Mathematical learning through actions on diagrams –

Reconstruction of learners' interpretations when acting on digital and analogue materials in primary school

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Deutsche Zusammenfassung der Forschungsarbeit

Dissertation von Lara Kristina Billion mit dem Titel

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Motivation der Forschungsarbeit

Unterschiedliche Materialien, sowohl digitale als auch analoge, sind für den Mathematikunterricht von großer Bedeutung und werden auch in der mathematikdidaktischen Forschung für das Mathematiklernen in der Grundschule untersucht (vgl. Kauthausen, 2012; Rink & Walter, 2020; Walter, 2018). Vor allem das Forschungsinteresse an digitalen Materialien, wie Apps und Programmen, die das mathematische Lernen unterstützen und ermöglichen sollen, ist in den letzten Jahren gestiegen. Larkin et al. (2019) stellen in ihrer Literaturübersicht Forschungsarbeiten zusammen, die sich mit der Evaluation bzw. Bewertung von Apps beschäftigen. In ihrer Übersicht weisen sie darauf hin, dass es nur wenig aktuelle Forschung sowie weitere Informationen zur Bestimmung der Qualität einer App gibt und dass die von den App-Entwickler*innen bereitgestellten Informationen meist als Werbung dienen (Larkin et al., 2019; Larkin, 2013). Darüber hinaus beruhen die meisten Bewertungen auf den Einschätzungen der Autor*innen selbst und nicht auf den Einschätzungen unabhängiger Gutachter*innen (Larkin et al. 2019). Larkin et al. (2019) verwenden die Artifact Centric Activity Theory (ACAT) als Rahmen für die Bewertung von Apps, die eine unabhängige Evaluation ermöglicht. Diese Theorie berücksichtigt die Aktivität der Lernenden mit der App zur Auseinandersetzung mit einem mathematischen Inhalt sowie das Design der App und ihre Verwendung in Unterrichtssituationen (Larkin et al., 2019).

Liegt der Schwerpunkt der Forschung auf der Bewertung des Materials, werden konkrete Handlungen von Lernenden mit verschiedenen Materialien nur in wenigen Forschungen fokussiert. Greiffenhagen (2014) und Sinclair und de Freitas (2014) argumentieren, dass sich die Forschung in der Regel auf die fertigen Produkte konzentriert und damit die mathematischen Resultate von den Aktivitäten (z.B. Handlungen oder Gesten) trennt, die sie hervorgebracht haben. Weiter führen Sinclair und de Freitas (2014) aus, dass solche Gewohnheiten der Fokussierung dazu führen, dass vernachlässigt wird, wie die Aktivität des Körpers die mathematische Tätigkeit beeinflusst. Radford (2019) schlussfolgert, dass die Aktivität von Lernenden in der Forschung stärker beleuchtet werden muss, da nur durch die Aktivität selbst (mathematisches) Wissen erworben werden kann.

Mit dem Blick auf die mathematikdidaktische Forschung wird daher deutlich, dass Handlungen selbst, die in mathematischen Situationen von Lernenden erzeugt werden, intensiver erforscht werden sollten. Ausgehend von diesem identifizierten Forschungsbedarf konzentriert sich die Studie *MatheMat – Mathematisches Lernen mit Materialien*, in der die vorliegende Forschungsarbeit entstanden ist, auf Handlungen von Lernenden in mathematischen Lernsituationen. Um Handlungen am Material fokussiert betrachten zu können, wird in dieser Forschungsarbeit eine semiotische Perspektive nach Charles Sanders Peirce (CP 1931-35) auf Mathematiklernen eingenommen. Diese wissenschaftstheoretische Perspektive hebt die Aktivität selbst, an sogenannten Diagrammen, als mathematische Tätigkeit hervor und ermöglicht so die Handlungen für das Mathematiklernen in den Vordergrund des Interesses zu stellen. Hoffmann (2010) beschreibt auf der Grundlage von Peirce, dass ein Diagramm in erster Linie eine Darstellung von Relationen ist, die mit Hilfe eines Systems von Darstellungen konstruiert wird. Ein solches System von Darstellungen wird durch eine Reihe von Regeln und Konventionen definiert (Hoffmann, 2010). Somit können eine bestimmte Anordnung von Materialien, aber auch etwas Geschriebenes auf einem Blatt Papier oder Gesten von Lernenden ein solches System von Darstellungen sein (vgl. Döfler, 2006; Huth, 2022; Schreiber, 2013). Das Erkunden mathematischer Relationen eines Systems von Darstellungen mit Hilfe von Papier und Bleistift kann sich vom Erkunden mittels digitaler Technologie unterscheiden. Moreno-Armella und Sriraman (2010) verweisen auf die Tatsache, dass ein (digitales) Werkzeug Einfluss auf das menschliche Handeln hat und dass mathematische Ideen und Prozesse daher völlig unterschiedlich sein können. Das menschliche Handeln verändert sich, da ein Werkzeug automatisch (mathematische) Relationen herstellen kann, die in den eigenen Handlungen dann nicht mehr hergestellt, aber nach der Handlung von der handelnden Person interpretiert werden müssen. Aus diesem Grund konzentriert sich die vorliegende Forschungsarbeit speziell auf Handlungen an digitalen und analogen Materialien, um herauszufinden, ob sich die Handlungen in mathematischen Lernsituationen an Materialien verschiedener Medialität unterscheiden. Es wird der Frage nachgegangen, ob durch mögliche unterschiedliche Handlungen an den verschiedenen Materialien auch unterschiedliche mathematische Deutungen der Lernenden rekonstruiert werden können. Außerdem ist von Interesse, ob das digitale oder analoge Material als Werkzeug fungiert und damit die Handlungen und die mathematischen Deutungen der Lernenden beeinflusst. Ausgehend von dem Forschungsinteresse der vorliegenden Forschungsarbeit werden Handlungen von Dritt- und Viertklässler*innen in digitalen und analogen Lernsituationen analysiert, in denen sie sich mit statistischen und geometrischen Aufgabenstellungen beschäftigen, um ihre mathematischen Deutungen der Diagramme zu rekonstruieren. Diese Deutungen der Lernenden der analog und digital gestalteten Diagramme werden anschließend verglichen, um mögliche Unterschiede und Gemeinsamkeiten zu identifizieren.

Theoretische Ausrichtung der Forschungsarbeit

Wie bereits beschrieben, wird in der vorliegenden Forschungsarbeit eine semiotische Perspektive nach Peirce auf Mathematiklernen eingenommen, die es ermöglicht, den Handlungen selbst eine größere Bedeutung für den mathematischen Lernprozess zuzuschreiben. Diagramme, die von Hoffmann (2010) als Systeme von Darstellungen zur Repräsentation von (mathematischen) Relationen beschrieben werden, haben als solche noch keine feste Bedeutung. Diagramme erhalten ihre Bedeutung erst, wenn entsprechend der Relationen, die von den Lernenden gedeutet werden, Aktivitäten (z.B. Handlungen oder Gesten) an ihnen ausgeführt werden (Dörfler, 2006). Das bedeutet, dass erst durch Handlungen an z.B. bestimmten Materialarrangements diese als Diagramme gedeutet werden können und sich die Diagramme je nach Handlung unterscheiden können. Dörfler (2016) gibt in diesem Zusammenhang ein alltägliches Beispiel, bei dem ein und dasselbe Kartenset in verschiedenen Kartenspielen durch den unterschiedlichen Gebrauch verschiedene Bedeutungen haben kann. Dabei ist das Aussehen der Karten (z.B. die Gestaltung des Aussehens des Königs auf den Karten) im Kartenset weniger relevant. Durch den Gebrauch der Karten, der durch die Regeln des Kartenspiels beeinflusst wird, gewinnen die Karten ihre Bedeutung und nicht durch ihr Aussehen (Dörfler, 2015). In diesem Sinne prägt Peirce den Begriff *diagrammatic reasoning*. Dieser Begriff umfasst mehrere Aspekte und enthält die Konstruktion von Diagrammen nach bestimmten Relationen und Regeln, das Experimentieren mit dem Diagramm nach den für die Konstruktion verwendeten Relationen und die Beobachtung der Ergebnisse des Experimentierens, die dann in allgemeinen Aussagen formuliert werden können (vgl. Hoffmann, 2010; Peirce, NEM IV; CP 1.54).

Bisher wurden die Diagramme nach Peirce in den Mittelpunkt der theoretischen Ausführungen dieser Forschungsarbeit gestellt. Aber auch die Peirce'sche Zeichentheorie ist für die Analyse von Handlungen an unterschiedlichen Materialien zur Rekonstruktion der mathematischen Deutungen der Lernenden von Bedeutung. Ein Zeichen, das erst durch dessen Wahrnehmung als Zeichen zum Zeichen wird, besteht im Peirce'schen Sinne aus einem wahrnehmbaren Zeichen, dem *Repäsentamen*, das für etwas steht, sein *Objekt*, und einem Zeichen, das im Kopf der zeichenlesenden Person entsteht, der *Interpretant* (CP 2.228). Der Interpretant kann durch die zeichenlesende Person wiederum als ein wahrnehmbares Zeichen geäußert werden, nämlich als eine Reaktion auf das erste Zeichen, wie etwa eine Handlung, Geste oder lautsprachliche Äußerung (Bakker & Hoffmann, 2005). Darüber hinaus kann nach Peirce ein Zeichen durch seine Beziehung zum Objekt definiert werden, wobei er drei mögliche Beziehungen beschreibt: Ein *Ikon* bezieht sich auf sein Objekt aufgrund seiner Ähnlichkeit zum Objekt, ein *Index* verweist auf das Objekt, ohne es weiter zu beschreiben, und ein *Symbol* bezieht sich auf sein Objekt aufgrund einer allgemeinen Idee, die durch den regelmäßigen Gebrauch ausgedrückt wird (CP 2.247–CP 2.249). Ausgehend von der Peirce‘schen Definition eines Zeichens wird ein Diagramm in der Literatur häufig als komplexes Zeichen beschrieben, das einen ikonischen Charakter hat, aber auch aus Symbolen und Indizes besteht (vgl. Wille, 2020). Diese Definition eines Diagramms kann im Einklang mit der oben genannten Definition verwendet werden, denn der ikonische Charakter des Diagramms ergibt sich aus der Tatsache, dass ein Diagramm in erster Linie Relationen darstellt (vgl. Bakker & Hoffmann, 2005).

Werden Handlungen an Diagrammen fokussiert, wird deutlich, dass Lernende die Relationen zwischen den Zeichen deuten müssen, um an einem Diagramm regelgeleitet handeln zu können. Auf diese Weise drücken Handlungen an Diagrammen die Deutung der Relationen zwischen den Zeichen aus, die die Lernenden vornehmen. Die Handlungen der Lernenden können damit als ein Index auf deren Diagrammdeutungen angesehen werden, da die Handlungen auf die zuvor gemachten Deutungen des Diagramms verweisen. Das durch die Handlung erzeugte neue (komplexe) Zeichen kann wiederum als Index auf die zuvor durchgeführte Handlung interpretiert werden (Sinclair & de Freitas, 2014). Das Lernen von Mathematik kann somit als ein Zyklus beschrieben werden: Die Lernenden deuten Relationen zwischen den Zeichen und beachten diese gedeuteten Relationen in ihren Handlungen am Diagramm. In ihren Handlungen können sie dann neue Relationen herstellen, indem sie das Diagramm manipulieren. Die neuen Relationen, die die Lernenden wiederum durch das Ergebnis der Handlung erkennen können, müssen in ihren weiteren Handlungen am neuen Diagramm beachtet werden. Auf diese Weise können sie zu neuen Erkenntnissen gelangen, die ihnen wiederum helfen, im Gebrauch von Diagrammen geübter zu werden.

Wie bereits beschrieben, kann davon ausgegangen werden, dass ein Werkzeug beeinflussen kann, welche Relationen von den Lernenden für ihre Handlungen am Diagramm gedeutet und wie in den Handlungen diese Relationen berücksichtigt werden müssen. Kadunz (2016) beschreibt, dass ein (digitales) Werkzeug die Handlungen verkürzen und diese vollständig von den Relationen zwischen den Zeichen trennen kann. Eine Verkürzung der Handlungen und die Trennung von Handlungen und Relationen zwischen den Zeichen kann dazu führen, dass wenige Relationen für Handlungen an Diagrammen gedeutet und beachtet werden müssen, da das Werkzeug automatisch Relationen herstellt. Kadunz (2016) vergleicht das Zeichnen einer Parallelen mit Stift und Papier mit dem Erstellen einer Parallelen mit dem Programm GeoGebra, das als Werkzeug fungiert: Beim Zeichnen einer Parallelen mit Stift und Papier müssen während des Zeichenvorganges immer wieder die gleichen Relationen zwischen der bereits gezeichneten und der zu zeichnenden Gerade hergestellt werden. Beim Zeichnen muss ständig sichergestellt werden, dass die Punkte auf beiden Geraden den gleichen Abstand zueinander haben. Bevor mit dem Zeichnen begonnen werden kann, muss sich die zeichnende Person im Klaren darüber sein, dass diese Relationen zwischen den Punkten auf den Geraden in ihren Handlungen hergestellt werden muss. Beim Erstellen einer Parallelen in GeoGebra kann mit einem Klick auf einen bestimmten Button und dem Anklicken einer bereits vorhandenen Geraden eine Parallele konstruiert werden. Das Werkzeug stellt automatisch die Relationen zwischen den Geraden her, sodass zwei parallele Geraden auf dem Bildschirm zu sehen sind. Um mit einem Werkzeug Handlungen ausführen zu können, muss die handelnde Person kaum mathematische Relationen im Voraus deuten und solche auch kaum während des Handlungsprozesses beachten. Aber um das auf dem Bildschirm erzeugte geometrische Diagramm als solches erkennen zu können, müssen die Lernenden die vom Werkzeug automatisch erzeugten Relationen neu deuten (vgl. Otte, 2003). Tut die handelnde Person dies nicht, bleiben die geraden Linien zwei voneinander unabhängige Striche.

Forschungsfokus

Wie in der theoretischen Ausführung der vorliegenden Forschungsarbeit beschrieben, müssen die Relationen zwischen den Zeichen von Lernenden gedeutet und berücksichtigt werden, um an dem Diagramm handeln zu können. Da Handlungen nur durch die Deutung von Relationen möglich sind, können Handlungen als Indizes auf die Deutung der Lernenden angesehen werden. Für die vorliegende Forschungsarbeit bedeutet dies, dass durch die Analyse der Handlungen die Diagrammdeutungen der Lernenden rekonstruiert werden können. Indem die Lernenden an dem Diagramm handeln, können sie durch die Beobachtung der Handlungsergebnisse neue Relationen erkennen und so auf der Grundlage einer neuen Deutung des Diagramms weitere Handlungen vornehmen. Eine Analyse der Handlungen kann damit die Rekonstruktion der Diagrammdeutungen der Lernenden in ihrem mathematischen Lernprozess ermöglichen.

Folgt man den Ausführungen von Peirce (NEM IV), Dörfler (2015) und Shapiro (1997) wird deutlich, dass die Materialität keinen Einfluss auf die Relationen zwischen den Teilen eines Diagramms hat. Bezogen auf die vorliegende Forschungsarbeit liegt die Vermutung nahe, dass die Materialität dann auch keinen Einfluss auf die Deutung der Relationen hat, die die Lernenden vornehmen, um am Diagramm handeln zu können. Folglich ist zu erwarten, dass in der Analyse der Handlungen gleiche Diagrammdeutungen rekonstruiert werden können, wenn die Lernenden an gleichen Diagrammen handeln, obwohl diese unterschiedlich medial realisiert wurden. Darüber hinaus kann davon ausgegangen werden, dass unterschiedliche Handlungen, in denen die gleichen Relationen zwischen Zeichen berücksichtigt werden, das Resultat von gleichen Diagrammdeutungen sind. Diese Vermutungen müssen in der vorliegenden Forschungsarbeit empirisch überprüft werden.

Aus der Literatur geht auch hervor, dass das Material einen bedeutenden Einfluss auf die Handlungen und die zu beachtenden Relationen haben kann, wenn es als Werkzeug fungiert. Einige Autoren verweisen darauf, dass Werkzeuge Handlungen verkürzen und sie von den Relationen zwischen Zeichen trennen können, die normalerweise vor der Handlung gedeutet werden müssen (vgl. Kadunz, 2016; Otte, 2003). Mit Bezug auf die theoretischen Ausführungen liegt die Vermutung nahe, dass in der vorliegenden Forschungsarbeit eine Rekonstruktion der Diagrammdeutungen durch die Analyse der Handlungen nicht möglich ist, wenn das Material als Werkzeug fungiert. Möglicherweise nehmen die Lernenden aber im Nachhinein eine Deutung der, von dem Werkzeug automatisch hergestellten, Relationen vor. Auf diese Weise wäre eine Rekonstruktion der Diagrammdeutungen durch nachfolgende Äußerungen (z.B. Gesten oder Lautsprache) der Lernenden möglich.

Auf Grundlage der Motivation und der theoretischen Ausrichtung der Forschungsarbeit lassen sich die folgenden drei Forschungsfragen formulieren:

1. Welche mathematischen Deutungen der Lernenden lassen sich anhand der Handlungen, die an den digital oder analog dargestellten Zeichen durchgeführt werden, rekonstruieren?

2. Inwieweit können mögliche Unterschiede zwischen den rekonstruierten Deutungen der Lernenden auf die unterschiedliche Materialität der Zeichen zurückgeführt werden?

3. Welchen Einfluss hat das Material auf die mathematischen Relationen, die von den Lernenden in ihren Handlungen berücksichtigt werden, um am Diagramm Manipulationen vorzunehmen?

Methodische Ausrichtung der Forschungsarbeit

Methoden der Datenerhebung

Zur Beantwortung der formulierten Forschungsfragen werden in der *MatheMat* Studie insgesamt acht verschiedene Lernsituationen für Dritt- und Viertklässler*innen zu zwei mathematischen Themen (Statistik und Geometrie) entworfen, die einmal mit digitalem und einmal mit analogem Material realisiert werden. Der semiotischen Theorie folgend, dass die mathematischen Relationen von größerer Bedeutung sind als das Aussehen der Zeichen (Dörfler, 2015; Shapiro, 1997), wird versucht, jede Lernsituation so zu gestalten, dass dieselben mathematischen Relationen zwischen den Zeichen einmal durch analoges und einmal durch digitales Material dargestellt werden. Auf diese Weise kann davon ausgegangen werden, dass die Lernenden, die an der gleichen Lernsituation arbeiten, in gleicher Weise mathematisch tätig werden, obwohl sich manche Lernenden mit digitalem und andere mit analogem Material beschäftigen.

Für die digital gestalteten geometrischen Lernsituationen wird die dynamische Geometriesoftware GeoGebra (Hohenwarter, 2001) genutzt. In den Lernsituationen werden Teile dieser Software verwendet und so konfiguriert, dass die Lernenden die Länge von parallelen Seiten bzw. Kanten von rechtwinkligen Figuren bzw. Körpern mit Schiebereglern einstellen können. Für die analog gestalteten geometrischen Lernsituationen wird eine Adaption des OrbiMath Materials (Huber, 1972) verwendet. Die Lernenden können mit Stäbchen unterschiedlicher Länge und rechtwinkligen Eckenverbindungen Figuren bzw. Körper zusammenbauen. Für die digitalen statistischen Lernsituationen wird die dynamische Statistiksoftware TinkerPlots (Konold & Miller, 2011) verwendet. In TinkerPlots sind 14 Datenkarten und ein Plot auf dem Bildschirm zu sehen. Auf den Datenkarten sind jeweils die Daten einer merkmalstragenden Person aufgelistet. Im Plot sind die merkmalstragenden Personen als Punkte dargestellt. Für die Bearbeitung der analog gestalteten statistischen Lernsituationen stehen den Lernenden analoge Datenkarten, Holzwürfel mit den Namen der merkmalstragenden Personen und Klebezettel zur Verfügung. Um das Sortieren der Daten nach mehreren Merkmalen zu erleichtern, stehen den Lernenden ebenfalls ein Quadratraster mit einer bereits eingezeichneten, beschrifteten x- und y-Achse zur Verfügung.

Die in der *MatheMat* Studie entworfenen Lernsituationen wurden als Erhebungsinstrumente eingesetzt. Im Sommer 2019 wurden die acht unterschiedlichen Lernsituationen an zwei deutschen Grundschulen erprobt. Insgesamt haben 16 Lernende der dritten und 16 Lernende der vierten Klasse an der Studie teilgenommen. Die Lernenden arbeiteten immer paarweise an zwei Lernsituationen. Es wurde darauf geachtet, dass jedes Paar einmal an einer digital und einmal an einer analog gestalteten Lernsituation sowie einmal an einer statistischen und einmal an einer geometrischen Aufgabe gearbeitet hat. Auf diese Weise wurden insgesamt 32 Bearbeitungen (jeweils ca. 45 Minuten) mit Videokameras aufgezeichnet. Die zwei Kameras wurden so im Raum positioniert, dass die Aktivität der Lernenden am Material und die gesamte Situation aufgenommen werden konnte. Bei der Bearbeitung der digital gestalteten Lernsituationen wurde auch der Bildschirm aufgezeichnet, so dass die Manipulationen, die die Lernenden im Programm vorgenommen haben, bei späterer Betrachtung leichter nachvollzogen werden können.

Methoden der Datenaufbereitung

Vergleichbare Stellen der videografierten Bearbeitung werden für die Analyse transkribiert. Im semiotischen Sinne werden für die Transkription Stellen ausgesucht, an denen die Lernenden bei der Bearbeitung der gleichen Aufgabe gleichermaßen versiert im Umgang mit dem Material sind und so eine Vergleichbarkeit zwischen dem Gebrauch des digitalen und analogen Materials besteht. Alle Handlungen der Lernenden am Material werden im Transkript detailliert wiedergegeben. Obwohl der Fokus auf Handlungen am Material liegt, sind sprachliche und gestische Äußerungen, die sich auf das digitale oder analoge Material beziehen, wichtig für eine annähernd vollständige Rekonstruktion der Diagrammdeutung der Lernenden. Denn Huth (2022) konnte zeigen, dass Gesten in der mathematischen Interaktion

Manipulationen am Diagramm anzeigen oder selbst diagrammatischen Charakter haben können. Aus diesem Grund werden auch Gesten am Diagramm und gleichzeitig zu Gesten und Handlungen geäußerte Lautsprache von den Lernenden im Transkript detailliert beschrieben. Um die Gesten und Handlungen der Lernenden im Transkript möglichst genau wiedergeben zu können, wird zunächst die Bewegung der Hand und anschließend die mögliche Auswirkung der Bewegung auf das Material beschrieben. Für die Beschreibung der Handlungen am digitalen Material werden in der vorliegenden Forschungsarbeit Teile des Touch Reference Guide (Villamor et al., 2010) verwendet und für die Arbeit mit den Programmen TinkerPlots und GeoGebra angepasst. Eine Anpassung an das jeweilig verwendete Programm ist notwendig, da die gleiche Bewegung über den Bildschirm in verschiedenen Programmen unterschiedliche Manipulationen hervorrufen kann. Beispielsweise kann eine Ziehbewegung in GeoGebra eine Einstellung des Schiebereglers bewirken. In TinkerPlots kann die gleiche Bewegung eine Sortierung nach einem bestimmten Merkmal zur Folge haben. Eine Ziehbewegung über den Bildschirm hat somit eine allgemeine Erscheinung (Andrén, 2010) und kann in vielen Programmen Anwendung finden.

Methoden der Datenanalyse

Für die Rekonstruktion der Diagrammdeutungen der Lernenden auf Grundlage ihrer Handlungen am Material wird eine semiotische Spezifikation von Vogels (2017) vorgenommener Anpassung der Kontextanalyse nach Mayring (2014) verwendet. Die semiotische Spezifikation bedient sich der triadischen Zeichentheorie nach Peirce, wobei vor allem der Interpretant für die Analyse von großer Bedeutung ist. Peirce versteht den Prozess der Interpretation nicht als einheitliches, stringentes Verfahren, sondern unterteilt ihn in verschiedene Aspekte und Phasen, die vollständig oder nur teilweise durchlaufen werden können (Rohr, 1993). Gemäß dieser Sichtweise auf den Prozess der Interpretation unterscheidet Peirce (CP 5.475; CP 5.486) drei unterschiedliche Typen von Interpretanten: den emotionalen, den energetischen und den logischen Interpretanten. Ausgehend von dem logischen Interpretanten wird in der Analyse ein forschungsbasierter Interpretant formuliert. Dieser Interpretant beschreibt eine gewohnheitsmäßige Deutung innerhalb der Community von Experten, hier also von mathematisch versierten Personen, und die daraus resultierende Verwendung eines mathematischen Diagramms, die auf bereits durchgeführten Forschungen basiert. Um die Diagrammdeutung der Lernenden zu rekonstruieren, wird in der Analyse der energetische Interpretant der Lernenden, der sich in Handlungen, Gesten und Lautsprache äußert, mit dem forschungsbasierten Interpretanten kontrastiert. In dieser Gegenüberstellung lassen sich die Diagrammdeutungen der Lernenden beschreiben.

Die Analyse ist so aufgebaut, dass zu Beginn ein energetischer Interpretant eines*einer Lernenden ausgesucht wird und im Verlauf der Analyse immer mehr energetische Interpretanten des*der Lernenden miteinbezogen werden. In Explikation 1 beginnt die Analyse mit einer kleinen Transkriptstelle, in der ein energetischer Interpretant beschrieben wird. Auf Grundlage dieser Transkriptstelle wird der forschungsbasierte Interpretant passend zu dem in der Transkriptstelle Verhandelten formuliert. In der Kontrastierung dieser beiden Interpretanten wird die Diagrammdeutung des*der Lernenden der ausgewählten Stelle beschrieben. Im Verlauf der Analyse (Explikation 2 und 3) werden weitere energetische Interpretanten des*der Lernenden ausgewählt, die ähnlich oder gleich zum ersten energetischen Interpretanten aus Explikation 1 sind, und erneut mit dem forschungsbasierten Interpretanten kontrastiert, um die weiteren Diagrammdeutungen zu beschreiben, die der*die Lernende vornimmt. In Explikation 2 werden energetische Interpretanten aus dem Transkript und in Explikation 3 aus der gesamten videografierten Bearbeitung in die Analyse einbezogen. Diese weiteren ähnlichen oder gleichen energetischen Interpretanten können vor oder nach dem ersten energetischen Interpretanten in der Bearbeitung des*der Lernenden gefunden werden. Durch den Einbezug immer weiterer energetischer Interpretanten und die Kontrastierung mit dem forschungsbasierten Interpretanten kann die annähernd vollständige Diagrammdeutung des*der Lernenden im Zeichenprozess beschrieben werden.

In der vorliegenden Forschungsarbeit werden die Diagrammdeutungen von insgesamt acht Lernenden der dritten und vierten Klasse rekonstruiert, wobei die Diagrammdeutung des Drittklässlers Nils und der Drittklässlerin Li einmal auf Grundlage deren Handlungen an einer geometrischen und einmal an einer statistischen Aufgabe rekonstruiert werden. Durch die Auswahl der Handlungen von Nils und Li an unterschiedlichen Aufgaben und verschieden medial gestalteten Materialien für die Analysen wird das oben beschriebene Erhebungsdesign der *MatheMat* Studie deutlich. Die rekonstruierten Diagrammdeutungen der Lernenden, die an der gleichen Lernsituation, aber mit unterschiedlichem Material gearbeitet haben, werden miteinander verglichen, um mögliche Unterschiede und Gemeinsamkeiten zwischen den Diagrammdeutungen zu identifizieren.

Zentrale Forschungsergebnisse zum Mathematiklernen durch Handlungen an digitalen und analogen Materialien

Zur Beantwortung der Forschungsfragen werden die vier Hauptergebnisse der vorliegenden Forschungsarbeit beschrieben, wobei bei jedem Hauptergebnis alle drei Forschungsfragen beantwortet werden. Die vier Hauptergebnisse werden auf drei Ebenen, der Analyseebene, der theoretischen und der praktischen Ebene, betrachtet und zusammengefasst.

Anhand aller Analysen wird deutlich, dass die Lernenden der dritten und vierten Klasse das analoge oder digitale Materialarrangement als Diagramm deuten. Durch die Analyse ihrer Aktivitäten an den statistischen und geometrischen Diagrammen, die Handlungen am Material, Gesten und lautsprachliche Äußerungen umfassen, wird deutlich, dass die Lernenden Relationen zwischen den Teilen der Diagramme erkennen, diese deuten und für ihre Manipulationen nutzen. Unabhängig davon, ob die Zeichen digital oder analog dargestellt werden, kann festgehalten werden, dass die Lernenden mit den Zeichen Mathematik treiben und sie für mathematische Erkenntnisse nutzen. In Bezug auf die methodische Anpassung, die in der Forschungsarbeit vorgenommen wurde, um Diagrammdeutungen von Lernenden rekonstruieren zu können, kann festgehalten werden, dass die Kontrastierung der energetischen Interpretanten der Lernenden mit dem forschungsbasierten Interpretanten eine Beschreibung der vorgenommenen Deutungen der Lernenden erlaubt. Die Verbindung der Kontextanalyse nach Mayring (2014) bzw. Vogel (2017) und der semiotischen Perspektive auf Mathematiklernen stellt eine neue Methode dar, die es ermöglicht durch die Analyse mathematischer Aktivitäten der Lernenden deren mathematische Deutungen rekonstruieren zu können. Betrachtet man die Ergebnisse der Analysen genauer, so lassen sich Unterschiede und Gemeinsamkeiten feststellen, die möglicherweise auf das Material zurückzuführen sind. Diese Hauptergebnisse werden im Folgenden zusammenfassend dargestellt:

Als *erstes Hauptergebnis* der Analysen kann festgehalten werden, dass die Lernenden trotz unterschiedlicher Handlungen an den digitalen und analogen Materialien die gleichen Relationen zwischen den Teilen der Materialanordnungen herstellen. Die Herstellung der gleichen Relationen ermöglicht die Rekonstruktion der gleichen Diagrammdeutungen. Es kann deswegen davon ausgegangen werden, dass die Lernenden die gleichen mathematischen Erkenntnisse gewinnen, auch wenn sie verschiedene Handlungen an unterschiedlichen Materialien ausführen. Wie in der Literatur beschrieben (vgl., Dörfler, 2015) spielt das Aussehen der Zeichen eine untergeordnete Rolle, da der Gebrauch der Zeichen deren Bedeutung bestimmt. Die Materialität der Zeichen gehört zu den unwesentlichen Merkmalen, da die wesentlichen Merkmale, wie die Relationen zwischen den Zeichen, nicht von der Materialität beeinflusst werden (Peirce, NEM IV). Der Literatur kann hinzugefügt werden, dass nicht nur das Aussehen der Zeichen keinen Einfluss auf die rekonstruierte Diagrammdeutung hat, sondern auch das Aussehen der Handlungen, wenn durch sie die gleichen Relationen zwischen den Teilen des Diagramms hergestellt werden. Das Ergebnis zeigt, dass die in der Literatur beschriebene Austauschbarkeit des Erscheinungsbilds von schriftlichen und materiellen Zeichen auch auf flüchtige Zeichen, wie Handlungen, angewendet werden kann. Denn das Aussehen von Handlungen, worunter in der vorliegenden Forschungsarbeit die haptische Bewegung verstanden wird, hat keinen Einfluss auf die diagrammatische Deutung der

Lernenden und damit auch nicht auf ihre möglichen mathematischen Erkenntnisse, wenn in den Handlungen die gleichen mathematischen Relationen hergestellt werden. Es kann festgehalten werden, dass die Erscheinung eines jeden Zeichens, ob flüchtig oder materiell manifestiert, den Relationen, die bereits bestehen oder zwischen den Zeichen hergestellt werden, untergeordnet ist. In Bezug auf die Praxis im Mathematikunterricht der Grundschule wird deutlich, dass es nicht notwendig ist zu Beginn des mathematischen Lernprozesses mit analogem Material zu arbeiten, um eine haptische Erfahrung zu ermöglichen. Es ist genauso empfehlenswert zu Beginn des mathematischen Lernprozesses mit digitalen Materialien zu arbeiten, wenn die Lernenden in ihren Handlungen alle Relationen zwischen den Teilen des Materialarrangements selbst herstellen müssen und das Material diese nicht automatisch herstellt, ohne dass die Lernenden verstehen können, wie diese Relationen zustande gekommen sind. Nicht die haptischen Erfahrungen, die mit dem Material gemacht werden können, entscheiden darüber, welches Material zuerst eingesetzt werden sollte, sondern die mathematischen Relationen, die die Lernenden mit dem Material herstellen. Aus mathematischer Sicht können die Lernenden die gleichen mathematischen Erkenntnisse gewinnen, auch wenn sie unterschiedliche Handlungen ausführen, in denen sie aber die gleichen mathematischen Relationen herstellen. Mathematisch gesehen hat damit die Wahl des Materials keinen Einfluss auf die mathematischen Deutungen, die die Lernenden vornehmen, sondern auf die mathematischen Relationen, die von den Lernenden in ihren Handlungen am Material beachtet werden müssen. Wenn das Lernziel nicht mathematischer Natur ist, sondern darin besteht, den motorischen Umgang mit einem Material zu erlernen, dann sollte die Wahl auf das Material fallen, dessen Umgang geübt werden soll.

Das *zweite Hauptergebnis* der vorliegenden Forschungsarbeit zeigt, dass anhand der Handlungen an den digitalen und analogen Materialien nicht immer die gleichen Diagrammdeutungen rekonstruiert werden können. Wenn durch eine Handlung ausgelöst automatisch mehrere Relationen zwischen den Zeichen hergestellt werden, dann ist die Rekonstruktion der Diagrammdeutungen der Lernenden anhand der Analyse ihrer Handlungen nur teilweise möglich. Das Material übernimmt die automatische Herstellung von Relationen und die Lernenden müssen diese Relationen in ihren Handlungen am Material nicht selbst herstellen. Dieses Ergebnis lässt sich vor allem bei Handlungen am digitalen Material feststellen. Das digitale Material verhält sich wie ein Werkzeug, indem es automatisch Relationen zwischen den Zeichen herstellt. Dabei hängt es von der Gestaltung der Lernsituation und der Nutzung des Materials ab, wie viele Relationen automatisch vom Werkzeug hergestellt werden. In den Analysen wird deutlich, je mehr Relationen vom digitalen Material automatisch hergestellt werden, desto mehr Relationen müssen die Lernenden anschließend wieder für sich deuten. Denn ohne eine Deutung der neuen Diagramme, die auf dem Bildschirm zu sehen sind, können die Lernenden nicht weiter mathematisch arbeiten. Die Lernenden stellen die Relationen zwischen den Zeichen, die nach der Manipulation des Diagramms sichtbar werden, durch Gesten und lautsprachliche Äußerungen her. Die Art und Weise, wie die Relationen ausgedrückt werden, ändert sich damit. Wenn die Relationen nicht automatisch hergestellt werden, dann müssen sie in den Handlungen der Lernenden hergestellt werden, um das Diagramm manipulieren zu können. Wenn die Relationen automatisch hergestellt werden, müssen sie im Nachhinein interpretiert und in Gesten und Lautsprache verdeutlicht werden. Aus semiotischer Sicht wird deutlich, dass die Zeichen und die Relationen, die zwischen den Zeichen auf dem Bildschirm sichtbar sind, keine Indizes auf die zuvor gemachten Handlungen sind, da in den Handlungen diese sichtbaren Relationen nicht hergestellt wurden. Die Handlungen wiederum sind auch keine Indizes auf die Diagrammdeutungen. Da in den Handlungen keine Relationen hergestellt wurden, kann auch nicht rekonstruiert werden, ob die Lernenden diese Relationen vor ihrer Handlung deuten. Vielmehr müssen die Lernenden die durch das Werkzeug automatisch hergestellten Relationen im Nachhinein deuten. Im Vergleich zur Theorie haben bereits Kadunz (2016) und Otte (2003) darauf hingewiesen, dass ein Werkzeug die Handlungen verkürzen und zu einer Trennung der Relationen im Diagramm und der Aktivität der Lernenden führen kann.

Die Folge ist die anschließende Deutung der automatisch hergestellten Relationen. Über diese Ergebnisse hinaus kann in der vorliegenden Forschungsarbeit bei näherer Betrachtung der Handlungen an den analogen und digitalen Materialien eine Verbindung zum *diagrammatic reasoning* nach Peirce hergestellt werden. Es wird deutlich, dass der Fokus auf das Diagramm beim Handeln mit den verschiedenen Materialien unterschiedlich gesetzt wird. Um mit dem analogen Material mathematisch arbeiten zu können, müssen die Lernenden in ihren Handlungen Relationen zwischen den Teilen der Materialanordnung herstellen. Im semiotischen Sinne liegt der Schwerpunkt also auf der Konstruktion und Manipulation von Diagrammen durch regelgeleitete Handlungen. Wenn die Handlungen und Relationen durch das digitale Material verkürzt werden, findet eine Deutung der Ergebnisse der Manipulationen, d.h. die bereits durch das Werkzeug hergestellten Relationen, durch die Lernenden in Gesten und gesprochener Sprache statt. Der Fokus liegt daher eher auf der Beobachtung der Ergebnisse der Manipulationen am Diagramm. In den Analysen wird deutlich, dass die Verkürzung der Handlungen unterschiedlich sein kann. Werden die Handlungen so verkürzt, dass viele Relationen automatisch hergestellt werden, dann liegt der Fokus stark auf der Beobachtung der Ergebnisse, die durch die Manipulationen sichtbar werden. Wenn die Handlungen nur minimal verkürzt werden und nur wenige Relationen automatisch hergestellt werden, dann liegt der Fokus weiterhin auf der Manipulation des Diagramms durch regelgeleitete Handlungen der Lernenden. In Bezug auf das Lehren und Lernen von Mathematik in der Grundschule zeigt das zweite Hauptergebnis der vorliegenden Forschungsarbeit, dass bei einer Verkürzung der Handlungen am Diagramm durch das Material die Lernenden bereits mit den Handlungen und den in den Handlungen hergestellten Relationen vertraut sein sollten, um die Ergebnisse der vom digitalen Material vorgenommenen Manipulationen des Diagramms richtig deuten zu können. Im Mathematikunterricht sollte zunächst mit Materialien gearbeitet werden, bei denen der Schwerpunkt auf den Manipulationen des Diagramms liegt und die Verwendung des Diagramms geübt werden kann, bevor die Lernenden ein Material verwenden, das als Werkzeug dient. Durch die Verkürzung der Handlungen und Relationen mittels eines Werkzeugs können die Lernenden sich anschließend mehr auf die Formulierung der Resultate der Manipulationen in allgemeinen Aussagen und weniger auf die Herstellung spezifischer Relationen konzentrieren. Hinsichtlich des ersten Hauptergebnisses wird deutlich, dass es nicht automatisch bedeutet, dass die Lernenden zuerst mit analogem Material arbeiten sollten, sondern dass die Lernenden zuerst alle Relationen in ihren Handlungen herstellen sollten. Dies kann sowohl mit digitalen als auch mit analogen Materialien erreicht werden.

Das *dritte Hauptergebnis* der vorliegenden Forschungsarbeit beschreibt das Phänomen, dass die Handlungen der Lernenden im Verlauf der Bearbeitung abnehmen. Die Analysen zeigen, dass ein Grund für die Verkürzung der Handlungen, die von den Lernenden in Zeichen notierten Erkenntnisse sind. Es konnte rekonstruiert werden, dass die Lernenden durch ihre Tätigkeiten am Materialarrangement Erkenntnisse gewinnen, die sie in Zeichen zusammengefasst notieren. Diese Zeichen verwenden sie im weiteren Verlauf der Bearbeitung, indem sie diese erneut deuten, um ihre Aktivitäten am Materialarrangement nicht wiederholen zu müssen und somit verkürzen zu können. Ein zweiter Grund für die Verkürzung der Handlungen ist die von den Lernenden vorgenommene lautsprachliche Formulierung einer Rechnung. Die lautsprachliche Formulierung der Relationen in kompakter Weise führt wiederum zu einer Verkürzung der Handlungen am Diagramm. Es wird deutlich, dass nicht nur das Material, sondern auch die Lernenden selbst die Handlungen verkürzen können und es so zu einer Verlagerung der Produktion der Relationen in den Handlungen zu einer Produktion der Relationen in Gesten und gesprochener Sprache kommt. Der Bezug zur Semiotik zeigt, dass die Schritte des *diagrammatic reasoning* auch beim dritten Hauptergebnis von Bedeutung sind. Mit den Handlungen an der Materialanordnung konstruieren und manipulieren die Lernenden das Diagramm, um mathematische Erkenntnisse zu gewinnen. Diese Ergebnisse der Manipulation, also die gewonnenen Erkenntnisse, halten sie in Zeichen fest und konstruieren gleichzeitig ein weiteres Diagramm. Das durch die notierten Zeichen entstandene

Diagramm können sie wiederum nutzen, um weitere Erkenntnisse zu gewinnen. Sie können die in Zeichen notierten Erkenntnisse aber auch in ihrem weiteren Bearbeitungsprozess verwenden, ohne die Manipulationen am Materialarrangement wiederholen zu müssen. Wenn die Lernenden dies tun, verkürzen sie die Manipulationen und konzentrieren sich mehr auf die Ergebnisse der Manipulation. Auch der zweite identifizierte Grund für die Verkürzung der Handlungen macht deutlich, dass keine Handlungen mehr nötig sind, wenn die Relationen in einer kurzen Rechnung schneller hergestellt werden können. Auch hier werden die Handlungen verkürzt, um sich mehr auf das Ergebnis der Manipulation zu konzentrieren. Diese Ergebnisse ähneln dem zweiten Hauptergebnis, mit dem Unterschied, dass es diesmal nicht das digitale Material ist, das den Fokus auf die Ergebnisse der Manipulation lenkt, sondern die Lernenden selbst. Es zeigt sich, dass je geschickter die Lernenden im Umgang mit den Diagrammen sind, desto mehr konzentrieren sie sich auf die letzten Schritte des *diagrammatic reasoning*. An verschiedenen Stellen wird deutlich, dass sich die Lernenden zunächst auf das Konstruieren und Manipulieren des Diagramms konzentrieren und mit zunehmender Versiertheit im diagrammatischen Arbeiten mehr auf die Ergebnisse der Manipulationen achten. Das dritte Hauptergebnis unterstützt die praktischen Überlegungen für den Mathematikunterricht in der Grundschule, die zuvor für Hauptergebnis zwei formuliert wurden. Da die Lernenden in ihrem mathematischen Bearbeitungsprozess selbst ihre Handlungen und die darin hergestellten Relationen verkürzen, um sich mehr auf die Ergebnisse der Manipulationen konzentrieren zu können, ist die oben vorgeschlagene Materialauswahl für die Lernenden unterstützend. Es hilft den Lernenden, zunächst Materialien zu verwenden, bei denen sie ihren Fokus auf die Herstellung von Relationen in ihren Handlungen legen können und später Materialien zu verwenden, die die Handlungen und Relationen verkürzen, um den Fokus auf die Ergebnisse der Manipulationen legen zu können. Auf diese Weise müssen die Lernenden ihre Handlungen nicht selbst verkürzen, um sich auf die letzten Schritte des *diagrammatic reasoning* zu konzentrieren.

Als *viertes Hauptergebnis* kann an einer Stelle der Analysen gezeigt werden, dass sowohl im digitalen als auch im analogen Material bereits vor der Handlung Relationen implementiert sind. Bei beiden Materialien müssen die Lernenden nicht darauf achten, diese Relationen in ihren Handlungen herzustellen. Beim analogen Material führen die Lernenden zwar Handlungen aus, bei denen sie aber keine Relationen herstellen, weil sie im Material verankert sind. Beim digitalen Material müssen die Lernenden nichts weiter tun, denn auf dem Bildschirm sind die Relationen automatisch sichtbar. Es wird deutlich, dass aus den Handlungen der Lernenden an den unterschiedlichen Materialien nicht rekonstruiert werden kann, ob sie die Relationen deuten können. Obwohl beim analogen Material weitere Handlungen ausgeführt werden, kann die Deutung der Relationen anhand der Handlungen nicht rekonstruiert werden. Es ist davon auszugehen, dass bei Handlungen, bei denen keine weiteren Relationen berücksichtigt werden als die, die bereits im Material vorhanden sind, keine Diagrammdeutungen rekonstruiert werden können. In Bezug auf die semiotische Theorie drücken die Handlungen nicht aus, ob die Lernenden diese Relationen erkannt haben, und deshalb kann die Handlung für die Rekonstruktion der Diagrammdeutung nicht als Index auf die Diagrammdeutung angesehen werden. An dieser Stelle wird deutlich, dass das digitale und analoge Material die Herstellung von Relationen in gleicher Weise verkürzen, da in beiden Materialien die Relationen bereits implementiert sind, im analogen Material aber noch Handlungen vorgenommen werden müssen, die jedoch keine Deutung der Relationen erfordern. In Bezug zur Praxis zeigt das vierte Hauptergebnis, dass für den mathematischen Lernprozess die Handlungen im Mathematikunterricht genauer betrachtet werden müssen, da nicht alle Handlungen gleich sind. Für den mathematischen Lernprozess sind vor allem die Handlungen relevant, bei denen die Lernenden Relationen zwischen den Teilen des Diagramms herstellen. Handlungen können gewiss zu Beginn des Lernprozesses experimentell oder nach einer gewissen Zeit mechanisch sein. Bei diesen Handlungen sind die Relationen zu Beginn nicht zielgerichtet oder nach einer gewissen Zeit nicht mehr im Fokus, aber sie werden vom Lernenden beachtet. Bei Handlungen, bei denen die Relationen bereits im Material verankert sind, müssen die Relationen vom Lernenden nicht explizit beachtet werden. Auf diese Weise werden die motorischen Fähigkeiten in den Vordergrund gestellt und nicht die mathematische Aktivität am Diagramm.

Mögliche, an die Forschungsarbeit anschließende, weiterführende Forschungsuntersuchungen

Ausgehend von den vier Hauptergebnissen der vorliegenden Forschungsarbeit können mögliche Anschlussforschungen identifiziert werden. In nachfolgenden Untersuchungen könnte die Forschungsarbeit auf andere Altersgruppen von Lernenden, andere Materialien und Aufgaben aus anderen mathematischen Bereichen ausgeweitet werden. Ein möglicher Untersuchungsschwerpunkt in Bezug auf das Material könnte die Frage sein, ob andere digitale Werkzeuge Handlungen auf die gleiche oder eine andere Weise verkürzen. Es wäre auch interessant zu untersuchen, ob sich Unterschiede zwischen Materialien zeigen, bei denen die Lernenden viele oder wenige diagrammatische Vorgaben haben. In der vorliegenden Forschungsarbeit waren viele diagrammatische Relationen bereits im Material und in den Aufgaben vorgegeben, die die Lernenden erkennen, untersuchen oder herstellen sollten. Es wäre aufschlussreich zu untersuchen, ob die gleichen Ergebnisse zu beobachten sind, wenn die Lernenden freier in ihrer diagrammatischen Arbeit sind. Es könnte auch untersucht werden, ob jüngere oder ältere Kinder die Materialien auf die gleiche Weise handhaben. Zum Beispiel könnte es sein, dass jüngere Lernende, die weniger Erfahrung mit mathematischen Diagrammen haben, ihre Handlungen später im Bearbeitungsprozess verkürzen als Lernende, die älter sind und bereits den Umgang mit verschiedenen mathematischen Diagrammen geübt haben.

Betrachtet man die Veränderung, wie die Lernenden die Relationen zwischen den Teilen des Diagramms während des Bearbeitungsprozesses und durch das digitale Werkzeug ausdrücken, wären weitere Forschungen interessant, die den Wechsel von Handlungen zu Gesten oder von Handlungen zu lautsprachlichen Äußerungen untersuchen. Erste Forschungen in diesem Bereich sind bereits durchgeführt worden. Zum Beispiel identifizieren Vogel und Huth (2020) in ihrer Forschung verschiedene Schnittstellen zwischen Handlungen und Gesten. Billion und Huth (2023) untersuchen, wie mathematische Relationen in Handlungen, Gesten und Sprache von Lernenden im Kindergartenalter ausgedrückt werden und wie insbesondere das Zusammenspiel von Gesten und Handlungen beschrieben werden kann. Anknüpfend an diese Forschung wäre es in einem nächsten Schritt fruchtbar, auch digital gestaltete mathematische Lernsituationen in diese Forschung einzubeziehen. Ein Vergleich zwischen dem analysierten Zusammenspiel von Handlungen zu Gesten bzw. Handlungen zu Lautsprache in digitalen und analogen Lernsituationen würde möglicherweise eine weiterführende Beschreibung der Verschiebung der Äußerung der Relationen ermöglichen. Darüber hinaus könnte es möglich sein, Unterschiede und Gemeinsamkeiten darin zu erkennen, wie die Lernenden in manchen Lernsituationen den Wechsel zwischen Handlungen und Gesten bzw. Lautsprache selbst vollziehen und wie dieser Wechsel durch das als Werkzeug fungierende Material erfolgt.

Ausgehend von den Ergebnissen der vorliegenden Forschungsarbeit und den darauf aufbauenden praktischen Überlegungen für den Mathematikunterricht in der Grundschule wäre es lohnenswert, konkrete Konzepte für den Mathematikunterricht mit digitalen und analogen Materialien zu entwickeln. Ähnlich wie Frischemeier (2018) Einsatzmöglichkeiten von Materialien im Statistikunterricht vorschlägt und das Zusammenspiel von digitalen und analogen Materialien begründet, könnte dies unter Berücksichtigung der Ergebnisse dieser Forschungsarbeit auf andere mathematische Bereiche, wie z.B. auf die Geometrie, ausgeweitet werden. Darüber hinaus könnten die Ergebnisse genutzt werden, um Apps oder Programme zu entwerfen, bei denen die Lernenden zu Beginn alle mathematischen Relationen in ihren Handlungen selbst herstellen müssen und diese Handlungen und Relationen dann im Laufe der Bearbeitung mathematischer Aufgaben mit der App oder dem Programm verkürzt werden. Durch eine solche adaptive App oder ein solches Programm könnten die Schritte des *diagrammatic reasoning* mit Hilfe von digitalem Material unterstützt werden.

Literatur

Andrén, M. (2010). *Children's gestures from 18 to 30 month*. Centre for Languages and Literature, Lund University. https://lucris.lub.lu.se/ws/files/3902148/4588138.pdf

Bakker, A., & Hoffmann, M. (2005). Diagrammatic reasoning as the basis for developing concepts: A semiotic analysis of students' learning about statistical distribution. *Educational Studies in Mathematics*, 60, 333–358. https://doi.org/10.1007/s10649-005-5536-8

Billion, L., & Huth, M. (2023). Mathematics in actions and gestures – a young learner's diagrammatic reasoning. *A Mathematics Education Perspective on early Mathematics Learning – POEM 2022*.

Dörfler, W. (2006). Inscriptions as objects of mathematical activities. In J. Maaz, & W. Schlögelmann (Hrsg.), *New Mathematics education research and practice* (S. 97–111). Sense Publishers. https://doi.org/10.1163/9789087903510_011

Dörfler, W. (2015). Abstrakte Objekte in der Mathematik. In G. Kadunz (Hrsg.), *Semiotische Perspektiven auf das Lernen von Mathematik* (S. 33–49). Springer-Verlag. https://doi.org/10.1007/978-3-642-55177-2

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In A. Bikner-Ahsbahs, A. Vohns, R. Bruder, O. Schmitt, & W. Dörfler (Hrsg.), *Theories in and of mathematics education*, ICME-13 Topical Surveys (S. 21–31). Springer. https://doi.org/10.1007/978-3-319-42589-4

Frischemeier, D. (2018). Design, implementation, and evaluation of an instructional Sequence to lead primary school students to comparing groups in statistical projects. In A. Leavy, M. Meletiou-Mavrotheris & E. Paparistodemou (Hrsg.), *Statistics in early childhood and primary education. Supporting early statistical and probabilistic thinking* (S. 217–238). Springer. https://doi.org/10.1007/978-981-13-1044-7_13

Greiffenhagen, C. (2014). The materiality of mathematics: presenting mathematics at the blackboard. *The British Journal of Sociology*, *65*(3), 502–528. https://doi.org/10.1111/1468-4446.12037

Hoffmann, M. (2010). Diagrams as scaffolds for abductive insights. *Proceedings of the twenty-forth AAAI conference on artificial intelligence*, 42–49.

Hohenwarter, M. (2001). *GeoGebra – Dynamic Mathematics for Everyone.* Austria & USA.

Huber, H. (1972). *OrbiMath. Mathematik konstruktiv.* Freiburg im Breisgau: Herder Verlag.

Huth, M. (2022). Handmade diagrams – Learners doing math by using gestures. *12th Congress of the European Society for Research in Mathematics Education (CERME).* Bozen.

Kadunz, G. (2016). Geometry, a means of argumentation. In A. Sáenz-Ludlow, & G. Kadunz (Hrsg.), *Semiotics as a tool for learning mathematics. How to describe the construction, visualisation, and communication of mathematical concepts* (S. 25–42). Sense Publishers. https://doi.org/10.1007/978-94-6300-337-7

Konold, C., & Miller, C. (2011). *TinkerPlots 2.0*. Key Curriculum Press.

Krauthausen, G. (2012). Digitale Medien im Mathematikunterricht der Grundschule. Spektrum. https://doi.org/10.1007/978-3-8274-2277-4

Larkin, K. (2013). Mathematics Education. Is there an App for that? In V. Steinle, L. Ball & C. Bardini (Hrsg.), *Mathematics Education: Yesterday, Today and Tomorrow.* Proceedings of the 36th annual Conference of the Mathematics Education Research Group of Australasia. file:///C:/Users/larab/Downloads/Larkin_MERGA36-2013-1.pdf

Larkin, K., Kortenkamp, U., Ladel, S., & Etzold, H. (2019). Using the ACAT framework to evaluate the design of two geometry apps: An exploratory study. *Digital Experiences in Mathematics Education, 5*(1), 59–92. https://doi.org/10.1007/s40751-018-0045-4

Mayring, Ph. (2014). *Qualitative content analysis: theoretical foundation, basic procedures and software solutions.* Klagenfurt. https://nbn-resolving.org/urn:nbn:de:0168-ssoar-395173

Moreno-Armella, L., & Sriraman, B. (2010). Symbols and mediation in mathematics education. In B. Sriraman, & L. English (Hrsg.), *Theories of mathematics education* (S. 213–232). Springer. https://doi.org/10.1007/978-3-642-00742-2_22

Otte, M. (2003). Does mathematics have objects? In what sense? *Synthese*, *134*(1/2), 181–216. https://www.jstor.org/stable/20117330

Peirce, Ch. S. (CP). *Collected Papers of Charles Sanders Peirce* (Volumes I-VI, ed. by C. Hartshorne, & P. Weiss, 1931-1935, Volumes VII-VIII, ed. by A. W. Burks, 1958). Harvard UP.

Peirce, C. S. (1976). *The new elements of mathematics* (NEM), vol. IV (Ed. C. Eisele). De Gruyter. https://doi.org/10.1515/9783110805888.313

Radford, L. (2019). On the epistemology of the theory of objectification. In U. T. Jankvist, M. V. D. Heuvel-Panhuizen, & M. Veldhuis (Hrsg.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (CERME11) (S. 3062–3069). ERME.

Rink, R., & Walter, D. (2020). Digitale Medien im Matheunterricht – Ideen für die Grundschule. Cornelsen.

Rohr, S. (1993). *Über die Schönheit des Findens. Die Binnenstruktur menschlichen Verstehens nach Charles S. Peirce: Abduktionslogik und Kreativität.* Springer.

Schreiber, C. (2013). Semiotic processes in chat-based problem-solving situations. *Educational Studies Mathematics, 82*(1), 51–73. https://doi.org/10.1007/s10649-012-9417-7

Shapiro, S. (1997). *Philosophy of mathematics: Structure and ontology.* Oxford University Press

Sinclair, N., & de Freitas, E. (2014). The haptic nature of gesture: Rethinking gesture with new multitouch digital technologies. *Gesture, 14*(3), 351–374. https://doi.org/10.1075/gest.14.3.04sin

Villamor, G., Willis, D., & Wroblewski, L. (2010). *Touch gesture reference guide*. Retrieved from https://www.lukew.com/ff/entry.asp?1071

Vogel, R. (2017). "wenn man da von oben guckt sieht das aus als ob…" – Die 'Dimensionslücke' zwischen zweidimensionaler Darstellung dreidimensionaler Objekte im multimodalen Austausch. In M. Beck, & R. Vogel (Hrsg.), *Geometrische Aktivitäten und Gespräche von Kindern im Blick qualitativen Forschens Mehrperspektivische Ergebnisse aus den Projekten erStMaL und MaKreKi* (S. 61–75). Waxmann-Verlag.

Vogel, R., & Huth, M. (2020). Modusschnittstellen in mathematischen Lernprozessen. Handlungen am Material und Gesten als diagrammatische Tätigkeit. In G. Kadunz (Hrsg.), *Zeichen und Sprache im Mathematikunterricht – Semiotik in Theorie und Praxis* (S. 215–255). Springer Spektrum. https://doi.org/10.1007/978-3-662-61194-4

Walter, D. (2018). Nutzungsweisen bei der Verwendung von Tablet-Apps: Eine Untersuchung bei zählend rechnenden Lernenden zu Beginn des zweiten Schuljahres. Springer. https://doi.org/10.1007/978-3-658-19067-5

Wille, A. (2020). Mathematische Gebärden in Österreichischen Gebärdensprache aus semiotischer Sicht. In G. Kadunz (Hrsg.), *Zeichen und Sprache im Mathematikunterricht – Semiotik in Theorie und Praxis* (S. 193–214). Springer Spektrum. https://doi.org/10.1007/978-3-662-61194-4_9

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Billion, L. K. (2021). Reconstruction of the Interpretation of Geometric Diagrams of Primary School Children based on Actions on Various Materials – A Semiotic Perspective on Actions. *International Electronic Journal of Mathematics Education, 16*(3), em0650. https://doi.org/10.29333/iejme/11068
Billion, L. K. (2022) Semiotic Analyses of Actions on Digital and Analogue Material when Sorting Data in Primary School. *Eurasia Journal of Mathematics, Science and Technology Education, 18*(7), em2126. https://doi.org/10.29333/ejmste/12138
Billion, L. K. (2023). A Semiotic Perspective on Learning Mathematics with Digital and Analogue Material – Primary School Children Acting on Statistical Diagrams. *Special Issue: Early Statistical and Probabilistic Thinking. Statistics Education Research Journal, 22*(2). https://doi.org/10.52041/serj.v22i2.420

Danksagung

"May some future student go over this ground again, and have the leisure to give his results to the world." (Charles Sanders Peirce, 1891)

Mit diesem Zitat beendet C. S. Peirce (1891) seinen Artikel „The Architecture of Theories" und fordert zukünftige Forschende auf, den von ihm beschrittenen Weg noch einmal zu beschreiten und die Ergebnisse der Forschung der Welt mitzuteilen. Ein solcher Weg kann aber nicht ohne die Unterstützung und Austausch mit der Doktormutter, Kolleginnen und Kollegen und Freunden und Familie beschritten werden. Allen diesen Personen, die mich auf meinem eigenen Forschungsweg begleitet haben, möchte ich von Herzen danken.

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Table of Contents

1 Introduction

In the introduction to this work, the motivation for conducting the *MatheMat* study is explained, insights into the theoretical perspective are given and the guiding research questions are clarified. Finally, the design of the study is briefly presented.

1.1 Motivation and Background of the *MatheMat* Study

Various materials, for example, digital and analogue materials, are of great importance for research in mathematics education (e.g. Krauthausen, 2012; Rink & Walter, 2020; Walter, 2018). In particular, digital material has gained much attention in research in recent years. In their literature review, Larkin et al. (2019) compile research that focuses on the evaluation of apps. In their review, they point out that there is little current research as well as further information available for the determination of the quality of an app and that the information provided by app developers mostly serves as an infomercial (Larkin et al., 2019; Larkin, 2013). Furthermore, most reviews are based on evaluations made by the authors themselves rather than on evaluations conducted by independent reviewers (Larkin et al. 2019). Therefore, Larkin et al. (2019) use Artifact Centric Activity Theory (ACAT) as a framework for app evaluation, which allows for independent evaluation and incorporates the activity of the learners with the app to engage with mathematical content as well as the design of the app and its use in classroom situations.

While the focus of the research is on the evaluation of the material, concrete actions of learners with different materials are only focused in a few research studies. For instance, Greiffenhagen (2014) and Sinclair and de Freitas (2014) argue that research usually focuses on the finished products and, thus, separates these outcomes from the activities that produced them, "[s]uch habits of focus have resulted in our neglect of how the activity of the body and various other material encounters factor in mathematical activity." (Sinclair & de Freitas, 2014, p. 356). Hence, it can be stated that the activity itself needs to be illuminated more in research since it is only through the activity that (mathematical) knowledge can be acquired (e.g. Radford, 2019).

When studies consider the actions on the material, they often do so from a psychological perspective. Actions on the material are used in this manner for developing action-based schemata to help learners apply them to new tasks and situations (Aebli, 1980; Floer, 1993; Lorenz, 1993; Piaget, 1998; Radatz, 1993; Wittmann, 1981). In this way, the action is used to build schemas and is, therefore, a means to an aim. The action itself is less central to the core of learning mathematics.

With a view to mathematics education research, it becomes clear that actions themselves should be researched more intensively. Based on this identified need for research, the study *MatheMat – Mathematical Learning with Materials*, in the context in which this work was written, focuses on actions with different materials. In order to concentrate on actions on the material, a semiotic perspective according to C. S. Peirce (CP 1931-35) on mathematics learning is used. This science-theoretical perspective highlights the activity itself on diagrams as mathematical work and, thus, allows the actions for learning mathematics to be brought to the forefront of interest. Diagrams according to Peirce can be seen as "a representation of relations that is constructed by means of 'system of representation'. Such a system is defined by a set of rules, conversations, and a certain ontology" (Hoffmann, 2010, p. 42). A certain arrangement of materials, but also something written on a piece of paper or the related gestures of the learners, can be such a 'system of representation' (e.g. Döfler, 2006, Schreiber, 2013).

Exploring mathematical relationships by acting on a 'system of representation' made of paper and pencil may differ from one made of digital technology. Moreno-Armella and Sriraman (2010) refer to the fact that a tool has an impact on human action and that mathematical ideas and processes can, therefore, be completely different. Human action changes because a tool can automatically establish (mathematical) relationships that are then no longer established in the own actions but must be interpreted by the acting person after the action. Thus, this work focuses on the actions made on the digital and analogue materials to see whether these actions on digital and analogue materials differ. The question is pursued of whether different mathematical interpretations can be reconstructed through possible different actions on the different materials, and it is also examined whether the digital or analogue material functions as a tool and, thus, influences the actions themselves. To address these research questions, four papers were written analysing the actions of learners in digital and analogue learning situations with statistical and geometrical tasks to reconstruct their mathematical diagram interpretations. These learners' diagram interpretations of the analogue and digital material arrangements were compared to identify possible differences and similarities. In the first paper (chapter 2), parts of the empirical example from the second paper (chapter 3) are used to illustrate the analysis method adapted in a semiotic way. In the second paper (chapter 3), the actions of the third-graders Marleen and Nils working on a geometrical problem are examined. Their task is to investigate the relationship between the perimeter and the area of similar squares. In chapters 4 and 5 two papers are presented in which the learners answer set questions on univariate and bivariate plots, respectively, that they have created themselves; the actions of creating plots from data cards are focused on in the analysis. In order to cover the mathematical learning situation that was investigated in the *MatheMat* study, the examples from the papers are supplemented by another example. In this example, Paul and Daniel examine the surface of different cuboids with a volume of twelve and a height of one. The results of all analyses are summarised and discussed in a concluding review. Implications for mathematics in practice are formulated based on the results and further possible follow-up research is listed.

1.2 Theoretical Perspective of the *MatheMat* Study

As mentioned above, the study uses a semiotic perspective on mathematics learning according to C. S. Peirce which allows the focus to be on the actions themselves. Mathematics learning from this perspective is the "social practice with, on, about, and through diagrams" (Döfler, 2006, p. 314). A diagram is described as a well-defined structure of inscriptions that has the aim of showing (mathematical) relationships (Dörfler, 2006). Inscriptions that are part of diagrams can be seen as a general result of activities (Latour, 2012); lines on paper, on the computer screen, in the sand, or on the blackboard can be understood as inscriptions (e.g. Döfler, 2006, Schreiber, 2013). The term inscriptions can be interpreted even more broadly, however, because the material can also be interpreted as being inscriptions (Gravemeijer, 2002a) and even gestures can have an inscriptional character (Huth, 2022). The characteristics of inscriptions can be found in section 3.2.1 (p. 20).

Diagrams in themselves do not yet have a fixed meaning but only acquire their meaning when activities (for example, actions or gestures) are carried out on them according to the relationships between the inscriptions (Dörfler, 2006). This means that it is only through the action on the inscriptions that diagrams can come into being, and so their meanings also differ due to the different actions. Dörfler (2016) gives an everyday example of this, where the same set of cards can have different roles in different card games through different usage. In this sense, Peirce (e.g. NEM IV) coins the term 'diagrammatic reasoning'. The definition of the term and the activities that Peirce includes in this term can be found in section 4.2.1. (pp. 35-36). Through the actions made on diagrams, new diagrams can be designed, and relationships can be represented in different ways (for example, geometrically or arithmetically). In section 3.2.1 (p. 20) an example is given of learners focusing on new relationships by transferring relationships from a geometrical to an arithmetical diagram.

Up to now, diagrams according to Peirce have been the focus of the *MatheMat* study, however, the Peircean sign theory is also of great importance here. A sign in the Peircean sense consists of a perceptible sign, the representamen, which stands for something, its object, and a sign that arises in the mind of the sign reader, the interpretant (CP 2.228). The interpretant, in turn, can be uttered by the sign reader as a perceptible sign, namely a reaction to the sign such as an action, gesture, or spoken language (Bakker & Hoffmann, 2005). More detailed descriptions of the structure of a sign according to Peirce can be found in section 2.2.2 (p. 12). Furthermore, according to Peirce, a sign is defined by its relationship to its object. He lists three possible relationships: an icon refers to its object because of its similarity, an index refers to the object without describing it further, and a symbol refers to its object because of a general idea expressed through regular usage (CP 2.247–CP 2.249). Section 3.2.2 (pp. 20-21) addresses the fine distinction between an index and a track. Based on the Peircean definition of the sign, a diagram is often described in the literature as a complex sign that has an iconic character but also consists of symbols and indices (e.g. Wille, 2020). This definition of a diagram can be used in harmony with the definition given above because the iconic character of the diagram results from the fact that a diagram primarily represents relationships (Bakker & Hoffmann, 2005).

Focusing on the actions on diagrams, it can be seen that learners need to interpret the relationships between the signs in order to act on a diagram. In this way, the action expresses the learner's interpretation of the diagram, and the action can be seen as an index to this interpretation. The sign produced by the action can, in turn, be interpreted as an index to the action previously performed (Sinclair & de Freitas, 2014). Thus, learning mathematics can be described as a cycle in which learners interpret relationships between the signs and observe them in their actions. In their actions, they may then establish new relationships by transforming the diagram, which they can observe again through the result of the action. In this way, they can arrive at new insights, which, in turn, help them to become more adept in their actions on the diagram.

Since learners mainly have to pay attention to and interpret the relationships between the signs, the appearance of signs plays a less important role in learning mathematics (Dörfler, 2015; Shapiro, 1997). A similar formulation can be found in Peirce (NEM IV):

> "One contemplates the Diagram, and one at once prescinds from the accidental characters that have no significance. […] one can contemplate the Diagram and perceive that it has certain features which would always belong to it however its insignificant features might be changed. What is true of the geometrical diagram drawn on paper would be equally true of the same Diagram when put on the blackboard." (p. 317).

Therefore, the nature of the signs is irrelevant to the relationships between the signs in the diagram. The card set can again be used as an example: the cards only acquire their meaning through their usage in the game. Thus, it is not the appearance of the cards that determines their meaning but their use in the game. Although appearance has no influence, a tool can influence which relationships need to be interpreted by the learner for action and how these are taken into account in the actions. A (digital) tool can shorten the actions and separate them from the relationships between the signs (Kadunz, 2016). If the actions are abbreviated and the actions and relationships are separated, no relationships need to be interpreted and observed for the actions since the tool automatically creates the relationships. Kadunz (2016) makes the comparison of drawing parallel lines with making the parallel lines in GeoGebra: when drawing a parallel line, one must repeatedly establish the relationships between the straight line already drawn and the straight line to be drawn during the drawing process. When drawing parallel lines, one has constantly to make sure that the points on the two straight lines have the same distance from each other. Thus, before one can start drawing, one must be aware that this relationship must be established. When creating a parallel line in GeoGebra, one can simply click on a button and click on the respective straight line to which a parallel is to be created. The tool then automatically creates the relationships between the straight lines so that two parallel straight lines are visible on the screen. To be able to act with the tool, barely any relationships need to be interpreted in advance and, likewise, barely any relationships need to be observed during the process of the action. In order to recognise the geometrical diagram created on the screen, as such, the learners have to re-interpret the relationships created by the tool (e.g. Otte, 2003); if the learners do not do so, the straight lines remain two inscriptions independent of each other.

1.3 Research Focus

As described in the theoretical perspective of the *MatheMat* study, relationships between the signs need to be interpreted and taken into account by the learners in order to execute actions on the diagrams. Since actions on a diagram are only possible through an interpretation of relationships, the actions are an index to the interpretations made by the learners. For the *MatheMat* study, it can thus be assumed that by analysing the actions, one can reconstruct the diagram interpretations made by the learners. By acting on the diagram, the learners can recognise new relationships by observing the results of the action and, therefore, further actions can be made based on the new interpretations. An analysis of actions can thus enable the reconstruction of the learner's diagram interpretation in their mathematical learning process. By delving into the literature, it can be seen that gestures can also indicate manipulations of the diagram (Huth, 2022). In order to show manipulations in gestures, learners must previously perform interpretations, as with actions, and gesture out based on the interpreted relationships. Thus, gestures on the diagram are also important for the *MatheMat* study in order to make an approximate complete reconstruction of the diagram interpretation.

Following Peirce (NEM IV), Dörfler (2015) and Shapiro (1997), it is evident that the materiality of the signs has no effect on the relationships between the signs (for an appropriate quote from Peirce see p. 4). Therefore, if the materiality of the signs does not influence the relationships between the signs, then this also does not influence the interpretation of the relationships that the learners make in order to be able to act on a diagram. Consequently, in the *MatheMat* study, it can be assumed that the same diagram interpretations of the learners are reconstructed, even if the materiality of the signs differs. Furthermore, it can be assumed that different actions that take into account the same relationships between the signs are the result of the same interpretation. These assumptions for the *MatheMat* study need to be empirically verified in this work.

However, it is also clear from the literature that the type of material can have a massive influence on the actions when it functions as a tool. Many authors refer to the fact that tools can shorten actions and separate them from the relationships that need to be interpreted beforehand (e.g. Kadunz, 2016; Otte, 2003). For example, if a tool separates the relationships between the signs from the actions, then one action can be done and the tool automatically takes into account the relationships that were previously implemented in the tool. In order to be able to perform actions on the tools, it is not necessary to interpret all the relationships between the signs since an action can be performed which the tool automatically translates into the desired manipulation of the diagram. Therefore, if only a few or even no relationships between the signs are required to be interpreted to perform the actions, it follows that a complete or partial diagram interpretation cannot be reconstructed based on these actions. Otte (2003) refers to the fact that the relationships automatically created by the tool have to be interpreted afterwards. In relation to the *MatheMat* study, it can, therefore, be assumed that a material that functions as a tool makes the reconstruction of the diagram interpretation by the actions impossible. However, it could be possible that the learner interprets afterwards that can be reconstructed through the analysis of subsequent actions, speech and gestures.

Furthermore, it can be surmised that when the action has been separated from the relationships between the signs then a completely different function may occur; for example, a drag movement in a digital learning situation may no longer have the function of moving an item from one place to another but may convey a completely different function depending on the software (e.g. Arzarello et al., 2002). A drag movement across the screen has become very general and depends strongly on the software in which it is applied. Therefore, in the *MatheMat* study, this means that the same action, i.e. a drag movement across the screen, can lead to the reconstruction of different diagram interpretations.

The research questions of the study can be formulated from the theoretical considerations above. These research questions are addressed in the papers in sections 2.3, p. 13; 3.6.1, pp. 26-27; 4.2.4.2, p. 38; 5.3.1, pp. 53-54 but always with slightly different wording and answered in relation to the respective example of the paper.

1. Which mathematical interpretations of the learners can be reconstructed based on the actions performed on the digitally- or analogue-represented signs?
2. To what extent can possible differences between the reconstructed interpretations of the learners be attributed to the various materiality of the signs?
3. What influence does the material have on the relationships considered in the learners' actions to manipulate the diagram?

1.4 Research Design and Methodological Considerations

In the following, the design of the study and the data gathering are discussed. Afterwards, it will be described how the data were selected for transcription, how the actions on the digital and analogue material were reproduced in the transcript, and, finally, the semiotic adaptation of the context analysis is addressed.

1.4.1 Design of the Study and Data Collection

To answer the research questions formulated, learning situations were designed in the *MatheMat* study for two mathematical topics (statistics and geometry) which were realised once with digital and once with analogue material. Following the semiotic theory that mathematical relationships are of greater importance than the appearance of the signs (Dörfler, 2015; Shapiro, 1997), an attempt was made to design the learning situations with the same theme in such a way that the same mathematical relationships between the signs were represented by different materiality. In this way, it can be assumed that the learners would do the same mathematics in the learning situations with the same topic, although they would be working once with the digital and once with the analogue material. This would ensure comparability between the digital and analogue material. A total of four learning situations with geometrical topics (two for third- and fourth-graders each) and four learning situations with statistical topics (two for third- and fourth-graders each) were designed which were then realised once with digital and once with analogue material.

	Statistical learning situations		Geometrical learning situations	
Third-grade class	Answering set questions about a univariate plot of nominal and ordinal data	Answering set questions about a bivariate plot of nominal and ordinal data	Investigating the relationship between the area and perimeter of similar squares	Investigating quadrilaterals with a constant area and different perimeters
Fourth-grade class	Answering set questions about a univariate plot of metric data	Answering set questions about a bivariate plot of metric data	Investigating the relationship between the volume and surface area of similar cubes	Investigating cuboids with a constant volume and various large surface areas

Table 1. Learning situations designed in the *MatheMat* study

In the selection of the learning situations considered in the individual chapters of this work, care was taken to ensure that each topic occurred once (see blue highlighting in Tab. 1). Thus, in chapters 4 and 5, the actions of third-graders on digital and analogue material were analysed as they dealt with univariate and bivariate plots. In chapter 3, the focus is on the actions of third-graders on digital and analogue materials in which they established the relationship between the area and perimeter of similar squares. For completeness, the last chapter of this work analyses the actions of fourth-graders on digital and analogue materials in which they examined cuboids with the same volume but different surface areas. In this way, each topic of the learning situations that either the third- or fourth-graders worked on was considered.

Each of the eight different learning situations (see Tab. 1) was realised once with digital and once with analogue material. The dynamic geometry software GeoGebra (Hohenwarter, 2001) was used for digital geometrical learning situations. In the learning situations for the third-graders, parts of this software were used and configured so that the learners could adjust the length of two parallel sides of a rectangular quadrilateral with two scrollbars. Furthermore, in GeoGebra a unit square is provided to help the learner determine the area of the quadrilateral in unit squares; the square grid that can be seen in the background is composed of smaller squares the size of this unit square (see sections 2.4.1, p. 14 and 3.5, p. 24). In the learning situations for the fourth-graders, the learners could adjust three scrollbars to change the height, breadth and length of the prisms with a rectangular quadrilateral base (see section 6.1.2, p. 80).

A further example of such settings in GeoGebra can be found in Billion & Vogel (2020). The analogue geometrical learning situations were realised with an adaptation of the OrbiMath material (Huber, 1972). The third-grade learners were provided with a unit square and an analogue square grid, squared in the size of the unit square. In addition, they could use rods of different lengths (the lengths of the rods were multiples of the length of the side of the unit square) and right-angled corner connections to put together squares of different sizes (see sections 2.4.1, p. 14 and 3.5, p. 24). The fourth-graders had a unit cube instead of a unit square and the corner connections allowed them to put the rods together to form geometrical solids (for the material see section 6.1.1, p. 69; Billion & Vogel, 2021).

For the digital statistical learning situations, the dynamic statistical software TinkerPlots (Konold & Miller, 2011) was used. For all the digital statistical learning situations for grades three and four, TinkerPlots presented data cards displaying the data of 14 children. There was also a plot open in the software showing 14 dots for the cases on the data cards (see sections 4.4.1, pp. 40-41 and 5.5.1, p. 57). The learning situations for the third- and fourth-graders only differed in the quality of the data shown on the data cards. The third-graders worked with ordinal and nominal scaled data, while the fourth-graders worked with metric data. Learners who were working on analogue statistical learning situations were provided with analogue data cards, wooden cubes with the names of the cases, and sticky notes (see section 4.4.1, p. 41). As with the digital learning situations, the learning situations of the third- and fourth-graders differed in the quality of the data on the data cards. The learners who worked with bivariate plots to answer set questions also had a square grid with an x- and y-axis available (see section 5.5.1, p. 56). Section 4.4.1 (p. 41) describes in detail, from a semiotic perspective, why the analogue material of the statistical learning situations was chosen in this way. In addition, in section 5.5.2 (pp. 57-58) an attempt was made to link the theory of emergent models (Gravemeijer, 1999; 2002b) with the plot and the data cards (which could be interpreted as a diagram in the semiotic sense).

The designed learning situations were used as data collection instruments in the study. In the summer of 2019, the learning situations with eight different investigation tasks were carried out at two German primary schools. A total of 16 learners from grade three and 16 learners from grade four participated in the study. The learners always worked together in pairs on a learning situation whereby each pair worked once on a digital and once on an analogue learning situation, and once on a geometrical and once on a statistical learning situation. This can easily be seen by comparing chapters 3 and 5. In section 3.6.1 (pp. 26-30), Nils works with his partner on a geometrical learning situation that was realised with digital material, whereas in section 5.5.3 (pp. 58-60), Nils works on a statistical learning situation designed with analogue material. In this way, a total of 32 processes of the learning situations (approximately 45 minutes each) were recorded with video cameras. One of the cameras focused on the learner's activity on the material, and the other recorded the whole situation. In the digital learning situation, the screen was also recorded so that the manipulations in the software could be easily followed.

In the digital and analogue learning situations, the learners worked with prompts. The prompts encourage the learners to engage with the mathematical content. An example of a geometric prompt, where the perimeter of similar squares has to be noted, can be seen in section 3.5.2. (p. 26). In all learning situations, the different prompts were spread out on the table so the learners had the possibility to design their working process freely, depending on which prompt they chose to work on first. The principles according to which the prompts were designed can be found in section 3.4.1 (p. 22).

1.4.2 Data Preparation

In order to find comparable passages in the video recordings that were suitable for transcription, the processing of the different prompts was shown graphically (see section 3.5.1, p. 25). In this way, it was possible to identify when the learners using the digital material and those using the analogue material had worked on the same prompt at the same point in their processing. Such points are suitable for transcription because, in a semiotic sense, one can assume that the learners are equally adept in using the material when they work on this prompt.

All the actions of the learners on the material that occurred in the passages selected for transcription were reproduced in detail in the transcript. Although the focus was on the actions, the spoken language and gestures were also important for an approximate complete reconstruction of the learner's diagrammatic interpretations. For example, Huth (2022) was able to show that gestures can also indicate manipulations on the diagram or become diagrams themselves in mathematical interactions. For this reason, all gestures and all spoken utterances in this study have been reproduced in the transcript. For clarity in the transcript, the learners' processing was divided into scenes. The time at which the scene began and the numbering of the scene are located on the far left of the first line (see section 3.6.1, p. 27). To the right is the name of the learner whose actions, gestures and spoken language are reproduced. In a scene, all spoken utterances were marked with letters, and all the gestural utterances and actions were numbered. When numbering the actions and gestures, the respective letter of an utterance could be noted to make clear which phonetic utterances occurred simultaneously with these actions or gestures (see section 3.6.1, p. 27).

To be able to depict the gestures and actions in the transcript as accurately as possible, first, the movement of the hand was described, and then the possible effect on the material. For the description of the actions on the digital material, excerpts of the Touch Reference Guide (Villamor et al., 2010) were used and adapted for the learning situations in which the learners worked with TinkerPlots or GeoGebra. An adjustment for the respective software was necessary as the identical movement across the screen can cause different manipulations in different software. For example, a drag movement in GeoGebra can cause an adjustment of the scrollbar, while in TinkerPlots it can result in sorting according to a certain attribute. Thus, such a drag movement across the screen has a generic appearance (Andrén, 2010) and can be applied in different software. For the statistical learning situations, the movements across the screen and their manipulations in TinkerPlots are listed in sections 4.3.2 (p. 39) and 5.4.2 (pp. 54-55). The movements across the screen and their manipulations in GeoGebra that are important for the geometrical learning situations can be found in section 3.4.2 (p. 23).

1.4.3 Data Analysis

For the reconstruction of the learners' diagram interpretations based on their actions, a semiotic specification of Vogel's (2017) adaptation of context analysis (or explication analysis) according to Mayring (2014) was used. For more information about the context analysis of Vogel (2017) and Mayring (2014) see sections 3.4.3 (pp. 23-24), 4.3.3 (pp. 39-40), and 5.4.3 (p. 55). The semiotic specification uses the theoretical background of the interpretant, according to the Peircean sign theory. Peirce does not regard the process of interpretation as a uniform, stringent procedure, but rather divides it into different aspects and stages that can be carried out completely or only partially (Rohr, 1993). In this way, Peirce (CP 5.475; CP 5.486) distinguishes three different types of interpretants: the emotional, the energetic and the logical interpretant. What exactly he understands by the different types of interpretants can be read in section 2.2.2 (pp. 12-13). Based on the logical interpretant, a research-based interpretant is formulated in the analysis. This research-based interpretant describes the habitual use of a mathematical diagram based on research that has already been done. In order to describe the learner's interpretations of the diagram in the analysis, the learner's energetic interpretant, such as actions or gestures made in the processing of the learning situation, is contrasted with the interpretant based on research. Further theoretical explanations of the semiotic adaptations of context analysis are described in section 2.2.2 (p. 12-13).

The analysis is structured in such a way that increasingly more energetic interpretants of the learner are included during the analysis. In Explication 1, a small transcript passage that describes a learner's energetic interpretant is chosen to begin the analysis. Then the research-based interpretant of this transcript passage is formulated and compared with the transcript passage. In the comparison, the learner's diagram interpretation is described. In the further course of the analysis (Explication 2 and 3), other energetic interpretants of the learner are selected that are similar or equal to the first energetic interpretant. In Explication 2, energetic interpretants from the transcript are included in the analysis, while in Explication 3 the entire videographed processing is included in the analysis. These further similar or equal energetic interpretants can be found before or after the first energetic interpretant in the learner's processing. All found interpretants of the learner are again compared with the research-based interpretant to describe the learner's diagram interpretation in the further sign process. For the exact sequence of the analysis steps, see section 3.4.3 (pp. 23-24).

In the first paper (chapter 2), the analysis method adapted in a semiotic sense is explained using extracts from the empirical examples in paper 2 (chapter 3). In the following papers (chapters 3-5), the diagram interpretations of a total of six learners were reconstructed, whereby Nils's and Li's diagram interpretation was reconstructed once when dealing with a geometrical learning situation and once with a statistical learning situation. In the papers, the reconstructed diagram interpretations of the learners who worked with digital material were compared with those who worked with analogue material. In this way, the research questions of the study were answered for every example. For full consideration of the data from the *MatheMat* study, the diagram interpretations of two other learners were reconstructed in chapter 6. These two fourth-graders were engaged in a geometrical problem in which they considered the surface areas of different cuboids with the same volumes. The results from all ten analyses are subsequently summarised and related to the theory to formulate implications for mathematics in practice and possible follow-up research investigations.

THE RECONSTRUCTION OF MATHEMATICAL INTERPRETATIONS – ACTIONS OF PRIMARY SCHOOL CHILDREN ON DIGITAL AND ANALOGUE MATERIAL

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This paper presents the results of the qualitative study MatheMat. It aims to analyse primary school children's actions on comparable digital and analogue materials to reconstruct their mathematical interpretation from a semiotic perspective on learning mathematics. For this purpose, a semiotic adapted qualitative analysis is applied to analyse the actions of two third graders in a geometrical learning situation to reconstruct their interpretation of the diagram realised with various materials. The comparison of the results shows that sometimes learners make the same interpretations of the digital and analogue material arrangements despite different actions because they recognise and interpret the same mathematical relationships as relevant to their actions.

MATHEMATICS LEARNING WITH DIGITAL AND ANALOGUE MATERIAL

Material is of great importance for mathematical learning processes and is often investigated in mathematics education research. However, the focus is often on the digital or analogue material itself rather than what learners do with it. For example, Larkin et al. (2019) use the Artifact Centric Activity Theory (ACAT) for evaluating digital materials to help teachers navigate the wide range of digital materials and understand their potential for mathematical learning. Such an instrumental approach, "[...] which is often used for research on technology learning settings, fails to attain insight into the epistemic process in all its aspects." (Behrens & Bikner-Ahsbahs, 2017, p. 2721). Therefore, the paper focuses on the results of the qualitative study *MatheMat – Mathematical Learning with Materials*, which aims to investigate learners' actions and their usage of the material. For the empirical investigation of learners' interpretations as they act on various materials a semiotic perspective on mathematical learning according to C. S. Peirce (1931-35) is adopted, which defines the actions on *diagrams* as the core of doing math (Dörfler, 2006). Specifically, this paper considers two cases in which the actions of two third-grade learners (9-year-olds) on comparable digital and analogue diagrams to solve a geometrical problem are analysed in order to reconstruct their mathematical interpretations. These interpretations are then compared to identify whether the learners make the same diagram interpretations in similarly designed digital and analogue learning situations and, thus, gain the same mathematical insights. The results of the comparison can be used to exploit the possibilities of the different materials for practice in mathematics teaching.

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 123-130). PME 46.

THEORETICAL FRAMEWORK

Semiotic Perspective on Learning Mathematics

Diagrams can be considered as relationally connected signs, namely, complex signs whose main function is to represent relationships (e.g. Wille, 2020). By this definition, a diagram is manifold: it can be a table, a function graph, a geometrical drawing, an arrangement of materials (digital or analogue), or an argument, as long as it is about the representation of the relationships. However, diagrams do not have a fixed reference that determines their meaning or significance (Dörfler, 2006). Instead, the activities on the complex signs ground them and make them meaningful (Roth & Bowen, 2001). According to Peirce (NEM IV), a diagram has certain features that would always belong to it, even if its non-essential features could be changed.

To illustrate this, Dörfler (2016) makes an everyday example: the same card can have different meanings due to the different activities in various card games. Thus, the meaning of a card is inseparable from its use and the rules and relationships that are recognised and established in the actions. If these are changed, the meaning of the cards also changes as they are among the essential features of the diagram. The cards' appearance, however, can be changed and is one of the non-essential features. In this sense, recognising and observing the relationships between the signs are constitutive of the activities on the diagram and require interpretations by the actor.

Mathematical Interpretation from a Semiotic Point of View

In order to be able to examine learners' interpretation of complex signs more precisely, it is useful to consider the Peircean definition of a sign in more detail. Here, a sign is something that stands for someone in some respect or quality and consists of a triadic structure that includes the representamen, the object, and the interpretant of the sign (CP 2.228). By using the word representamen, Peirce means the outwardly perceptible sign that stands for something meant; he calls it its object. The perceptible sign triggers in the mind of the sign reader an interpretation called the interpretant. This interpretant can be an "equivalent sign, or perhaps a more developed sign" (CP 2.228) of the perceptible sign. Looking at communication (with oneself and others), the sign reader's interpretant can be expressed in a reaction to the sign, a new representamen, and thus there is a continuous translation into new signs (Maffia & Maracci, 2019). The new representamen produced by the sign reader in communication can be, for instance, an action, a gesture, or a phonetic utterance.

Peirce distinguishes between three different types of interpretants: an emotional, an energetic, and a logical interpretant. He describes the *emotional interpretant* as the first effect evoked by a sign and describes this effect as a feeling (CP 5.475). It arises in the sign-reading person but does not have to be expressed as a perceptible sign. The *energetic interpretant*, on the other hand, can be seen as a spontaneous action that the sign-reader performs as an effect on the sign, which involves an effort on the part of the sign-reading person; this effort can be physical as well as mental (CP 5.475). However, the energetic interpretant is not an action that the sign-reading person has already repeated many times and that has become a habit with a specific goal. When it becomes habitual it is called the *logical interpretant* (CP 5.486).

The description of the different interpretants is important for the analysis of the data described in this paper, as they are used to reconstruct the mathematical interpretations that the learners make during their actions on the digital and analogue material. Based on the logical interpretant, an interpretant based on research is formulated, which can be described as the habitual reaction of experts in a community to a complex sign. The research-based interpretant describes relationships between the signs and the resulting rule-based actions that are necessary to establish these relationships. As Peirce highlights, the action must be described with specifications of the motive (CP 5.491). Therefore, in the description of the research-based interpretant, only the relationships and manipulations that are important regarding the task are dealt with. This research-based interpretant is compared with the learners' energetic interpretant. Through this comparison, it is possible to reconstruct which relationships between the signs the learners may have recognised and used to express their interpretant. In this way, it can be determined whether the learners working with digital material may recognise and focus on different relationships than those working with analogue material, even though they are working on comparable diagrams.

RESEARCH FOCUS

From the theoretical considerations, the objective of the *MatheMat* study is to reconstruct the learners' mathematical interpretation by analysing the actions on comparable diagrams realised with digitally and analogue represented signs. By comparing the reconstructed interpretations, it will be investigated whether the learners make the same interpretations even though they work with different materials. Geometrical and statistical learning situations with digital and analogue material were examined, which were developed especially for this study. The learning situations were designed so that, based on the same mathematical tasks and the same mathematics education considerations, the same mathematical relationships between the different materialised signs could be recognised. Following Dörfler (2016), diagrams that have the same mathematical structure and relationships are expected to enable the same mathematical engagement with them, so that the different materials should not interfere with this. This allows for comparison of material as the learners engage mathematically with the same diagrams of a different materiality.

This paper focuses on a geometrical example, where two third-grade learners investigate the relationship between the area and perimeter of similar squares digitally using GeoGebra (Hohenwarter, 2001) and analogue using an adaptation of the OrbiMath material (Huber, 1972) (see Figure 1). For the analysis, one part of Nils's and Marleen's work on the geometrical problem is considered, in which they create squares of different sizes with the material provided. These squares are the basis for completing various sub-tasks, which they have to solve.

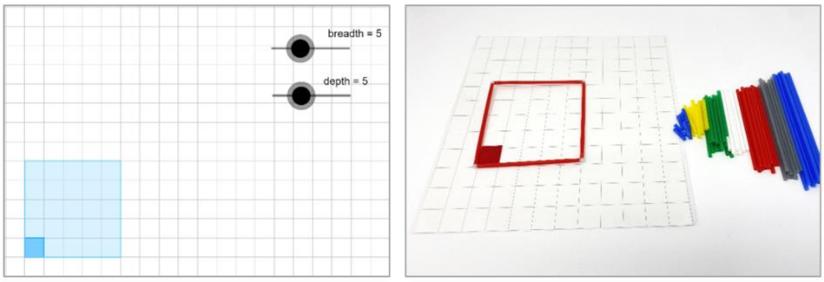
Considering the geometrical example, the following research question is addressed: Which mathematical interpretations of Nils and Marleen can be reconstructed from the actions on the digital or analogue geometrical diagrams, and which possible differences can be described between the reconstructed interpretations of the two learners?

METHOD AND DESIGN

Method of Data Generation

For data collection, material-based interviews were conducted in the summer of 2019 at two German primary schools with 16 learners at the end of grade 4 (10-11 year-olds) and with 16 learners at the beginning of grade 3 (8-9 year-olds). Each learner worked in a pair on two learning situations, once with digital and once with analogue material. In addition, each learner worked on a geometrical and statistical task. Working on the learning situations the learners themselves could choose which sub-tasks they worked on in which order (Billion, 2021). In this way, the learners could decide at which mathematical level they wanted to work and could put difficult sub-tasks aside first and work on them later after having dealt with other sub-tasks.

Educational goal
investigation of relationships between the area and perimeter of similar squares



Digital version (GeoGebra)
two scrollbars for breadth and depth, a quadrilateral with fixed right angles, a unit square and a square grid
Actions: adjusting the scrollbars to the right and left and measure the area/ perimeter with the unit square or the square grid

Analogue version
rods of different lengths, right-angled corner connectors, a unit square and a square grid
Actions: combining rods with right-angled corner connectors and measure the area/ perimeter with the unit square or the square grid

Figure 1: Geometrical learning situation

Method of Data Preparation and Data Analysis

The learners' processing was recorded with two cameras. One camera focused on the actions and gestures made on the digital and analogue material, the other recorded the whole scene. In the learning situations with the digital material, the manipulations on the screen were also recorded with a screencast. In the videos, passages are sought in which the learners work on the same sub-task and these are transcribed for analysis.

For the reconstruction of the learners' mathematical interpretations, a semiotic adaptation of the qualitative context analysis according to Mayring (2014) and Vogel (2017) is provided. As already described, the learners' energetic interpretant is compared with the research-based interpretant in order to reconstruct which relationships between the signs the learners have recognised, interpreted, and used. In the first step, an energetic interpretant (i.e. a spontaneous action) of the learner is selected from the transcribed passage. Then, in the second step, the research-based interpretant is developed for this selected spontaneous action and is compared with it.

This is where the first reconstruction of the learner's mathematical interpretation takes place, which he or she does at this point for acting on the diagram. In the third step, the narrow context analysis (Mayring, 2014), all the same, and similar energetic interpretants of the learner that can be found in the transcript are compared with the research-based interpretant formulated in the second step. Depending on whether the learner's energetic interpretants are actions on other diagrams, the research-based interpretant needs to be adjusted for the comparison of interpretants. The reconstruction of the learner's mathematical interpretation at further passages in the transcript can confirm, extend or discard the one already reconstructed. In the fourth step, the broad context analysis, all the same, and similar energetic interpretants to the first energetic interpretant from the entire videotaped processing are added. These are in turn compared with the (possibly adapted) research-based interpretant for further reconstruction. In this way, the reconstructed mathematical interpretation can be described across the advancing sign process. In the final step, the reconstructed mathematical interpretations of the learner are presented in summary.

Given the passages selected for analysis from Nils's and Marleen's treatment of the geometrical problem, the focus of the research-based interpretant is on the relationships necessary for the construction of a square. The two important relationships to consider in the actions on the digital and the analogue material are listed in Fig. 2.

Digital	**Analogue**
Relationship between the lengths of all sides: To establish an equal relationship between the side lengths, the learner has to set the two scrollbars (depth and breadth) to the same position. GeoGebra automatically changes the side length of one dimension.	*Relationship between the lengths of all sides:* To establish an equal relationship between the side lengths in actions, the learner needs to select four rods of the same length.
Relationship between the connections of two adjacent sides: The relationship does not need to be established in the learner's actions as the right angles of the quadrilateral are fixed and the square grid also represents the relationship.	*Relationship between the connections of two adjacent sides:* The relationship does not need to be established in the learner's actions as the right-angled corner connectors and the square grid already represent the relationship.

Figure 2: Research-based interpretant

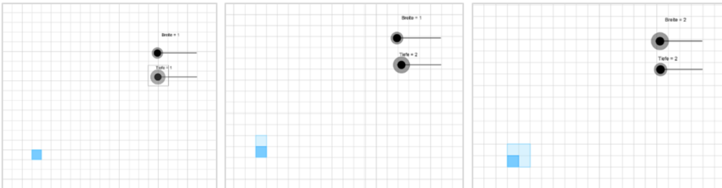
RESULTS

Analysis of the Actions on Digital Material

The focus is on the analysis excerpt where Nils sets the scrollbar *depth* and *breadth* to length 2. Therefore, he guides his finger towards the screen (see Fig. 3, Panel A), which shows a square with side lengths 1 (see Fig. 4, Panel A). Then he performs drag movements over the scrollbar *depth* (see Fig. 3, Panel B). Ultimately, the slider of this scrollbar changes to position 2, resulting automatically in a rectangle with a breadth of 1 and a depth of 2 (see Fig. 4, Panel B). Meanwhile, he utters "two\". Subsequently, Nils touches and makes drag movements over the scrollbar *breadth* (see Fig. 3, Panels C-D). Nils's actions move the slider to position 2, creating a square with side length 2 (see Fig. 4, Panel C). Nils then releases his finger from the screen (see Fig. 3, Panel E).

(A) (B) (C) (D) (E)

Figure 3: Nils's actions on the digital material



(A) (B) (C)

Figure 4: Manipulations in GeoGebra triggered by Nils's actions

By comparing the research-based interpretant (see Fig. 2) with Nils's actions, it can be reconstructed that he wants to establish an equal relationship between the lengths of the sides. Initially, Nils makes drag movements over the scrollbar *depth* to set the scrollbar *depth* to length 2. His phonetic utterance confirms this intention. His further actions suggest that he wants to make a square, as he also sets the scrollbar *breadth* to length 2. Since Nils withdraws his hand from the screen after he has set the second scrollbar, he is most likely finished with his action. He probably recognises that the relationships between the lengths of the sides are established, but it remains open whether he recognises the relationships between the connections of the sides, since these are already present in the material. Overall, Nils can interpret the parts of the material arrangement as a diagram, since he establishes new relationships in his actions and uses the relationships already implemented in the material to construct a square.

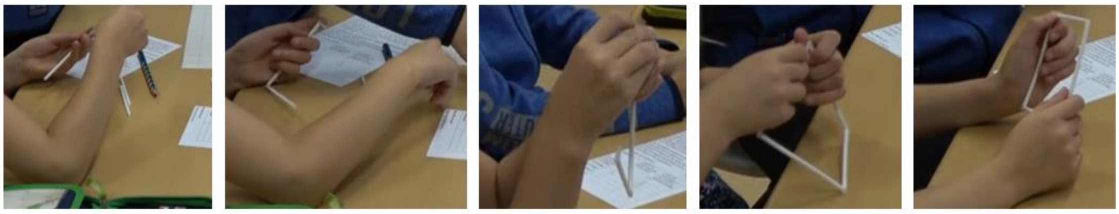
Analysis of the Actions on Analogue Material

The focus is on the analysis excerpt in which Marleen selects four rods of length 4 and joins them together to form a square. Initially, Marleen chooses three rods of length 4 and places them in the workspace in front of her (see Fig. 5, Panels A-B). She then selects another rod of length 4 and lays it alongside the others (see Fig. 5, Panel C).

(A) (B) (C)

Figure 5: Marleen selects four rods of length 4

Subsequently, she joins them together with the right-angled corner connectors to form a square (see Fig. 6, Panels A-E). Meanwhile, Marleen talks about her classmates. The spoken language does not refer to the mathematical content and will be neglected.



(A) (B) (C) (D) (E)

Figure 6: Marleen joins the four rods together to form a square

Comparing the research-based (see Fig. 2) and Marleen's energetic interpretant, it can be reconstructed that Marleen probably realises that she needs four rods of equal length to form a square. She uses the right-angled corner connections to put the selected four rods together. In her actions, she only establishes the relationship between the lengths of the rods; for the relationship between the connections of two sides, she uses the relationship already present in the material. The analysis of her actions does not reveal whether she is interpreting the relationship that is already visible in the material. She establishes the relationship between the lengths exclusively through her actions and does not refer to it linguistically. Overall, it can be reconstructed that Marleen interprets the material arrangement as a diagram by recognising relationships to create a square.

Comparison of the Analyses of the Actions on the Various Material

Comparing the extracts from the analysis results reveals that both learners interpret the material arrangement as a diagram. It can be assumed that Nils and Marleen interpret the relationship between the side lengths of the square in the same way and perform actions that correspond to the relationship. However, it is noticeable that they perform different actions to do so. Furthermore, it becomes clear that they both use the relationship between the connections of two sides that is present in the material arrangement, but it cannot be reconstructed whether they explicitly interpret it.

DISCUSSION AND OUTLOOK

Concerning the research question, the same mathematical interpretations can be reconstructed for Nils and Marleen, since they establish the same relationship despite different actions on different materials. Thus, the haptic of the action does not influence the mathematical interpretation, so the learners are likely to gain the same mathematical insights when working with digital and analogue material. Not only the appearance of the signs is insignificant to the meaning of a diagram (e.g. Dörfler, 2016), but also the appearance of the actions when the same relationship is established in these. Further results of the *MatheMat* study show that there are passages in the data where the digital material functions as a tool (by abbreviating actions and relationships to be established) and, thus, different mathematical interpretations can be reconstructed based on the actions on the digital and analogue materials (Billion, 2022).

REFERENCES

Behrens, D., & Bikner-Ahsbahs (2017). The perspective of indexicality: How tool-based actions and gestures contribute to concept-building. In T. Dooley, & G. Gueudet (Eds.), *Proceedings of the tenth CERME* (pp. 2721–2729). ERME.

Billion, L. (2021). Reconstruction of the interpretation of geometric diagrams of primary school children based on actions on various material - a semiotic perspective on actions. *International Electronic Journal of Mathematics Education, 16*(3), em0650.

Billion, L. (2022). Semiotic analyses of actions on digital and analogue material when sorting data in primary school. *Eurasia Journal of Mathematics, Science and Technology Education, 18*(7), em2126.

Dörfler, W. (2006). Inscriptions as objects of mathematical activities. In J. Maaz, & W. Schlögelmann (Eds.), *New Mathematics education research and practice* (pp. 97–111). Sense Publishers.

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In A. Bikner-Ahsbahs, A. Vohns, R. Bruder, O. Schmitt & W. Dörfler (Eds.), *Theories in and of Mathematics Education* (pp. 21–31). Springer.

Hohenwarter, M. (2001). *GeoGebra – Dynamic Mathematics for Everyone*. Austria & USA.

Huber, H. (1972). *OrbiMath. Mathematik konstruktiv*. Herder Verlag.

Larkin, K., Kortenkamp, U., Ladel, S., & Etzold, H. (2019). Using the ACAT framework to evaluate the design of two geometry apps: An exploratory study. *Digital Experiences in Mathematics Education, 5*(1), 59–92.

Maffia, A. & Maracci, M. (2019). Multiple artifacts in the mathematical class: A tentative definition of semiotic interference. In M. Graven, H. Venkat, A. Essien & P. Vale (Eds.). *Proceedings of the 43rd PME* (Vol. 3, pp. 57–64). PME.

Mayring, Ph. (2014). *Qualitative content analysis: theoretical foundation, basic procedures and software solutions.* https://nbn-resolving.org/urn:nbn:de:0168-ssoar-395173

Peirce, C. S. (1976). *The new elements of mathematics* (NEM). De Gruyter.

Peirce, C. S. (1931-1935). *Collected Papers of Charles Sanders Peirce* (CP). Harvard UP.

Rohr, S. (1993). *Über die Schönheit des Findens. Die Binnenstruktur menschlichen Verstehens nach Charles S. Peirce: Abduktionslogik und Kreativität*. Springer.

Roth, W.-M., & Bowen, M. (2001). Professionals read graphs: a semiotic analysis. *Journal for Research in Mathematics Education, 32*(2), 159–194.

Vogel, R. (2017). "wenn man da von oben guckt sieht das aus als ob…" – Die 'Dimensionslücke' zwischen zweidimensionaler Darstellung dreidimensionaler Objekte im multimodalen Austausch. In M. Beck, & R. Vogel (Eds.), *Geometrische Aktivitäten und Gespräche von Kindern im Blick qualitativen Forschens* (pp. 61–75). Waxmann.

Wille, A. (2020). Activity with signs and speaking about it: exploring students' mathematical lines of thought regarding the derivative. *International Journal of Science and Mathematics Education, 18*, 1587–1611.

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Reconstruction of the Interpretation of Geometric Diagrams of Primary School Children Based on Actions on Various Materials - A Semiotic Perspective on Actions

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ABSTRACT

This paper adopts a semiotic perspective on mathematical learning according to Peirce, in which the actions on material arrangements are considered the bases for diagrammatic work. The focus is on the learner's actions using digital and analogue material arrangements which are the starting point for the reconstruction of the learner's mathematical interpretations. In this paper a qualitative interpretative paradigm is adopted for the reconstruction. Specifically, a semiotic specification of the context analysis according to Mayring and an adoption of Vogel, is carried out. The reconstruction of the mathematical interpretation is presented with a focus on a geometrical problem that third graders are working on. Finally, the results of several cases are compared to identify possible differences between the analysed actions when using digital and analogue material arrangements.

Keywords: semiotics, mathematics education, actions on material, mathematical interpretation of diagrams, primary school

INTRODUCTION

In mathematics education, a cognitive psychological perspective largely determines the perception of actions on the learning material and the focus is on their function as means of illustration. Following this perspective, actions on the materials will develop action-based schemes which can be transferred to new elements in recurring situations (Aebli, 1980; Floer, 1993; Lorenz, 1993; Piaget, 1998; Radatz, 1993; Wittmann, 1981). For action-based schemes to emerge, materials with certain basic mathematical patterns which act as carriers of relationships – so-called "prototypes" (Rosch, 1978; Seiler, 2001) – are of central importance. Lorenz (1993) and Aebli (1980) emphasise that by acting on such materials, the function and, thus, the relationship between elements is discovered. The intensive action on the material and the resulting "objectified" (Aebli, 1980, p. 23, translated by the author) hand-based schemes enable the learners to recognise structurally identical or advanced tasks more easily (Aebli, 1980; Lorenz, 1993).

In this paper, a semiotic perspective of actions on the material is taken in order to analyse the actions made on different materials in a more effective way. From a semiotic point of view, mathematical rules and relationships can be constructed by actions on the material itself and the actions are not merely tools for establishing action-based schemes. The focus of this paper is on the actions themselves and, thus, in a semiotic sense, on the visible actions made on the material. Based on these actions, the relationships recognised by the learners will be reconstructed. For this purpose, the context analysis according to Mayring (2014) and an adaption by Vogel (2017), is further adapted based on the semiotic perspective. The aim of this paper is to show which possibilities of reconstruction become available by the semiotic adaptation of context analysis for the investigation of mathematical interpretations. In addition, the paper aims to answer the question of whether differences in the reconstruction of diagram interpretations can be traced back to the use of different materials. The comparison of the results of the analysis will eventually lead to the formulation of implications for the use of digital and analogue materials in primary school.

Firstly, a semiotic perspective on mathematical learning with a focus on actions, according to Charles Sanders Peirce, will be presented. In this context, implications of the semiotic perspective for the view of actions on the material are determined and logical conclusions for the reconstruction of the diagram interpretation are drawn. Secondly, the *MatheMat – Mathematical Learning with materials* study is briefly introduced in which the data for the analyses in this paper were generated. The data gathering focusses on working with prompts; these are used in data processing to identify comparable scenes in the videotaped processing of different child tandems. The transcript form developed for the analysis, as well as the specification of the context analysis according to Mayring (2014) in an adaptation for educational mathematics research made by Vogel (2017), are described. Thirdly, using the learning situation *Relationship between perimeter and area* as an example, a scene for transcription is selected and the prompt whose processing is transcribed is described in more detail. Finally, one of four analyses is presented in detail and compared with the results of another three analyses. The results of the analyses are compiled and discussed in the last section.

THEORETICAL BACKGROUND – ACTIONS ON MATERIAL FROM A SEMIOTIC PERSPECTIVE

The theoretical introduction focusses on the actions made when using the materials. From a semiotic perspective, it becomes clear that actions play a central role in learning mathematics and that they can be used as a starting point for the interpretation of diagrams.

Actions from a Semiotic Point of View

From a semiotic point of view, mathematical rules and their properties arise from thinking about acting with representations[1] (Dörfler, 2015). Concerning mathematical learning, this means that learning mathematics is the activity itself. Inscriptions can be the general result of activities and, thus, they can be lines on paper, on the screen of a tablet, on the board, in the sand, or can be tactilely experienced as a material (Dörfler, 2006a; Gravemeijer, 2002). Due to the nature of the inscriptions, inscriptions can be passed on, i.e., they are mobile without losing their properties, are readable and can be combined (Latour, 2012; Schreiber, 2013).

According to Peirce, diagrams are based on a well-defined structure of inscriptions (Dörfler, 2006a, 2006b) whose purpose is "[…] to represent certain relations in such a form that it can be transformed into another form representing other relations involved in those first represented and this transformed icon can be interpreted in a symbolic statement […]" (Peirce MS [R] 339:286r).

Representations or representation systems are, therefore, not mathematical rules themselves, but rather become mathematical rules only through their use and the actions taken with these representations. Dörfler (1988) understands mathematical rules as relationships that become clear through acting with the diagrams[2]. Some operations can be translated from one representation system into another representation system during the mathematical activity (Dörfler, 2015). If this translation of operations or relationships enables further mathematical reasoning and the recognition of further relationships, according to Gravemeijer (1999) this can be referred to as a model *for*. Nevertheless, each representation system is independent since the rules of the representation system are explained within the system itself (Dörfler, 2015).

As shown later in the empirical data, the change in perimeter and area of similar squares can be represented, for example, geometrically by squares of different sizes or by arithmetic patterns. In both diagrams, the same relationships can be revealed through the actions and, thus, the same mathematical rules can be recognised. The length of the perimeter of a square determined by actions on the geometric representation can be translated into an arithmetic sign. In this way, the arithmetic sign can be seen as a model *for* geometric representation. If the side length is described by 'a' and the perimeter by '4a', the algebraic representation shows fewer arithmetic signs, but more variables and, in Peirce's sense, more indices.

The appearance of the representations, i.e. the concrete fixations or inscriptions, are interchangeable (Dörfler, 2015) since they only gain their significance through rule-based action. Following the example focussed within the empirical data, it is, therefore, irrelevant whether similar squares are drawn or materialised. Dörfler (2015) introduces here a comparison to Wittgenstein's chess game: the appearance of the chess pieces has no meaning, only by the position on the board and the rules of moving do the pieces gain significance. Thus, the operation of a mathematician according to certain rules can be compared to the moving of a piece in chess, since both the role of the signs and the rules for their use are in the foreground (Dörfler, 2015; Shapiro, 1997; Thomas, 2009).

Actions and Signs

So far, Peirce's diagrammatic activity has been the focus of the semiotic perspective. For the analysis of actions on the materials, Peirce's types of signs, especially indices, are also of interest. In the semiotic sense, indices can provide indications of something absent in space or time, for example, a fingerprint can indicate the perpetrator (Krämer, 2007). They can also indicate something that is simultaneous and invisible, such as the weathervane as an indicator to the wind direction (Krämer, 2007). In Peirce's sense, the concept of indices can cover a wide variety of examples (Pape, 2007). According to Peirce, an index is a sign "[…] which refers to its object not so much because of any similarity or analogy with it, nor because it is associated with general characters which that object happens to possess, as because it is in dynamical (including spatial) connection […]" (CP 2.305).

At the same time there are also signs which are not clearly assignable. A proposal by Krämer (2007) draws attention to the distinction between indices and "tracks" (Krämer, 2007, translated by the author). Thus, in contrast to indices, tracks are always characterised by a time break and an unintentional nature, however, it becomes apparent in Krämer's explanations that a clear separation of indices and tracks is not possible (Krämer, 2007). Indices are characterised by temporal and spatial simultaneity, such as fever is an indication of an infection (Krämer, 2007). Krämer introduces a further distinction in that artificial indices are intentional and conscious; Krämer provides the example of a pointing gesture (Krämer, 2007). Natural indices, on the other hand, share their unintentional nature with tracks, such as smoke and fire (Krämer, 2007). Besides, the track of an animal can be read by

[1] According to Gravemeijer (1999, 2002), this paper assumes representations *for* and not representations *of*. This leads to a shift in thinking about modelled situations towards a focus on mathematical relationships (Gravemeijer, 1999). The changed perspective makes mathematics more accessible to learners (Gravemeijer, 2002).

[2] In this paper, diagrams in the sense of Charles Sanders Peirce are assumed. For more information, see Hoffmann (2005).

the hunter as an index which connects the past with the current presence of an animal, so that it results in a simultaneity of the hunter and the hunted (Krämer, 2007). In the semiotic sense, the sign, as the result of the action, can be seen as an index of the action (see Kadunz, 2015, 2016; Sinclair & de Freitas, 2014). Concerning the example from the empirical data given later, the arithmetic sign describing the length of the perimeter is an index to the previous action on the geometric representation. Thus, the sign, which is produced by the action, is characterised by non-simultaneity, but, as an artificial index, it is intentional and conscious since the actions are guided by objectives of the acting. In this case, the index, which refers to the actions, shows attributes of a track because of its non-simultaneity.

To learn concepts, defined relationships are observed during an action, where the view to the resulting sign reports back, if this relationship is mapped (Kadunz, 2015, 2016). The sign on the paper or screen can be referred to as the kind of action which was made to represent the relationship. However, in some situations there can be a complete separation between the action and the intended relationship; if a perpendicular is constructed by a mouse-click, the perpendicular cannot return to the action with the mouse. Only if the programme is used to vary a straight line, "and the behaviour of the drawn sign has to be interpreted again", can a connection between the signs and the relationship, which is considered in the action, be made (Kadunz, 2016, p. 35).

Actions on Various Materials

The use of digital tools may require a structural change in human activity (Dörfler, 1991). With the activities made possible and changed by the tool (e.g., the programme on the tablet), there is, inevitably, a shift in the attention of interest but also of the problems, difficulties and tasks (Dörfler, 1991). It is necessary to know the functions of the various instructions of the programme, for example, to produce a text on the tablet which can involve a "reorganisation" (Dörfler, 1991, p. 53, translated by the author) of the activity of writing. The reorganisation of activities must be defined and implemented educationally for each programme and each learning situation (Dörfler, 1991). Thus, the act of building with analogue material can mean the superimposition or juxtaposition of materials according to a certain construction plan. For the same action in GeoGebra (Hohenwarter, 2001), dragging a finger across the surface of the screen can result in the adjustment of a scrollbar, which, in turn, can affect a three-dimensional object displayed on the screen. In this context, Villamor, Willis and Wroblewski (2010) have compiled common 'touch gestures' on the tablet and resulting 'user actions'. Depending on the programme used, a movement on the screen can initiate several actions in the programme, but an action can also be initiated by several movements. The compilation by Villamor, Willis and Wroblewski (2010) is used, in a modified form, to describe the hand movements and the actions in the programme in the learning situation which is the focus in this paper.

Similar to the different modes, the Touch Gestures (Sinclair & de Freitas, 2014) are included in the analysis because they are directly linked to the impact of the programme. The touch of the screen and the subsequent manipulation of the inscriptions on the screen can be interpreted together as an action.

ACTIONS AS THE STARTING BASIS FOR THE INTERPRETATION OF DIAGRAMS

“When we perform manual actions with material things, for example in experiments, they are guided by ideas and aims and thus become integrative parts of thinking and observing.” (Dörfler, 2015, p. 43, translated by the author). The citation of Dörfler shows that actions on material arrangements are controlled by the interpretations of the learner. Through the actions, which are the expression of an individual’s interpretation process, the inscriptions can become a diagram in Peirce's sense, thus, the basis of mathematical thinking is always the sign or inscription (Dörfler, 2015). The rules for manipulation are expressed in the actions of the learners. If learners have often made manipulations and are, thus, well-versed in the use of inscriptions, they can interpret more flexibly which manipulations of the material and, thus, manipulations of the diagram are possible (Dörfler, 1988). Acting on the inscriptions consequently indicates which (mathematical) interpretation the learners make and which rules they may follow in their actions. Therefore, in a semiotic sense, the actions of the learners in the analysis can be interpreted in order to reconstruct possible diagram interpretations of the learners.

A semiotic specification of Vogel's (2017) adaptation of the context analysis according to Mayring (2014) should enable one to explicate the actions at the material arrangement by adding further transcript and video passages and, thus, to reconstruct the interpretation of learners based on their actions. The justified addition of occurring actions of the learners in the learning situation should enable one to grasp their interpretations over the course of the learning situation and, thus, ensure a solid reconstruction of the interpretation.

To reconstruct the interpretations of the diagram, modes other than the actions on the material, such as gestures and spoken language, are also used in the analysis. Huth (2018) refers to the relationship between actions and gestures and also describes the relationship between gestures and spoken language while considering mode-specific possibilities of expression. The functions of gestures in mathematical learning situations can be identified as the structuring of discourse, the clarification of mathematical interpretations, the presentation of mathematical ideas and the display of possible and impossible manipulations of diagrams and their inscriptional-diagrammatic use (Huth, 2020; Vogel & Huth, 2020). Thus, gestures, as well as actions, can indicate the interpretation of the diagram by the learners. Gestures that are similar to the actions or that clearly show manipulations on the diagram are also considered in the analysis due to their proximity to the actions on the material. Even spoken language that is expressed simultaneously with the action on the material, or which directly refers to the actions, is included.

METHODS AND METHODOLOGY

In the following, the data gathering and preparation for transcription are described. For the reconstruction of the interpretations of the diagrams, based on the actions made by the learners on the diagrams, the context analysis according to Mayring (2014) and the adaptation made by Vogel (2017) are specified.

Methods for Data Gathering

The research focus of the *MatheMat – Mathematical learning with materials* study is on the actions of the learners on different materials (digital and analogue materials). For this purpose of qualitative research, learning situations will be designed whose mathematical content and tasks are identical, but which will be realised once with digital and once with analogue material (Billion, 2018). In terms of the qualitative approach, these developed learning situations form the basis for the data collection and analysis. In four of the designed learning situations, primary school children deal with geometrical problems and, in four others, with stochastic problems. In the cross-sectional study a total of 32 third and fourth-grade students work on learning situations with different materials (digital and analogue). The learners always work together in tandem in a learning situation. Care is taken to ensure that a stronger student works together with a weaker one in mathematics. The assessment of achievement in mathematics is made by the respective teacher. Each tandem works, on the one hand, in a learning situation with digital material and, on the other hand, in one with analogue material. Besides, they each work on a stochastic and a geometric problem. The learners work on the problem in the time frame of one school lesson. The work is videotaped so that video excerpts can be transcribed for analysis.

All learning situations are initiated by so-called prompts; these are presented on paper cards on the table in front of the learners. "[…] Prompts are defined as recall and/or performance aids, which vary from general questions (e.g., “what is your plan?”) to explicit execution instructions […]” (Bannert, 2009). An example of a prompt from the field of mathematical learning could be *Calculate first and then check*. Prompts can stimulate cognitive, metacognitive, motivational and cooperative activities (Bannert, 2009). The prompts used in the learning situations of the *MatheMat* study mainly include the stimulation of cognitive and metacognitive activities: supporting information, questions to stimulate the processing and requests to stimulate the organisation of the learning activities. The learners can freely choose the order in which the prompts are processed. If the learners do not understand one prompt, they can put the processing of this prompt at the back and select another one. The learning situation on which this paper focusses is the relationship between area and perimeter for similar squares. Different thematic aspects are spotlighted when processing the following prompts: determining the area, determining the perimeter and the relationship between area and perimeter. The prompts for a thematic aspect have different linguistic levels in the description and are aimed at a different scope of results. The linguistic level is based on the different language registers of everyday language, educational language and technical language (Meyer & Prediger, 2012), but also on word, sentence and text levels (Wessel, Büchter, & Prediger, 2018). The different range of results is reflected in the varying number of squares to be examined. Based on these two characteristics of the prompts (linguistic level and mathematical thematic aspect), all prompts can be described in a cross table (**Table 1**).

Table 1. Description of the prompts in a cross table

	Level 1: Formulations in a table	**Level 2: Formulation of examples**	**Level 3: General formulations**
Perimeter	In the first column, squares with side lengths 1-5 are given. In the second column, the number of unit lengths on one side and in the third column the number of unit lengths on all sides should be entered.	Only those squares which are twice and three times as large as the unit square shall be considered. The question is, how many unit lengths are needed to be added to all sides of the squares.	The question is, how does the number of unit lengths on all sides change when the side length of the squares increase.
Area	In the first column, squares with side lengths 1-5 are given. In the second column, the number of unit squares which are needed to lay out the respective area of the square, should be entered.	Only those squares which are twice and three times as large as the unit square shall be considered. The question is, how many unit squares are needed to lay out these squares.	The question is, how does the number of unit squares change, when the side length of the squares increase.
Relationship between perimeter and area	In the first column, squares with side lengths 1-6 are given. In the second column, the number of unit squares (area) and, in the third column, the number of unit lengths on all sides (perimeter) should be entered.	___________	The question is about a pattern of changing the number of unit squares and unit lengths when the side length of the squares increases.

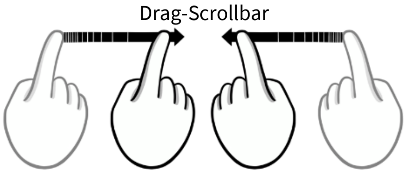
Horizontally, the linguistic level of the prompts is recorded, which increases from left to right. The thematic aspects of the prompts are arranged vertically. Due to the free choice of the processing of the prompts, there is a different processing sequence for each tandem. These sequences can be displayed within this cross table. In this way, each prompt can be assigned a processing number, which indicates at which point in the processing the prompt is processed. By connecting the processing numbers, the path of the processing sequence of a tandem can be displayed graphically (see **Table 3a**, **3b** and **Table 4a**, **4b**). With the help of the graphical representation, it is possible to find places in the processing where two or more tandems have the same processing number at the same prompt. To be able to compare the learner's interpretation of the diagram when working with digital and analogue diagrams, it is suitable to start the analysis at such a point in the process and then extend it to the entire process. It is possible that the order in which the children process the prompts can influence the actions on the diagram and, thus, the learner's interpretation of the diagrams to be reconstructed from their actions.

Methods for Data Preparation

In the transcript, all multimodal utterances of the learners are reproduced in detail. The focus is primarily on the actions on the material and, thus, on the manipulations of the diagram. As already discussed, other modes, such as spoken language and gestures, can refer to previous actions and are, therefore, also relevant for the reconstruction of the interpretations of the diagrams. For this purpose, a transcript form is developed that divides the videotaped processing into scenes. In this way, the multimodal utterances within a scene can be displayed in a coordinated manner. Every spoken utterance is marked with letters, while the actions and gestures that take place at the same time are also marked with the same letter. To distinguish gestures and actions from spoken language, they are numbered. The numbering is started anew in each scene.

The transcription of the actions on the digital and analogue diagrams differ. During the actions on the analogue diagram, all movements of the arms and hands are shown in detail, with or without material. When working with the digital diagram, the movements of the fingers on the tablet and the resulting manipulation in the GeoGebra scenario are described. The *Touch Gesture Reference Guide* compiled by Villamor, Willis and Wroblewski (2010) is adapted for the transcription of movements on the screen and their effects in the programme. **Table 2** shows the relevant movements on the screen and the resulting manipulations in the programme for the GeoGebra scenario presented in the paper. In contrast to other transcripts (Billion & Vogel, 2020a), only one movement on the screen is relevant for the transcription presented in the paper.

Table 2. Actions on the digital diagram[3]

Movement on the screen	Description of the movement	Manipulation in the programme
Drag-Scrollbar 	"Move fingertip over scrollbar without losing contact."	Adjusting the scrollbar

Methods for Data Analysis

For the reconstruction of the interpretations of the diagrams, the adaption of the context analysis according to Mayring (2014) made by Vogel (2017), and Vogel and Huth (2020) must be further specified (Billion & Vogel, 2020a, 2020b). In the context analysis according to Mayring (2014), the meaning of a term is worked out from its use by the participants and, thus, by the further addition of text passages. The adaptation of Vogel (2017) also includes the addition of transcript and video passages to contrast the multimodal expressions of learners with scientific mathematical concepts. Thus, in the context of the Conceptual Change approach (Carey, 1988; Posner et al., 1982), the individual mathematical concepts of the learners can be reconstructed.

To reconstruct the interpretation of the diagrams, the theoretical background of Conceptual Change used by Vogel (2017) is replaced by the diagrammaticity according to Peirce. Thus, the actions on the material, the interpretation of the diagram expressed by the action and the importance of the use through which a material arrangement becomes a diagram, come to the fore in the analysis. The quotation from Dörfler (2015, p. 43) above shows that the actions of learners express rules that allow conclusions to be drawn about the learner's interpretations. By analysing the rule-directed actions, diagrammatic interpretations of the learners can be reconstructed. The usage of the inscriptions can be employed to indicate which rules the learners take into account and, thus, how they interpret the inscriptions. Consequently, the usage of the inscriptions or signs determines the properties of the diagram. Furthermore, in context analysis, the addition of further passages, i.e., usage in context, renders the explication of a term, the reconstruction of a mathematical concept or a mathematical diagram interpretation possible. For these reasons, context analysis can be profitably linked to the semiotic theory presented above, since the usage has a special meaning in both. In this way a relationship between theory and method becomes clear.

As already pointed out in the theoretical background, further modes, such as gestures or spoken language, which refer to previous actions are taken into account in the analysis. For the reconstruction of the interpretations of the diagrams, the actions of the learners are contrasted with the relationships of the diagram resulting from rule-guided actions. Actions that can be derived from the relationships underlying the diagram can be described as the actions of persons experienced in the usage of the inscriptions. These descriptions are feasible, partly creative manipulations that can be used to determine the properties of the diagram. When contrasting the rule-directed actions and the actions of the learners, a discrepancy may become describable; this can also provide information on the interpretation of the learners.

Based on the described specifications, the following rules for the reconstruction of the learner's interpretations of the diagrams can be established (Billion & Vogel, 2020a, 2020b):

Step 1 - Determination of the evaluation unit – a passage of the transcript to be explicated: The search is for a passage in the transcript in which a mathematical (diagrammatic) action is described that is significant in this situation and which is expected to provide an answer to the research question. In this case, the passage in the transcript must be interesting for the reconstruction of the individual diagram interpretation made by a learner.

[3] The illustrations of the hands were created by Petra Tanopoulou. The movements on the screen, the description and the manipulations in the programme are partly documented literally, or phrased in the style of Villamor, Willis and Wroblewski (2010).

Step 2 - Explication 1 - Description of rule-guided actions based on the mathematical content of the passage in the transcript: (E1.1) Description of rule-driven mathematical actions or manipulations appropriate to the explicative passage in the transcript by persons experienced in usage, used to describe the properties of the diagram. (E1.2) Analysis of the selected passage in the transcript concerning the research question by contrasting the rule-directed actions with the actions exhibited by the learner. In the identified discrepancy, the interpretations of the focussed learner become clear and describable. (E1.3) Summary of the learner's previous mathematical interpretations from Explication 1.

Step 3 - Explication 2 - close context analysis: (E2.1) All passages in the transcript in which actions are directly related to the explicating passage in the transcript are compiled. (E2.2) Analysis of the identified passages in the transcript, that are directly related to the explicative passage in the transcript, by contrasting the rule-directed actions with the actions actions exhibited by the learner. (E2.3) Similar actions are searched for in the transcript which give further information on the interpretation by the learner. (E2.4) The description of the mathematically rule-guided actions of persons experienced in usage from Explication 1 may have to be extended at this point. (E2.5) The similar passages found in the transcript form the starting points for in-depth reconstructions of the mathematical interpretations. (E2.6) Summary of the learner's previous mathematical interpretations of the diagrams from Explication 2.

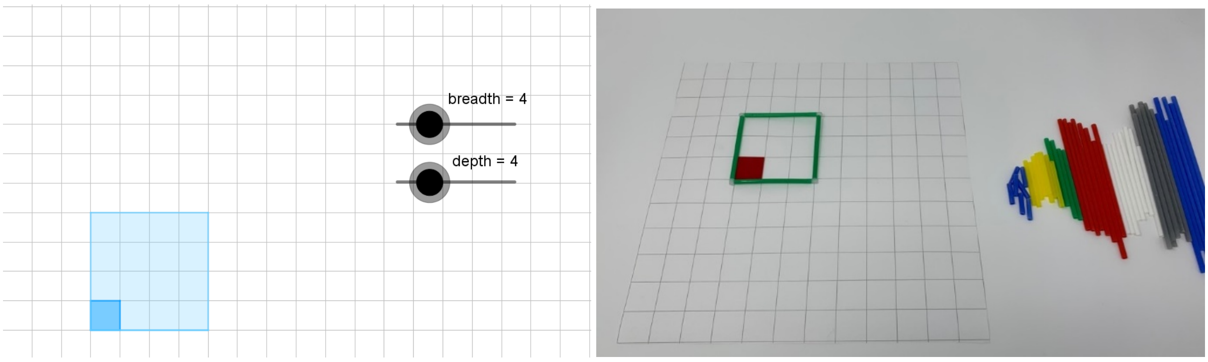
Step 4 - Explication 3 - broad context analysis: (E3.1) Further explicative material (e.g., non-transcribed excerpts from the videotaped learning situation) directly related to the explicative passage in the transcript is compiled and the relevance of this material is checked. (E3.2) Analysis of the compiled videotaped passages, which are directly related to the passage in the transcript, is carried out. The analysis may allow a more in-depth continuation of the reconstruction of the learner's interpretations of the diagrams. (E3.3) Further similar actions are searched for in the video, that are directly related to the actions from E2.3. (E3.4) The further passages from the video are the starting points for more in-depth reconstructions of the learner's interpretations of the diagrams. (E3.5) Summary of the learner's interpretations of the diagrams from Explication 3.

Step 5 - Summary: The aspects of the diagram interpretations reconstructed in the individual steps of the analysis are now described in summary.

A GEOMETRIC EXAMPLE

In this paper, a geometric example from the *MatheMat* study is presented in which learners deal with the relationship between area and perimeter of similar squares. The learning situation presented here was worked on by four student tandems from two third-grader classes of different schools. Two tandems worked with a scenario in GeoGebra (Hohenwarter, 2001), while the other two tandems worked with the material OrbiMath (Huber, 1972). In processing the prompts, the learners should make similar squares with the given materials.

To create a side model[4] of a square in the GeoGebra scenario (**Figure 1a**), learners can use their fingers to control the scrollbar in order to adjust the sides of the rectangular square to the same length. By dragging the scrollbar breadth, both sides of the dimension breadth are changed in the same way. By varying the length of the scrollbar, a connection between the visible signs and the corresponding relationship can be made (Kadunz, 2015). In the lower left-hand corner of the side model, a square with side length 1 remains visible, even during manipulations on the diagram. On the whole surface of the screen there is a flat square grid; this is divided into the size of the unit square.



(a) (b)

Figure 1. Digital (a) and analogue (b) materials for processing the described learning situation

To create a side model of a square using OrbiMath material (**Figure 1b**), learners can choose four rods of equal length from a larger number of rods of different lengths and assemble them with right-angled corner joints. Only right-angled corner joints are available. A right angle is given by the corner joint and, as in the GeoGebra scenario, the focus is exclusively on the side lengths of the two dimensions of breadth and depth. The manipulations of the OrbiMath material take place directly on the side model, unlike the digital material where the manipulation takes place on the scrollbar and the programme transfers this manipulation to the side model on the screen. In the case of the analogue material, learners also have a unit square and a square grid (which is divided into the size of the unit square) at their disposal. After the learners have created a side model of a square with the

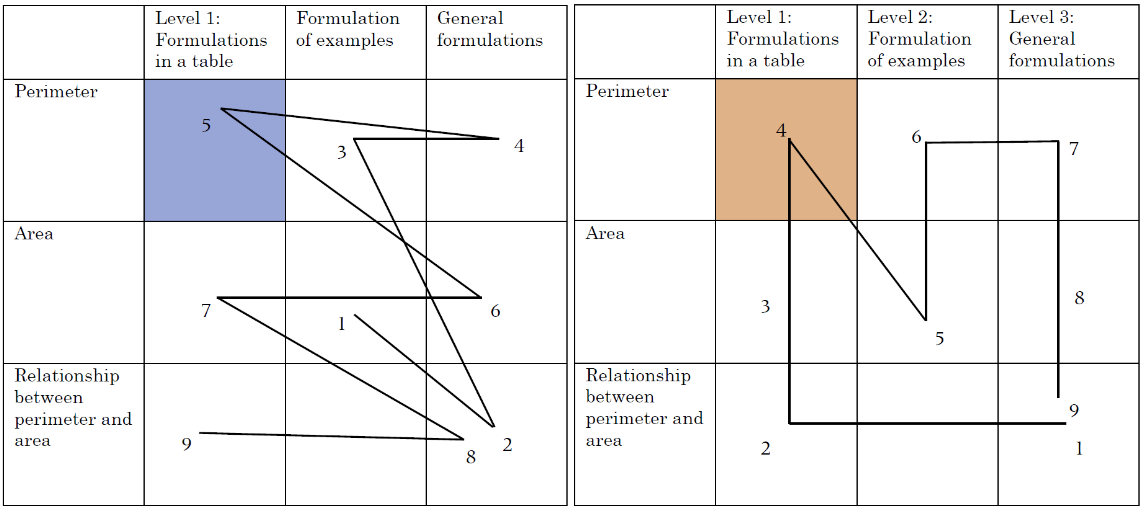
[4] In this case, a side model is understood as a flat figure in which only the sides are reproduced. The term is based on the edge model of geometric bodies.

respective material (larger than the unit square), they can use the unit square and the square grid to specify the area and perimeter in unit squares and unit lengths, respectively.

Selection of the Passages to be Transcribed

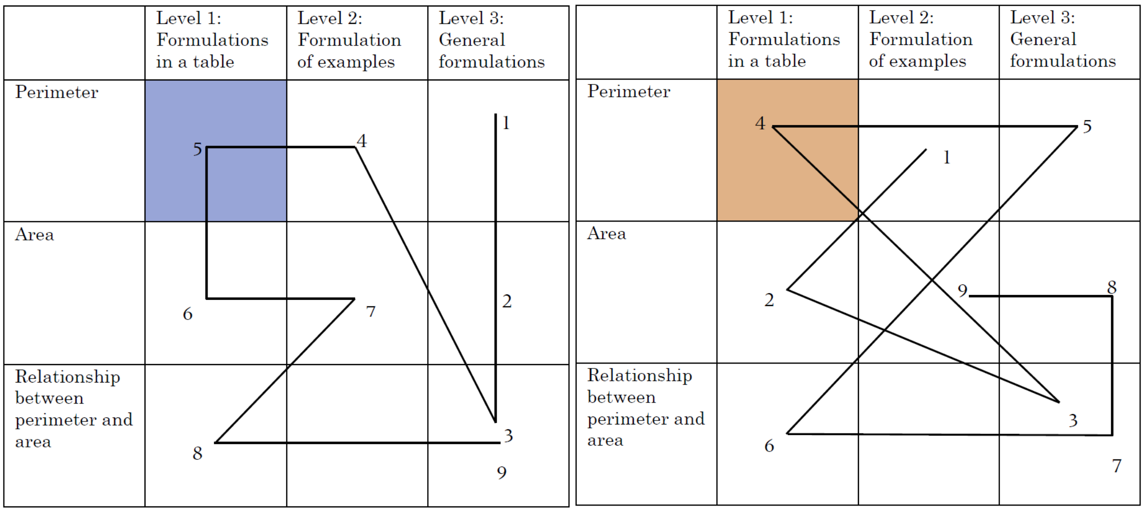
To identify comparable passages for transcription from the videotaped processing of the learners, the sequences of the processing of the prompts were graphically displayed. Tandems that process the same prompt at the same point in the processing have the same processing number in the table for the respective prompt. These passages, where two tandems (one with digital and one with analogue material) have the same processing number, are selected for transcription. In the analysis, these passages can be compared particularly well because the learners work at the same prompt at the same point in the processing and, thus, have the same experience in working with the material. It cannot be ruled out that the sequence of processing and in a semiotic sense, the experience in using the material has an influence on the actions made using the digital and analogue materials and, consequently, the reconstruction of the learner's interpretation of the diagram.

Figures 2a and **2b** show graphically the processing of two tandems when working with analogue material. It can be seen that the first tandem processes the prompt with the mathematical aspect *perimeter* and the language level 1 (*formulations in a table*) at the fifth, while the second tandem at the fourth position. In **Figures 3a** and **3b** the processing of two tandems with the digital GeoGebra scenario are shown graphically. Here it can also be seen that one tandem solves the prompt with the mathematical aspect *perimeter* and the language level 1 at the fifth position, while the other tandem solves the prompt at the fourth position of their processing. One tandem working with the digital material and one tandem working with the analogue material work on the prompt with the mathematical aspect of *perimeter* and language level 1 at the same position in the processing. The processing sequences show that it is useful to transcribe the processing of the prompt with the mathematical aspect of *perimeter* and the language level 1 of the four tandems and to analyse them comparatively. The results of this comparison of the reconstructed interpretations of the diagrams will then be used in a further step to generate a general statement.



(a) (b)

Figure 2. Processing sequences of the tandems when working with the OrbiMath material



(a) (b)

Figure 3. Processing sequence of the tandems when working with the GeoGebra scenario

Prompt with the Mathematical Aspect Perimeter and the Language Level 1

Based on the progressions of the editing process, it has been found that the processing of this prompt by all four tandems is suitable for transcription. The question on the prompt is generally aimed at changing the perimeter of similar squares. The question does not focus on special squares with a given side length. In the second step, concrete information about the side lengths of the squares is given in the table shown below the question. In the second column of the table, the learners should determine the unit lengths on one side length. In the third column, the unit lengths on all sides of the square (perimeter) should be determined (see **Table 1**). The table structures the processing of the prompt by systematically determining the unit lengths, firstly on one side and then on all sides of the square, for squares of different sizes. This prompt is intended to encourage learners to make side models of squares of different sizes and to use the side length of the unit square or square grid to determine the unit lengths on one or all sides of the square. Once the learners have determined the number of unit lengths by comparative actions on the geometric diagram, they can write the numerical value in the table using an arithmetic sign.

Perimeter – Prompt

A square with side length 1 is called a unit square. A square with side length 1 can be laid out exactly with 1 unit square. The area of the unit square is, therefore, 1 unit square. The unit square has 4 sides. Each side has the length 1. This length is called the unit length. If you put all unit lengths of a unit square together, you get a length of 4. This length is called the perimeter.

The side lengths of the squares get larger and the squares still look like squares.
How does the number of unit lengths change on one and all sides of these squares?

Fill out the table:

Square with	Number of unit lengths on one side of the square	Number of unit lengths on all sides of the square (perimeter)
side length 1 (unit square)	1	4
side length 2		
side length 3		
side length 4		
side length 5		

What do you notice?

Figure 4. Prompt with the mathematical aspect *perimeter* and the language level 1 (*formulations in the table*)

In this way, the students record the relationships of the geometric material arrangement or diagram with arithmetic signs. These inscriptions can be interpreted as a new arithmetic diagram and, thus, themselves become the focus of action and thought. The learners can discover the connections and rules of manipulation of the arithmetic pattern sequences that have been created. If the learners refer back to the geometric diagram, it can be seen that the increase of one side length by one unit length has a fourfold effect on the perimeter of the square. The interpretation of the arithmetic diagram can be reconstructed through the gestures and spoken language of the learners. Both modi refer to the act of writing down the signs necessary to create the diagram.

RECONSTRUCTION OF THE INTERPRETATION OF THE DIAGRAMS

In the following, the analysis of the actions of Nils is presented in excerpts to reconstruct his interpretations of the diagrams. Together with his partner, he works on the prompt with the mathematical aspect *perimeter* and the language level 1 at the fifth position with digital material. Subsequently, reference is made to the results of the analysis for Marleen's actions, a student who also processes this prompt as a fifth prompt but using analogue material. Finally, the results of the analysis of two learners (Emre and Li) who have worked on the same prompt at the fourth position are presented and compared.

Reconstruction of Interpretations of Diagrams when Working with Digital Material

In the first step of the context analysis, the reconstruction of Nils' mathematical interpretations of the diagrams refers to an action sequence in the created transcript (see **Table 3**). Subsequently, further passages in the transcript followed by the entire videotaped processing are included in the analysis. The focus of this analysis is the research question:

Which mathematical interpretations of the diagrams of Nils can be reconstructed based on his actions on the digital material while working on the geometrical problem?

Table 3. Transcript excerpt from processing with digital material

22:25	**Nils**		a	We already had that\
(5)		<	c	Two (*silent)*
			d	Actually we can
			e	there
			f	actually copy again \
			01	Looks at the tablet.
			02	Places the left forearm orthogonal to the edge of the table in front of him on the table top.
			03	Leads the right hand with outstretched index finger towards the tablet.
			04	Uses the right index finger to make a drag-scrollbar-movement to the left over the scrollbar *breadth*.
			05	The slider of the scrollbar takes the position of a unit length.
			06	The breadth of the square changes from five unit lengths to one unit length.
			07	Uses the right index finger to make a drag-scrollbar-movement to the left over the scrollbar *depth.*
			08	The scrollbar's slider takes the position of one unit length.
			09	The depth of the rectangle changes from five unit lengths to one unit length.
			10a	Now a square with the unit length one is visible.
			11b	Uses the right index finger to make a drag-scrollbar-movement to the right over the scrollbar *depth.*
			12b	The scrollbar's slider takes the position three unit length.
			13b	The depth of the square changes from one unit length to three unit lengths.
			14c	Uses the right index finger to make a quick drag-scrollbar-movement to the left over the scrollbar *depth*.
			15c	The scrollbar's slider takes the position of two unit lengths.
			16c	The depth of the rectangle changes from three to two unit lengths.
			17	Uses the right index finger to make a drag-scrollbar-movement to the right over the scrollbar *breadth*.
			18	The scrollbar's slider takes the position of three unit lengths.
			19	The breadth of the rectangle changes from one unit length to three unit lengths.
			20	Uses the right index finger to make a quick drag-scrollbar-movement to the left over the scrollbar *breadth*.
			21	The scrollbar's slider takes the position of two unit lengths.
			22	The breadth of the rectangle changes from three to two unit lengths.
			23	Now a square with the unit length two is visible.
			24	Looks at the prompt with the mathematical aspect *perimeter* and the language level 1 (*formulations in a table*).
			25	Removes the index finger of the right hand from the tablet.
			26d	Leads the right hand with outstretched index finger to this prompt.
			27	Touches the prompt with his index finger.
			28	Removes the left forearm from the table top.
			29	Leads the left hand with outstretched index finger to the prompt with the mathematical aspect *perimeter* and the language level 1.
			30e	Moves the index finger of his right hand up and down a few times.
			31	Touches the lower left corner of the prompt with the index finger of the left hand.
			32	Removes the right hand from the prompt with the mathematical aspect perimeter and the language level 1.
			33	Turns his gaze towards the prompt with the mathematical aspect *area* and the language level 1.
			34f	Leads the right hand to the prompt to this prompt.
			35f	Touches this prompt with his fingers.
			36	Leads the right hand back to himself.
			37	Places the right hand flat on the table top to the right of the prompt.
			38	Looks at the accompanying person.
			39	Looks at the prompt with the mathematical aspect *area* and the language level 1.
			40	Turns his gaze towards the prompt with the mathematical aspect *perimeter* and the language level 1.
			41	Bends the arms,
			42	so that the hands are facing each other.

In the transcribed processing (scenes 1-16) of the prompt with the mathematical aspect *perimeter* and the language level 1, the learners should determine the perimeter of similar squares using a scenario in GeoGebra and record their results in a table. The learners have read through the prompt and discussed with the accompanying person what the prompt is about. To start the analysis, the transcribed action of Nils in scene 5, lines 11b-16c is selected. The transcript excerpt from scene 5 displays, exclusively, the utterances of Nils so that the spoken utterance b "e x actly\" of the accompanying person is not performed. The analysis focusses on Nils' actions in combination with the phonetic utterances and gestures that match his actions.

Step 1: In the selected transcript passage (scene 5, lines 11b-16c), Nils changes the length of the square in the dimension depth at the scrollbar. He performs a drag-scrollbar-movement (see **Table 3**) to the right over the scrollbar *depth* and sets the length to 3. Shortly afterwards he makes a drag-scrollbar-movement to the left. During the second drag-scrollbar-movement, he quietly utters “two”.

Step 2 – Explication 1: (E1.1) In the learning situation, learners should examine side models of similar squares in terms of perimeter and area and their relationship. The focus of the prompt is to determine the number of unit lengths with which one or all sides of the different sized similar squares can be measured. To determine this, setting squares of different sizes in the given GeoGebra scenario would be a suitable action. In a square, all sides are of equal length and include an angle of 90°. In the GeoGebra scenario, a right angle, which is enclosed by the sides, is already given and cannot be changed. To create a square in the scenario, the sides in the dimension *breadth* and the sides in the dimension *depth* must be set to the same length. Two scrollbars, *breadth* and *depth*, allow to change or adjust the side lengths of the rectangular square. To do this, the slider of the scrollbar must be moved with the help of a finger. If the slider is moved to the left, the length of the corresponding sides of this dimension is reduced and, correspondingly, increased when moving to the right. To create a square, the sliders of both scrollbars must be set to the same number.

(E1.2) Nils moves the slider of the depth scrollbar to the right and, thus, aims at increasing the side length of this dimension. He sets the number 3. The hastily performed drag-scrollbar-movement of Nils to the left suggests that he is dissatisfied with the, possibly, unintended setting and wishes to correct it downwards. With this drag-scrollbar-movement he sets the number 2 and the side length in the dimension *depth* is reduced to 2. The phonetic utterance "two" during the drag-scrollbar-movement to the left supports this assumption. Setting the side lengths to two unit lengths is indicated by the number two in the table at the prompt. It can be assumed that he reads the sign and adjusts the scrollbar on the screen accordingly. Through his actions on the digital material and phonetic utterances, a relationship between the arithmetic diagram (the table on the prompt) and the geometrical diagram (the representations on the screen) is expressed. It can be assumed that Nils interprets both diagrams and translates relationships of the arithmetic diagram into relationships of the geometric diagram. It can also be assumed that Nils interprets the geometric diagram in such a way that the setting of the scrollbar *depth* affects the corresponding side lengths of the rectangular square and he uses this to lengthen the sides in the dimension *depth*.

(E1.3) The two actions that Nils takes on the scrollbar show that Nils consciously adjusts the slider, even if it takes him two attempts. If the desired length is not achieved, he tries to change it by another action. His actions and phonetic utterances suggest that he translates the relationships in the arithmetic diagram into relationships in the geometric diagram and recognises the diagrams as models *for* each other. It can be assumed that Nils has interpreted the rules for the lengthening of the sides into the geometrical diagram. In the narrow context analysis (below), it needs to be checked whether he uses these rules for the construction of a square.

Step 3 - Explication 2 - narrow context analysis: (E2.1) In scene 5, lines 17-23, Nils sets the slider of the scrollbar *breadth* to two unit lengths. Again, he first makes a drag-scrollbar-movement over the scrollbar *breadth* to the right to increase the side length. He sets the number 3. Afterwards, he performs another quick corrective action so that the scrollbar is set to the number 2. His manipulations on the diagram result in the construction of a square. In the same scene in lines 4-10, Nils sets the scrollbars *breadth* and *depth* to the number 1, so that, by doing this, setting a square with side length 1 can be seen on the screen. He states that they already had this square. In the following scenes (scenes 9, 11 and 14), he sets a square with side lengths 3, 4 and 5, one after the other.

(E2.2) The reconstructed diagram interpretation from Explication 1 can be supported by the further passages and extended by the transcript passages in scene 5 (lines 17-23) and the further passages in which Nils sets a square. The manipulations of the scrollbars suggest that Nils uses the geometric diagram to construct a square because he sets the depth and breadth to number 2 by his actions. When constructing the squares with a side length of 1-5, it becomes clear that he uses the signs on the prompt. The assumption can be supported that Nils translates the relationships of the arithmetic diagram into the relationships of the geometric diagram.

(E2.3) The focus of the prompt is on the relationship between side length and perimeter, so this aspect should be taken into account in similar passages. In scene 7, Nils looks at the tablet on which a square with side length 2 can be seen. When asked by the accompanying person how many unit lengths would fit onto one side, he answers "two". Additionally, in scene 9, Nils answers, after adjusting the square, that three unit lengths would fit onto one side. In scene 14, after setting the square, Nils records the unit lengths that fit onto one side of the square directly at the prompt, without any verbal utterances. In scene 8, in response to the accompanying person asking how many unit lengths fit onto all sides, Nils looks at the square with side length 2, which can be seen on the screen. Scarcely moving his head, twice up and down, he then utters "eight\". In scene 10, Nils looks at the square with side length 3, which can be seen on the screen. He asks himself the question "and (.) on all sides/". He answers his self-posed question with the statement "nine/". Immediately he revises his answer by saying "no\" while looking at the screen, which still shows the square with side length 3. He looks at the screen for 14 seconds without action, gestures or phonetic utterances. Subsequently, he pronounces "twelve/" while not taking his eyes off the screen. In scene 14, after a longer look at the screen, on which a square with a side length of 5 can be seen, Nils says "twenty/". In each scene, Nils notes the number of unit squares on one or all sides in the table on the prompt, following the consent of the accompanying person (scenes 8, 10 and 14). In scene 16, Nils notices that in the first column of the table, at the prompt, there is always one unit length added and in the second column there are always four unit lengths added.

(E2.4) At this point, the description of rule-guided mathematical actions by a person experienced in usage still needs to be supplemented. Until now, creating a side model of a square by moving the scrollbar was at the centre of the passage in the transcript to be explicated. Now the determination of the unit lengths is the focus for one side and for all sides of the square; this is also required in the prompt. To determine the perimeter, it can first be determined how many unit lengths can be used to measure one side of the side model and then how many are needed for all sides of the side model. To determine the number on one side, the lengths of the squares on the square grid bordering the sides of the square can be counted. Thus, the square grid can be interpreted as a measuring instrument. Comparative actions must be performed by the learners to determine a numerical value for the side length. The comparative actions, or the counting of the unit lengths, can be carried out gesturally. At this point, the gestural comparative actions are very closely connected with the measuring actions. In order to determine the unit lengths on all sides, the number of one side length can be added up four times, multiplied by four or counted from one side length.

The actions required to determine the number of unit lengths can be recorded as arithmetic signs on the prompt. If the numbers of unit lengths at one or all sides are expressed as arithmetic signs, they can be interpreted as an arithmetic diagram and the following relationships can be recognised: the number of unit lengths on one side always increases by one, whereas the number of unit lengths on all sides increases by four. If a reference to the geometric diagram is made, it can be seen that the enlargement of one side length affects the perimeter four times because the perimeter includes all four sides of the square. To formulate the relationships between the diagrams, the learners must recognise that the arithmetic diagram is a model *for* the geometric diagram. The interpretation of the relationships in the arithmetic diagram can be done by gestural or phonetic utterances; both modes refer to the action that was necessary to write down the arithmetic signs.

(E2.5) When determining the number of unit lengths on all sides, it can be seen that in scene 8 Nils scarcely notices his head moving up and down twice while looking at the square with side length 2 on the screen. According to de Ruiter (2000), gestures can also be performed with parts of the body other than the hands or arms, for example, when the learner sits on the hands or holds something in the hand. In this case, Nils only has a pen in his hand. It can be assumed that the head movement refers to a counting process. This suspicion can be supported by the longer periods in scenes 8, 10 and 14 (sometimes up to 17 seconds) that Nils looks at the screen. A counting process takes longer than, for example, quadrupling the unit lengths on one side of the square. However, it cannot be excluded that Nils might have added the number of unit lengths on all sides or multiplied it by four as he would need some time for this. With reference to the theories of Kadunz (2015, 2016) and Krämer (2007), the nodding of the head can be interpreted as an index or trace for the counting process. Since the counting process is an invisible sign, it can be a natural index if Nils does not want to make the counting process public. If he wishes to show the counting process to other people, this could be an artificial index. If the nodding of the head occurs after the counting process, this could be called a trace. In scene 16, Nils tries to interpret the relationships between the arithmetic signs on the prompt. It can be assumed from his phonetic utterances that he can interpret the relationships between the inscriptions of the arithmetic diagram as he notices that for the unit lengths on one side of the square, there is always one unit length added and, for the perimeter, there are always four. At this point he uses the arithmetic signs as indices to his previous actions to establish a relationship between the arithmetic and geometrical diagrams.

(E2.6) In the narrow contextual analysis, the reconstruction of Nils' interpretation, i.e., that he can interpret the rules of the geometric diagram for the extension of the sides, can be confirmed and extended. It can be assumed from his actions that he uses the rules of the geometric diagram interpreted by him to construct squares of different sizes. The further manipulations of the scrollbar's breadth and depth settings, in the same and subsequent scenes, support Nils' reconstructed diagram interpretation. In the similar passages to the explicative transcript passage, in comparison to the rule-guided mathematical actions or gestures of a person experienced in usage, it becomes clear that Nils does not use counting gestures with his hands to determine the number of unit lengths on one or all sides of the square. He just looks at the differently sized squares on the screen and, after a period of time, expresses a numerical value for the number of unit squares. It is noticeable that he looks at the screen for a shorter time for the number of unit lengths on one side than for the unit lengths on all sides. Due to the head movements and the longer time he looks at the screen, it can be assumed that he counts the unit lengths bordering the sides of the square. The head movements can be interpreted as an index or track. Nils can interpret the arithmetic signs on the prompt by naming the manipulation rules for changing the number of unit lengths. It is not evident whether Nils knows the reason for the relationships between the inscriptions of the arithmetic diagram. He recognises that the arithmetic diagram is a model *for* the geometric diagram, but does not express how the length of the perimeter changes in relation to the side length.

Step 4 - Explication 3 - broad context analysis: Due to lack of space, only the summary of the broad context analysis can be presented. For the broad context analysis, further passages from the processing that was videotaped were used. Based on the passages found in the broad context analysis, assumptions from the previous expositions can be confirmed. It can be confirmed from Explication 1 that Nils can interpret the effects of the scrollbar on the side model of the square. He transfers, as already described in Explication 2, the rules for constructing a square to the same setting of the scrollbars. The assumption that Nils counts the number of unit lengths due to the slight head movements up and down and the longer time he looks at the screen, could be supported by two video passages. In minutes 29 and 37, as in Explication 2, it can be assumed that Nils can interpret the related inscriptions on the prompt as an arithmetic diagram. In the broad context analysis, it can also be assumed why the manipulations of the geometric diagram decrease in the course of the learning situation. This can be attributed to the fact that Nils interprets the arithmetic signs on the prompt as indices of his previous actions on the geometric diagram. It can be reconstructed that he, thereby, recognises the arithmetic diagram as the model *for* the geometric diagram. On this basis, he can also interpret another arithmetic diagram at a different prompt as a model *for* the geometric diagram and, thus, establish a relationship between the two arithmetic diagrams. The relationship he recognises between the two arithmetic diagrams allows him to transfer the arithmetic signs from one arithmetic diagram to the other which makes manipulation of the geometric diagram unnecessary.

Step 5 - Summary: It can be reconstructed from Nils' actions at the scrollbar that Nils recognises the rules for lengthening the side length and interprets the relationship between the scrollbar and the square as a geometric diagram. He uses the effects of the scrollbar on the side length of the rectangular square to construct a square. He shows this by many actions on the scrollbars. In Explication 2, it can be seen that he establishes a relationship between the side length (which can be seen at the prompt in the table) and the geometric diagram. To determine the number of unit lengths on one or all sides of the square, gestures and spoken language close to the action were included. It becomes clear that Nils does not use finger-generated counting gestures to determine the number of side lengths. Based on the longer time Nils looks at the square on the screen and the barely noticeable head movements, it could be assumed in both Explication 2 and Explication 3 that Nils counts the unit lengths on all side lengths. The head movements can be interpreted as an index on or a track of the counting process. It can be assumed that Nils can interpret the arithmetic diagram after writing down the numbers of the unit squares on one or all sides of the square. He recognises that the arithmetic diagram is a model *for* the geometric diagram and uses it to talk about the geometric diagram. However, it is not clear whether Nils knows the reason for the relationship between the arithmetic signs he has recognised since he does not mention how the length of the perimeter changes in relation to the side length. In the course of the learning situation, it can be seen that the manipulations on the geometric diagram and, thus, the actions on the scrollbar decrease. It can be reconstructed that he interprets the arithmetic signs as indices to his previous actions at the geometric diagram. Therefore, it can be assumed that he recognises the arithmetic diagram as a model *for* the geometric diagram. Similarly, he can also interpret another arithmetic diagram at a different prompt as a model *for* the geometric diagram and establish a relationship between the two arithmetic diagrams. This allows him to transfer the arithmetic signs from one arithmetic diagram to another, thus, making manipulation of the geometric diagram unnecessary.

Reconstruction of Interpretations of Diagrams when Working with Analogue Material

It could be shown that two tandems processed the prompt with the mathematical aspect *perimeter* and the language level 1 in the fifth position of their processing. Firstly, Nils' diagram interpretation using digital materials was reconstructed. In the following, the diagram interpretation of Marleen is reconstructed, who processed the same prompt but with analogue material. Only the last step of the analysis (Step 5 - Summary) is shown.

It can be seen from Marleen's actions that she can interpret the rules for constructing a square (putting the sticks together with corner joints) and uses them to manipulate the geometric diagram. She appears to be aware that she needs four rods of equal length to construct the square; this can be reconstructed by selecting the four rods of equal length. It can be reconstructed from her actions of putting the rods together with the corner joints, or placing them on the lines of the square grid, that her actions are based on the rule that the rods must be at right angles to each other. It becomes clear in Marleen's activity of constructing that she has established a relationship between the arithmetic diagram (the side length in the table on the prompt) and the geometrical diagram (material arrangement). To determine the number of unit lengths, she does not need the unit square, but rather the square grid as a measuring instrument. From the data, it can be seen that Marleen uses gestures to determine the number of unit lengths on all sides of the square. It is striking that the gestures of counting are very much reduced in the narrow context analysis and occur only in two passages. Instead, only her gaze is directed towards the square grid and she expresses the result verbally or relates parts of an arithmetic diagram on another prompt to the arithmetic diagram that she is working on. She tries to transfer arithmetic signs from one arithmetic diagram to the other. In the broad context analysis, it can be seen that the arithmetic signs that Marleen adopts in the narrow context analysis were determined by actions of construction and gestures of counting on the geometric diagram. Therefore, it can be assumed, that she interprets the arithmetic signs as indices to the previous action and recognises that the arithmetic diagrams are models *for* the geometric diagram. She does not always succeed in this interpretation of the arithmetical signs and she uses partial gestures of counting to determine the number of unit lengths again.

Overall, it is noticeable that when constructing the squares at the beginning of the learning situation, the spoken language does not support her actions because Marleen is talking about other topics while she is constructing the squares. Here, Marleen's geometrical diagram interpretation can be reconstructed exclusively based on her actions. As soon as she no longer performs the actions explicitly, since she transfers the arithmetic signs from one arithmetic diagram to the other, she uses her phonetic utterances to refer to the actions on the geometric diagram. It can be observed that as the number of her actions decreases, Marleen's gestures and her spoken language become more mathematical.

Comparison of the Results of the Qualitative Analysis

By comparing the empirical results of the analysis, it can be seen that both learners interpret the material arrangement as a geometric diagram. By constructing the square, both learners establish a relationship between the side length indicated on the prompt (arithmetic diagram) and the geometric diagram. It can also be seen that both can interpret the square grid and use it to determine the number of unit lengths, although Marleen makes the counting gestures public in comparison to Nils. Both learners write down arithmetic signs for the number of unit lengths on one or all sides of the square. Nils expresses an interpretation of the arithmetic signs as a diagram by referring to the change of the number of unit lengths on one or all sides. He, thus, recognises the manipulation rules of the arithmetic diagram. The comparison of the analysis results also shows that the learners interpret the arithmetic diagram as a model *for* the geometric diagram and can correlate arithmetic diagrams on different prompts. Compared to Nils, Marleen does not always succeed in this. Due to the relationship between the arithmetic diagrams recognised by Nils and Marleen, both analyses clearly show that the actions on the diagram of geometry decrease and the importance of gestures and spoken language increases. Overall, the comparison of the analyses shows that although the actions on the digital and analogue materials differ, similar diagram interpretations can be reconstructed for both learners.

Inclusion of Further Analyses

The findings from the comparison of the diagram interpretations of Marleen and Nils will be compared with the results of two other analyses. As shown in **Figures 2a**, **2b** and **Figures 3a**, **3b**, two further tandems dealt with the prompt, focussing on the mathematical aspect *perimeter* and the language level 1, at the fourth position of their processing. As in the previous analyses, the diagram interpretation of one student was reconstructed from the tandems. Thus, the actions of Li and Emre were addressed; Li worked with the analogue material, while Emre worked with the digital material.

In all analyses, it becomes clear that the learners can interpret the rules for the construction of a square in the respective geometric diagram (analogue or digital) and use suitable manipulations for the construction of squares. It can be reconstructed that the learners establish a relationship between the arithmetic and the geometrical diagrams. The learners use counting gestures that make clear a comparative action of the square grid as a measuring instrument with the side of the square, or look at the square grid in order to determine the unit lengths, although here the comparative action is not explicit. Li first makes measurements on the analogue material by measuring the length of the other rods with the rods of length 1. In the course of the learning situation, she transfers this procedure to the square grid so that she moves from the measuring actions to comparative actions. The square grid is used by the other learners as a measuring instrument from the beginning. It becomes clear that the actions during the learning situations are reduced; this can be attributed to the fact that all learners recognise the arithmetic diagram as a model *for* the geometric diagram. In this way, they can establish relationships between arithmetic diagrams and transfer arithmetic signs from one arithmetic diagram to another. This could be reconstructed in the narrow context analysis and also in the broad context analysis. The learners interpret the arithmetic signs as an arithmetic diagram and recognise how the number of unit lengths changes on the sides of similar squares. Exclusively in the reconstruction of Emre's diagram interpretation, it becomes clear that he can speak arithmetically about the geometrical diagram. By establishing a relationship between the arithmetic signs and the geometric diagram, he succeeds in describing that the length of the sides occurs four times in the perimeter. He succeeds in the flexible translations between the arithmetic and geometric diagrams.

A comparison with the results of other analyses also shows that similar diagram interpretations of the learners can be reconstructed, although two learners worked with digital materials and two with analogue materials. It can be seen that the actions are different with using the various materials, but the same relationships are expressed with the actions. Thus, as a result, the same diagram interpretations can be reconstructed.

DISCUSSION

The paper aimed to carry out a semiotic adaptation of the context analysis and to illustrate it with an example. Also, by comparing the results of the analysis, it should be shown whether different interpretations of the learners can be reconstructed when qualitatively analysing their actions on the digital or analogue material arrangement. The results presented in the comparison are to be discussed in the following and, as a result, a perspective for the use of digital and analogue materials in the mathematics lessons of primary school children will be presented. Furthermore, the limitations of the results will be addressed and a perspective will be given.

Major Findings

When reconstructing the learner's diagram interpretations, many parallels can be seen. It can be reconstructed in all analyses that the learners can interpret the relationships of the geometric diagram, can design an arithmetic model *for* the geometric diagram and establish the relationships between the arithmetic and geometry. It can be shown by the reconstruction, that the learners can understand the arithmetic diagram as a model *for* the geometric diagram and translate the relationships from one diagram to the other. This allows the learners to connect the two diagrams and also enables the learners to speak arithmetically about the geometric diagram, or the other way round. Although the different materials result in a structural change of the actions, the same diagram interpretations can be reconstructed from the different actions. Despite the different actions, the same relationships are expressed and there is a reciprocal process between actions and signs (Kadunz, 2015, 2016). Even if two learners use the scrollbar and have two more plastic rods with corner joints at their disposal, they both use rule-based manipulations where the same relationships are observed to create a square and to determine the number of unit lengths. This shows that the appearance of the inscriptions, in this case the material arrangements in this learning situation, have no influence on the children's diagram interpretations. This can be incorporated into the theory, since, according to Dörfler (2015), and Shapiro (1997), the appearance of inscriptions is interchangeable and they only acquire their significance through the relationship between the inscriptions and the activity of the people.

In all analyses, the square grid is used by the learners as a measuring instrument by performing comparative actions between the sides of the square and the square grid. Through these comparative actions, learners can record a numerical value for the number of unit lengths (Dörfler, 1988). In the reconstructions of the diagram interpretations, it becomes clear that the learners use the square grid and, therefore, counting gestures, or may simply look at the square grid to determine the number of unit lengths.

Furthermore, in all analyses, it can be reconstructed that the learners record the number of unit lengths in arithmetic signs. By using arithmetic signs, it is possible to think arithmetically about the change of the perimeter of similar squares and it is also possible to speak arithmetically about the geometrical diagram. In this way, the relationship of the geometric diagram need no longer be constructed by actions, rather the arithmetic signs can be used for arithmetic considerations in the further course of the learning situation. It also makes it possible to recognise the relationships between arithmetic diagrams on multiple prompts.

All in all, concerning the aim of the paper formulated at the beginning, it can be stated that the diagram interpretations can be reconstructed by the actions of the learners and that the analysis with the presented semiotic specifications is suitable for reconstructing diagram interpretations. It can be shown that the appearance of the inscriptions, or more precisely the design of the material, has no effect on the diagram interpretation of the learners in this learning situation. For learning mathematics with different material arrangements, decisive new insights could be gained with the help of the analyses. Digital and analogue material arrangements, or diagrams, can evoke the same mathematical interpretations in learners if one material arrangement can be interpreted as a model *for* the other and the same relationships are expressed despite different actions. For learning mathematics in primary school this means that the same learning goal can be achieved through learning situations realised with digital or analogue materials. In order to achieve the same learning goal, it is important that the learners observe the same mathematical relationships when acting on the material, even if the movements differ. This is because if the learners recognise the same mathematical relationships between the inscriptions and observe them in their actions, the appearance of the inscriptions is interchangeable.

Limitations and Outlook

In this paper, only a small amount of data from geometry and arithmetic was considered. In the future, further analyses should provide a deeper insight into the diagram interpretations of learners who are dealing with a similar mathematical task. Besides, diagram interpretations of learners working on tasks on statistical topics will be reconstructed. Further analysis will show whether the design of the material has any influence on the diagram interpretation of the learners in the statistical field.

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REFERENCES

Aebli, H. (1980). *Denken, das ordnen des tuns. Band 1 Kognitive aspekte der handlungstheorie* [Thinking, ordering what is done. Volume 1 Cognitive Aspects of Action Theory]. Klett.

Bannert, M. (2009). Prompting self-regulated learning through prompts. *Zeitschrift für Psychologie*, *23*(2), 139-145. https://doi.org/10.1024/1010-0652.23.2.139

Billion, L., & Vogel, R. (2020a). Grundschulkinder arbeiten digital an einem geometrischen problem – Rekonstruktion mathematischer deutungen [Elementary school children work digitally on a geometric problem - reconstruction of mathematical interpretations]. In S. Ladel, C. Schreiber, R. Rink, & D. Walter (Eds.), *Aktuelle forschungsprojekte zu digitalen medien in der primarstufe* (pp. 135-150). WTM-Verlag. https://doi.org/10.37626/GA9783959871747.0

Billion, L., & Vogel, R. (2020b). Material as an impulse for mathematical activities in primary school – A semiotic perspective on a geometric example. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds), *Interim Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (pp. 28-36). PME.

Billion, L. (2018) Mathematical learning processes with varying types of material conditioning. In E. Bergqvist, M. Österholm, C. Granberg, & L. Sumpter (Eds.), *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 5, pp. 207). PME.

Carey, S. (1988). Conceptual differences between children and adults. *Mind and Language*, *3*(3), 167-181. https://doi.org/10.1111/j.1468-0017.1988.tb00141.x

De Ruiter, J. P. (2000). The production of gesture and speech. In D. McNeill (Ed.), *Language and Gesture* (pp. 284-311). University Press. https://doi.org/10.1017/CBO9780511620850

Dörfler, W. (1988). Die genese mathematischer objekte und operationen aus handlungen als kognitive konstruktion [The genesis of mathematical objects and operations from actions as a cognitive construction]. In W. Dörfler (Ed.), *Schriftreihe Didaktik der Mathematik Band 16. Kognitive Aspekte mathematischer Begriffsentwicklung* (pp. 55-126). Verlag Hölder-Pichler-Tempsky.

Dörfler, W. (1991). Der computer als kognitives werkzeug und kognitives medium [The computer as a cognitive tool and medium]. In W. Dörfler, W. Peschek, E. Schneider, K. Wegenkittl (Eds.), *Computer – Mensch – Mathematik* (pp. 51-75). Verlag Hölder-Pichler-Tempsky.

Dörfler, W. (2006a). Diagramme und mathematikunterricht [Diagrams and math lessons]. *Journal der Mathematikdidaktik*, 3/4, 200-219. https://doi.org/10.1007/BF03339039

Dörfler, W. (2006b). Inscriptions as objects of mathematical activities. In J. Maaz & W. Schlögelmann (Eds.), *New mathematics education research and practice* (pp. 97-111). Sense Publisher. https://doi.org/10.1163/9789087903510_011

Dörfler, W. (2015). Abstrakte objekte in der mathematik [Abstract objects in mathematics]. In G. Kadunz (Ed.), *Semiotische perspektiven auf das lernen von mathematik* (pp. 33-49). Springer-Verlag. https://doi.org/10.1007/978-3-642-55177-2

Floer, J. (1993). Lernmaterialien als stützen der anschauung im arithmetischen anfangsunterricht [Learning materials as support for the intuition in arithmetic beginners lessons]. In J.-H. Lorenz (Ed.), *Anschauung und mathematik* (Band 18) (pp. 106-121). Aulis Verlag Deubner & Co KG.

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning*, *1*(2), 155-177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K. (2002). Preamble: from models to modelling. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, modeling and tool use in mathematics education* (pp. 7-22). Kluwer Academic Publisher. https://doi.org/10.1007/978-94-017-3194-2

Hoffmann, M. (2005). *Erkenntnisentwicklung. Ein semiotisch-pragmatischer ansatz* [Knowledge development. A semiotic-pragmatic approach]. Vittorio Klostermann.

Hohenwarter, M. (2001). *GeoGebra – Dynamic mathematics for everyone.* Austria & USA.

Huber, H. (1972). *OrbiMath. Mathematik konstruktiv* [OrbiMath. Math constructive]. Herder Verlag.

Huth, M. (2018). Die bedeutung von gestik bei der konstruktion von fachlichkeit in mathematischen gesprächen junger lernender [The importance of gesture bi the construction of technicality in mathematical conversations of young learners]. In M. Martens, K. Rabenstein, K. Bräu, M. Fetzer, H. Gersch, I. Hardy, & C. Schelle (Eds.), *Konstruktion von fachlichkeit. Ansätze, erträge und diskussionen in der empirischen unterrichtsforschung* (pp. 219-231). Verlag Julius Klinkhardt.

Huth, M. (2020). Gestische darstellungen in mathematischen interaktionen [Gestural representations in mathematical interactions]. In H.-S. Siller, W. Weigel, & J. F. Wörler (Eds.), *Beiträge zum mathematikunterricht* (pp. 1381-1384). WTM-Verlag. https://doi.org/10.37626/GA9783959871402.0

Kadunz, G. (2015). Zum verhältnis von geometrischen zeichen und argumentationen [On the relationship between geometric signs and arguments]. In G. Kadunz (Ed.), *Semiotische perspektiven auf das lernen von mathematik* (pp. 71-88). Springer-Verlag. https://doi.org/10.1007/978-3-642-55177-2

Kadunz, G. (2016). Geometry, A means of argumentation. In A. Sáenz-Ludloy & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics. How to describe the construction, visualisation, and communication of mathematical concepts* (pp. 25-42). Sense Publishers. https://doi.org/10.1007/978-94-6300-337-7

Krämer, S. (2007). Immanenz und transzendenz der spur: Über das epistemologische doppelleben der spur [Immanence and transcendence of the trace: About the epistemological double life of the trace]. In S. Krämer, W. Kogge & G. Grube (Eds.), *Spur. Spurenlesen als orientierungstechnik und wissenskunst* (pp. 155-181). Suhrkamp.

Latour, B. (2012). *Visualisation and cognition: Drawing things together*. http://worrydream.com/refs/Latour%20-%20Visualisation%20and%20Cognition.pdf

Lorenz, J.-H. (1993). Veranschaulichungsmittel in arithmetischen anfangsunterricht [Illustrative means in beginning arithmetic lessons]. In J.-H. Lorenz (Ed.), *Anschauung und mathematik* (Band 18) (pp. 122-146). Aulis Verlag Deubner & Co KG.

Mayring, Ph. (2014). *Qualitative content analysis: theoretical foundation, basic procedures and software solutions.* Klagenfurt. https://nbn-resolving.org/urn:nbn:de:0168-ssoar-395173

Meyer, M., & Prediger, S. (2012). Sprachenvielfalt im mathematikunterricht – Herausforderung, chancen und förderansätze [Language diversity in mathematics lessons - challenges, opportunities and support approaches]. *Praxis der mathematik in der schule*, *PM 54*(45), 2-9.

Pape, H. (2007). Fußabdrücke und eigennamen: Peirces theorie des relationalen kerns der bedeutung indexikalischer zeichen [Footprints and proper names: Peirce's relational core theory of the meaning of indexical signs]. In S. Krämer, W. Kogge, & G. Grube (Eds.) *Spur. Spurenlesen als orientierungstechnik und wissenskunst* (pp. 37-54). Suhrkamp.

Peirce, C. S. (1865-1909). *The logic notebook*. MS [R] 339.

Peirce, C. S. (1901). Index (in exact logic). In J. M. Baldwin (Ed.), *Dictionary of philosophy and psychology* (Vol. I, pp. 531-532). Macmillan and Co.

Piaget, J. (1998). *Der aufbau der wirklichkeit beim kinde* [The construction of reality in the child] (2nd Ed.). Klett.

Posner, G. J., Strike, K. A., Hewson, P. W., & Gertzog, W. A. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. *Science Education*, *66*(2), 211-227. https://doi.org/10.1002/sce.3730660207

Radatz, H. (1993). MARC bearbeitet aufgaben wie 72 - 59. Anmerkungen zu anschauung und verständnis im arithmetikunterricht [MARC handles tasks like 72 - 59. Notes on visualization and understanding in arithmetic lessons]. In J.-H. Lorenz (Ed.), *Anschauung und mathematik* (Band 18) (pp. 14-24). Aulis Verlag Deubner & Co KG.

Rosch, E. (1978). Principles of categorization. In E. Rosch & B. Lloyd (Eds.), *Cognition and categorization* (pp. 27-48). Erlbaum.

Schreiber, C. (2013). Semiotic processes in chat-based problem-solving situations. *Educational Studies Mathematics*, *82*(1), 51-73. https://doi.org/10.1007/s10649-012-9417-7

Seiler, T. (2001). *Begreifen und verstehen. Ein buch über begriffe und bedeutungen* [Comprehend and understand. A book about terms and meanings] (pp. 129-154). Allgemeine Wissenschaft.

Shapiro, S. (1997). *Philosophy of mathematics. Structure and ontology.* Oxford University Press.

Sinclair, N., & de Freitas, E. (2014). The haptic nature of gestures. Rethinking gesture with new multitouch digital technologies. *Gesture*, *14*(3), 351-374. https://doi.org/10.1075/gest.14.3.04sin

Thomas, R. (2009). Mathematics is not a game but… *The Mathematical Intelligencer*, *31*(1), 4-8. https://doi.org/10.1007/s00283-008-9015-9

Villamor, G., Willis, D., & Wroblewski, L. (2010). *Touch gesture reference guide*. https://www.lukew.com/ff/entry.asp?1071

Vogel, R. (2017). "wenn man da von oben guckt sieht das aus als ob …“ – Die 'dimensionslücke‘ zwischen zweidimensionaler darstellung dreidimensionaler objekte im multimodalen austausch ["When you look from above, it looks like ..." - The 'dimension gap' between two-dimensional representation of three-dimensional objects in multimodal exchange]. In M. Beck & R. Vogel (Eds.), *Geometrische aktivitäten und gespräche von kindern im blick qualitativen forschens. Mehrperspektivische ergebnisse aus den projekten erStMaL und MaKreKi* (pp. 61-75). Waxmann-Verlag.

Vogel, R., & Huth, M. (2020). Modusschnittstellen in mathematischen lernprozessen. Handlungen am material und gesten als diagrammatische tätigkeit [Mode interfaces in mathematical learning processes. Actions on material and gestures as diagrammatic activity]. In G. Kadunz (Ed.), *Zeichen und sprache im mathematikunterricht – Semiotik in theorie und praxis* (pp. 215-255). Springer Spektrum. https://doi.org/10.1007/978-3-662-61194-4

Wessel, L., Büchter, A., & Prediger, S. (2018). Weil sprache zählt – Sprachsensibel mathematikunterricht planen, durchführen und auswerten [Because language counts - language-sensitive planning, implementation and evaluation of math lessons]. *Mathematik Lehren*, *206*, 2-7.

Wittmann, E. Ch. (1981). *Grundfragen des mathematikunterrichts* [Basic questions of math class] (6th Ed.). Vieweg. https://doi.org/10.1007/978-3-322-91539-9

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Semiotic analyses of actions on digital and analogue material when sorting data in primary school

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Abstract
This paper focuses on the actions of learners on digital and analogue materials while dealing with a statistical problem. To investigate the learners' actions, a semiotic perspective of mathematical learning according to C. S. Peirce is used, since in this perspective learning mathematics is described as visible activities on diagrams. Through a qualitative semiotic analysis of the actions of two third-graders working with given data, the statistical diagram interpretations of the learners can be reconstructed. A comparison of the reconstructed diagram interpretations reveals whether different movements lead to similar diagram interpretations. In addition, it is of interest whether the diagram interpretations are the same when acting on digital and analogue diagrams because the same mathematical relationships have to be observed. Through this comparison, conclusions can be drawn about the similarities and differences in working with digital and analogue materials and how these materials may be used profitably in statistical learning.

Keywords: diagrammatic reasoning, mathematical actions, primary school children, semiotic perspective on learning mathematics, statistical learning

INTRODUCTION

In recent years, the practice and research in mathematics education have shown that the use of digital material can play an important role in helping learners "represent, identify, and explore behaviors of diverse mathematical relationships" (Moreno-Armella & Sriraman, 2010, p. 221). Therefore, much research focuses on the digital itself. However, there is little research available on how such a process of exploring mathematical relationships with digital technologies differs from the processes with paper and pencil. Moreno-Armella and Sriraman (2010) provide an indication that a tool, whether analogue or digital, can have an impact on human action, while the mathematical ideas and the mathematical processes can be totally different.

Tools, such as a compass, change the mathematical action of drawing a circle; the tool establishes the mathematical relationship between the center and this relationship does not have to be considered in the actions when drawing a circle. Drawing a circle without a compass means that during the action one has to make sure that every point has the same distance to the center. With a tool, the relationships can be observed during and have to be interpreted after the action, whereas, without a tool, the relationships have to be recognized before to be able to perform the action. Thus, a tool, whether digital or analogue, can change the way one acts with the signs and the relationships between the signs.

The aim of this paper is to explain, by comparing actions on digital and analogue material, how learners' mathematical actions are influenced by digital and analogue materials, whether the digital or analogue material functions as tools and whether a possible change in actions has an impact on the learners' mathematical interpretations. This is achieved by using a Peircean (1931-1935) semiotic perspective on mathematical learning, which describes mathematics learning as a visible activity with diagrams. In this line, Dörfler (2006) describes learning mathematics as

> "something like a reflected handicraft of working productively with diagrams. This underlines the materiality of mathematics and mathematical activities versus its purported abstractness […]." (p. 105)

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Contribution to the literature

- In the paper, the actions of primary school children on digital and analogue materials are analyzed to reconstruct their mathematical interpretations. The compared analyses provide information about whether comparable digital and analogue learning situations lead to different interpretations.
- The analysis, which is specifically adapted to the semiotic perspective according to Peirce, enables a focus on mathematical activities on materials rather than the pure evaluation of didactically prepared materials.
- The results show how the materials function as tools that can influence the learners' actions and interpretations, and refer to how digital and analogue material can be used profitably in primary mathematics lessons.

A diagram, according to Peirce can be understood as "a representation of relations that is constructed by means of 'system of representation.' Such a system is defined by a set of rules, conversations, and a certain ontology" (Hoffmann, 2010, p. 42). A 'system of representation' can be a specific order of material, something written down on paper or related gestures. In this paper, adopting this semiotic view, the actions on digital and analogue material arrangements are analyzed to reconstruct which rules and relations the learners can recognize and establish in the 'system of representation'. These reconstructed mathematical diagram interpretations of the learners are compared to describe possible differences or similarities between them.

With this goal in mind, firstly, the theoretical background describing the semiotic perspective according to Peirce on the learning of mathematics is presented. Secondly, important terms required for teaching statistics in primary school are briefly highlighted. Thirdly, a link is made between the theory presented and the aims of this paper which are used to formulate the research questions. This is then followed by the description of the data collection and the presentation of the semiotic qualitative analysis. Subsequently, the diagram interpretations of two third-graders (9 year-olds) are reconstructed and compared. At the end of the paper, the results are summarized and discussed with regard to the research questions.

THEORETICAL FRAMEWORK

The theoretical framework clarifies what signs, their usage, and the term of diagrammatic reasoning mean and, thus, elaborates a semiotic definition of actions. It also relates to the mathematical content of the empirical example by focusing on statistical learning.

The Semiotic Perspective on Learning Mathematics

In the semiotic sense, with a mathematical activity, we perform "means of visible signs, and by interpreting and transforming signs we develop mathematical knowledge" (Hoffmann, 2006, p. 279). Thus, working with visible signs, which includes the interpretation and transformation of signs, is the core of doing mathematics. By using signs, we have the means to think about mathematical relations and thereby develop new signs. Peirce (CP 2.228), describes a sign as something that stands for someone in a certain sense. A sign is directed to someone and evokes in that person an *interpretant*, for example, an equivalent or perhaps more developed sign (CP 2.228). This "interpretant is determined by the concepts, theories, habits, and skills of the observer […]" (Schreiber, 2013, p. 57). Therefore, the recognition of relations between signs, or the perception of a sign as a sign, depends on the previous mathematical experiences of the learners (Billion, 2021a). In other words, "[P]attern recognition depends partly on the already memorized patterns" (Dörfler, 2006, p. 108).

Peirce distinguishes three types of signs: icons, indices and symbols. An icon is characterized by its similarity to its object and has the function of representing relationships (Bakker & Hoffmann, 2005). However, icons are not similar themselves; the impression of similarity is because one can do possible activities with the icons (Kadunz, 2006). An index draws the sign reader's attention to something, while the meaning of a symbol is determined by its usage or a rule (Bakker & Hoffmann, 2005). The described types of signs can be placed in relationship to each other and, in this way, signs that are more complex can be generated; diagrams are described as such, being complex signs which "are primarily of an iconic character but contain indexical and symbolic elements as well" (Dörfler, 2006). A definition that focuses less on the three types of signs states that signs which are in a contradiction-free context and have certain rules for usage are considered diagrams in the Peircean sense (Dörfler, 2016). Based on these definitions, diagrams are not only geometric figures but can also be equations, algebraic theorems or a sentence in spoken language (Dörfler, 2016).

Diagrams, themselves, do not have a fixed meaning or sense; the (mathematical) meaning unfolds through the activities with diagrams which include the learners' interpretations (Dörfler, 2006). In this sense, mathematical reasoning is diagrammatic and "all reasoning depends directly or indirectly upon diagrams" (Peirce, NEM IV, p. 314):

> "By diagrammatic reasoning, I mean reasoning which constructs a diagram according to a precept expressed in general terms, performs experiments upon this diagram, notes their results, assures itself that similar experiments performed upon any diagram constructed according to the same precept would have the same results, and expresses this in general terms" (Peirce, NEM IV, p. 47-48).

Mathematical rules and relations determine how to construct, read and experiment with a diagram (Hofmann, 2010). Depending on the activities, according to the different possible rules, the signs "might give rise to essentially different diagrams" (Dörfler, 2016, p. 25). For instance, Dörfler (2016) draws a comparison to an everyday example in which the same cards of a playing card deck perform different roles in different card games. Thus, the use of diagrams, which means activities according to certain rules and relations between the signs, is of great importance for learning mathematics.

In the empirical example of this paper, the learners have to construct a statistical display to answer given questions. In relation to the diagrammatic reasoning mentioned above, the construction of a diagram could be the creation of a plot with a sorting according to the values of a characteristic on the data card. Generating such a diagram must be motivated by the need to represent relations that are significant to the answer the given question. To explore the constructed diagram, actions defined by the relationships between the signs can be performed, for example, modifying the scale of the plot. Changing the scale of the plot is bound to certain relationships:

> "a dot has to be put above its value on the x-axis and this remains true if the scale is being changed." (Bakker & Hoffmann, 2005, p. 341).

When transforming the diagram, other relationships come to the fore and it becomes possible to make more detailed assumptions about the distribution. Diagrammatic reasoning also includes the observation of the results of exploring and reflecting on them. It is primarily through this reflection that new mathematical relationships can be recognized and expressed in general terms.

The Understanding of Actions in a Semiotic View

According to the semiotic theory, actions on diagrams are of major importance in learning mathematics (e.g., Dörfler, 2006; Peirce, NEM IV). These actions allow the learners to construct and explore diagrams and, once performed, the actions bring to light other mathematical relations that enable (mathematical) knowledge processes. The difficulty of performing the appropriate actions on a diagram depends on recognizing and observing the relationships that the diagram represents. Learning mathematics can be described as a mutual process in which learners notice already known relations and observe them in their actions, and can recognize further relations by observing the results of these actions. In this way, the more adept the learners are in using a diagram, the more mathematical relations they know and the better they can decide which actions are possible and which are not. This suggests that actions on the diagram are determined by the learner's interpretation, which, in turn, depends on their mathematical experiences and interaction with other learners (Dörfler, 2006). Huth (2022) shows that gestures, like actions, can also indicate possible transformations on diagrams and even the gestures themselves can have a diagrammatic character.

In terms of the types of signs described by Peirce, the focus is especially on indices as an index can be regarded as a reference to action,

> "An index represents an object by virtue of its connection with it. It makes no difference whether the connection is natural, or artificial, or merely mental" (Peirce, MS [R] 142).

For indices, the connection to an object is of great importance as it refers to the object through this connection. Concerning the actions on a material, the sign that arises from such actions can be regarded as an index of these actions. In the process of the action, the sign tells the actor whether he or she is right or wrong, thus, the actor can adjust their actions accordingly (Kadunz, 2016). Through this reciprocal process, the actor can notice relationships during the action, which, in turn, are expressed through the action. Tools can shorten the actions and, hence, the relationships may not be obvious in the actions. Kadunz (2016) argues that when a digital tool is used there can be a complete separation of the action and the relation. In the statistical example considered in this paper, during the action of assigning a characteristic carrier to a value on the scale, the resulting sign (arrangement of material) reports back to the actor whether the assignment was correct. In the action, the relationship between the characteristic carrier and its matching value is expressed by the positioning of this characteristic carrier in the plot.

By using a digital tool such as TinkerPlots (Konold & Miller, 2011), a click on the *separate* button assigns all characteristic carriers to the appropriate values on the scale. With the action of clicking, the relationships represented by the diagram are no longer observed. TinkerPlots also permits a drag-movement over the plot to assign a characteristic carrier to a value on the scale. By using this action, some relationships are established, such as the relationship between the characteristic and the axis on which the scale is plotted. However, the relationship between the characteristic carrier and its matching value is not expressed in the action.

Statistical Learning in Primary School

To live as an independent citizen in a data-driven community "today's students need to learn to work and think with data and chance *from an early age*" (Ben-Zvi, 2018, p. vii, emphasis in the original). It is primarily important that learners develop an understanding of data rather than teaching learners different skills independently of each other (Ben-Zvi & Garfield, 2004). To learn a comprehensive understanding of data, the terms of statistical literacy, statistical reasoning, and statistical thinking are often used in research (e.g., Ben-Zvi & Garfield, 2004; Frischemeier, 2020). Ben-Zvi and Garfield (2004) distinguished these three terms as follows:

> "**Statistical literacy** includes basic and important skills that may be used in understanding statistical information or research results. These skills include being able to organize data, construct and display tables, and work with different representations of data. […] **Statistical reasoning** may be defined as the way people reason with statistical ideas and make sense of statistical information. This involves making interpretations based on sets of data, representations of data, or statistical summaries of data. […] **Statistical thinking** involves an understanding of why and how statistical investigations are conducted […and] statistical thinkers are able to critique and evaluate results of a problem solved or a statistical study." (p. 7, emphasis in the original)

Sriraman and Chernoff (2020) concede that the described perspective on statistical and probabilistic learning, in distinguishing the three terms above, has a psychological origin and not an epistemological one. Subsequent to this, Kollosche (2021) draws attention to the following:

> "Thus, the questions how probabilistic and statistical reasoning relates to other forms of reasoning, how it justifies assertions, and how it contributes to our understanding of the world are not yet part of the academic reflections of the field" (p. 482).

Due to this need for research, in this paper an attempt is made to establish a link between diagrammatic reasoning and statistical learning. In addition, many researchers see modelling as a way to work with complex data (e.g., English, 2018; Gravemeijer, 2002; Wild & Pfannkuch, 1999). Modelling includes statistical processes such as

> "[...] posing, and refining questions; collecting and organizing data [...] and drawing conclusions and informal inferences from models generated [...]" (English, 2018, pp. 296-297).

Doerr and English (2003) define models as

> "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behavior of some other familiar system" (p. 112).

In a similar way, Hestenes (2013) defines a model as

> "a representation of structure in a given system. A system is a set of related objects, which may be real or imaginary, physical or mental, simple or composite. The structure of a system is a set of relations among its objects" (p. 17).

These definitions of a model are similar to the above definitions of a diagram in the Peircean sense, hence, this supports the idea of this paper in attempting to establish a link between statistical learning and diagrammatic reasoning.

Following diagrammatic reasoning (Peirce, NEM IV) described above, it can be assumed in this paper that statistical literacy is important for constructing a diagram, such as a plot. It is necessary to understand the statistical information to organize the data and construct a diagram. To experiment with or investigate a statistical diagram, such as a plot, learners must identify and interpret relations. In addition, different diagrams, such as the data card and the plot, can be combined to render further relations between values visible. For this, it is important that the learners can make sense of statistical information, thus, statistical reasoning is in the foreground when transforming diagrams. Observing the transformations of the diagrams, reflecting on the results and expressing these results in general terms can be described by statistical thinking. Statistical literacy, statistical reasoning and statistical thinking can be found beyond the proposed classification in all steps of diagrammatic reasoning. Attempting to establish a link between diagrammatic reasoning and statistical learning it can be assumed in this paper that diagrammatic reasoning contains the central terms of statistical learning. To continue in the semiotic perspective, this paper talks about diagrammatic reasoning because it encompasses all the important aspects of working with data.

In the example analyzed in this paper, the focus of the learners' task is on answering given questions using the statistical diagrams they have created with different materials. Questioning is an important aspect of understanding data (Friel et al., 2001). However, not every question is the same as another, so

> "[s]hallow questions address the content and interpretation of explicit material, whereas deep questions involve inference, application,

synthesis, and evaluation [...]" (Graesser et al., 1996, p. 23).

Friel et al. (2001) distinguish between three question levels: the elementary level focuses on extracting data from a plot (read the data), the intermediate level characterizes finding a relationship in the data (read between the data) and the advanced level requires analyzing the relationships implicit in the plot (read beyond the data). In the statistical learning situation considered in this paper, learners are asked questions that can be categorized into the first two levels. For example, the question '*How many children have named purple and how many children have named orange as their favorite color?*' posed to the third-graders can be classified as reading the data. Another question, the third-graders were asked to answer in another statistical learning situation is '*Do more boys or more girls attend the class?*' focusing on reading between the data.

Research Focus

In the following, implications for the research interest of this paper are described from the theoretical explanations and the research questions are formulated.

Relationship between the theory & the aim of the paper

The semiotic perspective on mathematical learning emphasizes the rule-governed (mathematical) activities with and on diagrams, which can be seen as the core of doing mathematics. In order to act with a diagram, learners need to consider the relationships between signs. This means that the perception of relationships influences the learners' actions on the diagram. For this paper, it is important that, by analyzing the learners' actions, it is possible to reconstruct the interpretation of the relations that the diagram shows. In this manner, different actions can be investigated to show whether different movements of the learners (e.g. on digital or analogue diagrams) arise from different diagram interpretations. Moreover, following the semiotic perspective, it becomes clear that the activities with and on the diagrams contain gestures that need to be analyzed to achieve an approximately complete description of the diagram interpretations.

Dörfler (2006) states that diagrams have an iconic character and that they are primarily characterized by representing relationships. Due to this main characteristic of diagrams, relationships between the signs and the resulting rules for using the signs are of greater importance than the appearance of the signs (Dörfler, 2015). Unless the appearance of the signs changes the relational structure of the diagram, the appearance is subordinate to the structure (Shapiro, 1997). If diagrams are constructed with the same relations differing only in the materiality of the signs, then the learners are likely to focus on the same relations rather than the materiality of the sign. When acting on digital and analogue diagrams, whose signs are connected by the same relations, these relations must be recognized and observed regardless of whether the signs are represented digitally or analogously. Concerning this paper, it is probable that the same diagram interpretations can be reconstructed when analyzing the learners' actions on the statistical digital and analogue diagrams.

Following Kadunz (2016) for the investigation in this paper, the assumption can be made that the digital tool leads to a shortening and separation of the actions and mathematical relationships. This means that the learners do not have to consider all the relationships of the diagram in their actions when manipulating the diagram. The tool automatically establishes the relationships that the learners do not have to do. In this case, the reconstruction of the diagram interpretation based on the actions alone may be impossible. Learners may re-establish the relationships established by the tool through subsequent actions, gestures or spoken language, which then provide information about the interpretation of the diagram.

In this paper, it is of major interest to determine whether the different actions lead to the reconstruction of different diagram interpretations. Another interest of the paper is whether the diagram interpretations are the same when acting on digital and analogue diagrams because the same relationships have to be noticed in the actions, or whether the digital tool shortens the actions and, thus, the learners do not have to notice all the relationships in their actions; in this case, the learners' reconstructed diagram interpretations would differ.

Research questions

With regard to the considerations of which implications arise from the theory for the research interest, research questions can be formulated as follows:

1. Which mathematical diagram interpretations can be reconstructed based on the actions on the statistical diagram implemented once with the digital and once with the analogue material?
2. Which possible differences exist between the reconstructed diagram interpretations, as it can be assumed that the actions are shortened with the digital tool?

METHOD AND DESIGN OF THE STUDY

In this section, first, the aims of the *MatheMat* study are described, followed by a detailed description of how the data collected in the study were analyzed.

Methods of Data Generation–The *MatheMat* Study

In the study *MatheMat–Mathematical learning with materials*, actions on the different materials (analogue or digital) form the center of interest. The goal of the study is to reconstruct diagram interpretations by analyzing the actions of the learners to draw conclusions about whether the different material has an influence on the learners' diagram interpretations. To achieve these goals, geometric and statistical learning situations are designed that deal with the same mathematical relationships; these are realized once with analogue and once with digital material. The learners work in pairs on one geometric and one statistical learning situation. When distributing tasks, care was taken to ensure that each pair worked once on a digital and once on an analogue learning situation. A total of 32 third- and fourth-graders (9-11 year-olds) from two German primary schools work on the learning situations for about 45 minutes. The pairs' work on the statistical and geometric problems is videotaped with two cameras; one focuses on the actions on the material, while the other camera records the overall situation. Comparable video passages in which the learners act on the material are transcribed for the analysis. In this way, the interpretations of the learners, reconstructed from the visible actions and gestures, can be compared in relation to the different materials used.

Preparation of Data

For the reconstruction of the diagram interpretations, all the learners' actions, gestures and spoken language are transcribed in detail. When working with analogue material, the separation of gestures and actions can be made through the definitions formulated by other researchers (e.g., Harrison, 2018; Kendon, 1984). However, when working with digital material, gestures and actions become more blurred, thus, the definitions of the actions and gestures can only help to distinguish between these with difficulty. Therefore, a definition of the actions on the digital material is provided in this paper. For the transcription of the actions on the digital material, the touch gestures on the tablet and the resulting manipulations in the program are interpreted together as actions. For the descriptions of the movements over the screen and the manipulations in the program, the touch gesture reference guide (Villamor et al., 2010) is adapted for the particular statistical learning situation. **Figure 1** describes the important movements for the analysis of the actions on the digital material.

Methods of Data Analysis

To reconstruct the learners' mathematical diagram interpretations, a semiotic specification (Billion, 2021b; Billion & Vogel, 2021) of Vogel's (2017) adaptation of Mayring's (2014) context analysis is made.

The context analysis according to Mayring (2014) aims to explain the meaning of a statement by adding further text passages from the data. Vogel's (2017; Vogel & Huth, 2020) adaptation for application in mathematics education focuses on the reconstruction of mathematical concepts by contrasting mathematical concepts and the learners' multimodal expressed individual concepts. For this purpose, Vogel (2017) uses the theoretical background of conceptual change (e.g., Carey, 1988).

For the semiotic specification of the qualitative analysis, the conceptual change theory is replaced by the Peircean theory of signs. As mentioned above, each sign evokes an interpretant in a person, and this interpretant depends on the knowledge, habits and experiences of the sign reader. The interpretant

> "can be a reaction to a sign or the effect in acting, feeling, and thinking […]" (Bakker & Hoffmann, 2005, p. 336).

The learners' actions and gestures working with the various material can be seen as the interpretant of them reading the signs. In the analysis, the learner's interpretant is contrasted with an interpretant based on current research that is close to the 'final logical interpretant'. Following Peirce, the 'final logical interpretant' can be defined as how "it comes out ideally 'in the long run' of scientific communication" (Bakker &

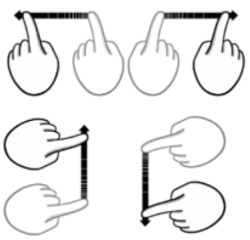
Movement on the screen	Description of movement	Manipulation in TinkerPlots
"Tap" 	"Briefly touch surface with fingertip" (Villamor et al., 2010)	Select value
"Drag" 	"Move fingertip over surface without losing contact" (Villamor et al., 2010)	Sorting data

Figure 1. Movements on the screen and triggered manipulations in TinkerPlots[1]

[1] Petra Tanoupoulo made the illustrations of the movements on the screen

Hoffmann, 2005, p. 336). Since this 'final logical interpretant' is something ideal that cannot be formulated, an interpretant is formulated based on research that should come close to the 'final logical interpretant'. The description of this interpretant includes rule-governed actions that can be performed on the signs based on the mathematical relations that the diagram shows. Firstly, the mathematical relations between the signs are described and then rule-governed actions are derived from them. Here, the focus is on the relations that are important with regard to the task to be solved. Other relations may also be recognized, but these are neglected. By the contrast of the interpretants, the learner's diagram interpretation can be formulated. In addition, by including increasingly more transcript and video passages, the learner's diagram interpretations can be made visible in the ongoing sign process.

The steps of analysis are structured in such a way that in explication 1, a very small transcript passage of a learner's action is focused on and contrasted with the research-based interpretant. In the course of analysis, the scope of the transcript or video passages considered is expanded. In explication 2, more of the same or similar passages are included in the analysis to reconstruct the learner's diagram interpretation. Explication 3 focuses on further video passages of the learning situation; these are similar to the transcript passages already found, and are, again, contrasted with the research-based interpretant. The passages that are important for analysis are those which are the same or similar to the first passage with which the analysis began. These same or similar passages can be found in the processing of the task both before and after the first transcript passage.

EMPIRICAL EXAMPLE

This section describes the statistical learning situation, outlines the diagram interpretations of Walerius and Matteo and compares their interpretations.

The Learning Situation

This paper focuses on a statistical learning situation in which German third-graders (9 year-olds) are asked to represent the values of one characteristic from different data cards in a plot to answer given questions. The focused question to answer is: *How many children have named purple and how many children have named orange as their favorite color?* This question can be classified as reading the data because the learners have to extract information from the data displayed in the plot. For the processing, the learners have data cards at their disposal. The data cards contain four nominally and ordinal scaled characteristics (gender, favorite color, grade in German and in mathematics) and the corresponding values of 14 children. In the *MatheMat* study, four third-graders use TinkerPlots to create a univariate plot with

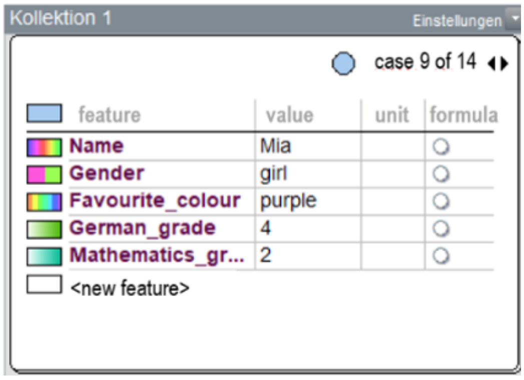


Figure 2. Data cards in TinkerPlots

these data cards, while another four third-graders use sticky notes and wooden cubes labelled with names.

To answer the question, the considered learner Walerius and his partner use TinkerPlots; this is a software toolkit for visualizing and simulating data (Konold & Miller, 2011). The software allows values of 14 children to be entered into data cards (see **Figure 2**). The data of the 14 children had already been entered into TinkerPlots at the beginning of the learning situation, allowing the learners to see available data on the data cards. The accompanying person explains to Walerius and his partner how to look at the data cards of the children entered into TinkerPlots.

In addition, a plot is opened in which the 14 children can be seen as dots (see **Figure 3**). At the beginning of the situation, all the dots are colored blue (see **Figure 3a**). By tapping on a characteristic listed on the data card, TinkerPlots colors the dots in the plot (see **Figure 3b**), thus the dots adopt the respective value of the clicked characteristic. However, the colors of the dots do not match the value of the characteristic *favorite color*. TinkerPlots automatically assigns different colors for nominally scaled data, therefore, unfortunately, it is not possible to set the colors to match the values of the characteristic *favorite color*. In the learning situation, the accompanying person discusses this with the learners. However, if one performs a drag-movement, starting from a dot, vertically or horizontally across the plot, TinkerPlots provides sorting of the dots according to the selected characteristic. TinkerPlots determines a scaling and allocates each dot to this scaling and, by performing another drag-movement in the same way, it provides a finer scaling. In relation to the question posed for the learners to answer, it can be seen that two children chose orange, and three children chose purple as their favorite color (see **Figure 3c**).

In the semiotic sense, the learners can interpret the data cards and the plot, each as a complex sign or a diagram. The plot itself is an icon because it represents relations between the values, the dots are indices to the 14 children whose values are considered and the signs on the scales can be seen as symbols or indices for the measurement or survey that was made. Similar to the

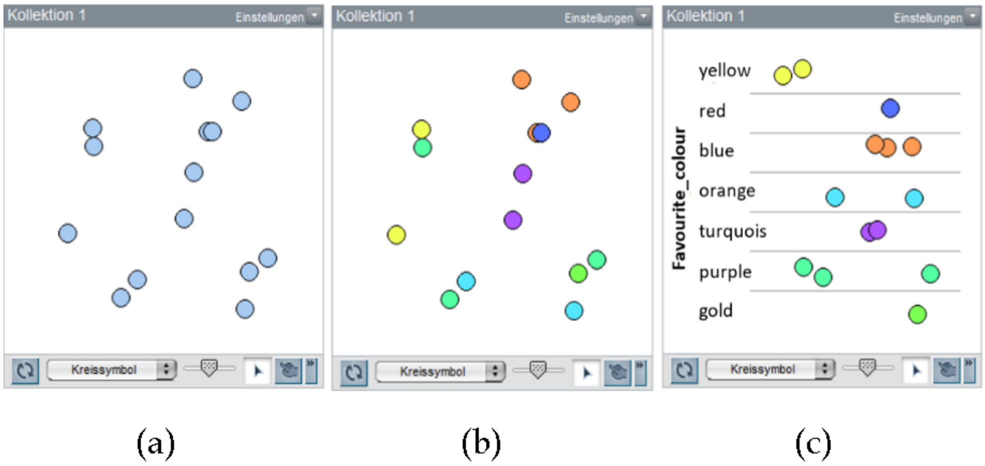


Figure 3. Data plots (a-c) in TinkerPlots

plot, all three types of signs according to Peirce can be found on the data card.

The other learner considered in this paper is Matteo; he and his partner work with analogue material. The learners receive 14 analogue data cards which they can order or sort independently according to certain relationships. Following Harradine and Konold (2006), the learners can sort the data cards flexibly by creating a plot from them. Such an idea gave rise to TinkerPlots, in which

> "[...] the construction of statistical displays is realized via the data card-operations stack, separate, order [...]" (Frischemeier, 2020, p. 42).

However, the semiotic view at TinkerPlots reveals that the diagram *plot* does not emerge from the data cards; it already exists. The learners who are working with the analogue data cards act exclusively on one diagram (data cards), establish relationships between the parts of that diagram and manipulate them, while the learners working with TinkerPlots manipulate two diagrams and have to recognize previously established relationships between the two diagrams (data cards and plot, see **Figure 2** and **Figure 3**).

To ensure that the learners who are working with the analogue material also manipulate two diagrams in a semiotic sense, they are provided with wooden cubes and blank sticky notes to accompany the analogue data cards (see **Figure 4**). Each wooden cube is marked with the name of a child whose values are noted on the data cards. As with TinkerPlots, relationships between the data cards and the wooden cubes are already predefined, making it possible to create a separate diagram (plot) with the wooden cubes and sticky notes. By labelling the sticky notes with the values of one characteristic and sticking them next to each other, the learners can determine a scale to which they can allocate the wooden cubes. In addition, by matching the cubes to the appropriate data cards, the learners can recognize the relationship between the cube and the data card. Subsequently, the learners can transfer a value from the data card to the cube and translate it into a position above the scale.

In **Figure 4**, above Matteo's hand, one can see sticky notes stuck next to each other as a scale. The wooden cubes above the sticky notes have been positioned according to their values on the data cards and the values on the sticky notes. This material arrangement can be interpreted as a plot. As in TinkerPlots, this plot helps the learner to see how many children have indicated orange or purple as their favorite color.

The analogue material has been chosen in such a way that the same relationships between the parts of the material are defined as in TinkerPlots. Even if different materials are used, the relations represented in the two digital and analogue diagrams (data cards and plots) are the same. To enable the reconstruction of the same mathematical diagram interpretations, the mathematical relationships observed when experimenting with the two digital or analogue diagrams must be the same. Nevertheless, when manipulating the diagrams, the assumption that the digital material can shorten the actions and separate them from the relationships must be considered.

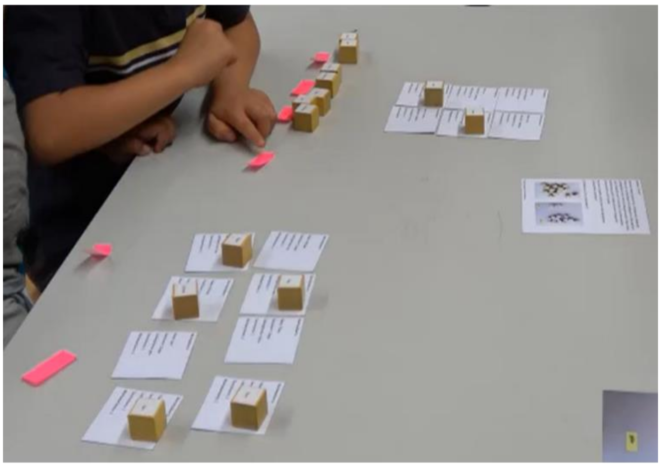
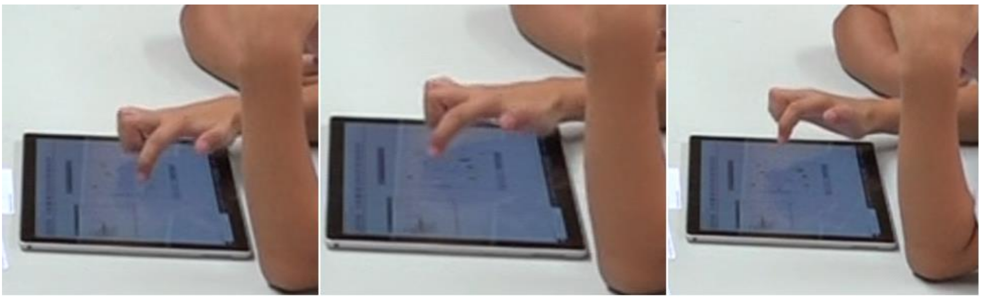
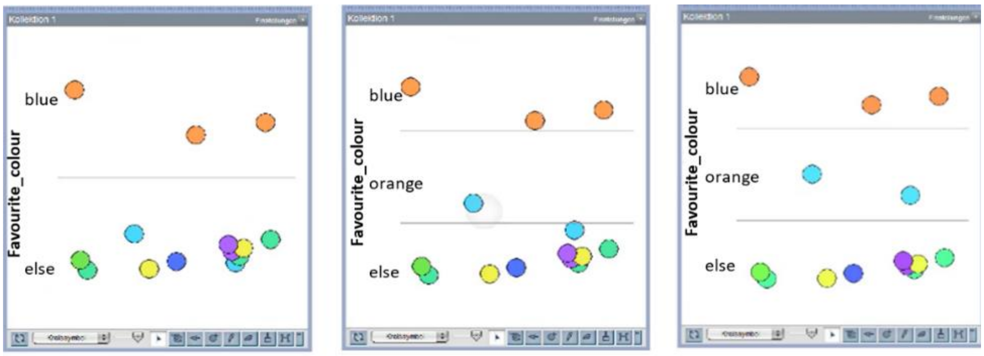


Figure 4. Analogue data cards, cubes marked with names and sticky notes



(a) (b) (c)

Figure 5. Drag-movement from the light blue dot upwards across the plot (a-c)



(a) (b) (c)

Figure 6. Manipulations in TinkerPlots initiated though the drag-movement (a-c)

Reconstruction of the Interpretation of Diagrams by Working with TinkerPlots

Walerius and his partner have not worked with TinkerPlots previously. They receive an introduction to the functions of the software relevant to the task from the accompanying person. The learners have also not previously sorted data with analogue material in their mathematics lessons.

For the analysis, mainly the actions, such as selecting a characteristic on the data card or making a drag-movement over the plot, but also the gestures and phonetic utterances made by Walerius are included. However, due to space restrictions here, the contrasts made in the steps of analysis between the learner's interpretant and the research-based interpretant are presented in summary. The analysis begins with the following transcribed video sequence:

Walerius:

1. *Makes a drag-movement with the right index finger, starting at a light blue dot and moving upwards* (see **Figure 5** and **Figure 6a**).

2. *The light blue dot moves upwards in the plot* (see **Figure 6b**).

3. *TinkerPlots separates the children in the plot who have indicated orange as their favorite color from the other children in the plot* (see **Figure 6c**).

4. Two\

5. *The plot now shows the children who indicated orange as their favorite color separately from the children who indicated another color* (see **Figure 6c**).

Explication 1: To sort the dots in the plot according to a characteristic, relationships between the signs have to be recognized. The research-based interpretant describes these relationships and the resulting actions on the material to contrast them with Walerius's actions. In this contrast, the diagram interpretations of Walerius can be formulated. The summarized research-based interpretant focuses on four main relationships and their resulting actions:

1. *Relationship between the data cards and the dots in the plot*: To establish this relationship, the learners have to make a tap-movement on a characteristic on the data card; consequently, TinkerPlots colors the dots according to the values of the characteristic. The relationship between a value and a dot is only partially expressed in the coloring of the dots. TinkerPlots does not color the dots according to the values for the characteristic *favorite color* but assigns the colors independently of the values.
2. *Relationship between the characteristic and the axes in the plot*: To establish this relationship, the learners have to make a drag-movement starting from one dot across the plot. Depending on the direction of

the drag-movement (horizontal or vertical), TinkerPlots plots the scaling on the x- or y-axis.

3. *Relationship between the values on one axis*: Since *favorite color* is a categorical characteristic, the individual values of the characteristic do not have to be placed in any particular order. Nevertheless, the distances between the individual values on the scale should be equal to enable a better interpretation of the plot. To establish this relationship, the learners have to make a drag-movement starting from one dot across the plot. TinkerPlots takes an equal distribution of the values on one axis.
4. *Relationship between the values on the scale and the positioning of the dot in the plot according to their values*: To establish this relationship, the learners have to make a drag-movement starting from one dot across the plot. TinkerPlots positions the dots in the plot according to the values on the scale.

Contrasting the research-based interpretant with Walerius's actions, his diagram interpretation can be summarized as follows: due to Walerius's drag-movement upwards across the screen, Walerius most likely establishes a relationship between the y-axis and the characteristic *favorite color*. Based on this action, TinkerPlots automatically scales the y-axis and positions the dots according to this scale. Thus, no relationship between the individual values of the scale or the positioning of the dots are observed in the actions. Accordingly, it cannot be reconstructed from Walerius's action whether he can interpret the relationship between the individual values of the scale, which TinkerPlots provided. Furthermore, it is unclear whether he can interpret the arrangement of the dots in the plot according to the scale given by TinkerPlots. Exclusively from the subsequent spoken language "two\", it can be reconstructed that he can interpret the positioning of the dots in the plot. He probably recognizes that two children have indicated *orange* as their *favorite color*. Following this assumption, Walerius can see that the two blue dots are children who have indicated orange as their favorite color. He recognizes the relationship between the scale and the dots and is not influenced by the color of the dots. Walerius has probably previously separated the orange dots from the others because he has assumed that the scale corresponds to the color.

Explication 2: In explication 2, Walerius's same and similar actions in the transcript are once again contrasted with the research-based interpretant. Walerius's diagram interpretations formulated in explication 1 can also be reconstructed in explication 2. In addition, by Walerius's tap-movement on the characteristic from the data card he likely recognizes a relationship between the diagram *data card* and the diagram *plot*. Since TinkerPlots automatically performs a translation of the values into coloring the dots, the relationship between a value on the data card and a dot in the plot cannot be explicitly reconstructed from Walerius's actions. Again, the relationships in the actions do not become fully clear because TinkerPlots, as a tool, shortens the action process. Since TinkerPlots does not adopt the favorite colors that are on the data cards when coloring the dots, the interpretation of the plot becomes more difficult for Walerius. Explication 1 shows that Walerius is unable to rely on the colors, but has to consider the relationship between the position of the dots and the scale to extract how many children have indicated *orange* as their *favorite color*.

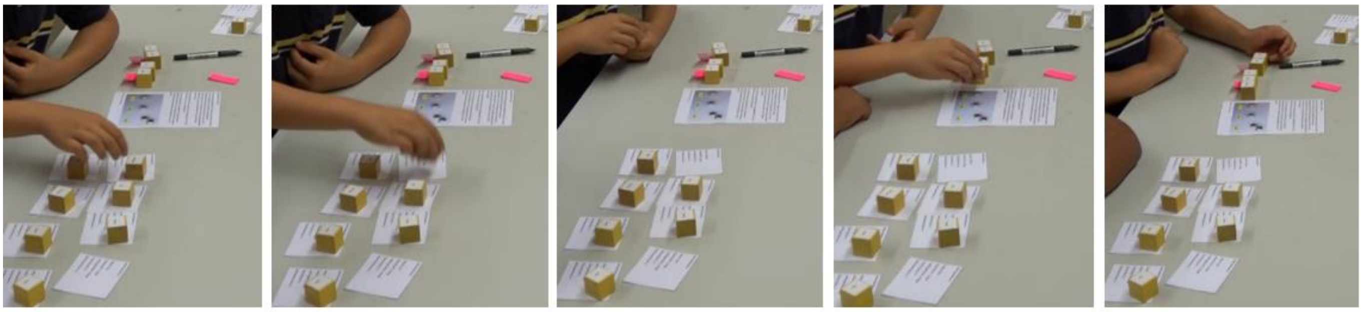
Explication 3: In explication 3, the same and similar actions of Walerius are identified in the recorded processing and contrasted with the research-based interpretant. In summary, Walerius's reconstructed diagram interpretation from explications 1 and 2 can be confirmed by certain passages. It is also possible to find actions where he assigns the values of a characteristic with his directions of movement to both axes. TinkerPlots sorts a dot on one axis according to the value *otherwise* and on the other according to a value of the characteristic *favorite color*. Walerius probably does not realize that the values of a characteristic can only be plotted on one axis and that the sorting made by TinkerPlots is not successful. The relationship between the characteristic and an axis does not become clear via his actions.

Summary of explications: In all steps of the analysis, TinkerPlots, which can be understood as a tool, shortens the actions. Not all of the relationships for creating the diagram *plot* have to be observed in separate actions and, thus, it is impossible to reconstruct from the one drag- or tap-movement whether or not Walerius recognizes all the relationships. Therefore, it remains open whether Walerius can already fully interpret the diagram *plot* during his actions on it. Only by looking at his subsequent phonetic utterances does it becomes clear that Walerius has succeeded in analyzing the diagram *plot*.

Reconstruction of the Interpretation of Diagrams by Working with Analogue Material

Matteo and his partner have not sorted data with digital or analogue material before this learning situation. Like Walerius, Matteo and his partner also receive a short introduction to the subject matter by the accompanying person.

For the second analysis, the focus is on Matteo's actions while working with the analogue material. As in the reconstruction of Walerius's diagram interpretation, Matteo's gestures and spoken language are included in the analysis.



(a) (b) (c) (d) (e)

Figure 7. Positioning of the cube labelled Wilhelm from the data card in the plot (a-e)

Matteo:

1. *Moves his right hand to the cube, which is marked with the name 'Wilhelm', and is lying on the corresponding data card* (see **Figure 7a**).

2. *Grabs the cube marked 'Wilhelm'* (see **Figure 7b**).

3. **purple**\

4. *Leads his right hand with the cube towards his left hand*.

5. *Grabs the cube marked 'Wilhelm' also with his left hand* (see **Figure 7c**).

6. *Moves the cube marked 'Wilhelm' with his right hand to the tabletop* (see **Figure 7d**).

7. *Places the cube marked 'Wilhelm' above the sticky note labelled 'purple' on the tabletop* (see **Figure 7d** and **Figure 7e**).

Explication 1: To sort cubes according to a certain characteristic in the plot, different relationships between the signs have to be considered. As in the work with the digital material, the summarized research-based interpretant focuses on four important relationships and the resulting actions:

1. *Relationship between the data cards and the cubes*: To establish the relationship between the data card and the cubes the learners have to relate the cubes to the corresponding data card. One such way is to place the cubes on the matching data cards.
2. *Relationship between the characteristic and the axes in the plot*: To establish the relationship between the characteristic and the axes, the learners can stick the sticky notes horizontally or vertically next to each other. It is important that the sticky notes be arranged in a line so that the distribution of the wooden cubes can be easily identified.
3. *Relationship between the values on one axis*: As already mentioned, for a categorical characteristic there is no particular order in which the values must be plotted on the scale. However, care should be taken to ensure that the values are equally spaced on the scale and, therefore, the sticky notes must be stuck next to each other at the same distance.
4. *Relationship between the values on the scale and the positioning of the cubes in the plot according to their values*: To establish a relationship between the cubes and the values on the scale, the learners have to recognize the scale as a part of the plot. To position the cube over the matching value, they need to read the value on the data card, transfer this value to the cube and position it over the matching value on the sticky note.

In contrasting the research-based interpretant with Matteo's actions, his diagram interpretation can be summarized as follows: Matteo has probably recognized the relationship between the data card and the cubes since the cube labelled *Wilhelm* is already placed on the corresponding data card before he started his action. With this assignment, it is later possible for Matteo to establish a relationship between the cube and a value on a sticky note. Based on his actions, he establishes a relationship between the data card and the positioning of the cube marked *Wilhelm*. He translates the value *purple* from the data card to the position of the cube above the sticky note labelled *purple*. To be able to place the cube, Matteo has to recognize the scale, which is part of the plot. By analyzing Matteo's actions, he probably recognizes the relationships between the *data card* and the *plot* and between the position of the cube and values on the scale. With regard to the question that Matteo and his partner are required to answer, he selects a suitable cube and makes a phonetic utterance to underline his choice. Due to the emphasis on "**purple**\", he likely deliberately chooses the cube marked *Wilhelm*.

Explication 2: In explication 2, the same and similar actions of Matteo are again contrasted with the research-based interpretant. Matteo's reconstructed diagram interpretation in explication 1 can also be reconstructed by analyzing his transcribed actions in explication 2. Further transcript passages substantiate the assumption that Matteo chooses the cubes based on the values noted on the data card. This can be reconstructed from his pointing gesture to the data card before choosing a cube and can be interpreted as an index in the semiotic sense. Matteo probably only selects cubes that can be assigned to the favorite color *orange* and *purple*. This can be interpreted as an effective selection of cubes with regard to the question to be answered.

Explication 3: By contrasting more of the same and similar video passages with the research-based interpretant, Matteo's reconstructed diagram interpretation from explication 1 and 2 can be confirmed. In the video passages, he probably translates the value of the data card into the position of the cube in the plot and succeeds by comparing the value on the data card and the sticky notes in pairs. At the beginning of the learning situation, Matteo already recognizes the relationship between the cubes and the data cards by assigning the cubes to the data cards. This diagram interpretation can be reconstructed very early in the situation based on Matteo's first actions. Furthermore, during the learning situation, Matteo writes down the values of a characteristic on the sticky notes and arranges them in a horizontal line with the same distance from each other. He can probably recognize the relationship between the characteristic *favorite color* and the x-axis and also the relationship between the values on the x-axis.

Summary of explications: Overall, Matteo can interpret the diagram *data card*, recognize a relationship between the cubes and the data cards and translate the values from the data card into the positioning of the cubes in the plot. Matteo succeeds in this translation by comparing the values on the data cards with the values on the sticky notes in pairs. He is also able to interpret the diagram *plot* and recognize a relationship between a characteristic and the x-axis, and the values among the others plotted on the x-axis. This relationship can be reconstructed based on his phonetic utterances but is also expressed in the actions made previously. Matteo observes all the necessary mathematical relationships in separate actions and, thus, Matteo's diagram interpretations can be reconstructed exclusively from his actions. The analogue material does not shorten Matteo's action as no relationships are established automatically.

Comparison of the Diagram Interpretations

By comparing the reconstructed diagram interpretations of Matteo and Walerius, it can be seen that the digital material leads to a shortening of Walerius's actions. With an action, TinkerPlots automatically observes several relationships. One drag-movement refers TinkerPlots to establish automatically the relationship between the characteristic and an axis, between the individual values on an axis and the dots in the plot and the values on the scale. TinkerPlots acts like a tool and, therefore, shortens the actions and separates them from the mathematical relationships.

Consequently, the reconstruction of Walerius's diagram interpretations through the analysis of the actions on the digital material is not successful in all places. When working with the digital material, only after TinkerPlots has performed the sorting does Walerius need to re-establish the relationship between the position of the dot and the value on the scale to interpret the diagram. In this way, there is a shift in when to interpret the diagram and how to express the relationships between the parts of the diagram. Walerius does not need to interpret the plot while sorting the dots according to a characteristic and, therefore, does not need to establish the relationships between the signs in his actions. After TinkerPlots has made a sorting, Walerius has to interpret the plot and re-establish the relationships between the signs; this is analyzed through his subsequent phonetic utterances and gestures.

In contrast, when working with analogue material, the relationships have to be established during the actions on the diagram because in *one* action, *one* relationship has to be observed. In this way, a reconstruction of Matteo's entire diagram interpretations is realizable through the actions on the analogue material. Matteo is required to form an interpretation during the sorting process, otherwise, he will not be able to do any sorting. In this way, the interpretations become visible in his actions and he observes the relationships between the signs to enable sorting.

However, it also becomes clear that in some places the same diagram interpretations could be reconstructed. Matteo and Walerius both show that they recognize a relationship between one axis and the characteristic *favorite color*: Matteo labels sticky notes with the different values of the characteristic *favorite color* and sticks them next to each other in a line, while Walerius makes a drag-movement from one point upwards. It is, therefore, possible to reconstruct the same diagram interpretations, although the actions on the digital and analogue material differ. This means that the actions can be different although the learners' have to observe the same relationships to act on the digital and analogue diagrams.

MAJOR FINDINGS AND DISCUSSION

This paper aimed to reconstruct mathematical diagram interpretations based on the actions on statistical diagrams that were carried out once with digital and once with analogue materials. For this purpose, the diagram interpretations of Matteo and Walerius were reconstructed with qualitative semiotic analysis. The goal of the comparison of the reconstructed interpretations was to find similarities or differences in the interpretations, as it was assumed that the digital material, as a tool, could shorten the actions.

In this paper, it could be shown that the diagram interpretations can be reconstructed by analyzing the actions of the learners. In contrast to the learners' interpretant expressed by gestures, actions and spoken language, and the research-based interpretant, the learners' diagram interpretations can be described.

In the comparison, the same diagram interpretations could be reconstructed, although the movements of the hands differed. It could be shown that different movements on analogue or digital material in which the same relationships were observed led to the reconstruction of the same diagram interpretation. In line with Dörfler (2015) and Shapiro (1997), analyses show that the different material has no influence on the diagram interpretations if the relationships between the signs are the same. Even if the same relationships must be recognized and observed when acting on the diagrams, the reconstructed diagram interpretations of Walerius and Matteo were not equal in all places.

The comparison shows that when working with analogue material, the necessary mathematical relationships in the actions become clear and the continuous reconstruction of the diagram interpretation is possible. This can be attributed to the fact that with the analogue material, one must already express the interpretation of the relations in the actions during the sorting. Due to the abbreviation of the actions by TinkerPlots, some diagram interpretations cannot be reconstructed by analyzing the actions on the digital material. As already mentioned, according to Kadunz (2016), it could be shown that the actions on the digital material were shortened because TinkerPlots functions as a tool. In this way, there is a shift between the expressions of the relationships interpreted by the learners. Working with the digital material, the relations have to be interpreted after TinkerPlots has made manipulations on the diagram; these results are comparable to the analogy of the compass at the beginning of the paper.

However, by shortening the actions, the digital material opens up the possibility of investigating more complex questions or large amounts of data. Consequently, in practice, the respective learning goal is significant for the choice of material type. If a basic understanding of sorting, as statistical literacy or statistical reasoning, is to be developed, this can be done through the actions on analogue material, as learners need to consider all the relationships between the signs in their actions. In the semiotic sense, the focus is in constructing and manipulating a diagram. For investigating large amounts of data or complex issues, requiring statistical reasoning or statistical thinking, it makes more sense to use digital material; TinkerPlots can help to sort through many data and, thus, learners can focus more effectively on interpreting the results of the manipulation of the diagram performed by TinkerPlots.

In this way, Biehler et al. (2013) make a suitable metaphor to these results, comparing statistical learning with travelling: "[W]hen travelling by plane or train we see fewer details along the road than when walking or cycling" but travelling by train or plane is faster and easier (Biehler et al., 2013, p. 678). This comparison takes into account that when acting on analogue material more relationships between parts of the diagram have to be observed, while some digital materials shorten the actions (and the relationships to be observed) to enable focusing on the entire diagram. However, learners should know by walking or cycling which way to take before they "arrive somewhere fast without knowing about all the decisions taken" (Biehler et al., 2013, p. 679). In relation to the results of the paper, it is important to know the relationships between the parts of the diagram, otherwise, the material arrangement cannot be recognized as a diagram and it is merely a picture for the learners.

Limitation and Outlook

In this paper, only a small data selection was considered. Further analyses of actions on statistical learning situations should show whether these results are also evident among other learners working with digital or analogue materials. In addition, the question of whether a shortening of actions can always be observed when working with TinkerPlots should be pursued.

If one compares the results with the analyses of actions in a geometric learning situation (Billion, 2021b), it is noticeable that no shortening of the actions by other programs was detected and the same diagram interpretations could be reconstructed at all points in the processing. These results can be attributed to the fact that in the geometric learning situation considered, the program did not function as a tool due to the design of the learning situation. If this program had been used differently in the learning situation, it would very likely have also functioned as a tool. It is likely that different programs and their different usage have different effects on the learners' diagram interpretations, as programs permit different actions. This assumption should be more precisely substantiated by undertaking further research.

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two schools where the survey took place were informed about the study in the school conference and approved it on February 21, 2019 and February 25, 2019.

Consent for publication: The consent forms from the parents of the two minor children in this paper were obtained and are available from the author. The consent forms take into account that data from the study may be published in scientific articles under anonymisation. In this paper, the names of the students are anonymised to avoid any conclusions about their identity.

Data sharing and reproducibility: The data of the study cannot be made available for other researchers and are not accessible for sharing.

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REFERENCES

Bakker, A., & Hoffmann, M. (2005). Diagrammatic reasoning as the basis for developing concepts: A semiotic analysis of students' learning about statistical distribution. *Educational Studies in Mathematics*, *60*, 333-358. https://doi.org/10.1007/s10649-005-5536-8

Ben-Zvi, D. (2018). Foreword. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education. Supporting early statistical and probabilistic thinking* (pp. vii-viii). Springer. https://doi.org/10.1007/978-981-13-1044-7

Ben-Zvi, D., & Garfield, J. (2004). Statistical literacy, reasoning, and thinking: Goals, definitions, and challenges. In D. Ben-Zvi, & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 3-15). Kluwer Academic Publisher. https://doi.org/10.1007/1-4020-2278-6

Biehler, R., Ben-Zvi, D., Bakker, A., & Makar, K. (2013). Technology for enhancing statistical reasoning at the school level. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. S. K. Leung (Eds.), *Third international handbook of mathematics education. Springer international handbooks of education* (pp. 643-689). Springer. https://doi.org/10.1007/978-1-4614-4684-2_21

Billion, L. (2021a). The usage of inscriptions – mathematical experiences of learners working with diagrams. In J. Novotná & H. Moravá (Eds.), *Proceedings of the International Symposium Elementary Mathematics Teaching. Broadening experiences in elementary school mathematics* (pp. 82-92). https://semt.cz/proceedings/semt-21.pdf

Billion, L. (2021b). Reconstruction of the interpretation of geometric diagrams of primary school children based on actions on various materials – a semiotic perspective on actions. *International Electronic Journal of Mathematics Education, 16*(3), em0650. https://doi.org/10.29333/iejme/11068

Billion, L., & Vogel, R. (2021). Material as an impulse for mathematical actions in primary school – A semiotic perspective on a geometric example. In M. Inprasitha, N. Changsri, & N. Boonsena (Eds.), *Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 81-88). PME. https://www.igpme.org/wp-content/uploads/2022/04/Volume-2_final.pdf

Carey, S. (1988). Conceptual differences between children and adults. *Mind and Language*, *3*(3), 167-181. https://doi.org/10.1111/j.1468-0017.1988.tb00141.x

Doerr, H., & English, L. (2003). A modeling perspective on students' mathematical reasoning about data. *Journal for Research in Mathematics Education*, *34*(2), 110-136. https://doi.org/10.2307/30034902

Dörfler, W. (2006). Inscriptions as objects of mathematical activities. In J. Maaz, & W. Schlögelmann (Eds.), *New mathematics education research and practice* (pp. 97-111). Sense Publishers. https://doi.org/10.1163/9789087903510_011

Dörfler, W. (2015). Abstrakte Objekte in der Mathematik [Abstract objects in mathematics]. In G. Kadunz (Ed.), *Semiotische Perspektiven auf das Lernen von Mathematik* [*Semiotic perspectives on learning mathematics*] (pp. 33-49). Springer-Verlag. https://doi.org/10.1007/978-3-642-55177-2

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In A. Bikner-Ahsbahs, A. Vohns, R. Bruder, O. Schmitt, & W. Dörfler (Eds.), *Theories in and of mathematics education* (pp. 21-31). Springer. https://doi.org/10.1007/978-3-319-42589-4

English, L. D. (2018). Young children's statistical literacy in modelling with data and chance. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education. Supporting early statistical and probabilistic thinking* (pp. 295-313). Springer. https://doi.org/10.1007/978-981-13-1044-7_17

Friel, S., Curcio, R., & Bright, G. (2001). Making sense of graphs: Critical factors influencing comprehension of instructional implications. *Journal of Research in Mathematics Education*, *32*(2), 124-158. https://doi.org/10.2307/749671

Frischemeier, D. (2020). Building statisticians at an early age–Statistical projects exploring meaningful data in primary school. *Statistics Education Research Journal*, *19*(1), 39-56. https://doi.org/10.52041/serj.v19i1.118

Graesser, A., Swamer, S., Baggett, W., & Sell, M. (1996). New models of deep comprehension. In B. Britton, & A. Graesser (Eds.), *Models of understanding text* (pp. 1-32). Erlbaum. https://doi.org/10.4324/9781315806143

Gravemeijer, K. (2002). Emergent modeling as the basis for an instructional sequence on data analysis. In B. Phillips (Ed.), *Proceedings of the 6th International*

Conference on Teaching Statistics, Developing a Statistically Literate Society. International Statistical Institute.

Harradine, A., & Konold, C. (2006). *How representational medium affects the data displays students make* [Paper presentation]. The 7th International Conference on Teaching Statistics.

Harrison, S. (2018). *The impulse to gesture. Where language, minds, and bodies intersect*. Cambridge University Press. https://doi.org/10.1017/9781108265065.002

Hestenes, D. (2013). Modeling theory for math and science education. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 13-42). Springer. https://doi.org/10.1007/978-94-007-6271-8_3

Hoffmann, M. (2006). What is a "semiotic perspective", and what could it be? Some comments on the contributions to this special issue. *Educational Studies in Mathematics*, *61*, 279-291. https://doi.org/10.1007/s10649-006-1456-5

Hoffmann, M. (2010). Diagrams as scaffolds for abductive insights. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence* (pp. 42-49).

Huth, M. (2022). Handmade diagrams–Learners doing math by using gestures. In *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education.*

Kadunz, G. (2006). Experiments with diagrams–A semiotic approach. *ZMD–Mathematics Education*, *38*(6), 445-455. https://doi.org/10.1007/BF02652781

Kadunz, G. (2016). Geometry, a means of argumentation. In A. Sáenz-Ludloy, & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics. How to describe the construction, visualisation, and communication of mathematical concepts* (pp. 25-42). Sense Publishers. https://doi.org/10.1007/978-94-6300-337-7

Kendon, A. (1984). Did gesture have the happiness to escape the curse at the confusion of babel? In A. Wolfgang (Ed.), *Nonverbal behaviour. Perspectives applications intercultural insights* (pp. 75-114). Hogrefe.

Kollosche, D. (2021). Styles of reasoning for mathematics education. *Educational Studies in Mathematics*, *107*, 471-486. https://doi.org/10.1007/s10649-021-10046-z

Konold, C., & Miller, C. (2011). *TinkerPlots 2.0*. Key Curriculum Press.

Mayring, P. (2014). *Qualitative content analysis: Theoretical foundation, basic procedures and software solutions.* Klagenfurt. https://doi.org/10.1007/978-94-017-9181-6_13

Moreno-Armella, L., & Sriraman, B. (2010). Symbols and mediation in mathematics education. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education* (pp. 213-232). Springer. https://doi.org/10.1007/978-3-642-00742-2_22

Peirce, C. S. (1899-1900[c.]). *Notes on topical geometry*. MS [R] 142.

Peirce, C. S. (1976). *The new elements of mathematics* (NEM), vol. IV (Ed. C. Eisele). De Gruyter. https://doi.org/10.1515/9783110805888

Peirce, C. S. (CP). *Collected papers of Charles Sanders Peirce* (Volumes I-VI, ed. by C. Hartshorne and P. Weiss, 1931-1935, Volumes VII-VIII, ed. by A.W. Burks, 1958). Harvard UP.

Schreiber, C. (2013). Semiotic processes in chat-based problem-solving situations. *Educational Studies Mathematics*, *82*, 51-73. https://doi.org/10.1007/s10649-012-9417-7

Shapiro, S. (1997). *Philosophy of mathematics: Structure and ontology*. Oxford University Press.

Sriraman, B., & Chernoff, E. J. (2020). Probabilistic and statistical thinking. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 675-681). Springer. https://doi.org/10.1007/978-3-030-15789-0_100003

Villamor, G., Willis, D., & Wroblewski, L. (2010). *Touch gesture reference guide*. https://www.lukew.com/ff/entry.asp?1071

Vogel, R. (2017). "wenn man da von oben guckt sieht das aus als ob…"–Die, Dimensionslücke' zwischen zweidimensionaler Darstellung dreidimensionaler Objekte im multimodalen Austausch ["when you look from above it looks as if..." –The 'dimensional gap' between two-dimensional representation of three-dimensional objects in multimodal exchange]. In M. Beck, & R. Vogel (Eds.), *Geometrische Aktivitäten und Gespräche von Kindern im Blick qualitativen Forschens Mehrperspektivische Ergebnisse aus den Projekten erStMaL und MaKreKi* [*Geometric activities and conversations of children in the perspective of qualitative research Multi-perspective results from the projects erStMaL and MaKreKi*] (pp. 61-75). Waxmann.

Vogel, R., & Huth, M. (2020). Modusschnittstellen in mathematischen Lernprozessen. Handlungen am Material und Gesten als diagrammatische Tätigkeit [Mode interfaces in mathematical learning processes. Actions on the material and gestures as a diagrammatic activity]. In G. Kadunz (Ed.), *Zeichen und Sprache im Mathematikunterricht–Semiotik in Theorie und Praxis* [*Signs and language in mathematics lessons–Semiotics in theory and practice*]. Springer. https://doi.org/10.1007/978-3-662-61194-4

Wild, C., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, *67*(3), 223-265. https://doi.org/10.1111/j.1751-5823.1999.tb00442.x

A SEMIOTIC PERSPECTIVE ON LEARNING MATHEMATICS WITH DIGITAL AND ANALOGUE MATERIAL: PRIMARY SCHOOL CHILDREN ACTING ON STATISTICAL DIAGRAMS

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ABSTRACT

This paper focuses on two third-grade students' work on the same statistical question whereby one acts with analogue material and the other with TinkerPlots™. The aim of the research was to find out whether different material influences the actions and, thus, possibly the mathematical interpretations of the learners. To investigate this research interest, a semiotic perspective on mathematical learning according to Peirce was adopted. Based on this perspective, a modification of Mayring's context analysis was made, which allowed the analysis of actions to reconstruct the learners' diagram interpretations. From the analyses, there is evidence that some materials can shorten actions and can automatically establish mathematical relationships and, thus, affect the mathematical interpretations of the learners. At times, however, other actions on different materials can also lead to the reconstruction of the same diagram interpretations. Using these insights, implications for mathematics teaching practice were formulated to assist teachers in selecting materials for designing learning environments to support early statistical thinking.

Keywords: *Statistics education research; Diagrammatic reasoning; Digital and analogue material; Primary school; Early childhood education*

1. INTRODUCTION

To be an empowered citizen in a data-driven community, it is important to work, think and reason with data from an early age (e.g., Ben-Zvi, 2018; Ben-Zvi & Garfield, 2004; Frischemeier, 2020). Leavy et al., (2018) indicated that the "[u]se of appropriate educational tools […], in combination with suitable curricula and other supporting material, can provide an inquiry-based learning environment through which genuine endeavours with data can start at a very young age […]" (p. ix–x). However, the question remains: What material can be used and how should the material be used profitably to create such 'an inquiry-based learning environment' in which young learners encounter endeavours with data? To create such a learning environment as a teacher, it is important to know how learners perform actions on different materials and how each material supports the learners to form mathematical interpretations. To provide such references for practice in mathematics lessons, mathematics education research often analyses or evaluates the material separate from the concrete actions of the learners on the material. In this paper, however, the activity of young learners and, thus, their mathematical interpretations are examined to determine how the learners use the material and, consequently, how and when which material supports the mathematical learning process of the young learners.

This paper reports on whether the mathematical actions of the learners are influenced by different materials (analogue and digital) and whether the potential influence has an impact on the mathematical interpretations of the learners when dealing with a statistical problem. Previous analyses of third-grade students' actions on digital and analogue materials when dealing with a statistical problem (Billion, 2022) indicated that digital materials can act as tools by shortening the learners' actions of sorting data and automatically creating mathematical relationships. This enables the learner to focus on different

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aspects of mathematical processing. Therefore, by analysing the actions of other third-grade students in this paper, the aim is to establish whether these indications can be confirmed.

To investigate this research interest, the research is theoretically framed by a semiotic perspective on statistical learning according to Peirce (1931–1935). This perspective highlights activities on diagrams, which are often creative, inventive, experimental and explorative, as doing mathematics (Dörfler, 2006) and therefore brings the actions of learners to the forefront of interest. This scientific approach fits well with the aim of the paper to investigate the usage of the different materials based on the learners' actions rather than to evaluate the material. Based on the theoretical considerations in the *MatheMat–Mathematical Learning with Materials* study (e.g., Billion, 2022), comparable learning situations with statistical questions were designed, which were used once with analogue material (wooden cubes, data cards, sticky notes, and a square grid) and once with digital material (*TinkerPlots*™, Konold & Miller, 2011). The learning situations can be understood as comparable since the same mathematical relationships can be constructed, even though the materiality differs. To achieve the formulated aim of the paper, the actions of two third-grade students (Nils and Li [pseudonyms]) in the analogue and digital learning situation were analysed when answering the question: *Did more boys or more girls indicate blue as their favourite colour?* For the analysis of the actions, a semiotic adaptation (Billion, 2021a; Billion, 2022) of the context analysis according to Mayring (2014) was used. This enabled the mathematical interpretations of the learners to be described by comparing the actions of the learners and actions resulting from a research-based interpretation of the material arrangement. The reconstructed mathematical interpretations of Nils and Li are compared to find out whether the material influences the actions and possibly also the mathematical interpretations of the learners. This comparison should enable a description of which material supports young learners at a particular step in the mathematical learning process. The findings of this paper in addition to findings from another paper (Billion, 2022) allow the formulation of interpretive hypotheses beyond the individual case for the use of digital and analogue materials in statistics classrooms. These interpretive hypotheses can be used by teachers to make reasoned decisions about when to use which material for designing an inquiry-based environment in which young learners can engage with data.

2. THEORETICAL FRAMEWORK

In this paper, a semiotic perspective according to Peirce (1931–1935) on learning mathematics is adopted. This perspective allows attention to be paid on the actual activities made on the material.

2.1. SEMIOTIC PERSPECTIVE ON LEARNING MATHEMATICS

From the semiotic perspective, learning mathematics is closely linked to working with diagrams. In this way, Peirce (1976, NEM IV) states that "[a]ll necessary reasoning is diagrammatic; and the assurance furnished by all other reasoning must be based upon necessary reasoning. In this sense, all reasoning depends directly or indirectly upon diagrams." (p. 314). Similarly, Dörfler (2006) formulated the importance of diagrams in learning mathematics: "[M]ath is understood (primarily and initially) as a social practice with, on, about, and through diagrams" (p. 105). It becomes clear that in a semiotic sense, mathematical learning is a reflected handicraft with diagrams in a social context rather than something abstract.

The concept of diagrams in the Peircean sense goes beyond the representations of data in a plot and is to be understood in a broader sense than usual. A diagram is a complex sign whose main interest is to represent relationships (Bakker & Hoffmann, 2005). The representation of relationships is constructed by means of a "system of representations" which is defined by a set of rules and conventions (Hoffmann, 2010, p. 42). A diagram can be described as an icon that represents things by imitation and can, thus, express relationships. The complex sign also includes indices whose main function is to direct attention to something and symbols whose meaning is determined by their use, habit or rule (Bakker & Hoffmann, 2005). Most importantly, a complex sign only becomes a diagram if it is used according to the defined rules and conventions (Dörfler, 2006). The usage of signs is a visible activity, for example, actions and gestures. Thus, actions and gestures are essential for mathematical learning and should be the focus of attention.

Reasoning with diagrams in the Peircean sense consists of several steps: constructing a diagram "by means of a consistent system of representations", experimenting with it "according to the rules of the chosen system of representation", reflecting upon the results of experimenting and, finally, to "express these results 'in general terms'" (Hoffmann, 2010, p. 47). Bakker (2004) refers to the importance of describing what is seen in the diagram after manipulating it; this forms the major component of reflecting on statistical diagrams. Looking at the empirical example considered in this paper in which Nils and Li investigate the correlation between the attributes of *favourite colour* and *gender*, diagrammatic reasoning means that the learners construct, in the semiotic sense, a statistical diagram by positioning cases in a plot according to values represented on data cards. By refining or coarsening the two scales of the plot, experiments can be carried out on this diagram and other relationships between the attributes can be recognised through the distribution of the cases. Thus, diagrammatic reasoning consists of a series of processes starting with the observations of the relationships that have come to the fore through experimenting with the diagram, reflecting on them and, as a final step, expressing these observations in general terms. In this, relationships between two attributes can be formulated through the distribution of the cases that are not necessarily expected, and generalisations can be made that are essential for mathematical reasoning.

2.2. ACTING ON DIAGRAMS

Complex signs, as every perceptible sign, evoke one or several ideas in the sign reader (Hoffmann, 2006). Peirce (1932, CP 2.228) termed the idea that is evoked by the sign in the mind an "interpretant", which be expressed as a perceptible sign (a representamen). In this way, "the interpretant can be a reaction to a sign or the effect in acting, feeling, and thinking […]" (Bakker & Hoffmann, 2005, p. 336). Thus, learners express their interpretant in their actions as well as in their gestures (Huth, 2022), which offers the researcher the opportunity to reconstruct the interpretant through the learners' expressions. If the learner has manipulated the diagram, this transformed diagram can be seen as the interpretant of the diagram and, at the same time, as a new diagram (Peirce, 1976, NEM IV). This results in an ongoing sign process that is rarely completed. The interpretant uttered by the learners in their action or construction of a new diagram is determined by the learners' experiences, understanding of concepts, and habits (e.g., Billion, 2021b; Dörfler, 2006; Schreiber, 2013).

As mentioned above, a diagram is a complex sign that is embedded in a habit, and it has to be interpreted according to the conventions that are anchored in the habit. Therefore, learners working with diagrams must first interpret the rules of the diagram to be able to recognise the permissible transformations. The rules and relationships of the diagram define the possible actions on the diagram, but, likewise, they can also restrict those (Bakker & Hoffmann, 2005). If persons are versed in the usage of diagrams, then they recognise that the diagram has certain attributes that are always associated with that diagram. In this context, Peirce (1976, NEM IV) gave the following example: "What is true of the geometrical diagram drawn on paper would be equally true of the same diagram when put on the blackboard?" (p. 317). This means that the material from which the diagram is made does not influence the relationships represented by the diagram. Similar explanations can be found in Dörfler (2015) and Shapiro (1997) who stated the structure of a diagram to be important in that it should describe the relationships between the individual parts rather than focusing on the individual parts which do not affect how they are connected in the structure.

Although the rules and relationships of the diagrams are the same and define the same possible actions, a tool may shorten the actions and distract them from the relationships between the signs by automatically creating multiple relationships triggered by one action (Kadunz, 2016). This means that a tool can automatically establish several mathematical relationships through one action that is performed on the material. Without the tool, the mathematical relationships would have to be established in several individual actions. In this sense, actions are shortened by the tool. Only when the new complex sign generated by the tool is re-interpreted by the learner can the relationship between the perceptible signs and the associated relationship be re-established (Kadunz, 2016). For example, if one draws a line parallel to another line, one must ensure that each point of the new line has the same distance to the existing line (Kadunz, 2016). In the action of drawing, relationships between all the points on both lines are considered. When dynamic geometry software is used, a parallel straight line can be constructed with one click (Kadunz, 2016). In the action of clicking, the relationships between the points on the two lines do not have to be considered. The action of drawing and the relationships to be observed in the action are shortened by the tool (software) to such an extent that no interpretation of the sign is necessary for the action of clicking. Only when the manipulation made by the software is to be interpreted does the connection between the perceptible sign and the relationship between the signs have to be re-established (Kadunz, 2016). A protractor can also be used as a tool to shorten the actions and the relationships that have to be established in the actions. Although a line still has to be drawn, the auxiliary lines on the protractor are parallel to the drawing line, thus, the relationships are created by the tool through the correct application of the protractor. Unlike working with dynamic geometry software, the relationships concerning the line and the protractor must be recognised and established before drawing. It becomes clear in the comparison that the software carries out a greater shortening than the protractor. For this paper, it can be assumed that the software *TinkerPlots*™ also shortens the actions and distracts the actions from the relationships between the signs. In this case, it may not be possible to reconstruct the learners' entire diagram interpretations from the abbreviated actions, but from the activity that the learners carry out after the action to re-interpret the new complex sign. This paper reports on the description and analysis of the actions on different materials. If there is a shortening of the action by a tool, this is not to be judged as better or worse, but it has an effect on when the learners make interpretations so that there is a different focus on the diagram in the sense of diagrammatic reasoning according to Peirce (1931–1935).

3. RESEARCH FOCUS

From the theory section, it becomes clear that to be able to act on a diagram, the individual must recognise the relationships beyond the diagram and observes those relationships when making manipulations on it. In the semiotic sense, the manipulations made on the diagram can be seen as the learners' interpretants. For this paper, this means that by analysing the actions performed on the material arrangement, it is possible to reconstruct the learners' interpretations of the arrangement as a mathematical diagram. However, the analysis of gestures also contributes to the reconstruction of an approximate complete diagram interpretation of the learners. As Huth (2022) and Chen and Herbst (2012) have found, gestures can indicate relationships between signs, generate mathematical ideas and, thus, can become diagrams themselves. A gesture can be described as an explicit communicative act addressed to someone else and guided by a specific intention (Kendon, 1984). Actions, on the other hand, are part of the performance of a specific task, such as the manipulation of things in the environment as part of the purposeful activity (Kendon, 1984). Nevertheless, it is sometimes difficult to decide whether an action does not also have a communicative purpose. Therefore, this paper does not attempt to separate gestures and actions from each other but to include their interplay in the analysis.

Looking at the diagrams that Nils and Li are working on to answer the given questions, which are realised with analogue and digital material, it can be assumed that the material plays a subordinate role as the relationships between the parts of the diagram are of greater importance than the materiality of the signs. It can be assumed that the learners have to observe the same relationships between the signs in their actions to be able to answer the question about the correlation between *favourite colour* and *gender* using a bivariate plot that they have constructed. However, if the material functions as a tool and shortens the actions, it may be that the relationships between the signs no longer need to be established in the actions as the tool establishes these relationships automatically, triggered by a short movement. This could affect the way that the learners interpret the diagram and, thus, the reconstruction of the learners' interpretations of the diagram.

3.1. RESEARCH QUESTIONS

Based on these theoretical considerations and extending the findings described from another paper (Billion, 2022), the following research questions arise for the comparability of the findings:

(1) Which mathematical interpretations of the learners can be reconstructed from the actions on the complex signs realised with digital and analogue material?

(2) What possible differences can be described between the learners' reconstructed interpretations due to the materiality of the signs, when the mathematical relationships between the signs are the same?

4. METHODOLOGY AND STUDY DESIGN

In the following, the design, data collection and data preparation of the *MatheMat* study are discussed. In addition, the semiotic adaption of the context analysis according to Mayring (2014) is described in detail.

4.1. STUDY DESIGN

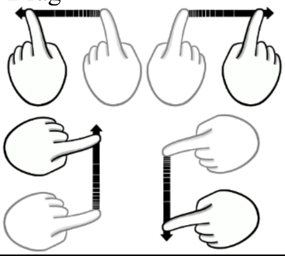
The study, *MatheMat–Mathematical Learning with Materials,* focused on the actions of primary school children while working on geometrical and statistical tasks with digital and analogue material. The study aimed to reconstruct the mathematical interpretations of learners based on their actions on the material to investigate the potential influence of the different materials on the learners' mathematical interpretations (Billion & Vogel, 2021). For this purpose, the learning situations were designed with different materials (digital and analogue), whereby the same mathematical relationships between the parts of the material arrangement can be established. Overall, 32 learners worked in pairs on the learning situations designed. The pairs of learners each worked on one statistical and one geometrical task, once with digital and once with analogue material. The learners' processing was videotaped so that every action, gesture, or phonetic utterance was available for analysis.

4.2. DATA COLLECTION AND PREPARATION

Four pairs of learners work on the learning situation considered in this paper in which the learners were to answer given questions with bivariate plots constructed by themselves. To create a bivariate plot, two pairs used *TinkerPlots*™, while the other two pairs worked with wooden cubes, data cards, sticky notes and a square grid. The learners' processing of the statistical task was recorded with two cameras. One camera focused on the actions and gestures on the material, while the other camera recorded the overall scene. From the recordings, comparable passages were singled out for transcription. These passages are characterised by the fact that the learners have worked with different materials on the same statistical question at the same point in the process. This means that only learners who worked on the same statistical question at the same time but with different material were selected for the analysis. In this case, it was Nils and Li both with their respective work partners. The learners Nils and Li were specifically selected for the analysis because their actions have already been analysed in a geometric learning situation (Billion, 2021a). In this way, comparability of the analysis findings is possible across topics. In a semiotic sense, it can be assumed that the learners show a comparable use of the potential diagram in the selected passages. Here, it is possible to compare whether the actions with the analogue and digital signs differ, although the same relationships between the signs may need to be observed in the actions. It can also be investigated whether different actions allow conclusions to be drawn about different diagram interpretations. The transcribed video passages formed the starting point for the qualitative analysis. Initially, the focus was only on the transcribed passages, then, in the course of the analysis, the entire processing of the learners' actions captured on video was included.

To analyse the actions on the digital and analogue material as accurately as possible, the learners' movements were described as precisely as possible during transcription. It is important to note that even with actions on the analogue materials, it is difficult to distinguish between actions and gestures (e.g., Andrén 2010; Harrison, 2018; Kendon, 1984; Vogel & Huth, 2020). In the case of actions on digital material, this distinction becomes even more difficult and is sometimes not possible. In this paper, therefore, all movements across the surface of the computer screen and the resulting manipulations of the software used were noted as actions. To describe these actions, parts of the *Touch Gesture Reference Guide* (Villamor et al., 2010) were used and adapted for *TinkerPlots*™. Table 1 (Billion, 2022) shows the movements on the screen, their descriptions and the resulting manipulations and relationships, which were established by *TinkerPlots*™. The movements were illustrated by Petra Tanopoulou (reproduced and adapted from Villamor et al. [2010]).

Table 1. Description of the movements and the resulting manipulations

Movement on the screen	Description of the movement	Manipulations and relationships
"Tap" 	"Briefly touch surface with fingertip" (Villamor et al., 2010)	Select an attribute and transfer the values to the dots in the plot. *TinkerPlots*™ establishes a relationship between each data card and the matching dot in the plot.
"Drag" 	"Move fingertip over surface without losing contact" (Villamor et al., 2010)	Sorting the dots in the plot according to a specific attribute. *TinkerPlots*™ establishes a relationship between an attribute and an axis, between the values plotted on an axis, and between the values and the dots in the plot.

4.3. ANALYTIC METHODS

For the reconstruction of the learners' mathematical diagram interpretations, a semiotic specification (Billion, 2023) of the context analysis (Mayring, 2014; Vogel, 2017) was applied. With the help of this analysis, the learner's interpretant was compared with an interpretant formulated based on current research to be able to describe the learner's diagrammatic interpretation.

As described in the theory section, each sign evokes an interpretant and this interpretant can be seen as a reaction to the sign, such as an action, a feeling, a thought, or a gesture. Unlike the spontaneous reaction of learners to a sign, the "'final logical interpretant' […] comes out ideally 'in the long run' of scientific communication" (Bakker & Hoffmann, 2005, p. 336). This final logical interpretant cannot be formulated because it is idealistic, therefore, an interpretant based on current research is formulated that comes close to this ideal (Billion, 2023). The interpretant based on research includes the description of relationships between the signs and the actions on the complex sign that are possible from these relationships (Billion, 2023). Only relationships and the resulting actions that are important for solving the task are described, although it is important to note that the learners may have recognised other relationships between the signs and performed actions based on those relationships.

The analysis consists of several explications. In each explication, by comparing the learner's actions with the actions of the research-based interpretant, the interpretations made by the learner for his or her actions are reconstructed (Billion, 2023). By adding more and more of the learner's actions and contrasting them with the research-based interpretant across the explications, the learner's diagrammatic interpretations were reconstructed throughout the processing. In Explication 1, a small transcript passage was selected in which the learner acted on the complex sign. This small transcript passage was compared with the research-based interpretant to describe the learner's diagram interpretation. In the further course of the analysis, similar or identical actions of the learner found in the transcript were selected in order to relate them again to the research-based interpretant (Explication 2). In this way, the learner's interpretations reconstructed in Explication 1 can be extended, confirmed or rejected. In Explication 3, non-transcribed passages of the learner's identical or similar actions in the learning situation are selected to compare them with the research-based interpretant. In the comparison, the learner's further interpretations of the diagram are described.

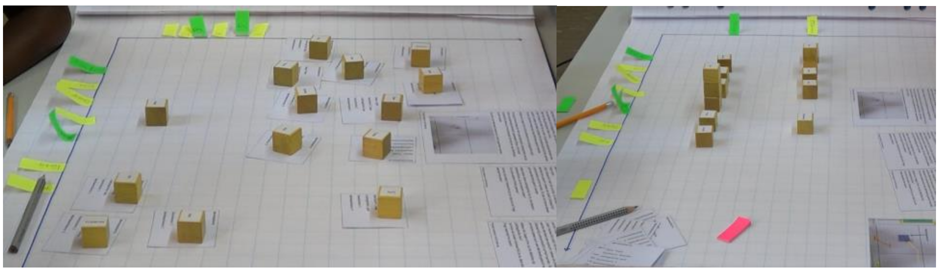
5. EMPIRICAL SNAPSHOT

In the following, the learning situation is described in which the learners investigated the relationship between the attribute *gender* and *favourite colour*. More specifically, the following question was answered by the learners: *Did more boys or more girls indicate blue as their favourite colour?* Friel et al. (2001) distinguished three different levels at which the understanding of a plot can emerge: at the first level, learners extract data from the plot, at the second level learners interpret relationships between the data as shown in the plot and, at the most advanced level, learners explore relationships that are implicitly evident in the plot. The question that the learners dealt with for this paper can be assigned to the two more advanced levels, in contrast to other research in the *MatheMat* study (Billion, 2022). In this case, the learners were required to identify relationships between the values of two attributes, compare the number of cases, and reduce the relationships identified in the data to a statement.

5.1. THE LEARNING SITUATION

The learning situation dealt with bivariate plots of nominal and ordinal data. The learners were to investigate data from 14 second-grade students. At the beginning of the learning situation, the learners explored the analogue and digital materials freely. In this way, they had the opportunity to interpret relationships between the parts of the material arrangement themselves. Afterwards, the researcher explained what could be investigated mathematically with the material and introduced the statistical questions the learners were to answer. The children had different statistical questions in front of them and were free to decide which question they wanted to answer. In this paper, the focus is on the sections of the processing in which the learners dealt with the question of whether more boys or more girls indicated *blue* as their *favourite colour*. The data were provided on data cards that included five attributes: name, gender, favourite colour, grade in mathematics, and grade in German (see the digital version in Figure 2a). The learners worked with data cards as it has been determined that learners produce more complex and informative representations by working with these cards than when using drawing representations (Harradine & Konold, 2006). In particular, the representation of multivariate data is easier to implement (Harradine & Konold, 2006). Based on the digital or analogue data cards, the learners can organise analogue cubes or dots in *TinkerPlots*™ "in small steps using simple actions such as separating them into groups and ordering them according to the value of one of the attributes" (Harradine & Konold, 2006, p. 4). Due to these simple actions, learners can easily reorganise the representations to create more informative representations.

Analogue materials. Nils and his partner worked with analogue material. The two learners had analogue data cards, cubes marked with the names of children (see Figure 1a), a square grid with squares having the same size as the sides of the cubes, and sticky notes to label them with different values (see Fig. 1a-b). To sort the cubes according to the attributes of *favourite colour* and *gender*, sticky notes must first be labeled with the relevant values of the two attributes. The learners then had to decide which attribute they wanted to mark on which axis. In Figure 1b, the attribute *favourite colour* was plotted on the x-axis and the attribute *gender* on the y-axis. To sort a cube by two attributes, the relevant values of the attributes had to be read from the data card and the cube positioned according to the values in the plot. When all the cubes were positioned in the plot, it showed that three boys and no girls chose *blue* as their *favourite colour*.

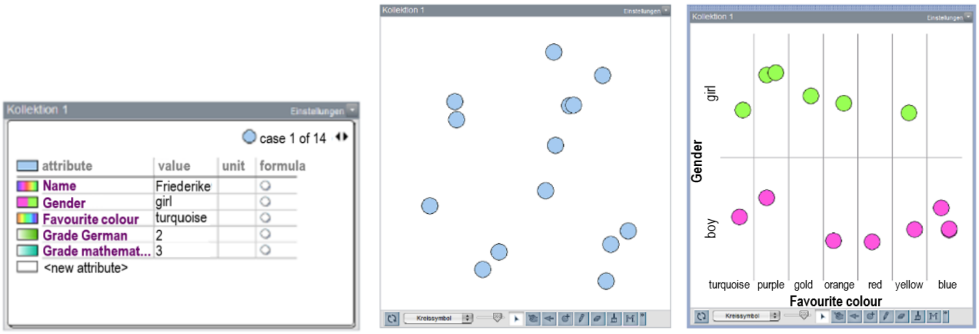


(a) (b)

Figure 1a-b. Analogue material used to sort the data according to two attributes

Digital materials. To solve the given question, *Did more boys or more girls indicate blue as their favourite colour?* Li and her partner were provided with *TinkerPlots*™. *TinkerPlots*™ is a software tool for simulating and visualising data (Frischemeier, 2018; Harradine & Konold, 2006). The software was set up to show the data cards in which the values of the 14 students were entered (see Figure 2a). In addition, a plot was opened in which the data were represented as dots. The dots in the plot were coloured blue at the beginning of editing (see Figure 2b). When an attribute is tapped on the data card, the dots in the plot change colour according to the value of the attribute (see Figure 2c). For the attribute *favourite colour*, *TinkerPlots*™ did not colour the dots in the plot according to the given favourite colour but coloured them independently of the values on the data card (see Figure 5). Thus, the learners could not rely on the colour of the dots but had to relate the value on the scale to the distribution of the dots. The researcher drew attention to this in the introduction to the material.

To sort the dots in the plot by two attributes in *TinkerPlots*™, an attribute must first be selected from the data card and then moved horizontally or vertically using a drag movement. If a horizontal drag movement is made across the plot, the values of the selected attribute are plotted on the x-axis. If a vertical drag movement is made, the attribute is plotted on the y-axis. The software simultaneously positions the dots in the plot according to the scaling. If an attribute is plotted already on the x-axis, the second attribute can be selected from the data card. Once selected, the dots can be dragged vertically to plot the attribute on the y-axis. In Figure 2c, the attribute *favourite colour* has been plotted on the x-axis and the attribute *gender* on the y-axis; *TinkerPlots*™ has positioned the dots according to the scale automatically. In Figure 2c the dots are coloured according to the attribute *gender*.



(a) (b) (c)

Figure 2a-c. Digital material with which sorting can be carried out according to two attributes

5.2. SEMIOTIC PERSPECTIVE ON THE LEARNING SITUATION

In the semiotic sense, the data cards and the bivariate plots in *TinkerPlots*™ can be interpreted as a diagram. Therefore, the analogue material was chosen based on the features of *TinkerPlots*™ in such a way that the data cards and the wooden cubes, with which a bivariate plot can be produced, can be interpreted as a diagram. This allows learners to work with comparable diagrams in both the analogue and the digital material. To describe these diagrams appropriately, the idea of a *model of* and *model for* by Gravemeijer (1999; 2002) is used. This approach focuses on emergent models that are created through an ongoing sign process. In this paper, the transition from *models of* context-specific situations to *models for* formal argumentation in the ongoing sign process is transferred to the work with diagrams in the semiotic sense according to Peirce (1931–1935). The diagram *data card* can be seen as a *model of* different measurements and surveys. To create a bivariate plot from the data cards, an ongoing sign process, that is, different manipulations on the material, is necessary. By positioning the cases in the plot, the relationships between the attributes become the focus. The position of the cases expresses how the values are related to a local position that depends on the scaling used. The positioning of all the cases displays the correlation or association of the data, which can be described across the individual values. If the positioning of the cases is recognised as a distribution, the context moves into the background and the relationships between the two attributes become the focus.

The actions on a plot can be compared to actions made on data cards. Positioning the cases in the plot is equivalent to sorting the data cards according to the values of the two attributes. Since the actions described are equivalent, the diagram *plot* can be interpreted as a *model of* the diagram *data card*. When the learner uses the diagram *plot* to talk about the relationship between two attributes that is made clear by the distribution, the diagram *plot* becomes a *model for* mathematical reasoning. The plot helps the learner deal with the relationships between attributes and derive a general meaning of the relationships through the positioning of the dots/cubes in the plot. Learners can start to see the distribution as a whole and make connections. Through this change from a *model of* to a *model for*, creating a plot and acting with it can become the intermediate between the concrete values of attributes and the general relationship between attributes. In this way, the number as a measurement or survey on the data card becomes a position in the plot and can, thus, be understood as being integrated into a network of relationships.

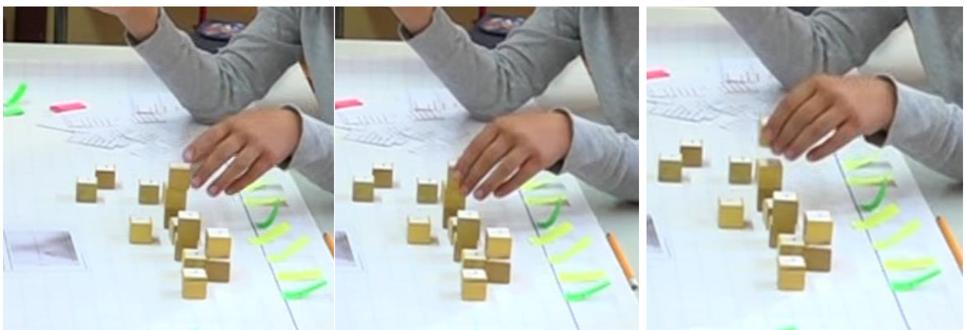
The transition from informal to formal mathematics, or the transition from a *model of* to a *model for*, is particularly important for researchers and teachers. For learners, this distinction is not necessarily of interest when working with diagrams. Following this line of reasoning, teachers or researchers can recognise with their trained eye whether the diagram is being used by the learner as a *model for* thinking about connections or remains a *model of* the data card. In this paper, the qualitative analysis of the learners' actions refers to how the learners use the diagrams.

After a closer look at the diagram *data card* and the diagram *plot,* it can be assumed that when working with digital or analogue material the same relationships have to be recognised to create a bivariate plot based on the values of the data card. For example, relationships between the axes and the values of an attribute must be established, or the relationship between an individual case and the values on the axes. In both the digital and the analogue material, the focus of the data card is initially on the individual; the data card is rather a *model of* different measurements than a *model for* mathematical reasoning. By creating the plot, the view can then be directed more towards the distribution of all cases instead of one individual. Therefore, the plot can more easily become a *model for* mathematical reasoning. Following Dörfler (2015) and Shapiro (1997), it can be assumed that the same relationships have more influence on the interpretation of the learners than the different materiality of the signs.

5.3. ANALYSIS OF ACTIONS ON ANALOGUE MATERIAL

The reconstruction of Nils's diagram interpretations began with the following transcribed actions. Nils and his partner were asked to answer the question of whether more girls or more boys selected *blue* as their *favourite colour*. Nils worked on similar questions previous to the transcribed passage. During the qualitative analysis, identical and similar actions, gestures, and phonetic utterances of Nils were integrated for the reconstruction of his diagram interpretations. The steps of the analysis (Explications) were summarised due to page length restrictions.

Nils: *Moves the left hand to the cubes with the value two of the attribute grade in mathematics and the value blue of the attribute favourite colour* (see Figure 3a).
These cubes are labeled Ray, Can, and Ogan.
Grabs with his left hand the top cube (see Figure 3b).
Lifts the top cube with the left hand about 3 cm high (see Figure 3c).



(a) (b) (c)

Figure 3a–c. Nils's actions on the analogue plot showing a sorting according to the attributes "grade in mathematics" and "favourite colour"

Explication 1. To answer the question asked, it was necessary to sort the cubes according to the attributes of *favourite colour* and *gender*. For such a sorting, many relationships must be recognised. For the research-based interpretant, the relationships between the signs and the resulting actions were formulated (Points 1–4). During the analysis, the research-based interpretant was contrasted with Nils's actions. The research-based interpretant is summarised and the four main relationships and the resulting actions are stated:

1. *Relationship between the data card and the cube:* the learners have to recognise that one cube matches one data card. In this way, learners can later transfer the values on the data cards to the cubes to sort them. To establish this relationship, the learners can match the cubes to the corresponding data cards (see Fig. 1b).
2. *Relationship between the attribute and the axis:* the learners have to recognise that one attribute can be plotted on one axis. Depending on which axis the first attribute is plotted, the second attribute must be plotted on the other axis. To establish such a relationship, learners need to label the axes on the square grid with their respective attributes.
3. *Relationship between the values on the axes:* learners need to recognise that all values of an attribute can be plotted on the axis marked with the corresponding attribute. To establish this relationship, learners have to transfer the values from the data cards to the sticky notes. The labeled sticky notes must then place next to each other on the axis. Since both attributes are nominal, the values do not have to position on the axis in a specific order. Nevertheless, leaving the same distances between the values increases the readability of the plot.
4. *Relationship between the values on the two axes and the cube:* to sort the cubes, learners establish relationships among the values on both axes and the cube. To establish such relationships, it would make sense to move a finger vertically and horizontally, starting from the desired value on each axis, whereby the meeting of the fingers marks the position of the cube. Thus, to establish the relationships, the values of the cubes must first be taken from the data cards.

In comparison to the interpretant based on research, it becomes clear that Nils did not sort the cubes according to the attribute *favourite colour* and *gender*. He selected the stack of cubes consisting of all the children who have indicated *blue* as their *favourite colour* from the sorting by *grade in mathematics* and *favourite colour* in front of him. As Nils performed a purposeful action towards the cases who indicated *blue* as their favourite colour after reading the statistical question, it can be assumed that Nils could interpret the relationship between the value *blue* of the attribute *favourite colour* and the cubes in the plot in front of him. It can also be assumed that Nils wanted to look at the names on the cubes to check whether they are girls' or boys' names. Following this assumption, he included the second attribute of *gender,* which was relevant to the question to be answered. Thus, he decided not to create a plot with the attributes that fit the answer to the question but used the already existing plot. Nils did not create another plot. This plot, if created, would still be a *model of* the data, which could have only become a *model for* mathematical reasoning after completion. Likely it was easier for Nils to use the plot that already existed as a *model for* answering the question and, thus, for mathematical argumentation. The diagram *plot* created already helped him to find relevant data and to think about new relationships among the data and other attributes.

Explication 2. Nils repeated the actions taken in Explication 1 to lift the second cube from the stack. With this and other movements, the reconstructed diagram interpretation can be confirmed.

Explication 3. As there are many passages where Nils acted on the diagrams, Nils's reconstructed diagram interpretation can be expanded. By matching the cubes to the data cards, it was assumed that he was implying that both the name on the cube and the name on the data card represent the same child. This supports the assumption that Nils recognised a relationship between the cubes and the data cards. In a further passage, Nils assigned a hypothetical attribute to the x-axis and the y-axis by suggesting to write the value *boy* on the x-axis and *colour* on the y-axis. It can be assumed that Nils established a relationship between the attributes and the axes. In further passages of the processing, Nils assigned all values of the attribute *favourite colour* to the x-axis by sticking all sticky notes on which different colours were written to the x-axis at approximately the same distance apart. He assigned all values of the attribute *gender* to the y-axis. Since the values of the two attributes were nominally, there was no evidence Nils established a relationship between the values on the axes. By including further passages where he assigned values of an ordinally scaled attribute to an axis in an ordered manner, it can be assumed that he can establish this relationship in his actions. In addition, passages were found where Nils simulated the positioning of a cube. To do this, he moved the index finger of his right hand to the right on the square grid, at the same height parallel to the x-axis, and moved his left index finger upon the square grid, at the same height parallel to the y-axis, so that his index fingers meet. Through these actions, it can be reconstructed that Nils established a relationship between the values of the x-axis and the y-axis. By tapping at the meeting point of his fingers on the square grid, it can be assumed that Nils marked the meeting of his fingers in the position of the cube. In other passages, Nils sorted the cubes according to the attributes of *gender* and *favourite colour*. He translated, in each case, two values on the diagram *data card* into the positioning of the cube in the diagram *plot*. Various passages suggest that Nils looked at the data cards before he positioned the cubes in the plot. By looking at the data card, it can be assumed that he recognised the values on the data card and expressed them by placing the corresponding cube in the diagram *plot*. The translation turns the diagram *plot* into a *model of* the diagram *data card*. Nils used the plot as a *model for* mathematical considerations and statistical reasoning.

5.4. ANALYSIS OF ACTIONS ON DIGITAL MATERIAL

Li and her partner worked on the same question as Nils and his partner. The reconstruction of Li's diagram interpretation begins with the following transcribed gestures and her phonetic utterances. In the further course of the analysis, Li's phonetic utterances, her gestures and her actions in which she deals with the relationship between the attributes of *favourite colour* and *gender* were included. Due to the lack of space, a summary of the reconstructed diagram interpretation was made in each step of the explications.

Li: *Removes the right forearm from the tabletop.*
Rests her right elbow on the table (see Figure 4a).
Moves the right hand towards the tablet with the index finger extended (see Figure 4b).
Holds the right index finger about 5 cm above the tablet (see Figure 4b).
The index finger is above the plot and points in the direction of the value "blue" and "boy" (see Figures 4b and 5).
"Here are three guys"
Raises the right hand about 2 cm upwards (see Figure 4c).

(a) (b) (c)

Figure 4a–c. Li's gestures above the plot

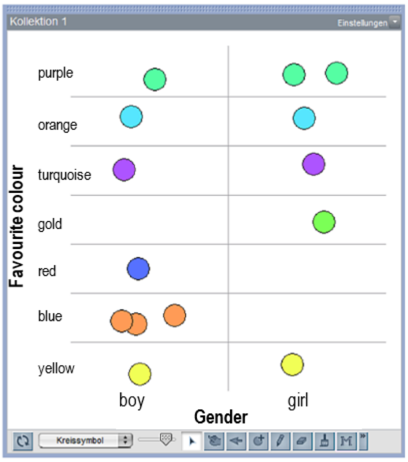


Figure 5. The positioning of the dots in the plot while Li gestures

Explication 1. To answer the question posed, it is useful to sort the dots in the plot according to the attributes of the *favourite colour* and *gender*. The relationships between the signs to be recognised and the resulting actions on the diagrams are described in the research-based interpretant. As in the case of working with analogue material, there are four main relationships in the foreground:

1. *Relationship between the data card and the dots in the plot:* in order to sort the data, the learners need to realise that the dots in the plot must take the value of the attribute to be sorted by. To establish this relationship, it is necessary to select an attribute with a tap movement on the data card. Triggered by the tap movement, all the dots in the plot automatically take on the value of the selected attribute, whereby in the case of the favourite colour, other colours are chosen by the software than the values on the data cards.
2. *Relationship between the attribute and the axes:* the learners recognise that one attribute has to be plotted on the x-axis and the other on the y-axis. To establish a relationship between an attribute and the x-axis the learners perform a drag movement horizontally across the plot. A relationship between an attribute and the y-axis is established with a drag movement vertically across the plot.
3. *Relationship between the values on the axes:* to establish a relationship between the values on one axis the learners need to make a drag movement starting from a dot horizontally across the plot. The software automatically performs a suitable scaling on the x-axis. A drag movement vertically across the plot triggers a scaling on the y-axis.
4. *Relationship between the values on the two axes and the position of the dot in the plot:* to be able to sort by two attributes, a relationship must be established between the values on the two axes and the dot. The learners perform a drag movement starting from one dot (vertically or horizontally) across the plot. The software automatically positions the dots according to the scaling of the axis. Once all the dots are positioned according to the scale on the x-axis (Attribute 1), the learners perform a drag movement vertically and the software positions the dots according to the scale on the y-axis (Attribute 2).

By comparing Li's interpretant with the research-based interpretant, it can be seen that Li did not act on the diagram *plot* but referred to it verbally and through gestures. The position of her index finger and the utterance "Here are three guys" can refer to her recognition that the boys who indicated *blue* as their *favourite colour* were found at that point in the plot. Regarding her phonetic utterances and gestures, Li most likely recognised that the dots in the plot were already positioned appropriately for the given question, and that she was able to read off the children who have indicated *blue* as their *favourite colour*. She, thus, established a relationship between the values on the two axes and the position of the dots in the plot. Li did not start from the colour of the dots in the plot (the dots representing boys who chose *blue* as their *favourite colour* are shown in orange) but from the scaling of the y-axis. She also recognised that three dots are assigned to the value *boy* on the x-axis. It can be assumed that Li was able to interpret the diagram *plot* because she recognised a relationship between the position of the dot and the values on the axes. The diagram *plot* as a *model of* the data card, thus, becomes a *model for* statistical reasoning.

Explication 2. The reconstructed diagram interpretation of Explication 1 can be confirmed since Li gestured above the plot where it showed the girls indicated *blue* as their *favourite colour*. Based on Li's phonetic utterance "… and there are no girls here", it can be assumed that she recognised that no girl indicated *blue* as her *favourite colour*. She again established a relationship between the two axes and the distribution of the dots in the plot. The following gestures can be used to reconstruct that Li wanted to compare the boys and the girls. The spoken conclusion that more boys like blue confirmed the reconstructed diagram interpretation. Explication 2 shows that Li used the diagram *plot* as a *model for* reasoning about the distribution of the dots. In Explications 1 and 2, Li's diagram interpretation was reconstructed exclusively from her gestures and phonetic utterances. Actions on the diagrams were not recognisable.

Explication 3. At the beginning of the processing, Li made rapid successive tap movements with which she selected several attributes on the data card and, thus, the software simultaneously colours the dots in the plot. Since she did not subsequently sort the dots in the plot, it can be assumed that Li could not interpret the relationship between the diagram *data card* and the diagram *plot*. Based on Li's later tap movements, it can be reconstructed that Li specifically selected an attribute and used it for sorting according to the values of the attribute. In these passages, it can be reconstructed that she was able to interpret the relationship between the data card and the plot, which can now be regarded as a *model of* the data card. In addition, other passages were found where Li performed a drag movement across the plot, starting at a purple dot to the left of the plot. Based on the direction in which the drag movement was executed, it can be reconstructed that Li established a relationship between the attribute and the x-axis. Based on Li's actions on the plot, *TinkerPlots*™ made a scale on the x-axis and positioned the dots according to this scale. Thus, it cannot be reconstructed from Li's actions whether she was able to interpret the relationship between the positioning of the dots and the values or the relationship between the values on the scale. Only through the following actions, in which Li performed another drag movement in the same direction several times in a row, it can be assumed that she interpreted the scaling and realised that it still needed to be refined. Further gestures and phonetic utterances illustrated that Li recognised the relationship between the position of the dots and the values on the scale. It was reconstructed through her gestures and spoken language that Li recognised the different relationships in the plot. A reconstruction based on her actions was not possible because one action initiated several manipulations in the software. This means that *TinkerPlots*™ established relationships automatically without separate actions from the learner. In this way, *TinkerPlots*™ functions as a tool and shortens the actions by establishing relationships automatically. By shortening the actions and establishing relationships, the plot changed quickly from a *model of* the data card to a *model for* mathematical reasoning. Li, however, must be practised in interpreting the relationships automatically created by *TinkerPlots*™. After all, other passages showed where Li's gestures and spoken language allowed for a different reconstruction. Those passages showed that Li only recognised the relationship between the positioning of the dots and one axis and neglected the relationship with the other axis. Thus, she did not consistently succeed in interpreting the positioning of the dots that *TinkerPlots*™ made.

5.5. COMPARISON OF THE RESULTS FROM THE ANALYSES

Overall, the reconstructed diagram interpretations of Li and Nils are similar. It was determined from the reconstruction of their actions that both students were able to establish a relationship between the diagram *data card* and the diagram *plot* and a relationship between the attributes and the axes. With the inclusion of the learners' gestures and phonetic utterances, it was reconstructed for both that they were able to establish a relationship between the values on one axis and between the values on both axes and the position of the cases. Furthermore, they use the diagram *plot* initially as a *model of* the data card and later as a *model for* statistical reasoning to answer the given question. It was shown that despite different materials, similar diagram interpretations were reconstructed, whereby it became obvious that by shortening the actions through the digital material the diagram *plot* can be used quickly as a *model for* mathematical reasoning by the learners. This analysis result was also reflected in the processing time, as Nils and his partner who worked with the analogue material needed about 41 minutes to process the learning situation, and Li and her partner who worked with the digital material worked for about 30 minutes on the learning situation. When using analogue material, the learners have to perform the manipulations themselves and, during the manipulation, the diagram *plot* is still a *model of* the data card. On this basis, it was shown in Nils's analysis that he used a plot that did not fit the question to answer the question. This may indicate that he used the diagram *plot* as a *model for* mathematical reasoning and skipped the process of manipulation where the plot was still a *model of* the data card. He succeeded in this because he understood a cube represented a boy or a girl by looking at the names on the cubes.

A closer comparison shows that Nils's interpretation of the diagrams can only be reconstructed based on his actions. This is not always the case with Li. It became clear that Nils established the relationships of the statistical diagrams in separate actions. Since Li worked with *TinkerPlots*™, the software abbreviated her actions as a tool and automatically established some relationships. In this way, the relationship between the values on an axis and the relationship between the values and the dots were established automatically by one drag movement. Conversely, it is not possible to reconstruct from Li's actions whether she was able to interpret the relationships because she did not establish them in separate actions. It was, however, recognisable from Li's gestures and phonetic utterances that she interpreted the relationships made by the software (see Explication 1). This means that she did not have to establish the relationships in her actions: she re-established them by interpreting the signs made by *TinkerPlots*™. In this way, there was a shift from expressing the relationships in the actions to expressing them in gestures and phonetic utterances. This shift means that with the analogue material, the interpretation of the diagram took place during the manipulation of the diagram, whereas with the digital material, firstly, the software generated the sign and then the result of this sign generation was interpreted by reconstructing the diagrammatic relationships. However, due to the abbreviation of the actions, the re-interpretation of the mathematical relationship was not always easy, thus, in Li's analysis, there are places where she did not recognise the relationships between the signs.

6. CONCLUSION AND DISCUSSION

This paper aimed to reconstruct the learners' diagram interpretations through the actions on diagrams that represented the same mathematical relationships but were realised with different materials. Furthermore, the question of whether the material influenced the reconstructed diagram interpretations was considered, even though the same mathematical relationships were shown in the diagrams. In the following, the main results of this paper are discussed and based on these, implications for teaching practice in mathematics lessons are formulated to aid teachers when choosing materials for supporting early statistical thinking.

6.1. MAJOR FINDINGS

The analyses showed that at some points in the mathematical learning process, despite different actions on the different materials, the same diagram interpretations could be reconstructed. In terms of semiotic theory, this means that the appearance of the signs and the appearance of the actions do not affect the learners' interpretations. Thus, appearance is subordinate to the mathematical relationships that are established by the actions of the learners. The mathematical interpretation of the learner is dependent on the mathematical relationships that are established with the material and less on the look or feel of the material. These findings are in line with previous analyses in which the learners Nils and Li worked on a geometric learning situation (Billion, 2021a). In contrast to the examples in this paper, Nils worked with digital material and Li with analogue material. The interpretive hypothesis that can be derived from the findings from the analyses is that the learners' interpretation depends on the relationships they establish and not on the look or feel of the material, which may also apply to engagement with other mathematical topics.

Furthermore, in the comparison of the reconstructed diagram interpretations, a shift between the expression of relationships by actions (analogue material) to an expression of relationships by language and gestures (digital material) becomes apparent. In contrast to analogue material, working with digital material leads to a shortening of the actions as the software automatically establishes the relationships as a tool. In this way, the relationships between the parts of the digital material arrangement do not have to be established in the actions, but the relationships created by the software have to be interpreted afterwards. These findings are consistent with results from analyses of actions on different materials in another statistical learning situation (Billion, 2022).

By comparing the results with the Peircean semiotic view of mathematics learning, it can be shown that through the material different steps of diagrammatic reasoning become focused. When working with analogue material, the relationships between the parts of the material arrangement must be considered, recognised and interpreted during the construction and manipulation of the diagram. Concerning the steps of diagrammatic reasoning, the focus is on creating and manipulating diagrams. When working with digital material, however, the relationships between the parts of the material arrangement do not have to be interpreted during the actions. Instead, these relationships must be re-established if the material arrangement produced by the software is to be interpreted. In this way, when working with digital material that shortens the action, the focus is on the other steps of diagrammatic reasoning: reflecting on the results of manipulating the diagram and formulating these results in general terms. This different focus on the diagram also affects how the diagram *plot* is used by the learners. In analogue material, the learners themselves construct and manipulate the diagram in their actions, thus, it can be seen as a *model of* the data card for a longer time in the action process. Only after manipulation can the diagram be used as a *model for* mathematical reasoning and for answering the question. By focusing on the reflection of the manipulations through the digital material, the diagram *plot* can become a *model for* mathematical reasoning after only a short move across the screen. If this result is related to the question that the learners were asked to answer in the analysed example, it appears that questions that can be assigned to a higher level according to Friel et al. (2001) could be answered seamlessly, as the shortening of actions helps to focus more quickly on relationships among data and the distribution of the data. In this way, the diagram *plot* can be used quickly as a *model for* statistical reasoning. Compared to other research in which the learners were asked to answer questions that could be assigned to a lower level (Billion, 2022), it is conjectured that the difficulty of the question also has an impact on the choice of material. This result is supported by the fact that Nils used a plot that he had already constructed to answer the question, even though it did not optimally fit the question he was supposed to answer. Probably, in this case, the fact that he could use this plot as a *model for* statistical reasoning outweighs the fact that the plot was not sorted according to the appropriate attributes. To be able to interpret a sign produced by a tool and to recognise the relationships among the parts of the new material arrangement, learners must become practised with the use of the diagrams.

6.2. IMPLICATIONS FOR USING MATERIALS TO SUPPORT EARLY STATISTICAL AND PROBABILISTIC THINKING

The findings of this paper can be used to help teachers make informed decisions about when to use what material to create an 'inquiry-based environment' in which young learners can engage with data. Since in the mathematical learning process the appearance of the actions and the materiality of the signs are subordinate to the mathematical relationships established by the learners, it is important for teachers not to make decisions for or against a material based on the appearance or feel of the material. Rather, choices should be based on three items identified as important for the choice of material to design a learning environment (Billion, 2021a; 2022).

Firstly, the choice of material should largely depend on the learners' experience with statistical material arrangements. If young learners are not yet familiar with working with data, it makes sense to use analogue materials at the beginning, where from the start they have to consider relationships in separate actions during the process of sorting the data. Of course, digital materials can also be used which do not function as a tool, so that there is no shortening of the actions and the relationships between parts of the material must be established when acting on these materials. In this way, learners need to establish the relationships between the parts of the diagram themselves and are then likely to be able to recognise and interpret them later when a material abbreviates the actions and establishes the relationships automatically.

Secondly, the learning objective plays a central significance in the selection of the material. If the focus is on constructing and manipulating diagrams when dealing with data, then care should be taken to ensure that the learners attain the mathematical relationships independently by their actions. If the focus is more on reflecting on the results of the manipulations on the diagram, shortening the learners' actions can support the learners to focus on interpreting the results. Such a focus can be used to identify relationships among large amounts of data and to address the distribution of the data.

Thirdly, it becomes clear in comparing the findings across papers (Billion, 2021a; 2022) that the statistical question to be answered by the learners also plays a role in the selection of the material. If the question can be assigned to a more complex level, it is advisable to use a material that shortens the actions so that the learners can use the diagram quickly as a *model for* statistical reasoning. As mentioned earlier, the diagram *plot* is used as a *model of* the data card during its manipulation and construction. Subsequently, once the results of the manipulation are focused, the diagram *plot* can be used as a *model for* statistical reasoning. Therefore, if complex statistical questions are to be answered with the diagram *plot*, it would be advisable for the diagram to become a *model for* statistical reasoning as quickly as possible. Here too, however, the learners must be trained in the usage of the diagram, otherwise, they will not be able to recognise the relationships in the diagram and will not be able to use it as a *model for* statistical reasoning. Suggestions on how to practise first the use of statistical diagrams with analogue materials and then to focus on interpreting larger amounts of data with digital materials can be found, for example, in Frischemeier (2018).

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REFERENCES

Andrén, M. (2010). *Children 's gestures from 18 to 30 month*. [Doctoral dissertation, Lund University]. https://lucris.lub.lu.se/ws/files/3902148/4588138.pdf

Bakker, A. (2004). Reasoning about shape as a pattern in variability. *Statistics Education Research Journal 3*(2), 64–83. https://doi.org/10.52041/serj.v3i2.552

Bakker, A., & Hoffmann, M. (2005). Diagrammatic reasoning as the basis for developing concepts: a semiotic analysis of students' learning about statistical distribution. *Educational Studies in Mathematics, 60*, 333–358. https://doi.org/10.1007/s10649-005-5536-8

Ben-Zvi, D. (2018). Forword. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education: Supporting early statistical and probabilistic thinking* (pp. vii–viii). Springer. https://doi.org/10.1007/978-981-13-1044-7

Ben-Zvi, D., & Garfield, J. (2004). Statistical literacy, reasoning, and thinking: Goals, definitions, and challenges. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 3–16). Kluwer Academic Publisher. https://doi.org/10.1007/1-4020-2278-6_1

Billion, L. (2021a). Reconstruction of the interpretation of geometric diagrams of primary school children based on actions on various materials: A semiotic perspective on actions. *International Electronic Journal of Mathematics Education*, *16*(3), em0650. https://doi.org/10.29333/iejme/11068

Billion, L. (2021b). The usage of inscriptions: Mathematical experiences of learners working with diagrams. In J. Novotná & H. Moravá (Eds.), *Proceedings of the international symposium* *elementary mathematics teaching*. *Broadening experiences in elementary school mathematics* (pp. 82–92). https://semt.cz/proceedings/semt-21.pdf

Billion, L. (2022). Semiotic analyses of actions on digital and analogue material when sorting data in primary school. *Eurasia Journal of Mathematics, Science and Technology Education, 18*(7), em2126. https://doi.org/10.29333/ejmste/12138

Billion, L. (2023). The reconstruction of mathematical interpretations: Actions of primary school children on digital and analogue material. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.), *Mathematics education for global sustainability.* Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education, Haifa (Vol. 2, pp. 115–122). https://pme46.edu.haifa.ac.il/conference-schedule/conference-proceedings

Billion, L., & Vogel, R. (2021). Material as an impulse for mathematical actions in primary school: A semiotic perspective on a geometric example. In M. Inprasitha, N. Changsri & N. Boonsena (Eds.), *Mathematics education in the 4th industrial revolution: Thinking skills for the future.* Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 81–88). https://www.igpme.org/wp-content/uploads/2022/04/Volume-2_final.pdf

Chen, C-L., & Herbst, P. (2012). The interplay among gestures, discourse, and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics, 83*, 285–307. https://doi.org/10.1007/s10649-012-9454-2

Dörfler, W. (2006) Inscriptions as objects of mathematical activities. In J. Maaz & W. Schlögelmann (Eds.), *New mathematics education research and practice* (pp. 97–111). Sense Publishers. https://doi.org/10.1163/9789087903510_011

Dörfler, W. (2015). Abstrakte objekte in der mathematik. In G. Kadunz (Ed.), *Semiotische perspektiven auf das lernen von mathematik.* Springer. https://doi.org/10.1007/978-3-642-55177-2

Friel, S., Curcio, R., & Bright, G. (2001). Making sense of graphs: Critical factors influencing comprehension of instructional implications. *Journal of Research in Mathematics Education, 32*(2), 124–158. https://doi.org/10.2307/749671

Frischemeier, D. (2018). Statistisches denken im mathematikunterricht der primarstufe mit digitalen werkzeugen entwickeln: Über lebendige statistik und datenkarten zur software *TinkerPlots*™. In B. Brandt & H. Dausend (Eds.), *Digitales lernen in der grundschule: Fachliche lernprozesse anregen* (pp. 73–102). Waxmann.

Frischemeier, D. (2020). Building statisticians at an early age: Statistical projects exploring meaningful data in primary school. *Statistics Education Research Journal*, *19*(1), 39–56. https://doi.org/10.52041/serj.v19i1.118

Gravemeijer, K. (1999). How emergent models may foster the constitution of formal mathematics. *Mathematical Teaching and Learning*, *1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K. (2002). Emergent modeling as the basis for an instructional sequence on data analysis. In B. Phillips (Ed.), *Developing a statistically literate society* The Sixth International Conference on Teaching Statistics (ICOTS6), Cape Town, South Africa. https://iase-web.org/documents/papers/icots6/2d5_grav.pdf?1402524960

Harradine, A., & Konold, C. (2006). How representational medium affects the data displays students make. In A. Rossman & B. Chance (Eds.), *Working cooperatively in statistics education*. Seventh International Conference on Teaching Statistics (ICOTS7), Salvador, Brazil. https://iase-web.org/documents/papers/icots7/7C4_HARR.pdf?1402524965

Harrison, S. (2018). *The impulse to gesture: Where language, minds, and bodies intersect*. Cambridge University Press.

Hoffmann, M. (2006). What is a "semiotic perspective", and what could it be? Some comments on the contributions to this special issue. *Educational Studies in Mathematics*, *61*, 279–291. https://doi.org/10.1007/s10649-006-1456-5

Hoffmann, M. (2010). Diagrams as scaffolds for abductive insights. In *Proceedings of the twenty-fourth AAAI-10 conference on artificial intelligence, Atlanta, Georgia.* (pp. 42–49).

Huth, M. (2022). Handmade diagrams: Learners doing math by using gestures. In J. Hodgen, E. Geraniou, G. Bolondi & F. Ferretti (Eds.), *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education (CERME12).* https://hal.archives-ouvertes.fr/hal-03745964

Kadunz, G. (2016). Geometry, a means of argumentation. In A. Sáenz-Ludlow & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics: How to describe the construction, visualisation, and communication of mathematical concepts* (pp. 25–42). Sense Publishers. https://doi.org/10.1007/978-94-6300-337-7

Kendon, A. (1984). Did gesture have the happiness to escape the curse at the confusion of babel? In A. Wolfgang (Ed.), *Nonverbal behaviour. Perspectives applications intercultural insights* (pp. 75–114). Hogrefe.

Konold, C., & Miller, C. (2011). *TinkerPlots 2.0.* Key Curriculum Press.

Leavy, A., Meletiou-Mavrotheris, M., & Paparistodemou, E. (2018). Preface. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education: Supporting early statistical and probabilistic thinking* (pp. ix–xxii). Springer. https://doi.org/10.1007/978-981-13-1044-7

Mayring, P. (2014). *Qualitative content analysis: Theoretical foundation, basic procedures and software solutions.* Social Science Open Access Repository. https://nbn-resolving.org/urn:nbn:de:0168-ssoar-395173

Peirce, C. S. (1931–1935). *Collected papers of Charles Sanders Peirce* (Volumes I-VI), C. Hartshorne & P. Weiss (Eds.). Harvard University Press.

Peirce, C. S. (1976). 18 (PAP) (293): Prolegomena for an apology to pragmatism. In C. Eisele (Ed.), *The new elements of mathematics (NEM): Volume 4 Mathematical philosophy* (pp. 313–330). De Gruyter. https://doi.org/10.1515/9783110805888.313

Schreiber, C. (2013). Semiotic processes in chat-based problem-solving situations. *Educational Studies Mathematics*, *82*, 51–73. https://doi.org/10.1007/s10649-012-9417-7

Shapiro, S. (1997). *Philosophy of mathematics: Structure and ontology*. Oxford University Press. https://doi.org/10.1093/0195139305.001.0001

Villamor, G., Willis, D., & Wroblewski, L. (2010). *Touch gesture reference guide*. https://www.lukew.com/ff/entry.asp?1071

Vogel, R. (2017). "wenn man da von oben guckt sieht das aus als ob…" Die, Dimensionslücke‘ zwischen zweidimensionaler Darstellung dreidimensionaler Objekte im multimodalen Austausch. In M. Beck & R. Vogel (Eds.), *Geometrische aktivitäten und gespräche von kindern im blick qualitativen forschens*. *Mehrperspektivische ergebnisse aus den projekten erStMaL und MaKreKi* (pp. 61–75). Waxmann.

Vogel, R., & Huth, M. (2020). Modusschnittstellen in mathematischen Lernprozessen. Handlungen am Material und Gesten als diagrammatische Tätigkeit. In G. Kadunz (Ed.), *Zeichen und sprache im mathematikunterricht: Semiotik in theorie und praxis* (pp. 215–255). Springer Spektrum. https://doi.org/10.1007/978-3-662-61194-4

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6 Conclusion and Discussion

In the last chapter of this work, another geometrical example from the *MatheMat* study is first discussed, and then the results of the analyses from chapters 3 to 5 are summarised. Based on this summary, the research questions are answered by describing the four main findings of this work. The main results are presented on three different levels: first, they are summarised at the level of analysis, then they are related to the theoretical considerations of this work, and, finally, they are related to practical considerations for mathematics teaching in primary school. Starting from the description of the results of this work, various possible follow-up research investigations are discussed.

6.1 A Further Geometrical Example from the *MatheMat* Study

The examples taken from the papers focus on the actions of third-graders in two digital and analogue statistical learning situations each and the actions of third-graders in one digital and analogue geometrical learning situation each. However, it can be seen that there is a desideratum in the analysis of actions in the geometrical learning situations and the analysis of actions of the fourth-graders. For this reason, two further analyses of the actions on an analogue and a digital geometrical learning situation of two fourth-grade learners were conducted. The topic of the considered geometrical learning situation is the investigation of cuboids with a constant volume and different surface areas. In this way, this work focuses on one example from each topic of the study and the actions of the third- and fourth-grade learners are analysed and included in the formulation of the results of the study.

In the geometrical learning situation, fourth-grade learners investigated the surface areas of different cuboids with a volume of twelve and a height of one. Using the material, they had available, the learners made different cuboids with such characteristics as having different edge lengths for breadth and depth. These cuboids, made by the learners, were the basis for the investigation of the different surface areas of the cuboids. For the beginning of the analysis, a task was chosen in which the learners had given the volume and height of the cuboid in a table and were to determine the breadth, depth and surface area of the cuboid that matched the information already printed. The processing of this prompt was chosen to begin the analysis because it was a comparable passage in the learner's processing. This comparable passage was found by comparing graphical representations of the learner's processing history in the situations. What such a graphical representation looks like can be seen in section 3.5.1 (p. 25). The processing of the prompt, which was identified as being a comparable passage, was transcribed. The transcript was structured in such a way that the actions and gestures could be described in detail, and these were related to the simultaneously uttered spoken language (see section 3.6.1, p. 27). In the analysis, the transcript formed the basis for Explication 1 and 2, while the remaining learner's processing was used in Explication 3 to identify further passages for analysis.

The steps of the semiotic adaptation of the context analysis were carried out as described in section 3.4.3 (pp. 23-24). The analysis of the actions on the digital and analogue material consists of a total of five steps each, which are, in turn, divided into sub-steps. These sub-steps are numbered consecutively throughout the analysis (e.g. E1.1). The contents of the individual steps and sub-steps of the analysis can be found in section 3.4.3 (pp. 23-24). For better comprehensibility, some sub-steps of the analysis were summarised. In addition, for better understanding, the actions of the learners were described in detail and illustrated with sequences of pictures. Actions that allowed the same reconstruction of the diagram interpretations as the actions that have already been described in detail were shortened in the analysis for reasons of space.

6.1.1 Analysis of the Actions on Analogue Material

Paul and his partner work with analogue material. The two learners have rods of different lengths, spatial right-angled corner connections, a square grid, and a unit cube at their disposal (see Fig. 1). The square grid is squared in the size of the side area of the unit cube. The length of the rods is also a multiple of the edge length of the unit cube. In this way, the unit cube and the square grid help the learner to determine the volume and surface area of the cuboids placed together. To put together a cuboid with a volume of twelve and a height of one, the learners have to adjust the breadth and depth of the cuboid so that multiplying the two dimensions results in product twelve. Of course, in such a calculation, it must not be neglected that the height of the cuboid is one, otherwise, it would not be a geometric solid. When plugging together the matching edge lengths, the spatial right-angled corner connections ensure that a cuboid is created. The learners only have to make sure that the four edges of one dimension have the same length and that the edges that meet in a corner have different lengths.

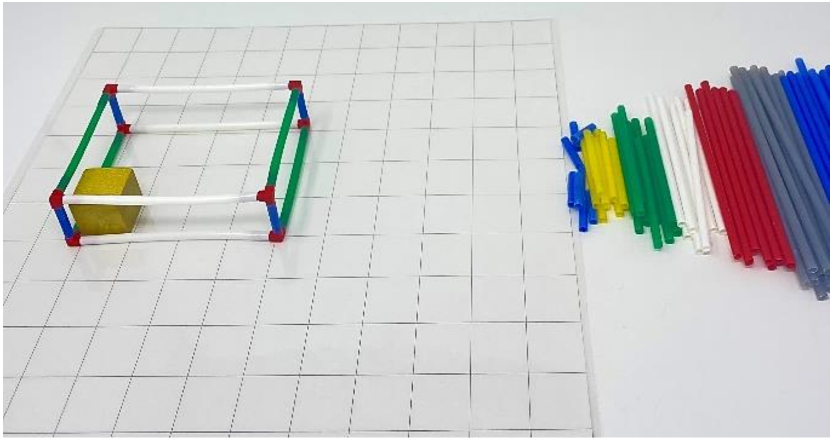


Figure 1. Analogue material arrangement

Step 1: The analysis begins with an action in which Paul notes the breadth and depth of a cuboid with a volume of twelve on a prompt. The prompt that Paul and his partner (Fabiana) are currently working on is in the workspace in front of Paul in the middle of two other prompts that have already been worked on. To the left of the current prompt is a prompt on which the edge lengths of the cuboids with a volume of twelve are indicated and on which the learners have previously noted the surface area of the respective cuboids. To the right of the current prompt, there is a prompt on which the constant area, despite the different edge lengths, has already been noted by the learners. Paul first turns his gaze to the left (see Fig. 2a; this cannot be shown further in the picture due to anonymisation). He then raises his left hand (see Fig. 2b) and lowers it again to place the writing end of the pencil on the current prompt (see Fig. 2c). Paul has now turned his gaze towards the prompt in the middle. After Paul has placed the writing end of the pencil on the paper, he moves the pencil onto the paper (see Fig. 2d). After moving the pencil across the paper, the column of the table on the prompt labelled 'breadth' now contains the number 'three' (see Fig. 4).

(a) (b) (c) (d)

Figure 2a-d. Paul notes the edge length of three for the breadth

Paul releases his pencil from the prompt (see Fig. 3a) and lets the writing end of the pencil sink a little further to the right onto the prompt (see Fig. 3b). He places the pencil on the prompt in the same row, but in the column marked ‘depth’. Paul moves the pencil on the prompt (see Fig. 3c-3d). In the same row where he previously noted ‘three’ for breadth, there is now the number ‘four’ for depth (see Fig. 4).

(a) (b) (c) (d)

Figure 3a-d. Paul notes the number four for the depth of the cuboid

Everything that Paul notes on the prompt in Explications 1 and 2 can be traced in Figure 4. The energetic interpretants, shown in the sequences of pictures (see Figs. 2a-d, 3a-d and 5a-f), lead to the symbols in the marked row. The symbols on the prompt are, in a semiotic sense, indices of Paul’s described actions (see Figs. 2a-d, 3a-d and 5a-f) and possibly also of actions on the geometrical material arrangement which can be found throughout Paul’s work. The actions, in turn, are indices of the relationships interpreted by Paul.

Number of unit cubes (volume)	Height	Breadth	Depth	Number of unit squares on all side faces (surface)
12	1	1	12	50
12	1	2	6	40
12	1	3	4	38
12	1	4	3	38
12	1	12	1	50
12	1	6 ~~2~~	~~6~~ 2	40

Figure 4. The prompt completed by Paul and Fabiana

Step 2 – Explication 1 (E1.1): At the beginning of the analysis, Paul’s energetic interpretant, previously described and illustrated in pictures, is contrasted with the research-based interpretant. The research-based interpretant describes, in line with the extract from Paul and his partner’s processing, the relationships between the signs that the learners have to interpret and establish in their actions to produce a cuboid with a volume of twelve. In comparing the two interpretants, Paul’s diagram interpretation is formulated.

- *Relationship between the length of the edges of a cuboid with the volume of twelve:*
  To build a cuboid with a given volume, the lengths of the different edges of the cuboid must be related. Since the task specifies a volume of twelve and a height of one, only the breadth and depth must be chosen in such a way that the multiplication of the breadth and depth gives the result of twelve. Possible choices for the breadth and depth would be the edge length twelve and one, six and two, or four and three. Beforehand, the learners have to determine the length of the rods available to them. They can do this by establishing a relationship between the rod and the unit length. To determine the length of the rods by actions, the learners can place a rod of length one several times along a longer rod and, thus, the number of times the rod of length one can be

placed along the longer rod indicates the length of the longer rod. The learners can also place the rods on the lines of the square grid and count the number of adjacent side lengths of the squares on the square grid. After the learners have determined the lengths of the rods and decided which length the breadth and depth of the cuboid should be, they then have to choose four rods of the same length for the edges of each dimension of the cuboid. For example, for a cuboid with a height of one, a breadth of three, and a depth of four, four rods each of lengths one, three, and four must be selected.

- *Relationship between three edges that meet in a corner:*
  To build a cuboid, of all the edges that meet in a corner, two of these edges must always enclose a right angle. This relationship between the three edges does not have to be established by the learners in their actions, as it is already established by the spatial, right-angled corner connections. The learners can use the material to put the selected edges together to form a cuboid. However, when putting the rods together, the learners have to establish the relationships between the lengths of the edges that meet in a corner. In one corner of a cuboid, the edges from all three dimensions meet, so that the three edges of different lengths come together. From this, it can be deduced that all parallel edges, to those that meet in a corner, have the same length. Therefore, in their actions, the learners must make sure that the edge lengths in the same dimension are of equal length and that the edge lengths between the dimensions are different.

(E1.2 and E1.3): In the comparison between the research-based interpretant and Paul's actions, it becomes clear that he does not establish any of the relationships between the parts of the geometrical material arrangement described in the research-based interpretant in his actions. By noting the edge length of the breadth and depth in one row of the table, it can be assumed that he is establishing a relationship between the edge lengths of these two dimensions. In addition, it can be assumed that he also relates the numbers he has noted down to the numbers already printed in the row of the table. Thus, it can be reconstructed that Paul recognises that a cuboid with a height of one, a breadth of three, and a depth of four has a volume of twelve. Since he establishes this relationship without further action, it is reasonable to assume that Paul already knows how to choose the breadth and depth of the cuboid to construct a cuboid with a volume of twelve. Based on Paul's glance at the left prompt, he probably uses the insights recorded in symbols that are already noted on the other prompt to edit the new prompt. Therefore, he establishes a relationship between the prompt that has been completed and the prompt that is to be processed. He succeeds because he probably interprets the symbols on the completed prompt, which are indices to his previously made actions and represent relationships between the edge lengths as an icon, as an arithmetical diagram. By interpreting the arithmetical diagram, Paul does not need to make any actions on the geometrical material arrangement to establish relationships between the lengths of the edges to build a cuboid with a volume of twelve.

Step 3 – Explication 2 (E2.1): In the further transcript, Paul's energetic interpretants can be found in which he also notes the edge length for the breadth and the depth of a cuboid with a volume of twelve. He also notes the numbers 'one' and 'twelve' as well as 'six' and 'two' for the depth and breadth of the cuboid.

(E2.2): The assumption from Explication 1 can be confirmed by the fact that Paul interprets the symbols, indices, and icons on the completed prompt as an arithmetical diagram. In this way, he does not have to perform actions on the geometrical material arrangement, and, at the same time, he does not have to establish any relationships that are created in the actions.

(E2.3): In addition, further energetic interpretants can be found in Explication 2 in which Paul notes the surface of the cuboids with a volume of twelve, but having different edge lengths. An energetic interpretant is described below and shown in a sequence of pictures: after noting 'three' for breadth and 'four' for depth, Paul turns his gaze back towards the prompt to the left of the one the learners are

working on. Paul raises and lowers his left hand, in which he holds the pencil, several times (see Fig. 5a-d). He then places the writing end of the pencil one column to the right in the row where he has already noted ‘three’ for the breadth and ‘four’ for the depth (see Fig. 5e). Paul moves the pencil on the prompt (see Fig. 5f) and notes the number ‘38’ in the column for the surface (see Fig. 4).

(a) (b) (c) (d) (e) (f)

Figure 5a-f. Paul makes notes for a cuboid with a volume of twelve, height of one, breadth of three, depth of four, and surface of 38

(E2.4): Since Paul puts a different focus on the relationships in the diagram in his further energetic interpretants, the relationships in the research-based interpretant must also be expanded. Therefore, in the following, the relationships that are important for determining the surface area of a cuboid with a volume of twelve are the focus of attention. Based on the relevant relationships, the resulting actions are formulated.

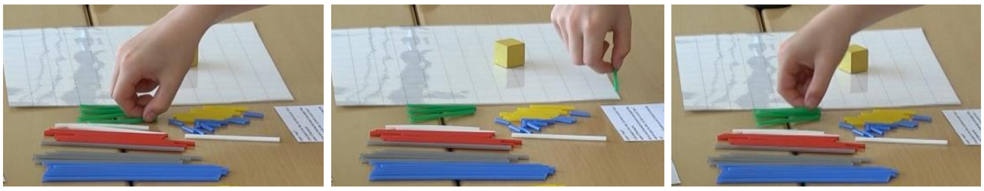
- *Relationship between the unit squares and a side face of the cuboid to determine the area of the side face:*
  The size of an area can be indicated by the number of smaller (standardised) areas. For this purpose, the smaller area, with which the area to be measured can be laid out, is placed in relationship to the area to be measured. The learners can establish this relationship by repeatedly touching the side face of the unit cube to the respective side faces of the cuboid, or by positioning the cuboid on the square grid. By placing the edges of the cuboid on the lines of the square grid, the number of unit squares located between the edges of the cuboid can be counted. By rotating the cuboid, each side face of the cuboid can be placed on the square grid, and the size of the side faces can be determined by counting the unit squares.
- *Relationship between the area of the different side faces of the cuboid to determine the area of the surface:*
  To determine the surface area, the determined number of unit squares on the different side faces must be placed in a relationship to each other. To be able to specify the entire surface with the number of unit squares, the numbers on all side faces must be added together. In the case of a cuboid, the opposite side faces are equal in size, so that the numbers of unit squares with which the opposite side faces can be laid out are equal. The learners can take advantage of this by doubling the number of opposite sides so that they do not have to measure each side individually. The usage of these relationships between the opposite side faces in a calculation shortens the actions in which the relationships are established between the unit squares and a side face of the cuboid.

(E2.5 and E2.6): By comparing the extended research-based interpretant with Paul’s energetic interpretant (see Fig. 5a-f), the reconstructed diagram interpretation from Explication 1 can also be confirmed for the determination of the surface. He does not establish any of the relationships described in the research-based interpretant between the parts of the geometrical material arrangement in his actions and gestures to determine the surface. By noting the number ‘38’ in the same row as the breadth ‘three’ and the depth ‘four’, it can be reconstructed that Paul establishes a relationship between these edge lengths and the surface. Furthermore, it can be assumed that he recognises through the numbers already printed for the volume and height of the cuboid that a cuboid with a height of one, a breadth of three, and a depth of four has a volume of twelve and a surface area of 38. Since he takes no further

action but keeps turning his gaze towards the left-hand prompt that has already been filled in, it is reasonable to assume that Paul interprets the symbols, indices, and icons on the completed prompt as an arithmetical diagram and transfers the relationships he recognises to the prompt he is currently working on. He, presumably, uses insights already gained through his actions on the geometrical material arrangement and, therefore, does not have to repeat the actions and the relationships he establishes with the actions between the parts of the material arrangement. This assumption, about the insights already made, needs to be explored by analysing further energetic interpretants in Explication 3.

Step 4 – Explication 3 (E3.1 and E3.2): In the continuing processing of the learning situation, further energetic interpretants made by Paul can be found in which he determines the edge lengths and the surface areas of cuboids with a volume of twelve. First, Paul's actions are focused on, which can be classified as the same actions as in Explication 1 due to the consideration of the same relationships (see Figs. 2a-d and 3a-d). These include actions in which Paul determines the length of the rods (see Figs. 6a-c and 7a-c), selects rods to build a cuboid (see Figs. 8a-d and 9a-c) and uses the selected rods to assemble a cuboid with a volume of twelve (see Figs. 10a-c and 11a-c).

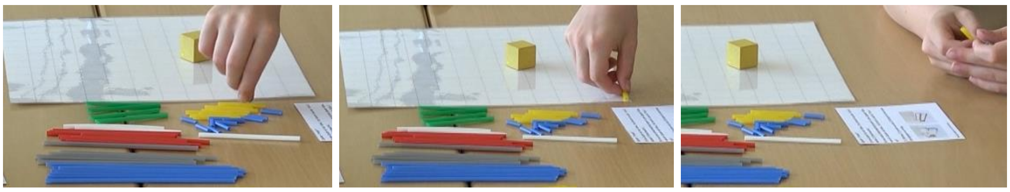
In minute 18 of the processing, Paul whispers "two", moves his right hand forward towards the green rods and, subsequently, grabs a green rod (see Fig. 6a). He holds this rod, which has a length of three unit lengths, against the square grid so that the rod comes to rest on the outer line of the square grid (see Fig. 6b). He then puts the green rod back on the pile with the other green rods (see Fig. 6c).



(a) (b) (c)

Figure 6a-c. Paul checks the length of the green rod

After Paul has put the rod, which has a length of three units, back on the pile, he picks up a yellow rod that is shorter than the green rod (see Fig. 7a). He places this rod on the square grid in the same way as the green rod (see Fig. 7b). While Paul moves his right hand, still holding the yellow rod, towards his upper body (see Fig. 7c), Fabiana (Paul's partner) says "that's two".



(a) (b) (c)

Figure 7a-c. Paul checks the length of the yellow rod

By comparing the research-based interpretant (see Explication 1) and Paul's energetic interpretants (see Figs. 6a-c and 7a-c), it can be reconstructed that Paul establishes a relationship between each rod and the unit lengths on the square grid. He probably recognises that the square grid is squared in the size of the unit squares and that the side length of one of these squares has the length of a unit length. Moreover, Paul can likely interpret that the lengths of the rods are multiples of the unit lengths. He uses these relationships between the parts of the material arrangement to determine the length of the rods. More precisely, it can be assumed from the phonetic expression "two" that he is looking for a rod with a length of two unit lengths. He probably chooses a rod that he thinks could have a length of two units (see Fig. 6a). He uses the relationships he has recognised between the rod and the square grid to check whether his assumption that the rod has the length of two unit lengths is correct (see Fig. 6b). From his action of putting the green rod back, it can be reconstructed that Paul determined that this rod was not two units in length (see Fig. 6c). The choice of a shorter rod suggests that Paul considers the first rod to be too long and, therefore, chooses a shorter rod (see Fig. 7a). Based on the action of placing the yellow rod against the square grid, it can be reconstructed that Paul, once again, exploits the relationships between the length of the rods and the square grid to identify whether the rod has a length of two unit lengths (see Fig. 7b). The action of pulling the yellow rod back towards his upper body suggests that he has discovered that this rod is two unit lengths long (see Fig. 7c). Fabiana's phonetic utterance also confirms the assumption that the two learners had wished to identify a rod that has the length of two unit lengths and that the yellow rod appears to be of that length.

In further actions during the processing, Paul checks the length of other rods in the same manner. Such actions can be used to reconstruct the same diagram interpretations made by Paul that were described earlier and are, therefore, not listed for reasons of space.

A few minutes later in the processing, Paul has two rods lying in front of him in the workspace, which are two and six unit lengths long, respectively. Starting from this situation, he moves his right hand forward towards the yellow rods whose length he has previously identified by placing them against the square grid (see Fig. 8a). He grabs two of the yellow rods from the pile with his right hand (see Fig. 8b) and guides them towards the two yellow rods in his working space (see Fig. 8c). He places the yellow rods from his right hand in front of him so that there are now four yellow rods in his workspace (see Fig. 8d).

(a) (b) (c) (d)

Figure 8a-d. Paul selects a total of four rods with the length of two unit lengths

After selecting two more yellow rods, Paul moves his right hand forward again towards the grey rods. He already has two grey rods in his working space whose lengths he has previously identified as six unit lengths by placing them on the square grid (minute 17 of processing). Paul initially grabs one grey rod and then a second grey rod with the thumb and index finger of his right hand (see Fig. 8a-b). He returns the right hand, holding the grey rods to his working space, and places the two grey rods next to the grey rods already there (see Fig. 8c).

(a) (b) (c)

Figure 9a-c. Paul selects a total of four rods with the length of six unit lengths

By comparing these energetic interpretants of Paul (see Figs. 8a-d and 9a-c) with the research-based interpretant (see Explication 1), Paul's interpretation of several relationships can be reconstructed. On the one hand, it can be reconstructed that Paul establishes a relationship between the edge lengths with which a cuboid with a volume of twelve can be built. Based on the selection of rods of lengths two and six, he probably realises that if the height of the cuboid is given as one, multiplying the other edge lengths must give the result twelve. On the other hand, the actions of selecting two more yellow rods (see Fig. 8a-d) suggest that Paul establishes a relationship between the cuboid to be built and the number of rods. He probably realises that, when building a cuboid, it is important to remember that four edges have the same length. It cannot be reconstructed from the selection whether Paul already recognises that four *parallel* edges must always have the same length; this must be verified by the actions of assembling the cuboid. By analysing Paul's further actions (see Fig. 9a-c), the assumption can be confirmed that Paul establishes relationships between the number of rods and the construction of a cuboid. As he again makes the selection of four rods, the assumption that he recognises that the four edges must always have the same length in a cuboid becomes more likely.

In the processing, further actions can be found in which Paul selects four rods in each action. These actions are comparable to the analysed actions described in Figs. 8a-d and 9a-c. For reasons of space, these are not listed in detail. In his further actions, Paul selects four rods of each length having three, and four unit lengths or one, and twelve unit lengths. From this, it can be reconstructed that Paul establishes further relationships between the edge lengths with which a cuboid with a volume of twelve can be built. He is likely to realise that multiplying twelve and one or three and four also produces the result of twelve, and with a height of one, this results in a cuboid with a volume of twelve. Since he always chooses four rods of the same length, the assumption is strengthened that Paul is establishing a relationship between the cuboid to be built and the number of selected rods.

In minutes 19 to 21 of the processing, Paul assembles a cuboid with four rods of length two, four rods of length six, and four rods of length one. First, he uses a corner connection to put together a yellow rod and a grey rod from his working space (see Fig. 10a). He holds the corner connection so that the two rods are on one plane. He then takes a rod of length one from the pile of rods (see Fig. 10a) and sticks it into the corner connection perpendicular to the two rods that are already in place. Afterwards, he takes another corner connection which he attaches to the rod of length two and assembles another rod of length one in such a way that this rod is parallel to the first assembled rod of length one (see Fig. 10b). He takes another rod, which has a length of six unit lengths and installs it parallel to the first rod, also with a length of six (see Fig. 10c). In addition, he assembles another rod of length one parallel to the other rods of length one that he has already put into two corner connections (see Fig. 10c).

(a) (b) (c)

Figure 10a-c. Paul builds the base of the cuboid with edge lengths of two, six, and one unit lengths

Although Paul first selects a rod of length one (see Fig. 10c), he puts it aside and selects a rod of length two which he assembles parallel to the first rod of length two (see Fig. 11a). He has built the base of the cuboid at this stage. He then takes the rod with the length of one again and builds it up to the height of the cuboid (see Fig. 11a). With four more corner connections, Paul assembles two more rods of lengths two and six parallel to the rods of the same length (see Fig. 11b-c). In this way, a cuboid with edge lengths two, six, and one is created (see Fig. 11c).

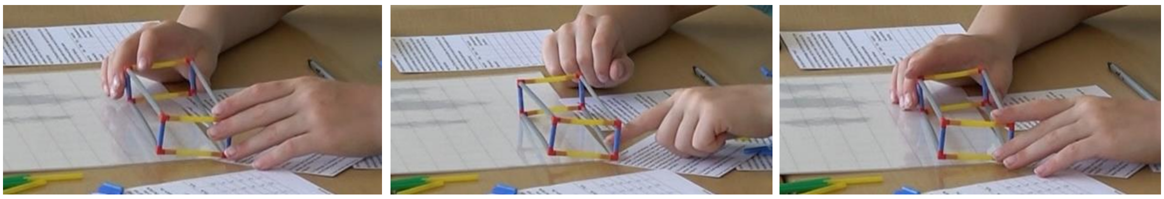
(a) (b) (c)

Figure 11a-c. Paul builds the cuboid with edge lengths of two, six, and one unit lengths

By comparing Paul's energetic interpretant described in Fig. 10a-c and Fig. 11a-c with the research-based interpretant (see Explication 1), it can be reconstructed that Paul makes further interpretations of the relationships between the parts of the geometrical material arrangement. Looking specifically at his actions of assembling, it can be reconstructed from the first action that he assembles three rods of different lengths at a corner connection (see Fig. 10a). He probably recognises that the edges from all three dimensions meet in one corner. He is also likely to recognise that these edges must be of different lengths since all the edges of one dimension have the same length. Starting from this corner, he installs each new rod that is the same length as a rod that has already been installed parallel to it. In this way, he first builds the base of the cuboid on which he installs the rods for the height and then completes the cuboid. Based on his actions, it can be reconstructed that Paul probably realises that all four parallel edges in one dimension are of equal length. It cannot be reconstructed from his actions, however, whether Paul can establish a relationship that refers to the angles between the three edges that meet in a corner, although it can be reconstructed that he probably recognises that three edges of different lengths meet in a corner of the cuboid. Nevertheless, it cannot be reconstructed how he chooses the angle between the edges since this relationship is already anchored in the material. He can use this material in his actions and, thus, does not have to be aware that he has to form a concrete angle between the edges.

(E3.3): So far, Explication 3 has focused only on actions (see Fig. 6a-c to Fig. 11a-c) that can be classified as the same actions as in Explication 1 (see Fig. 2a-d and 3a-d), based on the consideration of the same relationships. In the following, actions are included in the analysis that focus on the same relationships as the actions in Explication 2 (see Fig. 5a-f). In these actions, Paul determines the surface of cuboids with a volume of twelve (see Fig. 12a-c to 17a-d).

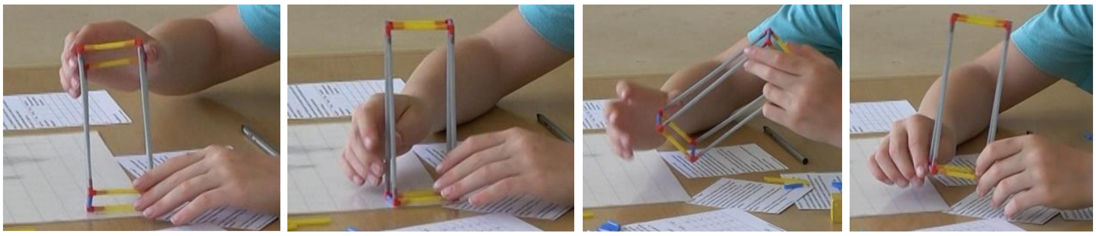
In minute 35 of the processing, Paul places the cuboid he assembled on the square grid and defines, with its positioning, that the breadth of the cuboid is six, the height one, and the depth two (see Fig. 12a). He releases his fingers from the edges of the cuboid and then guides the index finger of his left hand in the direction of the square grid (see Fig. 12b). He remains in this position for two seconds, and, after retracting his index finger and again touching the cuboid with the fingers of both hands, he utters, "twelve" (see Fig. 12c).



(a) (b) (c)

Figure 12a-c. Paul determines the number of unit squares with which the base area can be laid out

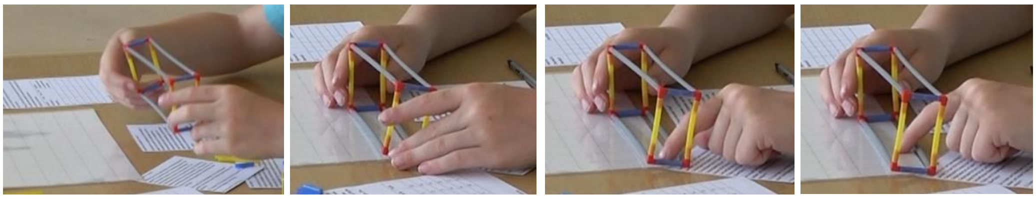
After Paul has determined the number of unit squares with which the base of the cuboid can be laid out, he tilts the cuboid 90° to the left so that it comes to rest on the side face resulting from the height and depth (see Fig. 13a-b). While this side face rests on the square grid, Paul utters "fourteen" phonetically. He then tilts the cuboid forward by 180° so that the parallel side face points downwards (see Fig. 13c-d). He does not place this side face on the square grid but holds it a few centimetres above the table top (see Fig. 13d). While holding the cuboid in this way, Paul utters “sixteen” phonetically.



(a) (b) (c) (d)

Figure 13a-d. Paul determines the number of unit squares with which the opposite side faces of height and depth can be laid out

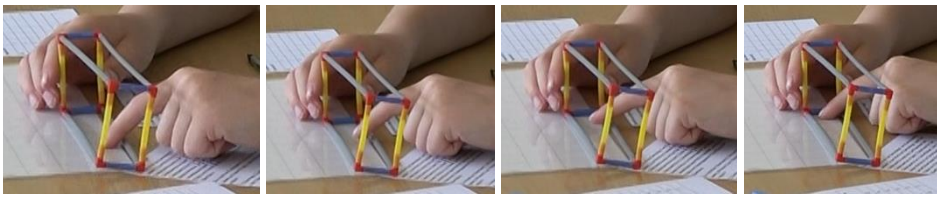
After Paul has determined the area of the two side faces that span between the height and depth, Paul turns the cuboid so that the side face that spans between the height and breadth faces downwards (see Fig. 14a). He places this side face on the square grid so that the edges of the cuboid lie on the lines of the square grid (see Fig. 14b). He releases his left fingers from the cuboid and touches with his left index finger the far left square on the square grid, which is between the edges of the cuboid (see Fig. 14c). While touching this square with his left index finger, Paul utters “seventeen” phonetically. He then releases his finger from the square grid, moves it one square to the right and touches the square to the right of the one he has just phonetically marked as seventeen (see Fig. 14d). During his touch, Paul utters “eighteen” aloud.



(a) (b) (c) (d)

Figure 14a-d. Paul counts the first unit squares with which the side face of height and breadth can be laid out

Paul releases his left index finger from the square grid, moves it one square to the right and touches the square on the square grid that is to the right of the square he previously marked aloud as eighteen (see Fig. 15a). While Paul is touching the square, he utters “nineteen” phonetically. Paul repeats the movement of his left index finger so that he touches a square that is to the right of the square he touched before (see Fig. 15b). He marks this square phonetically with the number “twenty”. In the same way, he touches the last two squares that are between the edges of the cuboid (see Fig. 15c-d). As he touches one square each, he first utters phonetically “twenty-one” and then “twenty-two”.

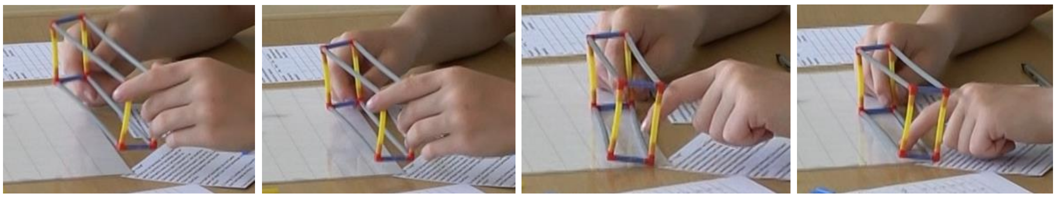


(a) (b) (c) (d)

Figure 15a-d. Paul counts the last unit squares with which the side face of height and breadth can be laid out

After Paul has gesturally and phonetically marked all the squares between the edges of the cuboid, he lifts the cuboid, tilts it forward 180° (see Fig. 16a) and places the opposite side face onto the square grid (see Fig. 16b). He releases his left fingers from the cuboid, as before, and extends his left index finger.

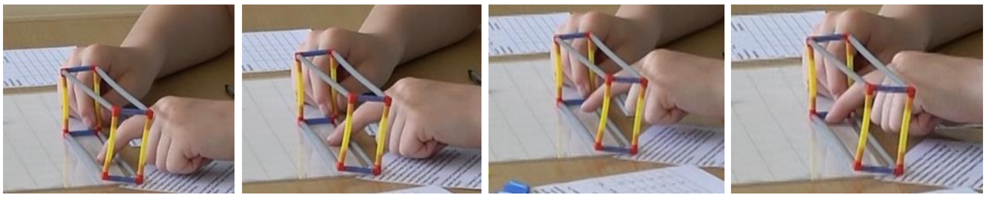
With his index finger, he touches the first square in the row between the edges of the cuboid resting on the square grid (see Fig. 16c). He utters “twenty-three” during the touch. Paul then releases his left index finger from the square grid again, moves it one square to the right and touches the second square from the left, which is between the edges of the cuboid (see Fig. 16d). Again, he makes a phonetic utterance. This time he says “twenty-four”.



(a) (b) (c) (d)

Figure 16a-d. Paul counts the first unit squares with which the opposite side face of height and breadth can be laid out

Paul again releases his index finger from the square grid, moves it again the same distance to the right, and touches the square on the square grid which is third from the right between the edges of the cuboid. He utters aloud “twenty-five”. Paul repeats the movement of his left index finger three more times. He touches with his index finger, one after the other, the last three squares located between the edges of the cuboid (see Fig. 17b-d). During the first touch, he utters, “twenty-six”, during the second, “twenty-seven”, and during the third, “twenty-eight”. Paul then lifts the cuboid and pulls his hands, in which he holds the cuboid, towards his upper body. He looks at the person accompanying them and says, “either I did it right or I did it wrong”.



(a) (b) (c) (d)

Figure 17a-d. Paul counts the last unit squares with which the opposite side face of height and breadth can be laid out

(E3.4 and E3.5): By comparing Paul’s energetic interpretant (see Figs. 12a-c to 17a-d) with the research-based interpretant (see Explication 2), it can be reconstructed that Paul establishes relationships between the parts of the material arrangement to be able to determine the surface area of the cuboid. Based on the actions of placing the cuboid on the square grid (see Figs. 12a, 13b, 14b and 16b), it can be assumed that Paul is establishing a relationship between the edges of the cuboid and the lines on the square grid. Paul probably realises that the length of the edges is a multiple of a unit length and he can, therefore, measure out the resulting area between the edges with unit squares. To determine the area of the respective side faces, Paul makes a 1:1 assignment between the number uttered by him phonetically and the square marked gesturally (see Figs. 14a-d to 17a-d). Presumably, in this way, he establishes a relationship between a number word and a unit square, thus enabling a relationship between the number of unit squares and the side face to be measured. Based on his continuous counting method, it can also be reconstructed that Paul probably establishes relationships between the side faces of the cuboid. He probably realises that to determine the surface area he has to add the number of unit squares on all the side faces. He does not formulate a calculation in phonetic terms, but he continues to count systematically to determine all the unit squares. From his actions in which he systematically determines the area of the opposite side faces (see, for example, Fig. 13a-d), it can be reconstructed that he probably recognises a relationship between the two opposite sides of the cuboid. He does not conclude that the

opposite sides of the cuboid can be laid out with the same number of unit squares, but he presumably uses the same appearance of the side faces to systematically determine the surface area of the cuboid. For the first side face, he forgets the opposite side face and, therefore, comes up with 28 instead of 40 unit squares with which the surface of the cuboid can be laid out.

Other actions made by Paul can be found in the processing in which he focuses on the same relationships to determine the surface of further cuboids. The analysis of these actions allows the same reconstruction of the diagram interpretation described earlier. In addition, the actions made by Paul found in minute 42 show that he does not position all the side faces of the cuboid on the square grid but immediately doubles the certain number of unit squares with which a side face can be laid out. Based on these actions, it can also be reconstructed that Paul establishes relationships between the sides of the cuboid. Paul probably realises that the opposite sides of a cuboid are the same size and, therefore, doubles the number of unit squares that fit on a side face. By formulating such a calculation, Paul abbreviates his actions on the material arrangement which he can interpret as a diagram. In this way, he no longer establishes the relationships he has recognised in his actions but announces them aloud.

In minute 37, Paul writes down the number 40 on a table printed on a prompt. He writes the number in the row where 'one' is written for the height of the cuboid, 'two' for the breadth, and 'six' for the length, under the column named *surface*. Paul records this insight about the surface of the cuboid (with edge lengths of one, two and six) which he has gained by acting and, thus, noticing the different relationships between the geometrical arrangement of materials, in a new, arithmetical sign on the paper. He can use this gained insight, by fixing it in a new sign, in his further processing without having to repeat the relationships previously established in his actions. In the course of processing, further actions can be found in which he records the insights he has made through new signs on the prompt. Explications 1 and 2 make clear that he uses the insights in the further processing of the activity in order not to have to repeat relationships between the parts of the material arrangement that have already been established by his actions.

Step 5 – Summary of the Explications: By comparing Paul's energetic interpretants and the research-based interpretants, different diagrammatic interpretations made by Paul could be reconstructed during the processing. In Explications 1 and 2, it can be reconstructed that Paul's actions do not establish any of the relationships between the parts of the geometrical material arrangement described in the research-based interpretant. By noting the numbers in the same row of the table, it can be assumed that Paul is establishing a relationship between volume twelve, the height one, the edge lengths that he noted for breadth and depth and the surface area. Since he makes no further actions and keeps glancing in the direction of a prompt that has already been worked on, it seems reasonable to assume that Paul is interpreting the previously made insights recorded in the signs and establishing the relationships between the prompt to be worked on and these signs. The reconstruction shows that he interprets the signs (which can be described in a semiotic sense as symbols, indices and icons) on the completed prompt and, therefore, interprets these as an arithmetical diagram and uses recognised relationships to complete the new prompt. Through this interpretation of the arithmetical diagram, Paul does not have to take any actions on the geometrical material arrangement to establish the relationships between the rods and the square grid, between the lengths of the edges of the cuboid, between the edges that meet in a corner of the cuboid, or between the side faces of the cuboid and the unit squares.

In Explication 3, by comparing Paul's actions (see Fig. 6a-c and Fig. 7a-c) and the research-based interpretant, it can be reconstructed that Paul establishes relationships between the rods and unit lengths on the square grid. Furthermore, the analysis of Paul's actions (see Fig. 8a-d and Fig. 9a-c) allows the reconstruction that he establishes different relationships between the edge lengths (e.g. edge lengths of two and six, or one and twelve) with which a cuboid with a volume of twelve can be built. In addition, by always selecting four rods of the same length, it can be assumed that he is establishing relationships between the cuboid to be built and the number of rods. Paul's energetic interpretant described in Figure 10a-c and Figure 11a-c suggests that Paul establishes relationships between the length of parallel edges and the length of edges that meet in a corner of the cuboid. However, it cannot be reconstructed whether Paul can establish relationships that relate to the angles between the edges in a corner; this was not possible due to these relationships already being anchored in the material, and so they did not have to be established in the actions. Based on the actions of placing the cuboid on the square grid (see Figs. 12a, 13b, 14b and 16b), it can be surmised that Paul establishes relationships between the edges of the cuboid and the lines on the square grid to determine the number of unit squares on a side face. He also establishes relationships between the side faces of the cuboid in the actions of rotating the cuboid and systematically placing it on the square grid. Similarly, Paul establishes relationships between the side faces of the cuboid by counting the unit squares aloud or by doubling the number of unit squares on a side face. By formulating the doubling aloud, Paul shortens his actions on the material arrangement since he no longer establishes the relationships he recognises in his actions but does so aloud. Paul records the insights he gains from acting on the geometrical material arrangement in a new sign on the prompt. In the further course of the processing, Paul can fall back on these insights without having to repeat the relationships previously established by his actions. In Explications 1 and 2, it becomes clear that he does this and, therefore, takes no more action on the geometrical material arrangement.

6.1.2 Analysis of the Actions on Digital Material

Daniel and his partner (Madison) work with digital material on the geometrical learning situation. Parts of the GeoGebra software (Hohenwarter, 2001) are available to the learners. The settings in the GeoGebra software are made in such a way that the learners see a cuboid in the middle of the screen, standing on a square grid, and a unit cube is placed within the lower left-hand corner of the cuboid. The edge lengths of the cuboid can be adjusted by the learners with scrollbars so that their lengths can be different multiples of the edge length of the unit cube. The squares on the square grid have the size of a side face of the unit cube. In this way, the unit cube and the square grid help the learner determine the surface area of the cuboid visible on the screen. To make a cuboid with a volume of twelve and a height of one in GeoGebra, the learner needs to set the scrollbar labelled *height* (bottom scrollbar on Fig. 18) to the position of one and to adjust the scrollbars labelled *breadth* (top scrollbar on Fig. 18) and *depth* (middle scrollbar on Fig. 18) so that multiplying the lengths of these two dimensions gives the product twelve.

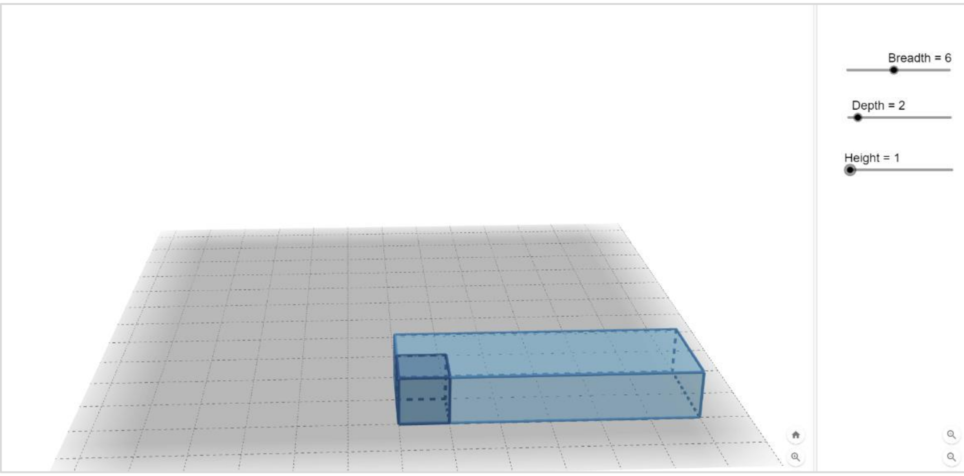
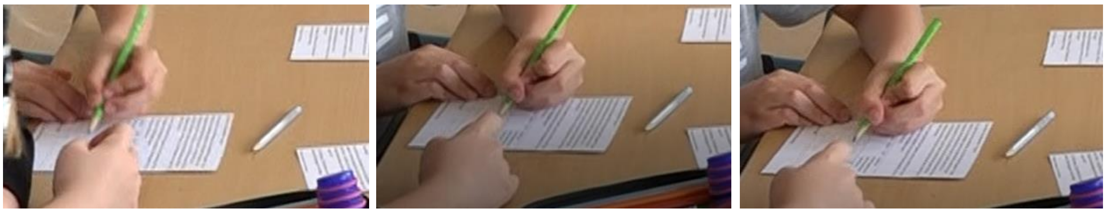


Figure 18. Digital material arrangement

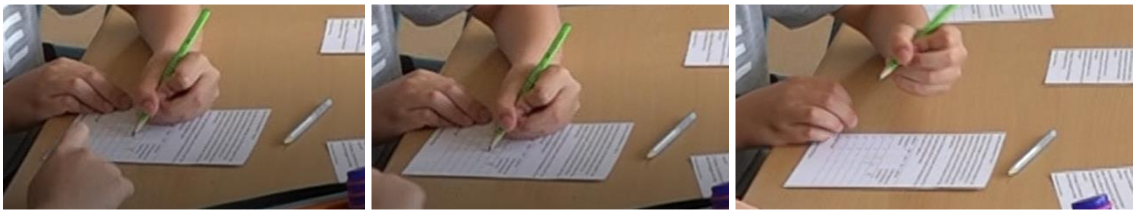
Step 1: The analysis begins with Daniel's actions, which are comparable to those of Paul. In his actions, Daniel notes the breadth and depth of a cuboid with a volume of twelve. The prompt on which Daniel writes down the numbers for the edge lengths of the respective dimension is in front of him. He first places his pencil on the prompt (see Fig. 19a). The tip of his pencil touches the table on the prompt in the second row and in the column named *breadth*. He moves the writing end of the pencil on the prompt (see Fig. 19b). During the movement, Daniel utters, “four”. He then releases the pencil from the prompt. The number four is now visible where the pencil touched the prompt (see Fig. 21).



(a) (b) (c)

Figure 19a-c. Daniel notes the number four for the breadth of a cuboid with a volume of twelve

Daniel moves his left hand to the right and places the pencil on the prompt again (see Fig. 20a). The tip of the pencil touches the table on the prompt again in the second row but this time in the column named *depth*. Daniel moves the pencil on the prompt (see Fig. 20b). During the movement of the pencil on the prompt, Daniel utters “three” phonetically. After the movement, he releases the writing end of the pencil from the prompt. The number three is now written in the cell where Daniel moved the pencil (see Fig. 21).



(a) (b) (c)

Figure 20a-c. Daniel notes the number three for the depth of a cuboid with a volume of twelve

Everything that Daniel and Madison noted on the prompt in Explications 1 and 2 can be traced in Figure 21. The energetic interpretants shown in the picture sequences 19a-c and 20a-c lead to the signs marked in row two. As in the analysis of the actions on the analogue material, the signs on the prompt can be understood in a semiotic sense as indices of the actions of writing down (see, for example, Fig. 19a-c) and of Daniel’s actions on the geometrical analogue material arrangement on the screen. The actions, in turn, are indices of the relationships between the signs as interpreted by Daniel.

Number of unit cubes (volume)	Height	Breadth	Depth	Number of unit squares on all side faces (surface)
12	1	3	4	38
12	1	4	3	38
12	1	2	6	40
12	1	6	2	40
12	1	1	12	50
12	1	12	1	56

Figure 21. The prompt completed by Daniel and Madison

Step 2 - Explication 1 (E1.1): In the analysis, Daniel's energetic interpretant, just described (see Figs. 19a-c and 20a-c), is first compared with the research-based interpretant to be able to reconstruct the diagram interpretation made by Daniel. As in the analysis of the actions made on the analogue material, relationships between the signs are described in the research-based interpretant, appropriate to the section of the processing, which must be interpreted by the learners and observed in their actions to produce a cuboid with a volume of twelve. The same relationships are described although the resulting actions on the digital material, in which these relationships are observed and established, differ from the actions on the analogue material.

- *Relationship between the length of the edges of a cuboid with the volume of twelve:*
  To make a cuboid with a given volume, the lengths of the different edges of the cuboid must be related. Since a volume of twelve and a height of one are already given in the task, only relationships between the edge lengths of the breadth and the edge lengths of the depth must be established. These edge lengths must be chosen in such a way that multiplying the lengths produces the number twelve. Possible lengths would be twelve and one, six and two, or three and four. To establish such a relationship between the lengths of the edges, the learners have to recognise which values the lengths of the cuboid edges can adopt on the screen. To do this, they have to establish a relationship between the setting of the scrollbar and the matching edge length. The learners have to realise that the number to which the scrollbar is set corresponds to the length of the respective edge. The learner can create this relationship actively by making a drag movement over the scrollbar. They can use the relationship to set the length that the breadth, height, and depth of the cuboid should take. To increase the length of the edges of a dimension, the learner must perform drag movements to the right over the scrollbar. A drag movement to the left will shorten the respective parallel edges.
- *Relationship between three edges that meet in a corner:*
  To build a cuboid, two edges of all the edges that meet in a corner must always enclose a right angle. This relationship between the edges does not need to be established by the learners in their actions as it is already implemented in GeoGebra. The learners only have to set the length of the edges by drag movements over the scrollbar and this automatically generates a cuboid. In a corner, the edges from all three dimensions always meet, so in the case of a cuboid, three edges of different lengths always meet. From this, it can be deduced that all parallel edges, to those that meet in a corner, have the same length. The learners do not need to consider these relationships in their actions because GeoGebra either automatically lengthens or shortens all the edges in a dimension in the same way, starting from the setting of the scrollbar, and the edges that meet in a corner are already predetermined. These relationships are also implemented in GeoGebra and are automatically established without any further action on the part of the learner.

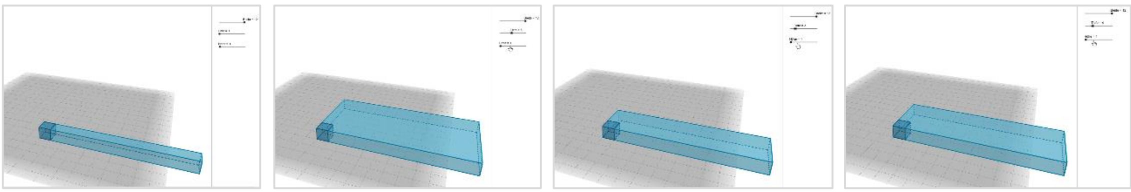
(E1.2 and E1.3): In the comparison between the research-based interpretant and Daniel's energetic interpretant (see Figs. 19a-c and 20a-c), it is clear that Daniel's actions do not establish relationships between the parts of the geometrical arrangement of the materials seen on the screen. The notation of the two edge lengths in one row of the table allows the reconstruction that Daniel establishes a relationship between these edge lengths. Due to the pre-filled columns in the row (see Fig. 21), in which Daniel has entered the breadth of four and the depth of three, it can be assumed that Daniel recognises that a cuboid with a height of one, a breadth of four and a depth of three has a volume of twelve. It can be assumed that Daniel interprets the signs that were already on the prompt and those that he wrote down as an arithmetic diagram. By interpreting the signs as an arithmetical diagram, Daniel does not have to take any actions on the geometrical material arrangement, with which he establishes the relationships between the lengths of the edges, to build a cuboid with a volume of twelve. Without further action and references to insights already made, it remains open whether Daniel can interpret the

numbers he has written down geometrically and whether he recognises the connection between the numbers he has written down and the geometrical arrangement of the materials.

Step 3 – Explication 2 (E2.1): In the transcript, further energetic interpretants made by Daniel can be found in which he determines the edge length of breadth and depth, and which can, thus, be seen as comparable actions to those in Figures 19a-c and 20a-c. Daniel moves his right hand forward towards the tablet and touches the border of the tablet with his right index, middle, ring, and little finger (see Fig. 22a). His right thumb is splayed out and over the tablet's screen (see Fig. 22a). The screen shows a cuboid with a height of one, a breadth of twelve, and a depth of one (see Fig. 23a). Daniel touches the screen with the thumb of his right hand at the position of the scrollbar labelled *depth* (see Fig. 22b). He performs a drag movement to the right with his right thumb over the scrollbar labelled *depth* (see Fig. 22c). The slider of the scrollbar *depth* changes from position one to position six. The screen now shows a cuboid with a height of one, a breadth of twelve, and a depth of six (see Fig. 23b). Without releasing his right thumb from the screen, Daniel makes a drag movement over the scrollbar to the left (see Fig. 22d). The scrollbar *depth* adjusts from position six to position three. The cuboid shown on the screen now has a height of one, a breadth of twelve, and a depth of three (see Fig. 23c). Without taking his right thumb off the screen, Daniel makes a drag movement over the scrollbar *depth* to the right (see Fig. 22e). The position of the scrollbar *depth* changes from three to four. The cuboid shown on the screen now has a height of one, a breadth of twelve, and a depth of four (see Fig. 23d). Daniel comments on the setting with the phonetic utterance "four". Daniel releases his right thumb from the surface of the tablet (see Fig. 22f) and utters the phonetic expression, "and then the breadth is three".

(a) (b) (c) (d) (e) (f)

Figure 22a-f. Daniel's drag movements over the scrollbar *depth*



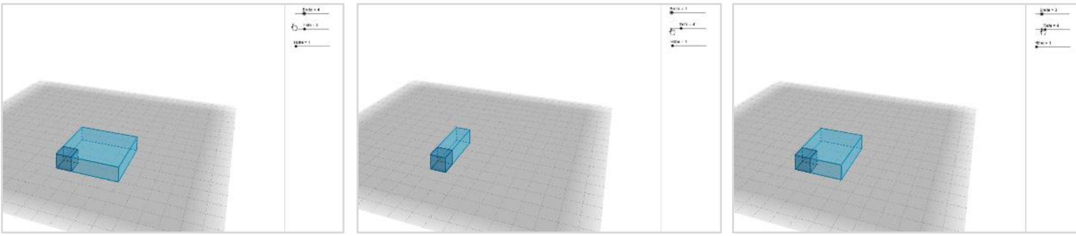
(a) (b) (c) (d)

Figure 23a-d. Settings of the scrollbar *depth*

Daniel moves his thumbs up above the tablet and touches the screen with his right thumb at the position of the scrollbar labelled *breadth* (see Fig. 24a). Daniel makes a drag movement to the left over the scrollbar *breadth* (see Fig. 24b). The setting of the scrollbar does not change. He releases his right thumb from the screen and places it again at the position of the scrollbar *breadth* on the screen. He makes another drag movement to the left over the scrollbar *breadth*. Again, the setting of the scrollbar *breadth* does not change. Daniel releases his finger from the surface of the screen a second time, places it again at the position of the scrollbar *breadth* on the screen, and performs a drag movement to the left over the scrollbar *breadth*. Again, GeoGebra does not change the setting of the scrollbar *breadth*. Daniel starts the fourth attempt by releasing his right thumb from the screen again, placing it at the position of the scrollbar *breadth* on the tablet (see Fig. 24c), and making a drag movement to the left over the scrollbar *breadth* (see Fig. 24d). This time the position of the scrollbar *breadth* changes from twelve to four. The screen now shows a cuboid with a height of one, a breadth of four, and a depth of four (see Fig. 25a). Daniel does not release his right thumb from the surface of the screen and makes another drag movement to the left over the scrollbar *breadth* (see Fig. 24e). The position of the scrollbar changes from four to one. The screen shows a cuboid with a height of one, a breadth of one, and a depth of four (see Fig. 25b). Without removing his right thumb from the screen, Daniel follows drag movements to the right over the scrollbar *breadth* (see Fig. 24f). The setting of the scrollbar changes from position one to position three. The screen now shows a cuboid with a height of one, a breadth of three, and a depth of four (see Fig. 25c).

(a) (b) (c) (d) (e) (f)

Figure 24a-f. Daniel's drag movements over the scrollbar *breadth*



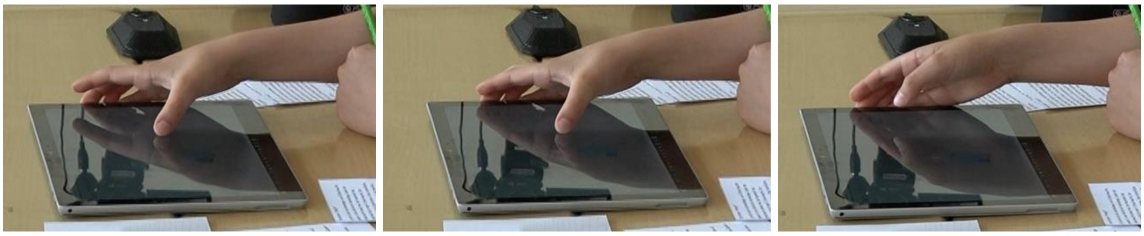
(a) (b) (c)

Figure 25a-c. Settings of the scrollbar *breadth*

(E2.2): In the comparison between Daniel's energetic interpretants (see Figs. 22a-f to 25a-c) and the research-based interpretant, it can be reconstructed that Daniel establishes relationships between the edge lengths. Based on his actions (see Figs. 22a-f to 25a-c), it is likely that he recognises that a height of one, a breadth of three, and a depth of four make a cuboid with a volume of twelve. To establish this relationship between breadth and depth in the geometrical material arrangement, it can be assumed that Daniel first establishes a relationship between the position of a scrollbar and the length of the respective edges. He likely recognises, for example, that the position of the scrollbar *breadth* affects the length of the edges in the dimension breadth. From Daniel's concrete actions at the scrollbar *depth*, it can be reconstructed that Daniel recognises that the direction of his drag movement is related to the change in the length of the edges. To lengthen the edges in the depth dimension, Daniel makes a drag movement to the right (see Fig. 22b). He probably realises that lengthening the edges of this dimension is made possible by a drag movement to the right via the matching scrollbar. Furthermore, it can be reconstructed from Daniel's following actions (see Fig. 22c-e) that he can shorten an edge length that is presumably too long (see Fig. 23b) with a drag movement to the left and, subsequently, lengthen an edge length that is presumably too short with a drag movement to the right (see Fig. 23c). By releasing the thumb from the screen (see Fig. 22f), it can be assumed that Daniel has set the desired edge length. This assumption is confirmed by his phonetic utterance "four". The immediate, phonetic utterance which follows that the breadth of the cuboid must be 'three', thus warrants the assumption that Daniel, in his subsequent actions, exploits the relationships between the scrollbar and the edge length to produce a cuboid with a breadth of three. From his actions over the scrollbar *breadth* (see Fig. 24a-f), it can again be reconstructed that Daniel establishes relationships between the direction of the drag movement and the change in edge length. Even if he does not succeed in shortening the lengths of the edges by a drag movement to the left, Daniel continues to interpret the relationships in the same way through the successive actions that are performed in the same manner. Based on the analysis of Daniel's actions for the reconstruction, it is likely that Daniel recognises relevant relationships between the parts of the material arrangement on the screen and interprets them as a geometrical diagram.

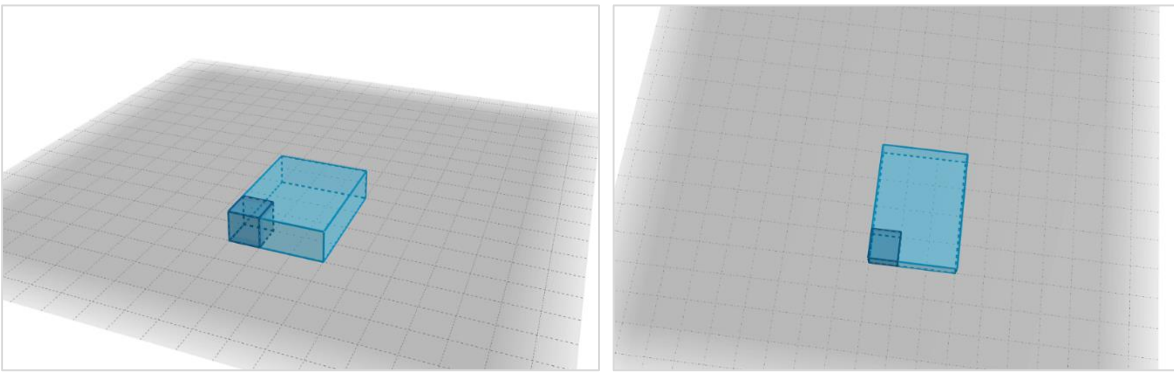
Other comparable interpretants of Daniel can be found in the transcript in which, for example, he sets edge lengths of two and six for breadth and depth, respectively, and notes edge lengths of one and twelve for the breadth and depth, respectively, on the prompt. The analysis of these actions allows the reconstruction of the same diagram interpretations and they are, therefore, not listed in more detail. Looking at all the interpretants of Daniel listed so far, it is likely that he establishes different relationships between the edge lengths of depth and breadth to construct a cuboid with a height of one and a volume of twelve. It becomes clear that he establishes these relationships in his actions with both geometrical and arithmetical signs.

(E2.3): In addition, further energetic interpretants can be found in Explication 2 in which Daniel considers the surface of the cuboids with a volume of twelve but having different edge lengths. Other energetic interpretants in which he takes this focus are shown in the following sequences of pictures. After Daniel has set the scrollbar *breadth* to position three, he releases his right thumb from the surface of the screen and moves it to the left so that his right thumb is above the cuboid. On the screen, the cuboid with a height of one, a breadth of three, and a depth of four can be seen diagonally above (see Fig. 27a). Daniel places his right thumb on the screen at the point where the cuboid can be seen (see Fig. 26a). From this position, Daniel makes a downward drag movement across the screen (see Fig. 26b). The perspective on the cuboid changes. The cuboid can now be seen from above (see Fig. 27b). Daniel then releases his right thumb from the surface of the screen (see Fig. 26c) and pulls his hand back towards his upper body. During his action, his gaze is turned towards the screen (not visible in the images due to anonymisation).



(a) (b) (c)

Figure 26a-c. Daniel's drag movement across the cuboid



(a) (b)

Figure 27a-b. Changes in the perspective of the cuboid

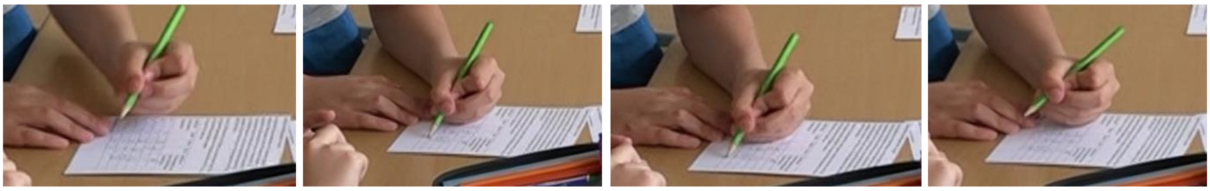
After Madison has noted the edge lengths of three for the breadth and four for the depth on the prompt, Daniel turns his gaze back to the cuboid on the screen. He brings his right hand forward towards the tablet. His right index finger is extended, while the remaining fingers of his right hand form a loose fist (see Fig. 28a). Meanwhile, Daniel utters, "then there are twelve at the top". Daniel then moves his right hand upwards, keeping the shape of his hand unchanged (see Fig. 28b). While Daniel says, "and twelve at the bottom", he moves his index finger closer to the surface of the tablet again (see Fig. 28c).

(a) (b) (c)

Figure 28a-c. Daniel's pointing gestures towards the cuboid

Daniel then makes further drag movements across the cuboid and, in this way, changes the perspective of the cuboid. He looks in the direction of the cuboid on the screen while he says aloud, "then you have four from the side twice, that's 28". He continues, "then you have six more (..) 34".

At the end of the transcript, actions can be found in which Daniel notes the surface of a cuboid with a height of one, a breadth of twelve, and a depth of one on the prompt. Daniel moves his left hand, in which he holds the pencil, towards the prompt (see Fig. 29a). He places the tip of the pencil on the last row of the table in the column labelled *surface* (see Fig. 29b). He moves the writing end of the pencil on the prompt (see Fig. 29c). He then removes the pencil from the prompt (see Fig. 29d). In the last line, where the volume is twelve, the height is one, the breadth is twelve, and the depth is one, the number 50 is now written for the surface (see Fig. 21). The actions in Figure 29a-d lead to the signs marked in the last line in Figure 21.



(a) (b) (c) (d)

Figure 29a-d. Daniel notes the number 50 on the prompt

(E2.4): Since Daniel focuses, in the further energetic interpretants, on other relationships between different parts of the diagram which then help him to determine the number of unit squares on all sides of the cuboid, the relationships in the research-based interpretant must also be expanded. Therefore, in the extension of the research-based interpretant, relevant relationships for determining the surface of a cuboid and the resulting actions are described.

- *Relationship between the unit squares and a side face of the cuboid to determine the area of the side face:*

  The size of an area can be indicated by the number of smaller (standardised) areas. For this purpose, the smaller area, with which the area to be measured can be laid out, is placed in relationship to the area to be measured. Since the unit cube is fixed in the lower left corner of the cuboid, the learners cannot establish this relationship by holding the side face of the unit cube against the respective side faces of the cuboid in a repeated fashion. However, they can estimate from the unit cube how many times the side face of the unit cube can be placed against the side face of the cuboid. The learners can also establish the relationship between the unit squares and a side face of the cuboid using the square grid. The cuboid is already placed on the square grid so that the edges of the cuboid lie on the lines of the square grid. In this way, the learners can see how many unit squares are between the edges of the base area of the cuboid and can determine them by counting. Since the cuboid is fixed on the square grid, the other side faces of the cuboid cannot be placed on the square grid. However, changing the perspective on the cuboid can help the learners recognise how many unit squares there are between the other edges of the cuboid. The learners can change their perspective on the cuboid by drag movements across the cuboid.

- *Relationship between the area of the different side faces of the cuboid to determine the area of the surface:*
  To determine the surface area, the determined number of unit squares on the different side faces must be placed in relationship to each other. To be able to indicate the total surface area with the number of unit squares, the numbers on all side faces must be added together. In the case of a cuboid, the opposite side faces are equal in size so that the number of unit squares with which the opposite side faces can be laid out are equal. The learners can take advantage of this by doubling the number of unit squares on each of the opposite side faces so that they do not have to measure each side face. Exploiting these relationships between the opposite side faces in a calculation shortens the actions of making relationships between the unit squares and a side face of the cuboid.

(E2.5 and E2.6): By comparing Daniel's interpretants (see Figs. 26a-c to 29a-d) and the research-based interpretant, it can be reconstructed that Daniel establishes several relationships in his actions to determine the surface area of cuboids with a volume of twelve. Based on the drag movement across the cuboid (see Fig. 26a-c), which results in an adjustment of the perspective of the cuboid, it is likely that Daniel is changing the perspective of the cuboid to establish, more effectively, a relationship between the unit squares and a side face of the cuboid. The setting of the perspective from above on the cuboid allows the assumption that Daniel wants to determine the unit squares with which the base or the top area of the cuboid can be laid out. The following gestures and the speech uttered at the same time (see Fig. 28a-c) confirm this assumption. Based on the first pointing gesture towards the cuboid and the phonetic utterance, "then there are twelve on top", it can be reconstructed that Daniel is establishing a relationship between the unit squares on the square grid and the top area of the cuboid. Exactly how he obtains the number of unit squares cannot be reconstructed from his gestures and spoken language. The following gestural and phonetic utterances allow the reconstruction that he is establishing a relationship between the unit squares and the base area. Furthermore, it can be assumed that Daniel is already establishing a relationship between the opposite side faces of the cuboid since he determines the area of these two faces, one after the other, and comes up with the same number of unit squares. The subsequent shift in perspective and the statement, "then you have four from the side twice and then 28" allows for several reconstructions. On the one hand, Daniel again uses the changed perspective to establish a relationship between the surface to be seen from the side and the unit squares. On the other hand, he establishes a relationship between the opposite faces by doubling the number of unit squares on one side face. The consequence of establishing this relationship is that he does not have to adjust the perspective on the respective opposite face but, simply, directly doubles it. These operative relationships formulated in phonetic language lead to an abbreviation of the actions and the relationships established in them. It is also clear from his phonetic utterance that he establishes a relationship between the unit squares on all faces by adding the number of unit squares. The subsequent spoken language, in which he immediately adds the number of six unit squares, makes clear that Daniel no longer produces the operation of doubling by spoken language but directly uses the result of the operation for further insights. This leads first to a shortening of the relationships produced in the actions and then to a further shortening of the relationships produced in the spoken language. The number 34, which he expresses at the end, can be interpreted as an insight that he gains from the relationships produced by his actions, gestures and spoken language.

The described actions of noting the number 50 (see Fig. 29a-d) allow the reconstruction that Daniel establishes relationships between the signs on the prompt. Due to the columns already being filled in the line in which Daniel notes the number (see Fig. 21), it can be assumed that Daniel recognises that a cuboid with a height of one, a breadth of twelve, and a depth of one has a volume of twelve and a surface of 50. It can be assumed that Daniel relates the signs already on the prompt to the number 50 and, thus, interprets these signs as an arithmetical diagram. Considering the already reconstructed interpretations for Daniel, it can be assumed that he establishes relationships between the geometrical signs in his actions, gestures, and spoken language and records the insights from his activity in arithmetical signs. Thus, it is likely that he can interpret the signs on the prompt not only as an arithmetical diagram but also geometrically.

There are further passages in the transcript where Daniel determines the surface of the cuboid with edge lengths of two and six, and twelve and one, respectively, for breadth and depth, in the same way as described in the actions in Figures 26a-c to 28a-c. Furthermore, a passage can also be found in which he notes the surface of a cuboid with a breadth of four and a depth of three. The analysis of these actions allows the reconstruction of the same diagram interpretations made by Daniel. For reasons of space, these actions are not listed in detail.

Step 4 – Explication 3 (E3.1 and E3.2): In the further processing of the learning situation, additional energetic interpretants made by Daniel can be found in which he determines the edge lengths of the breadth and depth and the surface area of cuboids with a height of one and a volume of twelve. First, the focus is on Daniel's actions which can be classified as the same actions as in Explication 1 (see Figs. 19a-c and 20a-c) due to the focus being on the same relationships. These include actions in which Daniel sets the scrollbar *breadth* and *depth* to a certain length.

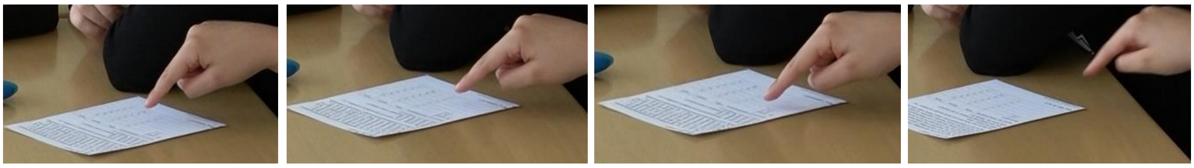
In many actions, Daniel sets the scrollbar *breadth* and *depth* in the same way as in Figures 22a-f to 25a-c. For example, in minute 20 of the processing, he sets the scrollbar *breadth* to position three and the scrollbar *depth* to position four. In minute 22, Daniel expresses, "then breadth six" and sets position six with drag movements across the scrollbar *breadth*. He then says aloud, "and depth two" and sets position two on the scrollbar *depth*. In minute 23, Daniel says, "ok breadth twelve" and sets the breadth to twelve. He then utters, "and depth one" and sets the depth to one. In minute 30, Daniel sets the scrollbar *breadth* to position four and the scrollbar *depth* to position three by performing drag movements to the right and left. The analysis of all these actions allows the reconstruction of the same diagram interpretations made by Daniel that were already described in the analysis step E2.2. As described above, it can be assumed that he is establishing a relationship between the length of the edges in the breadth and depth. He achieves this by using relationships between the position of a scrollbar and the length of the respective edges, and a relationship between the direction of the drag movement and the change in the length of the edges. Due to the reconstruction of the same diagram interpretation, these actions are not reproduced in detail.

Nevertheless, actions can also be found in which Daniel establishes a relationship between the edge lengths and uses other relationships between signs for this purpose. In minute 28 of the processing, Daniel looks at the prompt lying in front of the two learners. On this prompt, the edge lengths of the cuboids are already given and are chosen in such a way that the cuboid has a volume of twelve. The learners' task is to write down the volume of the cuboids with the given lengths. At this moment, Daniel and Madison have already written down a volume of twelve for a cuboid with a height of one, a breadth of one, and a depth of twelve. Daniel moves his right hand towards the prompt and has his right index finger pointing down towards the prompt. He has spread the other fingers of his right hand (see Fig. 30a). During his movement, he utters, "on the piece of paper you can already-". Daniel then forms the fingers of his right hand into a loose fist and points with his index finger to the right side of the table where the edge lengths of the cuboids are already printed (see Fig. 30b). Daniel moves his right index finger down over the right part of the table (see Fig. 30c) and says aloud, "already calculating multiplication problems like this".

(a) (b) (c)

Figure 30a-c. Daniel's movements of the right index finger across the right columns of the table

Daniel then moves his right index finger to the right along a row of the table (see Fig. 31a-c). As he then moves his right index finger down across the left column, labelled volume (see Fig. 31d), he utters, "and then you already have here what should be written everywhere".



(a) (b) (c) (d)

Figure 31a-d. Daniel's movements of the right index finger across the entire table

Comparing the research-based interpretant (Explication 1) and Daniel's energetic interpretant (see Figs. 30a-c and 31a-d), it can be reconstructed that Daniel establishes a relationship between the edge lengths that are on the prompt and the volume to be noted. Daniel has already established such a relationship between the signs of a line in Explication 1. What is new is that he describes this relationship in more detail. Specifically, it can be reconstructed from Daniel's gestures that he first marks the edge lengths already noted, then establishes a relationship between the numbers in a row and then marks the column of the prompt that still has to be filled in. He is likely expressing through the gestures that the left-hand side numbers can be inferred from the right-hand side numbers when reading from right to left. The simultaneously uttered spoken language allows for the further reconstruction that the relationship Daniel establishes in the gestures between the numbers of a row is multiplicative. It can be assumed that Daniel interprets the signs as an arithmetical diagram through the recognised relationships between the signs. In this passage, he not only uses the relationships between the signs in his activity but also expresses them explicitly. Through the multiplication he expresses, Daniel does not have to take any actions on the geometrical material arrangement, with which he establishes relationships between the lengths of the edges, to determine the volume.

(E3.3 to E3.5): Explication 3 has so far only focused on actions (see Figs. 30a-c and 31a-d) that can be classified as the same actions as in Explication 1 (see Figs. 19a-c and 20a-c), based on the consideration of the same relationships. In the following, actions are included in the analysis that focus on the same relationships as the actions in analysis step E2.3 (see Figs. 25a-c to 28a-d). In these actions, Daniel determines the surface of cuboids with a volume of twelve.

In many actions, Daniel adjusts the perspective of the cuboid (similar to Figs. 25a-c and 26a-b) and establishes relationships between the side faces of the cuboid on this basis (similar to Figs. 27a-c and 28a-d). For example, in minute 20 of the processing, Daniel uses a drag movement across the cuboid to adjust the perspective of the cuboid from above. He says aloud, "and now we have to look from above then twelve will fit here". He then says, "from the bottom it is also twelve (.) is 24". Daniel then adds the other numbers of unit squares that fit on the other side faces of the cuboid. In minute 21 of the processing, Daniel again first adjusts the perspective from above on the cuboid and then adjusts a lateral perspective. Based on these adjustments, he determines the numbers on all sides which he then adds up, one after the other. Daniel notes the result of the calculation in the table. He performs these actions in further processing for other cuboids with different edge lengths for breadth and depth. The analysis of all these actions allows the reconstruction of the same interpretations made by Daniel that were already described in the analysis steps E2.5 and E2.6. As described above, it can be assumed that he uses the changed perspective of the cuboid to establish relationships between the number of unit squares and a side face (how exactly he establishes this relationship remains open) and that he establishes relationships between the opposite side faces of the cuboid by doubling the number of unit squares, and that he establishes relationships between the signs on the prompt. Due to the reconstruction of the same diagram interpretation, these actions are not reproduced in detail.

However, it is also possible to find actions that can be used to reconstruct *how* Daniel establishes a relationship between the number of unit squares and a side face. In minute 23 of the processing, Daniel's gaze is turned towards the tablet. On the tablet, a cuboid with a breadth of six and a depth of two can be seen from above (see Fig. 32).

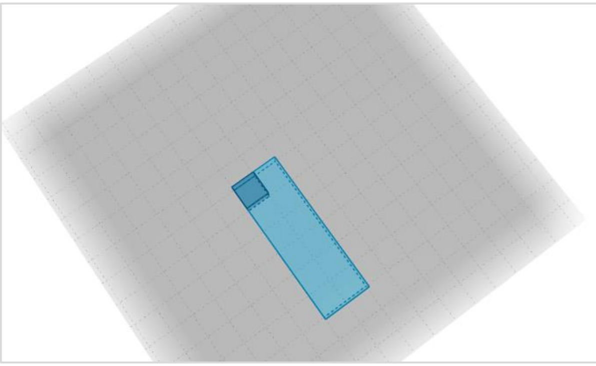
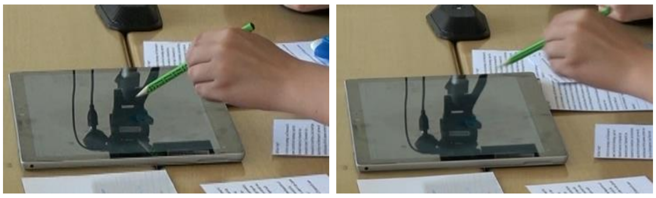


Figure 32. Perspective on the cuboid with a breadth of six and a depth of two

Daniel utters, "twelve" aloud. He then bends forward and brings the pencil, he is holding in his left hand, towards the tablet and places it over the cuboid on the tablet (see Fig. 33a). Starting from this position, he makes a movement with the pencil downwards to the right across the tablet (see Fig. 33b). Daniel says, "because here is six".



(a) (b)

Figure 33a-b. Daniel's first movement with the pencil across the tablet

Daniel then moves the tip of the pencil upwards to the left so that the pencil is in approximately the same position as in Figure 33a (see Fig. 34a). From this position, Daniel moves the pencil downwards to the right across the tablet (see Fig. 34b) so that the pencil is above the bottom right corner of the tablet (see Fig. 35c). While moving down to the right, Daniel says, "and here is six".

(a) (b) (c)

Figure 34a-c. Daniel's second movement with the pencil across the tablet

By comparing the research-based interpretant (Explication 2) and Daniel's energetic interpretant (see Figs. 33a-b and 34a-c), it can be reconstructed that Daniel determines the number of unit squares with which the base or top area can be laid out by establishing a relationship between the unit squares on the square grid and the side face. As already described, it can again be assumed that the adjustment of the perspective on the cuboid (see Fig. 32) helps Daniel to establish this relationship. In addition, it can be reconstructed, based on his gestures and spoken language, how Daniel establishes the relationship between the unit squares and the side face. It can be assumed that with his first gesture across the cuboid (see Fig. 33a-b), he marks the unit squares with which the surface can be laid out along the breadth. He names this marking with the number six. With the second gesture (see Fig. 34a-c), Daniel probably marks the other six squares that can be laid out along the breadth and names them again with the number six. Since he wants to use the gestural and phonetic utterances to justify why he previously uttered, "twelve" (see utterance of the word "because"), it can be assumed that Daniel determines the number of unit squares on this side face by counting the unit squares twice down the breadth.

Step 5 - Summary of the Explications: By comparing Daniel's energetic interpretants and the research-based interpretants, different diagram interpretations made by Daniel could be reconstructed during the processing. In Explications 1 and 2, it can be reconstructed that Daniel does not establish relationships between the parts of the geometrical material arrangement on the screen with the first actions. However, it can be reconstructed that Daniel establishes relationships between the arithmetical signs in a row of the table and interprets these signs as a diagram through the recognised relationships between the signs. What this relationship between the signs explicitly looks like cannot be reconstructed. Through this interpretation as an arithmetical diagram, Daniel does not have to take any actions on the geometrical material arrangement to establish relationships between the edge lengths of a cuboid with a volume of twelve. However, in further analysis, it becomes clear that Daniel also establishes relationships between the geometrical material arrangement and also interprets this as a diagram. He establishes relationships between the position of the scrollbar and the length of the edges, and between the direction of the drag movements across the scrollbar and the change in the length of the edges. Concerning the relationship between the lengths of the edges, Daniel chooses the breadth and depth in such a way that a cuboid with a height of one and a volume of twelve is created. He establishes this relationship between the edge lengths of breadth and depth in his actions between both geometrical and arithmetical signs. In the further course of the analysis, it can be reconstructed that Daniel changes the perspective of the cuboid to establish a relationship between the unit squares on the square grid and a side face. However, it remains unresolved, at present, how he establishes this relationship. In addition, it can be reconstructed that Daniel creates a relationship between the opposite side faces of the cuboid using spoken language and, in this way, no longer has to perform the actions for changing the perspective of the cuboid and the gestures for creating the relationship between the unit squares and a side face. Thus, the phonetically expressed relationship leads to an abbreviation of the action and gestures and the relationships established within them. Through the phonetically expressed relationships, it also becomes clear that Daniel establishes relationships between the numbers of unit squares on all side faces of the cuboid to determine the surface area. He even abbreviates the relationships in the spoken language in the calculation of the surface since he no longer expresses the calculation of the doubling of the number of unit squares on the opposite side faces but only expresses the result of the doubling. The number that Daniel finally expresses for the surface can be interpreted as an insight that he gains from the relationships between the geometrical signs he established in his actions, gestures, and phonetic language. However, Daniel also establishes relationships between the number for the surface area and the numbers of the edge lengths at an arithmetical level and interprets these as an arithmetical diagram. It can, therefore, be assumed that he establishes relationships between the geometrical signs in his actions, gestures, and spoken language and records the insights from his activity in arithmetical signs. It is, therefore, likely that he can interpret the signs on the prompt not only as an arithmetical diagram but also geometrically.

In Explication 3, two further passages from Daniel's processing can be used to reconstruct *how* he establishes relationships between the arithmetical signs of a row of the table on the prompt, and *how* he establishes relationships between the number of unit squares and a side face. On the one hand, it can be reconstructed that Daniel establishes a multiplicative relationship between the edge lengths to determine the volume of the cuboid. Following this reconstruction, it can be assumed that Daniel establishes a multiplicative relationship between the edge lengths of breadth and depth that he notes in Explication 1. On the other hand, it can be reconstructed that Daniel determines the number of unit squares on this side face by counting the unit squares twice down the breadth. Concerning Daniel's previously reconstructed interpretation of the diagram, it can be assumed that he establishes the relationship between the number of unit squares and a side face by counting the unit squares down the breadth or the depth even if he does not explicitly express this aloud.

6.1.3 Comparison of the Results from the Analyses

To be able to describe the differences and similarities between the reconstructed diagram interpretations of the fourth-graders, Paul and Daniel, the results of the analyses are compared. The focus is on how the learners establish the relationships between the signs and whether the different material influences the relationships they establish in their actions.

By comparing the reconstructed diagram interpretations for Paul, who works with analogue material, with those for Daniel, who works with digital material, it becomes clear that both learners interpret the signs on the prompt as an arithmetical diagram. They both establish relationships between the edge lengths, the surface area and the volume, which are noted in a row of the table on the prompt. It can be reconstructed that these signs noted on the prompt can be interpreted as insights from the activities on the geometrical material arrangement. In Paul's case, who works with analogue material, it becomes clear that he uses the insights he has already made for his further work and in this way does not have to repeat actions on the geometrical material arrangement to establish relationships between the geometrical signs.

It can also be reconstructed that Daniel interprets the digital and Paul the analogue geometrical material arrangement as geometrical diagrams. However, a closer look reveals that Paul and Daniel establish the relationships between the parts of the material arrangement differently. When making a cuboid with a volume of twelve, it can be reconstructed by placing the rods on the square grid, Paul first establishes a relationship between the rods and the unit lengths on the square grid to determine the length of the rods. In Daniel's case, it can be reconstructed from the drag movements over the scrollbar that he establishes a relationship between the setting of the scrollbar, the respective visible edge on the screen and the square grid to determine the length of an edge. Through the two different actions, relationships are established between the edges and the square grid, leading to the same mathematical insight about the length of the edges.

Paul's further reconstruction shows that he establishes different relationships between the edge lengths of the breadth and the depth for the construction of a cuboid with a volume of twelve and a height of one. This can be reconstructed from the selection of suitable rod lengths for the construction of a cuboid. Daniel also establishes relationships between the edge lengths of the breadth and depth by adjusting the scrollbars to the appropriate position by drag movements to the right and left. For the correct setting of the scrollbars, Daniel establishes a relationship between the direction of the drag movement and the change in edge length. With the different actions on the analogue and digital material, Paul and Daniel establish the same relationships between the edge lengths of the breadth and depth and, therefore, the same diagram interpretations can be reconstructed.

From Paul's choice of the four rods of equal length, it can be reconstructed that Paul probably recognises that the edges of a dimension are of equal length and that he always needs four edges of equal length specifically for the construction of a cuboid. In the concrete construction of the cuboid, it can then be further reconstructed that Paul establishes relationships between the length of the parallel edges and relationships between the length of the edges that meet in a corner of the cuboid. Daniel's actions do not allow for such a reconstruction. It becomes clear that the digital material establishes these relationships automatically since the scrollbar automatically changes all the edges of a dimension in the same way and the edges that meet in a corner are already fixed. Daniel does not have to establish these relationships in his actions, however, after GeoGebra has established the relationships, he is then required to re-interpret them to recognise that the geometrical solid is a cuboid. Thus, the digital material shortens the actions by automatically establishing certain relationships between the geometrical signs. This shortening leads to the fact that the relationships that GeoGebra has established have to be re-interpreted. Daniel can re-interpret these relationships only in his gestures and spoken language and not in his actions.

For both learners, it is not possible to reconstruct from their actions whether they establish a relationship in terms of the angles between the edges that meet in a corner. In both, the digital and the analogue material, these relationships are already predefined in the material and are automatically established. Even when Paul carries out another action of putting the rods together, he does not establish these relationships. It becomes clear that no relationships can be established in the actions if they are already predetermined by the material.

Regarding determining the surface area of cuboids with a volume of twelve and a height of one, it is clear that both Paul and Daniel make actions upon which the learners can establish a relationship between the unit squares on the square grid and a side face. Paul places the edges of the cuboid on the lines of the square grid and Daniel changes the perspective on the cuboid. While Paul is more flexible in placing the cuboid on the square grid, the actions on the digital and analogue material can be interpreted in the same way as preparation for establishing the relationship between the unit squares and a side face. Both, Paul and Daniel, establish the relationship between the unit squares and a side face in a counting manner. Furthermore, in both analyses, it can be reconstructed that Paul and Daniel establish relationships between the opposite side faces by doubling the number of unit squares on one side face for the inclusion of the opposite side face. By establishing this relationship between the opposite side faces in phonetic language, the actions of the two learners decrease. Daniel no longer changes the perspective for determining the area of each side face and Paul no longer places each side face on the square grid. The gestures for counting the unit squares also decrease as it is no longer necessary to determine the area of every side face by counting the unit squares. It can be assumed that a shortening of the actions, gestures and the relationships established in them leads to a shift in the way the relationships are expressed. The learners abbreviate the gestures and actions and express the relationships in spoken language. To determine the surface area of the cuboid, both learners establish a relationship between the number of unit squares on all side faces. Paul does this mainly by counting on the unit squares, while Daniel adds the number of unit squares on the respective side faces. Here, the same relationships are established phonetically in two different ways, and; thus, the same interpretations of the learners can also be reconstructed. Both learners write down their insights about the surface that they have gained from the relationships between the geometrical signs established in their actions, gestures, and spoken language in an arithmetical sign on the prompt. Thus, as already described above, they establish relationships between the arithmetical signs in a row of the table and interpret them as an arithmetical diagram.

Overall, the actions of both learners decrease in the course of the processing. In Paul's case, this is because he uses the insights he has already gained to complete another prompt and, thus, no longer has to repeat the actions on the geometrical material arrangement. He also establishes relationships between the opposite side faces phonetically by doubling the number of unit squares on one side face for the side opposite. This can also be reconstructed for Daniel; he even makes a further shortening in the spoken language by only uttering the result of the doubling. Although Daniel establishes a multiplicative relationship between the noted edge lengths in a row of the table on the prompt to determine the volume, he establishes the relationships between the edge lengths once again on the digital material arrangement. He voices that the multiplication can already provide insights into the volume, but he does not use calculations to determine the volume.

6.2 Summaries of the Results from the Analyses in Chapters 3 to 5

To be able to compare the results of the analyses of the actions made on the digital and analogue materials from the different examples in the most effective way, the results of the analyses from chapters 3 to 5 are briefly summarised again below.

6.2.1 Analysis Results of the Geometrical Example from Chapter 3

In chapter 3, the third-graders Nils and Marleen work on a geometrical learning situation in which they investigate the perimeter of similar squares. Nils works with features from GeoGebra, while Marleen, like Paul in the previous analysis, has rods of different lengths, a unit square, a square grid squared in the size of the unit square, and right-angled corner connections at her disposal (see sections 3.5, p. 24).

By comparing the results of the analysis of their actions on the analogue and digital materials, their gestures, and spoken language, it becomes clear that Marleen and Nils interpret the geometrical material arrangement as a diagram since they perform rule-governed activities on it. To construct squares of different sizes, both learners establish the same relationships between the lengths of all sides. Nils establishes the relationship between the side lengths by setting the scrollbar breadth and depth to the same position. As in Daniel's case, it can be assumed that Nils establishes a relationship between the position of the scrollbars and the length of the sides of the square and that he recognises that the direction of the drag movement is about the change in the respective side length. Marleen establishes the relationship between the side lengths by selecting rods of equal length; by choosing four rods of the same length, she probably realises that the two side lengths must have the same length in the dimensions of breadth and depth. In the comparison, it becomes clear that Nils only has to consider the relationship of the side lengths between the dimensions because by setting a scrollbar he automatically sets the sides of a dimension to the same length. Marleen has to explicitly select two rods of the same length for the same dimension. The two learners do not have to make sure that the sides enclose a right angle when constructing a square; this relationship is already implemented in both the digital and analogue material. Nevertheless, Marleen still has to put the rods together with the corner connections or has to place the rods on the square grid so that the sides enclose a right angle; Nils does not have to do this because the right angle is already taken into account when setting the scrollbars to the same position. However, Marleen does not establish any further relationships in these actions since the right angle is determined by the corner connection. This action can be shortened by the digital material without affecting the relationships that the learners have to establish in their actions. In addition, this does not affect the diagram interpretations that can be reconstructed by the learners. Although the two learners do different actions on the different materials, they establish the same relationships between the side lengths and; thus, the same diagram interpretations can be reconstructed.

Concerning the determination of a square's perimeter, it can be reconstructed from the actions of the two learners that they establish relationships between the unit lengths on the square grid and one side, and between the number of unit lengths on all sides of the square. Marleen explicitly establishes the relationship between the unit lengths and a side length by pointing gestures and assigning number words to each unit length marked in the gestures. Nils, on the other hand, expresses the number of unit lengths after a longer period in which he directs his gaze to the square seen on the screen. To determine the relationship between the number of unit lengths on all sides, the learners partially formulate calculations. This leads to a shortening of Marleen's counting gestures.

Marleen and Nils also record the insights they have gained from their work on the geometrical diagram in arithmetical signs on the prompt. By noting the arithmetical signs on the prompt, both learners establish the relationships between the signs in a row and a column. Therefore, it can be reconstructed that the two learners can interpret the signs on the prompt as an arithmetical diagram. Nils explicitly expresses what the relationships between the rows and columns look like. Both learners establish relationships between already completed prompts and the prompt to be worked on. They use the insights they have already gained for their further work and do not have to repeat actions on the geometrical material arrangement to establish relationships between the geometrical signs. Nils can consistently establish the relationships between the arithmetical diagrams on the prompts, whereas Marleen cannot always establish these relationships. By establishing relationships between the arithmetical diagrams and formulating the calculation of the number of unit lengths on all sides of the square, the learners' actions on the geometrical material arrangement decrease throughout processing the learning situation.

Including the analyses of Emre and Li, it becomes clear that similar diagram interpretations can be reconstructed as in the cases of Nils and Marleen. Exclusively in the reconstruction of Emre's diagram interpretation, it is evident that he can flexibly translate between the arithmetical and the geometrical diagram.

6.2.2 Analysis Results of the Statistical Example from Chapter 4

In the fourth chapter, the third-graders Walerius and Matteo work on a statistical learning situation in which they are to answer questions using the univariate plots they have created. Walerius creates these univariate plots with the help of the software TinkerPlots (see section 4.4, pp. 40-41). Matteo has analogue data cards, sticky notes, and cubes with names at his disposal (see section 4.4, p. 41).

In the comparison of the results of the analysis, it becomes clear that Matteo and Walerius can interpret the material arrangement as a statistical diagram. In Matteo's action of sticking the sticky notes labelled with the values of an attribute next to each other, he establishes a relationship between the attribute and the axis on which the values of the attribute are plotted. Walerius establishes this relationship between the attribute and the axis by a drag movement starting from a dot in the plot and moving upwards. The same relationships are established in the two different actions, and, therefore, the same diagram interpretations can be reconstructed. However, looking at the further reconstructed diagram interpretations, differences between the relationships established in the actions can be seen. Since Matteo sticks the sticky notes next to each other at almost the same distance, it can be reconstructed that he is establishing a relationship between the values that are plotted on an axis. It can also be assumed that Matteo recognises a relationship between the data cards and the cubes based on the assignment of the cubes to the matching data cards. He uses this relationship to position the cube over the sticky note with the appropriate value. In this way, Matteo establishes a relationship between the cube and the value on the scale. Walerius, on the other hand, does not establish these relationships by his actions since TinkerPlots establishes these relationships automatically by a drag movement starting from a dot in the plot and moving upwards. Thus, the reconstruction of Walerius's complete diagram interpretations is not possible based on this one drag movement. The reconstruction of his diagram interpretation only becomes possible through the analysis of gestures and spoken language that Walerius utters after his action on the digital material. Based on his gestures and spoken language, it can be reconstructed that Walerius probably recognises a relationship between the scaling of the axis and the position of the dots in the plot. Furthermore, Walerius's gestures and phonetic utterances allow the conclusion that he can interpret a relationship between the data cards and the dots in the diagram, as well as between the values on an axis.

In the comparison of the results of the analyses, it becomes clear that there is a shift in how the relationships are expressed by the learners. Matteo establishes the relationships between the parts of the diagram with his actions, otherwise sorting of the data would not be possible. He establishes *one* relationship between the parts of the diagram by *one* action. Walerius, on the other hand, barely establishes any relationships in his actions since TinkerPlots automatically establishes many relationships based on *one* action and, thus, enables the sorting of the data. However, to be able to continue working with the automatically created sorting, Walerius has to re-interpret the relationships created by TinkerPlots. He does this in subsequent gestures and spoken language that refer to the diagram.

6.2.3 Analysis Results of the Statistical Example from Chapter 5

In the fifth chapter, the third-graders Nils and Li work on a statistical learning situation in which they are to answer questions given on prompts using the bivariate diagrams they have created. Li works with the software TinkerPlots to create bivariate plots (see section 5.5.1, p. 57). Nils has analogue data cards, a square grid with an unlabelled x- and y-axis drawn on it, cubes labelled with names and sticky notes available to make a bivariate plot (see section 5.5.1, p. 56).

By comparing the analysed actions, gestures, and spoken language of Nils working on the analogue and Li on the digital materials, it becomes clear that both learners establish relationships between the parts of the respective materials and, in this way, interpret the material as a statistical diagram. Based on Li's tap movement on an attribute on the data card, it can be reconstructed that Li recognises that she has to select an attribute to sort the dots in the plot according to the values of that particular attribute. Since TinkerPlots automatically colours the dots in the plot, it cannot be reconstructed from the actions whether Li uses this movement to select the attribute she wants to look at, or whether she establishes relationships between the individual dots in the plot and the respective data cards. Nils establishes the relationship between the data cards and the corresponding cubes by placing the cubes on the matching data cards. Based on these actions, it can be reconstructed more clearly that Nils establishes the relationship between each data card and each cube, which can, thus, be interpreted as a case. Based on Li's vertical and horizontal drag movements across the plot, it can be reconstructed that she assigns different attributes to different dimensions. Presumably, Li establishes a relationship between the direction of the drag movement and the axis on which an attribute is to be plotted. Nils establishes the relationship between an attribute and an axis by assigning the values he has written on the sticky notes to an axis. As with the reconstructed diagram interpretations for Matteo and Walerius, it can also be seen with Nils and Li that the learners establish the same relationships by using different actions. In this way, the same diagram interpretations of the learners can be reconstructed even if they work with different materials. However, it becomes clear that TinkerPlots automatically colours the dots in the plot, and, therefore, Li's tap movement cannot be used to reconstruct exactly whether she recognises the relationship between the data cards and the dots in the plot.

In a further comparison of the reconstructed diagram interpretations for Li and Nils, it becomes even clearer that TinkerPlots automatically creates relationships between the parts of the digital material arrangement based on one action. By a drag movement starting from a dot in the plot, TinkerPlots automatically establishes relationships between the values on an axis and between the values on two axes and the position of the dots in the plot. In this way, this one drag movement cannot be used to reconstruct whether Li recognises or can interpret these relationships. However, it can be reconstructed from Li's gestures and spoken language following her actions, that she can interpret the relationships that TinkerPlots has automatically established and this enables her to continue working with the new diagram that has emerged. It becomes clear that Li barely has to establish any relationships during the action but has to re-interpret the relationships established by the software afterwards. Nils, on the other hand, establishes relationships between the parts of the diagram in each of his actions to be able to create a bivariate plot. For example, Nils sticks the values of an attribute at an approximate distance to an axis and establishes, with the approximate same distance between the values, a relationship between them. He establishes relationships between the cubes and the two axes by moving his fingers over the square grid and then positioning a cube.

As with the statistical example from chapter 4, it can be noted in the comparison of the results of the analyses from chapter 5 that, with the different material, there is a shift in how the learners express the relationships between the parts of the diagram. In the analogue material, the interpretation of the diagram takes place during the manipulation of the diagram, and, in this way, the relationships in the actions are established. In the digital material, TinkerPlots first generates the signs and then the result of this sign generation has to be interpreted by the learners using gestures and spoken language to recreate the existing relationships for themselves.

6.3 Conclusion of the Main Results of the *MatheMat* Study

The summarised results of the analyses form the basis for answering the research questions that have guided this work:

1. Which mathematical interpretations of the learners can be reconstructed based on the actions performed on the digitally- or analogue-represented signs?
2. To what extent can possible differences between the reconstructed interpretations of the learners be attributed to the various materiality of the signs?
3. What influence does the material have on the relationships considered in the learners' actions to manipulate the diagram?

The research questions are answered by summarising the four main findings of the study whereby the description of each main result always answers all three research questions. The main results are described at various levels. Firstly, they are summarised at the level of the analysis. Secondly, at the theoretical level, the main results are presented in the context of the theoretical considerations from a semiotic perspective, and, finally, they are described on a practical level by establishing connections to mathematics teaching at primary school.

6.3.1 Conclusion and Discussion of the Main Results of the Analyses

From all the analyses, it is clear that the third- and fourth-graders all interpret the analogue or digital material as a diagram. By analysing their activities (which include their actions, gestures and spoken language) on the statistical, geometrical, and arithmetical diagrams, it becomes clear that they recognise relationships between the parts of the diagrams, interpret them and then use them for manipulations. Regardless of whether the signs are represented digitally or by analogue means, it can be stated that the learners do mathematics with these signs and use them for further mathematical insights. Concerning the methodological adaptation made in the work to reconstruct the learners' diagram interpretations, it can be stated that the contrasting of the learners' energetic interpretants with the research-based interpretant allows a description of the interpretations made by the learners. The combination of the context analysis according to Mayring (2014) and Vogel (2017) and the semiotic perspective on mathematics learning represents a new method that enables the reconstruction of the mathematical interpretations of the learners through the analysis of their mathematical activities. If one takes a closer look at the results of the analyses, differences and similarities can be identified that may be due to the nature of the material. These main results are given in detail below.

As a *first main result* from the analyses, it can be stated that despite different actions being made on the digital and analogue materials the learners establish the same relationships between the parts of the material arrangements. By establishing the same relationships, the same diagram interpretations could be reconstructed in the analyses. If the learners establish the same relationships, then it can be assumed that they make the same mathematical insights even if they work with different materials. In the geometrical example, it becomes clear that the learners establish the relationships between the side or edge lengths with different actions. The learners who work with GeoGebra set the scrollbars to the appropriate positions whereas the learners who work with the analogue material select the appropriate lengths of the rods. In this way, the learners produce the same manipulations on the diagram with these different actions by producing the same relationships. However, it also becomes clear, independent of the digital and analogue material, that the learners sometimes establish the same relationships differently. Paul, for example, establishes the relationship between the unit squares on all side faces of the cuboid by continuing to count the unit squares. Daniel, on the other hand, adds up the number of unit squares with which the different side faces can be laid out. By adding and continuing to count, both learners can establish the relationship between the unit squares on all the side faces to determine the surface area of a cuboid.

As a *second main result* of the analyses, it could be shown that the same diagram interpretations could not always be reconstructed based on the actions of the learners. If several relationships are automatically created from one action, then the reconstruction of the diagram interpretation based on the actions is only partially possible. Thus, the material takes over the automatic establishment of the relationships and the learners do not have to establish them in their actions. This result was found, above all, in the actions on the digital material. The digital material behaves like a tool by automatically establishing relationships between the signs. In this work, a tool is understood to be a material that changes the structure of the action by automatically establishing relationships that would normally be established by the action; thus, it depends on the design of the learning situation as to how many relationships are established by the tool. In the geometrical example, where the learners have to examine equal squares, it becomes clear that Nils sets the scrollbars to the same position and only has to make sure that the parallel sides are the same length. Marleen, on the other hand, has to explicitly select four rods of the same length so that she can build a square. Looking at the second geometrical example, where the learners have to construct cuboids with a volume of twelve and a height of one, several relationships are automatically created by setting the scrollbars. Daniel, for example, sets the scrollbar for the breadth to a certain position and all edges in this dimension are automatically set to the same length. In addition, it is already implemented in the software that the edges from all three dimensions always meet in a corner. Paul, on the other hand, has to make sure when building the cuboid that all four edges of one dimension have the same length and that the edges from all three dimensions always meet in one corner. In the statistical examples, the shortening of the actions through the automatic creation of relationships by the software becomes even clearer. For example, Li makes a drag movement over the plot and the software automatically creates relationships between the values on an axis and relationships between the position of the dots and the values on the axis. Nils, on the other hand, has to establish these relationships in individual actions. However, it also becomes clear in the analyses that the more relationships are automatically created by the software, the more relationships the learners have to interpret again afterwards because, without an interpretation of the new diagrams that are on the screen, the learners cannot continue performing mathematics. The learners establish these relationships between the signs afterwards through gestures and phonetic utterances. Thus, there is a shift in how the relationships are expressed. If relationships are not established automatically then they must be established in the actions of the learners for the learners to then be able to manipulate the diagram. If relationships are established automatically, then the relationships must be interpreted afterwards and clarified in gestures and spoken language.

Due to the shortening of the action caused by the automatic creation of relationships by the digital material, different diagram interpretations can be reconstructed based on the same actions. It becomes clear that Daniel and Li both make drag movements across the screen, however, they establish different relationships between the signs on the screen. For this reason, different diagram interpretations can be reconstructed. Depending on the context in which the same action is made, different interpretations of the learner can also be reconstructed from the same actions.

As a *third main result*, it was found that the actions decrease in the course of the processing, especially in the geometrical examples. This may be because the learners in the geometrical learning situations, in contrast to the statistical learning situations, were asked to write down their insights in a table on the prompt. Here, the analyses show that one reason for the shortening of the learners' actions is the noting of signs on the prompt. It could be reconstructed that the learners gain insights through their activities on the geometrical diagrams which they note down in a sign on the prompt. They then used these signs in the course of the processing in order not to have to repeat their activities on the geometrical diagram. Furthermore, it can be reconstructed that the learners can interpret the signs on the prompt later again and, in this way, shorten the action. A second reason for the shortening of the actions was found to be that the learners formulate calculations in phonetic language, for example, to calculate the number of unit squares on the face opposite. The formulation of the relationships in the spoken language, in turn, leads to a shortening of the actions on the diagram. Thus, it becomes clear that it is not only the material but also the learners themselves that can shorten the actions and lead to a shift from the production of the relationships via the actions to the production of the relationships in gestures and spoken language.

As a *fourth main finding*, it can be shown at one point in the analyses that in the geometrical learning situations, relationships are already implemented in the digital as well as in the analogue material before the action. In both materials, the learners do not have to pay attention to the angles between the sides and edges of the figures or solids. In the analysis of Marleen's activity, for example, it becomes clear that she still makes actions, but with these, she does not establish relationships because they are anchored in the material. For example, she puts the rods together with the corner connections or places the rods on the lines of the square grid to make a square, while Nils, in his activity, does not have to take any further action because a square automatically appears on the screen. It becomes clear that it cannot be reconstructed from the actions of either learner whether they recognise that the sides of a square enclose a right angle. Although Marleen, in contrast to Nils, takes further actions, these relationships can still not be reconstructed. It can be assumed that in actions in which no further relationships are taken into account than those already present in the material, no diagram interpretations can be reconstructed.

6.3.2 Correlation between the Results and the Theoretical Considerations

The four main results of the study are placed in correlation with the theoretical considerations at the beginning of the work to enable the research questions to be answered on a theoretical level.

In the *first main result*, it can be shown that the same diagram interpretations of the learners are reconstructed when the learners establish the same relationships between parts of the diagram despite using different actions. It can be assumed, as the literature describes (e.g. Dörfler, 2015), that the appearance of the signs plays a subordinate role because it is the usage of the signs that determines their meaning. The materiality of the signs belongs to the insignificant characteristics since significant characteristics, such as the relationships between the parts of the diagram, are not influenced by the materiality (Peirce, NEM IV). Therefore, it is primarily relevant that the relationships between the parts of the diagram are the same. It can, thus, be added to the literature that it is not only the appearance of the signs that has no influence on the reconstructed diagram interpretations but also the appearance of the actions when the same relationships are established between the signs. These results show that the interchangeability of the appearance of written or material signs described in the literature can also be applied to fleeting signs such as actions. This can be deduced because the appearance of the action, which in this work is understood to mean the haptic movement, has no influence on the diagrammatic interpretation of the learners and, thus, their possible mathematical insights if the same (mathematical) relationships are established in the actions. It can be stated that the appearance of any sign, fleeting or manifested in material, is subordinate to the relationships that already exist or are established between these signs.

The *second main result* of the work shows that the digital material shortens the actions of the learners. The digital material automatically establishes the relationships between the signs, thus reducing the relationships that the learners can establish through their actions. Therefore, the learners' diagram interpretations can barely be reconstructed from their actions as there are no, or only a few, relationships between the signs in the actions to be considered. In terms of semiotic theory, it becomes clear that the signs and the relationships that are visible between the signs on the screen are not indices of the actions that have taken place before because these visible relationships were not established in the actions. The action is, in turn, not an index of the learner's interpretation of the diagram, and since no relationships are established in the actions, it cannot be reconstructed whether the learners can interpret these relationships to be able to perform the necessary actions. Rather, the learners have to re-interpret the relationships established by the software acting as a tool. As the learners interpret the relationships afterwards through gestures and phonetic utterances, a shift can be observed in how the relationships are established by the learners. Regarding the available literature, Kadunz (2016) and Otte (2003) have already referred to the fact that digital software can shorten the actions and can lead to a separation of the relationships in the diagram and the learner's activity. The shortening of the actions and the separation between the actions and the relationships lead to the actions on the digital material becoming increasingly more general. Thus, different diagram interpretations can be reconstructed based on the same movement across the screen. If the action is general, then it can be used flexibly in different situations and acquires its meaning through its usage. As mentioned above, the appearance of the action does not influence the relationships that are established by the actions. If no special relationships are connected with the action, different relationships can be created with the action in usage. It follows that the diagram interpretations that can be reconstructed from the same actions can also be different, depending on how the action is used.

Going beyond these results, a closer look at the actions on the analogue and digital materials can provide a connection to diagrammatic reasoning in the Peircean sense; it becomes clear that the focus on the diagram is set differently when acting with the different materials. To work mathematically with the analogue material, the learner must establish relationships between the parts of the material arrangement in their actions. Thus, in a semiotic sense, the focus is on constructing and manipulating diagrams through rule-governed actions. If the actions and relationships are abbreviated by the digital material, the results of the manipulations, in other words, the relationships already established by the tool, are traced by the learners in gestures and spoken language. Therefore, the focus is rather on observing the results of the manipulations on the diagram. In the analyses, it becomes clear that the abbreviation of the actions can be different. If the actions are shortened so that many relationships are automatically established, then the focus is strongly on observing the results that become visible through the manipulations. If the actions are only minimally abbreviated and only a few relationships are automatically established (see the geometrical example where the learners are asked to make squares of different sizes), then the focus is still on manipulating the diagram through the actions of the learners.

The steps of diagrammatic reasoning also become clear in the *third main result* in which it was possible to show that the actions decrease in the course of processing. With the actions on the geometrical material arrangement, the learners construct and manipulate the geometrical diagram. Through their counting gestures and, for example, the addition of the unit squares on all side faces, they make further manipulations built on the manipulations they have already made to gain insights about the surface. They record the results of the manipulation or, in other words, the insights gained in signs and at the same time construct another diagram. In turn, they can use the resulting arithmetical diagram to establish relationships between the surfaces of different cuboids and to gain further insights. However, they can also use the insights noted in signs independently in their further editing processes without having to repeat manipulations on the geometrical diagram. If the learners do so, they shorten the manipulations on the diagram to focus more on the results of the manipulation. Therefore, the expression of the relationships in the actions shifts again to an expression of the relationships in the language. This result is similar to the second main result except that, this time, it is not the digital material that draws the focus to the results of the manipulation but the learners themselves. It becomes clear that the more the learners are adept in the usage of the diagrams, the more they focus on the final steps of diagrammatic reasoning. Furthermore, during the manipulation of the geometrical diagram, it becomes clear that the learners shorten the actions in the course of their processing. For example, to determine the number of unit squares on the opposite face, they still have to make initial manipulations on the digital or analogue material and then later they double the number so that they no longer have to continue making the manipulations. This means that no more actions are necessary, and the relationships can be observed more quickly in a short calculation. Here, too, the actions are shortened to focus more on the result of the manipulation, namely the determination of the number of unit squares. At various places in the analyses, it is clear that the learners first focus on constructing and manipulating the diagram, and, as they become more proficient in the usage of the diagram, they then focus more on the results of the manipulations.

The *fourth main result* shows that learners do not always establish relationships in their actions if, for example, they have already been implemented in the material. Since the learners do not establish any further relationships between the parts of the diagram because these are already present, no further diagram interpretations can be reconstructed based on the actions. Thus, the actions do not express whether the learners have recognised these relationships and, therefore, for the reconstruction of the diagram interpretations, the action cannot be seen as an index to the diagram interpretations. At this point, it becomes clear that the digital material and the analogue material shorten the creation of relationships in the same way since, in both materials, the relationships are already implemented. However, in the analogue material, actions still have to be taken which do not require the interpretation of relationships.

6.3.3 Correlation between the Results and Practical Considerations for Teaching Mathematics in Primary School

The results of this work should help teachers decide when to use digital and when to use analogue materials in their mathematics lessons and also what they should consider when planning lessons with different media materials. In this way, the four main results are placed in correlation with practical considerations for teaching mathematics in primary school to be able to answer the research questions at a practical level.

The *first main result* shows that the same diagram interpretations of the learners can be reconstructed from the actions on the digital and analogue material if the same relationships are observed in the actions. Based on this result, it becomes clear that it is not necessary to work with analogue material at the beginning of the mathematical learning process to be able to work with digital material later. In this way, it is possible to work with digital materials right from the start of the learning process if all the relationships in the actions are created by the learners themselves and the material does not create them automatically without the learners being able to understand how the relationships come about. It is not the haptic experiences that can be made with the material that decide which material is used first but rather the mathematical relationships that the learners establish with the material. For example, if the learning objective is to create a perpendicular, this can be achieved by the learner in the same way with both digital and analogue material. However, care must be taken to ensure that the same relationships are observed in the actions which can have different appearances. If a perpendicular is created in GeoGebra by connecting the intersections of two circles whose centres lie on a straight line and whose radius is greater than half the distance between the two centres by a straight line, the same relationships will be established in the actions as learners who work with a compass and ruler. From a mathematical point of view, the learners can make the same mathematical insights even if they take different actions. Mathematically, the choice of material does not influence the mathematical interpretations that the learners make. If, however, the learning objective is to learn how to use the compass, then the choice must be analogue material.

The *second main result* shows that if the manipulations on the diagram are shortened by the material to focus on the results of the manipulations, then the learners should already know the manipulations to be able to interpret the results of the manipulations correctly. In other words, learners should already be familiar with the usage of the diagram before the actions and the relationships established in the actions are abbreviated by a tool. In mathematics lessons, this should first be worked within materials where the focus is on the manipulations of the diagram and the usage of the diagram can be practised, the learners then can use a material that acts as a tool as they can then better classify the results of the manipulations made. By shortening the actions and relationships, they can then focus more on formulating the observations made in general terms and establish less specific relationships in the actions. Concerning the first main result, it becomes clear that it does not automatically mean that the learners should work with analogue material first, but that the learners should first establish all relationships in their actions. This can be achieved with both digital and analogue materials.

The *third main result* supports the practical considerations for primary school mathematics teaching previously formulated for result two. As the learners, in their mathematical processing, process and shorten their actions themselves and the relationships established in them to be able to focus more on the results of the manipulations, the choice of material suggested above is supportive for the learners. It even helps the learner to use first the materials that focus on creating relationships in the actions and then, if they are skilled in manipulating the diagram, to use materials that shorten the actions and relationships to focus on the results of the manipulations. In this way, the learners do not have to shorten their actions themselves to focus on the final steps of diagrammatic reasoning.

The *fourth main result* shows that for the mathematical learning process, actions in mathematics lessons need to be looked at more closely because not all actions are the same. For the mathematical learning process, the actions in which the learners establish relationships between the parts of the diagram are most relevant. Actions can, of course, also be experimental at the beginning of the learning process or mechanical after a certain time, for example, when the algorithm for written arithmetic is applied. In these actions, the relationships are not purposeful at the beginning or are no longer in focus after a certain time, but they are still observed by the learner. In actions in which the relationships are already anchored in the material, the relationships do not have to be explicitly observed by the learner; in this way, motor skills are focused on rather than the mathematical activity on the diagram.

6.4 Possible Follow-up Research Investigations

Based on the results of this work, which have been presented on three different levels, the subsequent follow-up research investigations are of interest to mathematics education research.

The results of this work are limited to the tasks and materials used in the *MatheMat* study. However, there are places in the literature that support the results of this work (e.g. Döfler, 2015; Kadunz, 2016; Moreno-Armella & Sriraman, 2010; Otte, 2003), and therefore, these results can most likely be extended to other mathematical areas and materials, while subsequent research could extend the study to other age groups of learners, other materials and other tasks from different mathematical areas. A possible focus of investigation concerning the material could be whether other digital tools shorten the actions in the same or different ways. It would also be interesting to investigate whether differences can be seen between materials where the learners receive numerous or only a few diagrammatic instructions. In the *MatheMat* study, many diagrammatic relationships have already been specified in the material and tasks for the learners to recognise, investigate or establish. It would be interesting to focus on whether the same results are seen when the learners are freer in their diagrammatic work. It could also be investigated whether younger or older children handle the materials in the same way; for example, younger learners who have not had much experience with mathematical diagrams might shorten their actions later in the editing process than learners who are older and have already practised using different mathematical diagrams.

Looking at the change in the expression of the relationships throughout the editing process and through the digital tool, further research would be interesting in examining the change from actions to gestures or from actions to phonetic utterances. Initial research in this area has already been carried out, for example, Vogel and Huth (2020) have identified different interfaces between the actions and gestures in their research. In addition, Billion & Huth (2023) look at how mathematical relationships are expressed in actions, gestures, and language among learners of kindergarten age and how the interplay of gestures and actions, in particular, can be described. Following on from this research, it would also be fruitful in the next step to include digitally designed mathematical learning situations in this research. A comparison between the analysed interplay of actions to gestures or actions to spoken language in digital and analogue learning situations would possibly allow for a further description of the shift between expressions of relationships beyond those in this work. In addition, it may be possible to identify differences and similarities in how the learners in some learning situations make the shift between actions and gestures, or spoken language themselves, and how this shift happens as a result of the material acting as a tool.

Based on the results of this work and the practical considerations for teaching mathematics in primary school that build on them, it would be fruitful to develop concrete concepts for teaching mathematics with digital and analogue materials. Similar to how Frischemeier (2018) suggests possible uses of digital and analogue materials in statistical learning and justifies the interplay of materials, this could be extended to other mathematical areas regarding the results of this work. In addition, the results could be used to design apps or software in which the learners have to create all the relationships in their actions themselves at the beginning and these actions and relationships are then abbreviated by the app or software in the course of processing the mathematical tasks. Through such an adaptive app or software, the steps of diagrammatic reasoning could be supported by digital material.

7 References

Aebli, H. (1980). *Denken, das Ordnen des Tuns. Band 1 Kognitive Aspekte der Handlungstheorie*. Klett.

Andrén, M. (2010). *Children's gestures from 18 to 30 month*. Centre for Languages and Literature, Lund University. https://lucris.lub.lu.se/ws/files/3902148/4588138.pdf

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of fragging practises in Cabri environments. *ZMD, 34*(3), 66–72. https://doi.org/10.1007/BF02655708

Bakker, A., & Hoffmann, M. (2005). Diagrammatic reasoning as the basis for developing concepts: A semiotic analysis of students' learning about statistical distribution. *Educational Studies in Mathematics*, 60, 333–358. https://doi.org/10.1007/s10649-005-5536-8

Billion, L., & Vogel, R. (2020). Grundschulkinder arbeiten digital an einem geometrischen Problem – Rekonstruktion mathematischer Deutungen. In S. Ladel, C. Schreiber, R. Rink & D. Walter (Eds.), *Aktuelle Forschungsprojekte zu digitalen Medien in der Primarstufe* (pp. 135–150). WTM-Verlag. https://doi.org/10.37626/GA9783959871747.0

Billion, L., & Vogel, R. (2021). Material as an impulse for mathematical activities in primary school – A semiotic perspective on a geometric example. *Mathematics education in the 4th industrial revolution: Thinking skills for the future.* Proceedings of the 44th Conference of the International Group for the Psychology of Mathematics Education (Vol. 2, pp. 81–88). https://www.igpme.org/wp-content/uploads/2022/04/Volume-2_final.pdf

Billion, L., & Huth, M. (2023). Mathematics in actions and gestures – a young learner's diagrammatic reasoning. *A Mathematics Education Perspective on early Mathematics Learning – POEM 2022*.

Dörfler, W. (2006). Inscriptions as objects of mathematical activities. In J. Maaz, & W. Schlögelmann (Eds.), *New Mathematics education research and practice* (pp. 97–111). Sense Publishers. https://doi.org/10.1163/9789087903510_011

Dörfler, W. (2015). Abstrakte Objekte in der Mathematik. In G. Kadunz (Ed.), *Semiotische Perspektiven auf das Lernen von Mathematik* (pp. 33–49). Springer-Verlag. https://doi.org/10.1007/978-3-642-55177-2

Dörfler, W. (2016). Signs and their use: Peirce and Wittgenstein. In A. Bikner-Ahsbahs, A. Vohns, R. Bruder, O. Schmitt, & W. Dörfler (Eds.), *Theories in and of mathematics education*, ICME-13 Topical Surveys (pp. 21–31). Springer. https://doi.org/10.1007/978-3-319-42589-4

Floer, J. (1993). Lernmaterialien als Stützen der Anschauung im arithmetischen Anfangsunterricht. In J.-H. Lorenz (Ed.), *Anschauung und Mathematik* (Band 18) (pp. 106–121). Aulis Verlag Deubner & Co KG.

Frischemeier, D. (2018). Design, implementation, and evaluation of an instructional Sequence to lead primary school students to comparing groups in statistical projects. In A. Leavy, M. Meletiou-Mavrotheris, & E. Paparistodemou (Eds.), *Statistics in early childhood and primary education. Supporting early statistical and probabilistic thinking* (pp. 217–238). Springer. https://doi.org/10.1007/978-981-13-1044-7_13

Gravemeijer, K. (1999). How Emergent Models may Foster the Constitution of Formal Mathematics. *Mathematical Teaching and Learning, 1*(2), 155–177. https://doi.org/10.1207/s15327833mtl0102_4

Gravemeijer, K. (2002a). Preamble: from models to modelling. In K. Gravemeijer, R. Lehrer, B. van Oers, & L. Verschaffel (Eds.), *Symbolizing, Modeling and Tool Use in Mathematics Education* (pp. 7–22). Kluwer Academic Publisher. https://doi.org/10.1007/978-94-017-3194-2

Gravemeijer, K. (2002b). Emergent Modeling as the Basis for an Instructional Sequence on Data Analysis. In International Conference on Teaching Statistics, B. Phillips, International Statistical Institute & International Association for Statistical Teaching (Eds.), *ICOTS 6: the Sixth International Conference on Teaching Statistics: "Developing a Statistically Literate Society",* Cape Town, South Africa. International Assoc. for Statistical Teaching.

Greiffenhagen, C. (2014). The materiality of mathematics: presenting mathematics at the blackboard. *The British Journal of Sociology*, *65*(3), 502–528. https://doi.org/10.1111/1468-4446.12037

Hoffmann, M. (2010). Diagrams as scaffolds for abductive insights. *Proceedings of the twenty-forth AAAI conference on artificial intelligence*, 42–49.

Hohenwarter, M. (2001). *GeoGebra – Dynamic Mathematics for Everyone.* Austria & USA.

Huber, H. (1972). *OrbiMath. Mathematik konstruktiv.* Freiburg im Breisgau: Herder Verlag.

Huth, M. (2022). Handmade diagrams – Learners doing math by using gestures. *12th Congress of the European Society for Research in Mathematics Education (CERME).* Bozen.

Kadunz, G. (2016). Geometry, a means of argumentation. In A. Sáenz-Ludlow, & G. Kadunz (Eds.), *Semiotics as a tool for learning mathematics. How to describe the construction, visualisation, and communication of mathematical concepts* (pp. 25–42). Sense Publishers. https://doi.org/10.1007/978-94-6300-337-7

Konold, C., & Miller, C. (2011). *TinkerPlots 2.0*. Key Curriculum Press.

Krauthausen, G. (2012). Digitale Medien im Mathematikunterricht der Grundschule. Spektrum. https://doi.org/10.1007/978-3-8274-2277-4

Larkin, K. (2013). Mathematics Education. Is there an App for that? In V. Steinle, L. Ball & C. Bardini (Eds.), *Mathematics Education: Yesterday, Today and Tomorrow.* Proceedings of the 36th annual Conference of the Mathematics Education Research Group of Australasia. file:///C:/Users/larab/Downloads/Larkin_MERGA36-2013-1.pdf

Larkin, K., Kortenkamp, U., Ladel, S., & Etzold, H. (2019). Using the ACAT framework to evaluate the design of two geometry apps: An exploratory study. *Digital Experiences in Mathematics Education, 5*(1), 59–92. https://doi.org/10.1007/s40751-018-0045-4

Latour, B. (2012). *Visualisation and Cognition: Drawing Things Together.* Retrieved from http://worrydream.com/refs/Latour%20-%20Visualisation%20and%20Cognition.pdf

Lorenz, J.-H. (1993). Veranschaulichungsmittel in arithmetischen Anfangsunterricht. In J.-H. Lorenz (Ed.), *Anschauung und Mathematik* (Band 18) (pp. 122–146). Aulis Verlag Deubner & Co KG.

Mayring, Ph. (2014). *Qualitative content analysis: theoretical foundation, basic procedures and software solutions.* Klagenfurt. https://nbn-resolving.org/urn:nbn:de:0168-ssoar-395173

Moreno-Armella, L., & Sriraman, B. (2010). Symbols and mediation in mathematics education. In B. Sriraman, & L. English (Eds.), *Theories of mathematics education* (pp. 213–232). Springer. https://doi.org/10.1007/978-3-642-00742-2_22

Otte, M. (2003). Does mathematics have objects? In what sense? *Synthese*, *134*(1/2), 181–216. https://www.jstor.org/stable/20117330

Peirce, Ch. S. (CP). *Collected Papers of Charles Sanders Peirce* (Volumes I-VI, ed. by C. Hartshorne, & P. Weiss, 1931-1935, Volumes VII-VIII, ed. by A. W. Burks, 1958). Harvard UP.

Peirce, C. S. (1976). *The new elements of mathematics* (NEM), vol. IV (Ed. C. Eisele). De Gruyter. https://doi.org/10.1515/9783110805888.313

Piaget, J. (1998). *Der Aufbau der Wirklichkeit beim Kinde.* 2nd Edition. Klett.

Radatz, H. (1993). MARC bearbeitet Aufgaben 72 – 59. Anmerkungen zu Anschauung und Verständnis im Arithmetikunterricht. In J.-H. Lorenz (Ed.), Anschauung und Mathematik (Band 18) (pp. 14–24). Aulis Verlag Deubner & Co KG.

Radford, L. (2019). On the epistemology of the theory of objectification. In U. T. Jankvist, M. V. D. Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (CERME11) (pp. 3062–3069). ERME.

Rink, R., & Walter, D. (2020). Digitale Medien im Matheunterricht – Ideen für die Grundschule. Cornelsen.

Rohr, S. (1993). *Über die Schönheit des Findens. Die Binnenstruktur menschlichen Verstehens nach Charles S. Peirce: Abduktionslogik und Kreativität.* Springer.

Schreiber, C. (2013). Semiotic processes in chat-based problem-solving situations. *Educational Studies Mathematics, 82*(1), 51–73. https://doi.org/10.1007/s10649-012-9417-7

Shapiro, S. (1997). *Philosophy of mathematics: Structure and ontology.* Oxford University Press

Sinclair, N., & de Freitas, E. (2014). The haptic nature of gesture: Rethinking gesture with new multi-touch digital technologies. *Gesture, 14*(3), 351–374. https://doi.org/10.1075/gest.14.3.04sin

Villamor, G., Willis, D., & Wroblewski, L. (2010). *Touch gesture reference guide*. Retrieved from https://www.lukew.com/ff/entry.asp?1071

Vogel, R. (2017). "wenn man da von oben guckt sieht das aus als ob…" – Die ‘Dimensionslücke‘ zwischen zweidimensionaler Darstellung dreidimensionaler Objekte im multimodalen Austausch. In M. Beck, & R. Vogel (Eds.), *Geometrische Aktivitäten und Gespräche von Kindern im Blick qualitativen Forschens Mehrperspektivische Ergebnisse aus den Projekten erStMaL und MaKreKi* (pp. 61–75). Waxmann-Verlag.

Vogel, R., & Huth, M. (2020). Modusschnittstellen in mathematischen Lernprozessen. Handlungen am Material und Gesten als diagrammatische Tätigkeit. In G. Kadunz (Ed.), *Zeichen und Sprache im Mathematikunterricht – Semiotik in Theorie und Praxis* (pp. 215–255). Springer Spektrum. https://doi.org/10.1007/978-3-662-61194-4

Walter, D. (2018). Nutzungsweisen bei der Verwendung von Tablet-Apps: Eine Untersuchung bei zählend rechnenden Lernenden zu Beginn des zweiten Schuljahres. Springer. https://doi.org/10.1007/978-3-658-19067-5

Wille, A. (2020). Mathematische Gebärden in Österreichischen Gebärdensprache aus semiotischer Sicht. In G. Kadunz (Ed.), *Zeichen und Sprache im Mathematikunterricht – Semiotik in Theorie und Praxis* (pp. 193–214). Springer Spektrum. https://doi.org/10.1007/978-3-662-61194-4_9

Wittmann, E. Ch. (1981). *Grundfragen des Mathematikunterrichts*. 6th Edition. Vieweg. https://doi.org/10.1007/978-3-322-91539-9

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