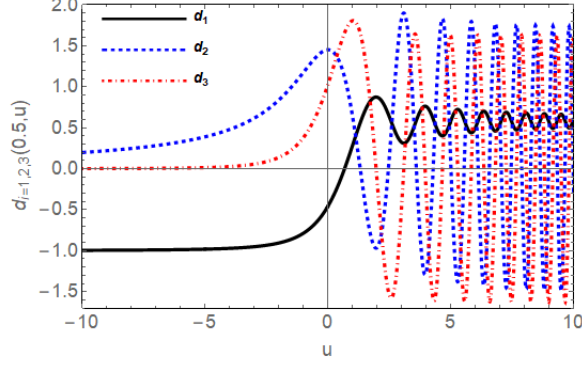


Figure 34: Dependence of the functions $d_i(\eta, u)$ for $i = 1, 2, 3$ with respect to u . Here $\eta = 0.5$ corresponds to $E_0 = 2m_T^2$.

the spatial homogeneity of the plane wave. However, we want to have a more realistic description. Thus, in this appendix we will introduce the wave-packet description of a particle. In quantum mechanics, the wave packet is interpreted as probability amplitude, whose square describes the probability of detecting a particle with given position and momentum.

The single-particle state and anti-particle state for the plane-wave case are given by acting with the creation operators onto the vacuum state,

$$|\mathbf{p}, s, +\rangle = a_{\mathbf{p},s}^\dagger |0\rangle, \quad |\mathbf{p}, s, -\rangle = b_{\mathbf{p},s}^\dagger |0\rangle, \quad (\text{C1})$$

where $|0\rangle$ is the vacuum state. Using the single-particle/anti-particle states, we can calculate the expectation values of energy, momentum, and polarization, respectively. Note that these states have fixed momentum \mathbf{p} , e.g. the uncertainty of momentum is zero. On the other hand, the wave packet for one particle is defined as a superposition of plane waves with different wave numbers,

$$|\mathbf{p}, s, +\rangle_{\text{wp}} = \frac{1}{N} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \exp\left[-\frac{(\mathbf{p} - \mathbf{p}')^2}{4\sigma_p^2}\right] a_{\mathbf{p}',s}^\dagger |0\rangle, \quad (\text{C2})$$

where the most probable momentum is \mathbf{p} and the uncertainty of momentum is described by the parameter σ_p . The normalization factor N is determined by ${}_{\text{wp}}\langle \mathbf{p}, s, + | \mathbf{p}, s, + \rangle_{\text{wp}} = 1$,

$$N = \sqrt{\frac{\sigma_p^3}{2\sqrt{2\pi^3}}}. \quad (\text{C3})$$

Now we calculate the energy of the wave packet. The total energy is given by

$$E_{\mathbf{p},\text{wp}} = {}_{\text{wp}}\langle \mathbf{p}, s, + | \hat{\mathcal{H}} | \mathbf{p}, s, + \rangle_{\text{wp}} = \frac{1}{N^2} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \exp\left[-\frac{(\mathbf{p} - \mathbf{p}')^2}{2\sigma_p^2}\right] \sqrt{m^2 + (\mathbf{p}')^2}. \quad (\text{C4})$$

We can compare the energy with the one $E_{\mathbf{p}} = \sqrt{m^2 + (\mathbf{p})^2}$ for the plane wave with the same momentum. We find that the ratio depends on the dimensionless parameters m/σ_p and $|\mathbf{p}|/\sigma_p$.

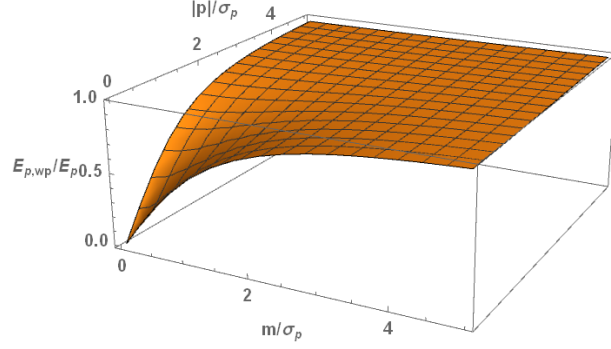


Figure 35: The ratio between the energy of a wave packet with the most probable momentum \mathbf{p} and that of a plane wave with the same momentum \mathbf{p} . The ratio depends on dimensionless variables, m/σ_p and $|\mathbf{p}|/\sigma_p$, where σ_p is the uncertainty of momentum.

The ratio is plot in Fig. 35. We can observe from this figure that if the mass and center momentum are much larger than the uncertainty σ_p , the ratio becomes 1. This indicates that a wave packet with the most probable momentum \mathbf{p} has the same energy as a plane wave with the same momentum \mathbf{p} when $E_{\mathbf{p}} \gg \sigma_p$.

On the other hand, the wavefunction of the wave packet in coordinate space can be obtained by superposition of the single-particle wavefunction, with the superposition coefficients equal to the ones for the state in Eq. (C2). For example, the particle's wavefunction in the wave-packet description is

$$\psi_{s,wp}^{(+)}(x) = \frac{1}{N} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \exp \left[-\frac{(\mathbf{p} - \mathbf{p}')^2}{4\sigma_p^2} - \frac{i}{\hbar} E_{\mathbf{p},wp}t + \frac{i}{\hbar} \mathbf{p}' \cdot \mathbf{x} \right] \begin{pmatrix} \sqrt{p'_\mu \sigma^\mu} \xi_s \\ \sqrt{p'_\mu \sigma^\mu} \xi_s \end{pmatrix}. \quad (C5)$$

In the limit $m, |\mathbf{p}| \gg \sigma_p$, this wavefunction can be expressed using the plane wave in Eq. (3.34)

$$\psi_{s,wp}^{(+)}(x) \simeq \exp \left(-\sigma_p^2 \frac{\mathbf{x}^2}{\hbar^2} \right) \psi_s^{(+)}(x). \quad (C6)$$

The overall factor suppresses the probability of detecting the particle in one point which is far from the original point. Thus the most probable position of the above wavefunction is the original point, while the uncertainty in spatial position is

$$\sigma_x = \frac{\hbar}{2\sigma_p}, \quad (C7)$$

which agrees with the uncertainty principle (1.5). Thus we conclude that the wave-packet description can be used for quantum particles with given center positions and average momentums.

Appendix D: PAIR PRODUCTION IN WIGNER-FUNCTION FORMALISM

In this appendix we will show the relation between the Schwinger pair-production process in a strong electric field and the Wigner function. This is helpful for the calculation of pair-production rate in subsections IIID and IIIE. In Quantum Kinetic Theory, the field operator is quantized in Heisenberg picture as

$$\hat{\psi}(t, \mathbf{x}) = \sum_s \int \frac{d^3 \mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \left[u_s(t, \mathbf{q}) \hat{a}_s(\mathbf{q}) + v_s(t, -\mathbf{q}) \hat{b}_s^\dagger(-\mathbf{q}) \right], \quad (\text{D1})$$

where $E_{\mathbf{q}}$ is the canonical energy, \mathbf{q} is the canonical momentum, and $u_s(t, \mathbf{q})$ and $v_s(t, -\mathbf{q})$ are normalized single-particle wavefunctions. On the other hand, we have

$$\hat{\psi}(t, \mathbf{x}) = \sum_s \int \frac{d^3 \mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \left[\tilde{u}_s(t, \mathbf{q}) \hat{a}_s(t, \mathbf{q}) + \tilde{v}_s(t, -\mathbf{q}) \hat{b}_s^\dagger(t, -\mathbf{q}) \right], \quad (\text{D2})$$

where \tilde{u}_s and \tilde{v}_s are adiabatic wavefunctions, which are chosen as $\tilde{u}_s(t, \mathbf{q}) = \tilde{u}_s(\mathbf{p})$ with the kinetic momentum $\mathbf{p} = \mathbf{q} - e\mathbf{A}(t)$, while for anti-fermions $\tilde{v}_s(t, -\mathbf{q}) = \tilde{u}_s(-\mathbf{p})$. Note that the wavefunctions should be normalized as

$$\begin{aligned} u_r^\dagger(t, \mathbf{q}) u_s(t, \mathbf{q}) &= \delta_{rs}, & v_r^\dagger(t, -\mathbf{q}) v_s(t, -\mathbf{q}) &= \delta_{rs}, \\ \tilde{u}_r^\dagger(t, \mathbf{q}) \tilde{u}_s(t, \mathbf{q}) &= \delta_{rs}, & \tilde{v}_r^\dagger(t, -\mathbf{q}) \tilde{v}_s(t, -\mathbf{q}) &= \delta_{rs}. \end{aligned} \quad (\text{D3})$$

Thus we can solve these adiabatic operators from the quantized field in Eq. (D2) using the normalization properties

$$\begin{aligned} \hat{a}_s(t, \mathbf{q}) &= \int d^3 \mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}} \tilde{u}_s^\dagger(t, \mathbf{q}) \hat{\psi}(t, \mathbf{x}), \\ \hat{b}_s^\dagger(t, -\mathbf{q}) &= \int d^3 \mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}} \tilde{v}_s^\dagger(t, -\mathbf{q}) \hat{\psi}(t, \mathbf{x}). \end{aligned} \quad (\text{D4})$$

Inserting the quantized field operator in Eq. (D1) into the above we obtain

$$\begin{aligned} \tilde{a}_s(t, \mathbf{q}) &= \tilde{u}_s^\dagger(t, \mathbf{q}) \sum_r u_r(t, \mathbf{q}) a_r(\mathbf{q}) + \tilde{u}_s^\dagger(t, \mathbf{q}) \sum_r v_r(t, -\mathbf{q}) b_r^\dagger(-\mathbf{q}), \\ \tilde{b}_s^\dagger(t, -\mathbf{q}) &= \tilde{v}_s^\dagger(t, -\mathbf{q}) \sum_r u_r(t, \mathbf{q}) a_r(\mathbf{q}) + \tilde{v}_s^\dagger(t, -\mathbf{q}) \sum_r v_r(t, -\mathbf{q}) b_r^\dagger(-\mathbf{q}). \end{aligned} \quad (\text{D5})$$

They give the relation between adiabatic operators and the ones in the Heisenberg picture. This relation is also known as Bogoliubov transformation. The particle number and anti-particle number for a system are defined as the expectation values

$$\begin{aligned} f_s^{(+)}(t, \mathbf{q}) &= \left\langle \Omega \left| \hat{a}_s^\dagger(t, \mathbf{q}) \hat{a}_s(t, \mathbf{q}) \right| \Omega \right\rangle, \\ f_s^{(-)}(t, \mathbf{q}) &= \left\langle \Omega \left| \hat{b}_s^\dagger(t, \mathbf{q}) \hat{b}_s(t, \mathbf{q}) \right| \Omega \right\rangle. \end{aligned} \quad (\text{D6})$$

Here we use $|\Omega\rangle$ to represent the quantum state for the considered system. Then the average pair number is defined as

$$n_{\text{pair}}(t) \equiv \frac{1}{2} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \sum_s \left[f_s^{(+)}(t, \mathbf{q}) + f_s^{(-)}(t, -\mathbf{q}) \right]. \quad (\text{D7})$$

Inserting the distribution functions into the average pair number we finally obtain

$$\begin{aligned} n_{\text{pair}}(t) = & \frac{1}{4} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \text{Tr} \left\{ \frac{\boldsymbol{\gamma} \cdot \mathbf{p} + m}{2E_{\mathbf{p}}} \sum_{r,r'} \bar{u}_{r'}(\mathbf{q}, t) \otimes u_r(\mathbf{q}, t) \langle \Omega | \hat{a}_{r'}^\dagger(\mathbf{q}) \hat{a}_r(\mathbf{q}) | \Omega \rangle \right\} \\ & + \frac{1}{4} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \text{Tr} \left\{ \frac{\boldsymbol{\gamma} \cdot \mathbf{p} + m}{2E_{\mathbf{p}}} \sum_{r,r'} \bar{v}_{r'}(-\mathbf{q}, t) \otimes v_r(-\mathbf{q}, t) \langle \Omega | \hat{b}_{r'}^\dagger(-\mathbf{q}) \hat{b}_r(-\mathbf{q}) | \Omega \rangle \right\} \\ & - \frac{1}{4} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \sum_r \langle \Omega | a_r^\dagger(\mathbf{q}) a_r(\mathbf{q}) + b_{r'}^\dagger(-\mathbf{q}) b_{r'}(-\mathbf{q}) | \Omega \rangle. \end{aligned} \quad (\text{D8})$$

Note that we are working in the Heisenberg picture, where the quantum state $|\Omega\rangle$ is independent of time and thus the last term is also independent of t . Using the equal-time Wigner function, we have

$$n_{\text{pair}}(t) = \frac{1}{4} \int d^3\mathbf{x} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \text{Tr} \left\{ \frac{\boldsymbol{\gamma} \cdot \mathbf{p} + m}{2E_{\mathbf{p}}} W(t, \mathbf{x}, \mathbf{p}) \right\} + \text{const.}, \quad (\text{D9})$$

where we have replaced the integration over canonical momentum by the one over kinetic momentum. From this formula we can derive the density of pairs,

$$n_{\text{pair}}(t, \mathbf{x}, \mathbf{p}) = \frac{\mathbf{p} \cdot \boldsymbol{\mathcal{V}} + m\mathcal{F}}{2E_{\mathbf{p}}} + \text{const.}, \quad (\text{D10})$$

where \mathcal{F} and $\boldsymbol{\mathcal{V}}$ are the scalar and vector components of the Wigner function, respectively, as shown in Eq. (2.20).

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Sheng Xinli

Add: Olof-Palame-Str. 31, 60439 Frankfurt am Main, Germany

Tel: +49 17675900692

E-mail: sheng@itp.uni-frankfurt.de

Personal Information

Family Name: Sheng

First Name: Xinli

Gender: Male

Date of Birth: 11. Dec. 1992

Place of Birth: Liangshan, Shandong, China

Education

2018.02 - Current, Goethe University Frankfurt, Frankfurt am Main, Germany

Degree: Doctoral candidate

Major: Physics

Major courses: Linear Accelerator, Physics of Particle Detectors, Analysis Methods of High Energy Physics

2015.09 - Current, University of Science and Technology of China, Hefei, China

Degree: Doctoral candidate

Major: Nuclear Physics and High Energy Physics

Major course: Supersymmetry

2013.09 - 2015.07, University of Science and Technology of China, Hefei, China

Degree: Master candidate

Major: Nuclear Physics and High Energy Physics

Major courses: Quantum Many Particle Physics, Advanced Quantum Mechanics, Advanced Electrodynamics, Advanced Statistical Physics

2009.09 - 2013.07, University of Science and Technology of China, Hefei, China

Degree: Undergraduate

Major: Theoretical Physics

Major courses: General Relativity, Quantum Field Theory, Group Theory in Physics, Thermodynamics and Statistical Physics, Astrophysical Frontier Lectures

Publications

1. Hui Li, Xin-li Sheng, and Qun Wang, Phys. Rev. C94(2016) no. 4, 044903.
2. Xin-li Sheng, Dirk H. Rischke, David Vasak, and Qun Wang, Eur.Phys.J. A54 (2018) no.2, 21.
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4. Nora Weickgenannt, Xin-li Sheng, Enrico Speranza, Qun Wang, and Dirk H. Rischke, e-Print: [arXiv:1902.06513](https://arxiv.org/abs/1902.06513).

Honors & Awards

2009, Bronze Prize of Scholarship for Outstanding Students, Hefei, China

2011, Silver Prize of Scholarship for Outstanding Students, Hefei, China

2012, Special Award of Experiment of College Physics in USTC, Hefei, China