Irreducibility of a universal Prym-Brill-Noether locus

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Abstract

For genus $g = \frac{r(r+1)}{2} + 1$, we prove that via the forgetful map, the universal Prym-Brill-Noether locus $\mathcal{R}_g^r$ has a unique irreducible component dominating the moduli space $\mathcal{R}_g$ of Prym curves.

1 Introduction

The moduli space $\mathcal{R}_g$ of Prym curves was brought to the attention of algebraic geometers by Mumford in his influential paper [Mum74], as a way of understanding principally polarized Abelian varieties. For an element $[C, \eta]$ of $\mathcal{R}_g$ we let $\pi\colon \widetilde{C} \to C$ be the associated double cover and let $\mathrm{Nm}_\pi\colon \mathrm{Pic}^{2g-2}(\widetilde{C}) \to \mathrm{Pic}^{2g-2}(C)$ be the norm map of this morphism of curves. In this situation, the preimage of ω_C consists of two disjoint Abelian varieties, namely

$$P^+ = \left\{ L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) = \omega_C \text{ and } h^0(\widetilde{C}, L) \equiv 0 \pmod 2 \right\}$$

and

$$P^- = \left\{ L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) = \omega_C \text{ and } h^0(\widetilde{C}, L) \equiv 1 \pmod 2 \right\}.$$

The intersection of $W^0_{2g-2}(\widetilde{C})$ with P^+ is twice a theta divisor, and this allows us to associate to $[C, \eta]$ a principally polarized Abelian variety.

Following this development, Welters pointed out in [Wel85] that Prym-Brill-Noether theory can be employed in order to understand the geometry of subvarieties of Prym varieties. More precisely, he considered the loci

$$V^r(C, \eta) \coloneqq \left\{ L \in \mathrm{Pic}^{2g-2}(\widetilde{C}) \mid \mathrm{Nm}(L) \cong \omega_C,\ h^0(\widetilde{C}, L) \geq r+1, \text{ and } h^0(\widetilde{C}, L) \equiv r+1 \pmod 2 \right\}$$

in order to study the singularities of the theta divisor of the associated Prym variety. The relation between Prym-Brill-Noether theory and the study of singularities of theta divisors attracted other mathematicians to this topic. The two papers [Wel85] and [Ber87] showed that when $g \geq \frac{r(r+1)}{2} + 1$, the locus $V^r(C, \eta)$ is non-empty of dimension at least $g - 1 - \frac{r(r+1)}{2}$. Moreover, for a generic $[C, \eta] \in \mathcal{R}_g$, the locus $V^r(C, \eta)$ has exactly this dimension when $g \geq \frac{r(r+1)}{2} + 1$ and is empty when $g < \frac{r(r+1)}{2} + 1$, see [Sch17]. Subsequently in [DCP95], De Concini and Pragacz viewed $V^r(C, \eta)$ as a Lagrangian degeneracy locus (cf. [Mum71]) and computed the class of $V^r(C, \eta)$ in the Prym variety when it has the expected dimension $g - 1 - \frac{r(r+1)}{2}$.

In recent years, two new perspective for the study of Prym-Brill-Noether theory emerged. On one hand, it has been studied from the point of view of tropical geometry, see [CLRW20] and [LU21], thus providing another proof for the dimension estimate of $V^r(C, \eta)$ for a generic $[C, \eta]$ and, on the other hand, from the perspective of moduli theory, in order to understand the birational geometry of $\mathcal{R}_g$ for small values of g. It is natural to ask when $g \geq \frac{r(r+1)}{2} + 1$ whether the universal Prym-Brill-Noether locus

$$\mathcal{V}_g^r \coloneqq \{[C, \eta, L] \mid [C, \eta] \in \mathcal{R}_g \text{ and } L \in V^r(C, \eta)\}$$

has a unique irreducible component dominating the moduli space $\mathcal{R}_g$. This is true for $g > \frac{r(r+1)}{2} + 1$ because the fibre above a general $[C, \eta] \in \mathcal{R}_g$ is irreducible, see [Deb00, Exemples 6.2]. However, as pointed out in [JP21], this was not known for $g = \frac{r(r+1)}{2} + 1$. The goal of this paper is to show that when $g = \frac{r(r+1)}{2} + 1$, the moduli space $\mathcal{V}_g^r$ has a unique irreducible component dominating $\mathcal{R}_g$. To prove this result, we will consider the compactification $\overline{\mathcal{R}}_g$ of the moduli space of Prym curves $\mathcal{R}_g$, see [BCF04] and [FL10]. Finally, we degenerate to the boundary locus of $\overline{\mathcal{R}}_g$ and employ the theory of limit linear series, adapted to our situation.

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2 Prym linear series

Let $[C,\eta] \in \mathcal{R}_g$ be a generic Prym curve. Then, we know from [Wel85, Lemma 3.2] that a generic element $L \in V^r(C,\eta)$ satisfies $h^0(\widetilde{C}, L) = r+1$. Moreover, when $g = \frac{r(r+1)}{2} + 1$ we know from [Sch17, Theorem 1.1] that all $L \in V^r(C,\eta)$ satisfy $h^0(\widetilde{C}, L) = r+1$. In particular, the line bundle L can be viewed as a g^r_{2g-2} on the curve $\widetilde{C}$. Furthermore, up to restricting to an open subset, we can view all irreducible components of $\mathcal{V}^r_g$ dominating $\mathcal{R}_g$ as contained in the moduli space $\mathcal{G}^r_{2g-2}(\mathcal{R}_g)$ parametrizing limit g^r_{2g-2} over double covers $[\pi\colon \widetilde{C} \to C]$ where $\widetilde{C}$ is of compact type. We ask what points can appear in the compactification of $\mathcal{V}^r_g$ inside this space.

Let $[\pi\colon \widetilde{C} \to C] \in \overline{\mathcal{R}}_g$ such that C is of compact type and admits a unique irreducible component X satisfying $\eta_X \not\cong \mathcal{O}_X$. For this component X, we denote by $p^X_1, \dots, p^X_{s_X}$ its nodes and by $g^X_1, \dots, g^X_{s_X}$ the genera of the connected components of $C \setminus X$ glued to X at these points. For an irreducible component Y of C, different from X, we denote by q^Y the node glueing Y to the connected component of $C \setminus Y$ containing X, and by $p^Y_1, \dots, p^Y_{s_Y}$ the other nodes of Y. We denote by $g^Y_0, g^Y_1, \dots, g^Y_{s_Y}$ the genera of the connected components of $C \setminus Y$ glued to Y at these points. Using the above notations, we can define the concept of a Prym limit g^r_{2g-2}:

Definition 2.1. A Prym limit g^r_{2g-2}, denoted L, is a crude limit g^r_{2g-2} on $\widetilde{C}$ satisfying the following two conditions:

1. For the unique component $\widetilde{X}$ of $\widetilde{C}$ above X, the $\widetilde{X}$-aspect $L_{\widetilde{X}}$ of L satisfies

$$Nm_{\pi_{|\widetilde{X}}} L_{\widetilde{X}} \cong \omega_X\Big(\sum_{i=1}^{s} 2g^X_i p_i\Big)$$

2. For a component Y of C different from X, we denote by Y_1 and Y_2 the two irreducible components of $\widetilde{C}$ above it. We identify these two components with Y via the map π. With this identification the Y_1 and Y_2 aspects of L satisfy:

$$L_{Y_1} \otimes L_{Y_2} \cong \omega_Y\big((2g-2+2g^Y_0)q^Y + \sum_{i=1}^{s} g^Y_i p^Y_i\big)$$

Because the points in the boundary need to respect the norm condition, we immediately obtain that:

Lemma 2.2. *Let $[\pi\colon \widetilde{C} \to C] \in \overline{\mathcal{R}}_g$ with $\widetilde{C}$ of compact type and let $\overline{\mathcal{V}}^r_g$ the closure of $\mathcal{V}^r_g$ inside $\mathcal{G}^r_{2g-2}(\mathcal{R}_g)$. Then the fibre of the map $\overline{\mathcal{V}}^r_g \to \overline{\mathcal{R}}_g$ over the point $[\pi\colon \widetilde{C} \to C]$ is contained in the locus of Prym limit g^r_{2g-2} on $[\pi\colon \widetilde{C} \to C]$.*

We are now ready to use a degeneration argument in order to prove our main result.

Theorem 2.3. *When $g = \frac{r(r+1)}{2} + 1$, the space $\mathcal{V}^r_g$ has a unique irreducible component dominating $\mathcal{R}_g$.*

Proof. Let $[Y_1 \cup_{x_1} \widetilde{E} \cup_{x_2} Y_2 \to Y \cup_x E]$ be the double cover associated to a generic element of Δ_1. We want to describe the locus of Prym limit g^r_{2g-2}'s on such a double cover.

Let L be a Prym limit g^r_{2g-2} on $[Y_1 \cup_{x_1} \widetilde{E} \cup_{x_2} Y_2 \to Y \cup_x E]$. The additivity of the Brill-Noether numbers implies:

$$\rho(2g-1, r, 2g-2) = -r \geq \rho(L_{Y_1}, x_1) + \rho(\widetilde{E}, x_1, x_2) + \rho(L_{Y_2}, x_2)$$

But we know from [EH87, Theorem 1.1] and [Far00, Proposition 1.4.1] that $\rho(L_{Y_1}, x_1) \geq 0$, $\rho(L_{Y_2}, x_2) \geq 0$ and $\rho(\widetilde{E}, x_1, x_2) \geq -r$. It is clear that these are in fact equalities and L is a refined limit g^r_{2g-2}.

We denote by $0 \leq a_0 < a_1 < \cdots < a_r \leq 2g-2$ and $0 \leq b_0 < b_1 < \cdots < b_r \leq 2g-2$ the vanishing orders for the Y_1 and Y_2 aspects respectively. The equality $\rho(\widetilde{E}, x_1, x_2) = -r$ implies that $a_i + b_{r-i} = 2g-2$ for all $0 \leq i \leq r$.

The genericity of $[Y_2, x_2] \in \mathcal{M}_{g-1,1}$ together with $\rho(L_{Y_2}, x_2) = 0$ imply that $h^0(Y_2, L_{Y_2}(-b_i x_2)) = r+1-i$ for all $0 \leq i \leq r$. Using that $L_{Y_1} \otimes L_{Y_2} \cong \omega_Y(2g \cdot x)$ and the Riemann-Roch theorem we obtain

$$h^0\big(Y_1, L_{Y_1}(-(2+a_{r-i})q)\big) = g + r - 1 - a_{r-i} - i$$

Choosing $i = 0$ we get $a_r = g + r - 1$. Inverting the roles of the a_i's and b_i's we obtain that $a_0 = g - r - 1$. Because we have the divisorial equivalences

$$a_i x_1 + b_{r-i} x_2 \equiv a_j x_1 + b_{r-j} x_2$$

on the elliptic curve E for every $0 \leq i, j \leq r$, we obtain that $a_i - a_{i-1} \geq 2$ for every $1 \leq i \leq r$. This implies that $a_i = g - r + 2i - 1$ for every $0 \leq i \leq r$.

We now view the moduli space $\mathcal{M}_{g-1,1}$ as embedded in $\overline{\mathcal{R}}_g$ via the map $\pi\colon \mathcal{M}_{g-1,1} \to \overline{\mathcal{R}}_g$ sending a pointed curve $[Y, x] \in \mathcal{M}_{g-1,1}$ to $[Y \cup_x E, \mathcal{O}_Y, \eta_E]$ where $[E, x]$ is a generic elliptic curve and η_E is a 2-torsion line bundle on E. For the ramification sequence $\alpha = (g-r-1, g-r, \ldots, g-1)$ associated to the vanishing orders $a = (a_0, \ldots, a_r) = (g-r-1, \ldots, g+r-1)$, we consider the locus $\mathcal{G}^r_{2g-2}(\alpha)$ parametrizing pairs $[C, p, L]$ where $[C, p] \in \mathcal{M}_{g-1,1}$ and L is a g^r_{2g-2} having vanishing orders greater or equal to a at the point p. Then the locus of Prym limit g^r_{2g-2} over $\mathrm{Im}(\pi)$ is birationally isomorphic to $\mathcal{G}^r_{2g-2}(\alpha)$.

We know from [EH89, Lemma 3.6] that $\mathcal{G}^r_{2g-2}(\alpha)$ has a unique irreducible component dominating $\mathcal{M}_{g-1,1}$. Moreover

$$\deg\big(\mathcal{G}^r_{2g-2}(\alpha) \to \mathcal{M}_{g-1,1}\big) = 2^{\frac{r(r-1)}{2}} \cdot (g-1)! \cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2i-1)!}$$

as stated on the second page of [FT16]. On the other hand we have from [DCP95, Theorem 9] that

$$\deg\big(\mathcal{V}^r_g \to \mathcal{R}_g\big) = 2^{\frac{r(r-1)}{2}} \cdot (g-1)! \cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2i-1)!}$$

.

We conclude that all dominant irreducible components of $\mathcal{V}^r_g$ contain $\mathcal{G}^r_{2g-2}(\alpha)$ in their closure. From this we get that each such component map to $\mathcal{R}_g$ with degree at least $2^{\frac{r(r-1)}{2}} \cdot (g-1)! \cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2i-1)!}$, implying unicity. $\square$

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