# Irreducibility of a universal Prym-Brill-Noether locus 

Andrei Bud


#### Abstract

For genus $g=\frac{r(r+1)}{2}+1$, we prove that via the forgetful map, the universal Prym-Brill-Noether locus $\mathcal{V}_{g}^{r}$ has a unique irreducible component dominating the moduli space $\mathcal{R}_{g}$ of Prym curves.


## 1 Introduction

The moduli space $\mathcal{R}_{g}$ of Prym curves was brought to the attention of algebraic geometers by Mumford in his influential paper Mum74, as a way of understanding principally polarized Abelian varieties. For an element $[C, \eta]$ of $\mathcal{R}_{g}$ we let $\pi: \widetilde{C} \rightarrow C$ be the associated double cover and let $\mathrm{Nm}_{\pi}: \operatorname{Pic}^{2 g-2}(\widetilde{C}) \rightarrow \operatorname{Pic}^{2 g-2}(C)$ be the norm map of this morphism of curves. In this situation, the preimage of $\omega_{C}$ consists of two disjoint varieties

$$
P^{+}=\left\{L \in \operatorname{Pic}^{2 g-2}(\widetilde{C}) \mid \operatorname{Nm}(L)=\omega_{C} \text { and } h^{0}(\widetilde{C}, L) \equiv 0(\bmod 2)\right\}
$$

and

$$
P^{-}=\left\{L \in \operatorname{Pic}^{2 g-2}(\widetilde{C}) \mid \operatorname{Nm}(L)=\omega_{C} \text { and } h^{0}(\widetilde{C}, L) \equiv 1(\bmod 2)\right\}
$$

isomorphic to the $\operatorname{Prym}$ variety in $\operatorname{Pic}^{0}(\widetilde{C})$.
Following this development, Welters emphasized in Wel85 that Prym-Brill-Noether theory can be employed in order to understand the geometry of subvarieties of Prym varieties. More precisely, he considered the loci

$$
V^{r}(C, \eta):=\left\{L \in \operatorname{Pic}^{2 g-2}(\widetilde{C}) \mid \operatorname{Nm}(L) \cong \omega_{C}, h^{0}(\widetilde{C}, L) \geq r+1, \text { and } h^{0}(\widetilde{C}, L) \equiv r+1(\bmod 2)\right\}
$$

in order to study the singularities of the theta divisor of the associated Prym variety. The relation between Prym-Brill-Noether theory and the study of singularities of theta divisors piqued the interest of other mathematicians. The two papers Wel85 and Ber87 showed that when $g \geq \frac{r(r+1)}{2}+1$, the locus $V^{r}(C, \eta)$ is non-empty of dimension at least $g-1-\frac{r(r+1)}{2}$. In addition, for a generic $[C, \eta] \in \mathcal{R}_{g}$, the locus $V^{r}(C, \eta)$ has exactly this dimension when $g \geq \frac{r(r+1)}{2}+1$ and is empty when $g<\frac{r(r+1)}{2}+1$, see Sch17. Subsequently in DCP95, De Concini and Pragacz viewed $V^{r}(C, \eta)$ as a Lagrangian degeneracy locus (cf. Mum71) and computed the class of $V^{r}(C, \eta)$ in the Prym variety when it has the expected dimension $g-1-\frac{r(r+1)}{2}$.

In recent years, two new perspectives for the study of Prym-Brill-Noether theory emerged. On one hand, it has been studied from the point of view of tropical geometry, see [CLRW20] and [LU21, thus providing another proof for the dimension estimate of $V^{r}(C, \eta)$ for a generic $[C, \eta]$ and, on the other hand, from the perspective of moduli theory, with a view to understanding the birational geometry of $\mathcal{R}_{g}$ for small values of $g$. It is natural to ask when $g \geq \frac{r(r+1)}{2}+1$ whether the universal Prym-Brill-Noether locus

$$
\mathcal{V}_{g}^{r}:=\left\{[C, \eta, L] \mid[C, \eta] \in \mathcal{R}_{g} \text { and } L \in V^{r}(C, \eta)\right\}
$$

has a unique irreducible component dominating the moduli space $\mathcal{R}_{g}$. This is true for $g>\frac{r(r+1)}{2}+1$ because the fibre above a general $[C, \eta] \in \mathcal{R}_{g}$ is irreducible, see Deb00, Exemples 6.2]. However, as pointed out in JP21, this was not known for $g=\frac{r(r+1)}{2}+1$. The present paper aims at showing that when $g=\frac{r(r+1)}{2}+1$, the moduli space $\mathcal{V}_{g}^{r}$ has a unique irreducible component dominating $\mathcal{R}_{g}$. In the interest of proving this result, we will consider the compactification $\overline{\mathcal{R}}_{g}$ of the moduli space of Prym curves $\mathcal{R}_{g}$, see BCF04 and FL10. Ultimately, we degenerate to the boundary locus of $\overline{\mathcal{R}}_{g}$ and employ the theory of limit linear series, adapted to our situation.

Acknowledgements: I would like to thank my advisor Gavril Farkas for suggesting this problem and for all his support and guidance along the way. I also want to thank the anonymous referee for the suggested improvements to this paper.

## 2 Prym linear series

Our goal in this section is to provide a suitable definition of Prym linear series and then use it to prove our main result. We will start by recalling some definitions regarding limit linear series, while referring the reader to EH86] for a throughout study.

First, recall that a $g_{d}^{r}$ on a smooth curve $Y$ is defined to be a pair $(V, L)$ of a degree $d$ line bundle $L$ together with an $(r+1)$-dimensional vector subspace $V \subseteq H^{0}(Y, L)$. This definition extends naturally to curves of compact type.

Let $Y$ be a genus $g$ curve of compact type. A crude limit $g_{d}^{r}$ consists of a $g_{d}^{r}\left(V_{i}, L_{i}\right)$ for every irreducible component $Y_{i}$ of $Y$, further satisfying the following property:

- Let $q$ be a node of $Y$ connecting two irreducible components $Y_{j}$ and $Y_{k}$ and let $0 \leq a_{0}<\cdots<a_{r} \leq d$ and $0 \leq b_{0}<\cdots<b_{r} \leq d$ the vanishing orders at $q$ of the sections in $V_{j}$ and $V_{k}$ respectively. Then for any $0 \leq i \leq r$ we have

$$
a_{i}+b_{r-i} \geq d
$$

If for any node $q$ all the inequalities are in fact equalities, the limit linear system is called refined.
Lastly, we define the Brill-Noether number associated to a smooth pointed curve and a $g_{d}^{r}$ on it. Let $\left(Y, p_{1}, \ldots, p_{n}\right)$ be a smooth genus $g$ curve together with $n$ points on it, and $l=(V, L)$ be a $g_{d}^{r}$ on $Y$. Let $0 \leq a_{0}^{i}<\cdots<a_{r}^{i} \leq d$ the vanishing orders at $p_{i}$ of the sections in $V$. The Brill-Noether number of the $g_{d}^{r}$ with respect to the points $p_{i}$ is defined as

$$
\rho\left(l, p_{1}, \ldots, p_{n}\right):=g-(r+1)(g-d+r)-\sum_{i=1}^{n} \sum_{j=0}^{r} a_{j}^{i}+n \cdot \frac{r(r+1)}{2} .
$$

Having these definitions, we are ready to particularize to our situation.
Let $[C, \eta] \in \mathcal{R}_{g}$ be a generic Prym curve. Then, we know from Wel85, Lemma 3.2] that a generic element $L \in V^{r}(C, \eta)$ satisfies $h^{0}(\widetilde{C}, L)=r+1$. Moreover, when $g=\frac{r(r+1)}{2}+1$ we know from [Sch17. Theorem 1.1] that all $L \in V^{r}(C, \eta)$ satisfy $h^{0}(\widetilde{C}, L)=r+1$. In particular, the line bundle $L$ can be viewed as a $g_{2 g-2}^{r}$ on the curve $\widetilde{C}$. Furthermore, up to restricting to an open subset, we can view all irreducible components of $\mathcal{V}_{g}^{r}$ dominating $\mathcal{R}_{g}$ as contained in the moduli space $\mathcal{G}_{2 g-2}^{r}\left(\mathcal{R}_{g}\right)$ parametrizing limit $g_{2 g-2}^{r}$ over double covers $[\pi: \widetilde{C} \rightarrow C]$ where $\widetilde{C}$ is of compact type. We ask what points can appear in the compactification of $\mathcal{V}_{g}^{r}$ inside this space.

Let $[\pi: \widetilde{C} \rightarrow C] \in \overline{\mathcal{R}}_{g}$ such that $C$ is of compact type and admits a unique irreducible component $X$ satisfying $\eta_{X} \not \neq \mathcal{O}_{X}$. For this component $X$, we denote by $p_{1}^{X}, \ldots, p_{s_{X}}^{X}$ its nodes and by $g_{1}^{X}, \ldots, g_{s_{X}}^{X}$ the genera of the connected components of $C \backslash X$ glued to $X$ at these points. For an irreducible component $Y$ of $C$, different from $X$, we denote by $q^{Y}$ the node glueing $Y$ to the connected component of $C \backslash Y$ containing $X$, and by $p_{1}^{Y}, \ldots, p_{s_{Y}}^{Y}$ the other nodes of $Y$. We denote by $g_{0}^{Y}, g_{1}^{Y}, \ldots, g_{s_{Y}}^{Y}$ the genera of the connected components of $C \backslash Y$ glued to $Y$ at these points.

Using the above notations, we can define the concept of a Prym limit $g_{2 g-2}^{r}$ :
Definition 2.1. A Prym limit $g_{2 g-2}^{r}$, denoted $L$, is a crude limit $g_{2 g-2}^{r}$ on $\widetilde{C}$ satisfying the following two conditions:

1. For the unique component $\widetilde{X}$ of $\widetilde{C}$ above $X$, the $\widetilde{X}$-aspect $L_{\tilde{X}}$ of $L$ satisfies

$$
N m_{\pi_{\mid \widetilde{X}}} L_{\tilde{X}} \cong \omega_{X}\left(\sum_{i=1}^{s} 2 g_{i}^{X} p_{i}^{X}\right)
$$

2. For a component $Y$ of $C$ different from $X$, we denote by $Y_{1}$ and $Y_{2}$ the two irreducible components of $\widetilde{C}$ above it. We identify these two components with $Y$ via the map $\pi$. With this identification the $Y_{1}$ and $Y_{2}$ aspects of $L$ satisfy:

$$
L_{Y_{1}} \otimes L_{Y_{2}} \cong \omega_{Y}\left(\left(2 g-2+2 g_{0}^{Y}\right) q^{Y}+\sum_{i=1}^{s} g_{i}^{Y} p_{i}^{Y}\right)
$$

Because the points in the boundary need to respect the norm condition, we immediately obtain that:
Lemma 2.2. Let $[\pi: \widetilde{C} \rightarrow C] \in \overline{\mathcal{R}}_{g}$ with $\widetilde{C}$ of compact type and let $\overline{\mathcal{V}}_{g}^{r}$ the closure of $\mathcal{V}_{g}^{r}$ inside $\mathcal{G}_{2 g-2}^{r}\left(\mathcal{R}_{g}\right)$. Then the fibre of the map $\overline{\mathcal{V}}_{g}^{r} \rightarrow \overline{\mathcal{R}}_{g}$ over the point $[\pi: \widetilde{C} \rightarrow C]$ is contained in the locus of Prym limit $g_{2 g-2}^{r}$ on $[\pi: \widetilde{C} \rightarrow C]$.

We are now ready to use a degeneration argument in order to prove our main result.
Theorem 2.3. When $g=\frac{r(r+1)}{2}+1$, the space $\mathcal{V}_{g}^{r}$ has a unique irreducible component dominating $\mathcal{R}_{g}$.
Proof. We consider the boundary divisor $\Delta_{1} \subseteq \overline{\mathcal{R}}_{g}$ whose generic point is of the form $\left[Y \cup_{x} E, \mathcal{O}_{Y}, \eta_{E} \neq \mathcal{O}_{E}\right]$, where $Y$ and $E$ are components of genus $g-1$ and 1 respectively. Let $\left[Y_{1} \cup_{x_{1}} \widetilde{E} \cup_{x_{2}} Y_{2} \rightarrow Y \cup_{x} E\right]$ be the double cover associated to a generic element of $\Delta_{1}$. We want to describe the locus of Prym limit $g_{2 g-2}^{r}$ 's on such a double cover.

Let $L$ be a Prym limit $g_{2 g-2}^{r}$ on $\left[Y_{1} \cup_{x_{1}} \widetilde{E} \cup_{x_{2}} Y_{2} \rightarrow Y \cup_{x} E\right]$. The additivity of the Brill-Noether numbers implies:

$$
\rho(2 g-1, r, 2 g-2)=-r \geq \rho\left(L_{Y_{1}}, x_{1}\right)+\rho\left(L_{\widetilde{E}}, x_{1}, x_{2}\right)+\rho\left(L_{Y_{2}}, x_{2}\right)
$$

But we know from [EH87, Theorem 1.1] and Far00, Proposition 1.4.1] that $\rho\left(L_{Y_{1}}, x_{1}\right) \geq 0, \rho\left(L_{Y_{2}}, x_{2}\right) \geq 0$ and $\rho\left(L_{\tilde{E}}, x_{1}, x_{2}\right) \geq-r$. It is clear that these are in fact equalities and $L$ is a refined limit $g_{2 g-2}^{r}$.

We denote by $0 \leq a_{0}<a_{1}<\cdots<a_{r} \leq 2 g-2$ and $0 \leq b_{0}<b_{1}<\cdots<b_{r} \leq 2 g-2$ the vanishing orders for the $Y_{1}$ and $Y_{2}$ aspects respectively. The equality $\rho\left(L_{\widetilde{E}}, x_{1}, x_{2}\right)=-r$ implies that $a_{i}+b_{r-i}=2 g-2$ for all $0 \leq i \leq r$.

The genericity of $\left[Y_{2}, x_{2}\right] \in \mathcal{M}_{g-1,1}$ together with $\rho\left(L_{Y_{2}}, x_{2}\right)=0$ imply that $h^{0}\left(Y_{2}, L_{Y_{2}}\left(-b_{i} x_{2}\right)\right)=r+1-i$ for all $0 \leq i \leq r$. Using that $L_{Y_{1}} \otimes L_{Y_{2}} \cong \omega_{Y}(2 g \cdot x)$ and the Riemann-Roch theorem we obtain

$$
h^{0}\left(Y_{1}, L_{Y_{1}}\left(-\left(2+a_{r-i}\right) q\right)\right)=g+r-1-a_{r-i}-i
$$

Choosing $i=0$ we get $a_{r}=g+r-1$. Inverting the roles of the $a_{i}$ 's and $b_{i}$ 's we obtain that $a_{0}=g-r-1$. Because we have the divisorial equivalences

$$
a_{i} x_{1}+b_{r-i} x_{2} \equiv a_{j} x_{1}+b_{r-j} x_{2}
$$

on the elliptic curve $E$ for every $0 \leq i, j \leq r$, we obtain that $a_{i}-a_{i-1} \geq 2$ for every $1 \leq i \leq r$. This implies that $a_{i}=g-r+2 i-1$ for every $0 \leq i \leq r$.

We now view the moduli space $\mathcal{M}_{g-1,1}$ as embedded in $\overline{\mathcal{R}}_{g}$ via the map $\pi: \mathcal{M}_{g-1,1} \rightarrow \overline{\mathcal{R}}_{g}$ sending a pointed curve $[Y, x] \in \mathcal{M}_{g-1,1}$ to $\left[Y \cup_{x} E, \mathcal{O}_{Y}, \eta_{E}\right]$ where $[E, x]$ is a generic elliptic curve and $\eta_{E}$ is a 2 -torsion line bundle on $E$. For the ramification sequence $\alpha=(g-r-1, \ldots, g-1)$ associated to the vanishing orders $a=\left(a_{0}, \ldots, a_{r}\right)=(g-r-1, \ldots, g+r-1)$, we consider the locus $\mathcal{G}_{2 g-2}^{r}(\alpha)$ parametrizing pairs [ $\left.C, p, L\right]$ where $[C, p] \in \mathcal{M}_{g-1,1}$ and $L$ is a $g_{2 g-2}^{r}$ having vanishing orders greater or equal to $a$ at the point $p$. Then the locus of $\operatorname{Prym}$ limit $g_{2 g-2}^{r}$ over $\operatorname{Im}(\pi)$ is birationally isomorphic to $\mathcal{G}_{2 g-2}^{r}(\alpha)$.

We know from [EH89, Lemma 3.6] that $\mathcal{G}_{2 g-2}^{r}(\alpha)$ has a unique irreducible component dominating $\mathcal{M}_{g-1,1}$. Moreover

$$
\operatorname{deg}\left(\mathcal{G}_{2 g-2}^{r}(\alpha) \rightarrow \mathcal{M}_{g-1,1}\right)=2^{\frac{r(r-1)}{2}} \cdot(g-1)!\cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2 i-1)!}
$$

as stated on the second page of [FT16]. On the other hand we have from [DCP95, Theorem 9] that

$$
\operatorname{deg}\left(\mathcal{V}_{g}^{r} \rightarrow \mathcal{R}_{g}\right)=2^{\frac{r(r-1)}{2}} \cdot(g-1)!\cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2 i-1)!}
$$

We conclude that all dominant irreducible components of $\mathcal{V}_{g}^{r}$ contain $\mathcal{G}_{2 g-2}^{r}(\alpha)$ in their closure. From this we get that each such component map to $\mathcal{R}_{g}$ with degree at least $2^{\frac{r(r-1)}{2}} \cdot(g-1)$ ! $\cdot \prod_{i=1}^{r} \frac{(i-1)!}{(2 i-1)!}$, implying unicity.

## References

[BCF04] E. Ballico, C. Casagrande, and C. Fontanari. Moduli of Prym curves. Documenta Mathematica, 9:265-281, 2004.
[Ber87] A. Bertram. An existence theorem for Prym special divisors. Inventiones mathematicae, 90:669671, 1987.
[CLRW20] S. Creech, Y. Len, C. Ritter, and D. Wu. Prym-Brill-Noether Loci of Special Curves. International Mathematics Research Notices, 2022:2688-2728, 2020.
[DCP95] C. De Concini and P. Pragacz. On the class of Brill-Noether loci for Prym varieties. Mathematische Annalen, 302:687-698, 1995.
[Deb00] O. Debarre. Théorèmes de Lefschetz pour les lieux de dégénérescence. Bulletin de la Société Mathématique de France, 128:283-308, 2000.
[EH86] D. Eisenbud and J. Harris. Limit linear series: Basic theory. Inventiones mathematicae, 85:337372, 1986.
[EH87] D. Eisenbud and J. Harris. The Kodaira dimension of the moduli space of curves of genus $\geq 23$. Inventiones mathematicae, 90:359-387, 1987.
[EH89] D. Eisenbud and J. Harris. Irreducibility of some families of linear series with Brill-Noether number -1. Annales scientifiques de l'École Normale Supérieure, 22:33-53, 1989.
[Far00] G. Farkas. The birational geometry of the moduli space of curves. Academisch Proefschrift, Universitet van Amsterdam, 2000.
[FL10] G. Farkas and K. Ludwig. The Kodaira dimension of the moduli space of Prym varieties. Journal of the European Mathematical Society, 12:755-795, 2010.
[FT16] G. Farkas and N. Tarasca. Pointed Castelnuovo numbers. Mathematical Research Letters, 23:389404, 2016.
[JP21] D. Jensen and S. Payne. Recent Developments in Brill-Noether Theory. Preprint, arXiv:2111.00351, 2021.
[LU21] Y. Len and M. Ulirsch. Skeletons of Prym varieties and Brill-Noether theory. Algebra \& Number Theory, 15:785-820, 2021.
[Mum71] D. Mumford. Theta characteristics of an algebraic curve. Annales Scientifiques de l'École Normale Supérieure, 4:181-192, 1971.
[Mum74] D. Mumford. Prym varieties I. Contributions to analysis, pages 325-350, 1974.
[Sch17] Irene Schwarz. Brill-Noether theory for cyclic covers. Journal of Pure and Applied Algebra, 221:2420-2430, 2017.
[Wel85] G. Welters. A theorem of Gieseker-Petri type for Prym varieties. Annales Scientifiques de l'École Normale Supérieure, 18:671-683, 1985.

