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The Existence of the Miyazaki-Wilson-Spence Equilibrium with Continuous Type Distributions

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Abstract

We prove the existence of an equilibrium in competitive markets with adverse selection in the sense of Miyazaki (1977), Wilson (1977), and Spence (1978) when the distribution of unobservable risk types is continuous. Our proof leverages the finite-type proof in Spence (1978) and a limiting argument akin to Hellwig (2007)'s study of optimal taxation.

JEL classification: D82, G22, D41.

Keywords: Asymmetric and private information; insurance market; adverse selection; equilibrium

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1 Introduction

Economists have long understood that private information can lead to adverse-selection-driven pathologies in many markets. One of the most well known and well-studied examples of such pathologies is the seminal non-existence result of Rothschild and Stiglitz (1976). Their paper demonstrated how private information-driven adverse selection can undermine the very existence of a (static Nash) equilibrium. This non-existence result spawned a wave of research considering alternative equilibrium concepts which would make predictions about outcomes in adversely selected and competitive markets in those cases when the Rothschild-Stiglitz candidate equilibrium fails to exist.

Two distinct concepts emerged and have endured. On the one hand, Riley (1979)'s so-called "reactive" equilibrium (henceforth Rothschild-Stiglitz-Riley or "RSR") always coincides with the Rothschild-Stiglitz equilibrium *candidate*, and is justified by a quasi-dynamic assumption about firms' reactions to entry. The "foresight" equilibrium attributed to Miyazaki (1977), Wilson (1977), and Spence (1978) (henceforth the "MWS" equilibrium), on the other hand, diverges from the Rothschild-Stiglitz candidate whenever that candidate is not an equilibrium in the sense of Rothschild and Stiglitz. In particular, the MWS equilibrium concept predicts that the market will implement Pareto-improving cross-subsidies across different types' contracts whenever such cross-subsidies exist. This prediction is justified via a (different) quasi-dynamic assumption about anticipated contract withdrawals by incumbents in response to entry or deviations.

Both concepts have been widely employed for studying competitive markets with adverse selection (including Hoy (1982), Crocker and Snow (1985), Puelz and Snow (1994) Crocker and Snow (2008) Finkelstein et al. (2009), and Mimra and Wambach (2017) for the MWS concept, and Besanko and Thakor (1987), Landers et al. (1996), Newhouse (1996), Inderst (2005), Handel et al. (2015), Mimra and Wambach (2017), and Boyer and Peter (2018) for the RSR concept). There has been a recent wave of the literature providing more formal foundations for both concepts (viz Netzer and Scheuer (2014), Picard (2014), Picard (2018), Mimra and Wambach (2018), and Diasakos and Koufopoulos (2018) for the MWS concept, and Engers and Fernandez (1987), Inderst and Wambach (2001), Dubey and Geanakoplos (2002), Mimra and Wambach (2016), and Azevedo and Gottlieb (2017) for the RSR concept).

In finite type models in the Rothschild and Stiglitz tradition, both the RSR and MWS equilibria are known to exist (see Riley (1979) and Spence (1978), respectively). Existence of the RSR equilibrium readily extends to a continuum of types model (Riley (1979)). This short essay shows that the MWS equilibrium can also be extended to the continuum of types case.

This extension is of more than just technical interest. Models with continuous distributions of types are common and natural in many contexts (see, for example Hendren (2013)'s important study of private information as a cause of insurance market rejections). Moreover, when there is a continuum of types, the sort of cross-subsidies implemented in the MWS equilibrium is typically *required* for insurance market outcomes to be non-trivial: as Hendren (2014) observes, whenever the distribution of risk types has full support (specifically, full support near a $p = 100\%$ risk of loss), the only possible outcome in the absence of cross-subsidies across types involves no insurance trade at all. The RSR equilibrium concept therefore implies no trade whenever the distribution of types has full support, even when the mass of very high risk types is vanishingly small. Indeed, the RSR equilibrium is typically discontinuous with respect to the introduction of an arbitrarily small measure of types at or near $p = 100\%$. In contrast, precisely because it allows Pareto-improving cross-subsidies, the MWS equilibrium is continuous in the distribution of types: when a small measure of very high risk types is introduced, the lower risk types simply cross-subsidize them and (because there are few of them) the equilibrium predictions for the lower risk types are essentially unaffected. Unless one can rule out the presence of very high risks with absolute *certainty*, then, the MWS equilibrium is both the less trivial and the more robust concept for competitive market equilibria with adverse selection in continuous type settings.

To extend the MWS equilibrium concept to the continuum of types case, we lean on Spence (1978)'s construction for the finite types case. Because Spence's construction is recursive, it does not directly extend to continuous type setting. Our first step (Section 2) is to reformulate Spence's construction in a non-recursive manner. Specifically, we show that Spence's construction is equivalent to establishing the existence of a "reservation utilities" function $\bar{V}(p)$ (on the space of types p). $\bar{V}(p)$ satisfies the property that the solutions to a particular family of constrained profit-maximization problems–specifically the family of problems which are dual to the ones used by Spence (1978)'s–

yield exactly zero profits and give at least utility $\bar{V}(p)$ to type p. This reformulation readily extends to a continuum of types. In Section 3, we then use a limit-of-finite-approximations argument akin to the one in Hellwig (2007) to show that there indeed exists a function $\bar{V}(p)$ defined on a continuum of types satisfying this zero-profit property, thereby showing that an MWS equilibrium exists. In Section 4, we argue, moreover, that our formulation of the MWS equilibrium in the continuum of types case can be justified by the same "anticipatory" logic used in Wilson (1977) and Spence (1978). Section 5 offers some brief conclusions.

2 Setup and definitions

We consider a natural extension of Rothschild and Stiglitz (1976)'s binary loss setting to one with a continuum of types. Types differ only in the probability p of experiencing a loss of size L from endowed wealth $W > L$. There is a continuous distribution $F(p)$ of types $p \in [\underline{p}, \bar{p}]$ (possibly with $\bar{p} = 1$ or $\underline{p} = 0$ or both), with a density $f(p)$ satisfying $f(p) \in [\underline{f}, \bar{f}]$, where $0 < \underline{f} < \bar{f} < \infty$.[1] Absent insurance, consumers will consume $c_N = W$ if they experience no loss, and $c_L = W - L$ if they experience a loss. If they purchase an insurance contract at a premium q which provides an indemnity payment I, then they will instead have state-contingent consumption $(c_N, c_L) = (W - q, W - q - L + I)$. In line with previous literature, we assume that firms impose an exclusivity condition, which requires customers to have no other insurance (see Jaynes (1978) and Hellwig (1988) for an analysis of the case that exclusivity is part of strategic interaction among firms). Preferences over state-contingent consumption vectors are given by $(1 - p)u(c_N) + pu(c_L)$, where u is a strictly concave utility function, which we normalize (without loss of generality), so that $u(W - L) = 0$ and $u(W) = 1$.[2]

2.1 Notation in utility space

Each consumption vector (c_N, c_L) (and hence each insurance contract (q, I)) is associated with a unique utility vector $\vec{U} = (u(c_N), u(c_L))$. It will be more convenient to formulate contracts

[1] As Hellwig (1992) notes, an equilibrium a la Spence (1978) might not exist for an unbounded type space.

[2] We follow the classical von Neumann-Morgenstern binary loss setting as, e.g., Rothschild and Stiglitz (1976). Many papers in the literature (e.g., Riley (1979) and Engers and Fernandez (1987)) use different and more general preferences. Our results extend, readily so to quasi-linear-in-premium settings such as in Spence (1978) and Azevedo and Gottlieb (2017).

and equilibrium outcomes in terms of these utility vectors rather than consumptions or premium-indemnity pairs. In this formulation, expected utility is linear (in $\vec{U} = (u_N, u_L)$ as well as p) and we denote it as

$$V(\vec{U}; p) \equiv V((u_N, u_L); p) \equiv (1-p)u_N + pu_L. \tag{1}$$

Risk neutral firms earn expected profits $q - pI$ from selling contracts (q, I) to a type p; equivalently, a firm selling a contract yielding utility vector $\vec{U} = (u_N, u_L)$ has expected profits

$$\Pi(\vec{U}; p) \equiv [W - pL] - (1-p)u^{-1}(u_N) - pu^{-1}(u_L), \tag{2}$$

which are strictly concave in $\vec{U}$.

2.2 MWS equilibrium

This section builds up to a definition of the MWS equilibrium in the continuum of types case, as outlined above. To that end, we first provide an alternative formulation of the MWS equilibrium in the finite type case considered in Spence (1978), adapted to our notation. Consider a discrete set of types $p_1 > p_2 > \cdots > p_n$ with probability masses $f(p_i) > 0$, $\sum_{i=1}^{n} f(p_i) = 1$. Spence (1978)'s approach defines a set of reservation utilities $\bar{V}(p_i)$ for each i recursively. Specifically, let $\bar{V}(p_1) \equiv u(W - p_1 L)$ the expected utility with full insurance for the highest risk type, and, for each $i > 1$, define:

$$\bar{V}(p_i) \equiv \max_{\{\vec{U}(p_j)\}_{j \leq i}} V(\vec{U}(p_i); p_i) \tag{3}$$

subject to

$$V(\vec{U}(p_j); p_j) \geq V(\vec{U}(p_k); p_j) \quad \forall\ j, k \leq i \text{ and} \tag{4}$$

$$\sum_{j=1}^{i} \Pi(\vec{U}(p_j); p_j) f(p_j) \geq 0 \text{ and} \tag{5}$$

$$V(\vec{U}(p_j); p_j) \geq \bar{V}(p_j) \quad \forall j < i. \tag{6}$$

The allocation $\{\vec{U}(p_j)\}_{j=1}^n$ solving the lowest risk type's sub-problem is called the MWS equilibrium allocation. Then, four properties of these programs are readily established. First, the same program can also be used as an equivalent definition of $\bar{V}(p_1)$. Second, for each $i = 1, \cdots, n$, constraint (5) binds. Third, for each i, the program is dual to the problem of maximizing profits $\sum_{j=1}^{i} \Pi(\vec{U}(p_j); p_j) f(p_j)$ subject to (4), (6), and $V(\vec{U}(p_i); p_i) \geq \bar{V}(p_i)$, and, by the second point, the solution to this dual problem yields
$\sum_{j=1}^{i} \Pi(\vec{U}(p_j); p_j) f(p_j) = 0$. Fourth—as an implication of the preceding two points and the fact that Π is concave and V a linear function of $\vec{U}$—there is a unique solution to these dual programs for each i. As such, describing the reservation utility function $\bar{V}(\cdot)$ fully (albeit implicitly) defines an MWS *allocation*, i.e., the solution to the preceding program for type p_n. These four observations together imply that it is equivalent to define an MWS equilibrium, in dual terms, as a real-valued function $\bar{V}(\cdot)$, on $\{p_i\}_{i=1,\cdots,n}$, with the property that, for all i:

$$0 = \max_{\{\vec{U}(p_j)\}_{j \leq i}} \sum_{j=1}^{i} \Pi(\vec{U}(p_j); p_j) f(p_j) \tag{7}$$

subject to

$$V(\vec{U}(p_j); p_j) \geq V(\vec{U}(p_k); p_j) \quad \forall\ j, k \leq i \text{ and} \tag{8}$$

$$V(\vec{U}(p_j); p_j) \geq \bar{V}(p_j) \quad \forall j \leq i. \tag{9}$$

Spence (1978)'s "primal" approach to the MWS equilibrium does not apply in the continuum of types case (since the natural ordering of risk types is not a well-ordering and recursion is impossible). The dual approach, on the other hand, generalizes naturally and as follows:

Definition 1. *A MWS equilibrium is a function $\bar{V}(p)$ such that for all $\hat{p} \in [\underline{p}, \bar{p}]$:*

$$0 = \sup_{\{\vec{U}(p)\}_{p \in [\hat{p}, \bar{p}]}} \int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}(p); p) dF(p) \tag{10}$$

subject to

$$V(\vec{U}(p); p) \geq V(\vec{U}(p'); p) \quad \forall\ p, p' \geq \hat{p} \ \text{ and} \tag{11}$$

$$V(\vec{U}(p); p) \geq \bar{V}(p) \quad \forall p \geq \hat{p}. \tag{12}$$

Note that $V(\vec{U};p)$, and hence the constraints, are linear in $\vec{U}$. The objective is concave. As such, the supremum in the objective (10) is always achieved and has an (essentially) unique solution.

The following section proves that an MWS equilibrium, so defined, always exists. Before turning to that section, we first provide a brief, non-technical summary of the key steps in the proof.

2.3 A non-technical summary of the proof

To prove the existence of an MWS equilibrium in the continuum of types case, we consider an increasingly fine sequence of finite approximations to the distribution of types F. As in Spence (1978), there is a well-defined MWS equilibrium set of utilities and corresponding allocations for each such discretization. We adapt an argument used by Hellwig (2007) in order to show that there is a subsequence of these allocations which converges on a dense set of types p. We use the completion of this convergent subsequence to define a candidate MWS function $\bar{V}(p)$, which we show is continuous in p.

We then verify, in two steps, that this candidate is indeed an MWS equilibrium in the sense of Definition 1. The first step involves a simple continuity argument which establishes that the appropriately-taken limits of *allocations* in the discrete MWS problems are feasible in the continuous problem (i.e., satisfy constraints (11) and (12)) and, moreover, yield zero profits at the limit. In the second step, we show that no *other* feasible allocation can yield positive profits in the limit problems. This second step is done by contradiction: if a feasible allocation did yield positive profits, then the continuity of $\bar{V}(p)$ could be used to construct a feasible allocation that would yield positive profits in some (sufficiently fine) discretization.

3 Equilibrium construction

In this section we formalize the non-technical proof summary described in the preceding section in order to construct an MWS equilibrium.

3.1 Discretizing the risk type distribution F

For each $n \in \mathbb{N}$, define the set of types P^n

$$P^n = \{p_0^n, p_1^n, \cdots, p_k^n, \cdots, p_{2^n+1}^n\} \tag{13}$$

$$= \left\{\underline{p}, \underline{p} + \frac{(\bar{p} - \underline{p})}{2^n}, \cdots, \underline{p} + k\frac{(\bar{p} - \underline{p})}{2^n}, \cdots, \bar{p}\right\} \tag{14}$$

and the corresponding cdf F^n via:

$$F^n(p) = \min_{p' \in P^n, p' \geq p} F(p).$$

The distribution F^n thus effectively collapses all types in the interval $[p_k^n, p_{k+1}^n)$ under F onto the point p_k^n, so that the probability mass at p_k^n is (for each $0 \leq k \leq 2^n$) given by:

$$f^n(p_k^n) = F(p_{k+1}^n) - F(p_k^n).$$

We define

$$\mathcal{P} = \bigcup_{n \in \mathbb{N}} P^n \tag{15}$$

as the set of all types which appear in any discretization.

As in Spence (1978), for each discretization $F^n, n \in \mathbb{N}$, an MWS equilibrium exists. In particular, the MWS equilibrium gives a well-defined and unique set of reservation utilities $\{\bar{V}^n(p_k^n)\}_{k=0,\cdots,2^n+1}$ that solves the MWS programs defined in (7)-(9) for each $\hat{p} \in P^n$. We refer to this program for type $\hat{p} \in P^n$ as "the MWS sub-problem for type $\hat{p}$".

Define $\vec{U}^n(p;\hat{p}) = (u_N^n(p;\hat{p}), u_L^n(p;\hat{p}))$ as the allocation of the p type in the solution to the MWS sub-problem for type $\hat{p} \leq p$ in discretization n (which is defined only if $p, \hat{p} \in P^n$).

3.2 Convergence of discretized allocations

The following lemma shows that there is a subsequence of discretizations that converges for all $p \in \mathcal{P}$.

Lemma 1. *There exists a sequence* $\{n_m\}_{m\in\mathbb{N}}$ *such that* $\lim_{m\to\infty}\vec{U}^{n_m}(p;\hat{p}) \equiv U^{\infty}(p;\hat{p})$ *exists for all* $p,\hat{p}\in\mathcal{P}$.

Proof. The argument is the same as for Helly (1921)'s Selection Theorem and as in Hellwig (2007)'s proof of Lemma B.3 (viz p. 812, where Hellwig cites Billingsley (1968)). Fix any $p,\hat{p}\in\mathcal{P}, p\geq\hat{p}$. Since the MWS equilibrium always features a consumption allocation with $W-L\leq c_L\leq c_N\leq W$ (i.e., each type buys non-negative insurance and does not overinsure) and since we have normalized $u(W-L)=0$ and $u(W)=1$, the components of $\vec{U}^n(p;\hat{p})$ are uniformly bounded. Hence, there exists a convergent subsequence $\{n_m\}$. By a standard diagonalization argument, we can, in fact, find a subsequence that converges for any countable set of such pairs $p,\hat{p}\in\mathcal{P}, p\geq\hat{p}$. The lemma follows from the fact that $\mathcal{P}\times\mathcal{P}$ is countable. □

Define:

$$\bar{V}^n(p)\equiv V(\vec{U}^n(p;p);p)\quad\forall p\in P^n \tag{16}$$

and

$$\bar{V}^*(p)\equiv V(\vec{U}^{\infty}(p;p);p)=\lim_{m\to\infty}\bar{V}^{n_m}(p)\quad\forall p\in\mathcal{P}, \tag{17}$$

where $\vec{U}^{\infty}$ is defined in Lemma 1. It is straightforward to show that $\bar{V}^n(p)$ decreases in p, so $\bar{V}^*(p)$ is non-increasing, and

$$\bar{V}^*(p)\equiv\lim_{\tilde{p}\nearrow p,\tilde{p}\in\mathcal{P}}\bar{V}^*(\tilde{p}) \tag{18}$$

is well-defined for all $p\in(\underline{p},\bar{p}]$. In fact, $\bar{V}^*(p)$ so defined is continuous in p, per the following Lemma.

Lemma 2. *The function* $\bar{V}^*(p)$*, defined in (17) and (18), is continuous.*

Proof. See Appendix A.1. □

We are now ready to state and prove our main theorem.

3.3 Statement and proof of main theorem

Theorem 1. *The function $\bar{V}^*(p)$ defined in (17) and (18), solves the program (10)-(12) for all $p \in [\underline{p}, \bar{p}]$, i.e., is an MWS equilibrium.*

Proof. The proof has two steps. In the first, we use a limiting argument for each type $\hat{p}$ to construct an allocation $\{\vec{U}^*(p;\hat{p})\}_{p\in[\hat{p},\bar{p}]}$ that is incentive compatible, has $V(\vec{U}^*(p;\hat{p});p) \geq \bar{V}^*(p)$ for all $p \in [\hat{p}, \bar{p}]$, and yields zero total profits. Second, we use Lemma 2 to show that there cannot be any $\hat{p}$ and any profitable, incentive compatible allocation $\{\vec{U}^\dagger(p)\}_{p\in[\hat{p},\bar{p}]}$ satisfying $V(\vec{U}^\dagger(p)) \geq \bar{V}^*(p)$ for all $p \in [\hat{p}, \bar{p}]$. Together, these imply that $\bar{V}^*(\cdot)$ is an MWS equilibrium per Definition 1.

Step 1: Constructing $\{\vec{U}^*(p;\hat{p})\}_{p\in[\hat{p},\bar{p}]}$.

We make extensive use of Lemma 1, which defines $\vec{U}^\infty(p;\hat{p})$ as a limit of a sequence $\{n_m\}_{m\in\mathbb{N}}$ of allocations $\vec{U}^{n_m}(p;\hat{p})$ for each $p, \hat{p} \in \mathcal{P}, p \geq \hat{p}$.

First, consider $\hat{p} \in \mathcal{P}$, and define $\vec{U}^*(p;\hat{p}) = \vec{U}^\infty(p;\hat{p})$ for all $p \in \mathcal{P} \cap [\hat{p}, \bar{p}]$. Each component of $\vec{U}^\infty(p;\hat{p})$ is monotonic in p (as higher risk types get weakly more insurance than lower risk types), and they are uniformly bounded by $[0,1]$ (by our normalization of the utility function). So $\lim_{\tilde{p}\nearrow p,\tilde{p}\in\mathcal{P}} \vec{U}^\infty(\tilde{p};\hat{p})$ and $\lim_{\tilde{p}\searrow p,\tilde{p}\in\mathcal{P}} \vec{U}^\infty(\tilde{p};\hat{p})$ are both well-defined and coincide except possibly at a countable number of points, which have measure 0 under the continuous distribution F. Extend $\vec{U}^*$ to $p \notin \mathcal{P}$ via

$$\vec{U}^*(p;\hat{p}) \equiv \lim_{\tilde{p}\nearrow p,\tilde{p}\in\mathcal{P}} \vec{U}^\infty(\tilde{p};\hat{p}).$$

$\{\vec{U}^\infty(p;\hat{p})\}_{p\in\mathcal{P}\cap[\hat{p},\bar{p}]}$ is incentive compatible for types $p \in \mathcal{P} \cap [\hat{p}, \bar{p}]$, so $\{\vec{U}^*(p;\hat{p})\}_{p\in[\hat{p},\bar{p}]}$ defined in this way is incentive compatible for all types $p \in [\hat{p}, \bar{p}]$. Similarly,
$V(\vec{U}^\infty(p;\hat{p});p) \geq \bar{V}^*(p)$ for all $p \in \mathcal{P} \cap [\hat{p}, \bar{p}]$, so $V(\vec{U}^*(p;\hat{p});p) \geq \bar{V}^*(p)$.

We will now show that $\int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}^*(p;\hat{p});p)dF(p) = 0$, thereby establishing feasibility for the $\hat{p} \in \mathcal{P}$ case. To that end, for each m, extend $\vec{U}^{n_m}(p;\hat{p})$ to all $p \in [\hat{p}, \bar{p}]$ via

$$\vec{U}^{m*}(p;\hat{p}) = \vec{U}^{n_m}\left(\max\{\tilde{p} \in P^{n_m} \cap [\hat{p}, p]\}; \hat{p}\right). \tag{19}$$

That is, "assign" types p outside of P^{n_m} to the allocation of the closest lower-risk type in P^{n_m}. Exactly as in Hellwig (2007)'s Lemma B.1, the (almost everywhere) pointwise convergence of $\vec{U}^{m*}(\cdot;\hat{p})$

to $\vec{U}^*(\cdot;\hat{p})$ and the setwise convergence (here, weak convergence) of F^{n_m} to F implies

$$\int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}^*(p;\hat{p});p)dF(p) = \lim_{m\to\infty}\int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}^{m*}(p;\hat{p});p)dF^{n_m}(p) = \lim_{m\to\infty} 0 = 0. \tag{20}$$

Second, consider $\hat{p} \notin \mathcal{P}$, and take any sequence $k = 1, \cdots, \infty$ of $p_k \in \mathcal{P}$ with $p_k \nearrow \hat{p}$. Use the associated sequence of (sub) allocations $\{\vec{U}^*(p;p_k)\}_{p\in[p_k,\bar{p}]}$ to construct (via a diagonalization argument as in Lemma 1) a subsequence $\{k_m\}$ that converges for each $p \in \mathcal{P} \cap [\hat{p},\bar{p}]$, and define $\vec{U}^*(p;\hat{p})$ as the limit for each such p. Complete the allocation by defining $\vec{U}^*(p;\hat{p})$ for $p \notin \mathcal{P}$ in terms of left-hand limits of $\vec{U}^*(p';\hat{p})$ for $p' \in \mathcal{P}$. The resulting allocation is incentive compatible and has $V(\vec{U}^*(p;\hat{p});p) \geq \bar{V}^*(p)$ for all $p \in [\hat{p},\bar{p}]$. Moreover, since

$$\lim_{m\to\infty}\int_{p=p_{k_m}}^{\hat{p}} \Pi(\vec{U}^*(p;p_{k_m});p)dF(p) = 0,$$

we have

$$\begin{aligned}
&\int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}^*(p;\hat{p});p)dF(p)\\
&= \int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}^*(p;\hat{p});p)dF(p) + \lim_{m\to\infty}\int_{p=p_{k_m}}^{\hat{p}} \Pi(\vec{U}^*(p;p_{k_m});p)dF(p)\\
&= \lim_{m\to\infty}\int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}^*(p;p_{k_m});p)dF(p) + \lim_{m\to\infty}\int_{p=p_{k_m}}^{\hat{p}} \Pi(\vec{U}^*(p;p_{k_m});p)dF(p)\\
&= \lim_{m\to\infty}\int_{p=p_{k_m}}^{\bar{p}} \Pi(\vec{U}^*(p;p_{k_m});p)dF(p) = \lim_{m\to\infty} 0 = 0,
\end{aligned} \tag{21}$$

where the last line follows from Equation (20). For each $p \in [\underline{p},\bar{p}]$, then, we have identified a feasible allocation $\{\vec{U}^*(p;\hat{p})\}_{p\in[\hat{p},\bar{p}]}$ which satisfies (11) and (12) and yields zero profits.

Step 2: Showing that $\{\vec{U}^*(p;\hat{p})\}_{p\in[\hat{p},\bar{p}]}$ is optimal.

Suppose, by way of contradiction, that there was some $\hat{p}$ and some other allocation $\{\vec{U}^\dagger(p)\}_{p\in[\hat{p},\bar{p}]}$ satisfying (11) and (12) with $\int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}^\dagger(p);\hat{p})dF(p) = \delta > 0$ (in which case $\{\vec{U}^*(p;\hat{p})\}_{p\in[\hat{p},\bar{p}]}$ would not be optimal in the MWS program). Then, for some sufficiently small $\varepsilon > 0$ and some $\tilde{p} \in \mathcal{P}$ sufficiently close to and at least as large as $\hat{p}$, the allocation $\{\vec{U}^\circ(p)\}_{p\in[\tilde{p},\bar{p}]}$ defined by

$$\vec{U}^\circ(p) \equiv \vec{U}^\dagger(p) + (\varepsilon,\varepsilon) \quad \forall p \in [\tilde{p},\bar{p}]$$

has profits $\int_{p=\tilde{p}}^{\bar{p}} \Pi(\vec{U}^\circ(p);p)dF(p) > \delta/2 > 0$, is incentive compatible, and has

$$V(\vec{U}^\circ(p);p) \geq \bar{V}^*(p) + \varepsilon, \tag{22}$$

i.e., minimum utility constraints are slack in the continuous problem.

Now consider the allocation $\{\vec{U}^\circ(p)\}_{p\in P^n\cap[\tilde{p},\bar{p}]}$ (the restriction of the continuous allocation $\vec{U}^\circ$ to the types in the nth discretization). This allocation is obviously incentive compatible, i.e., satisfies (8) in the nth discretization. Towards showing that it also satisfies minimum utility constraints (9) in the nth discretization, observe that $\bar{V}^n(p)$ converges pointwise to $\bar{V}^*(p)$. Since $\bar{V}^*(p)$ is monotonic and, per Lemma 2, continuous in p, $\bar{V}^n(p)$ in fact converges *uniformly* in n. That is, we can find N large enough so that $\bar{V}^n(p) < \bar{V}^*(p) + \varepsilon$ for all $p \in P^n \cap [\tilde{p}, \bar{p}]$ and $n \geq N$. It follows from (22) that $V(\vec{U}^\circ(p);p) > \bar{V}^n(p)$ for $n > N$, and hence that $\{\vec{U}^\circ(p)\}_{p\in P^n\cap[\tilde{p},\bar{p}]}$ is feasible in (the dual for) the $\tilde{p}$ type's MWS sub-program for the nth discretization for all $n \geq N$. It must (by definition of the MWS equilibrium utilities $\bar{V}^n$) therefore have non-positive profits. But

$$\lim_{n\to\infty} \int_{p=\tilde{p}}^{\bar{p}} \Pi(\vec{U}^\circ(p);p)dF^n(p) = \int_{p=\tilde{p}}^{\bar{p}} \Pi(\vec{U}^\circ(p);p)dF(p) > \delta/2 > 0,$$

so, in fact, $\vec{U}^\circ(p)$ is strictly profitable for sufficiently large n. This contradicts the optimality of the original MWS allocation $\vec{U}^n(p;\tilde{p})$ in the nth discretization. We conclude, for any $\hat{p}$, that *any* feasible allocation $\{\vec{U}^\dagger(p)\}_{p\in[\hat{p},\bar{p}]}$ satisfying (11) and (12) must have $\int_{p=\hat{p}}^{\bar{p}} \Pi(\vec{U}^\dagger(p);p)dF(p) \leq 0$. The allocation $\{\vec{U}^*(p;\hat{p})\}_{p\in[\hat{p},\bar{p}]}$ is therefore optimal in the MWS sub-problem (i.e., maximize (10) subject to (11) and (12)) for each $\hat{p} \in [\underline{p}, \bar{p}]$, and hence $\bar{V}^*(\cdot)$ is an MWS equilibrium.

□

4 Justifying the MWS concept

In this subsection, we argue that $\bar{V}(p)$ satisfying Definition 1 is a foresight equilibrium in the sense of Wilson (1977) and Spence (1978). Specifically, we show that the allocation $\vec{U}^*$ solving the MWS program (10) for type $\underline{p}$ given $\bar{V}(p)$ can be implemented in such a way that no entrant (or deviating incumbent) firm can offer a menu of deviating contracts that will be profitable once incumbent firms have withdrawn unprofitable contracts.

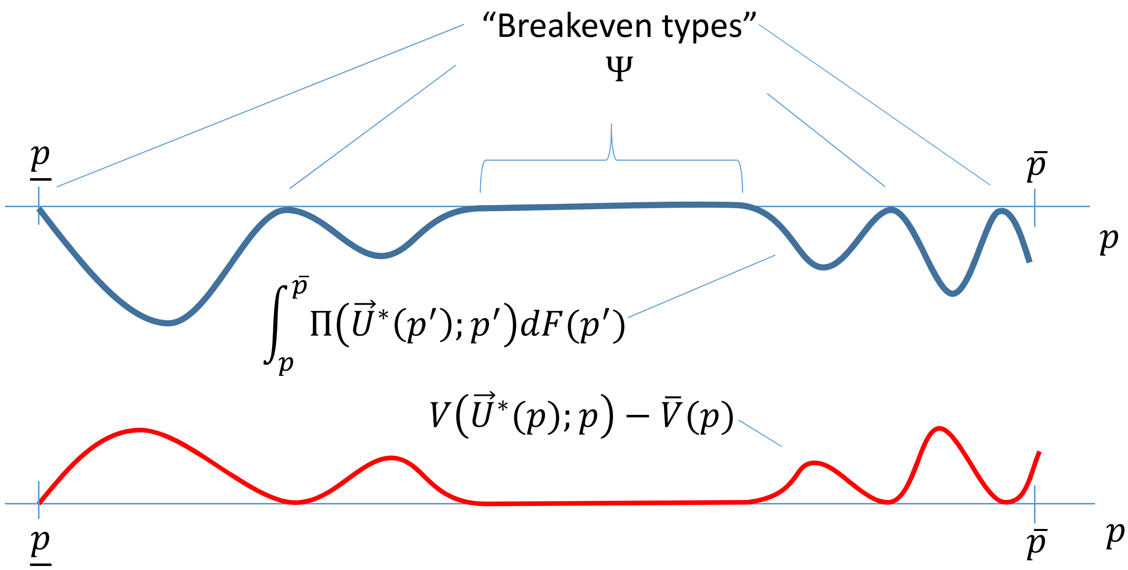

Figure 1: Breakeven types and utilities in the MWS equilibrium.

To that end, observe first that for all types p,

$$\int_{p'=p}^{\bar{p}} \Pi(\vec{U}^*(p'); p') dF(p') \leq 0, \tag{23}$$

since, otherwise, the allocation $\{\vec{U}^*(p')\}_{p' \in [p, \bar{p}]}$ would be strictly profitable while satisfying (11) and (12)—contradicting the definition of $\bar{V}(p)$. Define the set of *breakeven* types by:

$$\Psi = \{p | \int_{p'=p}^{\bar{p}} \Pi(\vec{U}^*(p'); p') dF(p') = 0\}. \tag{24}$$

The breakeven types are illustrated, qualitatively, in Figure 1. The figure also qualitatively plots the profit (23) for group $[p, \bar{p}]$ and the corresponding “slack” in the minimum utility constraints (12), i.e., $V(\vec{U}^*(p); p) - \bar{V}(p)$. As the figure indicates, the minimum utility constraints bind for a type p if and only if $p \in \Psi$.[3]

[3] To see why, observe that only downward incentive constraints bind, and by single crossing, if $p_1 > p_2 > p_3$ and if incentive compatibility is satisfied between the p_1 and p_2 and between the p_2 and p_3 types, then incentive compatibility is also satisfied between the p_1 and p_3 types. Fixing any p, consider the possibility of cross-subsidizing $[p, \bar{p}]$. Such a cross subsidy costs resources, so it is only desirable if it eases incentive constraints—i.e., by the preceding observations, only if it raises the utility of the p type. Consequently, if $p \notin \Psi$—so that $[p, \bar{p}]$ is cross-subsidized—p must receive a higher utility than $\bar{V}(p)$.

Because the objective (10) is concave and the constraints are linear, these observations imply that the allocation $\vec{U}^*$ in fact solves the problem of maximizing (10) subject to (11) and

$$V(\vec{U}(p);p) \geq \bar{V}(p) \quad \forall p \in \Psi. \tag{25}$$

In other words, dropping the non-binding minimum utility constraints (i.e., for $p \notin \Psi$) does not change the solution to the mathematical program underlying the MWS equilibrium allocations. We refer to the problem with the looser constraints (25) (in place of (12)) as the *relaxed MWS problem.*

For each $p \in \Psi$ with $p < \bar{p}$, define the associated breakeven *group* via

$$G(p) = \left\{p' | p \leq p' \leq \inf\{\hat{p} | \hat{p} \in \Psi, \hat{p} > p\}\right\}.$$

If $\bar{p} \in \Psi$, further define $G(\bar{p}) = \bar{p}$. Now implement the allocation $\{\vec{U}^*(p)\}$ by having any given firm sell contracts to a single breakeven group (i.e., offer the menu of contracts $\{\vec{U}^*(p')\}_{p' \in G(p)}$ for some p), and by having multiple firms sell to each breakeven group.[4]

Suppose now that a competitor attacks the hypothesized equilibrium by offering some menu of contracts. Without loss of generality, we can take this menu to be incentive compatible, describe it by $\{\vec{U}^\dagger(p)\}_{p \in [\underline{p}, \bar{p}]}$, and assume that $V(\vec{U}^\dagger(p);p) > V(\vec{U}^*(p);p)$ for some p. This attack will attract some types away from their incumbent, potentially causing the incumbent to anticipate becoming unprofitable. We assume, following Wilson (1977) and Spence (1978), that, in response, incumbents drop unprofitable contracts, starting with the highest risks they serve, and continue to drop until the remaining policies are profitable.[5] We will now argue that, under this assumption about responses, the final allocation $\{\vec{U}(p)\}_{p \in [\underline{p}, \bar{p}]}$ (after withdrawals) will have $V(\vec{U}(p);p) \geq \bar{V}(p)$ for all $p \in \Psi$.

For each $p \in \Psi$, there are three mutually exclusive and collectively exhaustive cases:

1. $V(\vec{U}^\dagger(p);p) \geq V(\vec{U}^*(p);p)$
2. $V(\vec{U}^\dagger(p);p) < V(\vec{U}^*(p);p)$ and $V(\vec{U}^\dagger(p');p') \leq V(\vec{U}^*(p');p')$ for all $p' \in G(p)$
3. $V(\vec{U}^\dagger(p);p) < V(\vec{U}^*(p);p)$ and there exists a $p' \in G(p)$ with $V(\vec{U}^\dagger(p');p') > V(\vec{U}^*(p');p')$.

[4]Note that this may require a continuum of firms.

[5]This concept of incumbents' reaction is precisely the withdrawal concept described by (Spence, 1978, p. 437).

We partition Ψ into three groups Ψ_1, Ψ_2, and Ψ_3 according to these three cases. (Note that singleton groups are in Ψ_1 or Ψ_2.) For $p \in \Psi_3$, define

$$r(p) \equiv \inf\{p' \in G(p) | V(\vec{U}^\dagger(p'); p') > V(\vec{U}^*(p'); p')\}$$

as the smallest risk type that strictly benefits from $\vec{U}^\dagger$ relative to $\vec{U}^*$. (Note that incentive compatibility implies $r(p) > p$.) Now consider what would happen if all incumbent firms offering to groups $G(p)$ with $p \in \Psi_2$ continued to offer their original menus (no withdrawal), and all incumbent firms offering to groups $G(p)$ with $p \in \Psi_3$ continued to offer *at least* the contracts $\{\vec{U}^*(p')\}_{p' \in [p, r(p)]}$. Under such a set of offers, each $p \in \Psi_1$ will optimally choose $\vec{U}^\dagger(p)$, and each $p \in \Psi_2 \cup \Psi_3$ will choose to remain with their incumbent; in either case, their expected utility will be at least $V(\vec{U}^*(p); p)$. In principle, an incumbent serving group $G(p)$ *could* end up attracting types who originally (before the attacker's deviation) were going to purchase from a different group. But, by single crossing, all *higher* risk types $p' > p$ will at least weakly prefer what $\tilde{p}$ prefers to $\vec{U}^*(p)$ for any $p' > \tilde{p} > p$, so the $G(p)$ incumbent will not attract types from any other higher risk group. Since attracting *lower risk* types than p will be strictly profitable for $G(p)$-incumbents, this immediately implies that incumbents serving groups in Ψ_1 and Ψ_2 will be at least weakly profitable. Similarly, an incumbent who was serving any group $G(p)$ with $p \in \Psi_3$ and offers *only* the contracts $\{\vec{U}^*(p')\}_{p' \in [p, r(p)]}$ will retain all types in $[p, r(p))$ and no types $p' > r(p)$. Since $\int_{p'=p}^{r(p)} \Pi(\vec{U}^*(p'); p') dF(p') > 0$, she will be strictly profitable.

Taken together, these observations imply that, after incumbent withdrawal, all incumbents serving groups in Ψ_2 will continue to sell to all types in their group, all incumbents serving groups in Ψ_3 will continue to sell to *at least* $p' \in [p, r(p))$. Hence, for $p \in \Psi_2 \cup \Psi_3$, $\vec{U}(p) = \vec{U}^*(p)$. For $p \in \Psi_1$, $\vec{U}(p) = \vec{U}^\dagger(p)$. In either case, $V(\vec{U}(p); p) \geq V(\vec{U}^*(p); p)$. In words: for *any* attack, the final allocation $\{\vec{U}(p)\}_{p \in [\underline{p}, \bar{p}]}$ after all incumbent withdrawals will feature $V(\vec{U}(p); p) \geq V(\vec{U}^*(p); p) \geq \bar{V}(p)$. It will (trivially) be incentive compatible, and it will have $\vec{U}(p) \neq \vec{U}^*(p)$ for some $p \in [\underline{p}, \bar{p}]$. Since $\vec{U}^*(\cdot)$ solves the *relaxed* MWS problem and yields zero profits, it must be that

$$\int_{\underline{p}}^{\bar{p}} \Pi(\vec{U}(p'); p') dF(p') < 0.$$

Since each incumbent remains at least weakly profitable, it must therefore be that the entrant loses money. We conclude that there is no scope for a deviator (or entrant) to offer contracts that will remain profitable after the anticipated withdrawal by incumbent firms.

5 Conclusions

We have shown how to extend the MWS equilibrium concept to models with a continuum of risk types, and we have shown that such an equilibrium always exists. The underlying argument and conclusions readily extend mixed-distribution models with a continuum of risks *and* a finite number of mass points. A natural example is models with a mass of zero-risk types in addition to a continuously distributed measure of positive-risk types.

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A Appendix: Proofs

A.1 Proof of Lemma 2

We will show that continuity of $\bar{V}^*$ holds at any $\tilde{p} < \bar{p}$. We omit the straightforward proof of continuity at $\bar{p}$.[6] To that end, we will show that for any δ, there exists a $p_1 < \tilde{p}$ and a $p_2 > \tilde{p}$ such that $\bar{V}^*(p_1) - \bar{V}^*(p_2) < \delta$. Since $\bar{V}^*(p)$ is non-increasing in p, it will follow that $|p - \tilde{p}| < \min\{\tilde{p} - p_1, p_2 - \tilde{p}\} \equiv \varepsilon$ implies $|\bar{V}^*(p) - \bar{V}^*(\tilde{p})| < \delta$, whence $\bar{V}^*$ is continuous at $\tilde{p}$.

In constructing such a p_1 and p_2 we will use the following definition extensively:

Definition 2. *For $\hat{p} \in P^n$ and any $T \geq 0$, define*

$$\tilde{V}^n(\hat{p}, T) \equiv \max_{\{\vec{U}(p)\}_{p \geq \hat{p}, p \in P^n}} V(\vec{U}(p); p) \tag{26}$$

subject to

$$V(\vec{U}(p); p) \geq V(\vec{U}(p'); p) \quad \forall\ p, p' \geq \hat{p} \text{ with } p, p' \in P^n \textit{ and} \tag{27}$$

$$V(\vec{U}(p); p) \geq \bar{V}^n(p) \quad \forall p \geq \hat{p}, p \in P^n \textit{ and} \tag{28}$$

$$\sum_{p' \in P^n \cap [\hat{p}, \bar{p}]} \Pi(\vec{U}(p); p) f^n(p) \geq -T. \tag{29}$$

This is the (primal) MWS program for the nth discretization, but with the budget constraint relaxed by T. In other words, $\tilde{V}^n(\hat{p}, T)$ is the maximum utility of type $\hat{p}$ in its sub-problem if there is a subsidy of size $T > 0$ available to the interval of types $p \in [\hat{p}, \bar{p}] \cap P^n$. By definition, $\bar{V}^n(p) = \tilde{V}^n(p, 0)$. For any $p_1, p_2 \in P^n$ with $p_1 \leq p_2$ we define

$$T^n(p_2; p_1) \equiv \sum_{p \in P^n \cap [p_1, p_2)} f^n(p) \Pi(\vec{U}^n(p; p_1); p) \tag{30}$$

[6] In fact, it is easy to show by a limiting argument that $\bar{V}^*(p)$ is left-continuous.

as the profit of types $[p_1, p_2)$ in the original solution $\{\vec{U}^n(p;p_1)\}_{p\in P^n\cap[p_1,\bar{p}]}$ to the p_1 sub-problem. Since the sum of profits over all types $[p_1,\bar{p}]\cap P^n$ is zero in the p_1 type's MWS sub-problem for the nth discretization, $T^n(p_2;p_1)$ is the cross subsidy to the group $[p_2,\bar{p}]$ in the solution to that sub-problem. It follows that the sub-allocation for the group $[p_2,\bar{p}]\cap P^n$ in the p_1 type's MWS sub-problem in the nth discretization coincides with the solution to the program defining $\tilde{V}^n(p_2, T^n(p_2;p_1))$. Hence:

$$V(\vec{U}^n(p_2;p_1);p_2) = \tilde{V}^n(p_2, T^n(p_2;p_1)).$$

For any $p_1, p_2 \in P^n$ with $p_2 \geq p_1$, we have:

$$\begin{aligned}|\bar{V}^n(p_1) - \bar{V}^n(p_2)| &= |\bar{V}^n(p_1) - \tilde{V}^n(p_2, T^n(p_2;p_1)) + \tilde{V}^n(p_2, T^n(p_2;p_1)) - \tilde{V}^n(p_2, 0)| \\ &\leq |\bar{V}^n(p_1) - \tilde{V}^n(p_2, T^n(p_2;p_1))| + |\tilde{V}^n(p_2, T^n(p_2;p_1)) - \tilde{V}^n(p_2, 0)|. \end{aligned} \tag{31}$$

By incentive compatibility in the p_1 MWS sub-problem in the nth discretization, p_2 types weakly prefer $\vec{U}(p_2;p_1)$ to $\vec{U}(p_1;p_1)$, implying that

$$V(\vec{U}^n(p_1;p_1);p_2) \leq V(\vec{U}^n(p_2;p_1);p_2) = \tilde{V}^n(p_2, T^n(p_2;p_1)). \tag{32}$$

Hence, for the first summand in Equation (31) it is

$$\begin{aligned}\left|\bar{V}^n(p_1) - \tilde{V}^n(p_2, T^n(p_2;p_1))\right| &\leq \bar{V}^n(p_1) - V(\vec{U}^n(p_1;p_1);p_2) \\ &= V(\vec{U}^n(p_1;p_1);p_1) - V(\vec{U}^n(p_1;p_1);p_2) \\ &= (p_2 - p_1)u_N^n(p_1;p_1) - (p_2 - p_1)u_L^n(p_1;p_1) \\ &\leq p_2 - p_1, \end{aligned} \tag{33}$$

where the last inequality follows from $0 \equiv u(W-L) \leq u_L^n \leq u_N^n \leq u(W) \equiv 1$. This gives us an upper bound that is independent of the discretization n.

Towards bounding the second summand in the last line of Equation (31), note that

$$T^n(p_2;p_1) = \sum_{\substack{p\in P^n \\ p\in[p_1,p_2)}} f^n(p)\Pi(\vec{U}^n(p;p_1);p) \leq (p_2-p_1)\bar{f}W. \tag{34}$$

Since the program defining $\tilde{V}^n(p,T)$ has linear constraints and a concave objective function, it follows that $\tilde{V}^n(p,T)$ is concave in T and hence that

$$\tilde{V}^n(p_2,T^n) - \tilde{V}^n(p_2,0) \leq T^n\frac{\partial V^n(p_2,0)}{\partial T} \leq (p_2-p_1)W\bar{f}\frac{\partial V^n(p_2,0)}{\partial T}. \tag{35}$$

The following Lemma (proved in Appendix A.2) allows us to bound $\frac{\partial V^n(p,0)}{\partial T}$ by some K uniformly in p and n (for $n > N$ for some sufficiently large N):

Lemma 3. *For any $p^* < \bar{p}$ there exists an N and a K such that $\frac{\partial \tilde{V}^n(p,0)}{\partial T} \leq K$ for all p in$[\underline{p},p^*]\cap P^n$ and $n > N$.*

By Lemma 3, for sufficiently large N, $n > N$ implies:

$$\tilde{V}^n(p_2,T^n) - \tilde{V}^n(p_2,0) \leq (p_2-p_1)W\bar{f}K. \tag{36}$$

Putting together (31), (33), and (36) and taking $p_2 - p_1 < \frac{\delta}{2(W\bar{f}K+1)}$ (with $p_1,p_2 \in \mathcal{P}$) shows that

$$|\bar{V}^n(p_1) - \bar{V}^n(p_2)| < \delta/2 \tag{37}$$

for all $n > N$, and hence that $|\bar{V}(p_1) - \bar{V}(p_2)| \leq \delta/2 < \delta$, completing the proof.

A.2 Proof of Lemma 3

We start by proving the following auxiliary lemma:

Lemma 4. *For any $p^* < \bar{p}$, there exists a $\bar{D} > 0$ and an N such that, for all $p \leq p^*$ and all $n \geq N$* either

1. $u^{-1}(u^n_N(p,p)) - u^{-1}(u^n_L(p,p)) > \bar{D}$ or

2. $\sum_{\{p'\in[\hat{p},\bar{p}]\cap P^n\}} f^n(p')\Pi(\vec{U}^n(p',p);p') < 0 \quad \forall \hat{p} > p.$

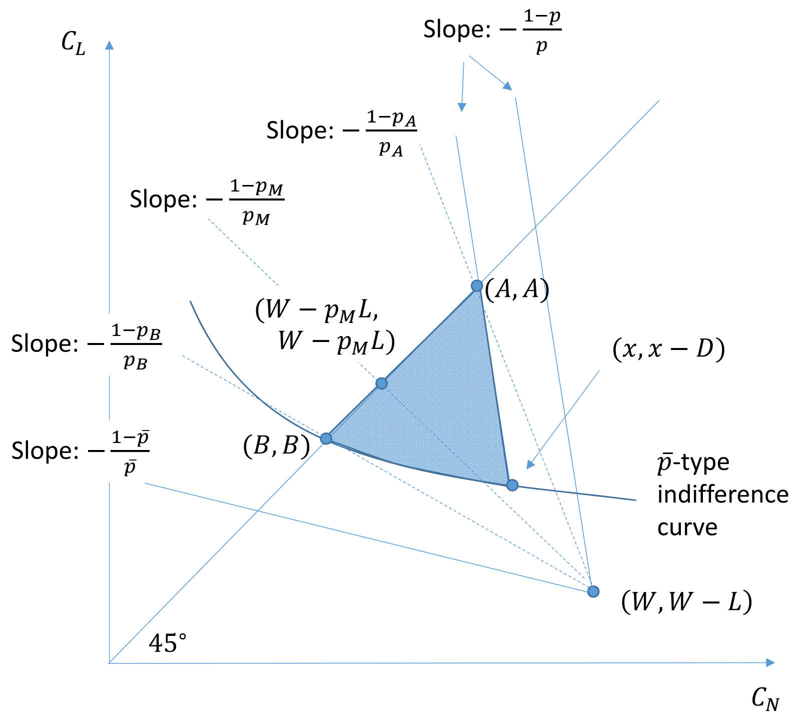

Figure 2: Constructing D for Lemma 4

In other words, for large enough n, *either* $p \leq p^*$ types face a deductible of at least $\bar{D}$ *or* the allocation for $p \leq p^*$-types' MWS sub-problem involves positive cross-subsidies for all subgroups $[\hat{p}, \bar{p}]$ (and hence non-binding minimum utility constraints).

Proof of Lemma 4. Let $p^* < \bar{p}$ and assume that type $p \leq p^*$, $p \in \mathcal{P}$ faces a deductible $D > 0$. The proof makes extensive use of Figure 2, which depicts a standard insurance diagram, in consumption space. If type p's allocation for her MWS sub-problem (for some n) is located at $(x, x - D)$, as shown in the diagram, then incentive compatibility implies that the allocations for *all* higher risk types $p' \in (p, \bar{p}]$ lie within the shaded area, depicted in Figure 2, with "corners" at (B, B), $(x, x - D)$, and (A, A), where $A \equiv x - pD$ (and thus is on the p type iso-profit line through $(x, x - D)$) and $B \equiv u^{-1}((1 - \bar{p})u(x) + \bar{p}u(x - D))$ (and thus lies on the $\bar{p}$ type's indifference curve through $(x, x - D)$).

For each type $p' \in [p, \bar{p}]$, the least profitable contract in the area is (A, A) and the most profitable is (B, B). Since the MWS sub-problem has exactly zero profits overall, this implies that B is below and A is above the joint pooling line

$$B \leq W - p_M^n(p)L \leq A$$

where

$$p_M^n(p) \equiv \mathbb{E}_{F^n}[p'|p' \in [p, \bar{p}] \cap P^n]$$

is the expected risk in p's sub-problem. Thus, $p_M^n(p)L$ is the cost of a pooling contract for types $[p, \bar{p}] \cap P^n$. It follows directly from the definition of A that

$$x \geq W - p_M^n(p)L + pD. \tag{38}$$

Similarly, from the definition of B:

$$u(W - p_M^n(p)L) \geq (1 - \bar{p})u(x) + \bar{p}u(x - D) \geq u(x - D)$$

and hence

$$x \leq W - p_M^n(p)L + D. \tag{39}$$

The preceding formalizes the simple observation that if D is small then x must be close to the pooled fair full insurance allocation $W - p_M^n L$. *At* full insurance, it is obvious that all subgroups $[\hat{p}, \bar{p}]$ receive cross-subsidies from the lower risk types $[p, \hat{p})$. We will now show that the same is true for sufficiently large N and for sufficiently small D.

To that end, define:

$$p_M(p) \equiv \lim_{n\to\infty} p_M^n(p),$$

$$p_H^n(p) \equiv \mathbb{E}[p'|p' \in [p_M(p), \bar{p}], F^n], \quad \text{and} \quad p_H(p) \equiv \lim_{n\to\infty} p_H^n(p)$$

$$p_L^n(p) \equiv \mathbb{E}[p'|p' \in [p, p_H(p)], F^n], \quad \text{and} \quad p_L(p) \equiv \lim_{n\to\infty} p_L^n(p),$$

and

$$Z \equiv \min\left\{\min\{p_H(p') - p_M(p'), p_M(p') - p_L(p')\}|p' \in [\underline{p}, p^*]\right\} > 0,$$

where $Z > 0$ follows from $p^* < \bar{p}$.

The fact that the CDF F^n converges uniformly to the continuous distribution F implies that $p_i^n(p)$ converges uniformly for each $i \in \{L, M, H\}$. Hence, one can choose N such that, for all p, and $i \in \{L, M, H\}$, $n > N$ implies $|p_i^n(p) - p_i(p)| < Z/3$. For such n, then, $p_H^n(p) - p_M^n(p) > Z/3$ and $p_M^n(p) - p_L^n(p) > Z/3$.

For $n > N$ and any p, the total profits accruing to the set of types below any given $\hat{p} \in P^n$ in the MWS-sub-problem for type p are:

$$\begin{aligned} \Pi^*(\hat{p}; p) = & \sum_{p' \in [p, \min\{\hat{p}, p_A^n\}) \cap P^n} f^n(p') \Pi(\vec{U}^n(p', p); p') \\ & + \sum_{p' \in [p_A^n, \min\{\hat{p}, p_B^n\}] \cap P^n} f^n(p') \Pi(\vec{U}^n(p', p); p') \\ & + \sum_{p' \in (p_B^n, \hat{p}] \cap P^n} f^n(p') \Pi(\vec{U}^n(p', p); p') \end{aligned} \tag{40}$$

where $p_A^n \equiv \frac{W-A}{L}$ and $p_B^n \equiv \frac{W-B}{L}$, and A and B are as in Figure 2 (and where we use the convention that the sum over an "interval" of the form $[x, y]$ with $y < x$ is zero). By definition, total profits in the p sub-problem are zero, $\Pi^*(p; p) = 0$. Thus, showing that $\Pi^*(\hat{p}; p) > 0$ for all $\hat{p} > p$ implies that profits of $(\hat{p}, \bar{p}]$ are negative which is equivalent to statement 2 in the lemma and will thus complete the proof. To that end, note first that all types $p' \in [p, p_A^n)$ have $\Pi(\vec{U}^n(p', p); p') > 0$ since all contracts in the shaded are with corners at (A, A), (B, B), $(x, x - D)$ are below the zero-profit lines for types $p' < p_A^n$. So $\Pi^*(\hat{p}; p) > 0$ for all $\hat{p} \leq p_A^n$. Similarly, all types $p' \in (p_B^n, \bar{p}]$ have $\Pi(\vec{U}^n(p', p); p') < 0$, since contracts in the shaded area are above p' types' zero-profit line. Hence, profits for the sub-group $(p_B^n, \bar{p}]$ are negative and that for $[p, p_B^n]$ positive. Thus, $\Pi^*(\hat{p}; p) > 0$ for all $\hat{p} \geq p_B^n$. It remains to establish $\Pi^*(\hat{p}, p) > 0$ for $\hat{p} \in [p_A^n, p_B^n]$.

Observe that for each type,

$$\Pi(\vec{U}^n(p', p); p') \geq W - p'L - A \geq p_M^n(p)L - p'L - (1 - p)D, \tag{41}$$

where we use the bounds on A derived above (and the fact that A is the least profitable allocation in the shaded area of Figure 2 for all types). Taking $D < ZL/3$, and taking an $f^n(p')$-weighted sum of expression (41) we have (using Equation (38), the definition of p_B^n and B, and the bound

$B \geq x - D$ to show $p^n_B \leq p^n_M + (1-p)D/L < p^n_M + Z/3 \leq p_H$):

$$\begin{aligned}
\Pi^*(\hat{p}, p) &\geq (F^n(\hat{p}) - F^n(p))L \left(p^n_M(p) - \mathbb{E}_{F^n}[p'|p' \in [p, \hat{p}]] - (1-p)D/L\right) \\
&> (F^n(\hat{p}) - F^n(p))L \left[(p_M - Z/3) - \mathbb{E}_{F^n}[p'|p' \in [p, p^n_B]] - Z/3\right] \\
&\geq (F^n(\hat{p}) - F^n(p))L \left[p_M - \mathbb{E}_{F^n}[p'|p' \in [p, p_H]] - 2Z/3\right] \\
&= (F^n(\hat{p}) - F^n(p))L \left[p_M - p^n_L - 2Z/3\right] \\
&\geq (F^n(\hat{p}) - F^n(p))L \left[p_M - p_L - Z\right] \\
&\geq (F^n(\hat{p}) - F^n(p))L \left[p_M - p_L - Z\right] \geq 0 \qquad (42)
\end{aligned}$$

which completes the proof. □

We can now prove the main lemma 3.

Choose N and D as in Lemma 4 and consider any $p \leq p^*$ and any $n > N$. If case 2. of that Lemma holds for this p and n, then none of the minimum utility constraints bind in the MWS sub-problem for type p in the nth discretization. By the envelope theorem, we can compute the welfare effects of a small increase in T via a uniform marginal increase $\Delta > 0$ in utility across all types and both states, so:

$$\begin{aligned}
\frac{\partial \tilde{V}^n(p, 0)}{\partial T} &= \frac{\Delta}{\sum_{p' \in P^n, p' \geq p} f^n(p')\Delta \left[\frac{1-p'}{u'(u^{-1}(u^n_N(p', p)))} + \frac{p'}{u'(u^{-1}(u^n_N(p', p)))}\right]} \qquad (43) \\
&\leq \frac{u'(W-L)}{1 - F^n(p)} \leq \frac{u'(W-L)}{1 - F^n(p^*)} \leq \frac{u'(W-L)}{1 - F(p^*)} \equiv K_1, \qquad (44)
\end{aligned}$$

where the denominator of (43) is the total resource cost of marginally increasing everyone's utility by Δ in both states.

If case 1. of that Lemma holds, on the other hand, we can compute the welfare consequences of a small increase in T by using that transfer to slide the p type down and to the right along the $p^{n+} \equiv p + \frac{1}{2^n}$ type's (the next lowest type's) indifference curve. A straightforward computation of this welfare consequences of this marginal increase yields:

$$\frac{\partial \tilde{V}^n(p,0)}{\partial T} = \frac{(1-p)/(1-p^{n+}) - p/p^{n+}}{f^n(p)\left(\frac{p}{u'(u^{-1}(u_L^n(p)))}(-1/p^{n+}) + \frac{1-p}{u'(u^{-1}(u_N^n(p)))}/(1-p^{n+})\right)} \tag{45}$$

$$\leq \frac{1}{\underline{f}(1-p)p\left(\frac{1}{u'(u^{-1}(u_N^n(p)))} - \frac{1}{u'(u^{-1}(u_L^n(p)))}\right)} \tag{46}$$

$$\leq \frac{1}{\underline{f}\min_{p'\in[\underline{p},\bar{p}]}(1-p')p'} \frac{1}{\min_{C\in[W-L,W]}\left[\frac{1}{u'(C+D)} - \frac{1}{u'(C)}\right]} \tag{47}$$

$$\equiv K_2 \tag{48}$$

Taking $K = \max\{K_1, K_2\}$ completes the proof.