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Massively parallel computation of tropical varieties, their positive part, and tropical Grassmannians ☆

Dominik Bendle [a], Janko Böhm [a], Yue Ren [b], Benjamin Schröter [c]

[a] *TU Kaiserslautern, Germany*
[b] *Durham University, United Kingdom*
[c] *Institut für Mathematik, Goethe-Universität, Frankfurt, Germany*

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We present a massively parallel framework for computing tropicalizations of algebraic varieties which can make use of symmetries using the workflow management system GPI-Space and the computer algebra system Singular. We determine the tropical Grassmannian $\mathrm{TGr}_0(3,8)$. Our implementation works efficiently on up to 840 cores, computing the 14763 orbits of maximal cones under the canonical S_8-action in about 20 minutes.
Relying on our result, we show that the Gröbner structure of $\mathrm{TGr}_0(3,8)$ refines the 16-dimensional skeleton of the coarsest fan structure of the Dressian $\mathrm{Dr}(3,8)$, except for 23 orbits of special cones, for which we construct explicit obstructions to the realizability of their tropical linear spaces. Moreover, we propose algorithms for identifying maximal-dimensional cones which belong to positive tropicalizations of algebraic varieties. We compute the

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E-mail addresses: bendle@rhrk.uni-kl.de (D. Bendle), boehm@mathematik.uni-kl.de (J. Böhm), yue.ren2@durham.ac.uk (Y. Ren), schroeter@math.uni-frankfurt.de (B. Schröter).

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Tropical varieties
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positive Grassmannian $\mathrm{TGr}^+(3,8)$ and compare it to the cluster complex of the classical Grassmannian $\mathrm{Gr}(3,8)$.

Contents

1. Introduction

While massively parallel computations are a standard method in numerical simulations, the use of large scale parallelism is still a fundamental challenge in many areas of computer algebra. In this paper, we develop a massively parallel algorithm for computing tropicalizations. The process of tropicalization associates to an algebraic variety a tropical variety, which is a polyhedral complex. This complex can be regarded as a combinatorial shadow of the original algebraic variety, since it retains a multitude of properties of the algebraic variety. Tropicalization thus constitutes the fundamental connection between algebraic and tropical geometry. This connection has very successfully been used, for example, to obtain enumerative results on the algebraic side, the first and most known being Mikhalkin's determination of the Gromov-Witten invariants of $\mathbb{P}^2$ (Mikhalkin, 2005). Another example which demonstrates the usage of the connection between algebraic and tropical geometry is Adiprasito's, Huh's and Katz's development of the combinatorial Hodge theory which allowed to prove many central conjectures in combinatorics; see for example Adiprasito et al. (2018). Tropical varieties also arise naturally in many areas beyond mathematics, such as optimization (Allamigeon et al., 2018), biology (Speyer and Sturmfels, 2004; Yoshida et al., 2019), economics (Tran and Yu, 2019; Baldwin and Klemperer, 2019), and theoretical physics, for example celestial mechanics (Hampton and Moeckel, 2006; Hampton and Jensen, 2011), mirror symmetry (Gross and Siebert, 2006; Böhm et al., 2017), and scattering amplitudes (Arkani-Hamed et al., 2016).

The process of tropicalization is often challenging due to the combinatorial complexity of the resulting tropical variety and the fact that known algorithms require a large number of Gröbner basis computations. Parallelization is hence a necessity to handle current research problems.

For parallelization, our approach builds on the Singular/GPI-Space framework for massively parallel computations in computer algebra, which has already been successfully used in various large scale applications in algebraic geometry, geometric invariant theory and high-energy physics

(Böhm et al., 2021b; Ristau, 2019; Böhm and Frühbis-Krüger, 2021; Böhm et al., 2018; Reinbold, 2018; Bendle et al., 2020a,b, 2021). This framework is based on the idea of separation of coordination and computation. The mathematical core algorithms are implemented in the computer algebra system Singular (Decker et al., 2021), while the coordination is handled by the workflow management system GPI-Space (Pfreundt and Rahn, 2019), which uses Petri nets for the modeling of concurrent algorithms.

Our implementation solves the so far unsolved problem of designing an efficient large scale parallel traversal of tropical varieties. We find that our approach is highly efficient and that the overhead of communication between threads is not an issue. Considering our main example, which is determining the tropical Grassmannian $\mathrm{TGr}_0(3,8)$ via tropicalization of the respective Grassmannian, we observe that our implementation scales well up to 840 cores (the largest number we have tried) and, in fact, shows a parallel efficiency significantly larger than one, see Section 3.4. We attribute this primarily to the algorithm finding a faster path to the final result when being executed on large amount of resources, relying on the randomized nature of the execution of a Petri net.

Our algorithm builds on the method originally developed by Bogart et al. (2007) and implemented, in a sequential way, by Jensen in Gfan (Jensen, 2017), and later (in more generality) in Singular, see Markwig and Ren (2019). The method is based on the fact that the tropicalization of an irreducible algebraic variety is the support of a polyhedral complex which is connected in codimension one and is a subcomplex of the Gröbner complex. Thus, the tropical variety can be determined via a hypergraph traversal with the vertices of the graph corresponding to the maximal polyhedra and the edges corresponding to their codimension one faces. To pass from one maximal cone to its neighboring cones, we compute tropical links using an algorithm described in Hofmann and Ren (2018), which relies on triangular decomposition and Puiseux expansions.

In addition to tropicalization, we also provide the first general algorithm that computes maximal-dimensional cones of *positive tropicalizations* as introduced by Speyer and Williams (2005). While tropical varieties arise from solutions of systems of polynomial equations, positive tropical varieties arise from solutions that are positive and real. They have been related to graphical models in algebraic statistics (Pachter and Sturmfels, 2004) and, more recently, semialgebraic sets in non-archimedian semidefinite programming (Allamigeon et al., 2020; Jell et al., 2022) and scattering amplitudes in high-energy physics for planar four-dimensional theories (Arkani-Hamed et al., 2016). It is also conjectured (Speyer and Williams, 2005, Conjecture 8.1) that they encode the combinatorics of cluster algebras of finite type. This was proven recently by Brodsky and Stump for a number of important cases (Brodsky and Stump, 2018).

One class of tropical varieties of particular interest and the main source of examples in this article are *tropical Grassmannians* $\mathrm{TGr}_0(k,n)$. In algebraic geometry, Grassmannians $\mathrm{Gr}(k,n)$ parametrize all k-dimensional linear spaces in K^n for a given field K. In tropical geometry, their tropicalizations $\mathrm{TGr}_p(k,n)$ parametrize all k-dimensional tropical linear spaces in $\mathbb{R}^n$ that are realizable over a field K of characteristic p. In both algebraic and tropical geometry, Grassmannians form one of the most important classes of moduli spaces and offer a strong basis for the understanding of general (tropical) varieties.

Aiming at a continuation of the work of Speyer and Sturmfels (2004) and Herrmann et al. (2009) on tropical Grassmannians, we use our implementation to compute the tropical Grassmannian $\mathrm{TGr}_0(3,8)$. Thus, we give a positive answer to Question 37 in Herrmann et al. (2014) on whether its computation is feasible. Furthermore, we compare $\mathrm{TGr}_0(3,8)$ to the *Dressian* $\mathrm{Dr}(3,8)$ described in Herrmann et al. (2014). The Dressian $\mathrm{Dr}(k,n)$ parametrizes all k-dimensional tropical linear spaces in $\mathbb{R}^n$, also known as valuated-matroids, independent of their realizability and is generally of higher dimension than the tropical Grassmannian $\mathrm{TGr}_p(k,n)$ it contains. We show that the Gröbner structure on $\mathrm{TGr}_0(3,8)$ refines the 16-dimensional skeleton of $\mathrm{Dr}(3,8)$ with exception of 23 *extended Fano cones* for which explicit obstructions for the realizability of tropical linear spaces are presented.

We then turn to the positive tropicalization of Grassmannians. In applications, the real points on $\mathrm{Gr}(k,n)$ and their tropicalization on $\mathrm{TGr}_0(k,n)$ are linked, for example, to the soliton solutions of the Kadomtsev-Petviashvili equation (Kodama and Williams, 2013a) with positivity corresponding to regularity at all times (Kodama and Williams, 2013b, 2014). We employ our algorithms to compute all maximal-dimensional cones of $\mathrm{TGr}^+(3,8)$ and compare them to the cluster complex of $\mathrm{Gr}(3,8)$. We verify that (Speyer and Williams, 2005, Conjecture 8.1) holds, which is proven to be true in Brodsky and Stump (2018). This serves as a verification of our computations, and as an alternative proof of the conjecture in this specific case. Recently, two groups of researchers, Arkani-Hamed et al. (2021) as well as Speyer and Williams (2021), have independently proven that the support of the positive tropicalization $\mathrm{TGr}^+(k,n)$ of the Grassmannian is the so-called positive Dressian $\mathrm{Dr}^+(k,n)$. The study of the positive part of tropical Grassmannians as well as positive flag Dressians is currently a very active research area; see for example Boretsky et al. (2022); Joswig et al. (2021). A general overview on positive geometries is presented in Lam (2022).

This article is organized as follows: In Section 2, we fix our notation, and recall some background on tropical geometry required in the subsequent sections. In Section 3, we present our massively parallel algorithm for computing tropical varieties with symmetry, which is described in terms of a Petri net. We give details on our implementation in the Singular/GPI-Space framework. Moreover, we discuss the computation of the tropical Grassmannian $\mathrm{TGr}_0(3,8)$ using our implementation, and provide timings on much time the computation required. In Section 4, we discuss the data produced on $\mathrm{TGr}_0(3,8)$, we analyze its natural fan structures using polymake (Gawrilow and Joswig, 2000), and compare the Grassmannian to the Dressian $\mathrm{Dr}(3,8)$. In Section 5, we propose general algorithms for computing all maximal-dimensional cones in a tropical variety $\mathrm{Trop}(I)$ which belong to the positive tropicalization $\mathrm{Trop}^+(I)$. These algorithms exploit the symmetry of $\mathrm{Trop}(I)$ even though $\mathrm{Trop}^+(I)$ itself need not be entirely symmetric. In Section 6, we compute all maximal-dimensional cones in $\mathrm{TGr}^+(3,8)$ and compare them to the cluster complex of $\mathrm{Gr}(3,8)$, verifying that Speyer and Williams (2005, Conjecture 8.1) holds. In Section 7, we discuss three open questions arising from this article on the fan structure of tropical varieties, the connection to cluster complexes and the topology of real tropicalizations.

All data and other auxiliary materials are available on the website of the Singular/GPI-Space project under the following URL:

https://www.mathematik.uni-kl.de/~boehm/singulargpispace/tropical.htm

Polynomial data is provided in Singular format, while polyhedral data is available in polymake format. The computations were performed using Singular 4-1-2 and polymake 4.3.

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2. Background

In this section, we fix our notation by briefly going over some basic concepts of immediate relevance to us. Our notation is largely compatible with that of Maclagan and Sturmfels (2015), with the exception that we will use polynomials instead of Laurent polynomials and the `max`-convention instead of the `min`-convention. This is because the software that we will be presenting in the latter sections is built on infrastructure using polynomials and the `max`-convention.

Convention 2.1. For the remainder of the section, let K be an algebraically closed field with a non-trivial valuation $\nu\colon K^* \to \mathbb{R}$, ring of integers $\mathcal{O}_K$, and residue field $\mathfrak{K}$. We fix a splitting $\mu\colon (\nu(K^*),+) \to (K^*,\cdot)$ and abbreviate $t^a := \mu(a)$ for $a \in \nu(K^*)$.We use $\overline{(\cdot)}$ to denote the canonical projection $\mathcal{O}_K \to \mathfrak{K}$, and we fix a multivariate polynomial ring $K[x] := K[x_1,\dots,x_n]$.

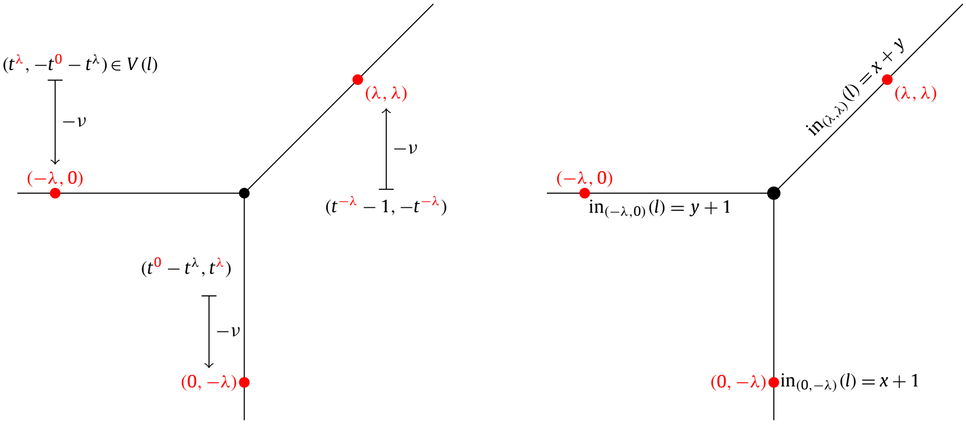

Fig. 1. Visualization of both definitions of the linear hypersurface $\mathrm{Trop}(\langle l \rangle)$ where $l = x + y + 1$.

Definition 2.2. The *initial form* of a polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha \in K[x]$ with respect to a weight vector $w \in \mathbb{R}^n$ is given by

$$\mathrm{in}_w(f) := \sum_{w\cdot\alpha - \nu(c_\alpha) \text{ maximal}} \overline{t^{-\nu(c_\alpha)} c_\alpha} \cdot x^\alpha \in \mathfrak{K}[x],$$

whereas the *initial ideal* of an ideal $I \trianglelefteq K[x]$ with respect to $w \in \mathbb{R}^n$ is given by

$$\mathrm{in}_w(I) := \langle \mathrm{in}_w(g) \mid g \in I \rangle \trianglelefteq \mathfrak{K}[x].$$

The equality in the next definition of a tropical variety is part of the *Fundamental Theorem of Tropical Algebraic Geometry* (Maclagan and Sturmfels, 2015, Theorem 3.2.3). Note that the Fundamental Theorem in Maclagan and Sturmfels (2015) is given for Laurent polynomial rings, but also holds in the setting of polynomial rings.

Definition 2.3. Let $I \trianglelefteq K[x]$ be an ideal and $V(I) \subseteq K^n$ its corresponding affine variety. The *tropical variety* of I is defined to be

$$\begin{aligned} \mathrm{Trop}(I) &:= \mathrm{cl}\Big(\big\{(-\nu(z_1), \ldots, -\nu(z_n)) \in \mathbb{R}^n \mid (z_1, \ldots, z_n) \in V(I) \cap (K^*)^n\big\}\Big) \\ &= \big\{w \in \mathbb{R}^n \mid \mathrm{in}_w(I) \text{ contains no monomial}\big\}, \end{aligned}$$

where $\mathrm{cl}(\cdot)$ denotes the closure in the euclidean topology.

Example 2.4. Let $K = \mathbb{C}\{\{t\}\}$ be the field of complex Puiseux series and ν its natural valuation. Consider the linear ideal $I = \langle x + y + 1 \rangle \subseteq K[x, y]$. Fig. 1 shows $\mathrm{Trop}(I)$ using both definitions, with valuations resp. weight vectors highlighted.

Tropical geometry usually involves Laurent polynomials $K[x^{\pm}]$ and takes place in the algebraic torus $(K^*)^n$. When working with polynomials, it is therefore important to assume all ideals to be saturated with respect to the product of all variables, or *saturated* in short.

Theorem 2.5 *(Structure theorem for tropical varieties (Maclagan and Sturmfels, 2015, Theorem 3.3.5)). Let $I \trianglelefteq K[x]$ be a saturated prime ideal of dimension d. Then* $\mathrm{Trop}(I)$ *is the support of a balanced polyhedral complex, pure of dimension d, connected in codimension* 1.

Definition 2.6. We define the *Gröbner polyhedron* of a homogeneous ideal $I \trianglelefteq K[x]$ around a weight vector $w \in \mathbb{R}^n$ to be

$$C_w(I) := \mathrm{cl}\Big(\{v \in \mathbb{R}^n \mid \mathrm{in}_v(I) = \mathrm{in}_w(I)\}\Big).$$

The *Gröbner complex* $\Sigma(I)$ is the collection of all Gröbner polyhedra of I.

Proposition 2.7 *(Maclagan and Sturmfels, 2015, Theorem 2.5.3). Let $I \trianglelefteq K[x]$ be a homogeneous ideal. Then $C_w(I)$ is a closed convex polyhedron for all $w \in \mathbb{R}^n$, and $\Sigma(I)$ is a finite polyhedral complex. In particular,* $\mathrm{Trop}(I)$ *is the support of the subcomplex of the Gröbner complex consisting of all Gröbner polyhedra whose initial ideals contain no monomials.*

For the sake of simplicity, we will restrict ourselves to what is commonly called the *constant coefficient case*, see Convention 2.8. While the parallel framework described in Section 3 works in full generality, the rest of the paper falls in this very special case.

Convention 2.8. From now on, assume that the field K is either $\overline{\mathbb{Q}}\{\{t\}\}$, $\mathbb{C}\{\{t\}\}$ or $\overline{\mathbb{F}}_p\{\{t\}\}$ for some prime number p, and that all ideals are homogeneous as well as generated by polynomials with coefficients in $\overline{\mathbb{Q}}$, $\mathbb{C}$ or $\overline{\mathbb{F}}_p$.

In particular, all Gröbner polyhedra will be conic, i.e., they are invariant under scaling with a positive real number, and we will therefore refer to the Gröbner polyhedra and Gröbner complexes as *Gröbner cones* and *Gröbner fans*.

Additionally, all Gröbner cones in this paper will be inside the tropical variety unless explicitly specified otherwise. This means that *maximal* or *maximal-dimensional* Gröbner cones will only be (inclusion) maximal or maximal-dimensional among the Gröbner cones on the tropical variety, i.e., a maximal-dimensional Gröbner cone is a Gröbner cone $C_w(I) \subseteq \mathrm{Trop}(I)$ with $\dim C_w(I) = \dim \mathrm{Trop}(I)$.

Moreover, we will use $\mathrm{Trop}(I)$ to denote both the set in Definition 2.3 and the subfan of the Gröbner fan covering it by Proposition 2.7. In written text, we will refer to the latter as the *Gröbner structure* on $\mathrm{Trop}(I)$.

Finally, let us recall the definition of Grassmannians.

Definition 2.9. Let $1 \leq k \leq n$. In the following, we will abbreviate the n-element set $\{1, \dots, n\}$ by $[n]$ and the set of all k-element subsets of $[n]$ by $\binom{[n]}{k}$. The *Grassmannian* $\mathrm{Gr}(k,n)$ is the variety defined by the ideal

$$\mathcal{I}_{k,n} := \Big\langle \mathcal{P}_{I,J} \;\Big|\; I \in \tbinom{[n]}{k-1}, J \in \tbinom{[n]}{k+1} \Big\rangle \subseteq K\Big[p_L \;\Big|\; L \in \tbinom{[n]}{k}\Big],$$

where

$$\mathcal{P}_{I,J} := \sum_{j \in J \setminus I} (-1)^{|\{i \in I \mid i<j\}| + |\{j' \in J \mid j' > j\}|} \cdot p_{I \cup j} \cdot p_{J \setminus j}.$$

The ideal $\mathcal{I}_{k,n}$ is commonly referred to as a *Plücker ideal* or a *Grassmann-Plücker ideal*, while the $\mathcal{P}_{I,J}$ are commonly called *quadratic Plücker relations*. Note that $\mathcal{P}_{I,J}$ is a trinomial if and only if $|I \cap J| = k-2$, in which case we will refer to them as *3-term Plücker relations*. The 3-term Plücker relations do not generate the Plücker ideal if $n \geq k+3 \geq 6$, but they always generate the Plücker ideal up to saturation, see Herrmann et al. (2009, Section 2).

The *tropical Grassmannian* is the tropicalization of the variety of the Plücker ideal, and we will denote it by $\mathrm{TGr}_p(k,n) := \mathrm{Trop}(\mathcal{I}_{k,n})$, where p is the characteristic of the field K. This is well-defined, as the tropical Grassmannian only depends on the characteristic of K, since the coefficients of the Plücker relations are integers.

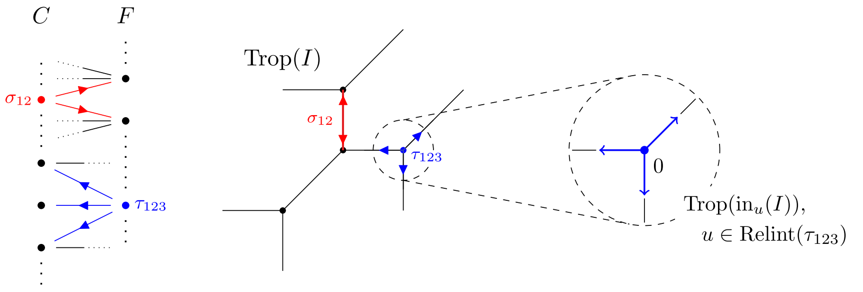

Fig. 2. The bipartite graph Γ illustrating the computation of the tropical variety $\mathrm{Trop}(I)$.

Similarly to the classical Grassmannian, its tropicalization $\mathrm{TGr}_p(k,n)$ is the easiest example of a non-trivial moduli space. Each point on $\mathrm{TGr}_p(k,n)$ corresponds to the tropicalization of a k-dimensional linear space in the projective space $\mathbb{P}^{n-1}$.

In this article, we will mainly focus on the case $p=0$, $k=3$, and $n=8$. This is a continuation of the two articles Speyer and Sturmfels (2004) and Herrmann et al. (2009), which discuss the tropical Grassmannians $\mathrm{TGr}_p(2,n)$, $\mathrm{TGr}_p(3,6)$ and $\mathrm{TGr}_p(3,7)$.

3. Parallel computation of tropical varieties

In this section, we discuss our algorithm for computing tropical varieties, its formulation in terms of Petri nets, and its technical realization relying on our infrastructure for massive parallelization.

We would like to stress that computing tropical varieties in this article means computing tropical varieties of an arbitrary polynomial ideal specified by a finite generating set. There are special cases for which specialized potentially more efficient algorithms exist; for example if the ideal is zero-dimensional (Görlach et al., 2022; Kulkarni, 2020), linear (Rincón, 2013; Hampe et al., 2019), if a tropical basis of the ideal is known (Jensen et al., 2017), or if the Gröbner fan coincides with a secondary fan, which allows to use software like TOPCOM or MPTOPCOM (Rambau, 2002; Jordan et al., 2018).

We start out by recalling the building block algorithms used in our approach for computing tropical varieties:

3.1. Computing tropical varieties

The general framework for computing tropical varieties of polynomial ideals has remained unchanged since its initial conception in Bogart et al. (2007) and implementation in GFAN (Jensen, 2017). Tropical varieties are computed by computing all Gröbner polyhedra of the Gröbner complex, which are contained in the tropical variety. In the special case described in Convention 2.8, the defining inequalities and equations can be read off classical Gröbner bases (Fukuda et al., 2007b, Proposition 2.6). The general case requires so-called tropical Gröbner bases (Chan and Maclagan, 2019) or, equivalently, standard bases (Markwig et al., 2017; Markwig and Ren, 2017).

Computing the tropical variety is best described as the traversal of the following bi-partite graph Γ illustrated in Fig. 2: Fix a polynomial ideal $I \trianglelefteq K[x]$. For the sake of simplicity, assume that I is equidimensional, so that $\mathrm{Trop}(I)$ is pure. Let C be the set of all maximal Gröbner polyhedra contained in $\mathrm{Trop}(I)$. Let F be the set of facets of the polyhedra in C. Let Γ be the bi-partite graph with vertices $C \cup F$ and edges $\{(\sigma,\tau) \in C \times F \mid \sigma \supseteq \tau\}$. Note that $\mathrm{Trop}(I)$ and consequently Γ need not be connected unless I is primary over the algebraic closure $\overline{K}$. To compute $\mathrm{Trop}(I)$, it suffices to compute the defining inequalities and equations of all Gröbner polyhedra in C.

The start of the traversal requires computing at least one maximal polyhedron $\sigma \in C$ for each connected component of Γ. This can be done by adding a generic linear ideal of complementary dimension to I. The tropicalization of the resulting zero-dimensional ideal then yields a point on each connected component of $\mathrm{Trop}(I)$.

Given $\sigma \in C$, computing all adjacent $\tau \in F$ is done by computing all facets of the maximal polyhedron σ. Given $\tau \in F$, computing all adjacent $\sigma \in C$ is more involved and done in the following two steps:

1. Tropical link: For any $u \in \mathrm{Relint}(\tau)$, $\mathrm{Trop}(\mathrm{in}_u(I))$ is called a *tropical link*, as it describes $\mathrm{Trop}(I)$ locally around u. It is also referred to as the star of $\mathrm{Trop}(I)$ (Maclagan and Sturmfels, 2015, Lemma 3.3.6). It has a lineality space of codimension 1, which allows it to be computed using specialized algorithms. The computation of tropical links has been the bottleneck of the algorithm for a long time. This bottleneck has been resolved through newer developments.

Newer versions of GFAN (v0.6 onward) rely on the algorithm in Chan (2013), which constructs the tropical link by computing sufficiently many and sufficiently generic projections. Our implementation relies on the algorithm in Hofmann and Ren (2018), which reduces the problem of constructing the tropical link to computing zero-dimensional tropical varieties by intersecting with tropical varieties of complementary dimension. The computation of the zero-dimensional tropical varieties is then done with polynomial system solving techniques over local fields. Both algorithms rely heavily on the simple combinatorial structure of the tropical link.

2. Gröbner walk: The Gröbner walk (Collart et al., 1997; Fukuda et al., 2007a) is a well-established technique for transforming Gröbner bases with respect to one ordering to another. It is used to compute Gröbner bases with respect to the different orderings required to read off the inequalities and equations of the Gröbner cones in the special case described in Convention 2.8. There is a straightforward generalization of the Gröbner walk to standard bases for the general case (Markwig and Ren, 2017), the only difficulty being the construction of the Gröbner polyhedra. Constructing Gröbner cones for the case in Convention 2.8 relies heavily on computing reduced Gröbner bases, and the absence of a well-ordering renders the reduction process impossible. If the ideal is homogeneous however, the ordering on it is still sufficiently close to a well-ordering that it allows for a partial reduction of the standard basis that is sufficient for constructing Gröbner polyhedra.

3.2. Massive parallelization in computer algebra

Our implementation builds on a framework for massively parallel computations in computer algebra (Böhm et al., 2021b; Ristau, 2019), which combines the computer algebra system SINGULAR with the workflow management system GPI-SPACE. This framework originated in work on a parallel smoothness criterion for algebraic varieties (Böhm et al., 2021b; Ristau, 2019), and has been used in Reinbold (2018); Böhm et al. (2021a) to realize a massively parallel traversal of a complete fan for computing GIT-fans. For an overview and more applications, see Böhm and Frühbis-Krüger (2021); Böhm et al. (2018) and Bendle et al. (2020a,b, 2021), respectively. The results of the current section extend the traversal of complete fans developed for the GIT-fan algorithm to pure fans connected in codimension with application to the computation of tropical varieties. The results are based on the thesis of the first author (Bendle, 2018).

The workflow management system GPI-SPACE is based on the idea of separation of coordination and computation (Gelernter and Carriero, 1992). In the coordination layer it uses the language of Petri nets (Petri, 1962) to model a computer program in the form of a concurrent system. It allows for running parallel computations on anything from a personal computer to large scientific computing clusters, and consists of the following three main components (Böhm et al., 2021b, Section 4):

(1) a distributed runtime system managing available resources and assigning jobs to resources,
(2) a virtual memory layer allowing processes to communicate and share data,
(3) a workflow manager tracking the global structure and state of a program formulated in terms of a so-called Petri net.

Definition 3.1. A *Petri net* is a finite bipartite directed graph $N = (P, T, F)$, where P and T are disjoint vertex sets called *places* and *transitions* respectively, and where the set of edges $F \subseteq (P \times T) \cup (T \times P)$ is called the set of *flow relations*. Given $p \in P$ and $t \in T$, we call p an *input* to t if $(p, t) \in F$ and p an *output* of t if $(t, p) \in F$.

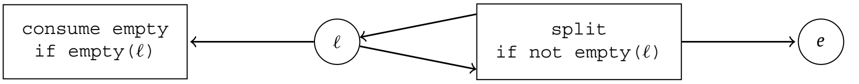

Fig. 3. Petri net unwrapping a list.

Petri nets depict a static model of the algorithm, with transitions representing processes and places which can hold data passed between them. The dynamics of the algorithm, i.e., its execution, is described via the notion of markings representing the data present on the places:

Definition 3.2. Let (P, T, F) be a Petri net. A *marking* M is a map $M\colon P \to \mathbb{N}_{\geq 0}$, and we say a place $p \in P$ *holds k tokens* under M if $M(p) = k$.

We call a transition $t \in T$ *enabled* if $M(p) > 0$ for all $p \in P$ with $(p, t) \in F$. In this case, the transition t may be *fired* by consuming one token of each input and returning one token in each output, which leads to a new marking M' given by

$$M'(p) := M(p) - |\{(p,t)\} \cap F| + |\{(t,p)\} \cap F|$$

for all $p \in P$. We denote the firing process by $M \xrightarrow{t} M'$. A Petri net with a marking is *executed* by firing a random enabled transition.

An important principle one usually adheres to in modeling the state of algorithms via markings in GPI-Space is locality: Firing enabled transitions should not block other enabled transitions from firing simultaneously.[1] To ensure this, one can, for example, impose restrictions on places in a way that any token that is in an input to multiple transitions can only be consumed by a single well-defined transition.

Example 3.3. In Fig. 3 we show an example of a Petri net, which unwraps a list of tokens. In illustrations of Petri nets, places are drawn as circles while transitions are shown as boxes. In the illustration, we specify conditions using the notation "`if (not)` *condition*". Note that conditions usually reference names of places while the conditions are actually imposed on the tokens held by the respective places. In the example, both `split` and `consume empty` have input ℓ. However, `split` only consumes non-empty lists from ℓ splitting off one entry and placing it on e, while `consume empty` only consumes empty lists. Thus, there is always a single well-defined transition that can fire.

While markings are the basic tool for describing the dynamics of algorithms (and in the fundamental theory the only tool), there are two important extensions of the concept of Petri nets realized in GPI-Space to facilitate the efficient use of Petri nets for practical programming purposes: Tokens are allowed to carry data (this corresponds in the theory to so called *colored* Petri nets). Moreover, transitions are allowed to take some time to execute, accommodating the situation that transitions have consumed their input, but have not yet produced their output (referred to as *timed* Petri nets).

As a consequence of locality, enabled transitions may fire simultaneously. This includes single enabled transitions with enough input tokens to fire multiple times (Fig. 4 top), which is referred to as *data parallelism*, and multiple enabled transitions with separate input tokens (Fig. 4 bottom), which is referred to as *task parallelism*.

3.3. Parallel traversals of tropical varieties

Fig. 5 shows the Petri net modeling the traversal of tropical varieties as outlined in Section 3.1. In addition to the standard Petri net arrows, dashed arrow indicate read-only access to places (typically containing a single token). Moreover, dotted arrows indicate read/write access of transitions to

[1] A typical exception to this principle is a Petri net in which multiple transitions utilizing different algorithms but producing identical results for identical input compete for a given amount of tokens on a common input place.

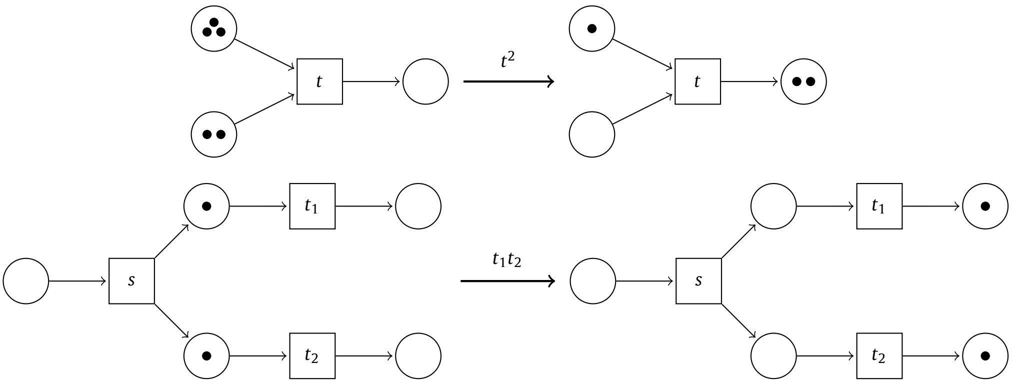

Fig. 4. Data parallelism (top) and task parallelism (bottom).

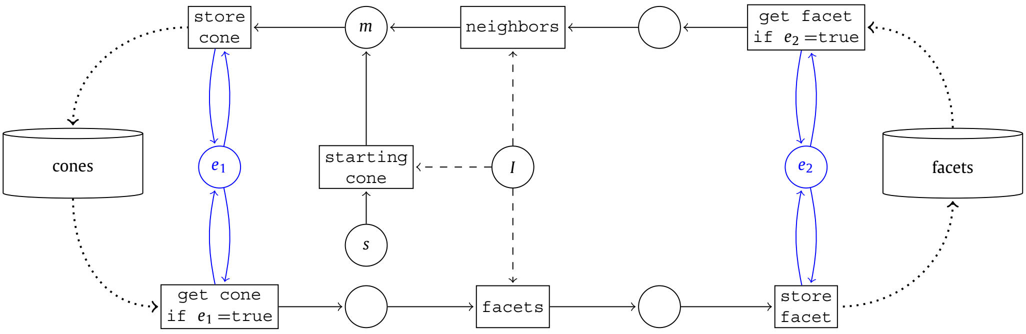

Fig. 5. Petri net modeling the traversal of a tropical variety.

storage units. A token consumed by the transition governs which data is written to or read from the storage unit. The determination of the maximal cones relies on the computation of Gröbner bases in combination with convex geometry, while the computation of the neighboring cones requires to find tropical links. We separate these fundamental task to accommodate different job sizes. The Petri net consists of the following places and transitions.

Place *s*: This place holds a single control token which allows the transition `starting cone` to fire exactly once at the start of the computation.
Place *I*: This place contains read-only input data including the given ideal and its symmetries, which are required by the transitions computing the starting cone, its facets and neighbors.
Transition `starting cone`: Computes a random maximal Gröbner cone of the tropical variety and places it on m. This happens exactly once, at the beginning of the computation, since the transition consumes the structureless token indicted as a black dot in the figure.
Place *m*: This place holds the maximal Gröbner cones of the tropical variety as they are produced in the course of the algorithm and are supposed to be inserted into the external storage.
Transition `store cone`: Takes a Gröbner cone and inserts it into the external storage as long as the cone is not already processed. Subsequently, it replaces the boolean token in e_1 by `true` or `false` depending on whether unprocessed cones remain in the storage.
Place e_1: Holds a single token with value `true` if the storage contains unprocessed cones, and `false` otherwise. This token is consumed and written by the transitions `store cone` and `get cone` thus preventing both transitions to access the storage simultaneously.
Transition `get cone`: Retrieves an unprocessed Gröbner cone and marks it as processed, provided that e_1 holds a token with value `true`.
Transition `facets`: Computes the facets of a Gröbner cone.

Table 1
Timings for computing the tropical Grassmannians $TGr_0(3,7)$ and $TGr_0(3,8)$ in parallel.

$TGr_0(3,7)$

nodes	cores	time [s]	speedup	eff.
1	1	792.8	1.000	1.000
1	2	382.8	2.070	1.035
1	4	191.1	4.147	1.037
1	8	98.1	8.080	1.001
1	12	74.1	10.691	0.891
1	16	58.0	13.653	0.853
2	24	42.8	18.522	0.772
2	32	39.7	19.942	0.623

$TGr_0(3,8)$

nodes	cores	time [s]	speedup	eff.
1	15	98926.1	*15.000	*1.000
2	30	37398.7	39.675	1.322
4	60	14486.3	102.435	1.707
8	120	6597.3	224.925	1.874
16	240	3297.9	449.955	1.874
24	360	2506.0	592.125	1.645
32	480	2001.7	741.285	1.544
40	600	1509.6	982.935	1.638
48	720	1267.3	1170.855	1.626
56	840	1188.2	1248.825	1.487

Transitions `store facet` and `get facet`: These are analogous to `store cone` and `get cone`, handling facets instead of cones.
Place e_2: Analogous to place e_1 for the case of facets.
Transition `neighbors`: Computes all maximal Gröbner cones of the tropical variety which are incident to a given facet.

The algorithm terminates if e_1 and e_2 each hold a token with value `false`, that is, no unprocessed cones and facets remain in the storages, and moreover the input places to `store cones` and `store facet` are empty. This is checked via token counters, which were omitted from Fig. 5 for clarity of presentation.

3.4. Timings for the tropical Grassmannian

Table 1 illustrates the timings (in seconds) for computing the tropical Grassmannians $TGr_0(3,7)$ and $TGr_0(3,8)$ in dependence of the number of CPU cores used in the respective computation. Moreover, the column `speedup` lists the ratios between the single- and multi-core computation times, while the column `efficiency` specifies the parallel efficiency obtained by dividing the speedup by the number of cores. Note that, due to the size of $TGr_0(3,8)$, a sequential (1-core) computation was not feasible. All `efficiency` numbers are thus based on the 15 core timing (marked with * in Table 1).

The computations were run on a cluster at the Fraunhofer ITWM (Fraunhofer ITWM, 2018). Each node of the cluster is fitted with two Intel Xeon E5-2670 processors and 64 GB of memory, amounting to 16 CPU cores per node at a base clock of 2,6 GHz (no hyperthreading). The computations of $TGr_0(3,8)$ were done with a fixed configuration of 15 compute jobs and one storage interface job per node.

As shown in Table 1, computing $TGr_0(3,7)$ scales favorably up to 8 CPU cores, whereafter a noticeable drop in efficiency can be observed. This is expected, since up to symmetry $TGr_0(3,7)$ is covered by only 125 cones, and hence eventually the number of cores exceeds the maximum queue size.

Fig. 6 shows the timings and the efficiency graph for $TGr_0(3,8)$. The timings scale very well, with no significant drop in efficiency, even to more than 800 cores. Due to the significantly larger size of $TGr_0(3,8)$ we do not encounter the queue size effect. There is no visible decrease in efficiency when increasing the core count (as one perhaps would have expected from classical multi-threaded settings of parallelism due to increased communication overhead).

In fact, we observe a surprising surge in efficiency around 60 cores. One should note that a comprehensive analysis of this behavior is non-trivial due to the interaction of various algorithmic and technical effects. We have identified two possible explanations the observed behavior: One reason we attribute to technical properties of the cluster hardware: In Böhm et al. (2021b), experiments on the same cluster with another algorithm have shown that distributing the number of used processor cores over more machines can lead to a speedup, which is arising from mitigating a memory bottleneck. However, leading to a speedup in a range of about 30%, this can only partially explain the behavior of the efficiency graph, which we attribute mainly to a different effect, which was also observed in

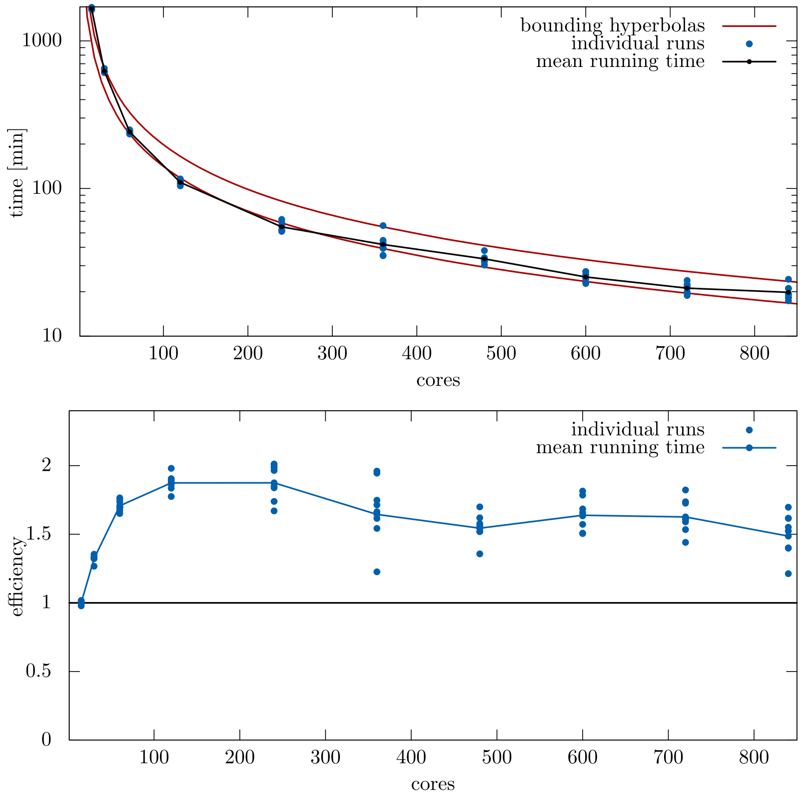

Fig. 6. Timings and efficiency of $\mathrm{TGr}_0(3,8)$.

Böhm et al. (2021b) in a different setting (testing smoothness of algebraic varieties): Massively parallel implementations can lead to a superlinear speedup by allowing the randomized algorithm to find a faster path to the final result. In our setting different paths correspond to different choices of distinct sets of representatives of the orbits under the finite symmetry group action. It is important to note that although in theory such an effect can also be realized by time-slicing on one or few cores, this is not at all practicable in our setting since continued loading and unloading of large amounts of data is highly inefficient. Taking advantage of the observed speedup thus requires a honestly massively parallel approach.

4. Tropical Grassmannians and Dressians

In this section, we compare the tropical Grassmannian $\mathrm{TGr}_0(3,8)$ to the Dressian $\mathrm{Dr}(3,8)$ described in Herrmann et al. (2014), i.e., we compare the moduli of realizable tropical linear spaces or realizable valuated matroids with the moduli of all tropical linear spaces or valuated matroids. The main difficulty stems from the fact that both are covered by thousands of cone orbits with respect to the sizeable group S_8. We begin with a brief introduction and some formal definitions.

Definition 4.1. Let $1 \leq k \leq n$ and recall the 3-term Plücker relations $\mathcal{P}_{I,J}$ from Definition 2.9. The *Dressian* $\mathrm{Dr}(k,n)$ is the intersection of their tropical hypersurfaces:

$$\mathrm{Dr}(k,n) := \bigcap_{I,J} \mathrm{Trop}(P_{I,J}),$$

where the intersection is taken over all sets $I \in \binom{[n]}{k-1}$, $J \in \binom{[n]}{k+1}$ with $|I \cap J| = k-2$. By definition, the Dressian is the support of the common refinement of the Gröbner subfans covering $\mathrm{Trop}(P_{I,J})$. We refer to this polyhedral fan as the *Plücker structure* and we will use $\mathrm{Dr}(k,n)$ to denote both the set and the polyhedral fan covering it.

Unlike the Gröbner structure on $\mathrm{TGr}_p(k,n)$, the Plücker structure is the coarsest possible structure on $\mathrm{Dr}(k,n)$: for any two vectors that lie in distinct maximal cones there is a tropical 3-term Plücker relation whose maximum is attained twice, but on different terms. Thus, a positive combination of these vectors attains the maximum only at a single term.

The Dressian is a *tropical prevariety*, i.e., it is the intersection of the tropical hypersurfaces of a finite generating set. Usually, tropical prevarieties and tropical varieties have little in common besides one being trivially contained in the other. In fact, merely testing both objects for equality is a hard task (Theobald, 2006; Görlach et al., 2021), and it is unclear what distinguishes a generating set for whom equality holds, commonly called a *tropical basis* (Joswig and Schröter, 2018).

Due to the inherent combinatorics of the Plücker ideal and the 3-term Plücker relations, the Dressian is interesting for many reasons:

- The Dressian is the moduli space of all tropical linear spaces, also known as valuated matroids. Similar to the Grassmannian in algebraic geometry, the Dressian can be regarded as one of the simplest moduli spaces in tropical geometry.
- The hypersimplex $\Delta(k,n)$ is the moment polytope for the torus action on the complex Grassmannian. The Dressian $\mathrm{Dr}(k,n)$ consists of all points in the secondary fan of $\Delta(k,n)$ which induce matroid subdivisions (Gel'fand et al., 1987; Maclagan and Sturmfels, 2015). Moreover, the Plücker structure coincides with the secondary fan structure on $\mathrm{Dr}(k,n)$ (Olarte et al., 2019).
- Recent work of Huh and Brändén (Brändén and Huh, 2020, Theorem 8.7) regard the Dressian $\mathrm{Dr}(d,n)$ as the tropicalization of the space of Lorentzian polynomials supported on $\Delta(d,n)$, i.e., on the basis of the uniform matroid of rank d on n elements.

There are several explicit computational and theoretical results which relate the Dressian to the tropical Grassmannian sitting inside it. In particular, all existing computations verify that the Plücker structure of the Dressian coarsens the Gröbner fan structure of the tropical Grassmannian in characteristic $p=0$, i.e., that there is a subfan of the Dressian supported on the tropical Grassmannian. This subfan is generally strictly coarser than the Gröbner fan restricted to it. A brief summary of these structures can be found in Remark 4.2 and Remark 4.3.

Remark 4.2. The tropical Grassmannian $\mathrm{TGr}_p(2,n)$ with the Gröbner structure is independent of the characteristic, has $2^{n-1}-n-1$ rays in $\lceil \frac{n-3}{2} \rceil$ orbits and $(2n-5)!!$ maximal cones. The number of S_n-symmetry classes equals the number of trivalent trees with n leaves (OEIS Foundation Inc., 2020, A000672). It is the moduli space of tropical lines, phylogenetic trees with n labeled leaves, and tropical rational curves of genus 0 with n marked points. This structure is the coarsest fan structure. The tropical Grassmannian and Dressian agree. The rays correspond to split hyperplanes and the tropical Grassmannian is isomorphic to the split complex. The connected matroids in the corresponding matroid subdivisions are sparse paving matroids. See Speyer and Sturmfels (2004), Herrmann and Joswig (2008) and Joswig and Schröter (2017) for further details.

Remark 4.3. The tropical Grassmannian $\mathrm{TGr}_p(3,6)$ is independent of the characteristic, the Dressian and tropical Grassmannian have the same support, but the Gröbner structure is a refinement of the Dressian. To be precise, there is a cone over a three-dimensional bipyramid in the Plücker structure which the Gröbner structure refines into three cones over tetrahedra. The tropical Grassmannian $\mathrm{TGr}_p(3,6)$ with the Gröbner structure is a simplicial fan.

The tropical Grassmannian $\mathrm{TGr}_p(3,7)$ depends on the characteristic. The Gröbner structure on $\mathrm{TGr}_2(3,7)$ is coarse, but not a subfan of the Dressian $\mathrm{Dr}(3,7)$. The Gröbner structure on $\mathrm{TGr}_p(3,7)$ is a refinement of the Plücker structure if $p \neq 2$. The number of rays and maximal cones is summa-

Table 2
The number of rays and maximal cones of the tropical Grassmannians $\mathrm{TGr}_p(3,n)$ for $n=6,7$ and all p with the Gröbner structure (G) inherited from the Gröbner fan or the coarsest Plücker structure (D) inherited from the Dressian. The columns "orbits" contain the number of S_n orbits of rays resp. maximal cones.

d	n	p	G/D	# rays	# orbits	# max. cones	# orbits
3	6	any	G	65	**3**	1035	**7**
3	6	any	D	65	**3**	1005	**7**
3	7	$p=2$	G	751	**7**	252 420	**125**
3	7	$p\neq 2$	G	721	**6**	252 000	**125**
3	7	$p\neq 2$	D	616	**5**	211 365	**94**

rized in Table 2. Further details can be derived from Speyer and Sturmfels (2004) and Herrmann et al. (2009).

We extend the previous results by combining our computation in Section 3 with the following corrected result of Herrmann et al. (2014) describing the Dressian $\mathrm{Dr}(3,8)$.

Proposition 4.4 *(Herrmann et al., 2014, Theorem 31). The Dressian* $\mathrm{Dr}(3,8)$ *is a non-pure* 17*-dimensional fan with an* 8*-dimensional lineality space, it consists of* 15 470 *rays in twelve* S_8*-orbits and* 117 595 485 *cones of dimension* 16 *in* 4 789 S_8*-orbits.*

Remark 4.5. Our computations showed that the data of Herrmann et al. (2014) misses a 16-dimensional simplicial cone of orbit size 840. The orbit is represented by a cone containing the corank vector of the sparse-paving matroid with non-bases 015, 024, 067, 126, 137, 235, 346, 457. It is a cone whose eight rays all correspond to vertex splits all lying in the same S_8-orbit. The 840 cones are the only 16-dimensional cones with that property.

The following theorem is a summary of a large scale computation using Singular, polymake and the framework described in Section 3.

Theorem 4.6. *The Gröbner subfan supported on the tropical Grassmannian* $\mathrm{TGr}_0(3,8)$ *is a* 16*-dimensional fan with an* 8*-dimensional lineality space, it consists of* 732 725 *rays in* 95 S_8*-orbits and* 278 576 760 *maximal cones in* 14 763 S_8*-orbits.*

Moreover, the coarsest fan structure supported on $\mathrm{TGr}_0(3,8)$ *is the Plücker structure, i.e., a subfan of the Dressian, consisting of* 15 470 *rays in twelve* S_8*-orbits and* 117 445 125 *maximal cones in* 4 766 S_8*-orbits.*

Proof. We computed the tropical Grassmannian $\mathrm{TGr}_0(3,8)$ with the Gröbner structure using the methods of Section 3.

In order to confirm that the Plücker structure on $\mathrm{TGr}_0(3,8)$ is well-defined, we tested that the relative interior of any 16-dimensional Dressian cone is either contained in or disjoint to $\mathrm{TGr}_0(3,8)$. For that, we verified that any maximal Gröbner cone on $\mathrm{TGr}_0(3,8)$ is contained in a 16-dimensional Dressian cone, and that every 15-dimensional Gröbner cone on $\mathrm{TGr}_0(3,8)$ intersecting the relative interior of a 16-dimensional Dressian cone is contained in exactly two maximal Gröbner cones. □

Remark 4.7. It is known that the Dressian $\mathrm{Dr}(3,8)$ has 14 S_8-orbits of 17-dimensional cones whose relative interior does not intersect the tropical Grassmannian $\mathrm{TGr}_0(3,8)$ as each matroid subdivision induced by such relative interior point contains a parallel extension of the Fano matroid as a cell; see Hampe et al. (2019, Remark 5.4).

Moreover, there are 23 S_8-orbits of 16-dimensional cones in the Dressian $\mathrm{Dr}(3,8)$ whose relative interior does not intersect the tropical Grassmannian $\mathrm{TGr}_0(3,8)$ as well. All of these cones sit in one of the 17-dimensional cones and each matroid subdivision induced by a relative interior point contains a cell that is either the matroid polytope of a parallel extension of the Fano matroid or a principle extension of a circuit hyperplane of the Fano matroid.

Thus, the following polynomial is a witness for the aforementioned of all these $37 = 14 + 23$ Dressian orbits:

$$f := 2\, p_{123}p_{467}p_{567} - p_{367}p_{567}p_{124} - p_{167}p_{467}p_{235} - p_{127}p_{567}p_{346} - p_{126}p_{367}p_{457} \\ - p_{237}p_{467}p_{156} + p_{134}p_{567}p_{267} + p_{246}p_{567}p_{137} + p_{136}p_{267}p_{457} \in \mathcal{I}_{3,7} \subset \mathcal{I}_{3,8},$$

i.e., for any of the aforementioned 37 orbits $\mathcal{D} \subseteq \mathrm{Dr}(3,8)$ there exists a Dressian cone $\sigma \in \mathcal{D}$ such that $\mathrm{in}_w(f) = 2 \cdot p_{123}p_{467}p_{567}$ for all relative interior points $w \in \mathrm{Relint}(\sigma)$.

This observation agrees with Hampe et al. (2019, Proposition 5.5), which states that regular matroid subdivisions that contain matroid polytopes of extensions of the Fano matroid lead to points in the Dressian that are not on the tropical Grassmannian.

As an immediate consequence, we get:

Theorem 4.8. *The quadratic Plücker relations together with the cubic polynomials in the S_8-orbit of f form a tropical basis of the Plücker ideal $\mathcal{I}_{3,8}$.*

Proof. By definition, the Plücker relations generate the Plücker ideal $\mathcal{I}_{3,8}$. Recall that $\mathrm{Dr}(3,8)$ is 17-dimensional. By Remark 4.7 or alternatively (Hampe et al., 2019, Remark 5.4), the polynomials in $S_8 \cdot f$ are witnesses for all 17-dimensional cones of $\mathrm{Dr}(3,8)$, i.e., for every point w inside a 17-dimensional cone of $\mathrm{Dr}(3,8)$ there is a $\sigma \in S_8$ with $w \notin \mathrm{Trop}(\sigma \cdot f)$. In Theorem 4.6 we verified that any 16-dimensional cone of $\mathrm{Dr}(3,8)$ either lies on $\mathrm{TGr}_0(3,8)$ or has a relative interior disjoint to it. In Remark 4.7, we verified that the polynomials in $S_8 \cdot f$ are witnesses for the latter. □

During our computations, we also encountered the following phenomenon, which will be relevant for Section 6.2. Unfortunately, this structural result does not hold for higher tropical Grassmannians as we discuss in Section 7.

Theorem 4.9. *For any $v \in \mathrm{TGr}_0(3,8)$ and $w \in \mathbb{R}^{56}$ we have*

$$v \text{ and } w \text{ lie in the relative interior of the same cone of } \mathrm{Dr}(3,8) \\ \iff \quad \mathrm{in}_w(\mathcal{I}_{3,8}) : p^\infty = \mathrm{in}_v(\mathcal{I}_{3,8}) : p^\infty,$$

where $(\cdot) : p^\infty$ denotes the saturation by the product of all Plücker variables.

Proof. The statement was proven through explicit computations in Singular. Note that it suffices to verify that weight vectors in the same Dressian cone have the same saturated initial ideal, because weight vectors in different Dressian cones have different saturated initial ideals:

Let $w, v \in \mathrm{TGr}_p(k,n)$ be in two distinct Dressian cones, which means there exist a three term Plücker relation $\mathcal{P} = s_0 + s_1 + s_2$ such that $\mathrm{in}_w(\mathcal{P}) \neq \mathrm{in}_v(\mathcal{P})$, and assume that $\mathrm{in}_w(\mathcal{I}_{k,n}) : p^\infty = \mathrm{in}_v(\mathcal{I}_{k,n}) : p^\infty$. We now distinguish between two cases.

The first case is $\mathrm{in}_w(\mathcal{P}) \neq \mathcal{P}$ and $\mathrm{in}_v(\mathcal{P}) \neq \mathcal{P}$, say $\mathrm{in}_w(\mathcal{P}) = s_0 + s_1$ and $\mathrm{in}_v(\mathcal{P}) = s_0 + s_2$. Then

$$s_0 = \mathrm{in}_w(\mathrm{in}_v(\mathcal{P})) \in \mathrm{in}_w(\mathrm{in}_v(\mathcal{I}_{k,n}) : p^\infty) = \mathrm{in}_w(\mathrm{in}_w(\mathcal{I}_{k,n}) : p^\infty) \subseteq \mathrm{in}_w(\mathcal{I}_{k,n}) : p^\infty$$

contradicting that $\mathrm{in}_w(\mathcal{I}_{k,n})$ and $\mathrm{in}_v(\mathcal{I}_{k,n})$ are monomial free. Note that the inclusion above holds because $\mathrm{in}_w(\mathrm{in}_w(\mathcal{I}_{k,n}) : p^\infty)$ is generated by elements of the form $\mathrm{in}_w(h)$ with $p^\alpha \cdot h \in \mathrm{in}_w(\mathcal{I}_{k,n})$, and because we have $p^\alpha \cdot \mathrm{in}_w(h) = \mathrm{in}_w(p^\alpha \cdot h) \in \mathrm{in}_w(\mathrm{in}_w(\mathcal{I}_{k,n})) = \mathrm{in}_w(\mathcal{I}_{k,n})$, i.e., $\mathrm{in}_w(h) \in \mathrm{in}_w(\mathcal{I}_{k,n}) : p^\infty$.

The second case is $\mathrm{in}_w(\mathcal{P}) = \mathcal{P}$ or $\mathrm{in}_v(\mathcal{P}) = \mathcal{P}$, say $\mathrm{in}_w(\mathcal{P}) = s_0 + s_1 + s_2$ and $\mathrm{in}_v(\mathcal{P}) = s_0 + s_2$. As $s_0 + s_1 + s_2$ and $s_0 + s_2$ are elements of $\mathrm{in}_w(\mathcal{I}_{k,n})$ and $\mathrm{in}_v(\mathcal{I}_{k,n})$, respectively, both polynomials are also elements of their saturation which coincide by assumption. Hence, $s_1 = (s_0 + s_1 + s_2) - (s_0 + s_2) \in \mathrm{in}_w(\mathcal{I}_{k,n}) : p^\infty = \mathrm{in}_v(\mathcal{I}_{k,n}) : p^\infty$, again contradicting that $\mathrm{in}_w(\mathcal{I}_{k,n})$ and $\mathrm{in}_v(\mathcal{I}_{k,n})$ are monomial free. □

```
> ring r = 0,(a,b,c,d,x,y),dp;
> ideal I = x*(a-b)-y*(c-d), x*(c-d)-y*(a-b);
> LIB "tropical.lib";
> tropicalVariety(I);
RAYS                              MAXIMAL_CONES
-1  0  0  0   0 0 # 0             {0 2} # Dimension 4
-1 -1  0  0  -1 0 # 1             {1 2}
-1 -1  0  0   0 0 # 2             {2 3}
-1 -1  0  0   1 0 # 3             {0 7}
 0  0  0  0  -1 0 # 4             {0 11}
 0  0  0  0   1 0 # 5             {1 4}
 0 -1  0  0   0 0 # 6             {3 5}
 0  0 -1  0   0 0 # 7             {2 6}
 1  1  0  0  -1 0 # 8             {4 8}
 1  1  0  0   0 0 # 9             {5 10}
 1  1  0  0   1 0 # 10            {6 7}
 1  1  1  0   0 0 # 11            {6 11}
                                  {7 9}
LINEALITY_SPACE                   {8 9}
 1  1  1  1   0 0 # 0             {9 10}
 0  0  0  0   1 1 # 1             {9 11}
```

Fig. 7. SINGULAR code for Example 4.10 (output cleaned up for clarity).

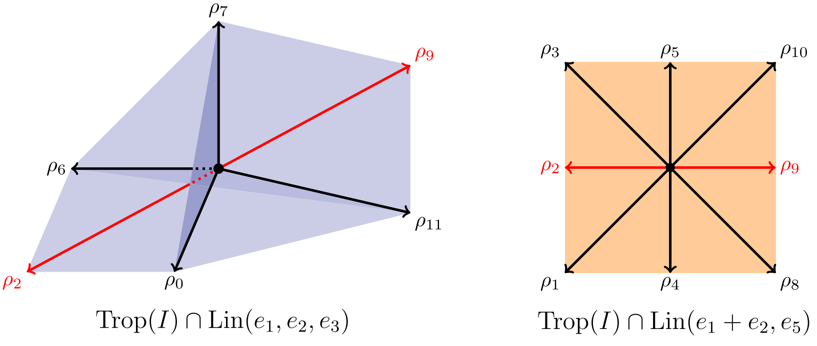

Fig. 8. Tropical variety in Example 4.10.

In general, the Gröbner structure on a tropical variety is far from being as coarse as possible, which incurs many iterations in the traversal of the tropical variety that might seem unnecessary. For example, there is a maximal cone in the tropical Grassmannian $\mathrm{TGr}_0(3, 8)$ equipped with the Plücker structure that is refined into 2620 maximal cones with the Gröbner structure. Therefore, an important open question is whether the tropical variety can be equipped with a natural polyhedral structure that is coarser than that of the Gröbner fan.

Theorem 4.9 states that saturated initial ideals provide such a structure for the tropical Grassmannian $\mathrm{TGr}_0(3, 8)$. While it is unknown whether it holds for all tropical Grassmannians over fields of characteristic 0, the result does not generalize to arbitrary tropical varieties. The following shows a tropical variety where gluing the Gröbner cones with the same saturated initial ideal yields a set of cones that do not fit together as a polyhedral fan:

Example 4.10. Consider the homogeneous ideal

$$I := \langle x(a-b) - y(c-d), x(c-d) - y(a-b)\rangle \trianglelefteq \mathbb{C}\{\{t\}\}[a, b, c, d, x, y].$$

Fig. 7 shows a SINGULAR computation which reveals that its tropical variety $\mathrm{Trop}(I)$ in $\mathbb{R}^6$ is four-dimensional with a two-dimensional lineality space. The Gröbner subfan supported on $\mathrm{Trop}(I)$ consists of 12 rays and 16 maximal cones. Fig. 8 shows the intersection $\mathrm{Trop}(I)$ with $\mathrm{Lin}(e_1, e_2, e_3, e_5)$, which removes the two-dimensional lineality space and whose maximal cones are contained in either

$\mathrm{Lin}(e_1, e_2, e_3)$ or $\mathrm{Lin}(e_1 + e_2, e_5)$. In other words, the following equality holds both in terms of sets and polyhedral complexes:

$$\begin{aligned}\mathrm{Trop}(I) = \Big(\big(\mathrm{Trop}(I) \cap \mathrm{Lin}(e_1, e_2, e_3)\big) \cup \big(\mathrm{Trop}(I) \cap \mathrm{Lin}(e_1 + e_2, e_5)\big)\Big) \\ + \mathrm{Lin}(e_1 + e_2 + e_3 + e_4, e_5 + e_6).\end{aligned}$$

The intersection $\mathrm{Trop}(I) \cap \mathrm{Lin}(e_1, e_2, e_3)$ resembles the standard tropical plane in $\mathbb{R}^3$ with two opposing maximal cones barycentrically subdivided, while the intersection $\mathrm{Trop}(I) \cap \mathrm{Lin}(e_1 + e_2, e_5)$ resembles $\mathbb{R}^2$ divided into octants. In the second intersection all saturated initial ideals with respect to relative interior points of maximal cones are generated by $a - b$ and $c - d$, which can easily checked by hand: I contains the polynomials $p = (x^2 - y^2)(a - b)$ and $q = (x^2 - y^2)(c - d)$, and for any weight vector of the form $w = \lambda_1 \cdot (e_1 + e_2) + \mu e_5$ with $\mu \neq 0$ the initial forms $\mathrm{in}_w(p)$ and $\mathrm{in}_w(q)$ will be monomial multiples of $a - b$ and $c - d$, respectively. Therefore, gluing all Groebner cones of $\mathrm{Trop}(I)$ that have the same initial ideal does not yield a polyhedral fan.

5. Computing positive tropicalizations

In this section, we recall the notion of positive tropicalization by Speyer and Williams (2005). Further, we introduce algorithms for testing which maximal-dimensional Gröbner cones lie in the positive tropicalization. These algorithms exploit the symmetry of the tropical variety even though the positive tropicalization inside of it may not be symmetric with respect to it.

We distinguish between cones whose initial ideals are binomial and cones whose initial ideals are not. For binomial cones, we state a simple combinatorial algorithm. For non-binomial cones, we reduce the problem to dimension zero which can then be tackled symbolically, numerically or with a mix of both. In Remark 5.18 we briefly discuss the challenges of testing whether lower-dimensional cones lie in the positive tropicalization.

Convention 5.1. For the remainder of the section, let $K := \mathbb{C}\{\{t\}\}$ be the field of complex Puiseux series and let $R_{>0}$ denote the set of complex Puiseux series whose lowest term is real and positive:

$$R_{>0} := \Big\{ \sum_{\alpha \geq \lambda} c_\alpha t^\alpha \in K \;\Big|\; 0 \neq c_\lambda \in \mathbb{R} > 0 \Big\}.$$

Fix an ideal $\mathcal{I} \trianglelefteq K[x] := K[x_1, \dots, x_n]$ that is generated over the subfield of real Puiseux series, i.e., Puiseux series with real coefficients. In particular, any Gröbner basis of I will consist of polynomials whose coefficients are real Puiseux series and any initial ideal of I will be generated over $\mathbb{R}[x]$.

We will further assume that there is a group S acting on $K[x]$ via signed permutation of the variables, i.e., for each group element $\sigma \in S$ there is a permutation $|\sigma| \in S_n$ and a sign vector $u_\sigma \in \{\pm 1\}^n$ such that $\sigma \cdot x_i = u_{\sigma,i} \cdot x_{|\sigma|(i)}$. The action of S on $K[x]$ induces an action of S on K^n acting via signed permutation of the coordinates. By considering the coordinatewise valuation $(K^*)^n \to \mathbb{R}^n$, we in turn obtain an action of S on $\mathbb{R}^n$ acting via (unsigned) permutations of the coordinates.

We will assume that $\mathcal{I} \trianglelefteq K[x]$ is invariant under the action of S on $K[x]$, so that both $V(\mathcal{I}) \subseteq K^n$ and $\mathrm{Trop}(\mathcal{I}) \subseteq \mathbb{R}^n$ are invariant under the action of S on their ambient spaces.

Example 5.2. Consider the following action of the symmetric group on n-elements S_n on the polynomial ring $K[x_I \mid I \in \binom{[n]}{k}]$:

$$S_n \times K\big[x_I \mid I \in \tbinom{[n]}{k}\big] \to K\big[x_I \mid I \in \tbinom{[n]}{k}\big], \quad (\sigma, x_I) \mapsto \sigma \cdot x_I := \mathrm{sgn}(\sigma(I)) \cdot x_{\sigma(I)},$$

where for $I = \{i_1, \dots, i_k\}$ with $i_1 < \dots < i_k$ the sign $\mathrm{sgn}(\sigma(I))$ is (-1) raised to the number of transpositions needed to sort the tuple $(\sigma(i_1), \dots, \sigma(i_k))$.

One can show that the Plücker ideal $\mathcal{I}_{n,k}$ is invariant under the S_n action above. Hence, for any p the tropical Grassmannian $\mathrm{TGr}_p(3, 8)$ is invariant under the action

$$S_n \times \mathbb{R}^{\binom{n}{k}} \to \mathbb{R}^{\binom{n}{k}}, \quad \big(\sigma, (w_I)_{I \in \binom{[n]}{k}}\big) \mapsto (w_{\sigma(I)})_{I \in \binom{[n]}{k}}.$$

Definition 5.3. We define the *positive tropicalization* of the variety of $I \trianglelefteq K[x]$ to be

$$\mathrm{Trop}^+(I) := \mathrm{cl}\Big(\nu\big(V(I) \cap (R_{>0})^n\big)\Big) \subseteq \mathbb{R}^n,$$

where again $\nu(\cdot)$ denotes coordinatewise valuation and $\mathrm{cl}(\cdot)$ denotes the closure in the euclidean topology.

For the sake of convenience, we call a weight vector $w \in \mathbb{R}^n$, an initial ideal $\mathrm{in}_w(I) \trianglelefteq \mathbb{C}[x]$, and a Gröbner cone $C_w(I) \subseteq \mathrm{Trop}(I)$ *positive* if $w \in \mathrm{Trop}^+(I)$.

Note that under the Fundamental Theorem of Tropical Geometry, positive tropical varieties also admit an algebraic description:

Proposition 5.4 *(Speyer and Williams, 2005, Proposition 2.2). Let $I \trianglelefteq K[x]$ be an ideal. Then*

$$\mathrm{Trop}^+(I) = \Big\{ w \in \mathbb{R}^n \;\Big|\; \mathrm{in}_w(I) \text{ monomial free and } \mathrm{in}_w(I) \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle \Big\}.$$

In particular, $\mathrm{Trop}^+(I)$ *is covered by all positive Gröbner cones if I is homogeneous.*

As an easy corollary, we get that positivity only depends on the saturated initial ideals, which will be relevant in Section 6.2.

Corollary 5.5. *Let $I \trianglelefteq K[x] = K[x_1, \dots, x_n]$ be an ideal and let $w, v \in \mathbb{R}^n$ be two weight vectors with $\mathrm{in}_w(I) : (\prod_{i=1}^n x_i)^\infty = \mathrm{in}_v(I) : (\prod_{i=1}^n x_i)^\infty$. Then*

$$w \in \mathrm{Trop}^+(I) \quad \Longleftrightarrow \quad v \in \mathrm{Trop}^+(I)$$

Proof. The statement follows directly from the following two easy equivalences:

- $\mathrm{in}_w(I)$ is monomial free if and only if $\mathrm{in}_w(I) : (\prod_{i=1}^n x_i)^\infty \neq \langle 1 \rangle$,
- $\mathrm{in}_w(I) \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle$ if and only if $(\mathrm{in}_w(I) : (\prod_{i=1}^n x_i)^\infty) \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle$. $\square$

5.1. Binomial cones

We decide positivity of binomial cones using the description of Proposition 5.4. Note that in this paper a *binomial* is a polynomial with exactly two distinct monomials, and a *binomial ideal* is an ideal generated by binomials. We begin by recalling a well-known result on the Gröbner bases of binomial ideals, and derive an easy test for positivity of binomial ideals from it.

Proposition 5.6 *(Eisenbud and Sturmfels, 1996, Proposition 1.1). Any reduced Gröbner basis of a binomial ideal consists of binomials or monomials.*

Lemma 5.7. *Let $J \trianglelefteq \mathbb{R}[x]$ be a monomial-free binomial ideal, $G \subseteq J$ a reduced Gröbner basis of J with respect to any ordering $>$. Then*

$$J \cap \mathbb{R}_{\geq 0}[x] = \langle 0 \rangle \quad \Longleftrightarrow \quad G \cap \mathbb{R}_{\geq 0}[x] = \emptyset.$$

Proof. $\Rightarrow$: Trivial, as $G \subseteq J$ and $0 \notin G$.
$\Leftarrow$: By Proposition 5.6 and because J is monomial-free, the Gröbner basis G consists solely of binomials. Since each element of G is normalized by definition, it contains only normalized binomials whose non-leading coefficient must be negative. Then the S-polynomial of any polynomial $f \in \mathbb{R}[x]$ with respect to a Gröbner basis element $g \in G$,

$$\mathrm{spoly}_>(f, g) := \frac{\mathrm{lcm}(\mathrm{LT}_>(f), \mathrm{LT}_>(g))}{\mathrm{LT}_>(f)} \cdot f - \frac{\mathrm{lcm}(\mathrm{LT}_>(f), \mathrm{LT}_>(g))}{\mathrm{LT}_>(g)} \cdot g,$$

will preserve the parity of f, i.e., if $f \in \mathbb{R}_{\geq 0}[x]$ then also $\mathrm{spoly}_>(f,g) \in \mathbb{R}_{\geq 0}[x]$, possibly $\mathrm{spoly}_>(f,g)=0$.

Now assume there is a non-zero polynomial $f \in J \cap \mathbb{R}_{\geq 0}[x]$. As G is a Gröbner basis, dividing f with respect to G will yield remainder 0. However, the division with respect to G is merely a nested chain of S-polynomials of f with respect to a sequence $(g_1,\dots,g_r)$ of possibly repeating elements $g_i \in G$:

$$\mathrm{spoly}_>(\underbrace{\mathrm{spoly}_>(\dots\mathrm{spoly}_>(f,g_1)\dots,g_{r-1})}_{=:f_r\neq 0},g_r)=0.$$

Abbreviating the penultimate non-zero S-polynomial with f_r, this implies two things: First, as $\mathrm{spoly}_>(f_r,g_r)=0$, f_r must be a multiple of g_r. Second, because spoly preserves the parity of f and $f \in \mathbb{R}_{\geq 0}[x]$, we also have $f_r \in \mathbb{R}_{\geq 0}[x]$. Both together contradict that G contains only binomials with a positive and a negative coefficient. $\square$

Proposition 5.8. *Let $C_w(I) \subseteq \mathrm{Trop}(I)$ be a maximal cone with $\mathrm{in}_w(I)$ binomial, and let $G \subseteq \mathrm{in}_w(I)$ be a reduced Gröbner basis of $\mathrm{in}_w(I)$ with respect to any ordering $>$. Then for any element $\sigma \in S$ we have*

$$\sigma \cdot C_w(I) \subseteq \mathrm{Trop}^+(I) \quad \Longleftrightarrow \quad \sigma \cdot G \cap \mathbb{R}_{\geq 0}[x] = \emptyset \text{ and } \sigma \cdot G \cap \mathbb{R}_{\leq 0}[x] = \emptyset.$$

Proof. As $G \subseteq \mathrm{in}_w(I)$ is a reduced Gröbner basis with respect to the ordering $>$, $\sigma \cdot G \subseteq \mathrm{in}_{\sigma\cdot w}(I)$ will be a Gröbner basis with respect to the ordering $>_\sigma$ defined by

$$x^\alpha >_\sigma x^\beta \quad :\Longleftrightarrow \quad x^{\sigma\cdot\alpha} > x^{\sigma\cdot\beta},$$

where σ acts on the exponent vectors as it does on the weight space $\mathbb{R}^n$.

By Proposition 5.6, G consists solely of binomials and hence so does $\sigma \cdot G$. Moreover, $\sigma \cdot G$ is reduced up to normalization. It is reduced up to multiplication by ± 1, since σ acts by signed permutation. The claim then follows from Lemma 5.7. $\square$

Example 5.9. Consider the Grassmannian $\mathrm{Gr}(2,5)$, whose Plücker ideal I is generated by the following 3-term Plücker relations:

$$\begin{aligned} I := \langle & p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23},\ p_{02}p_{34} - p_{03}p_{24} + p_{04}p_{23},\ p_{01}p_{34} - p_{03}p_{14} + p_{04}p_{13}, \\ & p_{01}p_{24} - p_{02}p_{14} + p_{04}p_{12},\ p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}\rangle \trianglelefteq \mathbb{C}\{\{t\}\}[p_{ij} \,|\, 0 \leq i < j \leq 4]. \end{aligned}$$

The Petersen Graph in Fig. 9 illustrates the combinatorics of the tropical Grassmannian $\mathrm{TGr}_0(2,5)$ modulo its 5-dimensional lineality space generated by

$$\sum_{\substack{0\leq i<j\leq 4\\ i\neq k\neq j}} e_{ij} \quad \text{for } k=0,\dots,4,$$

where e_{ij} denotes the unit vector in direction of p_{ij} in the weight space. Each vertex denotes a ray generated by the negative of the inscribed unit vector, and each edge denotes a maximal cone spanned by two rays. The edges in red are the maximal cones inside the positive tropical Grassmannian $\mathrm{TGr}^+(2,5)$.

The weight vector $w := -e_{01} - e_{23}$ lies in the interior of a maximal cone $C_w(I)$. Its corresponding initial ideal $\mathrm{in}_w(I)$ is generated by the following binomial reduced Gröbner basis:

$$\begin{aligned} G := \{ & p_{02}p_{13} - p_{12}p_{03},\ p_{02}p_{14} - p_{12}p_{04},\ p_{02}p_{34} - p_{03}p_{24},\ p_{03}p_{14} - p_{13}p_{04}, \\ & p_{12}p_{34} - p_{13}p_{24}\}. \end{aligned}$$

Thus, according to Lemma 5.7, w is contained in $\mathrm{TGr}^+(2,5)$. Moreover, consider the two transpositions $(14), (34) \in S_5$, which act on the coordinate ring as follows:

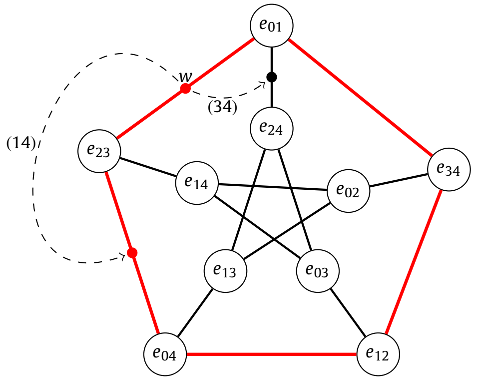

Fig. 9. The tropical Grassmannian $\mathrm{TGr}^+(2,5)$ and its positive cones. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$(14):\begin{cases} p_{01}\mapsto p_{04}, & p_{13}\mapsto -p_{34},\\ p_{02}\mapsto p_{02}, & p_{14}\mapsto -p_{14},\\ p_{03}\mapsto p_{03}, & p_{23}\mapsto p_{23},\\ p_{04}\mapsto p_{01}, & p_{24}\mapsto -p_{12},\\ p_{12}\mapsto -p_{24}, & p_{34}\mapsto -p_{13}, \end{cases} \quad\text{and}\quad (34):\begin{cases} p_{01}\mapsto p_{01}, & p_{13}\mapsto p_{14},\\ p_{02}\mapsto p_{02}, & p_{14}\mapsto p_{13},\\ p_{03}\mapsto p_{04}, & p_{23}\mapsto p_{24},\\ p_{04}\mapsto p_{03}, & p_{24}\mapsto p_{23},\\ p_{12}\mapsto p_{12}, & p_{34}\mapsto -p_{34}. \end{cases}$$

Applying them to G yields

$$(14)\cdot G=\left\{\begin{array}{l} -p_{02}p_{34}+p_{24}p_{03},\\ -p_{02}p_{14}+p_{24}p_{01},\\ -p_{02}p_{13}+p_{03}p_{12},\\ -p_{03}p_{14}+p_{34}p_{01},\\ p_{24}p_{13}-p_{34}p_{12} \end{array}\right\} \quad\text{and}\quad (34)\cdot G=\left\{\begin{array}{l} p_{02}p_{14}-p_{12}p_{04},\\ p_{02}p_{13}-p_{12}p_{03},\\ -p_{02}p_{34}-p_{04}p_{23},\\ p_{04}p_{13}-p_{14}p_{03},\\ -p_{12}p_{34}-p_{14}p_{23} \end{array}\right\}$$

Hence, by Proposition 5.8, $(14)\cdot w=-e_{04}-e_{23}$ lies on the positive tropical Grassmannian $\mathrm{TGr}^+(2,5)$, whereas $(34)\cdot w=-e_{01}-e_{24}$ does not.

From Proposition 5.8, we obtain the following simple algorithm:

Algorithm 5.10 *(Positivity of binomial cones).*

Input: (G,S), where
- $G\subseteq \mathrm{in}_w(I)\trianglelefteq \mathbb{C}[x]$, a reduced Gröbner basis of a binomial initial ideal $\mathrm{in}_w(I)$,
- S, a group as in Convention 5.1.

Output: $S_w^+(I):=\{\sigma\in S\mid \sigma\cdot C_w(I)\subseteq \mathrm{Trop}^+(I)\}$, the set of symmetries which map $C_w(I)$ into $\mathrm{Trop}^+(I)$.

1: **return** $\bigcap_{g\in G}\{\sigma\in S\mid \sigma\cdot g$ has coefficients with mixed parity$\}$

Remark 5.11. By Bossinger et al. (2017, Proof of Lemma 1), any maximal $C_w(I)\subseteq \mathrm{Trop}(I)$ of multiplicity one has a primary initial ideal $\mathrm{in}_w(I)$ and a binomial radical $\sqrt{\mathrm{in}_w(I)}$. And since

$$\left(\mathrm{in}_w(I)\text{ positive}\iff\sqrt{\mathrm{in}_w(I)}\text{ positive}\right)\quad\text{and}\quad \sigma\cdot\sqrt{\mathrm{in}_w(I)}=\sqrt{\mathrm{in}_{\sigma\cdot w}(I)}\ \forall\sigma\in S,$$

we can use Algorithm 5.10 to test positivity within their orbit.

5.2. Algorithm for general cones

We decide positivity of general maximal Gröbner cones using Definition 5.3. The idea is to reduce the problem to dimension zero, for which we can explicitly compute the signs of the finite number of roots.

We begin with recalling a central lemma for the proof of the Fundamental Theorem of Tropical Geometry, which allows us to read off positivity from the zeroes of the initial ideal.

Lemma 5.12 *(Maclagan and Sturmfels, 2015, Proposition 3.2.11). Let* $w = (w_1, \dots, w_n) \in \mathrm{Trop}(I)$*, then*

$$V(\mathrm{in}_w(I)) \cap (\mathbb{C}^*)^n = \{(\overline{t^{-w_1} z_1}, \dots, \overline{t^{-w_n} z_n}) \mid (z_1, \dots, z_n) \in V(I) \cap (\mathbb{C}\{\{t\}\}^*)^n \text{ with } \nu(z_i) = w_i\}.$$

In particular, $C_w(I) \subseteq \mathrm{Trop}^+(I)$ *if and only if* $V(\mathrm{in}_w(I)) \cap \mathbb{R}^n_{>0} \neq \emptyset$*.*

Proof. For the $\subseteq$ inclusion, consider Maclagan and Sturmfels (2015, Proposition 3.2.11) which proves the statement for prime ideals. To show that the same statement holds for non-prime ideals, assume without loss of generality that I is radical and consider a decomposition $I = P_1 \cap \dots \cap P_k$, where each P_i is prime. It suffices to show the following which will allow us to apply (Maclagan and Sturmfels, 2015, Proposition 3.2.11) to each P_i:

$$\sqrt{\mathrm{in}_w(I)} = \sqrt{\mathrm{in}_w(P_1)} \cap \dots \cap \sqrt{\mathrm{in}_w(P_k)}.$$

One can show that both sides of the equation above are generated by polynomials that are weighted homogeneous w.r.t. w. So let $h \in \sqrt{\mathrm{in}_w(I)}$ be weighted homogeneous. Then for some $\ell > 0$ we have $h^\ell \in \mathrm{in}_w(I)$. Since h^ℓ remains weighted homogeneous, there is an $f \in I$ such that $\mathrm{in}_w(f) = h^\ell$. The decomposition $I = P_1 \cap \dots \cap P_k$ then straightforwardly implies that $h \in \sqrt{\mathrm{in}_w(P_1)} \cap \dots \cap \sqrt{\mathrm{in}_w(P_k)}$. Now let $h \in \sqrt{\mathrm{in}_w(P_1)} \cap \dots \cap \sqrt{\mathrm{in}_w(P_k)}$ be weighted homogeneous. Then there is an $\ell > 0$ such that $h^\ell \in \mathrm{in}_w(P_i)$ for all i. Since h^ℓ remains weighted homogeneous, there are $f_i \in P_i$ such that $\mathrm{in}_w(f_i) = h^\ell$. By considering $f = f_1 \cdot \ldots \cdot f_k \in I$ we can straightforwardly follow that $h \in \sqrt{\mathrm{in}_w(I)}$.

For the $\supseteq$ inclusion, consider $z := (z_1, \dots, z_n) \in V(I) \cap (\mathbb{C}^*)^n$ with $\nu(z) = w$. Then for any $f \in I$ we have $f(z) = 0$ by definition, which necessarily implies $\mathrm{in}_w(f)(\overline{t^{-w_1} z_1}, \dots, \overline{t^{-w_n} z_n}) = 0$. Hence, $z \in V(\mathrm{in}_w(I)) \cap (\mathbb{C}^*)^n$.

The second part note Proposition 5.4 implies that $C_w(I) \subseteq \mathrm{Trop}^+(I)$ if and only if $w \in \mathrm{Trop}^+(I)$. By definition, the latter is equivalent to there being a $z \in V(I) \cap R^n_{>0}$ with $\nu(z_i) = w_i$. By the definition of $R^n_{>0}$ and the first part, this is then equivalent to $V(\mathrm{in}_w(I)) \cap \mathbb{R}^n_{>0} \neq \emptyset$. $\square$

The next lemma allows us to reduce the problem to dimension zero.

Lemma 5.13. *Let* $J \trianglelefteq \mathbb{R}[x]$ *be weighted homogeneous with respect to a weight vector* $0 \neq w = (w_1, \dots, w_n) \in \mathbb{Z}^n$*, say* $w_i \neq 0$*. Then*

$$V(J) \cap (\mathbb{R}_{>0})^n \neq \emptyset \iff V(J + \langle x_i - 1 \rangle) \cap (\mathbb{R}_{>0})^n \neq \emptyset$$

and moreover $\dim(J + \langle x_i - 1 \rangle) = \dim(J) - 1$.

Proof. $\Leftarrow$: Clear, as $V(J + \langle x_i - 1 \rangle) \subseteq V(J)$.
$\Rightarrow$: Note that the weighted homogeneity of J induces a torus action

$$\mathbb{C}^* \times V(J) \longrightarrow V(J), \quad (a, (z_1, \dots, z_n)) \longmapsto (a^{-w_1} z_1, \dots, a^{-w_n} z_n),$$

with $w_i \neq 0$. Hence, for any $z \in V(J) \cap (\mathbb{R}_{>0})^n$ there exists an $a \in \mathbb{R}^* \subseteq \mathbb{C}^*$ with $a^{-w_i} \cdot z_i = 1$. $\square$

Algorithm 5.14 *(`reduceDim`, Positivity reduced to dimension* 0*).*

Input: $G \subseteq \mathrm{in}_w(I)$ a reduced Gröbner basis where $C_w(I) \subseteq \mathrm{Trop}(I)$ is a maximal cone.
Output: H, generators of a zero-dimensional ideal $J \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$ such that

$$C_w(I) \subseteq \mathrm{Trop}^+(I) \quad \Longleftrightarrow \quad V(J) \cap (\mathbb{R}_{>0})^n \neq \emptyset.$$

1: Compute a basis $b_1, \dots, b_d \in \mathbb{R}^n$ of the d-dimensional vector subspace

$$C_0(\mathrm{in}_w(I)) = \{v \in \mathbb{R}^n \mid \mathrm{in}_v(g) = g \text{ for all } g \in G\} \subseteq \mathbb{R}^n$$

such that the matrix $B \in \mathbb{R}^{d \times n}$ with rows $b_1, \dots, b_d$ is in row-echelon form.
2: Let $\Lambda \subseteq \{1, \dots, n\}$ denote the column-indices of the pivots of B.
3: **return** $H := G \cup \{x_i - 1 \mid i \in \Lambda\}$.

Proof of correctness. By Lemma 5.12, we have $C_w(I) \subseteq \mathrm{Trop}^+(I)$ if and only if $V(\mathrm{in}_w(I)) \cap (\mathbb{R}_{>0})^n \neq \emptyset$. By (Bogart et al., 2007, Proposition 2.4), $C_0(\mathrm{in}_w(I))$ is the set of all vectors with respect to whom $\mathrm{in}_w(I)$ is weighted homogeneous. We can thus apply Lemma 5.13 iteratively d times to obtain $V(\mathrm{in}_w(I)) \cap (\mathbb{R}_{>0})^n \neq \emptyset$ if and only if $V(J) \cap (\mathbb{R}_{>0})^n \neq \emptyset$. $\square$

Additionally, we require an algorithm for computing the signs of the roots of a zero-dimensional ideal. We will treat this part as a black box, and discuss various possibilities in Remark 5.17.

Algorithm 5.15 *(`sgns`, black box for computing signs of real solutions).*
Input: $H \subseteq J$, a generating set of a zero-dimensional ideal $J \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$.
Output: $\mathrm{sgn}(V(H) \cap (\mathbb{R}^*)^n) \subseteq \{\pm 1\}^n$, where $\mathrm{sgn}(\cdot)$ denotes coordinatewise

$$\mathbb{R}^* \longrightarrow \{\pm 1\}, \quad z \longmapsto \begin{cases} +1 & \text{if } z > 0, \\ -1 & \text{if } z < 0. \end{cases}$$

Combining Algorithms 5.14 and 5.15, we obtain Algorithm 5.16 for positivity within orbits of maximal cones.

Algorithm 5.16 *(Positivity of maximal-dimensional cones).*
Input: (G, S), where

- $G \subseteq \mathrm{in}_w(I)$ a reduced Gröbner basis where $C_w(I) \subseteq \mathrm{Trop}(I)$ is a maximal cone,
- S, a group as in Convention 5.1.

Output: $S_w^+(I) := \{\sigma \in S \mid \sigma \cdot C_w(I) \subseteq \mathrm{Trop}^+(I)\}$, the set of symmetries which map $C_w(I)$ to $\mathrm{Trop}^+(I)$.
1: Apply Algorithm 5.14: $H := \mathtt{reduceDim}(G) \subseteq K[x]$.
2: Apply Algorithm 5.15: $B := \mathtt{sgns}(H) \subseteq \{\pm 1\}^n$.
3: Construct

$$P := \bigcup_{b \in B} \{\sigma \in S \mid \sigma \cdot b \geq 0\},$$

where S acts on $\{\pm 1\}^n$ as it acts on K^n and $\geq$ is coordinatewise comparison.
4: **return** P

Remark 5.17. Computing the signs of a finite set of points $V(J) \subseteq \mathbb{C}^n$ for a zero-dimensional ideal $J \trianglelefteq \mathbb{C}[x]$ as in Algorithm 5.15 can be done symbolically, numerically or with a mix of both.

One conceptually straightforward but not necessarily easy option is to approximate $V(J)$ using numerical algebraic geometry. Once a point in $V(J)$ is known with sufficient precision, there are algorithms for certifying reality (Hauenstein and Sottile, 2012) and its sign can simply be read off.

Alternatively, one can symbolically compute a triangular decomposition of J into factors of the form

$$\langle p_1(x_1), x_2^{d_2} - p_2(x_1), \ldots, x_n^{d_n} - p_n(x_1)\rangle, \qquad p_i \text{ univariate polynomials,}$$

from which one can proceed using numerical algorithms for the univariate case.

Remark 5.18. In this section, we focused on the task for Gröbner cones of maximal dimension in $\mathrm{Trop}(I)$ for containment in the positive tropicalization. Testing whether a lower-dimensional Gröbner cone $C_w(I) \subseteq \mathrm{Trop}(I)$, say $\dim C_w(I) = d - \ell$ for $d := \dim \mathrm{Trop}(I)$ and for some $\ell > 0$, is significantly more difficult.

By Definition 5.3 and Lemma 5.12, a possible method is to test whether the n-dimensional affine variety $V(\mathrm{in}_w I) \subseteq \mathbb{C}^n$ has a strictly positive solution. Using Lemma 5.13, this can be simplified into checking whether an ℓ-dimensional affine variety has strictly positive solution. This is a fundamental question in real algebraic geometry on the feasibility of (basic) semi-algebraic sets. See Basu (2017) for a recent comprehensive overview of existing algorithms.

6. The maximal-dimensional cones of $\mathrm{TGr}^+(3, 8)$

In this section, we verify (Speyer and Williams, 2005, Conjecture 8.1) for the Grassmannian $\mathrm{Gr}_0(3, 8)$, which relates the combinatorial structure of the positive tropicalization with the combinatorial structure of a cluster algebra. This serves as a test for the correctness of our computations, as the conjecture has been proven for $\mathrm{Gr}_0(3, 8)$ by Brodsky and Stump (2018, Remark 2.23).

Cluster algebras are algebras with a remarkable hidden combinatorial structure. First introduced by Fomin and Zelevinsky in Fomin and Zelevinsky (2002), cluster algebras are subrings of rational function fields $K(x_1, \ldots, x_n)$ generated by a union of overlapping algebraically independent n-subsets. These so-called *clusters* are connected through mutations, rules which transform one cluster to another, and together they form a simplicial complex called the *cluster complex*. The elements of the clusters are referred to as *cluster variables*. In Fomin and Zelevinsky (2003), Fomin and Zelevinsky completely classify all cluster algebras of *finite type*, i.e., cluster algebras with finite cluster complexes. Similar to the Cartan-Killing classification of semisimple Lie algebras, their classification associates any finite type cluster algebra to a Dynkin graph. One prominent family of cluster algebras are Grassmannians $\mathrm{Gr}_0(k, n)$, initially shown by Fomin and Zelevinksy for $k = 2$, later fully proven by Scott (2006).

The conjecture of Speyer and Williams is based on observations on the Grassmannians $\mathrm{Gr}_0(2, n)$, $\mathrm{Gr}_0(3, 6)$, and $\mathrm{Gr}_0(3, 7)$. By Scott (2006), this makes $\mathrm{Gr}_0(3, 8)$ the only remaining Grassmannian whose cluster algebra is of finite type, i.e., whose cluster complex is finite.

Conjecture 6.1 *(Speyer and Williams, 2005, Conjecture 8.1).* Let $\mathcal{A}$ be a cluster algebra of finite type, C its set of coefficient variables, and $\mathcal{S}(\mathcal{A})$ its associated cluster complex. If the lineality space of $\mathrm{Trop}^+ \mathrm{Spec}(\mathcal{A})$ has dimension $|C|$, then $\mathrm{Trop}^+ \mathrm{Spec}\, \mathcal{A}$ is abstractly isomorphic to the fan over the simplicial complex $\mathcal{S}(\mathcal{A})$. If the condition on the lineality space does not hold, then $\mathrm{Trop}^+ \mathrm{Spec}\, \mathcal{A}$ is abstractly isomorphic to a coarsening of the fan over the simplicial complex $\mathcal{S}(\mathcal{A})$.

The conjecture was proven by Brodsky and Stump (2018) for finite type cluster algebras that are either of type A, see Fomin and Zelevinsky (2003, Section 12.2), and of rank at most 8, which includes $\mathrm{Gr}_0(3, 8)$.

6.1. Computing the cluster complex $\mathcal{S}(\mathrm{Gr}_0(3, 8))$

Thanks to an implementation by Stump, SAGE (The Sage Developers, 2019) features functions for computing and analyzing cluster complexes. The algorithm is based on a work of Ceballos et al. (2014), and requires the root system of the cluster algebra. The root system for $\mathrm{Gr}_0(3, 8)$ is the exceptional group E_8 (Scott, 2006, Theorem 5):

```
C = ClusterComplex(['E', 8]);
```

SAGE returns an object of type `cluster complex`, whose maximal cells can be seen via

```
C.facets();
```

6.2. *Computing the positive tropicalization* $\mathrm{TGr}^+(3,8)$

Using the algorithms in Section 5 on Theorem 4.6, we obtain:

Proposition 6.2. *There is a Dressian subfan supported on the maximal-dimensional cones of the positive tropical Grassmannian* $\mathrm{TGr}^+(3,8)$*. It is a pure* 16*-dimensional subfan of the Dressian* $\mathrm{Dr}(3,8)$ *in* $\mathbb{R}^{56}$ *with an* 8*-dimensional lineality space and* f*-vector* $(120, 2072, 14\,088, 48\,544, 93\,104, 100\,852, 57\,768, 13\,612)$.

Proof. By Corollary 5.5, positivity only depends on the saturated initial ideals, and, by Theorem 4.9, the saturated initial ideals of $\mathrm{TGr}_0(3,8)$ only depend on the Dressian cones.[2] It therefore suffices to check the $4\,766$ S_8-orbits of Dressian cones in Theorem 4.6 instead of the $14\,763$ S_8-orbits of Gröbner cones.

Of the $4\,766$ saturated initial ideals of $\mathrm{Gr}_0(3,8)$ all but one are binomial and thus admissible for Algorithm 5.10. The unique non-binomial saturated initial ideal arises from the Dressian orbit containing $-e_{015}-e_{024}-e_{067}-e_{126}-e_{137}-e_{235}-e_{346}-e_{457}$. Applying Algorithm 5.13 to this ideal yields that there is no real solution. Thus, there is no positive cone it its orbit by Algorithm 5.15. All functions and data necessary for the computation can be found at the website linked in the introduction. □

Note that Speyer and Williams (2005) consider positive tropicalizations with the coarsest structure refined by the individual Gröbner fans of all cluster variables. For the cluster variables of $\mathrm{Gr}_0(3,8)$, recall the following result from Scott (2006):

Theorem 6.3. *(Scott, 2006, Theorem 8) The cluster algebra of* $\mathrm{Gr}_0(3,8)$ *possesses* 128 *cluster variables:*

48: *Plücker variables* p_{ijk} *where* $\{i,j,k\} \neq \{i,i+1,i+2\}$ *mod* 8.
56: *quadratic Laurent binomials with positive coefficients, inherited from* $\mathrm{Gr}_0(3,6)$*, describing six points in a special position:*

$$Y^{123456} = (p_{346})^{-1} \cdot \Big(p_{146}p_{236}p_{345} + p_{136}p_{234}p_{456}\Big) \quad \text{and} \quad X^{123456} = Y^{234561}$$

and their D_8*-translates.*
24: *cubic Laurent trinomials with positive coefficients describing eight points in a special position:*

$$A = (p_{578})^{-1} \cdot \Big(p_{178}p_{567} \cdot X^{123458} + p_{158}p_{678} \cdot X^{123457}\Big) \quad \text{and}$$
$$B = (p_{158})^{-1} \cdot \Big(p_{128}p_{567} \cdot X^{123458} + p_{258} \cdot A\Big)$$

and their D_8*-translates.*

Since the Gröbner fans of the Plücker variables consist of a single cone that is the whole space, refining with them does not change anything. Hence, it only remains the 80 polynomials X, Y, A and B.

Theorem 6.4. *The positive tropical Grassmannian* $\mathrm{TGr}^+(3,8)$ *endowed with the Plücker structure and refined by the Gröbner fans of all 120 cluster variables of* $\mathrm{Gr}(3,8)$ *is a* 16*-dimensional pure simplicial fan in* $\mathbb{R}^{56}$ *with an* 8*-dimensional lineality space and* f*-vector* $(128, 2\,408, 17\,936, 67\,488, 140\,448, 163\,856, 100\,320, 25\,080)$*. As an abstract simplicial complex, it is isomorphic to the cluster complex* $\mathcal{S}(\mathrm{Gr}(3,8))$.

[2] Instead of Corollary 5.5 and Theorem 4.9, we could also rely on the recent results of Arkani-Hamed et al. (2021) and Speyer and Williams (2021) that the positive tropical Grassmannian $\mathrm{TGr}_0^+(k,n)$ equals the positive Dressian $\mathrm{Dr}^+(k,n)$, which implies that the Plücker structure on $\mathrm{TGr}^+(k,n)$ is a coarsening of the Gröbner structure.

Proof. The refinement was straightforwardly computed by intersecting all maximal Dressian cones on $\mathrm{TGr}^+(3,8)$ with the maximal cones of the Gröbner fans of the cluster variables.

The isomorphism of the two simplicial complexes was tested using Nauty (McKay and Piperno, 2014) by McKay, which was called in Polymake through the function `fan::isomorphic`. The function takes two objects of type `IncidenceMatrix`, in our case:

(1) the output of SAGE's `C.facets()`, `C` being the cluster complex of type E_8,
(2) the output of Polymake's `$F->MAXIMAL_CONES`, `$F` being the polyhedral fan supported on $\mathrm{TGr}^+(3,8)$ described above. □

Remark 6.5. Positive tropical Grassmannians equipped with the coarsest structure are lifts of the normal fan of a polytope to higher dimensional space. This polytope is the $(n-3)$-dimensional associahedron in case of the positive tropical Grassmannian $\mathrm{TGr}^+(2,n)$; see Speyer and Williams (2005). In general this polytope is the Minkowski sum of Newton Polytopes that one derives from a parametrization of the positive part of the classical Grassmannian. This has been used to compute the positive tropical Grassmannians $\mathrm{TGr}^+(3,n)$ for $n \leq 10$, $\mathrm{TGr}^+(4,8)$ and $\mathrm{TGr}^+(4,9)$. See Arkani-Hamed et al. (2021) and He et al. (2020) for further details.

7. Open questions

A frequently arising question on the geometry of tropical varieties is whether they have a natural coarsest structure, i.e., whether there is a natural coarsest polyhedral complex supported on them. While it is long known that there is no unique coarsest structure (Sturmfels and Tevelev, 2008, Example 5.2) and that natural coarsenings of the Gröbner fan exist (Cartwright, 2012), the question remains largely open. For the tropical Grassmannians $\mathrm{TGr}_p(2,n)$, $\mathrm{TGr}_p(3,6)$, $\mathrm{TGr}_0(3,7)$ and $\mathrm{TGr}_0(3,8)$ is the Plücker structure the unique coarsest fan structure; see Section 4. This naturally gives rise to the following two problems:

Problem 7.1. Find a characterization or sufficient conditions on the cones C of the Dressian $\mathrm{Dr}(k,n)$ such that for any $v \in \mathrm{TGr}_p(k,n) \cap C$ and $w \in \mathbb{R}^{\binom{n}{k}}$ we have

$$\begin{aligned} &v \text{ and } w \text{ lie in the relative interior of the same cone of } \mathrm{Dr}(k,n) \\ &\qquad\qquad\qquad \iff \quad \mathrm{in}_w(\mathcal{I}_{n,k}) : p^\infty = \mathrm{in}_v(\mathcal{I}_{n,k}) : p^\infty, \end{aligned} \tag{1}$$

where $\mathcal{I}_{n,k}$ is the Plücker ideal and $(\cdot) : p^\infty$ denotes the saturation at the product of all Plücker variables.

Problem 7.2. For all cones $C \subseteq \mathrm{Dr}(k,n)$ for which the above equivalence (1) does not hold determine the (coarsest) polyhedral structure of the set $C \cap \mathrm{TGr}_p(k,n)$.

The following shows that such cones exist.

Example 7.3. Consider the two indicator vectors $v, w \in \mathbb{R}^{35} \cong \mathbb{R}^{\binom{7}{3}}$ of bases of the Fano matroid and its relaxation the non-Fano matroid, respectively. The vectors v and $\epsilon \cdot v + w$ do lie in the relative interior of the same cone of the Dressian $\mathrm{Dr}(3,7)$ for all $\epsilon > 0$, but $v \in \mathrm{TGr}_2(3,7)$ and $\epsilon \cdot v + w \notin \mathrm{TGr}_2(3,7)$; see Herrmann et al. (2009). Thus $\mathrm{in}_{\epsilon\cdot v+w}(\mathcal{I}_{n,k}) : p^\infty \neq \mathrm{in}_v(\mathcal{I}_{n,k}) : p^\infty$.

A similar pair of vectors is formed by the indicator vectors $v, w \in \mathbb{R}^{84} \cong \mathbb{R}^{\binom{9}{3}}$ of bases of the Pappus and non-Pappus matroid. Again the vectors v and $\epsilon \cdot v + w$ are contained in the relative interior of the same cone of the Dressian $\mathrm{Dr}(3,9)$, but $v \in \mathrm{TGr}_p(3,9)$ and $\epsilon \cdot v + w \notin \mathrm{TGr}_p(3,9)$ for small $\epsilon > 0$ and by Joswig and Schröter (2017, Theorem 35).

If C is a cone of the Dressian $\mathrm{Dr}(k,n)$ of dimension $(n-k)\cdot k+2$ or higher for which the equivalence (1) holds true then there is no realizable point in the relative interior of C. In addition to any theoretical insight that such a coarsest structure could offer, the question is of direct relevance for two practical reasons.

On the one hand, it will improve our understanding for the complexity of tropical varieties and consequently also the feasibility of computations in tropical geometry, especially with a view towards applications (Leykin and Yu, 2019). Current bounds on the f-vector of general tropical varieties are derived from universal Gröbner bases (Joswig and Schröter, 2018), and are thus expected to be far from optimal.

On the other hand, it will help with concrete large scale computations. For $\mathrm{TGr}_0(4,8)$ and even for $\mathrm{TGr}_2(4,8)$, i.e., working over the field with two elements, our implementation in Section 3 gets stuck on a handful of isolated Gröbner bases containing polynomials of degree 15, for whom simple division with remainder takes several days. Having a coarser structure might allow us to skip those problematic Gröbner cones which are still few and far in between.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The data is quite big (tens of GB) and is freely available under the link in the manuscript, hosted at TU Kaiserslautern.

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