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The Fair Surrender Value of a Tontine

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Abstract

A tontine provides a mortality driven, age-increasing payout structure through the pooling of mortality. Because a tontine does not entail any guarantees, the payout structure of a tontine is determined by the pooling of individual characteristics of tontinists. Therefore, the surrender decision of single tontinists directly affects the remaining members' payouts. Nevertheless, the opportunity to surrender is crucial to the success of a tontine from a regulatory as well as a policyholder perspective. Therefore, this paper derives the fair surrender value of a tontine, first on the basis of expected values, and then incorporates the increasing payout volatility to determine an equitable surrender value. Results show that the surrender decision requires a discount on the fair surrender value as security for the remaining members. The discount intensifies in decreasing tontine size and increasing risk aversion. However, tontinists are less willing to surrender for decreasing tontine size and increasing risk aversion, creating a natural protection against tontine runs stemming from short-term liquidity shocks. Furthermore we argue that a surrender decision based on private information requires a discount on the fair surrender value as well.

Keywords: Life Insurance, Tontines, Annuities, Life Insurance Surrender
JEL Classification: D81, D82, D86, G22, G23, H55, H75, I13, J14, J32, N23

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1 Introduction

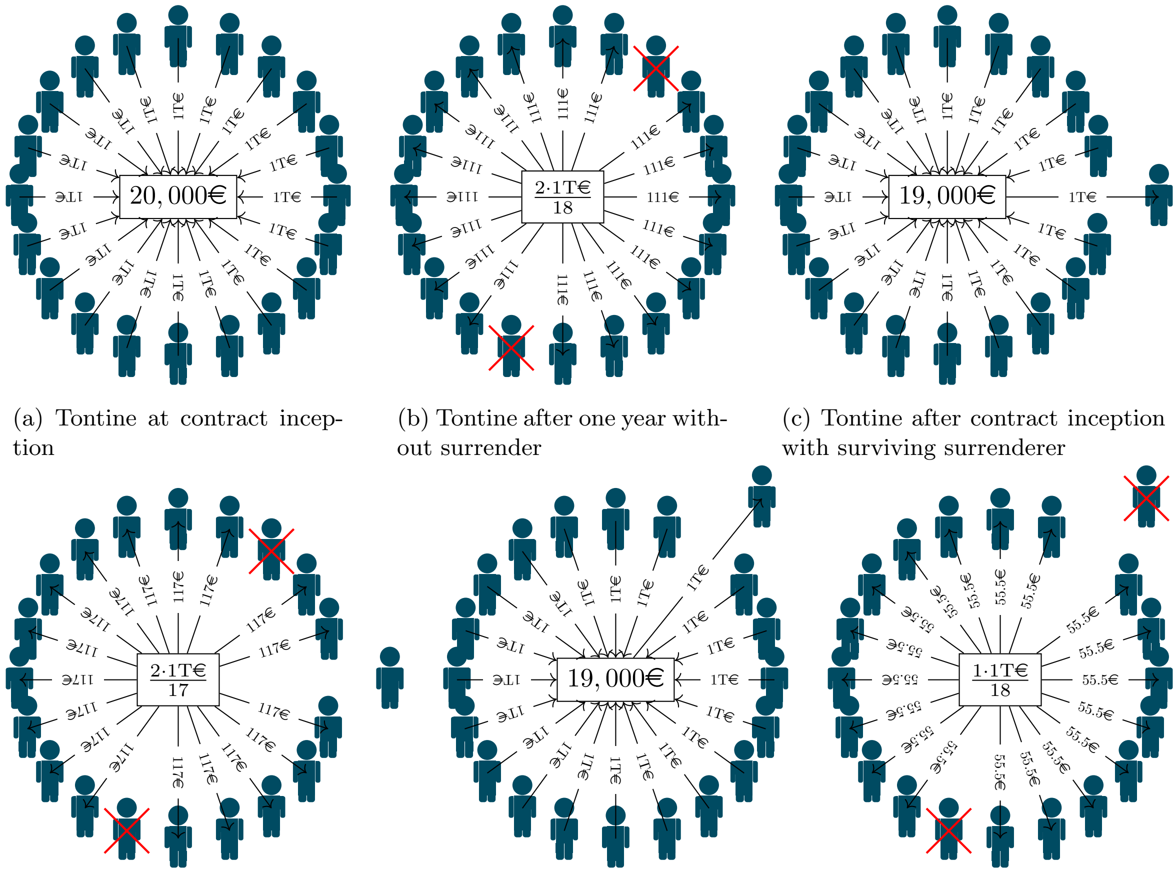
A tontine[1] provides a mortality driven, age-increasing payout structure through the pooling of mortality risk. However, the composition of a tontine has a direct influence on the payout structure of the tontinists. Hence, tontines provide a volatile payout structure generated through the pooling of the tontinists' investments and their individual characteristics. Individual decisions of single tontinists affect the payouts of all other tontinists. Although large tontines exhibit a low payout volatility and the decisions of single tontinists have hardly any impact on the tontine payouts, the tontine size diminishes over time and the decisions of single tontinists gain more and more impact on the other members. For example, the increase in standard deviation of a tontine consisting of 200 65-year-old tontinists, each investing EUR 1,000, is negligible if a tontinist decides to leave the tontine. After 30 years, the tontine size diminishes to approximately 14 surviving tontinists at the age of 95. If a tontinist then leaves the tontine, the standard deviation of tontine payouts increases by EUR 10, from EUR 226.6 to EUR 236.6[2]. In this case, the surrender decision has noticeable negative effect imposed on the remaining tontinists. Furthermore, such a tontine only is fair if tontinists with the same characteristics are pooled in a single tontine. This means that people have to be of the same age, gender and need to invest the same amount of money in the tontine. Therefore, the potential target group of tontinists for a tontine is limited and results in relatively small tontine sizes. Le Conservateur, the only European tontine provider today, runs many independent sub-tontines with a minimum size of 200 participants at tontine setup. Once invested in such a tontine, the invested money is tied in the tontine for 25 years[3]. In such small tontines, the volatility of tontine payouts is very large and the inflexibility to sell the individual tontine shares at short notice can lead to severe liquidity problems of the tontine holders and therefore lowers the attractiveness of such products. An important property of financial products is their liquidity and the ability to sell them on financial markets at short notice[4]. Unfortunately, the divestment of a tontine has wide financial consequences for the remaining tontinists if the tontine size is sufficiently small. In figure 1a we consider a stylized tontine consisting of 20 tontinists, each of them investing EUR 1,000. After one year, we assume that two of the tontinists die. As shown in figure 1b, the investment of the deceased tontinists of EUR 2,000 is distributed to the 18 surviving tontinists

[1] Throughout this paper, we use the tontine in the understanding of a tontine annuity, meaning that payouts occur on a regular basis and not just to the last survivor of the pool. See for example Sabin (2010), Milevsky and Salisbury (2015b) or Milevsky and Salisbury (2015a) wo consider a tontine annuity rather than a tontine.

[2] Results are based on own calculations.

[3] For more information, see `http://www.conservateur.fr`.

[4] See for example Guthrie (1960) and Donovan (1978).



(a) Tontine at contract inception

(b) Tontine after one year without surrender

(c) Tontine after contract inception with surviving surrenderer

(d) Tontine after one year with surviving surrenderer

(e) Tontine after contract inception with dying surrender

(f) Tontine after one year with dying surrenderer

Figure 1: Illustration of the impact of surrender on the tontine composition and payout structure

yielding to a payout of roughly EUR 111 per survivor. If we suppose that right after contract inception, one tontinist leaves the tontine and takes out his initial investment, then there are only EUR 19,000 left to distribute. We assume that the same tontinists die as in the previous example. This can lead to two situations. The first situation is shown in figure 1c. A tontinist who does not die after a year leaves the tontine. Then the remaining tontinists profit from the departure, because the investment of the deceased tontinists of EUR 2,000 is distributed to only 17 tontinists, yielding to a payout of roughly EUR 117 per survivor (figure 1d). The second situation is shown in figure 1e. One of the tontinists who die after one year leaves the tontine. This leads to the situation that only the investment of one tontinist can be distributed to 18 surviving tontinists after a year which results in a lower payout for the survivors of roughly EUR 55 (figure 1f). Therefore, a surrender decision directly affects the tontine composition and has a direct impact on the tontine payouts.

In this article we determine the fair surrender value of a tontine and analyze the impact and interdependencies of the surrender decision on the remaining tontinists. Furthermore, we provide implications on how to design tontines in a way that the impact of the surrender decision on the remaining tontinists is at a tolerable level.

While traditional life insurance products require an underwriting process at contract inception to reduce information asymmetries between the insured and the insurance provider at the expense of the provider, a tontine does not require an underwriting process at contract inception. The reason is that a tontine does not express any guarantees. If a life insurance policyholder has some private information that causes him to surrender the policy, no additional underwriting is required. This is because the private information finally results in a negative net present value for the policyholder, which then triggers a surrender and yields a positive net present value for the insurance provider. Since in that case, the interests of the insurer and the policyholder are aligned, the insurer will not prevent the policyholder from surrendering. However, the situation is the opposite in the case of a tontine. The purchase of a tontine does not require a medical underwriting process, but as one receives some private information that gives rise to a surrender decision, this would tremendously decrease the expected tontine payouts, and therefore a surrender on the basis of just the public information would yield to a positive net present value for the policyholder at the expense of the remaining tontinists. Therefore, a surrender decision in a tontine should require a medical underwriting process to determine the equitable surrender value.

In the literature there are only few contributions on the valuation of tontines. In contrast, there are various contributions on the fair surrender value of life insurance contracts. Sabin (2010) designs a fairly priced tontine with regard to age, gender and entry date with an equivalent to a common annuity scheme. His results exhibit a better payout pattern than a typical insurer-provided annuity not just on average, but for virtually every member who lived more than just a few years. Milevsky and Salisbury (2015b) derive an optimal tontine design by accounting for sensitivity of both the tontine size and the longevity risk aversion for each tontinist. By doing so, they raise the question whether an optimally designed tontine with low implications regarding capital requirements for the sponsor will gain more attention in times of risk-based capital standards and conclude that, although having higher volatility of the payouts, the tontine can provide a higher utility compared to a traditional life annuity when entailing safety loadings. Milevsky and Salisbury (2015a) extend the natural retirement income tontine by combining heterogeneous cohorts into one pool through an allocation of tontine shares at either a premium or a discount, depending on the individual characteristics and the amount to invest. Forman and Sabin (2014) construct a fair transfer plan (FTP) to guarantee a fair bet for all participating investors of a tontine by accounting for each age, death expectancy and investment level. They show that a fairly designed tontine is superior to defined benefit plans in terms of funding and sponsoring of the pension system. They illustrate that a fairly developed tontine model would improve the situation of pension providers while serving the retirement income demand of the tontine participants. Weinert and Gründl (2016) show that a tontine can be a reliable supplement to existing pension planning solutions and can extend and improve the prevailing privately funded pension solutions in a modern way when the society faces increasing costs of old-age provision. Regarding the fair surrender value Uzelac and Szimayer (2014) provide a numerical valuation method of equity-linked life insurance contracts with surrender guarantees. Bacinello (2003a) provide a fair valuation of a guaranteed life insurance participating policy with a surrender option using American-style put options. Bacinello (2003b) analyzes a life insurance endowment policy with annually adjusted benefits according to the performance of a special investment portfolio, a minimum guaranteed return and the availability of a surrender option. The author provides the conditions under which a fair premium exists. Bacinello (2005) proposes a model for pricing unit-linked life insurance policies with a surrender option using a backward recursive valuation. The author takes into account exogenous surrender values and provides necessary and sufficient conditions for well defined premiums. Nordahl (2008) uses American-style options related to life and pension insurance contracts and determines the value of a typical surrender option via Monte Carlo simulations. The author finds much lower fair surrender values than previously indicated. Siu (2005) determines the fair valuation of a participating life insurance policy with surrender options using Markov-modulated Geometric Brownian Motion market values of the assets, employing the Barone-Adesi-Whaley approximation of pricing American-style put options. However, to our knowledge, the determination of the fair surrender value of a tontine has not yet been carried out so far.

The paper is structured as follows: Section 2 first introduces the basic tontine model and specifies how payouts are generated, before we estimate the expected tontine return and derive the fair surrender value based on expected values. Thereafter, we estimate the variance of the tontine payouts and incorporate their variance into the analysis. We determine the utility of the remaining tontinists with and without surrender and estimate the utility of the surrenderer with and without surrender, assuming a short-term liquidity shock, which triggers surrender. Thereafter, we perform a sensitivity analysis regarding the main variables of the remaining tontinists as well as for the surrenderer. Subsequently, we introduce heterogeneous risk aversion into the model. Thereafter, we consider surrender decisions which are made on the basis of private information, before we discuss the effects of a secondary market for tontines. Thereafter, we simultaneously look in an exemplary illustration at the maximum fraction of the fair surrender value that can be paid to the surrenderer to hold constant the utility of the remaining tontinists and the minimum fraction of the fair surrender value that a potential surrenderer requires to receive to be willing to surrender. We will then draw implications and finally conclude.

2 The Tontine Model

The Italian Lorenzo de Tonti invented a product to consolidate the French public-sector deficit in the 1650s.[5] His ideas were based on the pooling of persons' mortality risk. The innovation was that, in exchange for a lump sum payment to the French government, one received the right to a yearly, lifelong pension. This pension increased over time because the yields were distributed among a decreasing number of surviving beneficiaries. The last survivor thus received the pensions of all others who died before. Here we consider a simple tontine model where the lump sum investments of the members are pooled and the investments of deceased tontinists are distributed to the surviving ones[6].

2.1 Basic Properties of the Model

We introduce a very simple tontine model without capital markets, where the investments of tontinists are pooled, and then are distributed to the surviving tontinists after one period. The tontine consists of initially $N_0 \geq 3$ ($N_0 \in \mathbb{Z}$) tontinists in $t = 0$. This is because the tontine requires to consist of at least two members even if one member surrenders. We assume that the tontinists have the same characteristics, meaning that they are equal in their survival prospects, which implies that the one-year survival probability from $t = 0$ to $t = 1$ is p for every person. After one year, in $t = 1$ there are N_1 people alive, yielding that the number of surviving tontinists in $t = 1$ is binomially distributed and the expected number of remaining tontinists in $t = 1$ is

$$E\left(N_1\right) = pN_0. \tag{1}$$

Every tontinist invests an amount I in the tontine at $t = 0$. We assume an intermediate point in time $t = \tau$ right after contract inception. We assume that up to this point in time, every initial

[5] See McKeever (2009) for an overview of the history of tontines.

[6] A contemporary treatment of a similar tontine can be found in the 1996 episode of "*The Simpsons - Raging Abe Simpson and His Grumbling Grandson in "The Curse of the Flying Hellfish""*, where a unit looted several paintings during World War II and agreed on a tontine, placing the paintings in a crate, and the final surviving member would inherit the paintings. As Mr. Burns wants the paintings as soon as possible, he orders Abe Simpson's assassination.

tontinist survives with certainty. At the intermediate point in time, one tontinist can decide to surrender and therefore to leave the tontine, yielding that there are $N_0 - 1$ tontinists left in the tontine. Therefore, there are $N_1^s \leq N_1$ tontinists left in the tontine with the expected number of tontinists after surrender being

$$E\left(N_1^s\right) = p\left(N_0 - 1\right). \tag{2}$$

The basic assumptions and the timeline of the tontine are shown in figure 2.

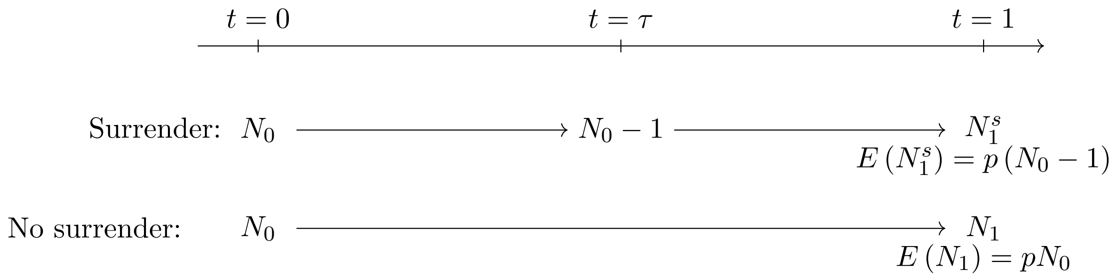


Figure 2: Basic assumptions and timeline of the tontine

We now estimate the fair surrender value that the surrenderer receives without worsening the lot of the remaining tontinists. If a tontinist surrenders, this has two counteracting effects. On the one hand, the number of people decreases to which the investments of deceased tontinists will be distributed. On the other, the volatility of the payouts increases, as we show in section A.3. To account for this, we take away that amount of money from the surrenderer and distribute it to the remaining tontinists such that they have the same utility as they have without surrender affecting the tontine.

2.2 The Expected Tontine Payout and the Fair Surrender Value

If no one surrenders, the total payout to all surviving tontinists in $t = 1$ is the sum of the initial investments of the deceased tontinists in $t = 1$. The number of deceased tontinists is the difference of people alive at tontine inception in $t = 0$, N_0 and the people alive after one year in $t = 1$. Furthermore, we assume a liquidation of the tontine in $t = 1$. The conditional payout per surviving tontinist in $t = 1$ then is the total investment of the initial tontinists which is distributed among all surviving tontinists N_1, yielding

$$PO_1|\text{alive} = \frac{\left(N_0 - N_1\right) I}{N_1} + I|\text{alive} = \frac{N_0}{N_1} I|\text{alive} \tag{3}$$

and in the case of death

$$PO_1|\text{dead} = 0. \tag{4}$$

Using equation (3), the expected conditional payout without surrender if one survives[7] is

$$\begin{aligned} E\left(PO_1|\text{alive}\right) &= E\left(\frac{N_0}{N_1}I|\text{alive}\right) \\ &= \frac{1-(1-p)^{N_0}}{p}I \end{aligned} \tag{5}$$

which even for small tontine sizes converges towards

$$E\left(PO_1|\text{alive}\right) = \frac{I}{p}. \tag{6}$$

This result is in line with the standard results for the payouts of a deferred annuity[8]. If one tontinist surrenders, we approach two steps to determine the fair surrender value. We assume in a first step that the investment I of the surrenderer remains in the tontine and will be distributed in addition to the investment of the deceased tontinists to the remaining tontinists, who therefore receive a hypothetical payout with surrender. In a second step, we then determine the fair surrender value as the aggregated difference of the payout without surrender and the hypothetical payout with surrender. Since no one dies until the interim period τ, there are $N_0 - 1$ people left after the surrenderer leaves the tontine, forming the new basis of the tontine and leading to N_1^s tontinists being left in $t = 1$ in the tontine. The available investment volume of the surrenderer and the deceased tontinists now is distributed to the remaining tontinists N_1^s. Therefore the total payout in $t = 1$ available for distribution with surrender is $\left(N_0 - N_1^s\right) I$. We assume for the hypothetical payout that the surrenderer receives nothing if he leaves the tontine. Again, we assume a liquidation of the tontine in $t = 1$. The conditional hypothetical payout with surrender in the case of survival for a tontinist then is

$$PO_1^{s*}|\text{alive} = \frac{\left(N_0 - N_1^s\right) I}{N_1^s} + I|\text{alive} = \frac{N_0}{N_1^s} I|\text{alive} \tag{7}$$

and in the case of death

$$PO_1^{s*}|\text{dead} = 0. \tag{8}$$

Accordingly, using equation (7), the expected conditional hypothetical payout with surrender if one survives[9] is

[7] See appendix A.6.1 for the proof of equation (5).
[8] See for example Cannon and Tonks (2008, p. 144).
[9] See appendix A.6.2 for the proof of equation (9).

$$
\begin{aligned}
E\left(PO_1^{s*}|\text{alive}\right) &= E\left(\frac{N_0}{N_1^s} I|\text{alive}\right) \\
&= N_0 \frac{1-(1-p)^{(N_0-1)}}{p\left(N_0-1\right)} I
\end{aligned} \tag{9}
$$

Since compared to equation (5) the same amount of money can be distributed among fewer people because of the departure of the surrenderer, the expected conditional hypothetical payout with surrender is larger than the the expected conditional payout without surrender[10], so $E\left(PO_1\right) < E\left(PO_1^{s*}\right)$.

In the second step we search that amount of money FSV which can be distributed to the surrenderer which yields the same payout for the remaining tontinists as without surrender:

$$
E\left(PO_1^s|\text{alive}\right) = E\left(\frac{\left(N_0 - N_1^s\right) I - FSV}{N_1^s} + I\right) \stackrel{!}{=} E\left(PO_1|\text{alive}\right) \tag{10}
$$

To do so, the amount of money that needs to be distributed to the surrenderer is the difference of the first step hypothetical payout with surrender PO_1^{s*} and the payout without surrender PO_1 for all the remaining tontinists in the tontine after surrender N_1^s, thus[11]

$$
\begin{aligned}
FSV =& E\left(\left(PO_1^{s*} - PO_1\right) N_1^s\right) \\
=& \left(1-(1-p)^{N_0}\right) I.
\end{aligned} \tag{11}
$$

Even for very small tontine sizes N_0 the fair surrender value converges towards the initial investment volume[12] of the surrenderer $FSV = I$.

As shown in appendix A.6.5, the expected amount of money which will be distributed to the remaining tontinists in $t = 1$ after the surrenderer has received the fair surrender value determined in equation (11) converges in the size of the tontine towards

$$
E\left(PO_1^s|\text{alive}\right) = E\left(PO_1|\text{alive}\right) = \frac{I}{p} \tag{12}
$$

even for a small tontine size N_0. This means that on the basis of expected values, the initial tontinists are not worse off after the surrender. However, as surrender unexpectedly decreases the size of the tontine, the volatility of future tontine payouts increases, and therefore changes

[10] As presented in appendix A.6.3 the expected conditional hypothetical payout with surrender is larger than the expected conditional payout without surrender.

[11] See appendix A.6.4 for the proof of equation (11).

[12] To support our results we simulate the FSV via Monte Carlo Simulation. The results are shown in appendix A.1. The simulation supports our theoretical result.

the initial, at tontine entrance contractually agreed conditions for the remaining tontinists. This should be reflected in the surrender value as well.

2.3 The Volatility of the Tontine Payouts

Although the expected tontine return does not change for the remaining tontinists through surrender, the increasing payout volatility can be a problem especially for risk averse tontinists, because there is a higher probability that tontine payouts are lower than expected, causing that the insurance character of the tontine to possibly be violated if the tontine payouts are not sufficient to cover expenses. This effect further intensifies for small tontine sizes, and becomes even more severe as time goes by and the tontine size naturally decreases due to mortality effects. Whereas the surrender of a tontine only has a minor impact on the volatility of the tontine payouts at the beginning of a tontine, as the tontinists naturally decrease over time, the surrender of a single participant can have a significant impact on the payout volatility. Therefore, we quantify the increase of the payout volatility through surrender. However, since the closed form solution for the variance of the tontine payout is relatively complex to employ, we furthermore approximate the variance using a Taylor approximation. This works very well for large tontine sizes and can be employed more easily, but is not that precise for smaller tontine sizes. In appendix A.3, we compare the variance approximation with the theoretical variance.
Using equation (3), we obtain for the conditional variance[13] of the tontine payout without surrender in the case of survival

$$\begin{aligned} Var\left(PO_1|\text{alive}\right) &= Var\left(\frac{N_0}{N_1}I|\text{alive}\right) \\ &= I^2N_0^2\left(\left(1-p\right)^{(N_0-1)}{}_iF_j\left(\overline{x};\,\overline{y};\,z\right)-\left(\frac{1-\left(1-p\right)^{N_0}}{N_0p}\right)^2\right) \end{aligned} \tag{13}$$

where ${}_iF_j\left(\overline{x};\,\overline{y};\,z\right)$ is the the generalized hypergeometric function with $\overline{x} = \{x_1, \ldots, x_i\} = \{1, 1, -\left(N_0-1\right)\}$, $\overline{y} = \{y_1, \ldots, y_j\} = \{2, 2\}$ and $z = \frac{p}{p-1}$. Accordingly, using equation (10) the

[13] See appendix A.6.6 for the proof of equation (13).

conditional variance of the tontine payout with surrender[14] is

$$Var\left(PO_1^s|\text{alive}\right) = Var\left(\frac{N_0 - 1 + (1-p)^{N_0}}{N_1^s} I|\text{alive}\right)$$
$$= I^2\left(N_0 - 1 + (1-p)^{N_0}\right)^2\left((1-p)^{N_0-2}\,{}_iF_j\left(\overline{x^s};\,\overline{y^s};\,z^s\right) - \left(\frac{1-(1-p)^{(N_0-1)}}{(N_0-1)\,p}\right)^2\right) \tag{14}$$

where ${}_iF_j\left(\overline{x^s};\,\overline{y^s};\,z^s\right)$ is the the generalized hypergeometric function with $\overline{x^s} = \{x_1,\ldots,x_i\} = \{1,1,-(N_0-2)\}$, $\overline{y^s} = \{y_1,\ldots,y_j\} = \{2,2\}$ and $z^s = \frac{p}{p-1}$. In figure 3, we graphically analyze the impact of surrender on the tontine volatility. We consider $Std\left(PO_1^s|\text{alive}\right) - Std\left(PO_1|\text{alive}\right)$ meaning that a positive Δ of the standard deviation is an increase in tontine payout volatility and a negative Δ is a decrease in tontine payout volatility. Since a tontine needs to consist of at least 2 tontinists even after surrender, we start with the smallest possible initial tontine size of $N_0 = 3$. If the survival probability p of the tontinists is high, surrender yields to an increase in tontine payout volatility irrespective of the tontine size. However, as p decreases, we can observe a decrease in tontine payout volatility for very small tontines (for example, if $p = 0.2$, up to a tontine size of $N_0 = 16$ tontinists, a surrender decreases the tontine payout volatility). This is because for low survival probabilities and low tontine sizes, there is a very high payout uncertainty. But if the available money is distributed to one person less after surrender with certainty, some uncertainty reduces. However, according to the Federal Statistical Office the survival probability of a 100-year-old male in Germany in 2010 was $p = 0.636$, meaning that only tontines consisting of 3 or 4 tontinists at the age of 100 would yield a decrease in payout volatility in the surrender case. Despite this, even without surrender, the payout volatility is very high for such a small tontine size of 4 members with a standard deviation of 66.27% of the initial investment. Such a tontine will never exist since it will be liquidated already for a much larger tontine size. Starting from the premise that small tontines will be liquidated at a tontine size considerably larger than 4 tontinists and that tontinists usually are younger than 100 years old, we can infer that surrender normally increases tontine payout volatility. To summarize, surrender usually increases the tontine payout volatility for the remaining tontinists and therefore reduces the utility of risk averse tontinists. Hence, we account for the payout volatility in addition to the expected tontine payout to determine an equitable surrender value. While an annuity expresses investment guarantees, backed up by equity capital, surrender does not directly influence the payout structure. Instead, surrender which is not ex-ante anticipated diminishes the risk pooling

[14] See appendix A.6.7 for the proof of equation (14).

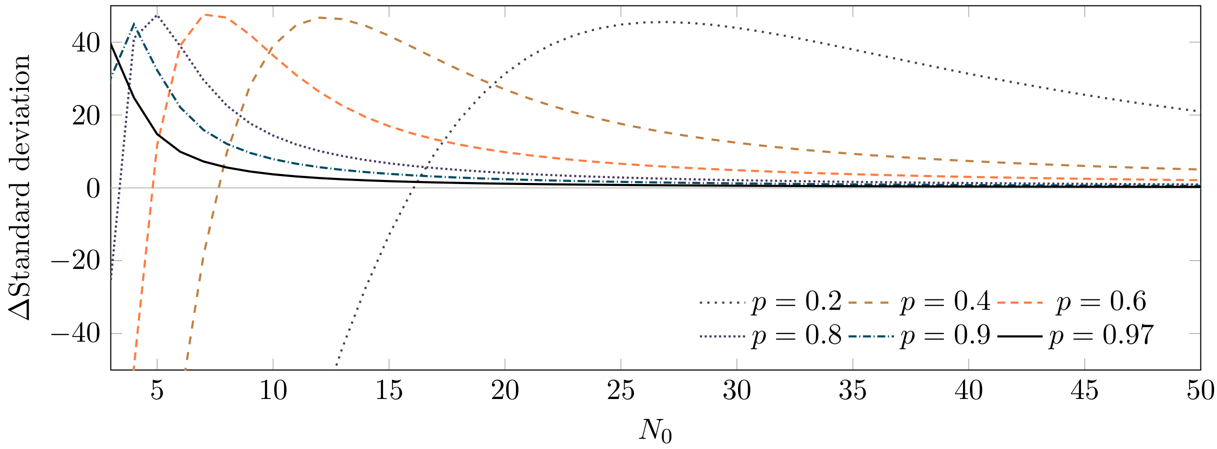


Figure 3: $Std\left(PO_1^s|\text{alive}\right) - Std\left(PO_1|\text{alive}\right)$ for varying tontine size N_0 and $I = 1,000$

effects of the providing insurer (and may cause liquidity problems, leading to default in extreme situations), in turn resulting in an increase in default probabilities of the insurer. Therefore, surrender in classical life insurance indirectly affects the policyholders, and not directly as in the tontine, where the payout structure itself is affected, making it necessary to account for the payout volatility.

2.4 The Utility of the Tontine

Surrender, although keeping constant the expected payout of the remaining tontinists, increases the tontine payout volatility from which risk averse tontinists suffer a reduction in utility. Therefore we estimate the utility of the remaining tontinists before and after surrender and balance the reduction in utility through an increase in the expected tontine payout. This is financed by a fraction of the fair surrender value and is taken away from the surrenderer and is distributed to the remaining tontinists in order to keep their utility constant. This means that the remaining tontinists face a risk-return trade-off and are compensated for bearing more risk with a higher expected return, which is in line with classical capital market theories[15].

2.4.1 The Perspective of the Remaining Tontinists

First, we analyze the utility of the remaining tontinists. We assume risk averse tontinists. Furthermore, we assume an expected-value-variance-utility framework. In our setting, this means that the utility increases in expected tontine payout and decreases in tontine payout volatility. Therefore, one can compensate bearing a higher risk with a higher return. We assume that a surrenderer receives the fair surrender value if the decision is based on the expected values,

[15] For example the Capital Asset Pricing Model assumes similar utility functions where only the first and second moment matter.

implying a risk aversion of zero. However, if the tontine payout volatility is incorporated into the analysis, a fraction $1-\alpha$ is deducted from the fair surrender value such that the surrenderer receives α of the fair surrender value. The utility for a tontinist without surrender is given by the preference function

$$U = E\left(PO_1|\text{alive}\right) - bVar\left(PO_1|\text{alive}\right) \tag{15}$$

with $E\left(PO_1|\text{alive}\right)$ coming from equation (5) and $Var\left(PO_1|\text{alive}\right)$ coming from equation (13) and where $b \geq 0$ is the risk aversion parameter. The utility for a tontinist with surrender if the surrenderer receives the fair surrender value is

$$U^s = E\left(PO_1^s|\text{alive}\right) - bVar\left(PO_1^s|\text{alive}\right) \tag{16}$$

with $E\left(PO_1^s|\text{alive}\right)$ coming from equation (54) and $Var\left(PO_1^s|\text{alive}\right)$ coming from equation (14). However, since surrender increases the payout volatility, the utility without surrender is higher than the utility with surrender if the surrenderer receives the fair surrender value, thus

$$U > U^s \tag{17}$$

To account for that we distribute only a fraction of α to the surrenderer to keep the utility of the remaining tontinists constant. We compensate bearing higher risk with a higher return, financed by a fraction of the fair surrender value remaining in the tontine. Therefore we search for the $\alpha^* \in (0,\, 1)$ which satisfies

$$U = U^s\left(\alpha^*\right) \tag{18}$$

where

$$U^s\left(\alpha\right) = E\left(PO_1^s\left(\alpha\right)|\text{alive}\right) - bVar\left(PO_1^s\left(\alpha\right)|\text{alive}\right) \tag{19}$$

with

$$E\left(PO_1^s\left(\alpha\right)|\text{alive}\right) = \frac{\left(N_0 - \alpha + \alpha\left(1-p\right)^{N_0}\right)\left(1-\left(1-p\right)^{\left(N_0-1\right)}\right)}{\left(N_0-1\right)p} I \tag{20}$$

and

$$Var\left(PO_1^s\left(\alpha\right)|\text{alive}\right)$$
$$= I^2\left(N_0-\alpha+\alpha\left(1-p\right)^{N_0}\right)^2\left(\left(1-p\right)^{(N_0-2)}{}_3F_2\left(1,1,-\left(N_0-2\right);2,2;\frac{p}{p-1}\right)-\left(\frac{1-\left(1-p\right)^{(N_0-1)}}{\left(N_0-1\right)p}\right)^2\right). \tag{21}$$

We solve for α^* implicitly[16].

2.4.2 The Perspective of the Surrenderer

Would a tontinist be willing to surrender if he only received α^* of the fair surrender value rather than the fair surrender value? To answer this question we assume a setting where a tontinist i has no further wealth than the initial investment I. In the interim period τ, i faces an ex-ante unexpected, immediate liquidity shock ε_τ which can be financed either by surrendering the tontine or borrowing an amount of money L at cost $c\left(L\right)$. If i surrenders, he receives an amount of $\alpha^* I$ in $t=\tau$ but forfeits his further tontine claims, thus receives nothing in $t=1$. If he does not surrender he receives an amount of $L-c\left(L\right)$ in $t=\tau$ where $c\left(L\right)>0$, $\frac{\partial c(L)}{\partial L}>0$, $\frac{\partial^2 c(L)}{\partial L^2}>0$ are the convex cost of borrowing. Furthermore, i has to repay the loan in $t=1$ at an amount of money of $\frac{L}{p}$. The lender sets the fair price because he receives the repayment only if the tontinist survives with probability p, yielding a net present value of zero. The timeline of the setting is presented in Figure 4. The expected payout if i surrenders is

$$\alpha^* I - \varepsilon_\tau. \tag{22}$$

If i surrenders, he receives a fraction a^* of the fair surrender value, but faces a liquidity shock of size ϵ_τ. The payout if i does not surrender is

$$L-c\left(L\right)-\varepsilon_\tau+p\left(PO_1-\frac{L}{p}\right). \tag{23}$$

If i does not surrender, he borrows an amount of L in the short term to finance the liquidity shock ϵ_τ which involves convex costs of borrowing $c\left(L\right)$. With probability p, i survives until $t=1$ and then receives a tontine payout PO_1. In addition, he has to return the fair amount repayable $\frac{L}{p}$. i is willing to surrender, if the utility with surrender is higher than the utility

[16] Although it is possible to solve equation (18) explicitly, the solution is far too complex to be interpretable in a reasonable way.

$$
\begin{array}{llll}
 & t=0 & t=\tau & t=1 \\
\text{Surrender:} & -I \longrightarrow & \alpha^* I - \varepsilon_\tau \longrightarrow & 0 \\
\text{No surrender:} & -I \longrightarrow & L - c(L) - \varepsilon_\tau \longrightarrow & PO_1 - \frac{L}{p}
\end{array}
$$

Figure 4: Timeline if there is a ex-ante unexpected liquidity shock ε at τ

without surrender, therefore

$$U_\tau^s + pU_1^s \geq U_\tau^L + pU_1^L. \tag{24}$$

Using equation (22) and equation (23) and applying the mean-variance utility principle to equation (24) yields for the minimum fraction the surrender needs to receive to be willing to surrender[17] $\overline{\alpha}$

$$\alpha^* \geq 1 - \frac{pb}{I} Var\left(PO_1\right) - \frac{c\left(L\right)}{I} = \overline{\alpha}. \tag{25}$$

Given that i faces an ex-ante unexpected liquidity shock at $t = \tau$, he will surrender if the utility of the lower-than-fair amount of money he receives is higher than the utility of staying in the tontine and facing an uncertain tontine payout and having costs of procuring a short-term loan to finance the liquidity shock. The tontinist will surrender when the maximum fraction that can be paid to the surrenderer to keep constant the utility of the remaining tontinists α^* is larger than the minimum fraction which has to be paid to the surrenderer to preserve the incentive to surrender $\overline{\alpha}$.

2.4.3 Sensitivity Analysis

We now examine the influence of the main variables on the surrender fraction. First we analyze the maximum fraction to keep the utility of the remaining tontinists constant before we come to the minimum fraction that needs to be paid to the tontinist to preserve the incentive to surrender.

[17] See appendix A.6.8 for detailed steps.

Sensitivity Analysis of the Maximum Fraction that can be paid to the Surrenderer

First, we approximate the tontine payout volatility using a Taylor approximation to receive a manageable explicit expression for the maximum fraction that can be paid to the surrenderer which keeps the utility for the remaining tontinists constant α^*. As shown in appendix A.2, the conditional Variance of the tontine payout without surrender can be approximated via

$$Var\left(PO_1|\text{alive}\right) \approx \frac{I^2\left(1-p\right)}{N_0 p^3}. \tag{26}$$

Similarly, the conditional variance of the tontine payout with surrender can be approximated via[18]

$$Var\left(PO_1^s|\text{alive}\right) \approx \frac{I^2\left(1-p\right)}{\left(N_0-1\right)p^3}. \tag{27}$$

As shown in appendix A.3, the Taylor approximation provides results which are very close to the exact solution of the payout volatility. Furthermore, we use the conditional expected tontine payout without and with surrender from equation (12). The utility considering that only a fraction α^* of the fair surrender value is paid to the surrenderer in equation (19) can be approximated via

$$\begin{aligned} U^s\left(\alpha\right) &= E\left(PO_1^s\left(\alpha\right)|\text{alive}\right) - bVar\left(PO_1^s\left(\alpha\right)|\text{alive}\right) \\ &\approx I\left(\frac{\left(N_0+1-\alpha\right)}{N_0 p} - 1\right) - b\frac{I^2\left(N_0-\alpha\right)^2\left(1-p\right)}{\left(N_0-1\right)^3 p^3}. \end{aligned} \tag{28}$$

Solving equation (18) using equation (26) and equation (28) finally yields for the maximum fraction that can be paid to the surrenderer without worsening the situation of the remaining tontinists α^*

$$\alpha^* = \frac{\left(\sigma_1 - p^2\right)\left(N_0-1\right)^3}{2IN_0 b\left(1-p\right)} + N_0 \tag{29}$$

where

$$\sigma_1 = \sqrt{\frac{4IN_0 b\left(Ib\left(p-1\right)^2 + \left(N_0-1\right)p^2\left(p-1\right)\right)}{\left(N_0-1\right)^3} + p^4} \tag{30}$$

Using the approximation, we perform a sensitivity analysis of the main variables on the maximum fraction that can be paid to the surrenderer α^* which keeps the utility of the remaining tontinists constant[19]. First we analyze the influence of the tontine size N_0 on the maximum fraction. The larger the tontine size, the closer to the fair surrender value is the maximum surrender fraction.

[18] As shown in appendix A.6.10 in equation (61), volatility increases with surrender as well if we employ a Taylor approximation.

[19] See Appendix A.4 for the detailed steps of the calculation.

This is because the larger the tontine is, the lower is the impact of surrender on the remaining tontinists, resulting in hardly any discount of the fair surrender value being needed to compensate the remaining tontinists. This situation is presented in appendix A.5 in figure 10a.

$$\frac{\partial \alpha^*}{\partial N_0} > 0 \tag{31}$$

Secondly, we consider the influence of the survival probability on the maximum surrender fraction. A higher survival probability leads to a higher survival fraction. This is because a higher survival probability reduces the uncertainty about the future expected number of tontinists and therefore, the payout volatility of the tontine decreases. Furthermore, a higher survival probability leads to a larger expected future tontine size and therefore has a similar impact on the maximum fraction of the fair surrender value that can be paid. This situation is presented in appendix A.5 in figure 10b.

$$\frac{\partial \alpha^*}{\partial p} > 0 \tag{32}$$

Thirdly, we consider the influence of the investment on the maximum surrender fraction. The larger the investment volume, the lower is the maximum surrender fraction. This is because a higher investment volume leads to a higher absolute cash-flow volatility, which has a negative impact on the utility yielding to a lower maximum α^*. This situation is presented in appendix A.5 in figure 10c.

$$\frac{\partial \alpha^*}{\partial I} < 0 \tag{33}$$

Finally, we consider the influence of the risk aversion on the maximum surrender fraction. The more risk averse the remaining tontinists are, the lower is the maximum surrender fraction. This is because a higher risk aversion leverages the impact of the volatility on the utility. This situation is presented in appendix A.5 in figure 10d.

$$\frac{\partial \alpha^*}{\partial b} < 0 \tag{34}$$

Sensitivity Analysis of the Minimum Fraction that must be paid to the Surrenderer

We perform a sensitivity analysis of the main variables on the minimum fraction $\overline{\alpha}$ which the surrendering policyholder at least requires to be willing to surrender. First we analyze the influence of the tontine size N_0 on the minimum required surrender fraction. The larger the tontine size, the higher is the minimum required surrender fraction, because of the less volatile tontine payouts.

$$\frac{\partial\overline{\alpha}}{\partial N_0} = \frac{bI\,(1-p)}{p^2 N_0^2} > 0 \tag{35}$$

Secondly, we determine the influence of the survival probability on the minimum required surrender fraction. The higher the survival probability is, the higher is the surrender fraction. This is because a higher survival probability reduces the volatility of the tontine payouts.

$$\frac{\partial\overline{\alpha}}{\partial p} = \frac{bI\,(2-p)}{p^3 N_0} > 0 \tag{36}$$

Thirdly, we determine the influence of the investment on the minimum required surrender fraction. Here, there is no clear result. The surrender fraction can either decrease or increase in I

$$\frac{\partial\overline{\alpha}}{\partial I} = \frac{c\,(L)}{I^2} - \frac{b\,(1-p)}{p^2 N_0} \lesseqgtr 0 \tag{37}$$

depending on the parameters. A large investment I supports $\frac{\partial\overline{\alpha}}{\partial I} < 0$ because of the larger influence of the then resulting tontine volatility, whereas a large tontine size N_0 fosters $\frac{\partial\overline{\alpha}}{\partial I} > 0$ because of the opposite effect on the volatility. High costs of a loan $c\,(L)$ encourage $\frac{\partial\overline{\alpha}}{\partial I} > 0$ because then a higher tontine investment needs to be compensated with an also higher secure amount if one surrenders, because of the costs incurred. On the other hand, a high risk aversion b facilitates $\frac{\partial\overline{\alpha}}{\partial I} < 0$ because one is more willing to replace the volatile tontine payout with the secure surrender value and is willing to accept this swap for a lower rate. One is willing to pay more for the security if one is more risk averse.

Fourthly, we determine the influence of the risk aversion on the minimum required surrender fraction. The more risk averse the surrenderer is, the lower is the minimum required surrender fraction. This is because a higher risk aversion leverages the impact of the volatility on the utility.

$$\frac{\partial\overline{\alpha}}{\partial b} = -\frac{I\,(1-p)}{p^2 N_0} < 0 \tag{38}$$

Finally, we determine the influence of the costs of borrowing money $c\,(L)$ on the minimum required surrender fraction. The higher the costs are when staying in the tontine, the lower is the minimum required fraction to surrender. One accepts a lower compensation in the surrender case because the opportunity costs of the tontine are higher.

$$\frac{\partial\overline{\alpha}}{\partial c\,(L)} = -\frac{1}{I} < 0 \tag{39}$$

2.5 Extension: Heterogeneous Risk Aversion

We now consider the case of heterogeneous risk aversion of the tontinists for a given tontine size N_0. We assume that some tontinists have a high risk aversion $b = h$ and the remaining tontinists have a low risk aversion $b = l$, where $h > l$. Because $\frac{\partial \alpha^*}{\partial b} < 0$ and $\frac{\partial \overline{\alpha}}{\partial b} < 0$, it holds that $\alpha_h^* < \alpha_l^*$ and $\overline{\alpha}_h < \overline{\alpha}_l$: the maximum fraction α^* that can be paid to the surrenderer to hold constant the utility of the remaining tontinists is higher for those tontinists with a lower risk aversion and the minimum fraction $\overline{\alpha}$, the surrenderer requires to be willing to surrender is also higher for those tontinists with a lower risk aversion. We now consider the possible cases.
The first situation is presented in appendix A.5 in figure 11a. Here, $\alpha_l^* > \overline{\alpha}_l$ and $\alpha_h^* > \overline{\alpha}_h$, which means that surrendering would be worthwhile for the low risk averse tontinists as well as for the high risk averse tontinists in general. Furthermore, $\alpha_h^* > \overline{\alpha}_l$, meaning that the maximum fraction that can be paid to a high risk averse surrenderer to hold constant the utility of the remaining high risk averse tontinists is larger than the minimum fraction a low risk averse surrenderer requires to be willing to surrender. If we suppose now that a low risk averse tontinist demands to surrender at a rate of α_l^*, then this would hold constant the utility of the remaining low risk averse tontinists, but would put the remaining high risk averse tontinists in a less favorable position. Therefore, any surrender is required to be at most at a level of α_h^*, which does not worsen the position of any tontinist. Since $\alpha_h^* > \overline{\alpha}_l$, even a low risk averse surrenderer would accept a rate of α_h^*. Therefore, in such a situation, surrender takes place for both cases of risk aversion at a rate of α_h^*.
The second situation is presented in appendix A.5 in figure 11b. Here, $\alpha_l^* > \overline{\alpha}_l$ and and $\alpha_h^* > \overline{\alpha}_h$, which means that surrendering would be worthwhile for the low risk averse tontinists as well as for the high risk averse tontinists in general. But now, $\alpha_h^* < \overline{\alpha}_l$ meaning that the maximum fraction that can be paid to a high risk averse surrenderer to hold constant the utility of the remaining high risk averse tontinists is smaller than the minimum fraction a low risk averse surrenderer requires to be willing to surrender. If we suppose now that a low risk averse tontinist wants to surrender at a rate of α_l^*, then this would hold constant the utility of the remaining low risk averse tontinists, but would put the remaining high risk averse tontinists in a less favorable position. Therefore, any surrender is required to be at most at a level of α_h^*, which does not worsen the position of any tontinist. However, since $\alpha_h^* < \overline{\alpha}_l$ a low risk averse surrenderer would not accept a rate of α_h^*. Therefore, in such a situation, surrender only takes place for high risk averse tontinists at a rate of α_h^*, whereas low risk averse tontinists are not willing to surrender at the offered rate.

The third situation is presented in appendix A.5 in figure 11c. Here, it holds again that $\alpha_l^* > \overline{\alpha}_l$, which means that surrendering would be worthwhile for the low risk averse tontinists in general. But now it holds that $\alpha_h^* < \overline{\alpha}_h$, which means that surrendering would not be worthwhile for the high risk averse tontinists at all. If we suppose now that a low risk averse tontinist wants to surrender at a rate of α_l^*, then this would hold constant the utility of the remaining low risk averse tontinist, but would put the remaining high risk averse tontinists in a less favorable position. To hold constant the utility of the high risk averse tontinists would require a rate of α_h^*. But because $\alpha_h^* < \overline{\alpha}_l$ the low risk averse tontinists are not willing to surrender at this rate. For the high risk averse tontinists surrender is not worthwhile because $\alpha_h^* < \overline{\alpha}_h$. Therefore, in such a situation, surrender does not take place at all.
The fourth situation is presented in appendix A.5 in figure 11d. Here, $\alpha_l^* < \overline{\alpha}_l$ and $\alpha_h^* < \overline{\alpha}_h$, which means that surrendering is not worthwhile for the low risk averse tontinists as well as for the high risk averse tontinists at all. Therefore, in such a situation, surrender does not take place.

2.6 Extension: Private Information

We now assume that a tontinist does not decide to surrender because of an immediate liquidity shock, but on the basis of private information. Furthermore, we consider the fair surrender value based on expected values of the payouts for simplicity. We suppose an investment of I in $t = 0$ in the tontine. From an ex-ante perspective, a tontinist has a net present value of

$$NPV = -I + p\frac{I}{p} = 0. \tag{40}$$

He invests I and in case of survival he expects to receive an amount of $\frac{I}{p}$, which he does with p, yielding to a fair bet. However, if the tontinist learns right after contract inception in the interim period τ that he will survive up to $t = 1$ only with a lower survival probability $\tilde{p} < p$, the tontinist would only receive a net present value of

$$\widetilde{NPV} = -I + \tilde{p}\frac{I}{p} < 0 \tag{41}$$

when staying in the tontine. If the tontinist now surrendered and received the fair surrender value calculated in equation (11), he would still yield a net present value of

$$\widetilde{NPV_s} = -I + FSV = -I + I = 0 = NPV. \tag{42}$$

Since $\widetilde{NPV_s} = NPV > \widetilde{NPV}$, if the tontinist has private information and still received the fair surrender value he would be better off if he surrendered. However, this would be at the expense of the remaining tontinists, because money would be taken out of the tontine that actually belongs to the pool. To prevent this, just the fraction κ of the fair surrender value which satisfies

$$\widetilde{NPV}_s^* = \widetilde{NPV} \tag{43}$$

should be distributed to the surrenderer. From this, it follows that $\kappa = \frac{\tilde{p}}{p} < 1$, yielding that in the case of private information, only κI should be distributed to the surrenderer. As a consequence. if one wants to surrender the tontine, a medical underwriting is necessary to determine whether the survival prospects have changed, and therefore the surrender value has to be adjusted. If one thinks of the extreme case that the tontinist learns in τ that he will not survive until $t = 1$. Then, through surrendering, he could recoup his whole investment, although he would not receive anything in $t = 1$ if he stayed in the tontine.

2.7 Extension: Secondary Market for Tontines

One possible way to mitigate the problems of surrender might be a secondary market for tontines. If a tontinist wants to surrender, he can sell his stake in the tontine, such that the tontine size does not decrease. However, this adds another layer of adverse selection to the tontine. The buyer needs to have similar characteristics as the surrenderer. Otherwise, the composition of the tontine changes. A buyer with worse characteristics than the surrenderer is unlikely to purchase the tontine at the fair surrender value because of the incentive problems discussed in the previous section, and a discount would be required. However, a discount equals a capital outflow out of the tontine at the expense of the remaining tontinists, because less capital is available for distribution. A buyer with better characteristics than the surrenderer would be willing to pay the fair surrender value, but this would also be at the expense of the remaining tontinists, because of the investment of the new buyer would be blocked for distribution for a longer time because of the high expected remaining lifetime. Also, the available money for distribution then is split up to a higher number of people, which further reduces the payout of the initial tontinists. One can expect to see a self-selection of buyers in the tontine. Only buyers with a potentially better health status than the average tontinist will join the tontine, which is not in the interest of the remaining tontinists. If the buyer has exactly the same characteristics as the surrenderer, a secondary market for tontines works frictionlessly. However, the assessment of the characteristics of the surrenderer as well as of the potential buyer is very costly and therefore might impede such a solution. Furthermore, if a tontinist wants to sell his share in the tontine, finding the right buyer might be a long process, due to limited demand for the specific policy of the surrenderer. This can cause severe opportunity and search costs. However, there is a trade off between the costs imposed by surrender and the costs imposed by a secondary market.

3 Exemplary Illustration

We now use the theoretically derived properties of the tontine to show the maximum required surrender fraction which keeps the utility of the remaining tontinists at a constant level α^* and the minimum required surrender fraction which the surrenderer would accept to surrender $\overline{\alpha}$ for different parameter values and their interaction. We assume that the survival probability of all tontinists is $p = 0.98$ and set the investment volume as well as the loan to $I = L = 1,000$. Table 1 summarizes the calibration.

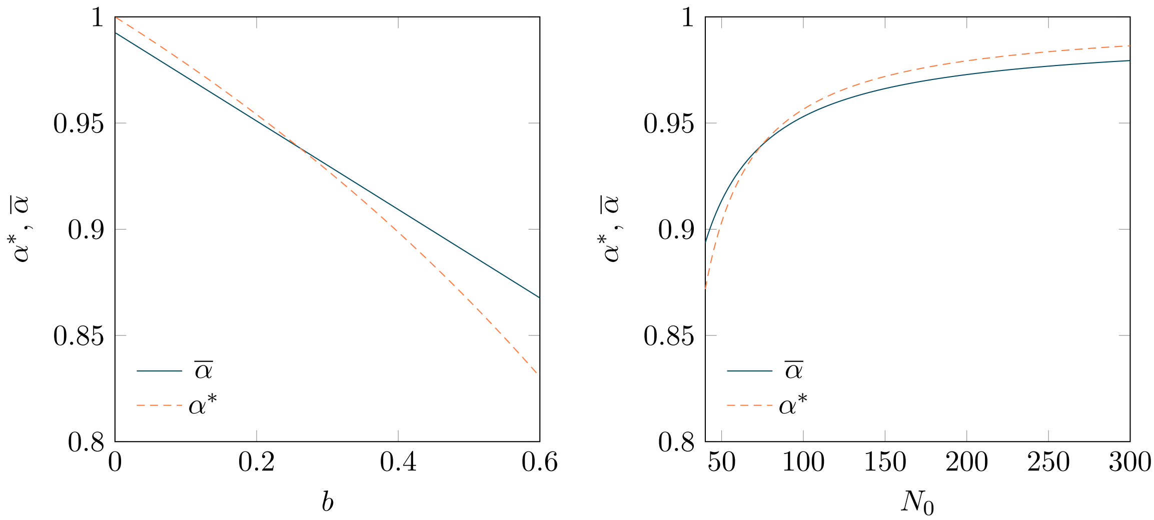
Parameter	**Notation**	**Value**
General		
Survival probability	p	0.98
Investment volume	I	1,000
Loan	L	1,000
Costs (1) of a loan	$c(L)$	7.389
Figure 5a		
Tontine size	N_0	100
Risk aversion	b	0,...,0.6
Figure 5b		
Risk aversion	b	0.19
Tontine size	N_0	50,...,300
Figure 7		
Costs 2 of a loan	$c_2(L)$	12.182
Costs 3 of a loan	$c_3(L)$	28.032
Costs 4 of a loan	$c_4(L)$	148.413

Table 1: Tontine parameters

In figure 5a we consider a tontine size of $N_0 = 100$ and determine α^* and $\overline{\alpha}$ for a changing risk aversion b. For a risk aversion of $b = 0$, the surrenderer is able to recoup $\alpha^* = 100\%$ of his initial investment I without worsening the utility of the remaining tontinists. As the risk aversion (of all tontinists) increases, the fraction α^* which can be distributed to the surrenderer without harming the remaining tontinists decreases. If we have a look at the minimum fraction $\overline{\alpha}$ which the surrenderer needs to receive in order to be willing to surrender, we can observe that for no risk aversion, the surrenderer would accept a lower secure payout than the initial investment I. This is because of the costs $c(L)$ to finance a loan which the surrenderer has to bear if he does not surrender. In such a situation, the surrenderer faces a liquidity need but has a lack of capital to finance the liquidity need, which he needs to fill up by costly money he needs to borrow. Therefore, for no risk aversion the surrenderer accepts a lower minimum surrender value than what could maximally be paid to him without worsening the position of the remaining tontinists. As a result, a surrender takes place. As the risk aversion (of all tontinists, including the surrenderer) increases, the minimum fraction $\overline{\alpha}$ the surrenderer requires to be willing to surrender declines less fast than the maximum fraction α^* that can be paid to the surrenderer to hold constant the utility of the remaining tontinists. Therefore, beyond a certain level of risk aversion, the surrenderer requires a higher fraction to be incentivized to surrender than one could take away from the remaining tontinists without harming them. Therefore, surrender will not take place for higher risk aversion. This is a somewhat surprising result, because the more risk averse one is, the higher is the probability to stay in the risky tontine, instead of taking the safe (but heavily discounted) payout.

In figure 5b we consider a risk aversion of $b = 0.19$ and determine α^* and $\overline{\alpha}$ for a changing tontine size $N_0 = 50, \ldots, 300$. We can observe that for low tontine sizes, the maximum fraction α^* that can be paid to the surrenderer to hold constant the utility of the remaining tontinists is below 1, because of the high volatility of the tontine payouts for low tontine sizes, which diminishes utility. Therefore, this has to be compensated through keeping the fraction of $1 - \alpha^*$ in the tontine. As the tontine size increases, the payouts become less volatile, resulting in a higher α^*. As the tontine size approaches infinity, the volatility of the tontine payouts disappears and the surrenderer can recoup the whole initial investment I. If we look at the minimum fraction $\overline{\alpha}$ the surrenderer requires to be willing to surrender, we can observe that for low tontine sizes $\overline{\alpha} < \alpha^*$. This means that for low tontine sizes, the surrenderer requires more money to be willing to surrender than it could be paid to him because otherwise, the remaining tontinists would be worse off. Therefore, for low tontine sizes, no surrender takes place. As the tontine size increases, $\overline{\alpha}$ increases slower than α^* making it worthwhile to surrender for the surrenderer. This is an interesting result. If there is a shock causing a general liquidity need, this might trigger a tontine run. However, as the tontine size then rapidly declines through the surrender of many tontinists, the tontine run stops at a certain tontine size where surrender is not favorable any more. Furthermore, as the tontine size naturally diminishes with time, surrender becomes more costly and at some threshold tontine size, there are no incentives to surrender any more.

In figure 6, we vary tontine size N_0 and risk aversion b simultaneously. In the top-left surface where $\alpha^* > \overline{\alpha}$, surrender takes place: a higher fraction can be distributed to the surrenderer than he requires to be incentivized to surrender. For large tontines and low risk aversion, an



(a) $\overline{\alpha}$ and α^* for $N_0 = 100$ and $c(L) = 7.389$

(b) $\overline{\alpha}$ and α^* for $b = 0.19$ and $c(L) = 7.389$

Figure 5: Minimum required $\overline{\alpha}$ to surrender and minimum α^* to keep constant the remaining participants' utility

almost fair surrender value can be paid. In the top-right surface, where $\alpha^* < \overline{\alpha}$, no surrender takes place because the surrenderer requires a higher fraction to surrender than it can maximally be offered to keep constant the utility of the remaining tontinists. We can observe the threshold $\alpha^* = \overline{\alpha}$ for all combinations of tontine size and risk aversion, which increases in N_0 and b. This means the lower the tontine size is, the less will be surrendered for higher risk aversion and as the tontine size increases, the higher the risk aversion can be, to still being incentivized to surrender. If we increase the costs of borrowing money $c(L)$, the surface of $\overline{\alpha}$ moves downwards. As a consequence, the costs of staying in the tontine increase; as a result, that one accepts a lower $\overline{\alpha}$ to surrender. In figure 7, the thresholds are depicted where $\alpha^* = \overline{\alpha}$ for varying tontine size N_0 and risk aversion b for different levels of $c(L)$, where $c_1(L) < c_2(L) < c_3(L) < c_4(L)$. In the area below each threshold line for a certain level of $c(L)$, surrender takes place, whereas in the area above the threshold line, surrender does not take place. As the costs increase, one is willing to surrender for a higher risk aversion given a level of tontine size and is willing to surrender for a given level of risk aversion still for a lower tontine size: as the costs of staying in the tontine increase, one is more willing to surrender, even if the conditions to surrender are not good in terms of the fraction of the fair surrender value one receives α^*.

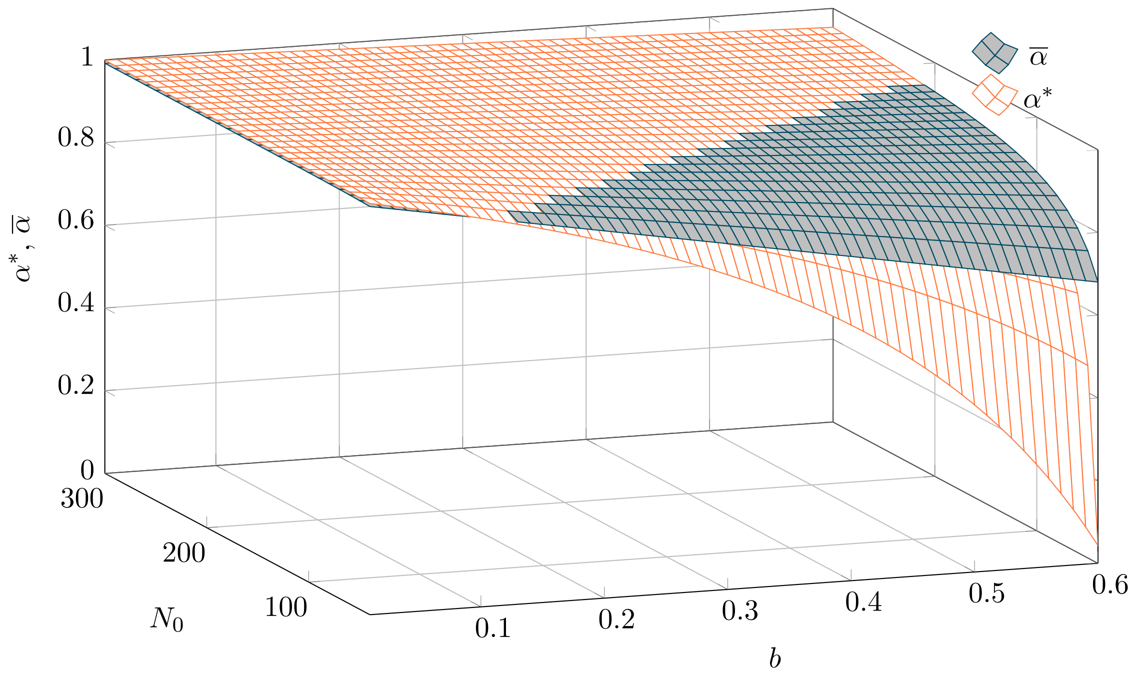


Figure 6: Minimum required $\overline{\alpha}$ to surrender and maximum α^* to keep constant the remaining tontinists utility for varying tontine size N_0 and risk aversion b with $p = 0.98$ and $c(L) = exp\{\frac{L}{1000}\} = 7.389$

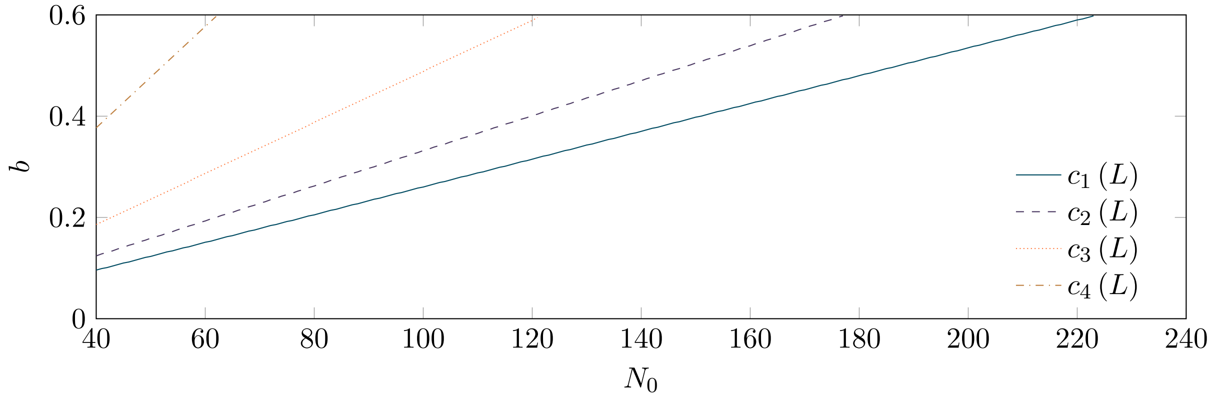


Figure 7: Threshold where $\alpha^* = \overline{\alpha}$ for varying tontine size N_0 and risk aversion b for different levels of $c(L)$ $(c_1(L) < c_2(L) < c_3(L) < c_4(L))$. In the area below the line, surrender takes place; in the area above, surrender does not take place

4 Conclusion

Because of the interlinkage in the tontine of all tontinists, there is a direct influence of the decisions of a single individual on the other tontinists. We show that the fair surrender value based on expected values equals the net present value of the expected future tontine payments. However, since the tontine does not entail any guarantees and tontine payouts thus are volatile, we furthermore incorporate the impact of a surrender decision on the tontine volatility. The larger the tontine is, the less severe is the impact of a single surrender decision, because the volatility increase of the payments of the remaining tontinists remains low. Nevertheless, the fair surrender value requires a discount to hold constant the utility of the remaining tontinists, dependent on the tontine size, the investment volume and the individual characteristics. Since the considered tontine decreases in its size over time due to mortality, the surrender decision is essentially influenced by the progress of the tontine. The fewer people are in the tontine, the higher the discount on the fair surrender value needs to be. Otherwise, surrender harms the remaining tontinists through an increased volatility. Furthermore, we determine under which conditions a tontinist would accept a reduced surrender value and when staying in the tontine provides a higher utility than to surrender, considering a short-term liquidity shock. We observe that for decreasing tontine size and for increasing risk aversion, tontinists are less willing to surrender, which provides a natural protection against a tontine run. Furthermore, we introduce heterogeneous risk aversion into the model, which further lowers the surrender fraction that can be paid to a surrenderer in order not to worsen any remaining tontininst. This can create situations when not every type of investor is willing to surrender. Finally, we argue that a surrender decision based on private information requires a discount on the fair surrender value as well, and we discuss the effects of a secondary market for tontines.

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A Appendix

A.1 Simulation of the Fair Surrender Value

We simulate the fair surrender value via Monte Carlo simulation with 100,000 paths for $p = 0.98$ and $I = 1,000$ for a varying initial tontine size of $N_0 = 5, \ldots, 2,000$. The results show that the observed fair surrender value is very stable around the theoretical expectation of $FSV = I = 1000$ for any tontine size.

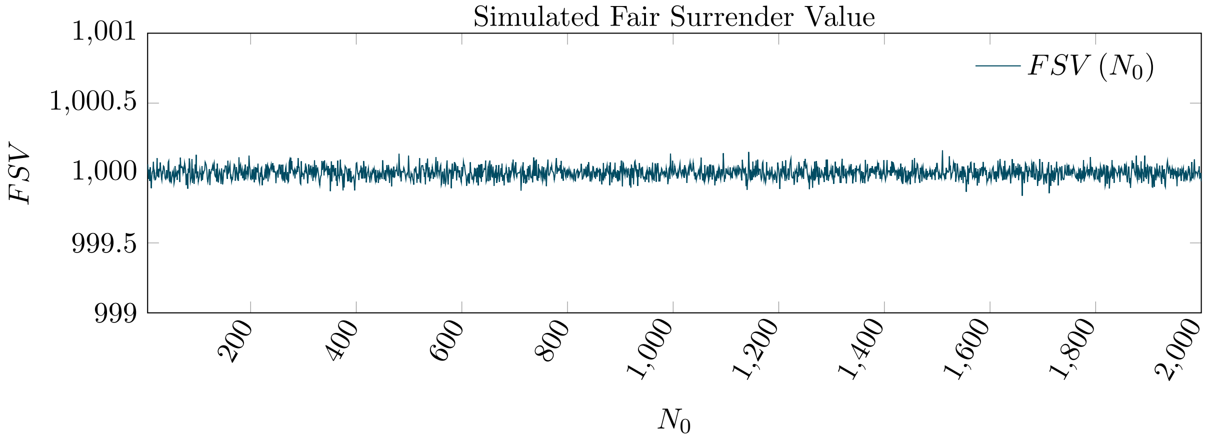


Figure 8: Simulated Fair Surrender Value (FSV) for $p = 0.98$, $I = 1,000$ and $N_0 = 5, \ldots, 2,000$, 100,000 paths per N_0

A.2 The Variance of the Tontine using the Taylor Approximation

Because the exact solution only works for relatively small tontine sizes because of computational restrictions, we approximate the variance of the tontine payout via Taylor approximation[20]. However, we proceed with a further simplification and determine the unconditional variance of the tontine payouts for $\widetilde{N_0} = N_0 - 1$ tontinists rather than the conditional variance for N_0 tontinists. This simplification reduces complexity and is still sufficient to capture the main effects of the model. Therefore, the variance of the tontine payout without surrender is

$$Var\left(PO_1|\text{alive}\right) = \left(I\widetilde{N_0}\right)^2 Var\left(\frac{1}{\widetilde{N_1}}\right). \tag{44}$$

[20] See Appendix A.6.9 for the taylor expansion for moments of functions.

Using Taylor series expansion yields for equation (44)

$$\begin{aligned}
Var\left(PO_1|\text{alive}\right) &\approx \left(I\widetilde{N_0}\right)^2 \frac{1}{E\left(\widetilde{N_1}\right)^4} Var\left(\widetilde{N_1}\right) \\
&\approx I^2\widetilde{N_0}^2 \frac{1}{\left(p\widetilde{N_0}\right)^4} p\left(1-p\right)\widetilde{N_0} \\
&\approx \frac{I^2\widetilde{N_0}^3 p\left(1-p\right))}{\widetilde{N_0}^4 p^4} \\
Var\left(PO_1|\text{alive}\right) &\approx \frac{I^2\left(1-p\right)}{\left(N_0-1\right)p^3}.
\end{aligned} \tag{45}$$

Accordingly, the variance of the tontine payout in t=1 with surrender is

$$\begin{aligned}
Var\left(PO_1^s|\text{alive}\right) &= Var\left(\frac{\left(\widetilde{N_0}-\widetilde{N_1^s}\right)I - FSV}{\widetilde{N_1^s}} + I\right) \\
&= I^2\left(\widetilde{N_0} - 1 + \left(1-p\right)^{\widetilde{N_0}}\right)^2 Var\left(\frac{1}{\widetilde{N_1^s}}\right).
\end{aligned} \tag{46}$$

Using Taylor series expansion, equation (46) yields approximately

$$\begin{aligned}
Var\left(PO_1^s|\text{alive}\right) &\approx I^2\left(\widetilde{N_0} - 1 + \left(1-p\right)^{\widetilde{N_0}}\right)^2 \frac{1}{E\left(\widetilde{N_1^s}\right)^4} Var\left(\widetilde{N_1^s}\right) \\
&\approx I^2\left(\widetilde{N_0} - 1 + \left(1-p\right)^{\widetilde{N_0}}\right)^2 \frac{1}{\left[p\left(\widetilde{N_0}-1\right)\right]^4}\left(\widetilde{N_0}-1\right)p\left(1-p\right) \\
Var\left(PO_1^s|\text{alive}\right) &\approx \frac{I^2\left(1-p\right)\left(\left(1-p\right)^{\left(N_0-1\right)} + N_0 - 2\right)^2}{\left(N_0-2\right)^3 p^3}.
\end{aligned} \tag{47}$$

Even for relatively small tontine sizes, the variance with surrender converges towards

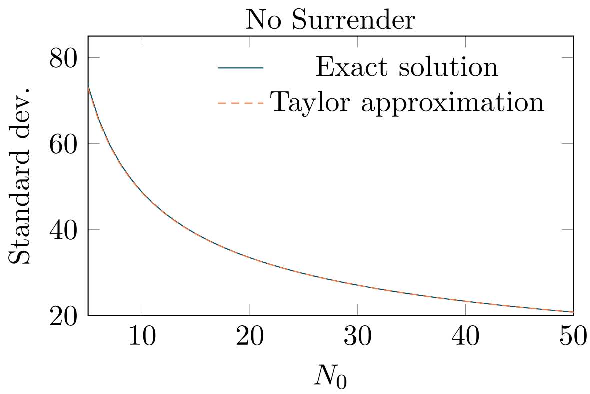
$$Var\left(PO_1^s|\text{alive}\right) \approx \frac{I^2\left(1-p\right)}{\left(N_0-2\right)p^3}. \tag{48}$$

As presented in appendix A.6.10 the variance approximation with surrender is larger than without surrender:

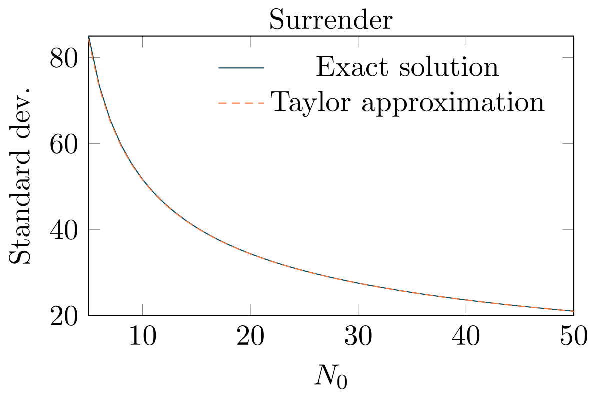
$$Var\left(PO_1|\text{alive}\right) < Var\left(PO_1^s|\text{alive}\right). \tag{49}$$

A.3 Evaluation of the Variance Taylor Approximation

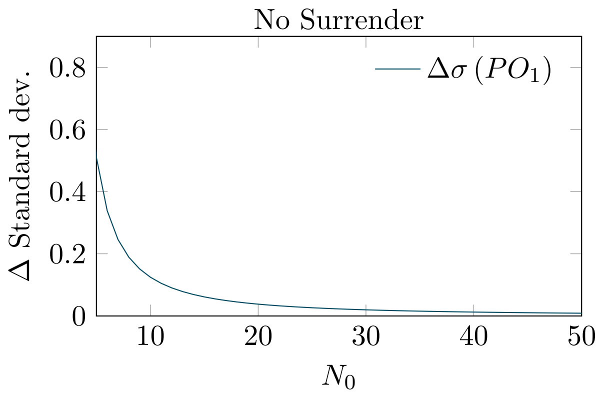
We calculate the volatility using the exact solution and the Taylor approximation varying $N_0 = 5, \ldots, 50$ for $p = 0.98$ and $I = 1000$ and compare the results. In figure 9a and figure 9c, we compare the volatility without surrender. The Taylor approximation is very precise compared to the exact solution. However, the Taylor approximation slightly overestimates the volatility, especially for low tontine sizes. In figure 9b and figure 9d, we compare the volatility with surrender. The Taylor approximation is again very precise compared to the exact solution. However, the Taylor approximation slightly overestimates the volatility, especially for low tontine sizes. If we compare the surrender and non-surrender case, it can be observed that surrender



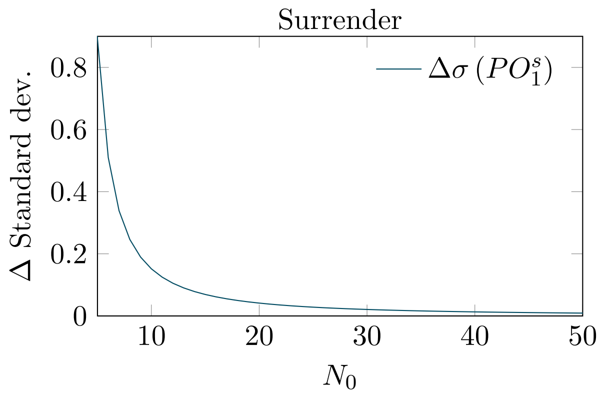
(a) Exact solution vs. Taylor Approximation of the tontine payout volatility without surrender



(b) Exact solution vs. Taylor Approximation of the tontine payout volatility with surrender



(c) Difference of standard deviations without surrender (Exact Solution − Taylor approximation)



(d) Difference of standard deviations with surrender (Exact Solution − Taylor approximation)

Figure 9: Comparison of the standard deviation with and without surrender for $N_0 = 5, \ldots, 50$, $p = 0.98$ and $I = 1000$ using the exact solution and the Taylor approximation

increases the variance of the tontine the smaller the tontine is. Furthermore, it can be observed that for small tontine sizes, the difference of both approximation methods is larger, the difference between the exact solution and the Taylor approximation converges toward zero in increasing N_0. Because the Taylor approximation is more easy to deal with, we use the Taylor approximation in the sensitivity analysis to approximate the volatility of the tontine payouts.

A.4 Sensitivity Analysis

Tontine Size

$$\frac{\partial \alpha^*}{\partial N_0} = -\frac{\begin{array}{c}\left(p^4 - p^2\sigma_1\right)(N_0-1)^3(2N_0+1) \\ + 2IN_0 b\left(\sigma_1(1-p)\left({N_0}^2 - N_0\right) + Ib\left(1+2N_0+2N_0p^2-2p^3+p^2-4N_0p\right) + \left(p^2-p^3\right)\left(1+2N_0-3{N_0}^2\right)\right)\end{array}}{2I{N_0}^2 b\sigma_1(N_0-1)(p-1)} \tag{50}$$

We consider a sufficiently large N_0. Then $\frac{\partial \alpha^*}{\partial N_0} > 0$ because $\sigma_1 < p^2$ yielding to $\frac{\partial \alpha^*}{\partial N_0} = -\frac{(>0)+(>0)}{(<0)} > 0$.

Survival Probability

$$\frac{\partial \alpha^*}{\partial p} = -\frac{p(N_0-1)(p-2)\left(-(N_0-1)^2\left(\sigma_1 - p^2\right) - 2IN_0b(1-p)\right)}{2IN_0b\sigma_1(p-1)^2} \tag{51}$$

We consider a sufficiently large N_0. Then $\frac{\partial \alpha^*}{\partial p} > 0$ because $\sigma_1 < p^2$ yielding to $\frac{\partial \alpha^*}{\partial p} = -\frac{(<0)(>0)}{(>0)} > 0$.

Investment

$$\frac{\partial \alpha^*}{\partial I} = \frac{p^2(N_0-1)\left(-(N_0-1)^2\left(\sigma_1 - p^2\right) - 2IN_0b(1-p)\right)}{2I^2N_0b\sigma_1(p-1)} \tag{52}$$

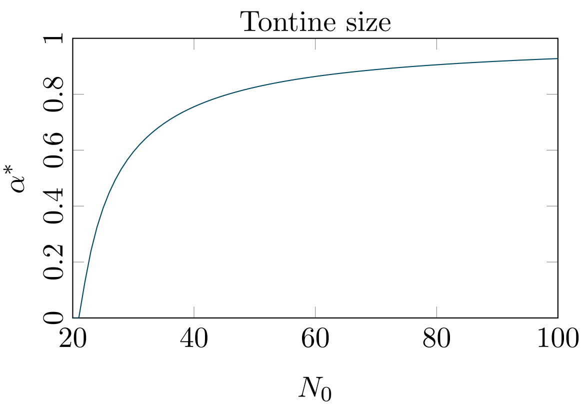
We consider a sufficiently large N_0. Then $\frac{\partial \alpha^*}{\partial I} < 0$ because $\sigma_1 < p^2$, yielding to $\frac{\partial \alpha^*}{\partial I} = \frac{(>0)(>0)}{(<0)} < 0$.

Risk Aversion

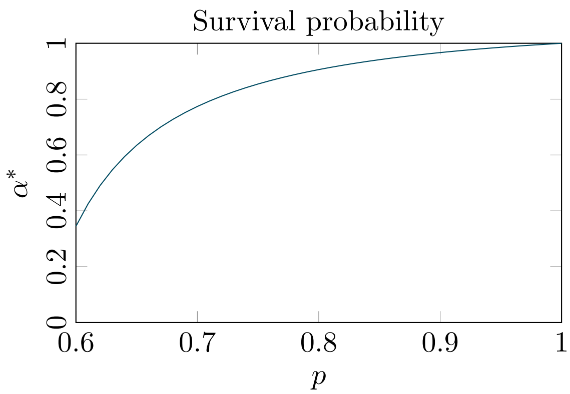
$$\frac{\partial \alpha^*}{\partial b} = \frac{p^2(N_0-1)\left(-(N_0-1)^2\left(\sigma_1 - p^2\right) - 2IN_0b(1-p)\right)}{2IN_0b^2\sigma_1(p-1)} \tag{53}$$

We consider a sufficiently large N_0. Then $\frac{\partial \alpha^*}{\partial b} < 0$ because $\sigma_1 < p^2$ yielding to $\frac{\partial \alpha^*}{\partial b} = \frac{(>0)(>0)}{(<0)} < 0$.

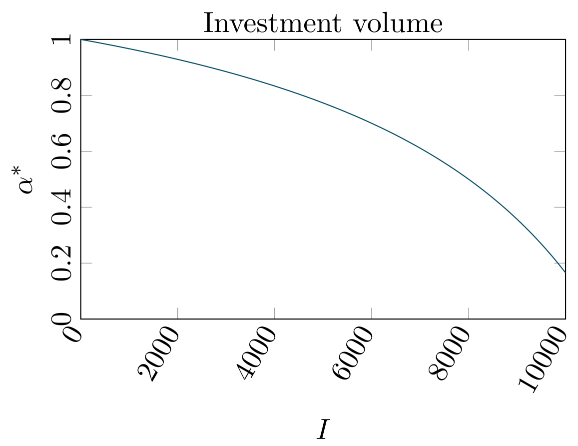
A.5 Additional Figures



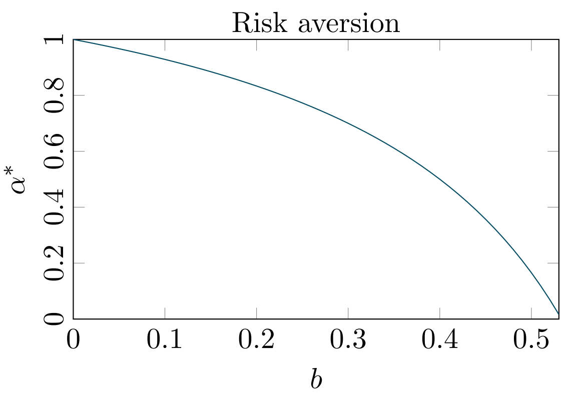
(a) Maximum α^* for increasing tontine size N_0, $p = 0.9$, $b = 0.05$ and $I = 1000$



(b) Maximum α^* for increasing survival probability p, $N_0 = 200$, $b = 0.05$ and $I = 1000$

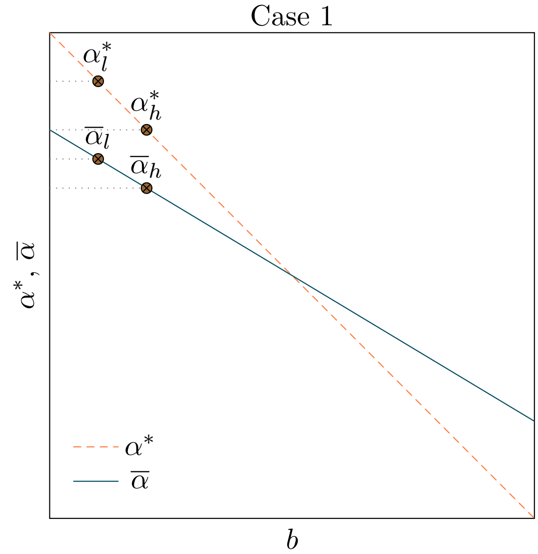


(c) Maximum α^* for increasing investment volume I, $N_0 = 200$, $p = 0.9$ and $b = 0.05$

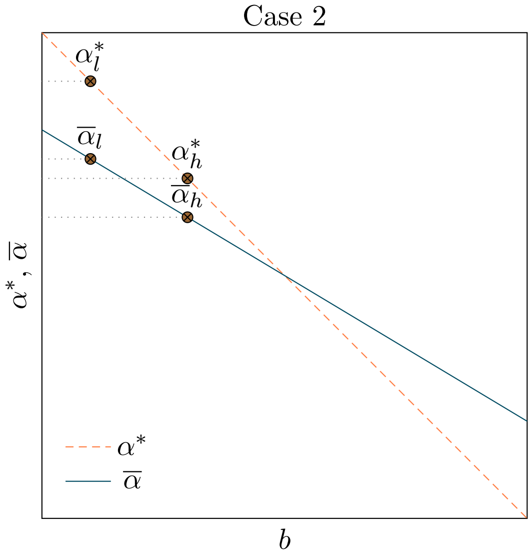


(d) Maximum α^* for increasing risk aversion b, $N_0 = 200$, $p = 0.9$ and $I = 1000$

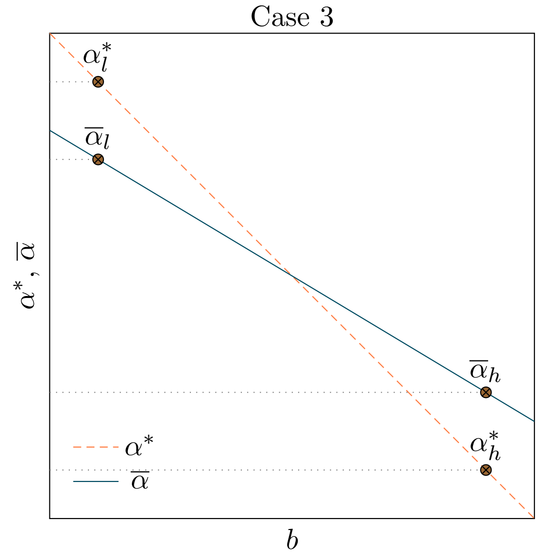
Figure 10: Sensitivity Analysis of the main variables on the maximum surrender fraction



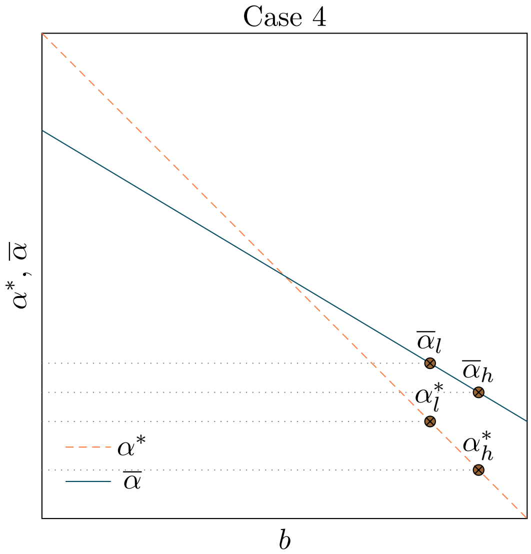
(a) Case 1: low and high risk averse tontinists surrender



(b) Case 2: Just high risk averse tontinists surrender



(c) Case 3: No one surrenders



(d) Case 4: No one surrenders

Figure 11: Heterogeneous risk aversion

A.6 Proofs

A.6.1 Proof of Equation (5)

$$\begin{aligned}
E\left(PO_1|\text{alive}\right) &= E\left(\frac{N_0}{N_1} I|\text{alive}\right) \\
&= IN_0 E\left(\frac{1}{N_1}|\text{alive}\right) \\
&= IN_0 E\left(\frac{1}{\widetilde{N_1}+1}\right)
\end{aligned}$$

where $\widetilde{N_1} \sim Bin\left(\widetilde{N_0}, p\right)$ with $\widetilde{N_0} = N_0 - 1$.

$$\begin{aligned}
E\left(PO_1|\text{alive}\right) &= IN_0 \sum_{k=0}^{\widetilde{N_0}} \frac{1}{1+k} \binom{\widetilde{N_0}}{k} p^k (1-p)^{\left(\widetilde{N_0}-k\right)} \\
&= IN_0 \frac{1}{\left(\widetilde{N_0}+1\right)p} \sum_{k=0}^{\widetilde{N_0}} \binom{\widetilde{N_0}+1}{k+1} p^{(k+1)} (1-p)^{\left(\widetilde{N_0}-k\right)}.
\end{aligned}$$

By substituting $l = k+1$

$$\begin{aligned}
E\left(PO_1|\text{alive}\right) &= IN_0 \frac{1}{\left(\widetilde{N_0}+1\right)p} \sum_{l=1}^{\widetilde{N_0}} \binom{\widetilde{N_0}+1}{l} p^l (1-p)^{\left(\widetilde{N_0}-l+1\right)} \\
&= IN_0 \frac{1}{\left(\widetilde{N_0}+1\right)p} \left[\sum_{l=0}^{\widetilde{N_0}+1} \binom{\widetilde{N_0}+1}{l} p^l (1-p)^{\left(\widetilde{N_0}-l+1\right)} - (1-p)^{\left(\widetilde{N_0}+1\right)}\right] \\
&= IN_0 \frac{1}{\left(\widetilde{N_0}+1\right)p} \left[(p+1-p)^{\left(\widetilde{N_0}+1\right)} - (1-p)^{\left(\widetilde{N_0}+1\right)}\right] \\
&= IN_0 \frac{1}{\left(\widetilde{N_0}+1\right)p} \left[1 - (1-p)^{\left(\widetilde{N_0}+1\right)}\right] \\
&= IN_0 \frac{1-(1-p)^{(N_0-1+1)}}{(N_0-1+1)p} \\
E\left(PO_1|\text{alive}\right) &= \frac{1-(1-p)^{N_0}}{p} I
\end{aligned}$$

A.6.2 Proof of Equation (9)

$$\begin{aligned}
E\left(PO_1^{s*}|\text{alive}\right) &= E\left(\frac{N_0}{N_1^s} I|\text{alive}\right) \\
&= IN_0 E\left(\frac{1}{N_1^s}|\text{alive}\right) \\
&= IN_0 E\left(\frac{1}{\widetilde{N_1^s}+1}\right)
\end{aligned}$$

where $\widetilde{N_1^s} \sim Bin\left(\widetilde{N_0^s}, p\right)$ with $\widetilde{N_0^s} = N_0 - 2$.

$$
\begin{aligned}
E\left(PO_1^{s*}|\text{alive}\right) &= IN_0 \sum_{k=0}^{\widetilde{N_0^s}} \frac{1}{1+k} \binom{\widetilde{N_0^s}}{k} p^k \left(1-p\right)^{\left(\widetilde{N_0^s}-k\right)} \\
&= IN_0 \frac{1}{\left(\widetilde{N_0^s}+1\right)p} \sum_{k=0}^{\widetilde{N_0^s}} \binom{\widetilde{N_0^s}+1}{k+1} p^{(k+1)} \left(1-p\right)^{\left(\widetilde{N_0^s}-k\right)}.
\end{aligned}
$$

By substituting $l = k+1$

$$
\begin{aligned}
E\left(PO_1^{s*}|\text{alive}\right) &= IN_0 \frac{1}{\left(\widetilde{N_0^s}+1\right)p} \sum_{l=1}^{\widetilde{N_0^s}} \binom{\widetilde{N_0^s}+1}{l} p^l \left(1-p\right)^{\left(\widetilde{N_0^s}-l+1\right)} \\
&= IN_0 \frac{1}{\left(\widetilde{N_0^s}+1\right)p} \left[\sum_{l=0}^{\widetilde{N_0^s}+1} \binom{\widetilde{N_0^s}+1}{l} p^l \left(1-p\right)^{\left(\widetilde{N_0^s}-l+1\right)} - \left(1-p\right)^{\left(\widetilde{N_0^s}+1\right)}\right] \\
&= IN_0 \frac{1}{\left(\widetilde{N_0^s}+1\right)p} \left[\left(p+1-p\right)^{\left(\widetilde{N_0^s}+1\right)} - \left(1-p\right)^{\left(\widetilde{N_0^s}+1\right)}\right] \\
&= IN_0 \frac{1}{\left(\widetilde{N_0^s}+1\right)p} \left[1 - \left(1-p\right)^{\left(\widetilde{N_0^s}+1\right)}\right] \\
&= IN_0 \frac{1-\left(1-p\right)^{\left(N_0-2+1\right)}}{\left(N_0-2+1\right)p} \\
E\left(PO_1^{s*}|\text{alive}\right) &= N_0 \frac{1-\left(1-p\right)^{\left(N_0-1\right)}}{p\left(N_0-1\right)} I
\end{aligned}
$$

A.6.3 Proof of $E\left(PO_1|\textbf{alive}\right) < E\left(PO_1^{s*}|\textbf{alive}\right)$

We check if the expected conditional payout in $t=1$ without surrender $E\left(PO_1|\text{alive}\right)$, or the expected conditional hypothetical payout with surrender $E\left(PO_1^{s*}|\text{alive}\right)$ is larger:

$$
\begin{aligned}
E\left(PO_1^{s*}|\text{alive}\right) &\gtreqless E\left(PO_1|\text{alive}\right) \\
N_0 \frac{1-\left(1-p\right)^{\left(N_0-1\right)}}{\left(N_0-1\right)p} I &\gtreqless \frac{1-\left(1-p\right)^{N_0}}{p} I \\
\underbrace{\frac{N_0}{N_0-1}}_{>1} &\gtreqless \underbrace{\frac{1-\left(1-p\right)^{N_0}}{1-\left(1-p\right)^{\left(N_0-1\right)}}}_{<1}
\end{aligned}
$$

from which follows that

$$
E\left(PO_1|\text{alive}\right) < E\left(PO_1^{s*}|\text{alive}\right)\cdot
$$

The expected conditional hypothetical payout with surrender is larger than the expected conditional payout without surrender.

A.6.4 Proof of Equation (11)

$$
\begin{aligned}
FSV_\tau &= E\left(\left(PO_1^{s*} - PO_1\right) N_1^s\right) \\
&= E\left(\left(\frac{N_0 - N_1^s}{N_1^s} I - \frac{N_0 - N_1}{N_1} I\right) N_1^s\right) \\
&= E\left(N_0 - N_1^s - \frac{N_0 N_1^s - N_1 N_1^s}{N_1}\right) I \\
&= E\left(N_0 - N_0 \frac{N_1^s}{N_1}\right) I \\
&= \left(N_0 - N_0 E\left(\frac{N_1^s}{N_1}\right)\right) I \\
&= \left(N_0 - N_0 E\left(\frac{\sum_{i=1}^{N_0-1} \mathbb{1}_{A_i}}{\sum_{i=1}^{N_0} \mathbb{1}_{A_i}}\right)\right) I \\
&= \left(N_0 - N_0 E\left(\frac{N_1^s}{N_1^s + \mathbb{1}_{A_{N_0}}}\right)\right) I \\
&= \left(N_0 - N_0 E\left(E\left(\frac{N_1^s}{N_1^s + \mathbb{1}_{A_{N_0}}} | N_1^s\right)\right)\right) I \\
&= \left(N_0 - N_0 E\left(p\left(\frac{N_1^s}{N_1^s + 1}\right) + (1-p)\left(\frac{N_1^s}{N_1^s}\right)\right)\right) I \\
&= \left(N_0 - N_0\left(pE\left(\frac{N_1^s}{N_1^s + 1}\right) + (1-p)\right)\right) I \\
&= \left(N_0 - N_0\left(p \sum_{k=0}^{N_0-1}\left(\frac{k}{k+1}\right)\binom{N_0-1}{k} p^k (1-p)^{N_0-k-1} + (1-p)\right)\right) I \\
&= \left(N_0 - N_0\left(p \sum_{k=0}^{N_0-1}\left(\frac{k}{k+1}\right)\frac{(N_0-1)!}{k!\,(N_0-k-1)!} p^k (1-p)^{N_0-k-1} + (1-p)\right)\right) I \\
&= \left(N_0 - N_0\left(p\frac{(1-p)^{N_0}\left((N_0 p - 1)\left(\frac{1}{1-p}\right)^{N_0} + 1\right)}{N_0 p} + (1-p)\right)\right) I \\
&= \left(N_0 - (1-p)^{N_0}\left((N_0 p - 1)\left(\frac{1}{1-p}\right)^{N_0} + 1\right) - N_0 (1-p)\right) I \\
&= \left(pN_0 - (1-p)^{N_0}\left((N_0 p - 1)\left(\frac{1}{1-p}\right)^{N_0} + 1\right)\right) I \\
&= \left(pN_0 - \left((1-p)^{N_0}(N_0 p - 1)\left(\frac{1}{1-p}\right)^{N_0} + (1-p)^{N_0}\right)\right) I \\
&= \left(pN_0 - \left(pN_0 - 1 + (1-p)^{N_0}\right)\right) I \\
&= \left(1 - (1-p)^{N_0}\right) I \\
&\approx I.
\end{aligned}
$$

A.6.5 Proof of Equation (12)

Using equation (11), the expected conditional payout with surrender if one survives is

$$\begin{aligned}
E\left(PO_1^s|\text{alive}\right) &= E\left(\frac{\left(N_0 - N_1^s\right)I - FSV}{N_1^s} + I|\text{alive}\right) \\
&= E\left(\frac{N_0 - N_1^s - 1 + (1-p)^{N_0}}{N_1^s} + 1|\text{alive}\right) I \\
&= \left(N_0 - 1 + (1-p)^{N_0}\right) E\left(\frac{1}{N_1^s}|\text{alive}\right) I \\
&= \left(N_0 - 1 + (1-p)^{N_0}\right) E\left(\frac{1}{\widetilde{N_1^s} + 1}\right) I
\end{aligned}$$

where $\widetilde{N_1^s} \sim Bin\left(\widetilde{N_0^s, p}\right)$ with $\widetilde{N_0^s} = N_0 - 2$. Therefore

$$\begin{aligned}
E\left(PO_1^s|\text{alive}\right) &= \left(N_0 - 1 + (1-p)^{N_0}\right) \frac{1 - (1-p)^{\left(\widetilde{N_0^s}+1\right)}}{\left(\widetilde{N_0^s} + 1\right)p} I \\
&= \frac{\left(N_0 - 1 + (1-p)^{N_0}\right)\left(1 - (1-p)^{(N_0-1)}\right)}{(N_0 - 1)\,p} I \qquad (54) \\
E\left(PO_1^s|\text{alive}\right) &= \frac{I}{p} - \epsilon
\end{aligned}$$

where $\epsilon = \left(\frac{(1-p)^{(N_0-1)}}{p} - \frac{(1-p)^{N_0}\left(1-(1-p)^{(N_0-1)}\right)}{(N_0-1)p}\right) I$ which converges even for very small tontine sizes towards 0 and therefore

$$E\left(PO_1^s|\text{alive}\right) = E\left(PO_1|\text{alive}\right) = \frac{I}{p}$$

for $N_0 \to \infty$.

A.6.6 Proof of Equation (13)

Using equation (3), we obtain for the conditional variance of the payout without surrender in the case of survival

$$\begin{aligned}
Var\left(PO_1|\text{alive}\right) &= Var\left(\frac{N_0}{N_1} I|\text{alive}\right) \\
&= I^2 N_0^2 Var\left(\frac{1}{N_1}|\text{alive}\right) \\
&= I^2 N_0^2 Var\left(\frac{1}{\widetilde{N_1} + 1}\right)
\end{aligned}$$

where $\widetilde{N_1} \sim Bin\left(\widetilde{N_0}, p\right)$ with $\widetilde{N_0} = N_0 - 1$. From the linearity of expected values and the definition of variance follows

$$\begin{aligned} Var\left(PO_1|\text{alive}\right) &= I^2 N_0^2 \left(E\left(\frac{1}{\left(\widetilde{N_1}+1\right)^2} \right) - E\left(\frac{1}{\widetilde{N_1}+1} \right)^2 \right) \\ &= I^2 N_0^2 \left(E\left(\frac{1}{\left(\widetilde{N_1}+1\right)^2} \right) - \left(\frac{1-(1-p)^{\widetilde{N_0}+1}}{\left(\widetilde{N_0}+1\right)p} \right)^2 \right). \end{aligned} \tag{55}$$

$E\left(\frac{1}{\left(\widetilde{N_1}+1\right)^2}\right)$ can be written as

$$\begin{aligned} E\left(\frac{1}{\left(\widetilde{N_1}+1\right)^2} \right) &= \sum_{k=0}^{\widetilde{N_0}} \frac{1}{(1+k)^2} \binom{\widetilde{N_0}}{k} p^k (1-p)^{\left(\widetilde{N_0}-k\right)} \\ &= \sum_{k=0}^{\widetilde{N_0}} \frac{k+2}{k+1} \frac{1}{\widetilde{N_0}+1} \frac{1}{\widetilde{N_0}+2} \binom{\widetilde{N_0}+2}{k+2} p^k (1-p)^{\left(\widetilde{N_0}-k\right)} \\ &= (1-p)^{\widetilde{N_0}} {}_iF_j\left(\overline{x};\, \overline{y};\, z\right) \end{aligned} \tag{56}$$

where ${}_iF_j\left(\overline{x};\, \overline{y};\, z\right)$ is the the generalized hypergeometric function with $\overline{x} = \{x_1, \ldots, x_i\} = \{1, 1, -(N_0 - 1)\}$, $\overline{y} = \{y_1, \ldots, y_j\} = \{2, 2\}$ and $z = \frac{p}{p-1}$. Plugging equation (56) in equation (55) yields

$$Var\left(PO_1|\text{alive}\right) = I^2 N_0^2 \left((1-p)^{(N_0-1)} {}_3F_2\left(1, 1, -(N_0-1);\, 2, 2;\, \frac{p}{p-1}\right) - \left(\frac{1-(1-p)^{N_0}}{N_0 p} \right)^2 \right). \tag{57}$$

A.6.7 Proof of Equation (14)

Using equation (10), we obtain for the conditional variance of the payout with surrender in the case of survival

$$\begin{aligned} Var\left(PO_1^s|\text{alive}\right) &= Var\left(\frac{N_0 - 1 + (1-p)^{N_0}}{N_1^s} I|\text{alive} \right) \\ &= I^2 \left(N_0 - 1 + (1-p)^{N_0}\right)^2 Var\left(\frac{1}{N_1^s}|\text{alive} \right) \\ &= I^2 \left(N_0 - 1 + (1-p)^{N_0}\right)^2 Var\left(\frac{1}{\widetilde{N_1^s}+1} \right) \end{aligned}$$

where $\widetilde{N_1^s} \sim Bin\left(\widetilde{N_0^s}, p\right)$ with $\widetilde{N_0^s} = N_0 - 2$. From the linearity of expected values and the definition of variance follows

$$
\begin{aligned}
Var\left(PO_1^s|\text{alive}\right) &= I^2\left(N_0 - 1 + (1-p)^{N_0}\right)^2\left(E\left(\frac{1}{\left(\widetilde{N_1^s}+1\right)^2}\right) - E\left(\frac{1}{\widetilde{N_1^s}+1}\right)^2\right)\\
&= I^2\left(N_0 - 1 + (1-p)^{N_0}\right)^2\left(E\left(\frac{1}{\left(N_1^s+1\right)^2}\right) - \left(\frac{1-(1-p)^{\left(\widetilde{N_0^s}+1\right)}}{\left(\widetilde{N_0^s}+1\right)p}\right)^2\right)
\end{aligned}
\tag{58}
$$

$E\left(\frac{1}{\left(\widetilde{N_1^s}+1\right)^2}\right)$ can be written as

$$
\begin{aligned}
E\left(\frac{1}{\left(\widetilde{N_1^s}+1\right)^2}\right) &= \sum_{k=0}^{\widetilde{N_0^s}} \frac{1}{(1+k)^2}\binom{\widetilde{N_0^s}}{k} p^k (1-p)^{\left(\widetilde{N_0^s}-k\right)}\\
&= \sum_{k=0}^{\widetilde{N_0^s}} \frac{k+2}{k+1}\frac{1}{\widetilde{N_0^s}+1}\frac{1}{\widetilde{N_0^s}+2}\binom{\widetilde{N_0^s}+2}{k+2} p^k (1-p)^{\left(\widetilde{N_0^s}-k\right)}\\
&= (1-p)^{\widetilde{N_0^s}}\, {}_iF_j\left(\overline{x^s};\, \overline{y^s};\, z^s\right)
\end{aligned}
\tag{59}
$$

where ${}_iF_j\left(\overline{x^s};\, \overline{y^s};\, z^s\right)$ is the the generalized hypergeometric function with $\overline{x^s} = \{x_1^s, \ldots, x_i^s\} = \{1, 1, -(N_0-2)\}$, $\overline{y^s} = \left\{y_1^s, \ldots, y_j^s\right\} = \{2, 2\}$ and $z^s = \frac{p}{p-1}$. Plugging equation (59) in equation (58) yields

$$
Var\left(PO_1^s|\text{alive}\right) =
$$

$$
I^2\left(N_0 - 1 + (1-p)^{N_0}\right)^2\left((1-p)^{(N_0-2)}\, {}_3F_2\left(1, 1, -(N_0-2); 2, 2; \frac{p}{p-1}\right) - \left(\frac{1-(1-p)^{(N_0-1)}}{(N_0-1)p}\right)^2\right)
\tag{60}
$$

A.6.8 Minimum Required Fraction to Preserve Incentives to Surrender

A tontinist is willing to surrender if the utility with surrender is higher than the utility without surrender

$$
U_\tau^s + pU_1^s \geq U_\tau^L + pU_1^L.
$$

Inserting equation (22) and (23) yields for the utility

$$
\begin{aligned}
E\left(\alpha^{*} I-\varepsilon_{\tau}\right)-b Var\left(\alpha^{*} I-\varepsilon_{\tau}\right) \geq E\left(L-c(L)-\varepsilon_{\tau}\right)-b Var\left(L-c(L)-\varepsilon_{\tau}\right)& \\
+p E\left(P O_{1}-\frac{L}{p}\right)-p b Var\left(P O_{1}-\frac{L}{p}\right)& \\
\alpha^{*} I-\varepsilon_{\tau} \geq L-c(L)-\varepsilon_{\tau}+\underbrace{p E\left(P O_{1}\right)}_{I}-L-p b Var\left(P O_{1}\right)& \\
\alpha^{*} I \geq I-p b Var\left(P O_{1}\right)-c(L)& \\
\alpha^{*} \geq 1-\frac{p b}{I} Var\left(P O_{1}\right)-\frac{c(L)}{I}=\bar{\alpha}.&
\end{aligned}
$$

A.6.9 Taylor Expansion for the Variance of Functions of Random Variables

$$
\begin{aligned}
g(x)= & g(\mu)+(x-\mu) g^{\prime}(\mu)+\frac{(x-\mu)^{2}}{2} g^{\prime \prime}(\mu)+\ldots \\
Var[g(x)]= & Var\left[g(\mu)+(x-\mu) g^{\prime}(\mu)+\frac{(x-\mu)^{2}}{2} g^{\prime \prime}(\mu)+\ldots\right] \\
= & Var\left[(x-\mu) g^{\prime}(\mu)+\frac{(x-\mu)^{2}}{2} g^{\prime \prime}(\mu)+\ldots\right] \\
= & g^{\prime}(\mu)^{2} Var[(x-\mu)]+2 g^{\prime}(\mu) Cov\left[(x-\mu) \frac{(x-\mu)^{2}}{2} g^{\prime \prime}(\mu)+\ldots\right]+Var\left[\frac{(x-\mu)^{2}}{2} g^{\prime \prime}(\mu)+\ldots\right]
\end{aligned}
$$

If we use only the first therm for the approximation, the variance is

$$
Var[g(x)] \approx g^{\prime}(\mu)^{2} Var(x)
$$

with $g(x)=\frac{1}{x}$ and $Var\left[\frac{1}{x}\right] \approx \frac{1}{\mu^{4}} Var(x)$.

A.6.10 Proof of Equation (49)

Assuming a sufficient large tontine size, we check whether the volatility without or with surrender is larger:

$$
\begin{aligned}
Var\left(P O_{1} \mid \text{alive}\right) & \gtreqless Var\left(P O_{1}^{s} \mid \text{alive}\right) \\
\frac{I^{2}(1-p)}{N_{0} p^{3}} & \gtreqless \frac{I^{2}(1-p)\left((1-p)^{N_{0}}+N_{0}-1\right)^{2}}{\left(N_{0}-1\right)^{3} p^{3}} \\
\frac{I^{2}(1-p)}{N_{0} p^{3}} & \gtreqless \frac{I^{2}(1-p)}{\left(N_{0}-1\right) p^{3}} \\
\frac{1}{N_{0}} & <\frac{1}{N_{0}-1} .
\end{aligned} \tag{61}
$$