



Collective strategy condensation towards class-separated societies

An envy-induced phase transition

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Abstract. In physics, the wavefunctions of bosonic particles collapse when the system undergoes a Bose–Einstein condensation. In game theory, the strategy of an agent describes the probability to engage in a certain course of action. Strategies are expected to differ in competitive situations, namely when there is a penalty to do the same as somebody else. We study what happens when agents are interested how they fare not only in absolute terms, but also relative to others. This preference, denoted envy, is shown to induce the emergence of distinct social classes via a collective strategy condensation transition. Members of the lower class pursue identical strategies, in analogy to the Bose–Einstein condensation, with the upper class remaining individualistic.

1 Introduction

Humans do not live in isolation. It is well established that the social context is important, influencing, beside others, memory, cognition, risk awareness, accountability and decision making [1–5]. An important example is the emergence of cooperation, for which non-monetary components to the utility are needed within a game-theoretical setting. Examples are reputation seeking [6–8], the desire to minimize risks [9, 10], or the willingness to socialize [11].

A generic non-monetary contribution to utility functions is a term arising from comparing rewards. Here we follow [12] and use ‘envy’ for contribution to game-theoretical utility function that models inter-agent comparisons [13]. Alternatively one could speak of an aversion to unfairness. Envy arises from the observation that the satisfaction individuals gain from making and spending money depends not only on absolute levels, but in good part also on how their own consumption level compares with that of others [14]. The importance of comparing ourselves to others is also at the core of relative income theory [15, 16]. Relative success is furthermore fundamental for competitions in general, being it a track race or evolutionary competition [17, 18].

For the study of envy, we examine games in which a psychological component is added to the classical, the monetary payoff [19]. Our focus is the question whether envy may have a qualitative impact on the structure of self-consistent multi-player Nash states. We find that this is the case, in the sense that that competitive soci-

eties split endogenously into distinct social classes when envy becomes relevant.

One-to-one inter-agent interactions are not present in our framework. This implies, that the sole determinant for behavior are the individual payoffs, which can be evaluated both in absolute and in relative terms. An analogy between agents and the distinguishable particles of classical physics can hence be made [20], with the game-theoretical space of available options corresponding to the physical state space. Strategy condensation occurs when the initially distinct strategies of a macroscopically large number of agents collapses into a single, encompassing strategy. This process can be considered as a classical correspondence of the Bose–Einstein condensation (BEC), which is characterized by the macroscopic occupation of a single state, usually the ground state [21]. At its core, BEC is however a quantum phenomenon. A similar analogy in term of a classical de-facto Bose–Einstein distribution has been pointed out in the context of complex networks [22].

2 Framework

In a first step we discuss our approach to model a society of competing agents, adding subsequently the psychological component, envy.

Agents have $i \in [1, N]$ possible options, with each options yielding a basic payoff, v_i . This is the monetary income the agent receives if nobody else would select the same option. For the modeling of the inter-agent competition, we define with $p_i^\alpha \geq 0$ the strategy of agent $\alpha \in [1, M]$, namely the probability to select

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option i , with the normalization condition $\sum_i p_i^\alpha = 1$. The number of options and agents is respectively N and M . Strategies are pure when p_i^α reduces to a discrete δ -function, as $p_i^\alpha \rightarrow \delta_{ik}$, being mixed otherwise. The real-world income I^α of an agent is given by

$$I^\alpha = \sum_i I_i^\alpha p_i^\alpha, \quad I_i^\alpha = v_i \left(1 - \kappa \sum_{\beta \neq \alpha} p_i^\beta \right), \quad (1)$$

where $\kappa \geq 0$ encodes the strength of the competition. Payoff reduction is proportional to $\sum_{\beta \neq \alpha} p_i^\beta$, which corresponds in lowest order (modulo higher-order combinatorial factors) to the probability to encounter other agents. Overcrowded options yielding negative payoffs will be avoided. In this study we set $\kappa = 1/2$, which describes the case that two agents with identical pure strategies share resources equally. More than two agents can opt for the same course of action i hence only when playing mixed strategies, with $p_i^\alpha < 1$.

The exact functional form of the bare utility v_i is of minor importance. Mapping options i to a scalar quantity, the quality $q_i \in [0, 1]$, allows to define a functional representation for $v_i = v(q_i)$. We use

$$v(q_i) = \frac{1 + q_i}{1 - \theta q_i}, \quad \theta \in [0, 1[, \quad (2)$$

with the parameter θ regulating the relative width of the distribution of bare utilities. One has $v(q_i) \in [1, 2/(1 - \theta)]$, which translate into $v(q_i) \in [1, 4]$ when $\theta = 0.5$.

2.1 Nash equilibria without envy

It is assumed throughout this study that the basic utility function v_i is non-generate, viz that $v_i \neq v_j$ for all option pairs $i \neq j$. Per se, without an additional psychological component, the utility defined by (1) has then a straightforward solution. Agents play exclusively pure strategies, with either one or two agents per option. Throughout this study we denote these two types of pure strategies as ‘pure-1’ and ‘pure-2’.

2.2 Information requirement

Formulating the competition $\sim \kappa$ in terms of strategies, as done in (1), demands at face sight a high amount of information, namely the knowledge of the strategies ρ_i^α of all participating agents. This is however not the case. For a proof we rewrite the probability to encounter other agents at a given option i as

$$\sum_{\beta \neq \alpha} p_i^\beta = \sum_{\beta} p_i^\beta - p_i^\alpha, \quad (3)$$

which is solely a function of the agent’s own strategy, p_i^α , and the overall mean number $N_i = \sum_{\beta} p_i^\beta$ of agents selecting the option in question. Agents need to

be aware hence only of their own strategy and of the mean occupations N_i , but not of the specifics of the strategies of everybody else.

2.3 Envy – comparing success

A basic reference level for success within a well-mixed population is the average income \bar{I} ,

$$\bar{I} = \frac{1}{M} \sum_{\alpha} I^\alpha, \quad (4)$$

with I^α being defined by (1). On the level of individuals, envy results from direct person-to-person comparison. Here we examine instead the population effect, for which the average income \bar{I} is taken as the yardstick. Both approaches, using \bar{I} as a reference, or pairwise comparisons, become identical, as shown in the appendix, when income differences are small.

In terms of the overall reward function R_i^α , the effect of envy is encoded by

$$R_i^\alpha = I_i^\alpha + \epsilon p_i^\alpha \log \left(\frac{I_i^\alpha}{\bar{I}} \right), \quad (5)$$

where the first term on the right-hand side corresponds to the monetary utility. The strength of second term, the psychological component, is regulated by $\epsilon \geq 0$.

The rationale for the functional form of the envy term in (5) is straightforward. The agent is happy when earning above average, when $I^\alpha > \bar{I}$, and unhappy otherwise. The log-dependency ensures that the envy term contributes positively in the first, and negatively in the second case. The envy term couples directly to the strategy, p_i^α , with the consequence that the current strategy is enforced when $I^\alpha > \bar{I}$. Below the average, when $I^\alpha < \bar{I}$, agents are motivated in contrast to search for alternatives. It is also worth mentioning that the log-dependency $\log(I^\alpha/\bar{I})$ of the envy term in (5) is in agreement with the Weber–Fechner law, which states that the brain discounts sensory stimuli [23], numbers [24], time [25], and data sizes [26] logarithmically.

In previous studies envy has been assumed to result from comparing rewards [20, 27], and not incomes, as done here. In that case the envy term involves $\log(R^\alpha/\bar{R})$, instead of (5), where \bar{R} is the mean reward. The resulting framework demands however that agents can estimate the psychological state of the other agents, which is included in R^α , but not in I^α , which seems to be less plausible. A well-defined large-size limit is attained in any case for constant ϵ when scaling M and N such that the ratio $\nu = M/N$, the fraction of agents per options, is retained [27].

2.4 Numerics

Agents maximize their expected rewards R^α ,

$$R^\alpha = \sum_i R_i^\alpha p_i^\alpha \quad (6)$$

viz the sum of the monetary utility I^α and of the envy term. Numerically, this is achieved using standard evolutionary dynamics [28],

$$p_i^\alpha(t + 1) = \frac{p_i^\alpha(t)R_i^\alpha(t)}{\sum_j p_j^\alpha(t)R_j^\alpha(t)}, \tag{7}$$

which describes the transition of strategies at time t to $t + 1$. Adding a constant offset R_0 to the bare rewards R_i^α leads to a smooth convergence. If not otherwise stated, $R_0 = 20$ has been used.

Once (7) has been iterated till self-consistency, the resulting payoffs R_i^α are positive. When negative values appear as an intermediate step, they are set to a lower bound R_0 , which has been taken to be zero. It is advantageous to use the representation (3) for the interaction term. We focused here on the case $M = N$, together with $\kappa = 0.5$ and $\theta = 0.5$. Testing extensively for alternative parameter settings we find that the overall picture emerging is remarkably robust.

For the results presented below the evolution dynamics (7) is iterated 10^5 times, if not otherwise stated. The initial strategies $p_i^\alpha(t = 0)$ are drawn randomly from $[0, 1]$, and normalized subsequently. We did test that the resulting strategy distributions are stable against small stochastic perturbations. The resulting states are hence ‘local’ Nash-equilibria, in the sense that it is disadvantageous for agents to make small changes to their p_i^α .

3 Results

We analyze our results in particular with respect to the size of the support S^α of the individual strategies p_i^α ,

$$S^\alpha = \{i | p_i^\alpha > 0\}, \tag{8}$$

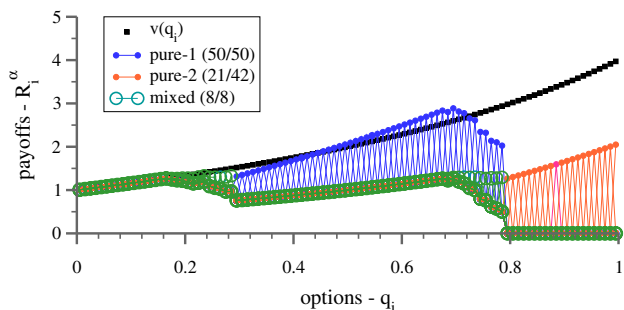


Fig. 1 Assorted competition. The payoffs of $\alpha \in [1, 100]$ agents (circles), $i \in [1, 100]$ options, and $\epsilon = 1$. The data points of the payoff functions R_i^α are connected by lines. 50 agents engage in pure strategies which are not contested (pure-1, blue), with 21 agents playing pure strategies that are pursued also by another agent (pure-2, red). The rest, 8 agents, follow distinct mixed strategies (green), with an average support of 2.9. Also given is the bare utility $v(q_i)$ (black). Shown in the brackets, (N_s/N_p) , is the number N_s of distinct strategies of a given type, together with the number N_p of engaged agents

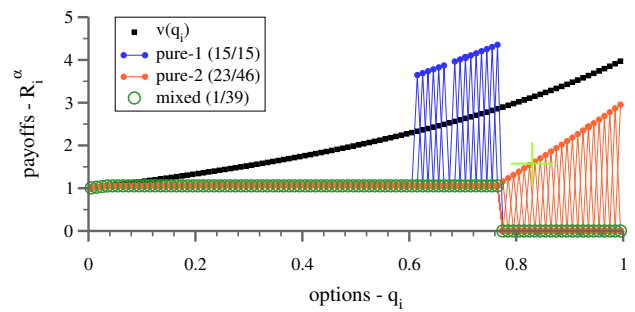


Fig. 2 Class-stratified state. The payoffs of $M = 100$ agents (circles), for $N = 100$ options q_i and $\epsilon = 4$. The lines connect payoffs R_i^α for one and the same agent α . There are 15 agents playing pure strategies which are not contested (pure-1, blue), with 23 agents playing pure agents that are played also by another agent (pure-2, red). The rest, 39 agents, play the identical mixed strategy (green) covering 62 options. Also shown is the bare utility $v(q_i)$ (black). The cross (light yellow) indicates the locus of vanishing envy, viz where $I^\alpha = \bar{I}$. The brackets, (N_s/N_p) , show the number N_s of distinct strategies of a given type together with the number of agents N_p playing the respective type of strategy

which is given by the set of options selected with finite probability. The smallest possible support is one, corresponding to the case of a pure strategy. Strategies with non-trivial support are mixed.

3.1 Low envy: assorted competitive state

In Fig. 1 a typical payoff configuration for $\epsilon = 1$ is given. Shown are the entirety of all $M = 100$ payoff functions R_i^α , viz the respective values for all $i = 1, \dots, N$ options, as defined by (5). Larger bare utilities $v(q_i)$ are taken each by two agents playing pure strategies, a configuration denoted with ‘pure-2’. This region cannot be invaded by competitors, given that I_i^α , as defined by (1), vanishes in the pure-2 region for third parties. Most agents are engaged in ‘pure-1’ strategies, which means that they play pure strategies that are contested, if at all, only by agents playing mixed strategies, but not by other agents pursuing pure strategies.

In the absence of envy, when $\epsilon = 0$, only pure-1 and pure-2 strategies are present. For small values of envy, as for the data presented in Fig. 1, mixed strategies start to invade the pure-1 region at the boundary to the pure-2 strategies, with additional support at low values of $v(q_i)$. Mixed strategies are distinct for low ϵ , which means that no two agents pursue identical mixed strategies. Supports are comparatively small. Overall, a competitive Nash equilibrium with assorted strategies is observed.

3.2 Large envy: condensed mixed strategies

The spectrum of payoffs functions reorganizes dramatically for larger value of ϵ . Not with regard to the two sets of pure-1 and pure-2 strategies, which change only

with respect to their sizes, but regarding the mixed strategies. All mixed strategies collapse into a single strategy pursued by a finite fraction of agents, the lower class. The situation is illustrated in Fig. 2 for $\epsilon = 4$. About 40% of all agents play one and the same mixed strategy, with the respective support covering about 60% of option space.

The details of the individual strategies are private information within the framework examined here. Only the cumulative weights, $\sum_{\beta} p_i^{\beta}$, are available, see (3). There is consequently no inbuilt mechanism allowing an agent to copy somebody else’s strategy. The incomes received, I^{α} , are however public, which means that they can be compared with each other, the core functionality of envy. The condensation of the strategies of a large number of agents is therefore a collective phenomenon in the sense of statistical physics [17]. It occurs when increased levels of envy lead to a substantial inter-agent coupling.

Agents are identical, apart from their strategies, which are drawn at the start from a flat random distribution. Chance, the starting strategy, determines therefore the outcome – whether the agent will belong in the end to the individualistic pure-1 or pure-2 clusters, or whether it will become part of the masses, loosely speaking, sharing the same mixed strategy with many others.

3.3 Social classes separated by income gaps

In Fig. 3 we present the player specific averages corresponding to the payoff functions shown in Figs. 1 and 2. For pure strategies, the rewards R^{α} correspond to the respective peaks of the payoff functions. When sorted by ascending values, the rewards of pure-1, pure-2 and mixed strategies are intertwined for $\epsilon = 1$, but not for $\epsilon = 4$. It is also noticeable that there is no gap in the $\epsilon = 4$ reward spectrum between mixed and pure-2 strategies, whereas a substantial gap is present for the corresponding income data. This phenomenon can be traced back to a discontinuity of the player specific envy E^{α} ,

$$E^{\alpha} = R^{\alpha} - I^{\alpha} = \epsilon \log \left(\frac{I^{\alpha}}{\bar{I}} \right) P_2^{\alpha}, \quad P_2^{\alpha} = \sum_i (p_i^{\alpha})^2, \tag{9}$$

compare (5). For pure strategies $P_2^{\alpha} \rightarrow 1$ holds, but not for mixed strategies. This is true in particular for the mixed strategy of the condensed state realized for $\epsilon = 4$, which has a substantial support and correspondingly small probabilities p_i^{α} . The envy term E^{α} jumps hence together with P_2^{α} at the boundary of the mixed and the pure-2 cluster, which have respectively low/large values of $|E^{\alpha}|$. Note that $I^{\alpha} < \bar{I}$ at the boundary, as indicated in Fig. 2, which implies a negative $E^{\alpha} < 0$ and that pure-2 strategies must have larger incomes at boundary to the lower class. The latter due to the continuity of the reward spectrum R^{α} .

The occurrence of substantial gaps in the income spectrum shows that the society of agents separates

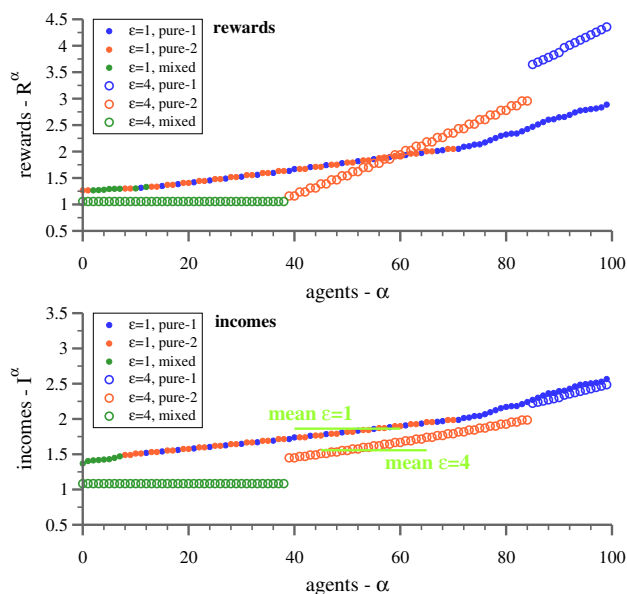


Fig. 3 Player specific rewards and incomes. For all agents, the rewards R^{α} (top panel) and the incomes I^{α} (bottom panel), as sorted by value. Shown are the results corresponding to the payoffs presented in Figs. 1 and 2, for both $\epsilon = 1$ (filled symbols) and $\epsilon = 4$ (open symbols). Also shown are the population averages \bar{I} for the incomes (light green). Note the income gaps for $\epsilon = 4$

spontaneously into distinct social classes under the influence of envy [20]. Incomes are distributed over a finite range also for low values of envy, as evident from the data presented in Fig. 3. The resulting distributions are however continuous, which implies that there is no unbiased criterion to subdivide the society into separated social classes. The situation changes beyond the strategy condensation transition, where three differentiated strategy clusters emerge:

- **Lower class:** A finite ϵ fraction of agents (‘the masses’) pursues the identical mixed strategy, with incomes that are below the average, $I^{\alpha} < \bar{I}$.
- **Middle class:** The middle class claims the options with the highest bare utilities $v(q_i)$. The price paid is that options are selected by pairs of agents, which halves the respective monetary yields. The resulting incomes I^{α} are both below and above the population average \bar{I} .
- **Upper class:** Agents playing pure strategies that are contested only by lower-class agents, but not by other pure strategies.

The here used definition of social classes in terms of separated income clusters is convenient for modeling studies, but less suitable for demographic investigations. Real-world income distributions do not show clear income gaps [29].

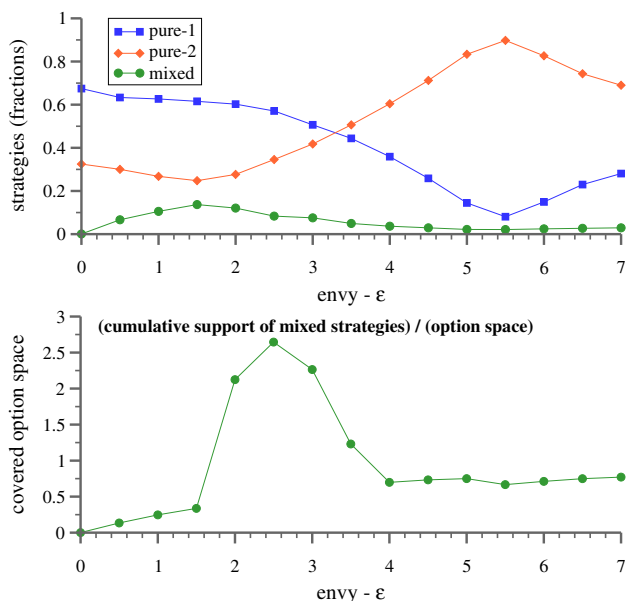


Fig. 4 Evolution of strategies played. Top: For $N = M = 200$ the fractions of observed strategies, as a function of envy. Shown are the relative fractions of different strategies, for pure-1 (blue), pure-2 (red) and mixed (green). Counted are the numbers of strategies, not agents. The data is averaged over ten random initial conditions. Bottom: The cumulative support of all mixed strategies divided by the number of available options, N . The ratio can exceed unity, as mixed strategies may partially overlap

3.4 Strategy condensation transition

In Fig. 4 the evolution of key strategy-related quantities is shown as a function of envy. Data for slightly larger numbers of agents and options, $N = 200 = M$, has been averaged over ten random initial conditions. The absolute system size is of minor importance, as long as the ratio M/N of agents per options is kept [27]. The fractions of strategies played are given relative to the total number of observed distinct strategies. For the Nash equilibrium of Fig. 1, to give an example, there are $50 + 21 + 8 = 79$ different strategies. The fractions of pure-1, pure-2 and mixed strategies would then be $50/79$, $21/79$ and $8/79$. One observes that mixed strategies are most important, in terms of relative numbers, around $\epsilon \approx 1.5$.

Beyond $\epsilon \approx 5.5$, the mixed strategy of the lower-class starts to invade intermittently the cluster of pure-2 strategies, which shrinks consequently in terms of relative importance. The strict ordering of the supports of the mixed and the pure-2 strategies in q_i -space, as presented in Fig. 2, is hence lost for very large values of ϵ .

Also included in Fig. 4 is the total support of all mixed strategies, with mixed strategies played by more than one agent counted only once. The total support has been divided by the overall number of options, N , with values above unity indicating that mixed strategies overlap on the average. The option space covered

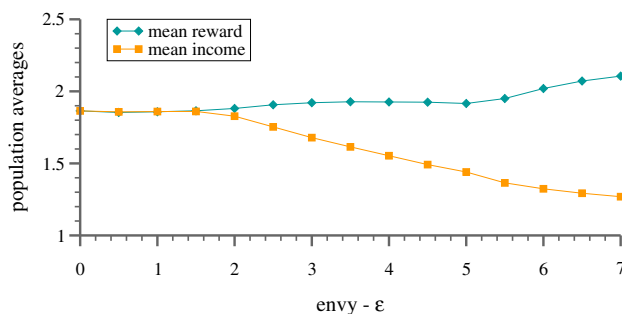


Fig. 5 Envy reduces monetary incomes Mean incomes \bar{I} and rewards, \bar{R} , as defined by (4) and (6). As a function of envy, mean incomes drop substantially beyond the strategy condensation transition, which starts between $\epsilon = 1.5$ and $\epsilon = 2$. Envy, a psychology component, affects real-world monetary incomes adversely

jumps substantially between $\epsilon = 1.5$ and $\epsilon = 2.0$, due to the emergence of mixed strategies with extended supports. Between $\epsilon = 1.5$ and $\epsilon = 2.0$ the average relative support of mixed strategies rises eight-fold, from 3.0 to 23.9. Total support drops beyond the peak, at $\epsilon \approx 2.5$, due to corresponding decrease in the number of mixed strategies. At this stage there is typically one large condensed mixed strategy, together with a few smaller mixed strategies, all leading to very similar incomes. The number of smaller mixed strategies becomes progressively smaller with increasing envy. For $\epsilon = 4$ and $\epsilon = 6$ there are, e.g., on the average 2.9 and respectively 1.0 mixed strategies, with actual numbers fluctuating between individual simulations and initial conditions. In Fig. 2 the case of a single mixed strategy has been shown for $\epsilon = 4$. Residual smaller mixed strategies would have been essentially indistinguishable on the scale.

It is important to point out, that the strategy condensation transition appears from replicator dynamics, see Eq. (7), which is the only way for agents to evolve their strategies. Other forms of learning, f.i. by joining other strategies, are not present. The formation of an extended class of agents with (numerically) identical strategies is hence due to a collective effect.

3.5 Social dilemma

In Fig. 5 the evolution of the population averages \bar{I} and \bar{R} with envy are presented, respectively for incomes and rewards. The mean reward varies comparatively little with ϵ , overall an increase from 1.86 to 2.11 can be observed between $\epsilon = 0$ to $\epsilon = 7$. The situation is different for \bar{I} , which drops noticeably within the class-stratified state. Once strategy condensation sets in, around $\epsilon = 1.5$ – 2.5 , incomes start to drop progressively.

The evolutionary benefits of envy as a multi-faceted human emotion are not fully understood [30]. It is however generically assumed that envy is a direct consequence of the core defining feature of Darwinian evo-

lution, namely that only the ‘fittest’ survives. To be among the top performers is defined in relative, and not in absolute terms [18,31]. Our results, that envy is detrimental to the overall welfare of societies, suggest in this perspective a social dilemma. As an operative human psychological trait [13], envy seems to have disparate effects on individual and societal levels. It is hence important to study, as done in the following sections, what happens when envy is not a uniform characteristic, but a heterogeneous, player-specific trait.

3.6 Player-specific envy levels

The stability of the phenomenology emerging from our framework, that a strategy condensation transition is induced by raising levels of envy, can be studied against various perturbations. Here we examine the case that agents have distinct degrees of envy. For this question we use a particularly straightforward protocol, namely that the player specific envy-parameters ϵ^α are uniformly distributed on the interval $[0, 2\bar{\epsilon}]$, where $\bar{\epsilon}$ is the mean envy level.

One finds that varying agent specific levels lead only to quantitative changes, leaving the basic processes intact. A typical outcome for $N = M = 100$ is shown in Fig. 6, where the payoff functions are given for $\bar{\epsilon} = 4$, together with the ordered incomes I^α . The latter is given for comparison also for $\bar{\epsilon} = 1$. Two observations are retained, the occurrence of a transition resembling the condensation of strategies, and that a substantial income gap opens between the lower class and the rest of the society. We notice that the mean income drops with raising $\bar{\epsilon}$, here from $\bar{I} = 1.85$ to $\bar{I} = 1.58$ when going from $\bar{\epsilon} = 1$ to $\bar{\epsilon} = 4$. A social dilemma is hence present also when agents are characterized not by uniform, but by player-specific levels of envy.

For $\bar{\epsilon} = 1$ one finds pure-1 (49/49), pure-2 (21/42) and mixed (9/9) strategies. For $\bar{\epsilon} = 4$ we have pure-1 (23/23), pure-2 (17/34) and “mixed (1/43)” strategies. Beyond the transition, a lower class emerges. At the resolution of the plot presented in Fig. 6, differences in respective rewards or incomes, R^α and I^α , of the 43 lower-class agents cannot be discerned. It is however not possible that agents with different envy ϵ^α satisfy (5) for identical strategies and incomes. On a fine scale it should hence be the case that the lower class splits into a mixed (43/43) state, albeit with only small differential payoffs. We verified that this concurs with the numerical results shown in Fig. 6 for $\bar{\epsilon} = 4$, for which the lower class has a small but finite bandwidth.

It is remarkable, that the strict condensation of strategies observed for identical $\epsilon^\alpha \equiv \bar{\epsilon}$, is retained to a very good approximation when envy becomes specific to the individual agents. For an understanding we recall that the expectation value (9) of the envy term involves with $P_2^\alpha = \sum_i (p_i^\alpha)^2$ the sum of the squared strategy. Strategies with macroscopic support obey the scaling $|S^\alpha| \sim N$, which implies $P_2^\alpha \sim N/N^2 \sim 1/N$. In this case, the envy term will vanish in the thermodynamic limit $N \rightarrow \infty$, with differences in the ϵ^α becoming irrelevant.

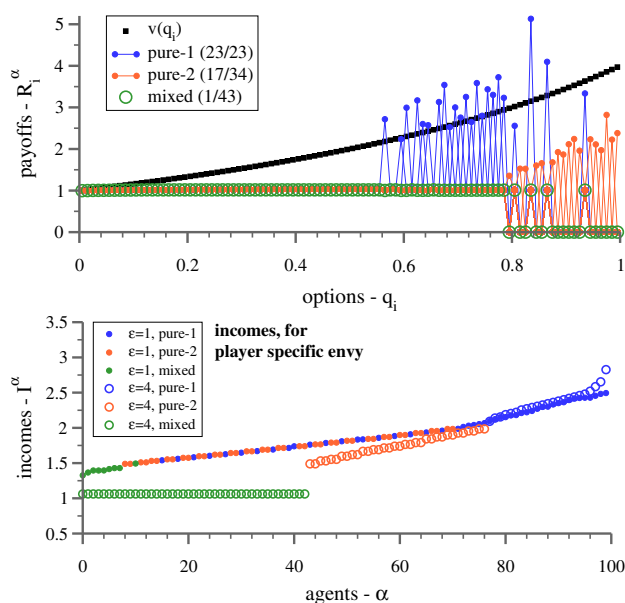


Fig. 6 Agents with personalized levels of envy. Top: As for Fig. 2, but for agents with player-specific envies ϵ^α that are distributed equally on the interval $[0, 2\bar{\epsilon}]$, with a mean $\bar{\epsilon} = 4$. The ‘scattering’ of the payouts for pure strategies is a consequence of the varying levels of envy. Denoting the mixed state (green) as (1/43), as done the legend, is an approximation. On a fine numerical level, the lower class is a (43/43) state, albeit with only very small payoff differences. **Bottom:** The respective average incomes, as in Fig. 3. Both for $\bar{\epsilon} = 1$ and $\bar{\epsilon} = 4$ (indicated by ϵ in the legend)

evant. The band of lower-class agents showing up in Fig. 6 will hence become degenerate in the thermodynamic limit.

3.7 Correlations between income levels and envy

We investigated whether the player specific incomes I^α are correlated with the respective envy-strengths, ϵ^α . We used $\bar{\epsilon} = 4$, as in Fig. 6, but this time for a larger system, with $M = N = 400$, which allows for a reliable statistics. In Fig. 7 the results are shown as a scatter plot in the $(\epsilon^\alpha, I^\alpha)$ plane. It is clear that the pure-1, pure-2 and mixed strategies form strategy-specific bands that are separated in the $(\epsilon^\alpha, I^\alpha)$ plane.

The incomes I^α of lower class members range from 1.058 to 1.067, which is just above the minimum of the bare utility, $v(0) = 1$. The width of the lower-class band of about 1% is proportional to sum over squared strategies, as discussed above, and hence small. The data presented in Fig. 7 shows furthermore that agents with low values of envy ϵ^α can be found with a somewhat increased probability close to the mean income, $\bar{I} = 1.525$, which was to be expected. Members of the lower class have in contrast substantial values of ϵ^α . When envy is large, the incentive to search for alternatives is substantial for agents with below the average incomes.

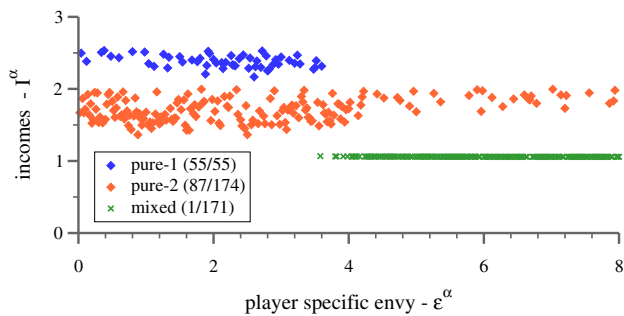


Fig. 7 Envy specific incomes. For distributed player specific values of envy, $\epsilon^\alpha \in [0, 8]$, the incomes I^α for a society with $M = N = 400$ agents and options. The numbers of agents playing pure-1, pure-2 and mixed strategies are 55, 174, and 171, which adds to $M = 400$. As for the data shown in Fig. 6, the denotation (1/171) is an approximation. On a fine numerical scale, the lower class (green) is composed of 171 distinct levels, being a (171/171) state. Lower-class agents play very similar, but not identical strategies. The three different types of strategies form non-overlapping clusters in the $(\epsilon^\alpha, I^\alpha)$ plane

3.8 Ultimatum game

Envy is a multi-faceted human trait that may act both on societal and on individual levels, with both types being equivalent to lowest order, as discussed in the appendix. The latter observation allows to gauge the magnitude of ϵ from lab studies, which are available for two-person games.

A reference game revolving around the notion of fairness is the ultimatum game [32, 33]. In the ultimatum game a given amount of monetary utility s_0 is partitioned according to what one of the players, the proposer, suggests. Say as 70–30% shares. The other player, the responder, may accept or decline. When declining, nobody gets anything. The expected outcome depends on whether the game is interpreted within a closed- or an open-room framework. Standard testing protocols presume the first case, namely that the game takes place in a closed room separated from everything else. It would then be rational for the responder to accept offers of every size, even then getting only 10%, which is however not observed.

Experimentally, people tend to reject offers that are too unfair. This makes sense in an open-room setting, namely for the case that both participants could potentially be competitors in the outside world. Unfair offers would provide the divider with a fitness advantage. An aversion to unfairness, which is functionally equivalent to envy, is then rational.

We specialize to $s_0 = 1$ and denote with $s \in [0, 1]$ the monetary utility the responder would receive when accepting, with the proposer getting $1 - s$. Applying our generic utility (5) to the ultimatum game we have

$$R(s) = s + \epsilon \log(s/\bar{s}), \quad \bar{s} = 1/2, \quad (10)$$

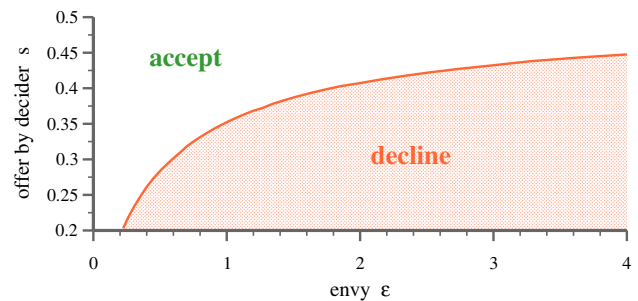


Fig. 8 Ultimatum game with envy. The decision space for the responder in the ultimatum game with envy, as defined by (10). Even substantially unfair offers are accepted when the envy parameter ϵ is small, but only nearly fair offers for large envy. Experimentally, offers below $s = 0.4$ are rare [34], which indicates that $\epsilon \approx 1.75$

for the reward function of the responder. We denoted with \bar{s} the average monetary utility and used that the responder plays pure strategies (accepting/declining). The responder accepts offers leading to positive rewards, declining negative $R(s)$.

The proposer maximizes its gains when the offer is as unfair as possible, namely close to s_ϵ , the lowest offer the responder will accept. It is, in effect, an judgment call. A large number of experiments have shown that $s_\epsilon \approx 0.4$ [34], which indicates an value of $\epsilon \approx 1.75$. This value can be extracted from the data presented in Fig. 8, which shows the solution of $R(s) = 0$.

The order of magnitude for the strength of envy, $\epsilon \approx 1.75$, is surprisingly large. In comparison, the range of the monetary utility is $s \in [0, 1]$. We can hence expect, that real-world envy parameters and monetary utilities are of the same order of magnitude, with envy being possible somewhat larger.

Here we used the concept of envy to model the experimental outcomes of the ultimatum games. An alternative would be, besides others, to postulate that both participants dispose of a ‘self-centered inequity aversion’ [34], which means that both players dislike unfair outcomes, to varying degrees, and not only the responder. In our case the strength of the envy parameter of the proposer does not enter. For the proposer, maximizing $1 - s$ or $R(1 - s)$ leads to the same conclusion, as both are monotonically increasing when $1 - s > 1/2$. Our view, that the proposer is mainly interested in profit maximization, draws support from experimental studies [35].

The fact that responder rarely accept offers below 40% implies that envy is impactful. Within our framework, this leads to a specific value of ϵ , which is about 1.75 times the overall bare utility. Transferring this value to the case of competitive societies, we would have $\epsilon \approx 3 \times 1.75 \sim 5.25$. This value arises when the range $v(1) - v(0) = 3$ is taken for the reference utility. At this point it is important to stress that the resulting ϵ is only a rough order of magnitude estimate. It suggests in any case that envy matters and that that our result, that

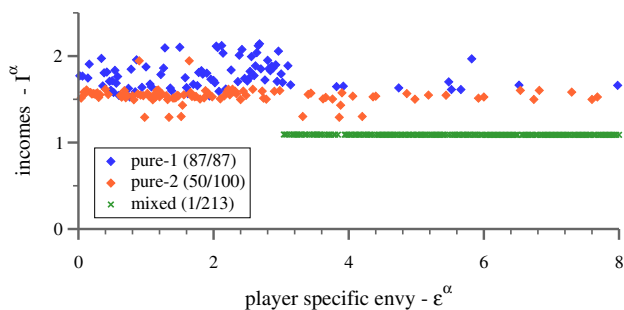


Fig. 9 Divide-the-cake competition. For player specific values of envy in the range $\epsilon^\alpha \in [0, 8]$, the incomes I^α , compare Fig. 7. Here we used $M = N = 200$ agents and options and the ‘divide-the-cake’ interaction, as specified by (11)

values of $\epsilon \approx 2.5$ and larger lead to a class-stratified state, is not just an academic exercise.

3.9 Alternative formulations

It is only rarely possible to fully analyze non-uniform games with arbitrary large numbers of options and agents. Examples are sealed auctions and animal conflict models with continuous bidding ranges [36,37]. The situation is more intricate for resource distribution games, like the framework examined here, as evidenced by the numerically obtained structure of the payoff functions shown in Fig. 1. The complication is in particular due to the presence of a *a priori* not known number of mixed strategies, for which the strategy functions p_i^α are to be determined maximizing M coupled self-consistency conditions for the individual rewards R^α .

An important point regards the robustness of our scenario, namely that competitive societies become unstable under the influence of envy. For example, we studied the ‘divide-the-cake’ type inter-agent competition

$$I_i^\alpha = \frac{v_i}{1 + \sum_{\beta \neq \alpha} p_i^\beta}, \tag{11}$$

which describes the situation that agents playing pure strategies equally divide the bare reward v_i when going for one and the same option. Only quantitative differences are found, as illustrated in Fig. 9.

A substantially different formulation is

$$R_i^\alpha = v_i - \kappa \sum_{\beta \neq \alpha} p_i^\beta + \epsilon p_i^\alpha \log \left(\frac{R^\alpha}{\bar{R}} \right) \tag{12}$$

which has been studied in the past [20,27]. Here $R^\alpha = \sum_i R_i^\alpha p_i^\alpha$ and $\bar{R} = \sum_\alpha R^\alpha / M$. Envy is now a function of the reward, and not of the monetary income. This implies a recursive functionality, given that rewards depend in turn on the envy term. We hence believe that that (12), which also leads to a class-stratified state with increasing ϵ , is less realistic. It has however been possible to treat the condensed state analytically [27].

Note that the κ -term in (12) is not proportional to v_i , as in (1).

For the framework investigated here, there is only a single references value for comparing incomes, the population average \bar{I} . Alternatively, agents could compare their incomes on a pairwise level, as discussed in the appendix. This would require $M - 1$ operations per agent. In between these two extremes, the average incomes of the peer community could be relevant, e.g. for societies of agents with a well defined network structure.

4 Conclusions

The evolutionary drivers of core human traits, like cooperation [38–40], altruism [41,42], charity [43] and revenge [44], to name a few, have been investigated intensively. To a certain extent, our propensity to value success not just as such, but also in relative terms, is set apart. Given that natural selection works on relative success [45], it benefits evolutionary success when being sensitive to the success of others. From this view it is not surprising that non-human animals may also be averse to inequalities [46,47]. As the closest available notion we used ‘envy’ to denote terms in reward functions based on relative success. As all psychological traits, envy has in addition an extended palette of distinct facets. Three key findings emerge.

The first observation is that strategies may condense collectively. This happens when a large number of small mixed strategies starts to exhaust option space, merging into a single big strategy. Small and big denote here the relative sizes of the respective supports, specifically when measured in units of overall option spaces. Small supports are intensive, viz not growing when increasing the number of options N , while keeping a constant density M/N of agents per available options. Supports are in contrast big when they are extensive, that is when they scale with N . When strategies condense, an extensive number of agents play the identical mixed strategy, which is characterized in turn by an extensive support.

Our second core result concerns the structure of the society once mixed strategies did condense. Clear gaps open in the spectrum of the monetary income between distinct clusters of agents. These gaps provide a well defined criterion for the subdivision of the society into different social classes. Our results lead consequently to the conjecture that envy may play a key role for the original emergence of class-stratified societies. Besides being separated by income gaps, the three emerging clusters of agents correspond to distinct strategy types, with lower, middle and upper class agents playing respectively mixed, pure-2 and pure-1 strategies, where the two types of pure strategies differ with respect to the amount of competition encountered.

It is possible to generalize our framework to the ultimatum game, which allows to estimate the real-world value of envy. Comparing with the experimental results

for the ultimatum game one finds that human societies are possibly located within the class-stratified phase.

Finally we observe that envy, as defined here, is counterproductive for general welfare, which can be regarded as a generalized social dilemma. The average income drops when envy becomes progressively more important, in particular once strategy condensation sets in. All in all we suggest that the study of games with mixed payoff functions, containing both monetary and psychological components, is a timely subject.

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Appendix

We used in (5) that envy is based on a comparison with the population mean. Alternatively, as in [34], one could start with a pairwise formulation for envy,

$$\text{Envy}|_{\alpha} = \frac{\epsilon}{M} \sum_{\beta \neq \alpha} \log \left(\frac{I^{\alpha}}{I^{\beta}} \right), \tag{13}$$

which expresses that agents make direct, one-to-one comparisons with everybody else. As before, we define with $\bar{I} = (\sum_{\alpha} I^{\alpha})/M$ the average monetary income. Expanding with respect to small relative deviations $(I^{\beta} - \bar{I})/\bar{I}$ from the mean, we obtain

$$\begin{aligned} \log \left(\frac{I^{\alpha}}{I^{\beta}} \right) &= \log(I^{\alpha}) - \log(I^{\beta}) \\ &= \log(I^{\alpha} - \bar{I} + \bar{I}) - \log(I^{\beta} - \bar{I} + \bar{I}) \\ &= \log \left(1 + \frac{I^{\alpha} - \bar{I}}{\bar{I}} \right) - \log \left(1 + \frac{I^{\beta} - \bar{I}}{\bar{I}} \right) \\ &\approx \frac{I^{\alpha} - I^{\beta}}{\bar{I}}. \end{aligned} \tag{14}$$

Noting that the vanishing term $\beta = \alpha$ could have been added to right-hand side of (13), we apply the sum $(\epsilon/M) \sum_{\beta}$ to the linearized expression (14). The result is

$$\text{Envy}|_{\alpha} \approx \epsilon \frac{I^{\alpha} - \bar{I}}{\bar{I}} \approx \epsilon \log \left(\frac{I^{\alpha}}{\bar{I}} \right). \tag{15}$$

which coincides in linear approximation with our original definition, as used in (5).

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