Tensor-Valued Valuations and Curvature Measures in Euclidean Spaces

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Abstract

Given an Abelian semi-group $(A,+)$, an A-valued curvature measure is a valuation with values in A-valued measures. If $A=\mathbb{R}$, complete classifications of Hausdorff-continuous translation-invariant $SO(n)$-invariant valuations and curvature measures were obtained by Hadwiger and Schneider, respectively. More recently, characterisation results have been achieved for curvature measures with values in $A=\operatorname{Sym}^p\mathbb{R}^n$ and $A=\operatorname{Sym}^2\Lambda^q\mathbb{R}^n$ for $p,q\geq 1$ with varying assumptions as for their invariance properties.

In the present work, we classify all smooth translation-invariant $SO(n)$-covariant curvature measures with values in any $SO(n)$-representation in terms of certain differential forms on the sphere bundle $S\mathbb{R}^n$ and describe their behaviour under the globalisation map. The latter result also yields a similar classification of all continuous $SO(n)$-module-valued $SO(n)$-covariant valuations. Furthermore, a decomposition of the space of smooth translation-invariant scalar-valued curvature measures as an $SO(n)$-module is obtained. As a corollary, we construct explicit bases of continuous translation-invariant scalar-valued valuations and smooth translation-invariant scalar-valued curvature measures.

Zusammenfassung

Falls $(A,+)$ eine Abelsche Halbgruppe ist, so ist ein A-wertiges Krümmungsmaß eine Bewertung mit Werten in A-wertigen Maßen. Für $A=\mathbb{R}$ wurden Hausdorff-stetige translationsinvariante $SO(n)$-invariante Bewertungen und Krümmungsmaße vollständig von Hadwiger und Schneider klassifiziert. Es gibt ähnliche Klassifizierungsresultate für Bewertungen und Krümmungsmaße mit Werten in $A=\operatorname{Sym}^p\mathbb{R}^n$ und $A=\operatorname{Sym}^2\Lambda^q\mathbb{R}^n$, wobei $p,q\geq 1$, und verschiedenen Invarianzeigenschaften bzgl. der Translationen und der Wirkung von $SO(n)$.

In der vorliegenden Arbeit klassifizieren wir alle glatten translationsinvarianten $SO(n)$-kovarianten Krümmungsmaße mit Werten in einer beliebigen Darstellung von $SO(n)$, indem wir jedem solchen Krümmungsmaß eine Äquivalenzklasse von bestimmten Differentialformen auf dem Sphärenbündel $S\mathbb{R}^n$ zuordnen. Außerdem beschreiben wir deren Verhalten unter der Globalisierungsabbildung, was uns erlaubt, eine ähnliche Klassifizierung von allen stetigen $SO(n)$-Modul-wertigen $SO(n)$-kovarianten Bewertungen zu erlangen. Darüber hinaus geben wir eine Zerlegung des Raumes von glatten translationsinvarianten skalarwertigen Krümmungsmaßen als ein $SO(n)$-Modul. Als ein Korollar konstruieren wir explizite Basen von stetigen traslationsinvarianten skalarwertigen Bewertungen und glatten translationsinvarianten skalarwertigen Krümmungsmaßen.

Contents

1 Introduction

Valuation theory is both old and new. It is old because a special sort of valuation, the so-called Dehn-invariant, was used by Max Dehn as early as in 1901 to solve one of Hilbert's problems [34]. It is new because the Irreducibility Theorem discovered by Alesker exactly one hundred years later [6] triggered a huge surge of research activity related to valuation theory that has not worn out even by present days.

In the following, we will concentrate on characterisation results for different classes of valuations. However, this should not evoke a false sense of valuations being just an insipid collection of some functionals. Far from it! Valuations have a very rich algebraic structure, including the convolution product [24], the Alesker-product [14], and the Alesker-Fourier-transform connecting the two products [12]. Furthermore, valuations satisfy several functorial properties [11] and are inherently connected to the heart of integral geometry, the *kinematic formulae* [37]. The reader is referred to [21] or [39] for a concise overview of key research areas in valuation theory and (algebraic) integral geometry.

1.1 Background

A valuation ϕ on a Euclidean space V with $\dim V = n$ is a functional on the set of convex bodies – i.e. convex compact sets – $\mathcal{K}(V)$ assuming values in an Abelian semi-group $(A, +)$ and satisfying the equation:

$$\phi(K) + \phi(L) = \phi(K \cup L) + \phi(K \cap L),$$

whenever $K \cup L$ is a convex body. Examples of valuations include the Euler-characteristic, the Lebesgue-measure or mixed volumes. The latter are examples of multivaluations, i.e. functionals that are valuations in each argument.

It is not possible to systematically study valuations without making some assumptions with regard to their regularity and invariance properties. Historically, the first valuations to become objects of systematic study were $\mathbb{R}$-valued valuations that additionally satisfy the following conditions:

- *continuity* with respect to the Hausdorff-topology on $\mathcal{K}(V)$;
- *invariance* under Euclidean motions, i.e. under the group action of $\overline{SO(n)} := SO(n) \ltimes \mathbb{R}^n$.

These valuations appear very naturally in the so-called Steiner formula for the volumes of tubes around a convex body. Blaschke established in [29] a set of integral relations called *kinematic formulae* between different such valuations. However, it had taken a rather long time until Hadwiger classified them. It turned out that they form an $(n+1)$-dimensional vector space $\mathrm{Val}^{SO(n)}(V)$ – also denoted simply by $\mathrm{Val}^{SO(n)}$ – spanned by the so-called *intrinsic volumes* μ_k, $k = 0, \ldots, n$, where μ_0 is a multiple of the Euler-characteristic and μ_n is a multiple of the Lebesgue-measure [45, Sec. 6.1.10] (see [54] for a more modern exposition of this topic).

At the same time, Weyl found a Steiner-like formula for r-tubes around compact submanifolds rather than convex bodies [92]. Trying to generalise the two, Federer extended the notion of valuations to the broader class of *sets of positive reach* [35] which comprises both convex sets and C^2-submanifolds in $\mathbb{R}^n$ [83].

Furthermore, Federer formulated his results for a family of functionals called *curvature measures* which, in a certain sense, generalise valuations (we will enlarge on this relationship later in this Section). The name "curvature measures" is not arbitrary. For a convex body

K with smooth boundary and $k < n$, the k-th intrinsic volume μ_k is given by integrating over ∂K the $(n-k-1)$-th elementary symmetric function of its principal curvatures. Hence, μ_k yields information about the total curvature of the body's boundary. Replacing ∂K by a Borel-subset $U \cap \partial K \subset \partial K$, where $U \in \mathcal{B}(V)$ is a Borel-subset in V, yields a functional $\Phi_k : \mathcal{K}(V) \times \mathcal{B}(V) \to \mathbb{R}$, such that $\Phi_k(\cdot, U)$ is a valuation for a fixed Borel-set U and $\Phi_k(K, \cdot)$ is a non-negative Borel-measure for a fixed convex body K. This functional yields the partially integrated curvature for any Borel-subset of the body's boundary and is called the *k-th Lipschitz-Killing curvature measures*. Note that Φ_k is *locally defined*: If $U \subset V$ is open and $K_1, K_2 \in \mathcal{K}(V)$ such that $K_1 \cap U = K_2 \cap U$, then $\Phi_k(K_1, U') = \Phi_k(K_2, U')$ for all open sets $U' \subset U$. Generalising this, any locally defined valuation with values in A-valued measures is often referred to as an A-valued curvature measure.

Federer's results are primarily based on one important observation that $\Phi_k(K_i, U)$ converges weakly as a measure, i.e.

$$\int_V f(x)\, d\Phi_k(K_j, x) \to \int_V f(x)\, d\Phi_k(K, x)$$

for any continuous function $f : V \to \mathbb{R}$ and any sequence of sets of positive reach K_i converging to a set of positive reach K. Naturally, this property is called the *weak continuity* of curvature measures [75, pp. 208ff.].

It had taken another twenty years – and an additional restriction back to the convex bodies – until Schneider showed in [74] that the space of weakly continuous $\overline{SO(n)}$-invariant curvature measures $\mathrm{Curv}^{SO(n)}$ is spanned precisely by k-th Lipschitz-Killing curvature measure Φ_k:

$$\mathrm{Curv}^{SO(n)} = \bigoplus_{k=0}^{n} \mathrm{span}(\Phi_k).$$

A similar decomposition on the sets of positive reach was made possible by Zähle [94] a decade later. She found for Φ_k, $k < n$, the following integral representation:

$$\Phi_k(K, U) = \int_{\mathrm{nc}(K) \cap \pi^{-1}(U)} \omega_k,$$

where $\mathrm{nc}(K) \subset SV$ is an $(n-1)$-dimensional Lipschitz-submanifold of SV called the *normal cycle* of K, $\pi : SV \to V$ is the natural projection, and ω_k is a certain $\overline{SO(n)}$-invariant differential form of bi-degree $(k, n-1-k)$ on the sphere bundle SV. We will call such valuations and curvature measures *smooth*.

If one relaxes $\overline{SO(n)}$-invariance to a mere translation-invariance, then the classical fact – shown by McMullen in [65] – states that the space of continuous translation-invariant scalar-valued valuations admits a decomposition by homogeneity degree and parity:

$$\mathrm{Val} = \bigoplus_{\substack{k=0 \\ \varepsilon \in \{-,+\}}}^{n} \mathrm{Val}_k^{\varepsilon},$$

where $\mathrm{Val}_k^{\varepsilon}$ are *infinite-dimensional* (Fréchet-)spaces unless $k \in \{0, n\}$, in which case Val_k is one-dimensional and spanned by the Euler-characteristic and the Lebesgue-measure, respectively.

These classical results suggest several directions, in which the study of valuations and curvature measures may be – and has been – expanded:

1. There have been numerous attempts to extend Federer's theory to even larger classes of sets. One obvious step was the natural generalisation to (locally finite) unions of sets of positive reach [95, 71]. Fu used Zähle's novel results to introduce valuations on sub-analytic sets [38]. However, he later admitted that this work "found only limited success" mainly for two reasons:

   - Not all sets of positive reach are sub-analytic sets;
   - One has to make several technical assumptions on the normal cycles of sub-analytic sets for the kinematic formulae to hold.

   In 2015, Fu, Pokorny, and Rataj naturally extended the theory of valuations and curvature measures to the class of the so-called WDC-sets which comprises all sets of positive reach [40].

   At the same time, Zähle's differential-form-based view of curvature measures led naturally to the theory of valuations on – sufficiently tame – subsets of abstract differentiable manifolds rather than of Euclidean spaces. It was developed mainly by Alesker and Bernig [8, 9, 14, 13]. In particular, Bernig and Bröcker described smooth valuations on manifolds by means of certain equivalence classes of differential forms whose structure is very well understood [22].

   Furthermore, the operations $\cup$ and $\cap$ on the set of convex bodies resemble the join $\vee$ and meet $\wedge$ operations on lattices [60]. Hence, one is tempted to study valuations on other lattices, such as various classes of functions with $\wedge = \min$ and $\vee = \max$. For example, there exists a rather well-developed theory of valuations on L^p- and Sobolev functions [84, 85, 63].

2. Motivated by the fact that $\overline{SO(n)}$-invariant valuations are smooth, one may ask for the relation between smooth and continuous valuations if the $\overline{SO(n)}$-invariance is relaxed or completely dropped. It turns out that smooth translation-invariant valuations comprise a dense subspace Val^{sm} of Val. Furthermore, the space Val^G of $\overline{G}$-invariant continuous valuations is finite-dimensional if and only if $G \subset SO(n)$ acts transitively on the unit sphere. In this case, any $\overline{G}$-invariant continuous valuation is also smooth.

   Such groups G have been classified in [30] and [68]: If we additionally assume the action of G to be effective, then G is either in one of the 6 regular series

$$SO(n), U(n), SU(n), Sp(n), Sp(n) \cdot U(1), Sp(n) \cdot Sp(1)$$

   or one of the 3 exceptional groups

$$G_2, Spin(7), Spin(9).$$

   Initiated by Alesker, the program of describing such smooth $\overline{G}$-invariant valuations resulted in a range of Hadwiger-type results for many of the above Lie-groups G [7, 10, 25, 20, 27]. The study was later expanded to space forms [28, 2].

   On the contrary, if G is not compact, then there are, in general, no smooth $\overline{G}$-invariant valuations with the exception of μ_0 and μ_n. In fact, even continuous valuations are scarce under the assumption of $\overline{G}$-invariance. For example, if $G = SO^+(p,q)$ is the connected component of the indefinite orthogonal group $O(p,q)$ and $p, q \geq 1$, then there are no continuous $\overline{G}$-invariant valuations of homogeneity degree $1 \leq k \leq n-2$ [23]. It is then even more astonishing that one obtains a quite rich theory of valuations by only slightly relaxing the continuity assumption to the so-called *Klain-continuity*.

For $G = SL(n)$, there are no $\overline{G}$-invariant continuous valuations of homogeneity degree $1 \leq k \leq n-1$ and only three linearly independent *upper semi-continuous* valuations, namely the affine surface area, the Euler-characteristic, and the volume [61, 62].

There also exists a theory of continuous $O(n)$-invariant – but not necessarily translation-invariant – valuations which originates from the study of geometric asymptotics on normed spaces [67], especially of the following $O(n)$-invariant functional:

$$\phi(K) = \int_K |x|^2 \, dx,$$

which is a valuation of K. Alesker characterised in [4] valuations that are $O(n)$-invariant and exhibit a *polynomial* behaviour with respect to translations. We will not explain the polynomial behaviour and refer instead to the original definition in [87] (In Russian. English translation in [86]). To our knowledge, there is no theory of *smooth* isometry-invariant valuations.

3. Several characterisation results were obtained for valuations with values in various semi-groups A with the property of G-invariance being usually replaced with the more natural notions of either *G-covariance* or *G-contravariance.*

   One of the oldest such generalisations is given by the so-called *Minkowski-valuations*, i.e. valuations with values in $A = (\mathcal{K}(V), +)$, where $+$ is the Minkowski-addition. Assuming $GL(n)$-contravariance, Petty found as early as in 1965 a continuous translation-invariant Minkowski-valuation called the *projection body* [70]. It was only 40 years later that the projection body was shown to be – up to a constant – the only such Minkowski-valuation [59]. Likewise, the well-known *difference body* turns out to be the only translation-invariant continuous Minkowski-valuation that satisfies the requirement of *$GL(n)$-covariance* [59]. A similar result exists for $GL(n, \mathbb{C})$-covariant Minkowski-valuations in complex spaces [1]. Dropping the $GL(n)$-covariance and assuming that a functional satisfy a certain Brunn-Minkowski-type or Rogers-Shephard-type inequality, Abardia, Colesanti and Gómez have obtained several interesting characterisation results [3].

   Both $GL(n)$-covariant and $GL(n)$-contravariant continuous Minkowski-valuations were characterised even without translation-invariance in [79] and [88], but their description is somewhat more involved. On the other hand, dropping co- and contravariance properties, one may ask whether there is still a McMullen-type decomposition for continuous translation-invariant Minkowski-valuations. The answer turns out to be negative, as was shown by Wannerer and Parapatits [69].

   Another area of active research with long tradition is $\mathrm{Sym}^p V$-valued valuations with different invariance assumptions. These comprise much more information about convex bodies than their scalar-valued counterparts and have several applications in the morphology and anisotropy analysis of cellular, granular, or porous structures [18, 77, 78, 76, 47] and in bio-imaging [57, 56], where they are often referred to as *Minkowski-tensors.*

   The case $p = 1$ of *vector-valued* valuations with isometry-covariance, i.e. $O(n)$-covariance and polynomial behaviour, was studied in 1971-1972 by Hadwiger and Schneider [46, 72, 73]. Generalising Schneider's results, McMullen introduced several families of isometry-covariant Sym^p-valued valuations, showed certain relations between them, and conjectured that they form a generating set of all $\mathrm{Sym}^p V$-valued isometry-covariant valuations [66]. McMullen sent these conjectures to Alesker after the latter had distributed the preprint of his paper [4]. Using many methods from his original work, Alesker was then able to confirm these conjectures in [5]. Later, Hug and Schneider showed that the relations on $\mathrm{Sym}^p V$-valued valuations that had been found by McMullen are essentially the only ones [51]. One year later, they developed the kinematic formulae for Sym^p-valued isometry-covariant valuations [52]. These formulae, although in a closed form, were somewhat cumbersome, as their computation involved taking fivefold sums. For the special case of translation-invariant $SO(n)$-covariant $\mathrm{Sym}^p V$-valued valuations, the formulae were much closer studied and considerably simplified in [26]. As Daniel Hug told us in a private conversation, a simplified version of kinematic formulae for the general case is being developed and will probably be published in 2016.

It is worth noting that there is a strong connection between $\overline{G}$-invariant scalar-valued and G-covariant G-module-valued valuations. As Val is a G-module itself, the space of G-covariant valuations with values in a G-module A are written as G-invariant elements of $\mathrm{Val} \otimes A$ or, equivalently, as G-invariant homomorphisms between Val and A:

$$\mathrm{TVal}_{k,A}^G \simeq (\mathrm{Val}_k \otimes A)^G \simeq \mathrm{Hom}_G(\mathrm{Val}_k^*, A). \tag{1.1}$$

By Schur's Lemma, such elements may be directly described if the structure of Val as G-module is known. Conversely, knowing all G-module-valued valuations for all G-modules directly yields the structure of the subspace of Val as a G-module. Following this argument, Alesker, Bernig, and Schuster were able to significantly refine McMullen's decomposition and classify all irreducible $SO(n)$-modules, in which non-trivial translation-invariant smooth (and, hence, continuous) valuations may assume values [15]. However, they did not explicitly describe the differential forms that represent these valuations.

4. Resuming Federer's and Schneider's tradition, one may study curvature measures rather than valuations. There are many reasons to do so:

   – Curvature measures reveal strictly more information about the underlying sets than valuations do. For example, the 0-th intrinsic volume is constantly equal to 1 on convex bodies whereas its localised version - the 0-th Lipschitz-Killing curvature measure – depends on $K \in \mathcal{K}(V)$ and $U \in \mathcal{B}(V)$ in a non-constant manner. Furthermore, seeing the r-tube as a set of normal vectors of length r at ∂K, curvature measures yield a Steiner-type formula for a tube containing normal vectors with *base points* from a Borel-subset $U \subset \partial K$. The original global Steiner-formula case is then easily obtained by setting $U = \partial K$.

   – By the same token, – at least under the smoothness assumption – every valuation is induced by some curvature measure Φ by means of the globalisation map $\mathrm{glob} : \mathrm{Curv}^{sm} \to \mathrm{Val}^{sm}$ which sends Φ to the valuation $\phi(\cdot) := \Phi(\cdot, V)$. However, the kernel of glob is not trivial, i.e. there may be linearly independent curvature measures globalising to the same valuation. Hence, there is, in general, no well-defined notion of *localisation* of a valuation. The only exception – at least among smooth valuations – is given by $\overline{SO(n)}$-invariant valuations, as they have canonical localisations given by the k-th Lipschitz-Killing curvature measures.

   – The space Curv is endowed in a natural way with the structure of a module over Val. The module structure is compatible with the structure of Val as an algebra in the sense that $\mathrm{glob}(\Phi \cdot \mu) = \mathrm{glob}(\Phi) \cdot \mu$ for all $\Phi \in \mathrm{Curv}$ and $\mu \in \mathrm{Val}$.

     Furthermore, the conditions on the Lie-groups G for Curv^G to be finite dimensional are the same as for valuations. Then a version of Fu's fundamental theorem [37] for curvature measures yields a comodule-structure $\mathrm{Curv}^G \to \mathrm{Curv}^G \otimes \mathrm{Val}^G$ called the *semi-local* kinematic operator. This co-product encodes the so-called semi-local kinematic formulae between a valuation and a curvature measures and it should not be surprising that it implies the classical global kinematic formulae which involve two valuations.

- There is also a co-product $K_G : \mathrm{Curv}^G \to \mathrm{Curv}^G \otimes \mathrm{Curv}^G$ which makes Curv^G into a co-algebra with some very remarkable properties. First, it encodes a very general version of kinematic formulae called the *local* kinematic formulae between two invariant curvature measures, from which both the global and the semi-local kinematic formulae may be recovered. Second, given two homogeneous spaces $M_i = G_i/G$, $i = 1, 2$, such that G_i are unimodular and G acts transitively on the sphere bundles SM_i, the co-algebras $\mathcal{C}^G(M_1)$ and $\mathcal{C}^G(M_2)$ of G-invariant curvature measures are canonically isomorphic. This property is called *Howard's transfer principle* after its discoverer [48], as it permits to "transfer" local kinematic formulae between different homogeneous spaces. Its special case was used in [28] to explicitly compute the global kinematic formulae for $U(n)$-invariant valuations on complex space forms. In contrast, there is no transfer principle for valuations.

Despite these advantages, there are several technical difficulties which complicate the study of curvature measures.

- Contrary to Val, the space Curv is not irreducible as an $SL(n)$-module and it is not known whether smooth curvature measures form a dense subset of weakly continuous curvature measures. In fact, to our knowledge, the notion of weak continuity has only been used in conjunction with $\overline{O(n)}$- or $\overline{SO(n)}$-covariant curvature measures [74, 49, 50].
- There is only a version of McMullen's theorem for smooth – but not weakly continuous – curvature measures.

As a result of these drawbacks, curvature measures are not so accessible as valuations. To our knowledge, there is no theory of localised Minkowski-valuations, i.e. $\mathcal{K}(V)$-valued curvature measures. In fact, even scalar-valued curvature measures are far from being completely understood. Apart from the classical $SO(n)$-invariant case, only $\overline{U(n)}$-invariant smooth curvature measures were studied in [28], where a Hadwiger-type result as well as local kinematic formulae were obtained. Furthermore, the author characterised $\overline{SU(n)}$-invariant curvature measures in an unpublished work.

On the other hand, the theory of curvature measures on (semi-convex) functions (which are called Hessian measures in this context) is virtually two decades old [31, 32]. Furthermore, $SO(n)$- and $O(n)$-covariant curvature measures with values in various vector spaces are understood rather well.

The local Minkowski-tensors – which in our parlance should be called $O(n)$-covariant Sym^p-valued curvature measures – were studied by Hug and Schneider in [49], where they first explicitly described all such curvature measures on the set of polytopes and then characterised all those with continuous extensions to $\mathcal{K}(V)$. On top of that, our present work has revealed that, unlike the (global) Minkowski-tensors, there are $SO(n)$-covariant local Minkowski-tensors that are not $O(n)$-invariant. This has entailed new efforts to classify such tensors on convex polytopes and to study their extensions to convex bodies [50].

Furthermore, a study of $\mathrm{Sym}^2\Lambda^q V$-valued curvature measures in the more general context of abstract manifolds was conducted by Bernig in [19]. He found a general family of such curvature measures which, when restricted to Euclidean spaces, yields a family of $SO(n)$-covariant translation-invariant curvature measures with some desirable additional properties. However, a full characterisation of $\mathrm{Sym}^2\Lambda^q V$-valued curvature measures was not achieved.

It is possible to localise Steiner's formula for the r-tube of normal vectors in a slightly different way by parametrising the *directions* in which they point – rather than their base points. They may be expressed as locally defined valuations with values in signed measures on the sphere $S(V)$, i.e. as functionals $\mathcal{K}(V) \times \mathcal{B}(S(V)) \to A$. Similarly to the curvature measures, the study of area measures had been initiated by classical convex geometers [16, 36] and has experienced a small Renaissance after the discovery of the normal cycle and the Irreducibility Theorem. For example, the structure and the integral-geometric properties of $U(n)$-invariant area measures were extensively described by Wannerer [90], [89]. Furthermore, a strong connection between the integral geometry of area measures and that of valuations has been established very recently by Solanes [82].

1.2 Main Results

In this work, we present three main results about the space of *smooth* translation-invariant curvature measures, which may be seen as a generalisation of [15], [19], and [51]:

1. We obtain a decomposition of translation-invariant *smooth* curvature measures Curv^{sm} as an $SO(n)$-module.

   Recall that finite-dimensional representations of linear Lie-group acting on V may be described up to an isomorphism by certain tuples of n numbers $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n)$. Each *integer-valued* tuple may best be visualised as a so-called *Young-diagram* which is just a left-aligned collection of boxes with λ_i boxes in i-th row. We denote by Γ_λ the representation that corresponds to the tuple λ. Then the following holds.

   Theorem 1.1. *Barring the exceptions which we omit here for the sake of clarity, the subspace* $\mathrm{Curv}^{sm}_k \subset \mathrm{Curv}^{sm}$ *of smooth translation-invariant k-homogeneous curvature measures is a direct sum:*

$$\mathrm{Curv}^{sm}_k \simeq \bigoplus_{\substack{p\in\mathbb{N} \\ 1\leq q\leq k'}} 2\,\Gamma^{q,p}_0 \oplus \bigoplus_{\substack{p\in\mathbb{N} \\ q=0,k'+1}} \Gamma^{q,p}_0 \oplus \bigoplus_{\substack{p\in\mathbb{N} \\ 1\leq q\leq k'}} \Gamma^{q,p}_1, \tag{1.2}$$

   where $k' := \min(k, n-k-1)$ and $\Gamma^{q,p}_0$ and $\Gamma^{q,p}_1$ are the irreducible $SO(n)$-modules of type $[q;p;0]$ resp. $[q;p;1]$, i.e. represented by the respective Young-diagrams

1	$q+1$	$2q+1$	$\ldots$	$2q+p$
2	$q+2$			
$\vdots$	$\vdots$			
q	$2q$			

   and

1	$q+1$	$2q+1$	$\ldots$	$2q+p$
2	$q+2$			
$\vdots$	$\vdots$			
q	$2q$			

   .

   Equivalently, the space of $\Gamma^{q,p}_0$-valued curvature measures is basically 2-dimensional while the space of $\Gamma^{q,p}_1$-valued curvature measures is 1-dimensional. These curvature measures are called regular, since they exist for any vector space.

   There are two exceptions, of which we mention here only one: If $\dim V = n = 2j+1$, $j \in \mathbb{N}$, then there exists another linearly independent family of curvature measures $\Theta^n_{[p]}$ of homogeneity degree j with values in $\Gamma_0^{j,p}$ for $p \in \mathbb{N}$. These curvature measures are $SO(n)$-covariant, but may additionally change sign if acted upon by $O(n)$. In particular, for $n = 3$, $\Theta^n_{[p]}$ is a family of Sym^p-valued curvature measures on $\mathbb{R}^3$. As previously mentioned, this was the starting point of the new work of Hug and Schneider [50].

This is in contrast to the Alesker-Bernig-Schuster decomposition of Val_k which is – again, ignoring some exceptions – a *multiplicity-free* direct sum:

$$\mathrm{Val}_k \simeq \bigoplus_{\substack{p \in \mathbb{N} \\ 0 \leq q \leq k''}} \Gamma_0^{q,p}, \tag{1.3}$$

where $k'' := \min(k, m - k)$.

The precise structure of Curv_k^{sm} as an $SO(n)$-module is elaborated in **Theorem 4.4**.

2. After having computed the dimension of the spaces $\mathrm{TCurv}_{k,\Gamma}^{SO(n)}$ of Γ-valued curvature measures, we now explicitly construct several different bases of them, each of them satisfying some distinguished properties.

   Several important remarks on our notation are in order before we proceed to the core results. Each basis essentially consists of three regular four-parameter families Φ, Ξ, Ψ of regular curvature measures as well as one exceptional two-parameter family Θ. The four parameters for the regular curvature measures are:

   - n stands for the dimension of the vector space a given curvature measure is defined in;
   - k indicates the curvature measure's homogeneity degree;
   - p and q parametrise its domain. The families Φ and Ψ take values in $SO(n)$-modules $\Gamma_0^{q,p}$ – or vector spaces related to it – whereas Ξ describes $\Gamma_1^{q,p}$-valued curvature measures.

   All regular families in all bases are defined for $n \in \mathbb{N}$, $0 \leq k \leq n - 1$. However, the allowed ranges for parameters p and q vary across the families: For Φ and Ξ, we require $p \geq 0$ and $0 \leq q \leq \min(k, n - k - 1) =: k'$ whereas Ψ is subject to the following constraints $p \geq 0$ and $1 \leq q \leq k' + 1$.

   The exceptional family Θ exists only for $n = 2j + 1$ and has always $k = q = j$. Thus, it depends only on parameters n and $p \geq 0$. To keep the notation from going overboard and to underline how families are related across the different bases, we use the same notation style $\Phi^n_{k,p,q}$, $\Xi^n_{k,p,q}$, $\Psi^n_{k,p,q}$, Θ^n_p for the four families and add special marks to distinguish the basis they belong to. Furthermore, for the sake of brevity, whenever we refer to all four families at once, we write $T^n_{k,p,q}$ even though Θ only depends on two – rather than four – parameters.

   The first basis is denoted by $T^n_{(k,p,q)}$ with the special mark being the parentheses around the subscript indices. Curvature measures from this basis assume values in restrictions of irreducible $SL(n)$-modules to $SO(n)$. Their advantage is the canonical way, in which they are constructed from scratch. All other bases are expressed in terms of it.

Theorem 1.2. *Let e_i, $i = 1, \dots, n$ be the standard orthonormal basis of $V \simeq \mathbb{R}^n$, dx^i, dy^i be the coordinates on the cotangential bundle $T^*\mathbb{R}^n$ and write $e_{i_1,\dots,i_q y} := e_{i_1} \wedge \dots \wedge e_{i_q} \wedge y$. Then the three regular families are defined pointwise at the point $(x, y) \in S\mathbb{R}^n$ as follows:*

$$\Phi^n_{(k,p,q)} = \sum_{\substack{i_1,\dots,i_q=1\\ \pi\in S_n}}^{n} \operatorname{sgn}\pi\ y_{\pi(1)}\, dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes e_{\pi(2)\dots\pi(q+1)} \cdot e_{i_1\dots i_q} \otimes y^p,$$

$$\Psi^n_{(k,p,q)} = \sum_{\substack{i_1,\dots,i_{q-1}=1\\ \pi\in S_n}}^{n} \operatorname{sgn}\pi\ y_{\pi(1)}\, dx^{i_1\dots i_{q-1}\pi(q+1)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes e_{\pi(2)\dots\pi(q)y} \cdot e_{i_1\dots i_{q-1}y} \otimes y^p,$$

$$\Xi^n_{(k,p,q)} = \sum_{\substack{i_1,\dots,i_q=1\\ \pi\in S_n}}^{n} \operatorname{sgn}\pi\ y_{\pi(1)}\, dx^{i_1\dots i_q\pi(q+1)\dots\pi(k+1)} \wedge dx^{\pi(k+2)\dots\pi(n)} \otimes e_{\pi(2)\dots\pi(q+1)y} \otimes e_{i_1\dots i_q} \otimes y^p,$$

$$\Theta^n_{(p)} = \sum_{\substack{i_1,\dots,i_q=1\\ j_1,\dots j_q=1}}^{n} dx^{i_1\dots i_q} \wedge dy^{j_1\dots j_q} \otimes e_{i_1\dots i_q} \cdot e_{j_1\dots j_q} \otimes y^p,$$

where the parameters k, p, q are subject to restrictions mentioned above and the term $\otimes y^p$ is replaced by $\cdot y^p$ for $q = 1$.

The families Φ, Ψ, and Θ assume values in the $SL(n)$-modules of type $[q; p; 0]$ while Ξ assumes values in $SL(n)$-modules of type $[q; p; 1]$. The following illustrations may be useful to distinguish the three regular families $\Phi_{(\cdot)}$, $\Psi_{(\cdot)}$, and $\Xi_{(\cdot)}$ by the $SL(n)$-modules Γ_λ they assume values in.

$$\Phi_{(k,p,q)} \sim \begin{array}{|c|c|c|c|c|}\hline \pi_2 & i_1 & y & \dots & y \\ \hline \pi_3 & i_2 \\ \cline{1-2} \vdots & \vdots \\ \cline{1-2} \pi_q & i_{q-1} \\ \cline{1-2} \pi_{q+1} & i_q \\ \cline{1-2} \end{array}, \quad \Psi_{(k,p,q)} \sim \begin{array}{|c|c|c|c|c|}\hline \pi_2 & i_1 & y & \dots & y \\ \hline \pi_3 & i_2 \\ \cline{1-2} \vdots & \vdots \\ \cline{1-2} \pi_q & i_{q-1} \\ \cline{1-2} y & y \\ \cline{1-2} \end{array}, \quad \Xi_{(k,p,q)} \sim \begin{array}{|c|c|c|c|c|}\hline \pi_2 & i_1 & y & \dots & y \\ \hline \pi_3 & i_2 \\ \cline{1-2} \vdots & \vdots \\ \cline{1-2} \pi_q & i_{q-1} \\ \cline{1-2} \pi_{q+1} & i_q \\ \cline{1-2} y \\ \cline{1-1} \end{array},$$

where we have replaced e_{i_j} with its subscript i_j and likewise $e_{\pi(j)}$ with just π_j.

The second basis – denoted by $T^n_{[k,p,q]}$ with the brackets instead of the parentheses – assumes values in *irreducible* $SO(n)$-modules and is particularly well adapted to the Alesker-Bernig-Schuster-type decomposition of Curv^{sm} described in the previous result.

The two bases are described more rigorously in **Theorem** 4.17 and **Corollary** 4.19. Constructing them is the main innovation of this work but their relationships are hard to study. Only the special case of $p = 0$ is explicitly computed in **Proposition** 4.20.

Note that any curvature measure $T^n_{(k,p,q)}$ or $T^n_{[k,p,q]}$, when restricted to $\mathbb{R}^{n-1}$, may be written as a sum of $T^{n-1}_{(k',p',q')}$ or $T^{n-1}_{[k',p',q']}$, respectively. Generally, the resulting sum depends on all four parameters k, n, p, q. However, we will obtain in **Theorem** 4.22 another basis whose restriction behaviour only depends on the parameter p just by rescaling $T^n_{(k,p,q)}$. Honouring

Peter McMullen who has discovered this behaviour for Sym^p-valued valuations (see Theorem 3.48), we call this basis the *McMullen-basis* and denote it by $T^n_{k,p,q}$.

As a matter of fact, we may obtain a basis – denoted by $\tilde{T}^n_{k,p,q}$ with a tilde above the letter – with even better restriction behaviour, namely such that the restriction of $\tilde{T}^n_{k,p,q}$ to $\mathbb{R}^{n-1}$ is $\tilde{T}^{n-1}_{k,p,q}$. We call curvature measures with this behaviour *horizontal* and the basis $\tilde{T}^n_{k,p,q}$ – the *horizontal basis.* The construction of this basis is carried out in **Theorem** 4.27.

3. Lastly, we compute in Theorem 4.3 the non-trivial curvature measures with trivial globalisations. Obviously the family $\Xi^n_{[k,p,q]}$ – or, for that matter, $\Xi^n_{(k,p,q)}$ – always yields trivial valuations, as Val does not contain the $SO(n)$-modules $\Gamma_1^{q,p}$. Recall that the multiplicity of $\Gamma_0^{q,p}$ is mostly 2 in Curv_k while Val_k is a multiplicity-free $SO(n)$-module. Hence, there must be, for most p and q, a unique up to scaling linear combination of $\Phi^n_{[k,p,q]}$ and $\Psi^n_{[k,p,q]}$ that globalises to a trivial $\Gamma_0^{q,p}$-valued valuation.

Theorem 1.3. *For any k,p,q, such that $\dim \mathrm{TCurv}^{SO(n)}_{k,\Gamma_0^{q,p}} \geq 2$, one has:*

$$q(n-k+p)\ \mathrm{glob}\,\Psi_{[k,p,q]} + (k-q+1)(c_{q-1}p+1)\ \mathrm{glob}\,\Phi_{[k,p,q]} = 0,$$

where $c_m = 1$ if m is even and $1/2$ if m is odd.

These results are re-formulated in **Theorem** 4.29 and **Corollary** 4.30.

There are several useful consequences of the three main results. In particular, the coefficients of the curvature measures $T^n_{[k,p,q]}$ turn out to be essentially a basis of Curv^{sm}. Lacking the density argument, we can not carry over this result to all weakly-continuous translation-invariant curvature measures. This problem will be tackled in the future research.

On the other hand, we will show that any tensor-valued $SO(n)$-covariant translation-invariant valuation is smooth. Hence, the space $\mathrm{TVal}^{SO(n)}$ of such valuations is spanned by the globalisations of $T^n_{[k,p,q]}$. We know all relations between the globalisations. Thus, we may explicitly describe a subset of $T^n_{[k,p,q]}$ which forms a basis of $\mathrm{TVal}^{SO(n)}$ and – by the same argument as for Curv^{sm} – a basis for Val^{sm}. As Val^{sm} is known to lie dense in Val, we thus obtain a Schauder-basis of Val.

1.3 The Proofs and the Plan

Re-phrasing Hermann Weyl, no mathematician is pleased by the prospect of going through a proof by virtue of a complicated chain of formal conclusions and computations, of finding his way in the dark by groping about from one chain link to another [93]. To facilitate the reader's journey, we now want to ignite several torches along the way by providing the outlines of proofs of the main results.

1. The proof of Theorem 4.4 is a simplified version of the proof of (1.3) in [15] and may be broken down into several steps.

   (a) The identification (1.1) holds for smooth curvature measures as well. Setting $A = \Gamma_{[\lambda]}$ to be an arbitrary irreducible $SO(n)$-module, one obtains (omitting the vector space V in the definition for the sake of clarity):

$$\mathrm{TCurv}^{sm,SO(n)}_{k,\Gamma_{[\lambda]}} \simeq \mathrm{Hom}_{SO(n)}(\mathrm{Curv}^{sm}_k, \Gamma_{[\lambda]}) \simeq ((\mathrm{Curv}^{sm}_k)^* \otimes \Gamma_{[\lambda]})^{SO(n)}. \tag{1.4}$$

(b) We establish an $SO(n)$-module-isomorphism between Curv_k^{sm} and the space $\Omega_p^{k,n-k-1}$ of *primitive* translation-invariant differential forms of bi-degree $(k, n-k-1)$ on the sphere bundle SV.

(c) Assuming without loss of generality that $k \leq (n-1)/2$, we will show that $\Omega_p^{k,n-k-1}$ is a $SO(n)$-representation induced by a direct sum of certain $SO(n-1)$-modules:

$$\Omega_p^{k,n-k-1} \simeq \mathrm{Ind}_{SO(n-1)}^{SO(n)} \bigoplus_{l=0}^{k} \bar{\Gamma}_0^{l,0}.$$

(d) Applying Frobenius' Reciprocity, we obtain the following identity:

$$\mathrm{Hom}_{SO(n)}\left(\mathrm{Ind}_{SO(n-1)}^{SO(n)} \bigoplus_{l=0}^{k} \bar{\Gamma}_0^{l,0}, \Gamma_{[\lambda]}^{SO(n)}\right) = \mathrm{Hom}_{SO(n-1)}\left(\bigoplus_{l=0}^{k} \bar{\Gamma}_0^{l,0}, \mathrm{Res}_{SO(n-1)}^{SO(n)} \Gamma_{[\lambda]}^{SO(n)}\right).$$

By virtue of the Branching Theorem for $SO(n)$, we may write $\mathrm{Res}_{SO(n-1)}^{SO(n)} \Gamma_{[\lambda]}^{SO(n)}$ a direct sum of several $SO(n-1)$-modules $\bigoplus_\mu \Gamma_{[\mu]}^{SO(n-1)}$.

(e) As, by Schur's Lemma, non-trivial $SO(n-1)$-invariant homomorphisms between irreducible $SO(n-1)$-modules exist if only if they are isomorphic, we need to identify how many $SO(n-1)$-modules $\Gamma_{[\mu]}$ are of type $\Gamma_{2[l]}$, $l = 0, \ldots, k$. The elaboration of this step concludes the proof.

2. In order to construct all four bases in an elegant way, we need to introduce the so-called λ-*operations* and several of their special cases: the λ-product, the λ-trace map, and the λ-embedding.

Suppose we have a fixed linear Lie-group G as well as two G-modules X, Y with a G-invariant embedding $\iota_X : X \to \Lambda_X V$ and a G-invariant projection $\pi_Y : \Lambda_Y W \to Y$, where $\Lambda_X V = \bigwedge^{x_1} V_1 \otimes \ldots \otimes \bigwedge^{x_k} V_k$ and $\Lambda_Y W = \bigwedge^{y_1} W_1 \otimes \ldots \otimes \bigwedge^{y_\ell} W_\ell$ are some tensor products of wedge-products and $V_1, \ldots, V_k, W_1, \ldots, W_\ell$ are – not necessarily identical – vector spaces. Given some G-invariant operation $F : \Lambda_X V \to \Lambda_Y W$, we may define another G-invariant operation $F_\lambda : X \to Y$ by the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\iota_X} & \Lambda_X V \\ {\scriptstyle F_\lambda}\downarrow & & \downarrow{\scriptstyle F} \\ Y & \xleftarrow{\pi_Y} & \Lambda_Y W. \end{array}$$

We will call such operations F_λ λ-*operations.* Informally, the λ operation F_λ may be described as "apply F and appropriately symmetrise afterward".

Bernig defined in [19] a commutative product on the graded ring $\bigoplus_k \left(\bigwedge^k V \otimes \bigwedge^k W\right)$ for fixed vector spaces V and W. To accommodate this product for our purposes, we extend it to the generalised wedge-product on $\Lambda_{X_1} V \otimes \Lambda_{X_2} V \to \Lambda_{X_1 \cup X_2} V$ for any two G-modules X_1, X_2, where G is a fixed linear group, and an appropriately defined *cup product* $X_1 \cup X_2$. The λ-product $\wedge_\lambda$ is then the λ-operation induced by this generalised wedge-product on the ring of all Young-diagrams.

Given a Lie-group G with a closed Lie-subgroup $H \subset G$, an H-module Γ_μ, and a G-module Γ_λ such that Γ_μ is a H-submodule of $\mathrm{Res}_H^G \Gamma_\lambda$, the λ-*embedding* of Γ_μ into Γ_λ is the following

H-invariant map

$$\begin{array}{rcl} \iota_{\bar{h}} : \Gamma_\mu & \hookrightarrow & \Gamma_\lambda \\ v & \mapsto & v \wedge_\lambda \bar{h}, \end{array}$$

where $\bar{h} \in \left(\bigwedge_{\lambda-\mu} V\right)^H$ is any H-invariant element from the "complementary space" $\bigwedge_{\lambda-\mu} V$ of $\bigwedge_\mu V := \bigwedge_{\Gamma_\mu} V$ in $\bigwedge_\lambda V := \bigwedge_{\Gamma_\lambda} V$ which is best described in terms of the so-called *skew* Young-diagrams $\lambda - \mu$.

In particular, restricting the standard action of $G := SO(n)$ on V to $H := SO(n-1)$, there is a unique element y fixed by H and, thus, we have a canonical $SO(n-1)$-invariant λ-embedding ι_y of X into Y by taking the λ-product with some tensor power of y.

Now, we are able to construct the first two bases $T^n_{(k,p,q)}$ and $T^n_{[k,p,q]}$. The main idea is this: By the proof of the above Theorem, a curvature measure $T^n_{[k,p,q]}$ with values in an irreducible $SO(n)$-representation $\Gamma_{[\lambda]} := \Gamma^{q,p}_r$ is uniquely identified by a canonical $SO(n-1)$-invariant element $\mathrm{Id}_{[\mu]} \in (\Gamma_{[\mu]} \otimes \Gamma^*_{[\mu]})^{SO(n-1)}$, where $\Gamma_{[\mu]} = \Gamma^{q',0}_0$. The main task is now to explicitly describe this identification, i.e. the embedding of $\mathrm{Id}_{[\mu]}$ into the space $\Omega^{k,n-k-1}_p \otimes \Gamma_{[\lambda]}$ of $SO(n)$-covariant translation-invariant primitive forms with values in $\Gamma_{[\lambda]}$. We proceed in several steps:

(a) The elements $\mathrm{Id}_{[\mu]}$ are best described as restrictions of the corresponding $SL(n-1)$-invariant elements $\mathrm{Id}_{(\mu)} \in (\Gamma_\mu \otimes \Gamma^*_\mu)^{SL(n-1)}$ which, in their turn, correspond to $SO(n)$-covariant curvature measures with values in the irreducible $SL(n)$(!)-representation Γ_λ of type $[q;p;r]$. Then we construct in Proposition 4.12 several interim embeddings of $\mathrm{Id}_{(\mu)}$ into $\bigwedge^q V^* \otimes \bigwedge^q V \otimes \mathrm{Sym}^{2q'} V$, then into $\bigwedge^k V^* \otimes \bigwedge^{n-k-1} V \otimes \mathrm{Sym}^{2q'} V$, and, finally, into $\bigwedge^k V^* \otimes \bigwedge^{n-k-1} V \otimes \Gamma_\lambda$. Interpreting the vector space V as a tangential space $T_y S^{n-1}$ for some generic point $y \in S^{n-1}$, we at last may obtain by some invariance arguments an embedding of $\mathrm{Id}_{(\mu)}$ into $(\Omega^{k,n-k-1}_p \otimes \Gamma_\lambda)^{SO(n)}$.

(b) The thus found differential forms are written as a closed sum over a set of multi-indices. However, their usefulness is limited, as this set of multi-indices is not known explicitly. Even worse, even if we fix the multi-indices, the computation of the corresponding summands is not at all a trivial task, as it contains some axiomatically introduced terms and involves multiple λ-operations whose explicit computation may be rather tedious process. Luckily, we may simplify the differential forms. First, we substitute the element $\mathrm{Id}_{(\mu)}$ with the canonical identity element $\mathrm{Id}_\mu \in (\bigwedge_\mu V \otimes \bigwedge_\mu V^*)^{SL(m)}$ – whose structure is very well known – and still obtain a multiple of the curvature measures induced by $\mathrm{Id}_{(\mu)}$. Second, we show that the symmetrisation in the λ-embedding into $\bigwedge^k V^* \otimes \bigwedge^{n-k-1} V \otimes \Gamma_\lambda$ is actually redundant. This further simplifies the formulas for the differential forms. These simplified forms – which we partially symmetrise for the sake of convenience – turn out to be essentially the families $T^n_{(k,p,q)}$.

(c) We now have to show that $T^n_{(k,p,q)}$ form a basis of TCurv. Define the families $T^n_{[k,p,q]} := \pi_{[\mathrm{tr}]}(T^n_{(k,p,q)})$ of $SO(n)$-valued curvature measures. Here, $\pi_{[\mathrm{tr}]}$ is a canonical projection from an arbitrary $SL(n)$-module to its trace-free subspace which is defined as a certain λ-operator in Proposition 2.62. By the nature of their construction, all curvature measures from $T^n_{[k,p,q]}$ are linearly independent (except for a small known subset of curvature measures) and there are exactly m distinct $\Gamma^{q,p}_r$-valued curvature measures for each module $\Gamma^{q,p}_r$ occurring in Curv_k with multiplicity m. Hence, $T^n_{[k,p,q]}$ forms a basis of TCurv.

(d) The correspondence between $T^n_{(k,p,q)}$ and $T^n_{[k,p,q]}$ given by $\pi_{[\mathrm{tr}]}$ is one-to-one and all curvature measures from the first family are linearly independent. Thus, the family $T^n_{(k,p,q)}$ forms a basis as well and we have constructed the first two bases of TCurv.

In order to compute the McMullen-basis $T^n_{k,p,q}$, we first compute the restriction behaviour of $T^n_{(k,p,q)}$ and then eliminate the dependency of this behaviour on n, k by appropriately rescaling the family. The key fact in constructing the horizontal base is Lemma 4.25, which states the necessary and sufficient condition for a family of curvature measures to be horizontalisable, i.e. so transformed as to acquire the horizontality property. We subsequently apply this Lemma to the three regular families $\Phi^n_{k,p,q}$, $\Xi^n_{k,p,q}$, $\Psi^n_{k,p,q}$.

3. The main idea behind the proof of Theorem 4.29 is to find an exact differential form and to express it in terms of the differential forms that induce the family $T^n_{(k,p,q)}$ of curvature measures. This is done by juggling with sums as well as making occasional use of some symmetry properties of $SL(n)$-modules and does not present any conceptional novelties. Since the globalisation of an exact form yields a trivial valuation, we obtain the result.

If the reader wants to refresh her memory before embarking on thoroughly understanding the above proofs, the relevant material is contained in Sections 2 and 3.

In Section 2.1, we quickly recall some basics of finite-dimensional representation theory of Lie-groups, including Schur's omnipresent Lemma. After that, we explain in Section 2.2 how any finite-dimensional representation of $GL(n,\mathbb{C})$ acting on V may be characterised up to an isomorphism by the corresponding Young-diagram. Furthermore, we outline the two classical ways of constructing a $GL(n,\mathbb{C})$-representation from its Young-diagram: the *Young-symmetriser* and the *Plücker-relations*. The former's symmetry properties is one of the key components in the proof step 2b.

In Section 2.3, we elucidate how representations (at least, those we need in the work) of a linear group, i.e. a closed Lie-subgroup G of $GL(n,\mathbb{C})$, still may be classified by Young-diagrams subject to some restrictions that depend on G. We then proceed to describe the interplay between representations of G and those of its closed subgroups, in particular, we state several Branching Theorems as well as the Frobenius Reciprocity used later to prove step 1d. Afterward, we explain in Section 2.4 how to recognise a representation by means of its *character* and prove with their help several interim results.

All of these sections are written in a survey-style, as most classical results are stated without proofs and with some very limited motivation. For more information, we refer the reader to the basic works on representation theory, such as [41] and [42].

The notion of the λ-operation and its special cases that play a central role in the proof steps 2a to 2c is introduced and thoroughly treated in Section 2.5.

In Section 3.1, we develop some basic notions of symplectic geometry: the primitive forms, the bi-degree of a form on SV, and certain partial Hodge-operators. Again, we barely graze the surface of this complex topic. A more circumstantial exposition can be found in [53]. The pinnacle of this section is Corollary 3.17, by which we prove the step 1c.

We then present a more rigorous introduction to valuation theory which we divide into classical and modern with the division line being Alesker's Irreducibility Theorem. In Section 3.2, we first provide the most basic definitions, explain the classical results of Hadwiger and Federer in a more detailed way, and conclude with the formal definition of – and some motivation for – smooth valuations and curvature measures. When introducing the basic notions from the valuation theory, we will make use of several notions from convex geometry that we mostly do not develop properly and instead refer to [75]. Section 3.3 then contains the recent characterisation results of translation-invariant curvature measures with values in different $SO(n)$-modules. In particular, we show the facts needed for the steps 1a and 1b.

Section 4 treats the three main results and their proofs: the decomposition of Curv_k^{sm} is given in Section 4.1, the bases are constructed in Section 4.2, and the globalisation map is analysed in Section 4.3. Section 4.4 contains several useful consequences of the main results.

Some Remarks

We would like to remark that the adjective "tensor-valued" is presently used to denote valuations with values in symmetric powers $\mathrm{Sym}^p V$ of a vector space V, $p \in \mathbb{N}$[51, 52]. We have carefully avoided using this adjective in the introduction so as not to spread confusion. However, throughout the present work – and in its title – we use it to denote valuations and curvature measures with values in *any* finite-dimensional (reducible or irreducible) $SO(n)$-module and not only in $\mathrm{Sym}^p V$.

Likewise, the term "curvature measures" is not uniformly accepted in the literature. In [49], $\mathrm{Sym}^p V$-valued curvature measures are called *local tensor valuations* or *local Minkowski tensors* whereas Bernig in [19] has coined the term *curvature tensors* to denote $\mathrm{Sym}^2\Lambda^q V$-valued curvature measures. In our paper, we have chosen to adhere to Federer's "curvature measures", as this term explains best their purpose and geometric interpretation.

Furthermore, the adjectives "G-equivariant" and "G-covariant" are used by different authors to denote the same invariance property (cf. [15] and [51]). We have opted for the notion of G-covariance more for reasons of personal taste than everything else.

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2 Representation Theory

2.1 Lie-Groups and Representations

Definition 2.1. A *Lie-group* G is a group endowed with the structure of a differentiable manifold such that the multiplication $G \times G \to G$ and the inverse map $G \to G$ are differentiable maps.

A (closed) *Lie-subgroup* of a Lie-group G is a subgroup of G equipped with the structure of a closed submanifold of G.

A *Lie-group homomorphism* is a smooth group homomorphism.

Definition 2.2. Let G be a Lie-group. A *representation* ρ of G on a finite-dimensional complex vector space V is a Lie-group homomorphism $\rho : G \to \mathrm{Aut}(V)$ to the group of invertible endomorphisms on V.

Remark 2.3. The notion of a group representation may be carried over *mutatis mutandis* to other categories of groups one of such natural modifications being to stipulate that the map $G \to \mathrm{Aut}(V)$ be a morphism of a given category. For example, we will need at some point a representation of the permutation group S_d which is a finite group. Other examples include complex Lie-groups with holomorphic representations or algebraic groups with regular representations [44, pp. 155, 523].

We will develop the theory in the category of Lie-groups. However, unless explicitly stated otherwise, the general results from this section – in particular, the notions of subrepresentation, character, induced representation and restriction of a representation – are valid for any of these categories.

Given a basis on $V \simeq \mathbb{C}^n$, where $n = \dim_{\mathbb{C}} V$ is its complex dimension, we may identify the group of invertible endomorphisms $\mathrm{Aut}(V)$ with the general linear group $GL(n, \mathbb{C})$ of invertible $n \times n$ complex matrices. In this sense, the representation of G on V is a group homomorphism $G \to GL(n, \mathbb{C})$, which will be the predominant way of regarding representations in this work. On several occasions, in order to emphasize or distinguish the underlying vector space, we will write $GL(V)$ instead of $GL(n, \mathbb{C})$.

Any representation of G on V gives to the latter the structure of a G-module. When there is no risk of ambiguity about the map $\rho : G \to GL(V)$, we sometimes call V itself the representation of G and write $g \cdot v$ or gv instead of $\rho(g)(v)$.

A map ϕ between two representations V and W of G is a linear map $\phi : V \to W$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi} & W \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V & \xrightarrow{\phi} & W \end{array}$$

commutes for every $g \in G$. We call such maps G-module homomorphisms or G-linear maps in contrast to any other linear maps that do not have this additional property. It is important to note that $\ker \phi$, $\mathrm{im}\, \phi$, $\mathrm{coker}\, \phi$ are also G-modules.

Definition 2.4. A subrepresentation of a representation V is a vector subspace $W \subset V$, which is invariant under G, i.e. $gw \in W$ for all $g \in G$ and $w \in W$. A representation V is called *irreducible* if there is no proper nonzero invariant subspace W of V.

If V and W are representations, then so are their direct sum $V \oplus W$ and their tensor product $V \otimes W$ with the induced operations being $g(v, w) = (gv, gw)$ and $g(v \otimes w) = gv \otimes gw$, respectively. Naturally, the group operation may be extended to any finite tensor-power $V^{\otimes p}$ of V making it a G-module, of which the exterior power $\bigwedge^p V$, and the symmetric power $\mathrm{Sym}^p V$ are subrepresentations. A representation $\rho : G \to V$ induces the natural structure of a G-module on the dual $V^* \simeq \mathrm{Hom}(V, \mathbb{C})$ by stipulating that the dual representation $\rho^* : G \to GL(V^*)$ respect the natural pairing denoted by $\langle \cdot, \cdot \rangle$ between V^* and V, i.e. that the following hold for all $g \in G$, $v \in V$, $v^* \in V^*$:

$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle .$$

Hence, the dual representation is defined as follows:

$$\rho^*(g) = (\rho(g^{-1}))^t : V^* \to V^*$$

for all $g \in G$.

By virtue of the identification $\mathrm{Hom}(V, W) \simeq V^* \otimes W$ for any vector spaces V, W, we may make $\mathrm{Hom}(V, W)$ into a G-module as well. Writing this out, if ϕ is a linear map from V to W, then:

$$(g\phi)(v) = \phi(g^{-1}v)$$

for all $v \in V$.

It is important to note that the tensor product of two irreducible representations is, in general, not irreducible. Decomposing it into irreducible components (the so-called Clebsch-Gordan problem) is not a trivial task. On the other hand, this decomposition is unique if the Lie-group in question is *completely reducible.*

Definition 2.5. G is called *completely reducible* if for any G-module V the following holds: If W is a G-submodule of V, then there is a complementary G-module W' of W in V so that $V = W \oplus W'$.

This follows directly from the following fundamental fact of finite-dimensional representation theory over algebraically closed fields, in particular over $\mathbb{C}$.

Theorem 2.6 (Schur's Lemma)**.** *If V and W are irreducible representations of G and $\phi : V \to W$ is a G-module homomorphism, then*

1. *Either ϕ is an isomorphism, or $\phi = 0$.*
2. *If $V = W$, then $\phi = \lambda \cdot \mathrm{Id}$ for some $\lambda \in \mathbb{C}$,* Id *being the identity.*

In fact, Schur's Lemma implies the uniqueness of decomposition of *any* finite-dimensional representation V (not just the tensor product of two irreducible ones), i.e.

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where V_i are distinct irreducible representations with multiplicities a_i. However, not every Lie-group is completely reducible.

Theorem 2.7. *A Lie-group is completely reducible if and only if it has a finite center.*

Such groups form a rather large subclass of Lie-groups, including all compact Lie-groups. Completely reducible Lie-groups are often called *semi-simple* for the reasons that become apparent after we have established the connection of a Lie-group to its Lie-algebra.

Definition 2.8. A *Lie-algebra* $\mathfrak{g}$ is a vector space equipped with a skew-symmetric bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

that additionally satisfies the Jacobi-identity;

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. This map is called the Lie-bracket or just the bracket.

A Lie-algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ is then a linear map that commutes with the Lie-brackets, i.e. $f[X, Y]_{\mathfrak{g}} = [f(X), f(Y)]_{\mathfrak{h}}$. With this definition in mind, the notion of a representation of a Lie-algebra becomes natural.

Definition 2.9. A *representation of a Lie-algebra* $\mathfrak{g}$ on a vector space V is a Lie-algebra homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(V) = \mathfrak{gl}(V)$, i.e. a linear map such that

$$[X, Y](v) = X(Y(v)) - Y(X(v))$$

for all $v \in V$, $X, Y \in \mathfrak{g}$.

Each Lie-group G has a Lie-algebra $\mathfrak{g}$ associated to it. It is defined to be the tangent vector space T_eG of G at the identity element $e \in G$ equipped with the following Lie-bracket. Define the map

$$\begin{array}{rcl} c_g : G & \to & G \\ h & \mapsto & g \cdot h \cdot g^{-1}. \end{array}$$

As c_g commutes with any Lie-group homomorphism, the induced map c, $g \mapsto c_g$ is, in fact, a map from G to the group of automorphisms on G. Furthermore, c fixes the identity element e, which enables us to consider the differential of c:

$$\mathrm{Ad}(g) := (dc_g)_e : T_eG \to T_eG.$$

It yields a new representation $\mathrm{Ad} : G \to \mathrm{Aut}(T_eG)$ of the group G on its own tangent space. This representation is called the *adjoint representation* of G. We now define:

$$\mathrm{ad} := d(\mathrm{Ad})_e : T_eG \to \mathrm{End}(T_eG).$$

For any $X \in T_eG$, $\mathrm{ad}(X)$ is an endomorphism on T_eG, meaning that $\mathrm{ad}(X)$, when applied to a vector $Y \in T_eG$, yields a third vector in T_eG. Hence, we may re-write ad as a map $T_eG \times T_eG \to T_eG$. Setting

$$[X, Y] := \mathrm{ad}(X)(Y),$$

we complete the construction of the above-mentioned associated Lie-algebra $\mathfrak{g}$ of G.

Many properties of a Lie-group G depend on the structure of its associated Lie-algebra $\mathfrak{g}$, which is a consequence of the the following claim.

Proposition 2.10. *Let G and H be Lie-groups and G be additionally connected and simply connected. A linear map $T_eG \to T_eH$ is the differential of a homomorphism $\rho : G \to H$ if and only if it preserves the bracket operation on the associated Lie-algebras, i.e.*

$$d\rho_e([X, Y]) = [d\rho_e(X), d\rho_e(Y)].$$

The above Proposition implies that the representations of a connected and simply connected Lie-group are in one-to-one correspondence with representations of its associated Lie-algebra. In particular, it allows us to characterise the complete reducibility of Lie-groups by means of algebraic structure of their Lie-algebra.

Definition 2.11. An *ideal* $\mathfrak{h}$ of a Lie-algebra $\mathfrak{g}$ is a Lie-subalgebra such that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$.

Definition 2.12. A Lie-algebra is called:

- *Abelian* if all its brackets vanish;
- *simple* if it does not contain any non-trivial ideals;
- *semi-simple* if it does not contain any non-trivial *Abelian* ideals.

Proposition 2.13. *A Lie-algebra is semi-simple if and only if all its finite-dimensional representations are completely reducible.*

As any semi-simple Lie-algebra is a direct sum of simple Lie-algebras, the characterisation of all semi-simple Lie-algebras is essentially reduced to the study of simple Lie-algebras whose characterisation is complete.

Proposition 2.14. *With five exceptions, every simple complex Lie-algebra is isomorphic to either* $\mathfrak{sl}_n\mathbb{C}$*,* $\mathfrak{so}_n\mathbb{C}$*, or* $\mathfrak{sp}_{2n}\mathbb{C}$ *for some* n*.*

One important remark is appropriate at this point. As the field $\mathbb{R}$ is not algebraically closed, classifying *real* semi-simple Lie-groups and Lie-algebras and their representations is a much harder problem compared to the case of complex Lie-groups and Lie-algebras. In fact, one often first obtains a desired result for complex Lie-groups and subsequently applies more refined methods in order to carry it over to the real case. In general, this process is not straight-forward[1]. However, there is a certain subclass of real Lie-algebras (and their Lie-groups) whose representations may be recovered directly from their complexifications.

Proposition 2.15. *If* $\mathfrak{g}_0$ *is a simple real Lie-algebra, such that its complexification* $\mathfrak{g} := \mathfrak{g}_0 \otimes \mathbb{C}$ *is a simple complex Lie-algebra, then there is one-to-one correspondence between the complex representations of* $\mathfrak{g}_0$ *and those of* $\mathfrak{g}$*.*

Proof. The correspondence is a special case of Weyl's unitary trick. We refer the reader to [55, Chapter 5.1] and [42, Chapter 26.1]. □

For example, the complexification of $\mathfrak{so}(n, \mathbb{R})$ is $\mathfrak{so}(n, \mathbb{C})$ and that of $\mathfrak{sl}(n, \mathbb{R})$ is $\mathfrak{sl}(n, \mathbb{C})$ which are both complex simple Lie-algebras. This yields a one-to-one correspondence between the complex representations of $\mathfrak{so}(n) \simeq \mathfrak{so}(n, \mathbb{R})$- resp. $\mathfrak{sl}(n) \simeq \mathfrak{sl}(n, \mathbb{R})$ and those of $\mathfrak{so}(n, \mathbb{C})$ resp. $\mathfrak{sl}(n, \mathbb{C})$. In particular, under this correspondence, the standard representation of $\mathfrak{so}(n, \mathbb{C})$ – or $\mathfrak{sl}(n, \mathbb{C})$ – is mapped *not* to the real standard representation of $\mathfrak{so}(n)$ resp. $\mathfrak{sl}(n)$ but rather its complexification $V'_{\mathbb{C}} := V' \otimes \mathbb{C}$ of complex dimension n. The same applies to the Lie-groups $SO(n) \simeq SO(n, \mathbb{R})$ and $SL(n) \simeq SL(n, \mathbb{R})$. Hence, although we will state all results for $SL(n, \mathbb{C})$ or $SO(n, \mathbb{C})$, the reader should bear in mind that they may be carried over to $SL(n)$ resp. $SO(n)$ by replacing the complex vector space V with the complexified space $V'_{\mathbb{C}}$ of the same complex dimension.

[1] For example, the Sataki-diagrams which characterise real semi-simple Lie-algebras are coloured versions of the Dynkin-diagrams used to classify complex semi-simple Lie-algebras.

2.2 $GL(n, \mathbb{C})$-Representations

Definition 2.16. Let V be a representation of a Lie-group G. The representation's character χ_V is the complex-valued smooth function on the group defined by

$$\chi_V(g) = \mathrm{Tr}(g|_V),$$

the trace of g on V.

In particular, it is easy to see that

$$\chi_V(hgh^{-1}) = \chi_V(g),$$

implying that $\chi_V \in \mathbb{C}_{\text{class}}(G)$, where $\mathbb{C}_{\text{class}}(G)$ denotes the space of class functions on G, i.e. functions that are constant on conjugacy classes of G. Furthermore, $\chi_V(1) = \dim V$.

Proposition 2.17. *Let V and W be representations of G. Then*

$$\chi_{V \oplus W} = \chi_V + \chi_W, \qquad \chi_{V \otimes W} = \chi_V \cdot \chi_W, \qquad \chi_{V^*} = \overline{\chi_V}$$

and

$$\chi_{\bigwedge^2 V}(g) = \frac{1}{2}\left[\chi_V(g)^2 - \chi_V(g^2)\right].$$

Define $\mathrm{Hom}_G(V, W)$ to be the space of G-module homomorphisms from V to W. If V is irreducible, then, by Schur's Lemma, $\dim \mathrm{Hom}_G(V, W)$ is the multiplicity $m(V, W)$ of V in W; similarly, if W is irreducible, $\dim \mathrm{Hom}_G(V, W)$ is the multiplicity of W in V. In particular, if both V and W are irreducible, then:

$$\dim \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \simeq W \\ 0 & \text{if } V \not\simeq W \end{cases}. \tag{2.1}$$

The character of a representation of G encodes much information about it.

Definition 2.18. A closed Lie-subgroup G of $GL(n, \mathbb{C})$ is called a linear (Lie-)group. If a linear group G is stable under conjugate transpose, it is called *reductive.*

Theorem 2.19. *If G is a compact Lie-group or linear reductive, then any finite-dimensional representation of G is determined – up to an isomorphism – by its character.*

Sketch of the proof. If G is compact, then one can easily show by using the above identity and the identification $\mathrm{Hom}(V, W) \simeq V^* \otimes W$ that

$$(\alpha, \beta)_G := \int_G \overline{\alpha(g)} \cdot \beta(g) dg,$$

where α, β are class functions, defines a scalar product on $\mathbb{C}_{\text{class}}(G)$. We often omit the subscript G if it is understood from the context. Then the characters of non-isomorphic irreducible representations of G are orthonormal in terms of the above scalar product. By the *Peter-Weyl Theorem*, characters span a dense subset of all continuous –and for that matter L^2 – class functions on G. This immediately implies the claim.

If G is linear reductive, the claim follows essentially from the fact that diagonalisable elements form a dense subset of $GL(n, \mathbb{C})$. The orthogonality is then relaxed to the linear independency. □

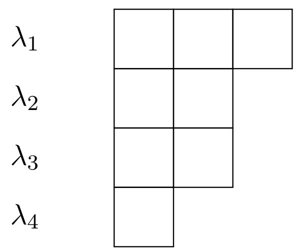
Remark 2.20. In fact, if G is a connected linear reductive group, then comparable results even hold for a large class of *infinite-dimensional* irreducible representations called *admissible representations* [55, Proposition 10.5, Theorem 10.6]. However, these results are of much more involved nature and exceed the scope of the present work.

Corollary 2.21. *Let G be a compact Lie-group. Then a representation V is irreducible if and only if $(\chi_V, \chi_V) = 1$, in which case the multiplicity $m(V, W)$ of V in any G-module W is equal to (χ_W, χ_V).*

Remark 2.22. The assertion in the above Corollary 2.21 may be extended to finite-dimensional $SL(n, \mathbb{C})$-representations by defining an appropriate scalar product on their characters (see [42, Appendix A.1] or [64, Section I.4]).

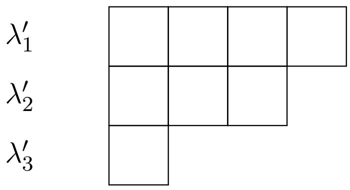
Corollary 2.23. *The number of irreducible representations of a finite group is equal to the number of its conjugacy classes.*

In order to describe representations of the above-mentioned Lie-groups, we need to introduce the notions of the Young-diagram and the Young-symmetriser associated to it. As we know from the above Corollary, the number of irreducible representations of the symmetric group S_d is equal to the number of its conjugacy classes which corresponds to the number $p(d)$ of partitions[2] of d: $d = \lambda_1 + \cdots + \lambda_k$, $\lambda_1 \geq \cdots \geq \lambda_k \geq 1$. To each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ is associated a Young-diagram[3].



, (2.2)

such that the number of boxes on i-th row is equal to λ_i and all boxes are left-aligned. The conjugate partition $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ to the partition λ is defined by the Young-diagram with number of boxes in the i-th *column* equal to λ_i.



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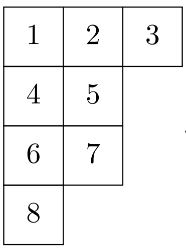
Now, define a *tableau* on a given Young-diagram as a numbering of the boxes by the integers $1, \ldots, d$.



. (2.3)

[2]Of course, we could stipulate $\lambda_k \geq 0$ and add an arbitrary number of zero-partitions at the end. In order to "harmonise" both definitions, we additionally stipulate that two sequences define the same partition if they differ only by zeroes at the end.

[3] Several authors also refer to it as Ferrers- (or Ferrars-)diagram, Ferrers-graph, or just the shape of a partition [44, p. 392] and may use dots instead of boxes [17, pp. 6-7].

The above tableau in (2.3) is an example of a so-called *semi-standard tableau*, i.e. a numbering of the boxes in such a way that the entries in each row are non-decreasing and the entries in each column are strictly increasing. In fact, it belongs to an important subclass of *standard tableaux*, in which the numbers $j \in \{1, \dots, d\}$ are ordered in a strictly increasing manner in both vertical and horizontal directions and occur exactly once.

For each Young-tableau, we may define two natural subgroups:

$$P = P_\lambda = \{g \in S_d : \ g \text{ preserves each row}\}$$

and

$$Q = Q_\lambda = \{g \in S_d : \ g \text{ preserves each column}\}.$$

Defining the *group algebra* $\mathbb{C}G$ to be a vector space spanned by vectors e_g for each $g \in G$, such that $e_g \cdot e_h = e_{gh}$, we set:

$$a_\lambda = \sum_{g \in P} e_g \in \mathbb{C}G \quad \text{ and } \quad b_\lambda = \sum_{g \in Q} \operatorname{sgn}(g) \cdot e_g \in \mathbb{C}G.$$

Note that if V is any vector space and S_d acts on the d-th tensor power $V^{\otimes d}$ on the right by permuting factors

$$(v_1 \otimes \cdots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)},$$

the image of the element $a_\lambda \in \mathbb{C}S_d \to \operatorname{End}(V^{\otimes d})$ is just the subspace

$$\operatorname{im}(a_\lambda) = \operatorname{Sym}^{\lambda_1} V \otimes \operatorname{Sym}^{\lambda_2} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_k} V \subset V^{\otimes d},$$

where the inclusion is obtained by grouping the factors of $V^{\otimes d}$ according to the rows of the Young-tableau. Similarly, the image of b_λ on the tensor power is

$$\operatorname{im}(b_\lambda) = \bigwedge^{\lambda'_1} V \otimes \bigwedge^{\lambda'_2} V \otimes \cdots \otimes \bigwedge^{\lambda'_r} V \subset V^{\otimes d},$$

where λ' is the conjugate partition and the inclusion is obtained by grouping factors according to the columns of the Young tableau. We now define the *Young-symmetriser* c_λ to be

$$c_\lambda := a_\lambda \cdot b_\lambda \in \mathbb{C}S_d.$$

Theorem 2.24. *Some scalar multiple of c_λ is idempotent, i.e. $c_\lambda^2 = n_\lambda c_\lambda$, and the subspace $V_\lambda := c_\lambda \cdot \mathbb{C}S_d$ is an irreducible representation of S_d. Every irreducible representation of S_d can be obtained in this way for a unique partition.*

Remark 2.25. We may use a numbering different from the canonical one in (2.3). In this case, the groups P and Q would be different but the representations constructed in this way would be isomorphic.

The above-mentioned right action of S_d on $V^{\otimes d}$ commutes with the standard left action of $GL(n, \mathbb{C})$, hence, the image of c_λ on $V^{\otimes d}$ – which we denote by $\mathbb{S}_\lambda V$ – is a representation of $GL(n, \mathbb{C})$. This construction is called the *Schur-functor*[4], as it has functorial properties: any linear map $\phi : V \to W$ induces a linear map $\mathbb{S}_\lambda(\phi) : \mathbb{S}_\lambda V \to \mathbb{S}_\lambda W$ with $\mathbb{S}_\lambda(\phi \circ \psi) = \mathbb{S}_\lambda(\phi) \circ \mathbb{S}_\lambda(\psi)$ and $\mathbb{S}_\lambda(\mathrm{Id}_V) = \mathrm{Id}_{\mathbb{S}_\lambda V}$.

[4]This notion has several other names, of which the most prominent are Weyl-module and Weyl's construction [42, Chapters 6, 15].

However, these are not all irreducible representations of $GL(n,\mathbb{C})$. Setting $D_\ell := (\bigwedge^n V)^{\otimes \ell}$ and $D_{-\ell} := (D_\ell)^*$ for $\ell \geq 0$, it may be shown that

$$\mathbb{S}_{(\lambda_1+\ell,\ldots,\lambda_k+\ell)}V = \mathbb{S}_{(\lambda_1,\ldots,\lambda_k)}V \otimes D_\ell,$$

wherever λ and $\lambda+\ell$ are defined, i.e both λ_i and $\lambda_i+\ell$ are non-negative for all $i = 1,\ldots,k$. This allows us to define the Schur-functor for any tuple $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ by taking

$$\mathbb{S}_\lambda := \mathbb{S}_{\lambda+\ell} \otimes D_{-\ell}$$

for a sufficiently large ℓ. This definition may be shown to be independent from ℓ.

Proposition 2.26. *Every irreducible complex representation of $GL(n,\mathbb{C})$ is isomorphic to $\mathbb{S}_\lambda V$ for a unique tuple $\lambda = (\lambda_1,\ldots,\lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.*

Corollary 2.27. *For any such tuple $\lambda = (\lambda_1,\ldots,\lambda_n)$, one has:*

$$(\mathbb{S}_\lambda V)^* \simeq \mathbb{S}_\lambda(V^*) \simeq \mathbb{S}_{(-\lambda_n,\ldots,-\lambda_1)}V.$$

For non-negative λ, $\mathbb{S}_\lambda V$ may be realized as a subspace of $V^{\otimes d}$ as follows. Set $\mathbf{a} := \mathbf{a}(\lambda) := (a_1(\lambda),\ldots,a_n(\lambda))$, where $a_i(\lambda)$ is the number of columns of length i in the Young-diagram associated to λ. For example, the Young-diagram from (2.2) has the associated tuple $\mathbf{a} = (1,0,1,1)$. Set

$$W_\lambda V := \operatorname{Sym}^{a_n}(\textstyle\bigwedge^n V) \otimes \operatorname{Sym}^{a_{n-1}}\left(\bigwedge^{n-1} V\right) \otimes \cdots \otimes \operatorname{Sym}^{a_1}(V), \tag{2.4}$$

where $\operatorname{Sym}^a \bigwedge^b V$ is naturally embedded into $V^{\otimes ab}$ by mapping a symmetric product of exterior products

$$(v_{1,1} \wedge \cdots \wedge v_{b,1}) \cdot \cdots \cdot (v_{1,a} \wedge \cdots \wedge v_{b,a})$$

to

$$\sum \operatorname{sgn} q \ (v_{q_1(1),p(1)} \otimes \cdots \otimes v_{q_1(b),p(1)}) \otimes \cdots \otimes (v_{q_a(1),p(a)} \otimes \cdots \otimes v_{q_a(b),p(a)}),$$

and the sum is over $p \in S_a$, $q = (q_1,\ldots,q_a) \in S_b \times \cdots \times S_b$. We may also sum only over $p \in S_a$ and keep the wedge-products, thus obtaining an embedding $\operatorname{Sym}^a \bigwedge^b V \hookrightarrow \left(\bigwedge^b V\right)^{\otimes a}$, which yields a map:

$$W_\lambda V \hookrightarrow \Lambda_\lambda V := (\textstyle\bigwedge^n V)^{\otimes a_n} \otimes \left(\bigwedge^{n-1} V\right)^{\otimes a_{n-1}} \otimes \cdots \otimes V^{\otimes a_1}. \tag{2.5}$$

To embed $\mathbb{S}_\lambda V$ into $W_\lambda V$ in a manner compatible with the above embedding of $W_\lambda V$ into $V^{\otimes d}$, we must use the Young-tableau, in which the cells are numbered from top to bottom, then from left to right:

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 8 \\ \hline 2 & 6 & \\ \cline{1-2} 3 & 7 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array}\;.$$

Set $\lambda' = (\lambda'_1,\ldots,\lambda'_r)$ to be the conjugate of λ. The Young-symmetriser may be written as $c_\lambda = a_\lambda \cdot b_\lambda$, where $a_\lambda = \sum e_p$, the sum over all p in the subgroup $P = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ of

S_d preserving the rows, $b_\lambda = \sum \operatorname{sgn}(q) e_q$, the sum over the subgroup $Q = S_{\lambda'_1} \times \cdots \times S_{\lambda'_r}$ preserving the columns. Define

$$R := S_{a_k} \times \cdots \times S_{a_1} \subset P,$$

consisting of all elements that move all entries of each column to the same position in some column of the same length, i.e. permute columns of the same length. Then $a_\lambda = a''_\lambda \cdot a'_\lambda$, where $a'_\lambda \in R$ and $a''_\lambda \in P/R$ is the sum over any set of representatives in P for the left cosets P/R. Then one has:

$$\mathbb{S}_\lambda V = (V^{\otimes d} \cdot a''_\lambda) \cdot a'_\lambda \cdot b_\lambda. \tag{2.6}$$

But $V^{\otimes d} \cdot a'_\lambda \cdot b_\lambda$ is precisely $W_\lambda V$ and $V^{\otimes d} \cdot a''_\lambda \subset V^{\otimes d}$, hence $\mathbb{S}_\lambda V \subset W_\lambda V \subset \Lambda_\lambda V \subset V^{\otimes d}$. In particular, we may write

$$\begin{aligned} \pi_\lambda : W_\lambda V &\to \mathbb{S}_\lambda V \\ v &\mapsto a''_\lambda \cdot v \end{aligned} \tag{2.7}$$

the canonical projection from $W_\lambda V$ to $\mathbb{S}_\lambda V$.

We may describe all $\mathbb{S}_\lambda V$ at once by considering the symmetric algebra

$$W^\cdot V = \operatorname{Sym}^\cdot \left(V \oplus \textstyle\bigwedge^2 V \oplus \cdots \oplus \bigwedge^n V \right) = \bigotimes_{a_1,\ldots,a_n} W_{\lambda(a_1,\ldots,a_n)} V,$$

the sum over all n-tuples $(a_1, \ldots, a_n)$ of non-negative integers. Define the ring $\mathbb{S}^\cdot V$ to be the quotient of $W^\cdot V$ modulo the graded, two-sided ideal $I^\cdot$ generated by the so-called *Plücker-relations*, i.e. by the elements of the form:

$$\begin{aligned} &(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q) \\ &\quad - \sum_{i=1}^{p} (v_1 \wedge \cdots \wedge v_{i-1} \wedge w_1 \wedge v_{i+1} \wedge \cdots \wedge v_p) \cdot (v_i \wedge w_2 \wedge \cdots \wedge w_q) \end{aligned} \tag{2.8}$$

for all $p \geq q \geq 1$. Note that these elements lie in $\operatorname{Sym}^2(\bigwedge^p V)$ if $p = q$ and in $\bigwedge^p V \otimes \bigwedge^q V$ if $p > q$. The algebra $\mathbb{S}^\cdot V$ is then the direct sum of the images $\mathbb{S}^{\mathbf{a}} V$ of the summands $W_{\lambda(\mathbf{a})} V$ under this quotient.

For any semi-standard tableau T, we write $T(i,j)$ for the entry of T in the i-th row and the j-th column. Then the equation above yields the following Bianchi-type relations on $\mathbb{S}^{\mathbf{a}} V$ [41, §7.4], [64, §I.5, (5.12)].

Lemma 2.28. *Let $e_1, \ldots, e_n$ be a basis of V, $\tilde{\pi}_\lambda$ be the canonical projection from $W_\lambda V$ to $\mathbb{S}^{\mathbf{a}} V$, and:*

$$e_T := \prod_{j=1}^{r} e_{T(1,j)} \wedge \cdots \wedge e_{T(\lambda'_j, j)} \in W_\lambda V. \tag{2.9}$$

for any Young-tableau T. Then

$$\tilde{\pi}_\lambda \left(e_T - \sum_S e_S \right) = 0,$$

where the sum is over all S obtained from T by exchanging the top k elements of one column with any k elements of the preceding column, maintaining the vertical orders of each set exchanged.

There is one such relation for each numbering T, each choice of adjacent columns, and each k at most equal to the length of the shorter column.

We now can describe a natural system of generators of $\mathbb{S}^{\mathbf{a}}V$.

Proposition 2.29. *Let λ be a fixed partition. Then*

1. *The elements $\tilde{\pi}_\lambda(e_T)$ for T any valid semi-standard Young-tableau and e_T defined in (2.9) generate $\mathbb{S}^{\mathbf{a}}V$ as a vector space.*
2. *The subspace $\mathbb{S}_\lambda V = \pi_\lambda(W_\lambda V)$, where π_λ is from (2.7), is isomorphic to the quotient $\mathbb{S}^{\mathbf{a}}V = \tilde{\pi}_\lambda(W_\lambda V)$.*

Remark 2.30. We write Γ_λ instead of $\mathbb{S}_\lambda V$ or $\mathbb{S}^{\mathbf{a}}V$ if the embedding or the construction method of an irreducible representation corresponding to a tuple λ are not important or are understood from the context.

2.3 Restrictions, Reciprocity, and Classifications

One of the crucial notions in the present work are those of an induced and a restricted representation. Given a representation V of a Lie-group G, any closed Lie-subgroup $H \subset G$ inherits from G the action on V so that V may also be regarded as a representation of H. The thus constructed representation of H is denoted by $\operatorname{Res}_H^G V$ or, when no confusion is possible, simply $\operatorname{Res} V$.

There is also a canonical way to "extend" a representation of H to a representation of G. Let V be some representation of G and $W \subset V$ be an H-invariant subspace. For any $g \in G$, the subspace $g \cdot W = \{g \cdot w \mid w \in W\}$ depends only on the left coset $\sigma := gH$ of g modulo H, as $gh \cdot W = g \cdot (h \cdot W) = g \cdot W$, thus, justifying the notation $\sigma \cdot W$ for this subspace. The representation V of G is then said to be induced by W and denoted by $V = \operatorname{Ind}_H^G W = \operatorname{Ind} W$ if

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

It is not hard to show that $\operatorname{Ind} W$ exists for any Lie-group G and any closed Lie-subgroup $H \subset G$. In fact, there are multiple generic constructions of $\operatorname{Ind} W$. The one that we will make later use of has analytical character. Consider the space $C^\infty(G, W)$ of all smooth functions from G to W. Letting G act on $C^\infty(G, W)$ by left translations: $gf(x) := f(g^{-1}x)$, we obtain a representation of G. Then the G-invariant subspace of $C^\infty(G, W)$:

$$\mathcal{I}_H := \{f \in C^\infty(G, W) \mid f(gh) = h^{-1} f(g), \quad \forall h \in H, \forall g \in G\}. \tag{2.10}$$

is isomorphic to $\operatorname{Ind}_H^G W$.

Note that $\operatorname{Ind}_H^G W$ is, in general, not finite-dimensional. Nevertheless, the formulae for $\operatorname{Res}(\operatorname{Ind} W)$ and $\operatorname{Ind}(\operatorname{Res} W)$ are known and can be found in [81]. Although both constructions are generally not equal to W, the well-known Frobenius' Theorem shows that Ind and Res are, in some sense, reciprocal to each other.

Theorem 2.31 (Frobenius' Reciprocity Theorem)**.** *Let G be a compact Lie-group and $H \subset G$ a closed Lie-subgroup. Given a representation U of G and a representation W of H, there is a canonical vector space isomorphism*

$$\operatorname{Hom}_G(U, \operatorname{Ind} W) \simeq \operatorname{Hom}_H(\operatorname{Res} U, W).$$

Remark 2.32. The above theorem implies that we may characterize G-module homomorphisms on $\operatorname{Ind} W$ uniquely by H-module homomorphisms on W. Hence, $\operatorname{Ind} W$ is defined uniquely up to a canonical isomorphisms.

The above theorem together with Schur's Lemma and Corollary 2.21 directly imply the following result.

Proposition 2.33. *If W is a representation of H and U a representation of G, then*

$$\left(\chi_{\operatorname{Ind} W}, \chi_U\right)_G = \left(\chi_W, \chi_{\operatorname{Res} U}\right)_H .$$

In particular, if W and U are irreducible, then $m(U, \operatorname{Ind} W) = m(W, \operatorname{Res} U)$.

While classifying irreducible finite-dimensional representations of a closed Lie-subgroup H of $GL(n,\mathbb{C})$, it is helpful to study restrictions to H of irreducible $GL(n,\mathbb{C})$-representations. For example, as any element of $GL(n,\mathbb{C})$ is a scalar multiple of an element of $SL(n,\mathbb{C})$, $\operatorname{Res}^{GL(n,\mathbb{C})}_{SL(n,\mathbb{C})} \Gamma_\lambda$ is an irreducible $SL(n,\mathbb{C})$-representation for any λ, which we again denote by Γ_λ. Furthermore, $\bigwedge^n V$ is a trivial $SL(n,\mathbb{C})$-representation, which yields $V^* \simeq \bigwedge^{n-1} V$ and $\mathbb{S}_\lambda V \simeq \mathbb{S}_{\lambda+\ell} V$.

Proposition 2.34. *Every irreducible $SL(n,\mathbb{C})$-representation is characterized by a unique tuple λ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$.*

We may restrict $SL(n,\mathbb{C})$-representations to $SO(n,\mathbb{C})$-representations. One cannot, in general, expect such restriction to be an irreducible $SO(n,\mathbb{C})$-representation any more [58]. However, due to the fact that there exists a symmetric bilinear form Q on V preserved by $SO(n,\mathbb{C})$, we may find certain maps on $V^{\otimes d}$ induced by Q such that the intersections of their kernels with given $SL(n,\mathbb{C})$-representations yield irreducible $SO(n,\mathbb{C})$-representations. More precisely, for each pair $p < q$, define a contraction:

$$\begin{array}{rccl} \operatorname{tr}_{p,q} : & V^{\otimes d} & \to & V^{\otimes d-2} \\ & v_1 \otimes \cdots \otimes v_d & \mapsto & Q(v_p, v_q)\, v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes \hat{v}_q \otimes \cdots \otimes v_d \end{array} \tag{2.11}$$

Let $V^{[d]}$ define the intersection of the kernels of all these contractions[5]. These subspaces are mapped to itself by permutations, so $V^{[d]}$ is an S_d-subrepresentation of $V^{\otimes d}$. Set

$$\mathbb{S}_{[\lambda]} V := V^{[d]} \cap \mathbb{S}_\lambda V. \tag{2.12}$$

Theorem 2.35. *If $\lambda_{n+1} > 0$ or $\lambda'_1 + \lambda'_2 > n$, then the $SO(n,\mathbb{C})$-module $\mathbb{S}_{[\lambda]} V$ is trivial. Otherwise:*

1. *If $n = 2k+1$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \geq 0)$ or $n = 2k$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{k-1} \geq \lambda_k = 0)$, then $\mathbb{S}_{[\lambda]} V$ is an irreducible $SO(n,\mathbb{C})$-module.*
2. *If $n = 2k$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k > 0)$, then $\mathbb{S}_{[\lambda]} V$ is a direct sum of two irreducible $SO(n,\mathbb{C})$-modules that are dual to each other.*

Similarly to Remark 2.30, we may explicitly write $\Gamma_{[\lambda]}$ for $\mathbb{S}_{[\lambda]} V$ in the first case. In the second case, $\mathbb{S}_{[\lambda]} V$ may be written as $\Gamma_{[\lambda]} \oplus \left(\Gamma_{[\lambda]}\right)^*$, where $\Gamma_{[\lambda]}$ may be chosen canonically so as to satisfy Weyl's character formula. It is, however, easy to see that $\bigwedge^k V$ is not irreducible. The bilinear form $Q \in \operatorname{Sym}^2 V^* \simeq \operatorname{Sym}^2 V$ preserved by $SO(n,\mathbb{C})$ extends to a

[5] The bracket notation is the classical notation for objects related to the Lie-group $SO(n, C)$ as opposed to the bracket-free notation for $SL(n,\mathbb{C})$. For $Sp(2n)$, we use the angled-brackets notation $\langle\cdot\rangle$. Alternatively, if the same object may be considered under the actions of two different Lie-groups, we usually denote them as superscripts.

scalar product on $\bigwedge^j V$, which we denote by the same symbol Q[6]. On the other hand, there is another bilinear form on $\bigwedge^k V$ defined by $\tilde{Q}(v,w)\cdot \mathrm{vol} := Q(v,*w)\cdot \mathrm{vol} = v\wedge w$, where $*:\bigwedge^k V \to \bigwedge^k V$ is the Hodge-star operator. We, thus, have two distinct bilinear forms preserved by $SO(n,\mathbb{C})$, which implies that $\bigwedge^n V$ has at least two irreducible components.

It is sometimes more convenient to work directly with $\mathbb{S}_{[\lambda]}V$ which we denote by $\bar{\Gamma}_{[\lambda]}$.

$$\bar{\Gamma}_{[\lambda]} = \bar{\Gamma}_{[\lambda_1,\ldots,\lambda_{\lfloor n/2\rfloor}]} := \begin{cases} \Gamma_{[\lambda_1,\ldots,\lambda_{\lfloor n/2\rfloor}]} \oplus \Gamma_{[\lambda_1,\ldots,-\lambda_{\lfloor n/2\rfloor}]} & \text{if } n \text{ is even and } \lambda_{\lfloor n/2\rfloor}\neq 0,\\ \Gamma_{[\lambda_1,\ldots,\lambda_{\lfloor n/2\rfloor}]} & \text{otherwise}\end{cases}$$

The module $\mathbb{S}_{[\lambda]}V \subset \mathbb{S}_\lambda V$ is still canonically embedded into $W_\lambda V$, hence $W_{[\lambda]}V = W_\lambda V$.

The irreducibility of the above modules is proved on the level of Lie-algebras first and carried over to Lie-groups in the spirit of Proposition 2.10. As $SO(n,\mathbb{C})$ has the fundamental group $\mathbb{Z}/2$ and is, thus, not simply-connected, the $SO(n,\mathbb{C})$-representations constructed above constitute only roughly a half of all irreducible $\mathfrak{so}(n,\mathbb{C})$-representations. The other half comes from the universal covering group of $SO(n,\mathbb{C})$ called the *Spin*$(n,\mathbb{C})$-group. Correspondingly, these representations are called Spin-representations. However, we do not construct nor describe them in this work, as they play no role in the classification of curvature measures.

There are two other natural restrictions that we will make heavy use of in the following sections: $\mathrm{Res}^{SO(n,\mathbb{C})}_{SO(n-1,\mathbb{C})}\Gamma_{[\lambda]}$ and $\mathrm{Res}^{SL(n,\mathbb{C})}_{SO(n,\mathbb{C})}\Gamma_\lambda$. They are the subject of the so-called Branching Theorems that describe this restriction in a closed form.

Theorem 2.36 ($SO(n,\mathbb{C})$-branching)**.** *Let λ be a tuple of integers satisfying conditions from Theorem 2.35. Then*

$$\mathrm{Res}^{SO(n,\mathbb{C})}_{SO(n-1,\mathbb{C})}\Gamma^{SO(n,\mathbb{C})}_{[\lambda]} = \bigoplus_\mu \Gamma^{SO(n-1,\mathbb{C})}_{[\mu]},$$

where μ runs over all partitions $\mu = (\mu_1,\ldots,\mu_k)$, $k=\lfloor (n-1)/2\rfloor$, such that

$$\begin{cases} \lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \ldots \ge \mu_{k-1} \ge \lambda_{\lfloor n/2\rfloor} \ge |\mu_k| & \text{for odd } n,\\ \lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \ldots \ge \mu_k \ge |\lambda_{\lfloor n/2\rfloor}| & \text{for even } n.\end{cases}$$

Theorem 2.37 ($SL(n,\mathbb{C})$-branching)**.** *Let $\lambda = (\lambda_1 \ge \lambda_2 \ge \lambda_n \ge 0)$ be a non-negative tuple of integers. Then*

$$\mathrm{Res}^{SL(n,\mathbb{C})}_{SO(n,\mathbb{C})}\Gamma^{SL(n,\mathbb{C})}_{\lambda} = \bigoplus_\mu N_{\mu\lambda}\Gamma^{SO(n,\mathbb{C})}_{[\mu]},$$

where $N_{\mu\lambda} = \sum_\delta N_{\delta\mu\lambda}$ is the sum of the so-called Littlewood-Richardson coefficients *(see, for example, [42, pp. 455f.]) over $\delta = (\delta_1 \ge \delta_2 \ge \ldots)$ with all δ_i even. In particular, Γ_λ contains only $SO(n,\mathbb{C})$-modules $\Gamma_{[\nu]}$ for which $|\lambda| - |\nu|$ is even.*

If $\dim V = 2n$, we may construct irreducible $Sp(2n,\mathbb{C})$-modules in a similar fashion as in the $SO(n,\mathbb{C})$-case. The restriction $\mathrm{Res}^{SL(2n,\mathbb{C})}_{Sp(2n,\mathbb{C})}\Gamma_\lambda$ is again not irreducible in general [58]. However, there exists a *skew-symmetric* $Sp(2n,\mathbb{C})$-invariant bilinear form[7] Q'. Correspondingly, replacing the symmetric form Q with Q', we may again define the contractions Φ'_I

[6] We still use the classical notation $\langle\cdot,\cdot\rangle$ for the scalar product induced by the standard bilinear form $Q = \sum_{r=1}^n (e_r^*)^2$.

[7] This bilinear form had motivated the group's older name "complex line group". However, in order to avoid possible confusion with complex numbers, Weyl ([91, p. 165]) proposed in 1939 to replace the Latin word complex (*com-plexus* "braided together") with its Greek translation ($\sigma\upsilon\mu$-$\pi\lambda\varepsilon\kappa\tau\iota\kappa o\varsigma$ "sym-plektikos"). Correspondingly, the form is also called symplectic.

and set $V^{\langle d\rangle}$ to be the intersection of these contractions' kernels. In contrast to $SO(n,\mathbb{C})$, $Sp(n,\mathbb{C})$ is simply-connected, which is reflected the following classification result.

Theorem 2.38. *The space $\mathbb{S}_{\langle\lambda\rangle}V := V^{\langle d\rangle} \cap \mathbb{S}_\lambda V$ is nonzero if and only if $\lambda_{n+1} = 0$. Furthermore, every irreducible $Sp(2n,\mathbb{C})$-representation is isomorphic to $\mathbb{S}_{\langle\lambda\rangle}V$ for a unique tuple $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq \lambda_{n+1} = 0)$.*

2.4 Character Formulae

So far, we were using characters to describe several abstract results in the theory of representations. In this section, we will compute characters of irreducible representations of all Lie-groups with simple Lie-algebras. Although there exists a very powerful means – called Weyl's character formula – of computing the character of any irreducible representation of any (compact) Lie-group, this formula is too general for our purposes. Instead, we will introduce several of its corollaries and special cases.

For $GL(n,\mathbb{C})$- and $SL(n,\mathbb{C})$-representations, the character of $\Gamma_\lambda = \mathbb{S}_\lambda V$ may be obtained by using rather elementary algebraic means for any λ satisfying the conditions from Proposition 2.26. It turns out that $\chi_{\mathbb{S}_\lambda V}$ can be written as the so-called *Schur-polynomial*

$$\chi_{\mathbb{S}_\lambda V}(g) = S_\lambda(x_1,\ldots,x_n) := \frac{\det\left(x_j^{\lambda_i+n-i}\right)}{\det\left(x_j^{n-i}\right)}, \tag{2.13}$$

where x_i are eigenvalues of the endomorphism $g \in GL(n,\mathbb{C})$. In particular, $\chi_{\bigwedge^k V} = E_k(x_1,\ldots,x_n)$ is the k-th elementary symmetric polynomial:

$$E_k(x_1,\ldots,x_n) = \sum_{i_1<\cdots<i_k=1}^{n} x_{i_1}\cdot\ldots\cdot x_{i_k}. \tag{2.14}$$

The elementary symmetric polynomials play a prominent role in the theory of finite-dimensional representations, as they form a basis of all homogeneous symmetric polynomials of degree d in k variables and, hence, can be used to express χ_{Γ_λ} for any λ, which is the subject of the second determinantal – or Giambelli – formula.

Theorem 2.39 (Second determinantal formula)**.** *Let λ be a non-negative tuple $\lambda = (\lambda_1 \geq \ldots \geq \lambda_n \geq 0)$ and $\mu = (\mu_1,\ldots,\mu_\ell) = \bar{\lambda}$ its conjugate partition. Then:*

$$S_\lambda = \det(E_{\mu_i+j-i}) = \det\begin{pmatrix} E_{\mu_1} & E_{\mu_1+1} & \cdots & E_{\mu_1+\ell-1} \\ E_{\mu_2-1} & E_{\mu_2} & \cdots & E_{\mu_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ E_{\mu_\ell-l+1} & E_{\mu_\ell-l} & \cdots & E_{\mu_\ell} \end{pmatrix}.$$

Remark 2.40. The *first* determinantal formula expresses χ_{Γ_λ} in terms of the so-called complete symmetric polynomials, which form yet another basis of homogeneous symmetric polynomials. We will not make use of the first determinantal formula in the work. Hence, whenever we refer to the determinantal or Giambelli formula without specifying its ordinal number, we mean the second, and not the first, one.

The characters of irreducible $SO(n,\mathbb{C})$- and $Sp(2n,\mathbb{C})$-representations may be represented in terms of the above bases as well. For example, for any λ, for which there is a non-trivial irreducible $Sp(2n,\mathbb{C})$-representation $\Gamma_{\langle\lambda\rangle}$

$$\chi_{\Gamma_{\langle\lambda\rangle}} = \frac{\det(x_j^{\lambda_i+n-i+1} - x_j^{-(\lambda_i+n-i+1)})}{\det(x_j^{n-i+1} - x_j^{-(n-i+1)})}.$$

There is also a modified version of the second determinantal formula for $Sp(2n,\mathbb{C})$. As $\bigwedge^k V$ is not an irreducible $Sp(2n,\mathbb{C})$-representation, it is natural to replace the elementary symmetric polynomials from the $SL(n,\mathbb{C})$-case with the characters of

$$\Gamma_{\langle 1[k]\rangle} = \ker\left(\bigwedge^k \mathbb{C}^{2n} \to \bigwedge^{k-2}\mathbb{C}^{2n}\right).$$

From the above equations, one quickly sees that

$$E'_k := \chi_{\Gamma_{\langle 1[k]\rangle}} = E_k(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}) - E_{k-2}(x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}),$$

where $d[k]$ denotes a k-tuple with all elements equal to d. Then the formula reads

Corollary 2.41. *The character of a non-trivial irreducible $Sp(2n)$-representation $\Gamma_{\langle\lambda\rangle}$ is given by the determinant of the $\ell\times\ell$-matrix whose i-th row is*

$$(E'_{\mu_i-i+1} \quad E'_{\mu_i-i+2}+E'_{\mu_i-i} \quad E'_{\mu_i-i+3}+E'_{\mu_i-i-1} \quad \cdots \quad E'_{\mu_i-i+\ell}+E'_{\mu_i-i-\ell+2}),$$

where $\mu=(\mu_1,\ldots,\mu_\ell)=\bar\lambda$ is again the conjugate partition of λ.

Relinquishing the desire to only use characters of irreducible representations, the above result may be written in terms of E_k – rather than of E'_k – as

$$\chi_{\Gamma_{\langle\lambda\rangle}} = \det(E_{\mu_i-i+j} - E_{\mu_i-i-j}).$$

Note that $E_{n+k} = E_{n-k}$, which corresponds to the isomorphism $\bigwedge^{n+k}\mathbb{C}^{2n} \simeq \bigwedge^{n-k}\mathbb{C}^{2n}$. Consequently, $E'_{n+k} = -E'_{n-k+2}$.

For $SO(n,\mathbb{C})$-representations, the character $\chi_{\bar\Gamma_{[\lambda]}}$ may be written as follows:

$$\chi_{\bar\Gamma_{[\lambda]}} = \begin{cases} \frac{\det(x_j^{\lambda_i+n-i+1/2} - x_j^{-(\lambda_i+n-i+1/2)})}{\det(x_j^{n-i+1/2} - x_j^{-(n-i+1/2)})} & \text{if } n=2m+1; \\ \frac{\det(x_j^{\lambda_i+n-i} - x_j^{-(\lambda_i+n-i)})}{\det(x_j^{n-i} - x_j^{-(n-i)})} & \text{if } n=2m. \end{cases}$$

However, the determinantal formula is formally identical for both cases:

Corollary 2.42. *The character of a non-trivial $SO(n,\mathbb{C})$-representation $\bar\Gamma_{[\lambda]}$ is given by the determinant of the $\ell\times\ell$-matrix whose i-th row is*

$$(E_{\mu_i-i+1} \quad E_{\mu_i-i+2}+E_{\mu_i-i} \quad E_{\mu_i-i+3}+E_{\mu_i-i-1} \quad \cdots \quad E_{\mu_i-i+\ell}+E_{\mu_i-i-\ell+2}),$$

where $\mu=(\mu_1,\ldots,\mu_\ell)=\bar\lambda$ is again the conjugate partition of λ.

However, the definition of E_k and its symmetry depend on n's parity. For even $n = 2m$, E_k is defined as $E_k(x_1, \dots, x_m, x_1^{-1}, \dots, x_m^{-1})$ with $E_{m+k} = E_{m-k}$ due to the isomorphism $\bigwedge^{m+k}\mathbb{C}^{2m} \simeq \bigwedge^{m-k}\mathbb{C}^{2m}$. For odd $n = 2m+1$, $E_k = E_k(x_1, \dots, x_m, x_1^{-1}, \dots, x_m^{-1}, 1)$. The symmetry $E_{m+k} = E_{m+1-k}$ is slightly different in this case, too, as the isomorphism is given by $\bigwedge^{m+k}\mathbb{C}^{2m+1} \simeq \bigwedge^{m-k+1}\mathbb{C}^{2m+1}$.

The formal similarity of the Giambelli formulas for even and odd dimensions allows us to formulate and prove the following crucial result in a dimension-independent manner.

Lemma 2.43. *Let $i, j \in \mathbb{N}$ such that $0 \le i, j \le n$ and set*

$$i' := \max(\min(i, n-i), \min(j, n-j)), \qquad j' := \min(\min(i, n-i), \min(j, n-j)).$$

Then the following $SO(n, \mathbb{C})$-modules are isomorphic:

$$\bigwedge^i V \otimes \bigwedge^j V = \left(\bigoplus_{l=0}^{j'} \bar{\Gamma}_{[2[l],1[i'-j']]}\right) \oplus \left(\bigwedge^{i'+1} V \otimes \bigwedge^{j'-1} V\right) = \bigoplus_{k=0}^{j'} \bigoplus_{l=0}^{j'-k} \bar{\Gamma}_{[2[l],1[2k+i'-j']]}. \tag{2.15}$$

Proof. Since $\bigwedge^i V \simeq \bigwedge^{n-i} V$ and $\bigwedge^i V \otimes \bigwedge^j V \simeq \bigwedge^j V \otimes \bigwedge^i V$, we may assume without loss of generality $i = i' \le n/2$ and $j = j' \le n/2$. If $\lambda = (\lambda_1, \dots, \lambda_m)$ is a non-negative tuple, as specified in the middle term of the above identity, then the conjugate $\bar{\lambda} = \mu = (l+k, l)$, where $k := i - j$. By virtue of the determinantal formula above,

$$\chi_{\bar{\Gamma}_{[\lambda]}} = \det \begin{pmatrix} E_{l+k} & E_{l+k+1} + E_{l+k-1} \\ E_{l-1} & E_l + E_{l-2} \end{pmatrix}.$$

Summing over all λ as in the middle term yields then

$$\begin{aligned} \sum_\lambda \chi_{\bar{\Gamma}_{[\lambda]}} &= \sum_{l=0}^{j} \left(E_{l+k}(E_l + E_{l-2}) - E_{l-1}(E_{l+k+1} + E_{l+k-1})\right) \\ &= E_i E_j - E_{i+1} E_{j-1}. \end{aligned}$$

The second identity in (2.15) is obtained by applying the first one recursively until $j' = 0$. □

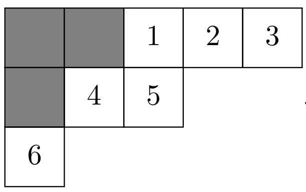
Remark 2.44. The above Lemma is a slight generalisation of Corollary 3.4 in [15].

2.5 G-invariant λ-Maps

Before proceeding to the main topic of this section, G-invariant λ-maps, we need to slightly extend several notions from Young-diagrams to the so-called skew Young-diagrams.

Definition 2.45. If λ and μ are partitions with $\mu_i \le \lambda_i$, then the *skew Young-diagram* $\lambda - \mu$ is the complement of the Young diagram for μ in that for λ.

The notion of a – standard or semi-standard – Young-tableau carries over one-to-one to skew Young-diagrams. For example, if $\lambda = (5, 3, 1)$ and $\mu = (2, 1)$, then the numbered part of the following diagram is a standard Young-tableau for the skew Young-diagram $\lambda - \mu$:



. (2.16)

With the skew (semi-)standard Young-tableaux at hand, one can further extend the definitions of $a_{\lambda-\mu}$, $b_{\lambda-\mu}$, and the skew Young-symmetrisers $c_{\lambda-\mu} := a_{\lambda-\mu} \cdot b_{\lambda-\mu}$ in $\mathbb{C}S_{d'}$ to the skew setting. The number $d' = \sum \lambda_i - \mu_i$ is the *weight* of the skew Young-diagram and the number of its columns is called its *length.* By the same token, we write the skew Schur-functor $\mathbb{S}_{\lambda-\mu}$, which takes a vector space V to the image $\mathbb{S}_{\lambda-\mu}V$ of $c_{\lambda-\mu}$ on $V^{\otimes d'}$. The characters of $\mathbb{S}_{\lambda-\mu}V$ are the skew Schur-polynomials [64, §1.5]:

$$S_{\lambda-\mu}(g) = S_{\lambda-\mu}(x_1, \ldots, x_n) := \det\left(E_{\lambda'_i-\mu'_j-i+j}\right),$$

where x_i are the eigenvalues of g as an endomorphism on V and E_k are k-th elementary symmetric polynomials as in (2.14). Even the generating vectors may be written down as in (2.9). For example, the projection of $v_T := v_6 \otimes v_4 \otimes (v_1 \wedge v_5) \otimes v_2 \cdot v_3$ is the canonical basis vector that corresponds to the skew tableau displayed above. If $\lambda'_i - \mu'_i = 0$ for some i, then we replace the corresponding vector by the scalar 1, as $\bigwedge^0 V \simeq \mathbb{F}$. For example, $(1^2 \otimes v)$ is an element of $\Gamma_{(3)-(2)}$.

The notions of $W_\lambda V$ and $\Lambda_\lambda V$ may be extended to skew Young-diagrams as well. To that purpose, we modify the definition of $\mathbf{a}(\lambda-\mu)$ and introduce a new tuple $\mathbf{b}(\lambda-\mu)$. Both are defined in a step-by-step manner. Set a_1 to be the number of consecutive columns with the same numbers μ'_ℓ and $\lambda'_\ell - \mu'_\ell$ starting from the left edge of the diagram. The number b_1 is then the number of boxes in each row, i.e. $\lambda'_\ell - \mu'_\ell$ for any $\ell = 1, \ldots, a_1$. Accordingly, set a_i to be the number of consecutive columns starting from the $(\sum_{\ell=1}^{i-1} a_\ell) + 1)$-th whereas b_i is the number of boxes in each such row. For example, for the above skew Young-diagram of $\lambda-\mu$, $\mathbf{a}(\lambda-\mu) = (1,1,1,2)$ and $\mathbf{b}(\lambda-\mu) = (1,1,2,1)$. Similarly to $\mathbb{S}_\lambda V$, the (not necessarily irreducible) representation $\mathbb{S}_{\lambda-\mu}V$ may be naturally embedded into

$$W_{\lambda-\mu}V := \mathrm{Sym}^{a_1}\left(\bigwedge^{b_1} V\right) \otimes \ldots \otimes \mathrm{Sym}^{a_m}\left(\bigwedge^{b_m} V\right)$$

and

$$\Lambda_{\lambda-\mu}V := \left(\bigwedge^{b_1} V\right)^{\otimes a_1} \otimes \ldots \otimes \left(\bigwedge^{b_m} V\right)^{\otimes a_m},$$

where $m = |\mathbf{a}(\lambda-\mu)| = |\mathbf{b}(\lambda-\mu)|$ is the length of both tuples.

Let $k \in \mathbb{N}$, $G \subset SL(n,\mathbb{C})$ be a closed linear subgroup, and $\lambda_1, \ldots, \lambda_k, \mu$ be skew Young-diagrams. Furthermore, let $X_1, \ldots, X_k$ be some finite-dimensional G-modules with G-invariant embeddings $\iota_i : X \to \Lambda_{\lambda_i}V$ and $\pi_\mu : \Lambda_\mu V \to Y$ be a G-invariant projection to a finite-dimensional G-module Y.

Definition 2.46. Given a G-invariant homomorphism $F : \Lambda_{\lambda_1}V \times \ldots \times \Lambda_{\lambda_k}V \to \Lambda_\mu V$, the *$G$-invariant λ-map* F_λ is defined by the following commutative diagram:

$$\begin{array}{ccc} X_1 \times \ldots \times X_k & \xrightarrow{(\iota_1,\ldots,\iota_k)} & \Lambda_{\lambda_1}V \times \ldots \times \Lambda_{\lambda_k}V \\ {\scriptstyle F_\lambda}\big\downarrow & & {\scriptstyle F}\big\downarrow \\ Y & \xleftarrow{\pi_\mu} & \Lambda_\mu V. \end{array} \tag{2.17}$$

Complying with the tradition, if $G = SO(n,\mathbb{C})$, the induced λ-map is denoted by $F_{[\lambda]}$.

Note that the same map F may induce different λ-maps F_λ depending on the embeddings ι_i, $i = 1, \ldots, k$, and the projection π_μ. In particular, the λ-map F_λ induced on $\Lambda_\lambda V$-type spaces by using trivial embeddings and projections coincides with the original G-invariant homomorphism F.

If not otherwise specified, we will assume for the $SL(n,\mathbb{C})$-modules $\Gamma_\lambda \simeq \mathbb{S}_\lambda V$ that ι_λ be the canonical embedding:

$$\iota_\lambda : \mathbb{S}_\lambda V \hookrightarrow \Lambda_\lambda V$$

given by composing the embedding $\mathbb{S}_\lambda V \hookrightarrow W_\lambda V$ and the embedding $W_\lambda V \hookrightarrow \Lambda_\lambda V$ defined in (2.5). Similarly, the projection π_λ will be assumed to be the canonical projection $\pi_\lambda : \Lambda_\lambda V \to \mathbb{S}_\lambda V$ given by the projection $W_\lambda V \twoheadrightarrow \mathbb{S}_\lambda V$ from Proposition 2.29 and the natural projection $\Lambda_\lambda V \twoheadrightarrow W_\lambda V$. By the above, both ι_λ and π_λ may be extended to skew Young-diagrams.

One example of $SL(n,\mathbb{C})$-invariant λ-maps – which will be used quite extensively in the present work – is the Kulkarni-Nomizu product.

Definition 2.47. The *(generalised) wedge product* is a product on the ring of all tensor products $\bigwedge^{a_1} V_1 \otimes \ldots \otimes \bigwedge^{a_k} V_k$, where $a_1, \ldots, a_k \in \mathbb{N}$ and $V_1, \ldots, V_k$ are arbitrary finite-dimensional vector spaces defined as follows:

$$\begin{array}{rcl} \bigwedge^{a_1} V_1 \otimes \ldots \otimes \bigwedge^{a_k} V_k \times \bigwedge^{b_1} V_1 \otimes \ldots \otimes \bigwedge^{b_k} V_k & \to & \bigwedge^{a_1+b_1} V_1 \otimes \ldots \otimes \bigwedge^{a_k+b_k} V_k \\ \left(v^{1,1}_{\mathbf{a}_1} \otimes \ldots \otimes v^{1,k}_{\mathbf{a}_k}, v^{2,1}_{\mathbf{b}_1} \otimes \ldots \otimes v^{2,k}_{\mathbf{b}_k} \right) & \mapsto & v^{1,1}_{\mathbf{a}_1} \wedge v^{2,1}_{\mathbf{b}_1} \otimes \ldots \otimes v^{1,k}_{\mathbf{a}_k} \wedge v^{2,k}_{\mathbf{b}_k}. \end{array}$$

where $v^{i,j}_{\mathbf{a}_j} = v^{i,j}_1 \wedge \ldots \wedge v^{i,j}_{a_j}$ and $v^{i,j}_1, \ldots, v^{i,j}_{a_j} \in V_j$, $i = 1, 2$, $j = 1, \ldots, k$.

Define the cup-product $\lambda_1 \cup \lambda_2$ for Young-diagrams λ_1, λ_2 as the Young-diagram whose conjugate is the componentwise sum of the conjugates $\lambda' + \mu$ [64, p. 6]. This operation may be extended to skew Young-diagrams $\gamma_1 := (\lambda_1 - \mu_1)$, $\gamma_2 := (\lambda_2 - \mu_2)$ as follows:

$$(\lambda_1 - \mu_1) \cup (\lambda_2 - \mu_2) := (\lambda_1 - \mu_1) \cup (\lambda_2 - \mu_2).$$

Then the generalised wedge-product restricts to a product

$$\wedge : \bigwedge_\lambda V \times \bigwedge_\mu V \to \bigwedge_{\lambda \cup \mu} V.$$

for λ,μ arbitrary (skew) Young-diagrams and V some finite-dimensional vector space. We will denote this generalised wedge-product simply by $\wedge$, since it coincides with the classical wedge-product whenever the latter is defined.

Definition 2.48. Assuming $V_1 = \cdots = V_k = V$, the *Kulkarni-Nomizu (wedge-)product* $\wedge_{KN}$ on the ring of $W_\lambda V$-type spaces is the λ-operation of the generalised wedge product via the canonical embeddings of $W_\lambda V$ into $\Lambda_\lambda V$.

Remark 2.49. This product is called after Kulkarni and Nomizu who used its special case $\mathrm{Sym}^2 V \times \mathrm{Sym}^2 V \to \mathrm{Sym}^2 \bigwedge^2 V$ to characterise the space of curvature tensors on differentiable manifolds [43, §3.K.1].

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_\ell)$ be arbitrary Young-diagrams and set $\lambda + \mu := (\lambda_1 + \mu_1, \ldots, \lambda_{\max(k,\ell)} + \mu_{\max(k,\ell)})$, where the shorter Young-diagram is filled with zeroes if necessary. Recall that any $w = w_1 \otimes \ldots \otimes w_k \in \bigwedge_\lambda V$, where $w_j = w_{j,1} \wedge \ldots \wedge w_{j,\lambda'_j} \in \bigwedge^{\lambda'_j} V$, may be written as $b_\lambda \cdot w'$, where $w' := w'_1 \otimes \ldots \otimes w'_k$, where $w'_j := w_{j,1} \otimes \ldots \otimes w_{j,\lambda'_j} \in V^{\otimes \lambda'_j}$.

Definition 2.50. The product

$$\begin{array}{rcl} \cdot : \bigwedge_\lambda V \times \bigwedge_\mu V & \to & \bigwedge_{\lambda+\mu} V, \\ (v, w) & \mapsto & v \wedge (b_\nu \cdot w'), \end{array}$$

where ν is the skew Young-diagram $(\lambda + \mu) - \mu$ and $b_\mu \in \mathbb{C}S_d$ is the partial symmetriser from Section 2.2, is called the *generalised dot-product.* The corresponding λ-operation

$$\begin{aligned} \cdot_{KN} : W_\lambda V \times W_\mu V &\to W_{\lambda+\mu} V, \\ (v, w) &\mapsto v \wedge_{KN} (b_\nu \cdot w'), \end{aligned}$$

on $W_\lambda V$-type spaces given by the canonical embeddings of $W_\lambda V$ into $\bigwedge_\lambda V$ is called *the Kulkarni-Nomizu dot-product.*

Remark 2.51. The notion "Kulkarni-Nomizu product" without the specifier "dot" or "wedge" always refers to the Kulkarni-Nomizu *wedge*-product.

Example 2.52. Let $\lambda = (p_1)$ and $\mu = (p_2)$. Then $\lambda + \mu = (p_1 + p_2)$ and ν is given by its conjugate $\nu' = (\underbrace{0, \dots 0}_{p_1 \text{ times}}, \underbrace{1, \dots 1}_{p_2 \text{ times}})$. Then, given $v \in \bigwedge_\lambda V = \mathrm{Sym}^{p_1} V$ and $w \in \bigwedge_\mu V = \mathrm{Sym}^{p_2} V$, $b_\nu \cdot w' = 1^{p_1} \otimes w$ and $v \cdot_{KN} w = v \wedge (1^{p_1} \otimes w) = v \cdot w = w \cdot_{KN} v$ is the usual symmetric product. Hence, we are justified to drop the KN subscript, which we will often do.

Replacing $v \in \mathrm{Sym}^{p_1} V$ with $v \in W_\lambda V$ for a Young-diagram $\lambda = (\lambda_1, \dots, \lambda_\ell)$ and $w \in \mathrm{Sym}^{p_2} V$ with $w \in W_\mu V$ for any $\mu = (\mu_1, \dots, \mu_m)$ such that $\lambda'_{\lambda_1} \geq \mu'_1$ (i.e. the first column of μ is not longer than the last column of λ), we obtain

$$v_\lambda \cdot_{KN} v_\mu = v_\lambda \wedge_{KN} (1^{\lambda_1} \otimes v_\mu) \in W_\nu V,$$

where ν is a Young-diagram with $\nu' := (\lambda'_1, \dots, \lambda'_{\lambda_1}, \mu'_1, \dots, \mu'_{\mu_1})$. As this example shows, the Kulkarni-Nomizu dot-product is not symmetric any more.

As a special case of this dot-product, consider $v_1 \in \mathrm{Sym}^{p_1} \bigwedge^i V$ and $v_2 \in \mathrm{Sym}^{p_2} \bigwedge^j V$, $p_1, p_2 \in \mathbb{N}$, $i \geq j$. Then

$$v_1 \cdot_{KN} v_2 := \begin{cases} v_1 \otimes v_2 \in \mathrm{Sym}^{p_1} \bigwedge^i V \otimes \mathrm{Sym}^{p_2} \bigwedge^j V & \text{if } i > j; \\ v_1 \cdot v_2 \in \mathrm{Sym}^{p_1+p_2} \bigwedge^i V & \text{if } i = j, \end{cases}$$

where $v_1 \cdot v_2$ in the case $i = j$ is the usual symmetric product.

Definition 2.53. The $SL(n, \mathbb{C})$-invariant λ-*wedge product* – or just the λ-*product* – $\wedge_\lambda$ on the ring $\bigoplus_\lambda \Gamma_\lambda$, where the sum is over all skew Young-diagrams λ, is given by the following commutative diagram:

$$\begin{array}{ccc} \mathbb{S}_\lambda V \times \mathbb{S}_\mu V & \xrightarrow{(\iota_\lambda, \iota_\mu)} & \Lambda_\lambda V \times \Lambda_\mu V \\ \wedge_\lambda \big\downarrow & & \wedge \big\downarrow \\ \mathbb{S}_{\lambda \cup \mu} V & \xleftarrow{\pi_{\lambda \cup \mu}} & \Lambda_{\lambda \cup \mu} V, \end{array} \tag{2.18}$$

where $\wedge$ represents the generalised wedge-product. As $\mathbb{S}_{\lambda \cup \mu} V \subset \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$, it follows by Theorem 2.6 that the λ-product is onto.

Invariant λ-maps are also used to embed a H-module X into a G-module Y if $H \subseteq G$ is a closed Lie-subgroup. If X has an H-invariant embedding into $\Lambda_\mu V$ and there exists a G-invariant projection to Y from $\Lambda_\lambda V$ such that $\lambda - \mu$ is a valid skew Young-diagram, then we may embed X into Y by means of the following map

$$\begin{aligned} \iota_{\bar{h}} : X &\hookrightarrow Y \\ v &\mapsto v \wedge_\lambda \bar{h}, \end{aligned} \tag{2.19}$$

where $\bar{h}$ is some element from the space of H-invariant elements $(\Lambda_{\lambda-\mu} V)^H$. Naturally, we call such embeddings λ-*embeddings*.

Remark 2.54. If $X = \Gamma_\lambda$ and $Y = \Gamma_\mu$, some authors speak about "filling up" boxes in $\lambda - \mu$ with the element $\bar{h}$ [41]. Equation (2.19) is an attempt to give this expression a geometrical meaning.

Being dependent on the chosen H-invariant element $\bar{h}$, the embedding $\iota_{\bar{h}}$ is not canonical in general, since there are no canonical H-invariant elements in $(\Lambda_{\lambda-\mu}V)^H$ for an arbitrary Lie-subgroup H. However, in the cases that are important for the present work, $G = SO(n, \mathbb{C})$ and the invariant spaces $(\Lambda_{\lambda-\mu}V)^H$ have a particularly simple structure for all relevant Lie-subgroups H.

Lemma 2.55. *Let X, Y be (not necessarily irreducible) $SO(n, \mathbb{C})$-modules. Then there exists a λ-embedding $\iota_\lambda : X \hookrightarrow Y$ if and only if there exist partitions $\mu \leq \lambda$ such that:*

- *there are an embedding $\iota_{[\mu]} : X \hookrightarrow \Lambda_\mu V$ and a projection $\pi_{[\lambda]} : \Lambda_\lambda V \twoheadrightarrow Y$ both invariant under $SO(n, \mathbb{C})$;*
- *the weight $|\lambda - \mu| = 2k$ is even and $b_i(\lambda - \mu) \leq k$ for all $i = 1, \dots, m$, where m is the length of $\lambda - \mu$.*

Proof. If $G = H = SO(n, \mathbb{C})$, then $(\Lambda_{\lambda-\mu})^G$ is either empty or generated by the symmetric bilinear form Q from (2.11). As Q is bilinear, it fills up two boxes of the diagram at once. Hence, $(\Lambda_{\lambda-\mu})^G$ for odd-weighted diagrams is empty and there is no $SO(n, \mathbb{C})$-invariant embedding from X into Y.

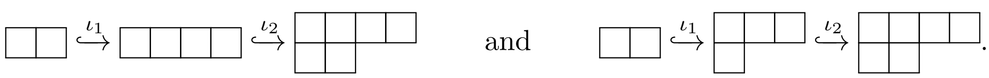
Furthermore, $e_T = -e_{T'}$, where T' is a skew Young-diagram obtained from T by interchanging two adjacent boxes in one column. On the other hand, Q is symmetric and, thus, may not fill up any two boxes in the same column of $\lambda - \mu$. Hence, if there is a column with more than k boxes, then we cannot split the diagram into k pairs of boxes with no pair being in the same column. Consequently, $(\Lambda_{\lambda-\mu})^G$ is empty and there is, again, no $SO(n, \mathbb{C})$-invariant embedding from X into Y. $\square$

If $X = \Gamma_\mu$ is irreducible and contained in Y with multiplicity m', then, by Schur's Lemma, there are m' linearly independent $SO(n, \mathbb{C})$-invariant embeddings $\Gamma_\mu \hookrightarrow Y$. Thus, imagining these embedding as a sequence of "interim" embeddings ι_i into $SO(n, \mathbb{C})$-modules X_i given by λ-wedge products with exactly one Q:

$$X \overset{\iota_1}{\hookrightarrow} X_1 \overset{\iota_2}{\hookrightarrow} \dots \overset{\iota_{k-1}}{\hookrightarrow} X_{k-1} \overset{\iota_k}{\hookrightarrow} Y,$$

each embedding is identified by the sequence of the interim modules $(X_1, \dots, X_{k-1})$. We will denote any such composite embedding by $\wedge_{[(i_1\, j_1),\dots,(i_k\, j_k)]}Q$, where $(i_p\, j_p)$ stands for the column indices of the boxes filled by Q at p-th step.

Example 2.56. Let $Y = \mathrm{Res}^{SL(n,\mathbb{C})}_{SO(n,\mathbb{C})}\Gamma_{(4,2)}$ and $X = \Gamma_{[2,0]}$. By Theorem 2.37, $m(X, Y) = 2$ and there exist two linearly independent $SO(n, \mathbb{C})$-invariant embeddings given by the sequences:



The left embedding sequence is denoted as $\wedge_{[(3\,4),(1\,2)]}$ whereas the one on the right is $\wedge_{[(1\,3),(2\,4)]}$.

Example 2.57. By Lemma 2.43, the space $\bigwedge^i V \otimes \bigwedge^j V$ is multiplicity-free as an $SO(n,\mathbb{C})$-module. Hence, for each $SO(n,\mathbb{C})$-submodule $X := \bigwedge^{i-k} V \otimes \bigwedge^{j-k} V$, $k \leq \min(i,j)$, there exists a unique $SO(n,\mathbb{C})$-invariant embedding $\iota : X \to \bigwedge^i V \otimes \bigwedge^j V$. We drop the subscript notation and write $\wedge Q^k$ in this case.

Lemma 2.58. *Let $H = SO(n-1,\mathbb{C})$, $G = SO(n,\mathbb{C})$, and $\Gamma^G_{[\lambda]}$ be a G-module. If $\Gamma^H_{[\mu]}$ is an irreducible H-submodule in $\mathrm{Res}^G_H \Gamma^G_{[\lambda]}$, then there exists a canonical – up to scaling – H-invariant embedding:*

$$\begin{array}{rcl} \iota_y : \Gamma^H_{[\mu]} & \to & \mathrm{Res}^G_H \Gamma^G_{[\lambda]} \\ v & \mapsto & v \wedge_{[\lambda]} y^{\lambda-\mu}, \end{array} \tag{2.20}$$

where $y^{\lambda-\mu} := b_{\lambda-\mu} \cdot y^{\otimes|\lambda-\mu|} \in (\Lambda_{\lambda-\mu} V)^H$.

Proof. Let $\Gamma^H_{[\mu]}$ be an irreducible H-submodule in $\mathrm{Res}\, \Gamma^G_{[\lambda]}$. By Theorem 2.36, $\mu_i \leq \lambda_i$. Hence, $\lambda - \mu$ is a valid skew Young-diagram and we may again construct embeddings by means of the λ-product. Let $y \in V$ be the element fixed by H. This gives us three H-invariant elements: Q, y, and Q_y, the latter being the H-invariant bilinear form on the orthogonal complement of the span of $y \in V$. As $Q = y^2 + Q_y$, the space $(\Lambda_{\lambda-\mu} V)^H$ is effectively generated by y and Q_y. We have just seen that Q_y is used to embed H-representations into other H-representations. Hence, we may only use y to construct the claimed embedding. Again, by the Branching Theorem, the multiplicity of $\Gamma^H_{[\mu]}$ in $\mathrm{Res}^G_H \Gamma_{[\lambda]}$ is one, hence, this embedding is canonical and is given by (2.20).

Note that this embedding is not trivial. First, $y \wedge v \neq 0$ for all $v \in \Gamma^H_{[\mu]}$, as y is linearly independent from v (otherwise, we would have a trivial H-submodule in $\Gamma^H_{[\mu]}$ spanned by y). On top of that, by the above-mentioned Branching Theorem, there is at most one box in every column of $\lambda - \mu$, hence, at most one y is put in each column. This concludes the proof. □

Remark 2.59. Of course, if $\Gamma^H_{[\mu]}$ is not a submodule of $\mathrm{Res}^G_H \Gamma^G_{[\lambda]}$, then, by Schur's Lemma, there are no H-invariant embeddings into $\mathrm{Res}^G_H \Gamma^G_{[\lambda]}$.

As the last application of λ-maps, we will study the construction of $\mathbb{S}_{[\lambda]} V$ in (2.12) in a more detailed way. Let λ be an arbitrary Young-diagram with the conjugate $\lambda' = (\lambda'_1, \ldots, \lambda'_\ell)$ and weight d. Recall that

$$\mathbb{S}_{[\lambda]} V := \mathbb{S}_\lambda V \cap V^{[d]},$$

where $V^{[d]}$ is the intersection of kernels of all trace maps $\mathrm{tr}_{q_1 q_2} : V^{\otimes d} \to V^{\otimes d-2}$ given for $1 \leq q_1 < q_2 \leq d$ by contracting the vectors in the q_1-th and the q_2-th factor with the metric tensor Q:

$$v_1 \otimes \ldots \otimes v_d \mapsto Q(v_{q_1}, v_{q_2})\, v_1 \otimes \ldots \otimes \hat{v}_{q_1} \otimes \ldots \otimes \hat{v}_{q_2} \otimes \ldots \otimes v_d.$$

By using the canonical embedding $\iota_\lambda : \Lambda_\lambda V \hookrightarrow V^{\otimes \lambda'_1} \otimes \ldots \otimes V^{\otimes \lambda'_\ell}$, we see that $\mathrm{tr}_{q_1 q_2}|_{\Lambda_\lambda V} = 0$, whenever the boxes q_1, q_2 are in the same column, as Q is symmetric (compare the proof of Lemma 2.55). Hence, the non-vanishing trace maps on $\Lambda_\lambda V$:

$$\mathrm{tr}^{q_i\, q_j}_{i\, j} : \bigwedge^{\lambda'_1} V \otimes \ldots \otimes \bigwedge^{\lambda'_i} V \otimes \ldots \otimes \bigwedge^{\lambda'_j} V \otimes \ldots \otimes \bigwedge^{\lambda'_\ell} V \to \bigwedge^{\lambda'_1} V \otimes \bigwedge^{\lambda'_i - 1} V \otimes \ldots \otimes \bigwedge^{\lambda'_j - 1} V \otimes \ldots \otimes \bigwedge^{\lambda'_\ell} V, \tag{2.21}$$

are given by contracting the vector in the q_i-th row of the i-th column and the vector in q_j-th row of the j-th column with Q:

$$\begin{aligned}
&\mathrm{tr}_{i\,j}^{q_i\,q_j}(t_1 \otimes v_1 \wedge \ldots \wedge v_{\lambda'_i} \otimes t_2 \otimes w_1 \wedge \ldots \wedge w_{\lambda'_j} \otimes t_3) \\
&\quad = \sum_{\sigma \in S_{\lambda'_i},\ \pi \in S_{\lambda'_j}} \mathrm{sgn}\,\sigma\ \mathrm{sgn}\,\pi\, Q(v_{\sigma(q_i)}, w_{\pi(q_j)}) \\
&\qquad\qquad t_1 \otimes v_{\sigma(1)\otimes\ldots\otimes\widehat{\sigma(q_1)}\otimes\ldots\otimes\sigma(\lambda'_i)} \otimes t_2 \otimes\ w_{\pi(1)\otimes\ldots\otimes\widehat{\pi(q_2)}\otimes\ldots\otimes\pi(\lambda'_j)} \otimes t_3 \\
&\quad = (-1)^{i+j-(q_i+q_j)} \sum_{\substack{r=1,\ldots,\lambda'_i,\, \sigma(\lambda'_i)=r \\ s=1,\ldots,\lambda'_j,\, \pi(\lambda'_j)=s}} \mathrm{sgn}\,\sigma\ \mathrm{sgn}\,\pi\, Q(v_r, w_s)\, t_1 \otimes v_{\sigma(1)\otimes\ldots\otimes\sigma(\lambda'_i-1)} \otimes t_2 \otimes w_{\pi(1)\otimes\ldots\otimes\pi(\lambda'_j-1)} \otimes t_3 \\
&\quad = (-1)^{q_i+q_j} \sum_{\substack{r=1,\ldots,\lambda'_i \\ s=1,\ldots,\lambda'_j}} (-1)^{r+s} Q(v_r, w_s)\, t_1 \otimes v_1 \wedge \ldots \wedge \widehat{v}_r \wedge \ldots \wedge v_{\lambda'_i} \otimes t_2 \otimes w_1 \wedge \ldots \wedge \widehat{w}_s \wedge \ldots \wedge w_{\lambda'_j} \otimes t_3,
\end{aligned}$$

where $t_1 \in \bigwedge^{\lambda'_1} V \otimes \ldots \otimes \bigwedge^{\lambda'_{i-1}} V$, $t_2 \in \bigwedge^{\lambda'_{i+1}} V \otimes \ldots \otimes \bigwedge^{\lambda'_{j-1}} V$, and $t_3 \in \bigwedge^{\lambda'_{j+1}} V \otimes \ldots \otimes \bigwedge^{\lambda'_\ell} V$. Furthermore, we use the shorthand notation $v_{i_1\otimes\ldots\otimes i_j} := v_{i_1} \otimes \ldots \otimes v_{i_j}$. Thus, for fixed i, j, all traces $\mathrm{tr}_{i\,j}^{q_i\,q_j}$ differ only by the sign $(-1)^{q_i+q_j}$. We may therefore omit the superscripts and assume $q_i = q_j = 1$ going forward, in which case $(-1)^{q_i+q_j} = 1$.

Definition 2.60. Let λ be a skew Young-diagram. The $SO(n,\mathbb{C})$-invariant trace map $\mathrm{tr}_{i\,j,\lambda}$ on $\Gamma_\lambda^{SL(n,\mathbb{C})}$ is defined to be the λ-map to the trace map $\mathrm{tr}_{i\,j}$ on $\Lambda_\lambda V$ introduced in (2.21).

May now refine (2.12):

$$\mathbb{S}_{[\lambda]} V = \bigcap_{i<j} \ker(\mathrm{tr}_{i\,j}) \cap \mathbb{S}_\lambda V \tag{2.22}$$

Let us denote the natural projection $\mathbb{S}_\lambda V \to \mathbb{S}_{[\lambda]} V$ by $\pi_{[\mathrm{tr}]}$. For $\ell = 2$, one can write it in a closed form, as there is only one trace map $\mathrm{tr}_\lambda = \mathrm{tr}_{1\,2,\lambda}$ whose kernel is to be computed. To this end, one needs the following interim result.

Lemma 2.61. *Given an element* $\tilde{\tau} \in \bigwedge^i V \otimes \bigwedge^j V$, $\mathrm{tr}(\tilde{\tau} \wedge Q)$ *is again in* $\bigwedge^i V \otimes \bigwedge^j V$ *and*

$$\mathrm{tr}(\tilde{\tau} \wedge Q^k) = (\mathrm{tr}\,\tilde{\tau}) \wedge Q^k + k\, C_{n+k-1,i+j}\, \tilde{\tau} \wedge Q^{k-1} \in \bigwedge^{i+k-1} V \otimes \bigwedge^{j+k-1} V.$$

Here, $C_{n,m} = (-1)^m (n-m)$ *and* $k \geq 1$.

Proof. The maps $\tilde{\tau} \mapsto \tilde{\tau} \wedge Q^k$ and tr are both linear, hence, it suffices to show the identity for

$$\tilde{\tau} := e_{\ell_{1,1}} \wedge \ldots \wedge_{\ell_{1,i}} \otimes e_{\ell_{2,1}} \wedge \ldots \wedge e_{\ell_{2,j}},$$

with $\ell_{s,t}$ being some positive integers, and e_ℓ, $\ell = 1, \ldots n$ being basis vectors of V orthonormal with respect to Q, i.e. such that $Q(e_r, e_s) = \delta_{rs}$ for all $r, s = 1, \ldots n$. Then $Q = \sum_{q=1}^n e_q \otimes e_q = \sum_{q=1}^n e_q^2$.

We show the first result inductively. For $k=1$, the following holds:

$$\begin{aligned}
\operatorname{tr}(\tilde\tau\wedge Q) &= \sum_q \operatorname{tr}(e_{11}\wedge\ldots\wedge e_{1i}\wedge e_q\otimes e_{21}\wedge\ldots\wedge e_{2j}\wedge e_q)\\
&= \sum_{q,r,s}(-1)^{r+s}\,Q(e_{1r},e_{2s})\,e_{11}\wedge\ldots\wedge\widehat{e}_{1r}\wedge\ldots\wedge e_{1i}\wedge e_q\otimes\; e_{21}\wedge\ldots\wedge\widehat{e}_{2s}\wedge\ldots\wedge e_{2j}\wedge e_q\\
&\quad+\sum_{q,s}(-1)^{i+1+s}\,Q(e_q,e_{2s})\,e_{11}\wedge\ldots\wedge e_{1i}\otimes\; e_{21}\wedge\ldots\wedge\widehat{e}_{2s}\wedge\ldots\wedge e_{2j}\wedge e_q\\
&\quad+\sum_{q,r}(-1)^{r+j+1}\,Q(e_{1r},e_q)\,e_{11}\wedge\ldots\wedge\widehat{e}_{1r}\wedge\ldots\wedge e_{1i}\wedge e_q\otimes e_{21}\wedge\ldots\wedge e_{2j}\\
&\quad+\sum_q(-1)^{i+j}Q(e_q,e_q)\,e_{11}\wedge\ldots\wedge e_{1i}\otimes e_{21}\wedge\ldots\wedge e_{2j}\\
&= \operatorname{tr}(\tilde\tau)\wedge Q+(-1)^{i+j+1}j\tilde\tau+(-1)^{i+j+1}i\tilde\tau+(-1)^{i+j}n\tilde\tau=(\operatorname{tr}\tilde\tau)\wedge Q+C_{n,i+j}\tilde\tau,
\end{aligned}$$

where $e_{st}=e_{\ell_{s,t}}$, $r=1,\ldots,i$, $s=1,\ldots,j$, $q=1,\ldots n$ for the sake of clarity. It follows by induction:

$$\begin{aligned}
\operatorname{tr}(\tilde\tau\wedge Q^k) &= \operatorname{tr}((\tilde\tau\wedge Q^{k-1})\wedge Q)=C_{n+2(k-1),i+j}\tilde\tau\wedge Q^{k-1}+\operatorname{tr}(\tilde\tau\wedge Q^{k-1})\wedge Q\\
&= C_{n+2(k-1),i+j}\tilde\tau\wedge Q^{k-1}+(k-1)\,C_{n+k-2,i+j}\tilde\tau\wedge Q^{k-1}+(\operatorname{tr}\tilde\tau)\wedge Q^k\\
&= k\,C_{n+k-1,i+j}\,\tilde\tau\wedge Q^{k-1}+(\operatorname{tr}\tilde\tau)\wedge Q^k.
\end{aligned}$$

□

Proposition 2.62. *For $0\le i,j\le n$ such that $i+j\le n+1$, the map*

$$\begin{aligned}
\pi_{\mathrm{tr}}:\textstyle\bigwedge^i V\otimes\bigwedge^j V &\to \textstyle\bigwedge^i V\otimes\bigwedge^j V \\
\tilde\tau &\mapsto \sum_{k=0}^{\min(i,j)}\frac{(-1)^k}{k!\prod_{r=1}^k C_{n+3r-1,i+j}}(\operatorname{tr}^k\tilde\tau)\wedge Q^k.
\end{aligned}\tag{2.23}$$

is a projection to the trace-free subspace of $\bigwedge^i V\otimes\bigwedge^j V$. Here, $\operatorname{tr}^k\tau$ is the k-fold application of the trace map tr.

In particular, given the tuple λ with $\lambda'=(i,j)$, the $SO(n,\mathbb{C})$-invariant λ-map

$$\pi_{[\mathrm{tr}]}:=\pi_{[\mathrm{tr},\lambda]}:\Gamma_\lambda^{SL(n,\mathbb{C})}\to\Gamma_\lambda^{SL(n,\mathbb{C})}$$

induced by π_{tr} is an $SO(n,\mathbb{C})$-invariant projection $\Gamma_\lambda^{SL(n,\mathbb{C})}\to\bar\Gamma_{[\lambda]}^{SO(n,\mathbb{C})}$.

Proof. Let $\tilde\tau\in\bigwedge^i V\otimes\bigwedge^j V$ be an arbitrary element and assume $j\le i$ without loss of generality. It is clear that $\prod_{r=1}^k C_{n+3r-1,i+j}\neq 0$, as $n\ge(i+j)$ by definition. This allows for the following computation:

$$\operatorname{tr}\left(\sum_{k=0}^{j}\frac{(-1)^k}{k!\prod_{r=1}^k C_{n+3r-1,i+j}}(\operatorname{tr}^k\tilde\tau)\wedge Q^k\right)=\sum_{k=0}^{j}\frac{(-1)^k}{\prod_{r=1}^k r\,C_{n+3r-1,i+j}}\operatorname{tr}((\operatorname{tr}^k\tilde\tau)\wedge Q^k)$$

$$=\operatorname{tr}\tilde\tau+\sum_{k=1}^{j}\frac{(-1)^k}{\prod_{r=1}^k r\,C_{n+3r-1,i+j}}\left(k\,C_{n+k-1,i+j-2k}(\operatorname{tr}^k\tilde\tau)\wedge Q^{k-1}+(\operatorname{tr}^{k+1}\tilde\tau)\wedge Q^k\right).$$

Since $C_{n+k-1,i+j-2k} = (-1)^{2k} C_{n+3k-1,i+j} = C_{n+3k-1,i+j}$, we obtain

$$= \sum_{k=1}^{j} \frac{(-1)^k}{\prod_{r=1}^{k-1} r\, C_{n+3r-1,i+j}} (\mathrm{tr}^k\, \tilde{\tau}) \wedge Q^{k-1} + \sum_{k=0}^{j-1} \frac{(-1)^k}{\prod_{r=1}^{k} r\, C_{n+3r-1,i+j}} (\mathrm{tr}^{k+1}\, \tilde{\tau}) \wedge Q^k$$
$$= 0.$$

If v is in the trace free subspace of $\bigwedge^i V \otimes \bigwedge^j V$, then $\mathrm{tr}^k\, v = 0$ for all $k > 0$. Thus, $\pi_{\mathrm{tr}} v = v$ and $\pi_{\mathrm{tr}}^2 = \pi_{\mathrm{tr}}$, which proves the first claim.

Now, the λ-map $\pi_{[\mathrm{tr}]} := \pi_\lambda \circ \pi_{\mathrm{tr}} \circ \iota_\lambda$ projects $\Gamma_\lambda \subset \bigwedge^i V \otimes \bigwedge^j V$ to its trace-free subspace which is precisely the $SO(n, \mathbb{C})$-module $\bar{\Gamma}_{[\lambda]} = \mathbb{S}_{[\lambda]} V$ by (2.22). This proves the second claim. □

For $\ell > 2$, the situation is more complicated. There are many linearly independent projections $\pi_{i\,j}$ which correspond to the trace maps $\mathrm{tr}_{i\,j}$ and which are incompatible with each other in the sense that $\mathrm{tr}_{i\,j} \circ \pi_{k\,l} \circ \pi_{i\,j} \neq 0$ if $(i, j) \neq (k, l)$. This renders the explicit computation of π_{tr} intractable. However, one has the following property of $\pi_{[\mathrm{tr}]}$.

Lemma 2.63. *Let Γ_λ be an irreducible $SL(n, \mathbb{C})$-module, $\Gamma_{[\mu]}$ be an $SO(n, \mathbb{C})$-submodule of* $\mathrm{Res}\, \Gamma_\lambda$*, and $\pi_{\mathrm{tr}} : W_\lambda V \to W_\lambda V$ be the projection from $W_\lambda V$ to its trace-free subspace. Then*

$$\pi_{[\mathrm{tr}]}(v \wedge_\lambda Q^{\lambda-\mu}) = 0$$

for all $v \in \Gamma_{[\mu]}$ and all $\mu \neq \lambda$.

Proof. If $\iota : \Gamma_{[\mu]} \to \Gamma_\lambda$ is any $SO(n, \mathbb{C})$-invariant embedding, then $\pi_{[\mathrm{tr}]} \circ \iota : \Gamma_{[\mu]} \to \Gamma_{[\lambda]}$ is an $SO(n, \mathbb{C})$-invariant map between $SO(n, \mathbb{C})$-modules. By Schur's Lemma, this map is not trivial if and only if $\lambda = \mu$. By the Branching Theorem 2.37 and Lemma 2.55, there always exists an $SO(n, \mathbb{C})$-invariant embedding $\iota_\mu : \Gamma_{[\mu]} \to \Gamma_\lambda$, $v \mapsto v \wedge_\lambda Q^{\lambda-\mu}$. The result now follows, as ι_μ is just a special case of ι. □

3 Valuations and Curvature Measures

In the previous Section, V was a complex vector space of complex dimension n. In this Section and throughout the entire remaining work, V will stand for a real vector space of real dimension n. Correspondingly, $SL(n) \simeq SL(n,\mathbb{R})$ and $SO(n) \simeq SO(n,\mathbb{R})$ will denote the special linear resp. orthogonal group with real coefficients.

3.1 Contact and Symplectic Geometry

Definition 3.1. An even-dimensional vector space V equipped with a bilinear alternating non-degenerate form ω is called a *symplectic* vector space. The form is called the *symplectic* or *fundamental* form.

Example 3.2. Consider a complex vector space $\mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$, $1 \leq j \leq n$. Then it is easy to see that the form

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j \tag{3.1}$$

is symplectic on $\mathbb{C}^n$ and $(\mathbb{C}^n, \omega)$ is, thus, a symplectic vector space. This symplectic structure is called the *standard – or Kähler – symplectic structure* of $\mathbb{C}^n$.

Definition 3.3. The operator

$$\begin{aligned} L : \textstyle\bigwedge^*(V^*) &\to \textstyle\bigwedge^*(V^*) \\ \tau &\mapsto \tau \wedge \omega \end{aligned} \tag{3.2}$$

is called the *Lefschetz operator*. L is of degree 2, i.e. $L(\bigwedge^k(V^*)) \subset \bigwedge^{k+2}(V^*)$.

Definition 3.4. Fixing an Euclidean scalar product $\langle \cdot, \cdot \rangle$ on V, the operator Λ uniquely determined by

$$\langle \Lambda\tau, \beta \rangle = \langle \tau, L\beta \rangle\,, \quad \forall \beta, \tau \in \textstyle\bigwedge^*(V^*) \tag{3.3}$$

is called the *dual Lefschetz operator*.

Lemma 3.5. *The operator Λ is of degree -2. Furthermore, one has $\Lambda = *^{-1} \circ L \circ *$.*

Proof. Let β be of degree k. Then $\beta \wedge \omega$ is of degree $k+2$. $\bigwedge^l(V^*)$ and $\bigwedge^k(V^*)$ are orthogonal to each other if $k \neq l$. Hence, in order for the right-hand side of (3.3) to be non-zero, α must be of degree $k+2$, too. On the other hand, $\Lambda\alpha$ must be of degree k. This proves the first assertion.

The second assertion follows from the definition of the Hodge $*$-operator, since

$$\begin{aligned} \langle \alpha, L\beta \rangle \,\mathrm{vol} &= \langle L\beta, \alpha \rangle \,\mathrm{vol} = L\beta \wedge *\alpha = \omega \wedge \beta \wedge *\alpha \\ &= \beta \wedge (\omega \wedge *\alpha) = \beta \wedge (* \circ *^{-1})(\omega \wedge *\alpha) \\ &= \left\langle \beta, (*^{-1} \circ L \circ *)\alpha \right\rangle. \end{aligned}$$

□

Definition 3.6. A k-linear form $\alpha \in \bigwedge^k(V^*)$ is called *primitive* if $\Lambda\alpha = 0$. The subspace of all primitive elements in $\bigwedge^k(V^*)$ is denoted by $\bigwedge_p^k(V^*) \subset \bigwedge^k(V^*)$.

The operators L, Λ, as well as the notion of (smooth) primitive forms can easily be generalised to the so-called *symplectic manifolds* in the similar pointwise fashion as the Hodge $*$-operator is extended from vector spaces to real differential manifolds.

Definition 3.7. A differentiable manifold is called *symplectic* if it is equipped with a global closed 2-form that is pointwise non-degenerated. This obviously implies that the dimension of symplectic manifolds is always even as it was the case for symplectic vector spaces. The space of (smooth) primitive forms on a symplectic manifold M is denoted by $\Omega_p^k(M)$

Closely related to symplectic manifolds are contact manifolds. In fact, some authors call contact geometry an odd-dimensional counterpart of symplectic geometry (cf. [33, p. 57]).

Definition 3.8. A *contact element* on a manifold M is a point $p \in M$, called the contact point, together with a tangent hyperplane at p, $Q_p \subset T_pM$, i.e. a codimension 1 subspace of T_pM.

A hyperplane $Q_p \subset T_pM$ is completely determined by a linear form $\alpha_p \in T_p^*M \setminus \{0\}$ that is unique up to some non-zero scalar. Indeed, if (p, Q_p) is a contact element, then $Q_p = \ker \alpha_p$. On the other hand, $\ker \alpha_p = \ker \alpha'_p$ if and only if $\alpha_p = \lambda \alpha'_p$. Now, let Q be a smooth field of contact hyperplanes on M defined by $Q(p) := Q_p$. Then $Q = \ker \alpha$ for an open subset $U \in M$ and some 1-form α called a locally defining 1-form for Q. This form is again unique up to a smooth nowhere vanishing function $f \in C^\infty(U)$.

Definition 3.9. A *contact structure* on M is a smooth field of tangent hyperplanes $Q \subset TM$ such that, for any locally defining 1-form α, $d\alpha|_Q$ is non-degenerate, i.e. symplectic. The pair (M, Q) is called a *contact manifold* and α is called a *local contact form*.

The restriction $d\alpha_p|_{Q_p}$ is symplectic on Q_p, which implies immediately that $\dim Q_p = 2n$ is even and $d\alpha_p^n|_{Q_p} \neq 0$ is a volume form on Q_p. Since $T_pM = \ker \alpha_p \oplus \ker d\alpha_p$, one has $\dim T_pM = 2n + 1$ is odd. Furthermore, if α is a global contact form, then $\alpha \wedge d\alpha^n$ is a volume form on M. An easy computation shows that the converse is also true.

Proposition 3.10. *Q is a contact structure if and only if $\alpha \wedge d\alpha^n \neq 0$ for every locally defining 1-form α.*

If there is a globally defined form α, one can obtain a unique vector field T on M such that the contraction $\iota_T(d\alpha) = 0$ and $\iota_T(\alpha) = 1$. Indeed, $\iota_T(d\alpha) = 0$ implies that $T \in \ker d\alpha$, which is one-dimensional, and $\iota_T\alpha = 1$ just normalises T.

Definition 3.11. The vector field T defined as above on a contact manifold M is called the *Reeb vector field.*

Example 3.12. $\mathbb{R}^{2n+1}$ with coordinates $(x^1, y^1, \cdots, x^n, y^n, z)$ and a contact form $\alpha := \sum_{i=1}^n x^i dy^i + dz$ is a contact manifold.

Another example is the sphere bundle $SV := V \times S(V)$ over a vector space V, where $S(V)$ is the unit sphere on V.

Lemma 3.13. *Let $p = (x, v) \in SV$. Then the form α defined pointwise by*

$$\alpha|_{(x,v)}(w) := \langle v, d\pi(w) \rangle,$$

where $\pi : SV \to V$ is the projection, is a contact form on SV.

Proof. Consider the embedding $SV \subset V \times V$ with coordinates $(x^1, \cdots, x^n, y^1, \cdots, y^n)$. Then $x = (x^1, \cdots, x^n)$, $y = (y^1, \cdots, y^n)$ and we write:

$$\alpha|_{(x,y)}(\cdot) = \langle y, d\pi(\cdot)\rangle = \sum_{i=1}^{n} y^i dx^i. \tag{3.4}$$

Taking the exterior derivative yields

$$d\alpha|_{(x,y)} = \sum_{i=1}^{n} dy^i \wedge dx^i. \tag{3.5}$$

One may now verify that

$$\alpha \wedge d\alpha^{n-1} = C_n \sum_{i_1,\dots i_n=1}^{n} y_{i_1} dx^{i_1} \wedge \ldots \wedge dx^{i_n} \wedge dy^{i_2} \wedge \ldots \wedge dy^{i_n},$$

where $C_n \neq 0$ is some constant dependent on n, does not vanish on SV. This fact, together with Proposition 3.10, concludes the proof. □

Lemma 3.14. *For any given point $p = (x, y) \in SV$, the Reeb vector field on $T_{(x,y)}SV = T_xV \times T_vS(V)$ is given by $T|_{(x,y)} = \sum_{i=1}^{n} y^i \frac{\partial}{\partial x^i}$.*

Proof. One calculates easily $\iota_T(\alpha) = \sum_{j=1}^{n}(y^i)^2 = 1$ and $\iota_T(d\alpha) = \sum_{j=1}^{n} y^j dy^j = 0$, since $v \in S(V)$. □

Definition 3.15. A (complex-valued) form $\omega \in \Omega^*(SV)$ is called *horizontal* if $\iota_T\omega = 0$. A form ω that can be written as $\tau \wedge \alpha$ is called *vertical.* The algebras of horizontal or vertical forms on SV are denoted by $\Omega_h^*(SV)$ and $\Omega_v^*(SV)$, respectively.

Going forward, we always assume differential forms on SV to be complex-valued. A smooth translation-invariant form ω on SV is said to be of bi-degree (i, j) if ω can be written as $\sum_a \tau_a \otimes \phi_a$ with $\tau_a \in \Omega^i(V)^V$ and $\phi_a \in \Omega^j(S(V))$. Clearly, $\omega \in \Omega^{i+j}(SV)^V$ and

$$\Omega^k(SV)^V = \bigoplus_{i+j=k} \Omega^i(V)^V \otimes \Omega^j(S(V)). \tag{3.6}$$

If the bi-degree of a form is important, we emphasize this by replacing the superscript $i + j$ with a pair of numbers (i, j) in the $\Omega^*(SV)^V$-notation. Furthermore, to lighten up the notation, we will write $\Omega^{i,j}$ rather $\Omega^{i,j}(SV)^V$ for the space of translation-invariant differential forms of bi-degree (i, j) on SV. In the same fashion, $\Omega_p^{i,j}$ will stand for the space of *primitive* translation-invariant horizontal forms of the same bi-degree. Since $\alpha \in \Omega_v^{1,0}$, the Lefschetz-operator L is of bi-degree $(1, 1)$ in this notation, hence:

$$\Omega_p^{i,j} = \Omega_h^{i,j} / L\Omega_h^{i-1,j-1}, \tag{3.7}$$

whenever $i + j \leq n$. Furthermore, observe that Hodge-$*$-operator on SV induces two finer operators on Ω^*: $*_1 : \Omega^{i,j} \to \Omega^{n-i,j}$ and $*_2 : \Omega^{i,j} \to \Omega^{i,n-j-1}$ given by applying the Hodge-$*$-operator on the $\Omega^i(V)^V$- resp. $\Omega^j(S(V))$-part of a differential form. Since, for any vertical translation-invariant form ω, both $*\omega$ and $*_1\omega$ are translation-invariant and horizontal, and vice versa, both operators yield isomorphisms $*_1 : \Omega_h^{i,j} \to \Omega_h^{n-1-i,j}$ and $*_2 : \Omega_h^{i,j} \to \Omega_h^{i,n-j-1}$.

To reduce a vertical form $\tau \wedge \alpha$ to a horizontal form, we use a contraction with the Reeb vector field ι_T. Indeed, $\iota_T(\tau \wedge \alpha) = (\iota_T \tau) \wedge \alpha + \tau \wedge (\iota_T \alpha) = \tau$ for any horizontal form τ. Hence, we may write for $\omega \in \Omega^{i,j}$:

$$\omega|_{(x,y)} \in \left(\Lambda^i T_y^* S^{n-1} \otimes \Lambda^j T_y^* S^{n-1}\right) \oplus \left(\Lambda^{i-1} T_y^* S^{n-1} \otimes \Lambda^j T_y^* S^{n-1} \otimes \mathbb{R}\right),$$

where the term $\mathbb{R}$ stands for the linear span of α. In particular, if $\omega \in \Omega_h^{i,j}$, then:

$$\omega|_{(x,y)} \in \Lambda^i T_y^* S^{n-1} \otimes \Lambda^j T_y^* S^{n-1}.$$

Therefore we will write in the following $\omega|_y$ instead of $\omega|_{(x,y)}$, whenever $\omega \in \Omega_h^{i,j}$ and $(x, y) \in SV$.

Let $SO(n-1)$ be a stabilizer of $SO(n)$ at any fixed point $y_0 \in S^{n-1}$. For $y \in S^{n-1}$, denote $W_y := T_y S^{n-1} \otimes \mathbb{C}$ the complexification of the tangent space $T_y S^{n-1}$. Then the following holds.

Lemma 3.16. *For all $i, j \in \mathbb{N}$,*

$$\Omega_h^{i,j} \simeq \mathrm{Ind}_{SO(n-1)}^{SO(n)} (\Lambda^i W_y^* \otimes \Lambda^j W_y^*)$$

Proof. The claim has been proved – with a slightly different notation – as Lemma 4.1 in [15]. For the reader's convenience, we reproduce the proof here.

Recall that the natural action of $SO(n)$ on the sphere bundle $SV \subset TV$ is given by:

$$g(x, v) := l_g(x, v) := (gx, gv)$$

for all $g \in SO(n)$, $(x, v) \in SV$. The induced action of $SO(n)$ on $\Omega^{i,j}$ is then $g\omega = l_{g^{-1}}^* \omega$ for $\omega \in \Omega^{i,j}$. Furthermore, the differential of the map $l_g : SV \to SV$ induces a linear isomorphism

$$\widehat{d_y l_g} := (d_y l_g)^* : \bigwedge^i W_{gy}^* \otimes \bigwedge^j W_{gy}^* \to \bigwedge^i W_y^* \otimes \bigwedge^j W_y^*,$$

which yields a natural representation of the group $SO(n-1)$ on the space $\bigwedge^i W_y^* \otimes \bigwedge^j W_y^*$ given by $\eta \mapsto \widehat{d_y l_{\eta^{-1}}}$.

Now, let $\omega \in \Omega_h^{i,j}$ and define:

$$\begin{aligned} f_\omega : SO(n) &\to \bigwedge^i W_y^* \otimes \bigwedge^j W_y^* \\ g &\mapsto \widehat{d_y l_g}(w|_{gy}). \end{aligned}$$

Then, for every $h \in SO(n-1)$,

$$f_\omega(gh) = \widehat{d_y l_{gh}}(\omega|_{(gh)y}) = \widehat{d_y l_g \, d_y l_h}(\omega|_{(gh)y}) = \widehat{d_y l_h}\, \widehat{d_y l_g}(\omega|_{gy}) = h^{-1} f_\omega(g).$$

By the definition of (the analytical form of) the induced representation, $f_\omega \in \mathrm{Ind} \bigwedge^i W_y^* \otimes \bigwedge^j W_y^*$. Conversely, if $f \in \mathrm{Ind} \bigwedge^i W_y^* \otimes \bigwedge^j W_y^*$, then we may define a horizontal form $\omega_f \in \Omega_h^{i,j}$ by:

$$\omega_f|_{gy} := \widehat{d_y l_g}(f(g)).$$

This form is well-defined: If $gy = g'y$ for $g, g' \in SO(n)$, then $\omega|_{gy} = \omega|_{g'y}$.

Now, observe that both $SO(n)$-invariant linear maps $\omega \mapsto f_\omega$ and $f \mapsto \omega_f$ are clearly inverse to each other. This finishes the proof. □

Corollary 3.17. *If $i+j \leq n-1$ and $\max(i,j) \geq (n-1)/2$, then there is an isomorphism of $SO(n)$-modules*

$$\Omega_p^{i,j} \oplus \mathrm{Ind}_{SO(n-1)}^{SO(n)}(\Lambda^{i-1}W_y^* \otimes \Lambda^{j-1}W_y^*) = \mathrm{Ind}_{SO(n-1)}^{SO(n)}(\Lambda^i W_y^* \otimes \Lambda^j W_y^*), \tag{3.8}$$

or, equivalently,

$$\Omega_p^{i,j} = \mathrm{Ind}_{SO(n-1)}^{SO(n)} \Lambda_p^{i,j} W_y^*, \tag{3.9}$$

where $\Lambda_p^{i,j}W_y^* := \bigoplus_{l=0}^{j} \bar{\Gamma}_{[2[l],1[n-1-(i+j)]]}$ is the space of primitive (i,j)-linear skew-symmetric forms on W_y.

Proof. We obtain the claims immediately from (3.7), the above Lemma, and Lemma 2.43 applied to $n-1$ instead of n. Note that, for $0 \leq j \leq i < (n-1)/2$, one has:

$$\bigwedge^i V \otimes \bigwedge^j V = \left(\bigoplus_{l=0}^{j} \bar{\Gamma}_{[2[l],1[i-j]]}\right) \oplus \left(\bigwedge^{i+1} V \otimes \bigwedge^{j-1} V\right),$$

and the decomposition from Lemma 2.43 is not compatible with (3.7), as the second term $\left(\bigwedge^{i+1} V \otimes \bigwedge^{j-1} V\right)$ may not be embedded into $\left(\bigwedge^i V \otimes \bigwedge^j V\right)$ by taking the wedge-product with $d\alpha$ which is of bi-degree $(1,1)$. Hence, the condition $\max(i,j) \geq (n-1)/2$ is essential for the claim's validity. □

3.2 Classical Valuation Theory

We denote by $\mathcal{K}(V)$ the set of convex bodies, i.e. compact convex sets, equipped with the Hausdorff-metric, in a vector space V of dimension $n < \infty$. Furthermore, we write $s_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ for the volume of an n-dimensional sphere and ω_n for the volume of an n-dimensional ball.

Using the usual notation $K+L$ for the Minkowski-addition of two convex bodies K, L and $(-K)$ for the reflection of a body K through the origin, we start with the following definition.

Definition 3.18. A *valuation* is a functional $\phi : \mathcal{K}(V) \to (A,+)$, where $(A,+)$ is an Abelian half-group, with the property that, whenever $K, L, K \cup L \in \mathcal{K}(V)$, then

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L). \tag{3.10}$$

Valuations with values in $SO(n)$-modules are called *tensor-valued valuations*. If $(A,+)$ is a trivial representation of $SO(n)$, the valuation is also referred to as a *scalar-valued* valuation.

In this work, valuations are always assumed to be continuous with respect to the Hausdorff-topology. There are several other properties that play a significant role in the course of the present work.

Definition 3.19. Let ϕ be a continuous valuation. Whenever the corresponding operations are defined, we say that:

1. ϕ is *k-homogeneous* – or *of degree k* - if $\phi(tK) = t^k\phi(K)$ with $K \in \mathcal{K}(V)$, $t > 0$, $0 \leq k \leq n$.

2. ϕ is said to be *G-covariant* for a Lie-group G if $\phi(gK) = g\,\phi(K)$, $\forall K \in \mathcal{K}(V)$, $\forall g \in G$. A similar property is the *G-invariance* stipulating that $\phi(gK) = \phi(K)$ for all K and g. In particular, ϕ is called *translation-invariant* if $\phi(K+x) = \phi(K)$, $\forall K \in \mathcal{K}(V)$, $\forall x \in V$.

3. ϕ is called *even* if $\phi(-K) = \phi(K)$, $\forall K \in \mathcal{K}(V)$ and *odd* if $\phi(-K) = -\phi(K)$, $\forall K \in \mathcal{K}(V)$.

4. ϕ is called *simple* if $\phi(K) = 0$, $\forall K \in \mathcal{K}(V), \dim K < n$.

Remark 3.20. For trivial G-representations $(A, +)$, the notions of G-covariance and G-invariance coincide.

It is clear that the scalar-valued translation-invariant valuations constitute a vector space which we denote by $\mathrm{Val}(V)$ or just Val if there is no risk of confusion. Denoting by $\overline{G} := G \ltimes T(V)$ the semi-direct product of G extended by the group of translations $T(V)$ on V, the subspace of $\overline{G}$-invariant valuations is referred to as Val^G. Further, we write Val_k for the space of translation-invariant valuations of degree k. The subspaces of even or odd valuations in Val_k are denoted by Val_k^+ or Val_k^-, respectively. General translation-invariant tensor-valued valuations are denoted by TVal while valuations with values in a fixed $SO(n)$-module Γ are denoted by TVal_Γ. One has the following identification of $SO(n)$-modules:

$$\mathrm{TVal}_{k,\Gamma} = \mathrm{Val}_k \otimes \Gamma. \tag{3.11}$$

Example 3.21. 1. The Euler-characteristic χ is an even, $\overline{SO(n)}$-invariant scalar-valued valuation of degree 0 while the volume vol is of degree n.

2. Let Δ be an equilateral triangle in $\mathbb{R}^2$. Then $\mu(K) := \mathrm{vol}(K + \Delta) - \mathrm{vol}(K + (-\Delta))$ is a simple odd scalar-valued valuation on $\mathbb{R}^2$ of degree 1. It is not $SO(n)$-invariant, however.

Definition 3.22. A map $\Phi : \mathcal{K}(V) \times \mathcal{B}(V) \to A$ is called a *curvature measure* of K if

- Φ is *locally defined*, i.e. if $U \subset V$ is open and $K_1, K_2 \in \mathcal{K}(V)$ such that $K_1 \cap U = K_2 \cap U$, then $\Phi(K_1, U') = \Phi(K_2, U')$ for all open sets $U' \subset U$.
- $K \mapsto \Phi(K, S)$ is a valuation for a fixed Borel set $S \subset V$.
- $S \mapsto \Phi(K, S)$ is a Borel measure for a fixed $K \in \mathcal{K}(V)$.

A range of valuation-inherent properties, such as translation-invariance and G-covariance, can be generalised to curvature measures in a natural way. Recall that, for a fixed convex compact body K, $\Phi(K, \cdot)$ is an A-valued measure. We may then define the weak continuity of Φ as follows.

Definition 3.23. Let A be an $\mathbb{R}$-vector space. A curvature measure $\Phi : \mathcal{K}(V) \times \mathcal{B}(V) \to A$ is called *weakly continuous* if, for each sequence $(K_i)_{i \in \mathbb{N}}$ of convex bodies in $\mathcal{K}(\mathbb{R}^n)$ converging to a convex body K, the relation

$$\lim_{i \to \infty} \int_{S\mathbb{R}^n} f \, d\Phi(K_i, \cdot) = \int_{S\mathbb{R}^n} f \, d\Phi(K, \cdot)$$

holds for all continuous functions $f : S\mathbb{R}^n \to \mathbb{R}$. The integrals are defined component-wise.

We write Curv for the space of all weakly continuous translation-invariant and Curv^G for the subspace of $\overline{G}$-invariant curvature measures. Similarly to the above, Curv_k is the space of weakly continuous translation-invariant curvature measures of degree k and TCurv stands for weakly continuous translation-invariant tensor-valued curvature measures. Again, we have the following isomorphism between the $SO(n)$-modules:

$$\mathrm{TCurv}_{k,\Gamma} = \mathrm{Curv}_k \otimes \Gamma. \tag{3.12}$$

Furthermore, there exists a natural map between curvature measures and valuations.

Definition 3.24. The map

$$\begin{aligned} \text{glob} : \text{Curv} &\to \text{Val} \\ \Phi(\cdot,\cdot) &\mapsto \Phi(\cdot, V) \end{aligned} \tag{3.13}$$

is called the *globalisation map.*

Following the examples above, a natural question arises whether one can characterise all $\overline{SO(n)}$-invariant valuations. It turns out that only a finite number of linearly independent valuations span the vector space $\text{Val}^{SO(n)}$.

Theorem 3.25 (Hadwiger's Theorem)**.**

$$\text{Val}^{SO(n)} = \text{span}(\mu_0^V, \cdots, \mu_n^V). \tag{3.14}$$

The valuation $\mu_k^V \in \text{Val}_k$ is the well-known *k-th intrinsic volume* that may be defined by the following *projection formula*

$$\mu_k^V(K) := \begin{bmatrix} n \\ k \end{bmatrix} \int_{Gr_k(V)} \text{vol}_E(\pi_E K)\, dE, \tag{3.15}$$

where $Gr_k(V)$ is the manifold of all k-flats in V and $\begin{bmatrix} n \\ k \end{bmatrix} := \binom{n}{k} \frac{\omega_n}{\omega_k \omega_{n-k}}$ is the so-called flag coefficient. The intrinsic volumes are so normalised as to yield

$$\mu_k^V(K) = \mu_k^W(K) = \text{vol}_k(K) \tag{3.16}$$

for all $K \subseteq W \subseteq V$ with $\dim K = k$, which explains the adjective "intrinsic", i.e. independent of the embedding.

Relaxing the requirements to mere translation-invariance yields an infinite-dimensional vector space Val that can additionally be graded by the homogeneity degree and the parity.

Theorem 3.26 (McMullen's decomposition)**.**

$$\text{Val} = \bigoplus_{k=0}^{n} \text{Val}_k = \bigoplus_{\substack{k=0 \\ \varepsilon \in \{-,+\}}}^{n} \text{Val}_k^{\varepsilon}. \tag{3.17}$$

Note that only the spaces Val_0 and Val_n are finite-dimensional with $\dim \text{Val}_0 = \dim \text{Val}_n = 1$. They are spanned by the Euler-characteristic and the Lebesgue-measure, respectively.

The theory of weakly continuous curvature measures is somewhat less developed. For $\overline{SO(n)}$-invariant curvature measures, there exists a Hadwiger-type theorem asserting that:

$$\text{Curv}^{SO(n)} = \bigoplus_{k=0}^{n} \text{Curv}_k^{SO(n)}, \tag{3.18}$$

where $\dim \text{Curv}_k^{SO(n)} = 1$ for all k in the sum. The element Φ_k that spans $\text{Curv}_k^{SO(n)}$ is called the k-th *(intrinsic) curvature measure.* Hence, intrinsic volumes are precisely the globalisations of intrinsic curvature measures. On the contrary, a McMullen-type decomposition is only known for a subclass of weakly continuous translation-invariant curvature measures called *smooth* curvature measures. Before defining them, we need to introduce several concepts.

Consider an arbitrary n-dimensional compact convex set $K \in \mathcal{K}(V)$ with a smooth boundary and a Borel set $U := W \times S(V)$ with $W \in \mathcal{B}(V)$. Recall from elementary differential geometry that the map

$$\begin{aligned} n_K : \partial K &\rightarrow S(V) \\ x &\mapsto n(x) \end{aligned} \tag{3.19}$$

that assigns an outer normal vector to each point $x \in \partial K$ is called the *Gauss map*. It is continuously differentiable and induces the self-adjoint map

$$\begin{aligned} W_K : T_x\partial K &\rightarrow T_{n_K(x)}S(V) \simeq T_x\partial K \\ v &\mapsto dn_K|_x(v) \end{aligned} \tag{3.20}$$

called the *Weingarten map*. The $n-1$ eigenvalues $k_1, \cdots, k_{n-1}$ of W_K are then called the *principal curvatures* of K. Then one can show by applying the area formula:

$$\Phi_i(K, W) = \frac{1}{(n-i)\omega_{n-i}} \int_{\partial K \cap W} E_{n-i-1}(k_1, \cdots, k_{n-1}) \, \mathrm{vol}_{n-1}(x),$$

where E_j is the j-th elementary symmetric function. Thus, $\Phi_i(K, W)$ is the result of symmetrising the principal curvatures of K. Furthermore, one obtains by globalising:

$$\mu_i(K) = \frac{1}{(n-i)\omega_{n-i}} \int_{\partial K} E_{n-i-1}(k_1, \cdots, k_{n-1}) \, \mathrm{vol}_{n-1}(x).$$

Clearly, the above equations cannot be generalised to convex bodies, since their boundary is in general not smooth enough to integrate differential forms over it. However, there is a means of "mollifying" the boundary.

Definition 3.27. The *normal cycle* $\mathrm{nc}(K) \subset SV$ is defined as follows:

$$\mathrm{nc}(K) := \{(x, y) \in SV | \langle x - x', y \rangle \geq 0, \forall x' \in K\}. \tag{3.21}$$

Single elements $(x, y) \in \mathrm{nc}(K)$ are then called *support elements* of K.

Proposition 3.28 ([94])**.** *For a convex body K, $\mathrm{nc}(K)$ is an $(n-1)$-dimensional Lipschitz manifold.*

Remark 3.29. The normal cycle $\mathrm{nc}(K)$ is a so-called *Legendrian manifold*, i.e. the global contact form defined for SV in Lemma 3.13 vanishes identically on $\mathrm{nc}(K)$. Consider a curve $c : \mathbb{R} \rightarrow \mathrm{nc}(K)$, $s \mapsto (x(s), y(s))$. As $c'(s) = (x'(s), y'(s)) \in T_{(x(s), y(s))} \mathrm{nc}(K)$, one has $\alpha_{c(0)}(c'(0)) = \langle y(0), x'(0) \rangle$. At every point $(x, y) \in \mathrm{nc}(K)$, y is a normal vector to the boundary whereas x' is a tangent vector. Hence, $x' \perp y$ and α vanishes identically on $\mathrm{nc}(K)$.

Consider the sets of type $K_r := K + rB$ for $r > 0$ that are called *r-tubes* or *parallel sets* of K. Defining the exponential map by:

$$\begin{aligned} \exp : SV \times \mathbb{R} &\rightarrow V \\ ((x, y), t) &\mapsto x + ty, \end{aligned}$$

one easily sees that $K_r \setminus K = \exp(\mathrm{nc}(K) \times (0, r])$. In particular, by Proposition 3.28, exp is a bi-Lipschitz homeomorphism $\mathrm{nc}(K) \times (0, \infty) \rightarrow V \setminus K$. Likewise, one may define localised r-tubes $M_r(K,U) := \exp(\mathrm{nc}(K) \cap \pi^{-1}(U)) \times (0,r])$ for any Borel-set $U \subset V$. Here, $\pi : SV \simeq V \times S(V) \to V$ is the projection on the first factor. Then one obtains:

$$\mathrm{vol}\, M_r(K,U) = \mathrm{vol}(K \cap U) + \int_{(\mathrm{nc}(K)\cap\pi^{-1}(U))\times(0,r]} \exp^* \mathrm{vol}\,. \tag{3.22}$$

Computing the pullback $\exp^* \mathrm{vol}$ in local coordinates (x^i, y^i) yields an n-form on SV:

$$\exp^* \mathrm{vol} = d(x^1 + ty^1) \wedge \cdots \wedge d(x^n + ty^n),$$

The form is $SO(n)$-invariant, since vol is $SO(n)$-invariant and exp commutes with $g \in SO(n)$. Since $\overline{SO(n)}$ acts transitively on SV, an $\overline{SO(n)}$-invariant form on SV is determined by its value at any point $(x,y) \in SV$. Choose $x = 0$ and $y = (0, \cdots, 0, 1) =: e_n$. Then $dy^n = 0$ and $y^i = 0, \forall 0 \leq i \leq n-1$. As $\alpha_{(0,e_n)} = dx^n$, one has:

$$\begin{aligned}\exp^* \mathrm{vol}_{(0,e_n)} &= (dx^1 + t dy^1) \wedge \cdots \wedge (dx^{n-1} + t dy^{n-1}) \wedge (\alpha + dt) \\ &= \sum_{i=0}^{n-1} t^i \rho_{n-i-1} \wedge (\alpha + dt)\end{aligned}$$

for some $\overline{SO(n)}$-invariant $(n-1)$-forms ρ_{n-i-1}. By the remark above, α vanishes identically on any normal cycle. Thus, (3.22) may be rewritten as

$$\mathrm{vol}\, M_r(K,U) = \mathrm{vol}(K \cap U) + \sum_{i=0}^{n-1} \frac{r^{i+1}}{i+1} \int_{\mathrm{nc}(K)\cap\pi^{-1}(U)} \rho_{n-i-1}.$$

By homogeneity arguments, one can show that

$$\begin{aligned}\Phi_i(K,U) &= \frac{1}{(n-i)\omega_{n-i}} \int_{\mathrm{nc}(K)\cap\pi^{-1}(U)} \rho_{n-i-1}, \quad 0 \leq i \leq n-1, \\ \Phi_n(K,U) &= \mathrm{vol}(K \cap U).\end{aligned}$$

The form ρ_{n-i-1} is often called the *i-th Lipschitz-Killing curvature form.*

Definition 3.30. A translation-invariant valuation ϕ is called *smooth* if, for all $K \in \mathcal{K}(V)$,

$$\phi(K) = \int_K \beta + \int_{\mathrm{nc}(K)} \omega,$$

where $\beta \in \Omega^n(V)^V$ and $\omega \in \Omega^{n-1}(SV)^V$. Likewise, a translation-invariant curvature measure Φ is called smooth if, for all $K \in \mathcal{K}(V)$ and all $U \in \mathcal{B}(V)$,

$$\Phi(K,U) = \int_{K\cap U} \tilde{\beta} + \int_{\mathrm{nc}(K)\cap\pi^{-1}(U)} \tilde{\omega},$$

where $\tilde{\beta} \in \Omega^n(V)^V$, $\tilde{\omega} \in \Omega^{n-1}(SV)^V$, and $\pi : SV \simeq V \times S(V) \to V$ is the projection on the first factor.

Proposition 3.31. *Denoting the space of smooth curvature measures by* Curv^{sm}*, one has the following decomposition:*

$$\mathrm{Curv}^{sm} = \bigoplus_{k=0}^{n} \mathrm{Curv}_k^{sm} = \bigoplus_{\substack{k=0 \\ \varepsilon\in\{-,+\}}}^{n} \mathrm{Curv}_k^{sm,\varepsilon}, \tag{3.23}$$

where $\mathrm{Curv}_k^{sm,\varepsilon}$ *is the space of all smooth translation-invariant k-homogeneous curvature measures of parity ε.*

3.3 Modern Valuation Theory and $SO(n)$-Representations

Theorem 3.32 (Alesker's Irreducibility Theorem). *Let $\varepsilon \in \{-,+\}$ and $\phi : GL(n) \to GL(\mathrm{Val}_k^\varepsilon)$ be the representation of $GL(n)$ on $\mathrm{Val}_k^\varepsilon$ with $(\phi(g)\mu)(K) := \mu(g^{-1}K)$, where $\mu \in \mathrm{Val}_k^\varepsilon$ and $g \in GL(n)$. Then this representation is irreducible.*

It follows from the definition of a smooth valuation that a $(k, n-k-1)$-form on SV induces a valuation of degree k. Since $g\beta \in \Omega^{k,n-k-1}$ for any $g \in GL(n)$, we immediately obtain by applying Theorem 3.32.

Proposition 3.33. Val_k^{sm} *is dense in* Val_k.

Unfortunately, there is no analogon of the Irreducibility Theorem for weakly continuous curvature measures. Neither is it known whether smooth curvature measures lie dense in Curv. Hence, whereas we often may – and will – replace Val_k^{sm} with Val_k by density arguments, one must carefully specify whether a claim is true for smooth or for weakly continuous curvature measures. Going forward, all curvature measures will be assumed to be smooth unless stated otherwise. Correspondingly, the following results hold exclusively for *smooth* curvature measures.

We have seen by Remark 3.29 that the integration operator

$$\mathrm{integ} : \Omega^n(V)^V \times \Omega^{n-1} \to \mathrm{Val}$$

which yields a valuation from a given pair of forms has a non-trivial kernel whose structure may be described by rather elementary means.

Definition 3.34. For each $(n-1)$-form ω on SV, there is a unique vertical form $\alpha \wedge \tau$ such that $d(\omega + \alpha \wedge \tau)$ is vertical. The operator $D\omega := d(\omega + \alpha \wedge \tau)$ is called the *Rumin-differential.*

Proposition 3.35. *A form $\omega \in \Omega^{n-1}$ induces a trivial valuation if and only if $D\omega = 0$ and ω does not induce a multiple of the Euler-characteristic.*

Sketch of a proof. Since $\mathrm{nc}(K)$ is a manifold without boundary, it follows by virtue of Stokes' Theorem that $\mathrm{nc}\, K(d\omega) = 0$ for any exact form $\omega \in \Omega^{n-1}$. More generally, we may modify ω by a vertical form $\alpha \wedge \tau$ without changing the value of the original integral. If $d(\omega + \alpha \wedge \tau) = 0$, then ω induces a trivial valuation as well unless $\omega + \alpha \wedge \tau$ is the representative of the non-trivial $(n-1)$-th deRham-cohomology class on SV. By the Künneth-formula, the representatives of the non-trivial cohomology class are given by multiples of the volume form on the spherical part of $SV = V \times S(V)$, i.e. multiples of the form that induces the Euler-characteristic.

Finally, $\mathrm{nc}(K)(d\alpha \wedge \tau) = 0$ for all $\tau \in \Omega^{n-3}(SV)^V$, since

$$\int_{\mathrm{nc}(K)} (d\alpha \wedge \tau) = \int_{\mathrm{nc}(K)} d(\alpha \wedge \tau) + \int_{\mathrm{nc}(K)} \alpha \wedge d\tau = 0.$$

The "only if" part is harder to show and we refer the reader to [22] instead. ☐

The following result is one of the key ingredients in proving the following decomposition of Val.

Theorem 3.36 (Harmonic Decomposition of Val). *Let $0 \le k \le n$. The space Val_k is the direct sum of the irreducible representations of $SO(n)$ corresponding to the integer tuples $(\lambda_1, \ldots, \lambda_{\lfloor n/2 \rfloor})$ such that:*

1. $\lambda_j = 0$ *for* $j > \min\{k, n-k\}$*;*

2. $|\lambda_j| \neq 1$ *for* $1 \leq j \leq \lfloor n/2 \rfloor$*;*

3. $|\lambda_2| \leq 2$.

In particular, under the action of $SO(n)$, the space Val_k *is multiplicity-free.*

Definition 3.37. An irreducible $SO(n)$-representation satisfying the above conditions is called *valuative.*

The isomorphism (3.11) allows us to reformulate the above Theorem in terms of tensor-valued $SO(n)$-covariant valuations.

Theorem 3.38. *Let $0 \leq k \leq n$. There exists a unique up to scaling non-trivial translation-invariant and $SO(n)$-covariant valuation of degree k with values in an irreducible $SO(n)$-representation Γ_λ if and only if λ satisfies all the conditions from Theorem 3.36.*

Proof. See Theorem 1′ in [15]. □

For curvature measures, the kernel of the similar integration map Integ – written with a capital 'I' – has an even simpler structure being spanned exactly by vertical forms and multiples of $d\alpha$.

Lemma 3.39. *The spaces* Curv_k^{sm} *and* $\Omega_p^{k,n-1-k}(SV)$ *from equation (3.9) are isomorphic as $SO(n)$-modules.*

Two families of $SO(n)$-covariant tensor-valued valuations and curvature measures are understood particularly well: those with values in $\Gamma = \mathrm{Sym}^2 \Lambda^q V$ and in $\Gamma = \mathrm{Sym}^p V$.

Theorem 3.40 ([19])**.** *Let $0 \leq q \leq k \leq n-1$. Define for each point $(x, y) \in SV$ a differential form $\Psi_{k,q}^n := \tilde{C}_{n,k,q} \overline{\Psi}_{k,q}^n$, where $\tilde{C}_{n,k,q} = \frac{-1}{(n-k-1)!q!(k-q)!\, s_{n-k-1}}$ and*

$$\overline{\Psi}_{k,q}^n = \sum_{\substack{i_1,\ldots,i_q=1 \\ \pi \in S_n}}^{n} \mathrm{sgn}(\pi)\, y_{\pi(1)}\, dx^{i_1 \cdots i_q \pi(q+2) \cdots \pi(k+1)} \wedge dy^{\pi(k+2) \cdots \pi(n-1)} \otimes e_{i_1 \otimes \cdots \otimes i_q} \otimes e_{\pi(2) \otimes \cdots \otimes \pi(q+1)},$$

where $e_{i_1 \otimes \cdots \otimes i_q} := e_{i_1} \otimes \cdots \otimes e_{i_q}$. Then the form $\Psi_{k,q}^n$ induces an $SO(n)$-covariant curvature measure in $\mathrm{Curv}_k^{sm} \otimes \mathrm{Sym}^2 \Lambda^q V$ which we denote by the same symbol.

The space $(\mathrm{Val} \otimes \mathrm{Sym}^p V)^{SO(n)}$ has been studied even more extensively. There exists a full characterisation of all valuations under the even weaker assumption of *isometry covariance.* We only state here the special cases of translation-invariant and $SO(n)$-covariant valuations.

Theorem 3.41 ([51])**.** *The space of $\mathrm{Sym}^p V$-valued valuations of degree $0 \leq k \leq n$ is spanned by the valuations $Q^m \phi_k^{0,s}$ with $s + 2m = p$. These are induced by the differential forms defined pointwise at $(x, y) \in SV$ as*

$$Q^m \Phi_k^{0,s} := c_{n,k}^{0,s}\, \rho_k \otimes Q^m y^s,$$

where $c_{n,k}^{0,s} := \frac{1}{k!(n-k-1)!s!s_{n-k+s-1}}$ and $Q = \sum_{i=1}^n e_i^2$ is the metric tensor preserved by $SO(n)$.

Remark 3.42. The valuations $Q^m\Phi_k^{0,s}$ from the above Theorem are subject to the so-called *McMullen-relations* named after Peter McMullen who proved them in [66]. For $k > 0$, their formulation involves isometry-covariant valuations. Therefore we omit them here and refer to [51] instead. For $k = 0$, these relations boil down to the identity:

$$2\pi s\, \phi_0^{0,s} = Q\phi_0^{0,s-2}$$

for all $s > 1$. For $s = 1$, they state that $\phi_0^{0,1} = 0$ while for $s = 0$ the relations are trivial.

As the following result shows, $Q^m\Phi_k^{0,s}$ turn out to be linearly independent as Sym^p-valued curvature measures and form a basis – rather than just a set of generators – of $\mathrm{Curv} \otimes \mathrm{Sym}^p$.

Theorem 3.43 ([49])**.** *The structure of the space of* $\mathrm{Sym}^p V$*-valued curvature measures of degree* $0 \le k \le n-1$ *depends on the homogeneity degree* k*:*

1. *if* $k \in \{0, n-1\}$*, then the basis is given by the curvature measures induced by the forms* $Q^m\Phi_k^{0,s}$ *from Theorem 3.41;*

2. *if* $0 < k < n-1$*, then the basis is given by* $Q^m\Phi_k^{0,s} =: Q^m\Phi_k^{0,s,0}$ *as above* and *by curvature measures* $Q^m\Phi_k^{0,s,1}$*,* $s + 2m = p - 2$*, induced by the differential forms*

$$C_{n,k}^{0,s}\, S(\overline{\Psi}_{k,1}^n \otimes Q^j y^s),$$

where $C_{n,k}^{0,s} := kc_{n,k}^{0,s} = \frac{1}{(k-1)!(n-k-1)!s!s_{n-k+s-1}}$ *and* S *is the symmetrisation operator applied to the form's tensor-part.*

Remark 3.44. Hug and Schneider constructed in [49] more general *support measures* by means of Federer's geometric measure theory. Theorem 3.43 is a re-interpretation of their original results in a more elementary differential-geometric language for the case of curvature measures which is a special case of support measures.

With the above normalisation, the curvature measures $\Psi_{k,q}$ may be shown to have a similar "intrinsic" property (3.16) as the intrinsic volumes μ_k[8]. Let us formulate this property in a more precise manner.

Alesker introduced in [12] the notion of the pullback of valuations which we now slightly generalise for curvature measures.

Definition 3.45. Let $f : V \to W$ be a linear map of vector spaces V, W. Then, f induces a map called *pullback*

$$f^* : \mathrm{Curv}(W) \to \mathrm{Curv}(V)$$

by $(f^*\Phi)(K, U) = \Phi(f(K), f(U))$ for any $K \in \mathcal{K}(V)$, $U \in \mathcal{B}(V)$.

The intrinsic property of intrinsic volumes may now be expressed as follows: For any isometric embedding $\iota : V \to W$ of vector spaces, one has:

$$\iota^*\mu_k^W = \mu_k^V.$$

The situation is not as straight-forward for tensor-valued valuations. For example, the domains of $\Psi_{k,q}^n$ and $\Phi_{k,q}^m$ for $V \simeq \mathbb{R}^n$ and $W \simeq \mathbb{R}^m$ are $\mathrm{Sym}^2 \bigwedge^q \mathbb{R}^n$ and $\mathrm{Sym}^2 \bigwedge^q \mathbb{R}^m$, respectively, and, thus, cannot be compared with each other directly. Fortunately, we may construct pullbacks of tensor-valued curvature measures by isometric embeddings.

[8] In fact, the form $\Psi_{k,0}$ is the Lipschitz-Killing form ρ_k that induces the intrinsic curvature measures Φ_k.

Definition 3.46. Let $f : V \to W$ is an isometric embedding of an Euclidean vector space V into another Euclidean vector space W. Furthermore, let $\Gamma(W)$ be an irreducible representation of $SO(W)$ and Φ^W be an $SO(W)$-covariant curvature measure on W with values in $\Gamma(W)$. Then the *pullback* of Φ^W by f is defined as follows:

$$f^*\Phi^W(K, U) = \operatorname{Res}_{SO(V)}^{SO(W)} \Phi^W(f(K), f(U)), \tag{3.24}$$

where $\operatorname{Res}_{SO(V)}^{SO(W)}$ is the restriction of $\Gamma(W)$ to $SO(V)$ (see Section 2.3). We often will call such pullbacks the *restrictions* of Φ^W to V and denote them by $\Phi^W|_V$.

Now let Φ^V be an $SO(V)$-covariant curvature measure on V and let Φ^V assume values in an irreducible $SO(V)$-module $\Gamma(V)$ with the same Young-diagram as $\Gamma(W)$. Applying the Branching Theorem, one sees that $\Gamma(V)$ is an irreducible $SO(V)$-submodule of $\operatorname{Res}_{SO(V)}^{SO(W)} \Gamma(W)$ and there is a unique canonical $SO(V)$-invariant isometric embedding ι_Γ of $\Gamma(V)$ into the $SO(V)$-module $\operatorname{Res}\Gamma(W)$ induced by the isometric embedding ι. The intrinsic property of the pair (Φ^V, Φ^W) is now expressed as

$$\iota_\Gamma(\Phi^V) = \iota^*\Phi^W. \tag{3.25}$$

This definition may be extended to pairs of any (not necessarily irreducible) representations of $SO(W)$ resp. $SO(V)$ by regarding them as direct sums of irreducible ones.

More generally, let (V, ι) be a category of Euclidean vector spaces with isometric embeddings and $(\mathrm{Mod}_{SO}, \iota_\Gamma)$ the category of representations of $SO(V)$ for all Euclidean spaces V. Given two Euclidean vector spaces V, W, the set of morphisms between an $SO(W)$-representation Θ^W and an $SO(V)$-representation Γ^V is given by all $SO(V)$-invariant isometric embeddings $\Gamma^V \hookrightarrow \Theta^W$ if $\dim V \le \dim W$ and is empty otherwise.

Definition 3.47. Let $\Gamma : (V, \iota) \to (\mathrm{Mod}_{SO(V)}, \iota_\Gamma)$ be a functor assigning to each Euclidean vector space V an $SO(V)$-representation $\Gamma(V)$ with a fixed Young-diagram and to each isometric embedding $\iota : V \to W$ an induced $SO(V)$-invariant isometric embedding $\iota_\Gamma : \Gamma(V) \to \Gamma(W)$.

Then a one-parameter family of curvature measures Φ assigning to each $V \in \{V, \iota\}$ a curvature measure Φ_V with values in $\Gamma(V)$ such that the property (3.25) holds for all pairs (Φ^V, Φ^W) of curvature measures is called a family of *horizontal curvature measures.*

Then the family $\Psi^n_{k,q}$ of $\operatorname{Sym}^2 \bigwedge^q V$-valued curvature measures is horizontal in the sense of the above definition. Contrary to that, the $\operatorname{Sym}^p V$-valued curvature measures from the above result are not, in general, horizontal. However, their behaviour under pullbacks by embeddings still may be controlled.

Theorem 3.48 (McMullen's Theorem)**.** *Let $\Phi_k^{0,s,i}$ with fixed i, k, s be a family of curvature measures from Theorem 3.43 and denote $\Phi_{V,k}^{0,s,i}$, $\Phi_{W,k}^{0,s,i}$ be the representatives of this family for Euclidean vector spaces V, W. Then, with the notation from (3.25), one has:*

$$\iota^*\Phi_{W,k}^{0,s,i} = \sum_{j=0}^{\lfloor s/2 \rfloor} \frac{1}{(4\pi)^j j!} \Phi_{V,k}^{0,s-2j,i}\, Q(\operatorname{coker} \iota)^j.$$

4 Tensor-Valued Curvature Measures

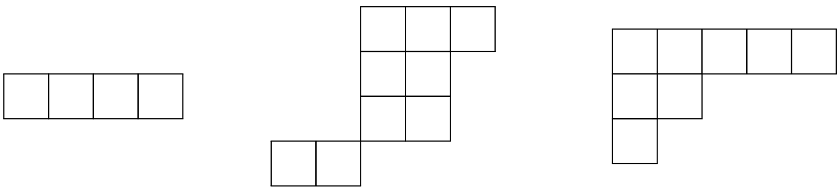
4.1 Harmonic Decomposition of Smooth Curvature Measures

Inspired by Theorem 3.36, one may ask if a similar harmonic decomposition may be obtained for Curv_k^{sm}. As the kernel of Integ is smaller than that of integ, one may expect the conditions for $SO(n)$-representations, in which *curvature measures* may assume values in, to be less restrictive than those stated in Theorem 3.36. First, let us introduce a more efficient notation for the appropriate class of $SO(n)$-modules.

Definition 4.1. For $q > 0$, $p \geq 0$, and $r = -2, 0, 1$, we write $\Gamma_r^{q,p}$ for an $SO(n)$-module represented by the tuple λ such that $\lambda_1 = p + 2$, $\lambda_2 = \ldots = \lambda_q = 2$ and $\lambda_{q+1} = r$. Additionally, we denote the trivial representation by $\Gamma_0^{0,0}$ and the standard representation by $\Gamma_1^{0,0}$. The tuple λ for any representation covered by this notation is uniquely written as $\lambda = [q; p; r]$ with semicolons instead of commas between the numbers. More generally, a representation Γ_λ^G of a Lie-group G which corresponds to the Young-diagram $\lambda = [q; p; r]$ is called of type $[q; p; r]$.

Remark 4.2. Obviously, if $q = 0$, the only allowed value for p is 0 and for $r - \{0, 1\}$.

Example 4.3. The following diagrams



are represented by $\Gamma_0^{1,2}$, $\Gamma_{-2}^{1,3}$, and $\Gamma_1^{2,3}$, respectively.

Given a vector space V of dimension n, Theorem 3.36 may now be restated more concisely as follows:

$$\mathrm{Val}_k = \bigoplus_{p,q} \Gamma_0^{q,p} \oplus \left(\bigoplus_{p} \Gamma_{-2}^{k-1,p} \right)$$

and the sum is over $0 \leq q \leq \min(k, n-k)$ and all values of $p \in \mathbb{N}$ admissible for q. The term in parentheses is only added if n is even and $k = n/2$. We now state and prove the similar harmonic decomposition for curvature measures.

Theorem 4.4. *Let V be an n-dimensional space, where $n < \infty$ and $0 \leq k \leq n-1$. Then the space Curv_k^{sm} can be decomposed as follows:*

$$\mathrm{Curv}_k^{sm} = \mathrm{Curv}_{k,R}^{sm} \oplus \mathrm{Curv}_{k,X}^{sm},$$

where

$$\mathrm{Curv}_{k,R}^{sm} = \bigoplus_{p,q \in \mathbb{N}} m_{k,n,q} \, \Gamma_0^{q,p} \oplus \bigoplus_{p,q \in \mathbb{N}} m'_{k,n,q} \, \Gamma_1^{q,p}$$

and

$$\mathrm{Curv}_{k,X}^{sm} = \begin{cases} \bigoplus_{p \in \mathbb{N}} \Gamma_{-2}^{j-1,p} & \text{if } n = 2j \text{ is even and } k = j-1, j; \\ \bigoplus_{p \in \mathbb{N}} \Gamma_0^{j,p} & \text{if } n = 2j+1 \text{ is odd and } k = j \text{ for all } j \in \mathbb{N}. \end{cases}$$

The coefficients

$$m_{k,n,q} = \begin{cases} 2 & \text{if } 1 \le q \le k'; \\ 1 & \text{if } q \in \{0, k'+1\}; \\ 0 & \text{otherwise}, \end{cases} \qquad m'_{k,n,q} = \begin{cases} 1 & \text{if } 0 \le q \le k'; \\ 0 & \text{otherwise} \end{cases},$$

where $k' := \min(k, n-k-1)$*, are the* multiplicities *of the corresponding* $SO(n)$*-modules.*

The space $\mathrm{Curv}_{k,R}^{sm}$ stands for *regular* curvature measures. Conversely, $\mathrm{Curv}_{k,X}^{sm}$ denotes the space of *exceptional* curvature measures. Similarly to Theorem 3.36:

Definition 4.5. The irreducible $SO(n)$-modules $\Gamma_r^{q,p}$ such that $\dim(\mathrm{Curv}_k^{sm} \otimes \Gamma_r^{q,p})^{SO(n)} > 0$ are called *locally valuative.*

Remark 4.6. If $n = 2j+1$, the $SO(n)$-modules $\Gamma_0^{j,p}$ occur twice in the regular part and once in the exceptional part, thus, yielding the overall multiplicity of 3.

Proof. Let $\Gamma_{[\lambda]}$ be an arbitrary irreducible $SO(n)$-module. By (2.1), we know that the total multiplicity of $\Gamma_{[\lambda]}$ in Curv_k^{sm} is the dimension of $SO(n)$-invariant homomorphisms $\mathrm{Hom}_{SO(n)}(\mathrm{Curv}_k^{sm}, \Gamma_{[\lambda]})$. Assume without loss of generality that $k \le (n-1)/2$. Then

$$\begin{aligned} \mathrm{Hom}_{SO(n)}(\mathrm{Curv}_k^{sm}, \Gamma_{[\lambda]}) &= \mathrm{Hom}_{SO(n)}(\Omega_p^{k,n-k-1}(SV), \Gamma_{[\lambda]}) && (\textit{Lemma 3.39}) \\ &= \mathrm{Hom}_{SO(n)}\left(\mathrm{Ind}_{SO(n-1)}^{SO(n)} \Lambda_p^{k,n-k-1} W_y^*, \Gamma_{[\lambda]}\right) && (\textit{Corollary (3.17)}) \\ &= \mathrm{Hom}_{SO(n)}\left(\mathrm{Ind}_{SO(n-1)}^{SO(n)} \bigoplus_{l=0}^{k} \bar{\Gamma}_0^{l,0}, \Gamma_{[\lambda]}\right) \\ &= \mathrm{Hom}_{SO(n-1)}\left(\bigoplus_{l=0}^{k} \bar{\Gamma}_0^{l,0}, \mathrm{Res}_{SO(n-1)}^{SO(n)} \Gamma_{[\lambda]}\right) && (\textit{Theorem 2.31}). \end{aligned}$$

The Branching Theorem 2.36 now yields:

$$\begin{aligned} &\mathrm{Hom}_{SO(n)}(\mathrm{Curv}_k^{sm}, \Gamma_{[\lambda]})^{SO(n)} \\ &\quad = \mathrm{Hom}_{SO(n-1)}\left(\bigoplus_{l=0}^{k} \bar{\Gamma}_0^{l,0}, \bigoplus_{\mu} \Gamma_{[\mu]}\right) = \bigoplus_{l=0}^{k} \bigoplus_{\mu} \mathrm{Hom}_{SO(n-1)}\left(\bar{\Gamma}_0^{l,0}, \Gamma_{[\mu]}\right). \end{aligned} \tag{4.1}$$

By Schur's Lemma, $\mathrm{Hom}_{SO(n-1)}\left(\bar{\Gamma}_0^{l,0}, \Gamma_{[\mu]}\right)$ is not trivial if and only if $\mu = [0; l; 0]$ for some l in the range. Let's call these irreducible $SO(n-1)$-modules *restricted valuative (RV) modules.* Hence, the multiplicity of $\Gamma_{[\lambda]}$ in Curv_k^{sm} is equal to the number of RV modules in $\mathrm{Res}_{SO(n-1)}^{SO(n)} \Gamma_{[\lambda]}$, hence an $SO(n)$-module is locally valuative if and only if it contains at least one RV module. We now study the classes $\Gamma_{[\lambda]}$ on a case-by-case basis:

- The $SO(n)$-modules $\Gamma_1^{q,p}$ contain exactly one RV module $\Gamma_0^{q,0}$ if $0 \le q \le k'$ and none otherwise. This yields the multiplicities $m'_{n,k,q}$;
- Let us turn to the case $\lambda = [q, p, 0]$.
  - The case $\lambda = [0; 0; 0]$ corresponds to intrinsic curvature measures, of which we know the multiplicity to be 1.

- For $0 < q \leq k'$, $\Gamma_0^{q,p}$ contains exactly two RV modules: $\Gamma_0^{q,0}$ and $\Gamma_0^{q-1,0}$. The only exception is for $n = 2j+1$, $k = q = j$, $j \in N$, in which case $\bar{\Gamma}_0^{q,0} = \Gamma_0^{q,0} \oplus \Gamma_{-2}^{q-1,0}$ is not an irreducible $SO(n-1)$-module, whence follows that the the total multiplicity of $\Gamma_0^{q,p}$ in Curv_k^{sm} is 3 and not 2.
- For $q = k' + 1$, the only RV module is $\Gamma_0^{q-1,0}$.
- For higher values of q, no RV modules are contained in $\Gamma_0^{q,p}$.

Thus, we have computed the multiplicities $m_{k,n,q}$.

- Most of the irreducible $SO(n)$-modules are self-dual. The only exception is for $n = 2j$ and $\lambda_j \neq 0$ for $j \in \mathbb{N}$. Then $\left(\Gamma_0^{j,p}\right)^* = \Gamma_{-2}^{j-1,p}$. However, by the Branching Theorem 2.36, their restrictions are the same, hence the multiplicity of $\Gamma_{-2}^{j-1,p}$ in Curv_k^{sm} is equal to the multiplicity of $\Gamma_0^{j,p}$, which is 1 for $k = j-1, j$ and 0 otherwise.

The irreducible representations of $SO(n)$ not mentioned in the above list do not contain RV modules, hence their multiplicity in Curv_k^{sm} is zero. This concludes the proof. □

Remark 4.7. From the above result, we may immediately obtain the dimension of $\mathrm{TVal}_{\Gamma_{[\lambda]}}^{SO(n)}$:

$$\begin{aligned} \dim \mathrm{TVal}_{\Gamma_{[\lambda]}}^{SO(n)} &= \dim(\mathrm{Curv}_k^{sm} \otimes \Gamma_{[\lambda]})^{SO(n)} = \dim \mathrm{Hom}_{SO(n)}(\mathrm{Curv}_k^{sm}, \Gamma_{[\lambda]}^*) \\ &= m(\Gamma_{[\lambda]}^*, \mathrm{Curv}_k^{sm}) = m(\Gamma_{[\lambda]}, \mathrm{Curv}_k^{sm}). \end{aligned}$$

4.2 The Four Bases of Curvature Measures

4.2.1 $SL(n)$- and $SO(n)$-Module-Valued Curvature Measures

Let Γ_μ be a fixed irreducible $SL(m)$-module for a partition μ and set $r := \mu_1$ be the number of columns of the conjugate partition μ'. Furthermore, let $(e_1, \ldots, e_m)$ be the canonical basis of the standard representation V with the dual basis $(e_1^*, \ldots, e_m^*)$ and use the following shorthand notation

$$e_I := e_{i_1 \ldots i_\ell} := e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_\ell}$$

for a multi-index $I = (i_1, \ldots, i_\ell)$.

Before proceeding to the main results of this section, we first need to study the symmetriser a''_μ from (2.7) and its more general counterpart a_μ more thoroughly. Proposition 2.29 together with Lemma 2.28 give a hint that a''_μ resp. a_μ must act on $W_\mu V$ resp. $\bigwedge_\mu V$ by some sort of Bianchi-type permutations. Let us elaborate this action directly.

Example 4.8. Let $\mu = (2, 1)$. Then the corresponding standard Young-tableau is $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$.

Furthermore, as there are no columns of identical width, $a'_\mu = \mathrm{Id}$ is trivial and $a_\mu = a''_\mu = \mathrm{Id} + e_{(12)}$, where we employ the usual cycle notation to describe permutations.

Let $v = (v_1 \wedge v_3) \otimes v_2 \in \bigwedge_\mu V$. The action of Id on v is clear: $\mathrm{Id}(v) = v$. Let us investigate the action of the element $e_{(12)}$. First of all, it must be well-defined on $\bigwedge_\mu V$, i.e.

$$e_{(12)}[(v_1 \wedge v_3) \otimes v_2] = -e_{(12)}[(v_3 \wedge v_1) \otimes v_2]. \tag{4.2}$$

To this end, we re-write v as:

$$v = \frac{1}{2}[(v_1 \wedge v_3) \otimes v_2 - (v_3 \wedge v_1) \otimes v_2]$$

and let $e_{(12)}$ act naturally on this sum by exchanging the boxes 1 and 2 and re-scaling the result by the factor 2 (which is the number of summands in the above expansion of v), i.e.

$$\begin{aligned} e_{(12)}(v) &= \frac{1}{2}e_{(12)}[(v_1 \wedge v_3) \otimes v_2 - (v_3 \wedge v_1) \otimes v_2] = (v_2 \wedge v_3) \otimes v_1 - (v_2 \wedge v_1) \otimes v_3 \\ &= (v_2 \wedge v_3) \otimes v_1 + (v_1 \wedge v_2) \otimes v_3. \end{aligned}$$

One sees that the action of $e_{(12)}$ is given by a Bianchi-type permutation, i.e. a sum over all permutations exchanging v_2 with each of the vectors in the first column. Furthermore, the condition (4.2) can be readily verified. Finally, as $e_{(12)}\left(\mathrm{Id} + e_{(12)}\right) = \mathrm{Id} + e_{(12)}$, we have $e_{(12)}(a''_\mu(v)) = a''_\mu(v)$, i.e.:

$$a''_\mu[(v_2 \wedge v_3) \otimes v_1 + (v_1 \wedge v_2) \otimes v_3] = a''_\mu[(v_1 \wedge v_3) \otimes v_2],$$

which is a special case of the Bianchi-type relations from Lemma 2.28.

Generalising the above Example for arbitrary Young-diagrams μ, we postulate the following Lemma:

Lemma 4.9. *Let $v = v_{I_1} \otimes \ldots \otimes v_{I_r} \in \bigwedge_\mu V$ and $v_{I_j} = v_{j,1} \wedge \ldots \wedge v_{j,\mu'_j} \in \bigwedge^{\mu'_j} V$. Then every summand in a_μ acts on v as a composition of elements*

$$\pi_j = \sum_{r,s} S_{j,r,s},$$

where $2 \leq j \leq r$, $1 \leq r \leq \mu'_{j-1}$, $1 \leq s \leq \mu'_j$ and $S_{j,r,s}$ is an element which exchanges the vector $v_{j,s}$ in v_{I_j} with the vector $v_{j-1,r}$ from the preceding wedge-vector $v_{I_{j-1}}$.

Proof. Directly generalising the above Example, we may write the j-th wedge-vector v_{I_j} as follows:

$$v_{I_j} = v_{j,1} \wedge \ldots \wedge v_{j,\mu'_j} = \frac{1}{|S_{\mu'_j}|} \sum_{\sigma \in S_{\mu'_j}} \operatorname{sgn} \sigma \, v_{j,\sigma(1)} \wedge \ldots \wedge v_{j,\sigma(\mu'_j)}.$$

Thus, any transposition in a_μ that acts on the Young-tableau by exchanging two adjacent boxes in columns j and $j+1$ in the same row acts on $W_\mu V$ as a sum of permutations which exchange *any* vector $v_{j+1,s}$ in the $(j+1)$-th column with any vector $v_{j,r}$ in the j-th column, $1 \leq r \leq \mu'_j$, $1 \leq s \leq \mu'_{J+1}$.

Now, setting $d := |\mu|$ to be the weight, i.e. the number of boxes, of the Young-diagram μ, the subgroup $P = P_\mu \subset S_d$ of elements preserving the rows in the standard Young-tableau μ is obviously generated by $\pi \in P_\mu$ which permute boxes in only one row and do not affect others. As the permutation group S_d is a Coxeter group, i.e. it is generated by transpositions, every permutation $\pi \in P_\mu$ may be written as a composition of transpositions that exchange two adjacent boxes in the same row, which proves the assertion. □

Remark 4.10. Obviously, every summand in a''_μ acts on $W_\mu V$ by a composition of the above Bianchi-type permutations excluding those exchanging the entire wedge-vector v_{I_j} from one column with another wedge-vector v_{I_k} of the same length, i.e. those belonging to a'_μ. One has, however, to take into account the additional symmetries introduced by a'_μ while defining the action of a''_μ on $W_\mu V$.

Furthermore, the following Lemma will be of much assistance later in this Section.

Lemma 4.11. *Denote by* $\mathrm{Id}_{(\mu)} \in (\Gamma_\mu \otimes \Gamma_\mu^*)^{SL(m)}$ *the canonical* $SL(m)$*-invariant element and define*

$$\mathrm{Id}_\mu := \sum_{I_1,\dots I_r} e_{I_1} \otimes \ldots \otimes e_{I_r} \otimes e_{I_1}^* \otimes \ldots \otimes e_{I_r}^* \in \bigwedge_\mu V \otimes \bigwedge_\mu V^*,$$

where the sum is over all multi-indices I_j *of length* μ'_j, $j = 1, \dots, r$. *If* $\bigwedge_\mu V = \Gamma_\mu \oplus \bigoplus_\nu \Gamma_\nu$ *is a multiplicity-free* $SL(m)$*-module, then*

$$\mathrm{Id}_\mu = C_{\mu,m} \left(\mathrm{Id}_{(\mu)} + \sum_\nu \mathrm{Id}_{(\nu)} \right),$$

where $C_{\mu,m}$ *is some non-zero constant dependent only on* μ *and* $m = \dim V$.

Proof. The proof is elementary. First, recall that the identity $\mathrm{Id}_W \in \mathrm{Hom}(W, W) = W \otimes W^*$ of an arbitrary finite-dimensional vector space W with basis $(v_1, \dots, v_m)$ and the dual basis $(v_1^*, \dots, v_m^*)$ is expressed as an element $\sum_{i=1}^m v_i \otimes v_i^* \in W \otimes W^*$. Now, $e_{I_1} \otimes \ldots \otimes e_{I_r}$ for multi-indices $I_j = (i_{j,1} < i_{j,2} < \ldots i_{j,\mu'_j})$ form a canonical orthogonal basis of $\bigwedge_\mu V$ and, hence, Id'_μ is a $C_{\mu,m}$-multiple of the $SL(m)$-invariant identity map $\bigwedge_\mu V \to \bigwedge_\mu V$.

As $\bigwedge_\mu V$ is multiplicity-free, any $SL(m)$-invariant homomorphism $\bigwedge_\mu V \to \bigwedge_\mu V$ is written as $c_{\mu,m} \mathrm{Id}_{(\mu)} + \sum_\nu c_{\nu,m} \mathrm{Id}_{(\nu)}$ for some constants $c_{\mu,m}$, $c_{\nu,m}$. When restricted to any submodule Γ_ν or the submodule Γ_μ of $\bigwedge_\mu V$, Id_μ yields $C_{\mu,m}$ times the respective identity elements $\mathrm{Id}_{(\nu)}$ or $\mathrm{Id}_{(\mu)}$. Hence, $c_{\mu,m} = c_{\nu,m} = C_{\mu,m}$ and the proof is finished. □

We will employ the $\equiv$-symbol to underline that the equation holds up to some non-zero constant. For example,

$$\mathrm{Id}_\mu \equiv \mathrm{Id}_{(\mu)} + \sum_\nu \mathrm{Id}_{(\nu)}.$$

Proposition 4.12. *Let* $0 \le k \le n/2$ *and* $\Gamma_{[\lambda]}$ *be an irreducible* $SO(n)$*-module. If* $\Gamma_{[\lambda]}$ *contains the RV* $SO(n-1)$*-module* $\bar{\Gamma}_{[\mu]} := \bar{\Gamma}_0^{q,0}$, *then the* $SO(n)$*-covariant translation-invariant form* $\mathrm{Id}_{k,q,\lambda} \in \Omega_p^{k,n-k-1} \otimes \Gamma_\lambda \subset \Omega_p^{k,n-k-1} \otimes \bigwedge_\lambda V$ *defined pointwise as follows:*

$$\begin{aligned}
\mathrm{Id}_{k,q,\lambda}|_{(x,y)} &:= \sum_{I,J} y \wedge *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes e_I \cdot e_J \wedge y^{\lambda-\mu} \\
&\equiv \sum_{\substack{i_1,\dots,i_q=1 \\ \pi \in S_n}}^{n} \mathrm{sgn}(\pi)\, y_{\pi(1)}\, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes e_{i_1\dots i_q} \cdot e_{\pi(2)\dots\pi(q+1)} \wedge y^{\lambda-\mu}
\end{aligned}$$

induces a non-trivial k*-homogeneous* $\Gamma_\lambda^{SL(n)}$*-valued* $SO(n)$*-covariant curvature measure.*

If $n = 2j+1$ *and* $k = q = j$, $j \in \mathbb{N}$, *then there is another linearly independent* k*-homogeneous* $SO(n)$*-covariant curvature measure given by the differential form* $\mathrm{Id}^*_{k,q,\lambda}$:

$$\mathrm{Id}^*_{k,q,\lambda}|_{(x,y)} = \sum_{I,J} dx^I \otimes dy^J \otimes (e_I \cdot e_J) \wedge y^{\lambda-\mu}.$$

Proof. Let $\Gamma_{[\lambda]}$ be a locally valuative $SO(n)$-module and $\bar{\Gamma}_{[\mu]} := \bar{\Gamma}_0^{q,0}$ be the RV module contained in $\Gamma_{[\lambda]} := \Gamma_r^{q',p}$. Set Γ_λ to be the irreducible $SL(n)$-module of type $[q'; p; r]$ and, likewise, Γ_μ the irreducible $SL(n)$-module of type $[q; 0; 0]$. Furthermore, denote by $\iota_\mu : \Gamma_\mu \to W_\mu V = \mathrm{Sym}^2 \bigwedge^q V$ the canonical embedding.

Assume that the canonical identity element $\mathrm{Id}_{(\mu)}$, when embedded into $W_\mu V^* \otimes W_\mu V$, is written as follows

$$\iota_\mu^* \otimes \iota_\mu(\mathrm{Id}_{(\mu)}) \equiv \sum_{I,J} (v_I^* \cdot v_J^*) \otimes (v_I \cdot v_J), \tag{4.3}$$

where $v_I := v_{i,1} \wedge \ldots \wedge v_{i,q}$ are wedge-vectors and the sum is over some multi-indices I, J of length q such that $v_I \cdot v_J$ span Γ_μ (Going forward, we will omit the embedding maps for the sake of clarity). Then we may even embed $\mathrm{Id}_{(\mu)}$ into $(\bigwedge^q V^* \otimes \bigwedge^q V^*) \otimes \mathrm{Sym}^2 \bigwedge^q V$ by adding an $SL(n)$-invariant element

$$\sum_{I,J} (v_I^* \wedge v_J^*) \otimes (v_I \cdot v_J),$$

i.e. replacing the symmetric product $v_I^* \cdot v_J^*$ by the ordinary tensor product $v_I^* \otimes v_J^*$. However, as $\left(\bigwedge^2(\bigwedge^q V^*) \otimes \mathrm{Sym}^2 \bigwedge^q V\right)^{SL(n)} = 0$, the above element is equal to zero, and we have:

$$\mathrm{Id}_{(\mu)} \quad \equiv \quad \sum_{I,J} (v_I^* \otimes v_J^*) \otimes (v_I \cdot v_J)$$

Let us call this argument the Schur-type argument, as we will use it later in the proof.

We may further embed $\mathrm{Id}_{(\mu)}$ into $\left(\bigwedge^k V^* \otimes \bigwedge^k V^*\right) \otimes \left(\mathrm{Sym}^2 \Lambda^q V\right)$ for any $q \le k \le \lfloor m/2 \rfloor$ by taking the Kulkarni-Nomizu-product with Q^{k-q}. Using the isomorphism between $\bigwedge^k V^* \otimes \bigwedge^k V^*$ and $\bigwedge^k V^* \otimes \bigwedge^{n-k} V^*$ given by applying the Hodge-star operator on the second factor, we obtain the following form of $\mathrm{Id}_{(\mu)}$ as an element of $\left(\bigwedge^k V^* \otimes \bigwedge^{n-k} V^*\right) \otimes \left(\mathrm{Sym}^2 \Lambda^q V\right)$:

$$\mathrm{Id}_{(\mu)} \quad \equiv \quad \sum_{I,J} *_2((v_I^* \otimes v_J^*) \wedge Q^{k-q}) \otimes v_I \cdot v_J.$$

Note that $*_2$ is $SO(m)$- – but not $SL(m)$- – invariant and, hence, so is the above embedding of $\mathrm{Id}_{(\mu)}$ into $\left(\bigwedge^k V^* \otimes \bigwedge^{n-k} V^*\right) \otimes \left(\mathrm{Sym}^2 \Lambda^q V\right)$. Setting $W_y^* = T_y^* S^m$, $y = e_n$, $m = n-1$ and letting V be the standard $SO(m)$-representation, we may now embed $\mathrm{Id}_{(\mu)}$ into $\bigwedge^k W_y^* \otimes \bigwedge^{m-k} W_y^* \otimes \mathrm{Sym}^2 \Lambda^q V$. It is then written as follows:

$$\mathrm{Id}_{(\mu)} \quad \equiv \quad \sum_{I,J} *_2((\tau_x^I \otimes \tau_y^J) \wedge d\alpha^{k-q}) \otimes v_I \cdot v_J$$

for some $\tau_x^I \in \bigwedge^k W_y^*$ and $\tau_y^J \in \bigwedge^{m-k} W_y^*$. As $\Gamma_0^{q,0} \subset \bigwedge_p^{k,m-k} W_y^*$ by Corollary 3.17, $\mathrm{Id}_{(\mu)}$ is, in fact, an element of $\bigwedge_p^{k,m-k} W_y^* \otimes \mathrm{Sym}^2 \Lambda^q V$ regarded as a subspace of $\bigwedge^k W_y^* \otimes \bigwedge^{m-k} W_y^* \otimes \mathrm{Sym}^2 \Lambda^q V$.

Now, set $y = e_n$ and $\tilde{V} := V \oplus \langle y \rangle$ the standard representation of $SO(n)$ such that $SO(n-1)$ fixes y. By (2.20), we may now embed the $SL(n-1)$-module Γ_μ in an $SO(n-1)$-invariant manner by taking the λ-wedge product with y. This yields the following embedding of $\mathrm{Id}_{(\mu)}$ into $\bigwedge_p^{k,m-k} W_0^* \otimes \Gamma_\lambda$:

$$\mathrm{Id}_{(\mu)} \equiv \sum_{I,J} *_2((\tau_x^I \otimes \tau_y^J) \wedge d\alpha^{k-q}) \otimes v_I \cdot v_J \wedge_\lambda y^{\lambda-\mu}.$$

Now, the translation-invariant differential form defined pointwise at an arbitrary point $(x, y) \in S\tilde{V}$ as

$$\widehat{\mathrm{Id}}_{(k,q,\lambda)} := \sum_{I,J} y \wedge *_2((\tau_x^I \otimes \tau_y^J) \wedge d\alpha^{k-q}) \otimes v_I \cdot v_J \wedge_\lambda y^{\lambda-\mu},$$

where the sum is over the same indices as in (4.3) is equal to $\mathrm{Id}_{(\mu)}$ at the point $(0, e_n)$ and $SO(n)$-covariant. As it is primitive and non-trivial at $(0, e_n)$, it remains so on the entire $S\tilde{V}$ by $SO(n)$-covariance. Hence, $\widehat{\mathrm{Id}}_{(k,q,\lambda)}$ induces a non-trivial $\Gamma_{[\lambda]}$-valued curvature measure.

The above form, although written as a closed sum, is not very useful, as we neither explicitly know the τ_x^I and τ_y^J nor which multi-indices I, J appear in the sum. On top of that, we still have the implicit – and potentially tedious – symmetrisation that comes along with the λ-product with $y^{\lambda-\mu}$. Luckily, we may apply the above-mentioned Schur-type argument twice in order to simplify the resulting differential form.

1. Instead of $\mathrm{Id}_{(\mu)}$, we may take the identity element $\mathrm{Id}_\mu \in (\bigwedge_\mu V^* \otimes \bigwedge_\mu V)^{SL(m)}$ from Lemma 4.11. Let $\widehat{\mathrm{Id}}_{k,q,\lambda} \in \Omega^{k,m-k} \otimes \Gamma_\lambda$ be the $SO(n)$-covariant differential form equal to Id_μ in $(x,y) \in SV$. The vectors $e_I \otimes e_J$ which appear in the definition of Id_μ correspond to dx^I, dx^J under the embedding into $\bigwedge^k W_y^* \otimes \bigwedge^{m-k} W_y^*$. Thus, we may write the form $\widehat{\mathrm{Id}}_{k,q,\lambda}$ explicitly as follows:

$$\widehat{\mathrm{Id}}_{k,q,\lambda} := \sum_{I,J} y \wedge *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes (v_I \otimes v_J) \wedge_\lambda y^{\lambda-\mu}$$

and the sum is simply over *all* multi-indices I, J. Now, the space $\bigwedge^q V \otimes \bigwedge^q V$ is multiplicity-free by Lemma 2.43. Writing $\bigwedge^q V \otimes \bigwedge^q V = \Gamma_\mu \oplus \bigoplus_\nu \Gamma_\nu$, Corollary 3.17 implies that none of the $SO(n)$-modules $\operatorname{Res}\Gamma_\nu$ are in $\bigwedge_p^{k,n-k-1} W_y^*$. Then, by Lemma 4.11, $\mathrm{Id}_\mu \equiv \mathrm{Id}_{(\mu)} + d\alpha \wedge \tau$, where $\tau \in \bigwedge^{k-1} W_y^* \otimes \bigwedge^{m-k-1} W_y^* \otimes \Gamma_\lambda$. Of course, this implies:

$$\operatorname{Integ} \widehat{\mathrm{Id}}_{k,q,\lambda} \equiv \operatorname{Integ} \widehat{\mathrm{Id}}_{(k,q,\lambda)}.$$

2. The second Schur-type argument is somewhat more involved. Define the element $\mathrm{Id}_{\mu,g} \in \bigwedge^k W_y^* \otimes \bigwedge^{m-k} W_y^* \otimes \bigwedge_\lambda V$ as follows:

$$\begin{aligned} \mathrm{Id}_{\mu,g} &:= \sum_{I,J} *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes e_I \otimes e_J \wedge y^{\lambda-\mu} \\ &\equiv \sum_{\substack{i_1,\dots,i_q=1 \\ \pi \in S_m}}^{m} \operatorname{sgn}(\pi)\, dx^{i_1\dots i_q \pi(q+1)\dots\pi(k)} \wedge dy^{\pi(k+1)\dots\pi(m)} \otimes e_{i_1\dots i_q} \otimes e_{\pi(1)\dots\pi(q)} \wedge y^{\lambda-\mu}. \end{aligned}$$

As $\mathrm{Id}_{\mu,g}$ is just an embedding of Id_μ into $\bigwedge^k W_y^* \otimes \bigwedge^{m-k} W_y^* \otimes \bigwedge_\lambda V$ by means of the generalised wedge-product rather than the λ-product, we have:

$$\mathrm{Id}_\mu \;=\; \pi_\lambda(\mathrm{Id}_{\mu,g}) \overset{(2.7)}{=} a_\lambda(\mathrm{Id}_{\mu,g}).$$

According to the proof of Theorem 4.4, there are three generic cases, in which $\Gamma_{[\mu]} = \Gamma_0^{q,0}$ may be a RV module to a module $\Gamma_{[\lambda]} = \Gamma_r^{q',p}$, namely if:

1. $q' = q$, $p \geq 0$, $r = 1$, then
$$\mathrm{Id}_{\mu,g} = \sum_{I,J} *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes (e_I \wedge y) \otimes e_J \otimes y^p;$$

2. $q' = q$, $p \geq 0$, $r = 0$, then
$$\mathrm{Id}_{\mu,g} = \sum_{I,J} *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes e_I \otimes e_J \otimes y^p;$$

3. $q' = q+1$, $p \geq 0$, $r = 0$, then

$$\mathrm{Id}_{\mu,g} = \sum_{I,J} *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes (e_I \wedge y) \otimes (e_J \wedge y) \otimes y^p.$$

Although it is possible to formulate the Schur-type argument in a general way, we would like to demonstrate it in a more concrete manner on the first case. The others may be shown by the same token. Let now Γ_λ be of type $[q;p;1]$ and Γ_μ be of type $[q;0;0]$. Recall from Lemma 4.9 that a_λ is a sum of compositions of permutations $\pi_j = \sum_{r,s} S_{j,r,s}$.

As $\Gamma_1^{q+1,p}$ has only 1 box in every column beginning from the third and all of them are filled with y, the permutations π_j, $j \geq 4$ act trivially on $\mathrm{Id}_{\mu,g}$. Let us consider the permutation

$$\pi_2 = \sum_{r,s=1}^{q} S_{2,r,s} + \sum_{s=1}^{q} S_{2,q+1,s} =: \pi_2' + \sum_{s=1}^{q} S_{2,q+1,s}.$$

The permutation π_2' is a summand of a_μ, hence $\pi_2'(\mathrm{Id}_{(\mu)}) = \pi_2'(a_\mu(\mathrm{Id}_\mu)) = C_{\mu,n}\,\mathrm{Id}_{(\mu)}$. Correspondingly, $\pi_2'(\mathrm{Id}_{\mu,KN}) \overset{\mathrm{mod}\ d\alpha}{\equiv} \mathrm{Id}_{\mu,g}$. On the other hand, the permutation $S_{2,q+1,s}$ yields the element

$$A_2 \equiv \sum_{I,J} *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes (e_I \wedge e_{2,s}) \otimes (e_{J'} \wedge y) \otimes y^p,$$

where J' is the multi-index obtained by dropping the s-th index from J. However, A_2 is obviously an $SO(m)$-invariant embedding of a multiple of the element:

$$\sum_{I,J} *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes (e_I \wedge e_{2,s}) \otimes e_{J'} \in (\textstyle\bigwedge^k W_y^* \otimes \bigwedge^{m-k} W_y^*) \otimes (\bigwedge^{q+1} V \otimes \bigwedge^{q-1} V).$$

Again, by Lemma 2.43, none of the $SO(m)$ submodules of $\bigwedge^{q+1} V \otimes \bigwedge^{q-1} V$ are isomorphic to submodules of $\bigwedge_p^{k,m-k} W_y^*$, thus, A_2 is a multiple of $d\alpha$. We obtain:

$$\pi_2(\mathrm{Id}_{\mu,g}) \overset{\mathrm{mod}\ d\alpha}{\equiv} \mathrm{Id}_{\mu,g}\,.$$

Similarly, the permutation $\pi_3 = \sum_{r=1}^{q} S_{r,1}$ exchanges the vector y from the third column with the r-th vector $e_{2,r}$, thus, yielding an embedding of the $SO(m)$-invariant element

$$A_3 = \sum_{I,J} *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes (e_I \otimes e_{J'}) \otimes e_{2,r}$$

from the space $(\bigwedge^k W_y^* \otimes \bigwedge^{m-k} W_y^*) \otimes (\bigwedge^q V \otimes \bigwedge^{q-1} V \otimes V)$. We again apply Lemma 2.43 to see that the only RV-module in this tensor product is $\Gamma_0^{q,0} = \Gamma_\mu$. Hence, A_3 is modulo $d\alpha$ a – possibly zero – multiple of $\mathrm{Id}_{\mu,g}$.

All in all, one has $\mathrm{Id}_\mu - C_{n,k,\mu,\lambda}\,\mathrm{Id}_{\mu,g} = d\alpha \wedge \tau$, where $\tau \in \bigwedge^{k-1} W_y^* \otimes \bigwedge^{m-k-1} W_y^* \otimes \bigwedge_\lambda V$ and $C_{n,k,\mu,\lambda}$ is some non-zero constant. Denoting by $\mathrm{Id}_{k,q,\lambda} \in \Omega^{k,m-k} \otimes \bigwedge_\lambda V$ the $SO(n)$-covariant differential form induced by $\mathrm{Id}_{\mu,g}$, we obtain:

$$\mathrm{Integ}\,\mathrm{Id}_{k,q,\lambda} \equiv \mathrm{Integ}\,\widehat{\mathrm{Id}}_{(k,q,\lambda)}.$$

According to the proof of Theorem 4.4, there is another linearly independent curvature measure if $m := n-1$ is even and $q = m/2$. By arguments similar to those for the regular curvature measures, we may canonically define another $SO(m)$-invariant element:

$$\mathrm{Id}^*_\mu \quad \equiv \quad \sum_{I,J} (e_I \otimes *e_J) \otimes e_I \otimes e_J,$$

where $*$ is the Hodge-star operator. Id_μ and Id^*_μ are linearly independent. To see this, observe that $\mathrm{Id}_{[\mu]} := \pi_{\mathrm{tr}}(\mathrm{Id}_\mu)$ and $\mathrm{Id}^*_{[\mu]} := \pi_{\mathrm{tr}}(\mathrm{Id}^*_\mu)$ are $SO(m)$-invariant elements in $\bar{\Gamma}^{m,0}_0 \otimes \bar{\Gamma}^{m,0}_0$. By the argument after Theorem 2.35, $\bar{\Gamma}^{m,0}_0$ consists of two irreducible $SO(m)$-modules. We set $\tau : \bigwedge^m V \to \bigwedge^m V^* \simeq \bigwedge^m V$ the duality map induced by Q and $\tilde{\tau}$ the one induced by $\tilde{Q}$. Then $\mathrm{Id}_{[\mu]}$ corresponds to the map $\tau \cdot \tilde{\tau}$, whereas $\mathrm{Id}^*_{[\mu]} = \tilde{\tau} \cdot \tilde{\tau}$, implying that both elements are linearly independent and, hence, so are Id^*_μ and Id_μ. □

Instead of studying $\mathrm{Id}_{k,q,\lambda} \in \Omega^{k,n-k-1}_p \otimes \Gamma_\lambda \subset \Omega^{k,n-k-1}_p \otimes \bigwedge_\lambda V$, we work with their partial symmetrisations $a'_\lambda(\mathrm{Id}_{k,q,\lambda}) \in \Omega^{k,n-k-1}_p \otimes \Gamma_\lambda \subset \Omega^{k,n-k-1}_p \otimes W_\lambda V$

$$a'_\lambda(\mathrm{Id}_{k,q,\lambda}) \equiv \sum_{I,J} y \wedge *_2((dx^I \otimes dy^J) \wedge d\alpha^{k-q}) \otimes e_I \cdot e_J \wedge_{KN} y^{\lambda-\mu},$$

where we use the Kulkarni-Nomizu product instead of the generalised wedge-product. Writing out all cases of $a'_\lambda(\mathrm{Id}_{k,q,\lambda})$ yields the following distinguished families of primitive differential forms:

1. For $k, p, q, n \in \mathbb{N}$ with $0 \le k \le n-1$ and $0 \le q \le k' := \min(k, n-k-1)$:

$$\Phi^n_{(k,p,q)} := \sum_{\substack{i_1,\ldots,i_q=1 \\ \pi \in S_n}}^{n} \operatorname{sgn}\pi \; y_{\pi(1)}\, dx^{i_1\ldots i_q \pi(q+2)\ldots\pi(k+1)} \wedge dy^{\pi(k+2)\ldots\pi(n)} \otimes e_{\pi(2)\ldots\pi(q+1)} \cdot e_{i_1\ldots i_q} \cdot y^p$$

   taking values in the $SL(n)$-module of type $[q; p; 0]$.

2. For $k, p, q, n \in \mathbb{N}$ with $0 \le k \le n-1$ and $1 \le q \le k'+1$, set:

$$\Psi^n_{(k,p,q)} := \sum_{\substack{i_1,\ldots,i_{q-1}=1 \\ \pi \in S_n}}^{n} \operatorname{sgn}\pi \; y_{\pi(1)}\, dx^{i_1\ldots i_{q-1} \pi(q+1)\ldots\pi(k+1)} \wedge dy^{\pi(k+2)\ldots\pi(n)} \otimes e_{\pi(2)\ldots\pi(q)y} \cdot e_{i_1\ldots i_{q-1}y} \otimes y^p$$

   taking values in the $SL(n)$-module of type $[q; p; 0]$.

3. For $k, p, q, n \in \mathbb{N}$ with $0 \le k \le n-1$ and $0 \le q \le k'$, set:

$$\Xi^n_{(k,p,q)} := \sum_{\substack{i_1,\ldots,i_q=1 \\ \pi \in S_n}}^{n} \operatorname{sgn}\pi \; y_{\pi(1)}\, dx^{i_1\ldots i_q \pi(q+2)\ldots\pi(k+1)} \wedge dx^{\pi(k+2)\ldots\pi(n)} \otimes e_{\pi(2)\ldots\pi(q+1)y} \otimes e_{i_1\ldots i_q} \cdot y^p$$

   taking values in the $SL(n)$-module of type $[q; p; 1]$.

4. For $p, n \in \mathbb{N}$ such that $n = 2j+1$ and $q = j$:

$$\Theta^n_{(p)} \quad := \quad \sum_{\substack{i_1,\ldots,i_q=1 \\ j_1,\ldots j_q}}^{n} dx^{i_1\ldots i_q} \wedge dy^{j_1\ldots j_q} \otimes e_{i_1\ldots i_q} \cdot e_{j_1\ldots j_q} \cdot y^p.$$

   taking values in the $SL(n)$-module of type $[j; p; 0]$.

Important Remark 4.13. Going forward, we will drop the subscript KN in the notation of the Kulkarni-Nomizu product, as we are going to only work with these partially symmetrised forms. Hence, from now on, any occurrence of $\wedge$ in the tensor-part of differential forms should be understood as the Kulkarni-Nomizu wedge-product and **not** the generalised wedge-product.

Remark 4.14. Observe the usage of the Kulkarni-Nomizu dot-product $\cdot y^p$ in the above formulas. According to Example 2.52, $\cdot y^p$ evaluates to $\otimes y^p$ for $q > 1$ and to the usual symmetric product $\cdot y^p$ for $q = 1$. As the tensor part of $\Psi^n_{(k,p,1)}$ is written as $y^2 \cdot y^p = y^2 \otimes y^p$, we may replace $\cdot y^p$ with $\otimes y^p$ in the general definition of $\Psi^n_{(k,p,q)}$.

This irregularity for $q = 1$ is responsible for the following linear dependencies between the families:

$$\Phi^n_{(k,1,0)} = \Xi^n_{(k,0,0)} \qquad \text{and} \qquad \Psi^n_{(k,p,1)} = \Xi^n_{(k,p+1,0)} = \Phi^n_{(k,p+2,0)} \tag{4.4}$$

for all k and p. These dependencies are, however, the only ones.

We will use the families $T^n_{(\cdot)}$ to derive different bases of curvature measures and valuations. Whenever a result is only applicable for $T^n_{(\cdot)}$ – or its derivatives – as a basis, we explicitly exclude $\Psi^n_{(k,p,1)}$ and $\Xi^n_{(k,p,0)}$. However, many results are easier to state and prove if we assume the families to contain the redundant forms. For the same reason, we also trivially extend all families for all p, q, k not explicitly mentioned in the list.

Remark 4.15. For fixed k, n, and p, the differential forms in these families assume values in the irreducible $SL(n)$-module Γ_λ of type $[q; p; r]$ – or rather the (reducible) $SO(n)$-module $\operatorname{Res}\Gamma_\lambda^{SL(n)}$. Ignoring this subtle distinction, we call $T_{(\cdot)}$ the families of $SL(n)$-module-valued forms and sometimes denote them $T^n_{(k,\lambda)}$ instead of $T^n_{(k,p,q)}$ if p and q can be inferred from λ.

Inspired by this, we often refer to the *entire family* of such differential forms as $SO(n)$-covariant or $SL(n)$-module-valued even if n is not fixed, although the dimension n is, strictly speaking, not a well-defined notion in this context.

Remark 4.16. The curvature measures $\Theta^n_{(p)}$ are $SO(n)$- but not $O(n)$-covariant. As an element in $\Omega_p^{k,n-k-1} \otimes W_\lambda V$, $\Theta^n_{(p)}$ is directly seen to be $O(n)$-invariant. As, however, $\operatorname{nc}(gK) = \det(g) g \operatorname{nc}(K)$, one has:

$$\int_{\operatorname{nc}(gK)} \Theta^n_{(p)} = \det(g) \int_{g \operatorname{nc}(K)} \Theta^n_{(p)} = \det(g) \int_{\operatorname{nc}(K)} g^* \Theta^n_{(p)} = \det(g) \int_{\operatorname{nc}(K)} \Theta^n_{(p)}.$$

By setting element-wise $T^n_{[k,p,q]} := \pi_{[\mathrm{tr}]}(T^n_{(k,p,q)})$, where $T \in \{\Phi, \Psi, \Xi, \Theta\}$ and the projection $\pi_{[\mathrm{tr}]}$ from (2.23) is applied to the tensor-part of the differential form, we obtain, for a fixed n, four new families of $SO(n)$-covariant differential forms with values in the irreducible $SO(n)$-modules $\Gamma_{[\lambda]}$. Theorem 4.4 then yields the following result.

Theorem 4.17. *Let V be an n-dimensional space with $n < \infty$, $0 \le k \le n-1$, and $\Gamma_r^{q,p}$ be a locally valuative $SO(n)$-module.*

1. *If $q = p = 0$, then $\Phi_{[k,0,0]}$ is a multiple of the k-th Lipschitz-Killing curvature form known to be the unique, up to scalars, form that induces a non-trivial curvature measure, namely the k-th Lipschitz-Killing curvature measure.*

2. *Every $\Gamma_1^{p,q}$-valued curvature measure can be written as*

$$X_{[k,p,q,1]} = a \operatorname{Integ} \Xi^n_{[k,p,q]}, \quad a \in \mathbb{R}.$$

3. If $q > 0$, then every $\Gamma_0^{p,q}$-valued curvature measure can be written as

$$X_{[k,p,q,0]} = \operatorname{Integ}\left(a_1\, \Psi^n_{[k,p,q]} + a_2\, \Phi^n_{[k,p,q]} + a_3\, \Theta^n_{[k,p,q]}\right), \quad a_1, a_2, a_3 \in \mathbb{R},$$

where $a_1 = 0$ if $q = 0$, $a_2 = 0$ if $m_{k,n,q} < 2$, and $a_3 = 0$ if $m_{k,n,q} < 3$.

4. Every $\Gamma_{-2}^{p,q}$-valued curvature measure is written as:

$$X_{[k,p,q,-2]} = a \operatorname{Integ} *\Phi^n_{[k,p,q]}, \quad a \in \mathbb{R}.$$

Remark 4.18. We will often omit the operator Integ if there is no risk of confusion as to whether a curvature measure or its underlying form is referred to.

Note that the above – or any other – basis is unique *up to embedding maps.* Let Γ be a functor that assigns to each Euclidean vector space of dimension n an $SO(n)$-module $\Gamma = \bigoplus_\lambda \Gamma_{[\mu]}$ that can be written as a direct sum of mutually irreducible $SO(n)$-modules $\Gamma_{[\mu]}$ with embeddings $\iota_\mu : \Gamma_{[\mu]} \to \Gamma$. Then every family of Γ-valued $SO(n)$-covariant curvature measures X of a fixed homogeneity degree k can be written as a sum:

$$X = \sum_\mu \iota_\mu \left(a_1\, \Phi_{[k,\mu]} + a_2\, \Psi_{[k,\mu]} + a_3\, \Xi_{[k,\mu]} + a_4\, \Theta_{[k,\mu]}\right).$$

If there exist a partition λ such that $\Gamma \subset \Lambda_\lambda V$ and the conditions of Lemma 2.55 are fulfilled for all μ, then we may re-write the above identity as follows:

$$X = \sum_\mu \left(a_1\, \Phi_{[k,\mu]} + a_2\, \Psi_{[k,\mu]} + a_3\, \Xi_{[k,\mu]} + a_4\, \Theta_{[k,\mu]}\right) \wedge_\lambda Q^{|\lambda-\mu|/2}.$$

Recall that these embeddings are not, in general, canonical if $m(\Gamma, \Gamma_\mu) > 1$, hence, we may need to additionally specify the entire sequence of "interim" embeddings as in the argument after Lemma 2.55. In this sense, the families $T^n_{(k,p,q)}$ defined after Proposition 4.12 induce another basis of the space of $(\mathrm{Curv}_k^{sm} \otimes \Gamma)^{SO(n)}$.

Corollary 4.19. *The $SO(n)$-covariant $SL(n)$-module-valued curvature measures given by the forms $\Phi^n_{(k,p,q)}$, $\Xi^n_{(k,p,q)}$ for $q > 0$, $\Psi^n_{(k,p,q)}$ for $q > 1$, and $\Theta^n_{(p)}$ comprise a basis of k-homogeneous $SO(n)$-covariant tensor-valued curvature measures on $\mathbb{R}^n$.*

Proof. The correspondence between $T_{(\cdot)}$ and $T_{[\cdot]}$ is one-to-one and the forms $T^n_{(k,\lambda)}$ are all linearly-independent – modulo the lower-rank relations from (4.4) – for all k, n, λ, for which $T^n_{(k,p,q)}$ are not trivial. □

The relations between $T_{[\cdot]}$ and $T_{(\cdot)}$ are not quite trivial, as the following special case of $p = 0$ shows. The general relations for arbitrary p are still not known.

Proposition 4.20. *If $T \in \{\Phi, \Xi\}$, then:*

$$T^n_{[k,0,q]} = \sum_{j=0}^{q} \frac{(-1)^j j!}{\prod_{r=1}^{j}(n-2q-1+3r)} \binom{q}{j}^2 T^n_{(k,0,q-j)} \wedge Q^j.$$

For the forms $\Psi^n_{[k,0,q]}$, one has the following identity:

$$\Psi^n_{[k,0,q]} = \sum_{j=0}^{q-1} \frac{(-1)^j j!}{\prod_{r=1}^{j}(n-2q-1+3r)} \left(\binom{q-1}{j}^2 \Psi^n_{(k,0,q-j)} + \binom{q-1}{j-1}^2 \Phi^n_{(k,0,q-j)}\right) \wedge Q^j + \frac{(-1)^q q!}{\prod_{r=1}^{q}(n-2q-1+3r)} \Phi^n_{k,0,0}.$$

This result is a direct consequence of Proposition 2.62 and the following Lemma.

Lemma 4.21. *With the notation from (2.23), one has for all $q \geq 0$:*

$$\operatorname{tr}^j T^n_{(k,0,q)} = \prod_{r=q-j+1}^{q} r^2 \;\; \Phi^n_{(k,0,q-j)} = \frac{(q!)^2}{(q-j)!^2}\, \Phi^n_{(k,0,q-j)}$$

if $T \in \{\Phi, \Xi\}$ and

$$\operatorname{tr}^j \Psi^n_{(k,0,q)} = j\,\frac{(q-1)!^2}{(q-j)!^2}\, \Phi^n_{(k,0,q-j)} + \frac{(q-1)!^2}{(q-j-1)!^2}\, \Psi^n_{(k,0,q-j)}.$$

Proof. The expressions for general j may be recursively obtained from the following identities in a similar fashion as in the proof of Lemma 2.61:

$$\operatorname{tr}\Phi_{(k,0,q)} = q^2\,\Phi_{(k,0,q-1)}, \quad \operatorname{tr}\Xi_{(k,0,q)} = q^2\,\Xi_{(k,0,q-1)},$$
$$\operatorname{tr}\Psi_{(k,0,q)} = \Phi_{(k,0,q-1)} + (q-1)^2\Psi_{(k,0,q-1)}.$$

To show these identities, we compute by the definition of the trace map:

$$\begin{aligned}
\operatorname{tr}\Phi_{(k,0,q)} &= \sum_{i_1,\dots,i_q,\pi} \operatorname{sgn}\pi\; y_{\pi(1)}\, dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes \operatorname{tr}(e_{\pi(2)\dots\pi(q+1)} \cdot e_{i_1\dots i_q}) \\
&= \sum_{r,s=1}^{q} (-1)^{r+s} \sum_{i_1,\dots,i_q,\pi} y_{\pi(1)}\, dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \\
&\qquad\qquad \otimes Q(e_{i_r}, e_{\pi(s+1)})\, e_{\pi(2)\dots\widehat{\pi(s+1)}\dots\pi(q+1)} \cdot e_{i_1\dots\widehat{i_r}\dots i_q}.
\end{aligned}$$

As $Q(e_{i_r}, e_{\pi(s+1)}) = \delta_{i_r,\pi(s+1)}$, we obtain:

$$\begin{aligned}
\operatorname{tr}\Phi_{(k,0,q)} &= \sum_{r,s,i_1,\dots,\widehat{i_r},\dots,i_q,\pi} (-1)^{r+s} \operatorname{sgn}\pi\; y_{\pi(1)}\, dx^{i_1\dots\pi(s+1)\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \\
&\qquad\qquad \otimes e_{\pi(2)\dots\widehat{\pi(s+1)}\dots\pi(q+1)} \cdot e_{i_1\dots\widehat{i_r}\dots i_q}.
\end{aligned}$$

Moving $dx^{\pi(s+1)}$ to the right of dx^{i_q} and renaming indices $i_{r+1},\dots,i_q$ into $i_r,\dots,i_{q-1}$ yields:

$$\begin{aligned}
\operatorname{tr}\Phi_{(k,0,q)} &= \sum_{\substack{i_1,\dots,i_{q-1}\\ r,s,\pi}} (-1)^{q+s} \operatorname{sgn}\pi\; y_{\pi(1)}\, dx^{i_1\dots i_{q-1}\pi(s+1)\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \\
&\qquad\qquad \otimes e_{\pi(2)\dots\widehat{\pi(s+1)}\dots\pi(q+1)} \cdot e_{i_1\dots i_{q-1}}.
\end{aligned}$$

The last step is to permute every term in the sum over π by the cycle $(s{+}2\; s{+}3 \dots q{+}1\; s{+}1)$ with the sign $(-1)^{q-s}$. As $(-1)^{q-s+q+s} = 1$, it follows:

$$\begin{aligned}
\operatorname{tr}\Phi_{(k,0,q)} &= \sum_{\substack{i_1,\dots,i_{q-1}\\ r,s,\pi}} \operatorname{sgn}\pi\; y_{\pi(1)}\, dx^{i_1\dots i_{q-1}\pi(q+1)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes e_{\pi(2)\dots\pi(q)} \cdot e_{i_1\dots i_{q-1}} \\
&= \sum_{r,s=1}^{q} \Phi_{(k,0,q-1)} = q^2\,\Phi_{(k,0,q-1)}.
\end{aligned}$$

To compute the trace for $\Xi_{(k,0,q)}$ and $\Psi_{(k,0,q)}$, recall that the trace map is $SO(n)$-invariant, hence, it suffices to compute these identities at the point $(0, e_1) \in S\mathbb{R}^n$. In this case, $\Xi^n_{(k,0,q)}\Big|_{(0,e_1)} = \Phi^n_{(k,0,q)}\Big|_{(0,e_1)} \wedge e_1$ and $\Phi^n_{(k,0,q)}\Big|_{(0,e_1)}$ may be written as:

$$\Phi^n_{(k,0,q)}\Big|_{(0,e_1)} = \sum_{\substack{i_1,\dots,i_q=2 \\ \pi(1)=1}}^{n} \operatorname{sgn}\pi \; dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes e_{\pi(2)\dots\pi(q+1)} \cdot e_{i_1\dots i_q}.$$

Since $Q(e_1, e_j) = 0$ for all $j = 2, \dots, n$, one has:

$$\operatorname{tr}\left(\Xi^n_{(k,0,q)}\Big|_{(0,e_1)} \right) = \left(\operatorname{tr} \Phi^n_{(k,0,q)}\Big|_{(0,e_1)} \right) \wedge e_1 = q^2 \, \Phi^n_{(k,0,q-1)}\Big|_{(0,e_1)} \wedge e_1 = \Xi^n_{(k,0,q-1)}\Big|_{(0,e_1)}.$$

By the same token, $\Psi^n_{(k,0,q)}\Big|_{(0,e_1)} = \Phi^n_{(k,0,q-1)}\Big|_{(0,e_1)} \wedge_\lambda e_1^2$ and the trace is now given by:

$$\begin{aligned} \operatorname{tr}\left(\Phi^n_{(k,0,q-1)}\Big|_{(0,e_1)} \wedge e_1^2 \right) &= \left(\operatorname{tr} \Phi^n_{(k,0,q-1)}\Big|_{(0,e_1)} \right) \wedge e_1^2 + (-1)^{2q} \, \Phi^n_{(k,0,q-1)}\Big|_{(0,e_1)} \wedge \operatorname{tr}(e_1^2) \\ &= (q-1)^2 \, \Phi^n_{(k,0,q-2)}\Big|_{(0,e_1)} \wedge e_1^2 + \Phi^n_{(k,0,q-1)}\Big|_{(0,e_1)}. \end{aligned}$$

□

4.2.2 McMullen's Theorem and Horizontality

In addition to the two bases from the previous section, we may find a basis that generalises the behaviour of the $\operatorname{Sym}^p V$-valued curvature measures from Theorem 3.48. To simplify the computations, we assume $V = \mathbb{R}^{n-1}$ and $W = \mathbb{R}^n$ with the canonical embedding, such that $W = V \oplus \langle e_n \rangle$. The same notation applies to differential forms. Let

$$\begin{aligned} H : S\mathbb{R}^{n-1} \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] &\to S\mathbb{R}^n \\ (x, v, \phi) &\mapsto ((x, 0), \cos\phi v + \sin\phi e_n) \end{aligned}$$

be the parametrisation of $S\mathbb{R}^n$. Then the pullback via an isometric embedding $\iota : V \to W$ of a differential form $\phi \in \left(\Omega^{n-1}_p(S\mathbb{R}^n) \right)^{\mathbb{R}^n} \otimes \Gamma(\mathbb{R}^n) \big)^{SO(n)}$ to a lower-dimensional space:

$$\iota^*(\phi) := \operatorname{Integ} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} H^*\phi \in \left(\Omega^{n-2}(S\mathbb{R}^{n-1})^{\mathbb{R}^{n-1}} \otimes \operatorname{Res}^{SO(n)}_{SO(n-1)} \Gamma(\mathbb{R}^n) \right)^{SO(n-1)},$$

where pullback and integration are only applied on the differential form and the tensor-part is restricted in the sense of (3.24). Set

$$\Phi^n_{k,p,q} := C_{\Phi,n,k,p} \, \Phi^n_{(k,p,q)}, \quad \Psi^n_{k,p,q} := C_{\Psi,n,k,p,q} \, \Psi^n_{(k,p,q)}, \quad \Xi^n_{k,p,q} := C_{\Xi,n,k,p,q} \, \Xi^n_{(k,p,q)}$$

with the following constants:

$$C_{\Phi,n,k,p} := \frac{1}{p!k!(n-k-1)!s_{n-k+p-1}}, \quad C_{\Xi,n,k,p,q} := \frac{1}{p_{\Xi,q} k!(n-k-1)!s_{n-k+p}},$$

$$C_{\Psi,n,k,p,q} := \frac{1}{p_{\Psi,q} k!(n-k-1)!s_{n-k+p+1}},$$

where $p_{\Xi,q} = p!$ if $q > 1$ and $(p+1)!$ if q=1 and, similarly, $p_{\Psi,q} = p!$ if $q > 1$ and $(p+2)!$ if $q = 1$. Then the relations for $q < 2$ are still satisfied:

$$\Phi^n_{k,0,1} = \Xi^n_{k,0,0} \qquad \text{and} \qquad \Psi^n_{k,1,p} = \Xi^n_{k,0,p+1} = \Phi^n_{k,0,p+2}. \tag{4.5}$$

and $\Phi^n_{k,0,0} = \Phi^n_k$ is the k-th intrinsic curvature measure in $\mathbb{R}^n$. Furthermore, the following generalisation of McMullen's Theorem 3.48 holds.

Theorem 4.22. *Let $\Phi^n_{k,p,q}$, $\Xi^n_{k,p,q}$, $\Psi^n_{k,p,q}$ be as defined above. Then*

a) For all $q \in \mathbb{N}$:

$$\iota^*\Phi^n_{k,p,q} = \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{1}{(4\pi)^j j!} \Phi^{n-1}_{k,p-2j,q} \cdot e_n^{2j}.$$

b) For $q > 0$:

$$\iota^*\Xi^n_{k,p,q} = \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{1}{(4\pi)^j j!} \Xi^{n-1}_{k,p-2j,q} \cdot e_n^{2j} + \frac{1}{2\pi} \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{1}{(4\pi)^j j!} \Phi^{n-1}_{k,p-2j-1,q} \wedge (e_n \otimes 1^{q-1} \otimes e_n) \cdot e_n^{2j}$$

c) For $q > 1$:

$$\begin{aligned} \iota^*\Psi^n_{k,p,q} = & \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{1}{(4\pi)^j j!} \tilde{\Psi}^{n-1}_{k,p-2j,q} \cdot e_n^{2j} + \frac{1}{2\pi} \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{2j+1}{(4\pi)^j j!} \Phi^{n-1}_{k,p-2j,q-1} \cdot e_n^{2j} \wedge e_n^2 \\ & + \frac{1}{\pi} \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{1}{(4\pi)^j j!} \Xi^{n-1}_{k,p-2j-1,q-1} \cdot e_n^{2j+1} \cdot (e_n \otimes 1^{q-2} \otimes e_n). \end{aligned}$$

Remark 4.23. Due to the irregular nature of the exceptional curvature measures, we will not formulate McMullen's Theorem for them.

Definition 4.24. Setting $\Theta^n_p := \Theta^n_{(p)}$, we obtain another family of differential forms which we call the *McMullen*-family. Since all elements $T^n_{k,\lambda}$ are just multiples of $T^n_{(k,\lambda)}$, they obviously form – excluding $\Psi^n_{k,p,1}$ and $\Xi^n_{k,p,0}$ – another basis of tensor-valued $SO(n)$-covariant curvature measures. Naturally, we call it the *McMullen*-basis.

Proof. It suffices to prove the claim at an arbitrary point $(x, y) \in S\mathbb{R}^n \subset T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$. If x_i, y_i are standard coordinates on $S\mathbb{R}^{n-1} \subset S\mathbb{R}^n \subset T\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n$, then:

$$H^*x_i = \begin{cases} 0, & \text{if } i = n, \\ x_i, & \text{if } i < n, \end{cases} \qquad H^*y_i = \begin{cases} \sin\phi, & \text{if } i = n, \\ \cos\phi \, y_i, & \text{if } i < n \end{cases}$$

and

$$H^*dx^i = \begin{cases} 0, & \text{if } i = n, \\ dx^i, & \text{if } i < n, \end{cases} \qquad H^*dy^i = \begin{cases} \cos\phi \, d\phi, & \text{if } i = n, \\ -\sin\phi \, y_i \, d\phi + \cos\phi \, dy^i, & \text{if } i < n. \end{cases}$$

In particular, one obtains:

$$H^*y^p = (\cos\phi\,\bar{y} + \sin\phi\, e_n)^p = \sum_{j=0}^{p} \binom{p}{j} \cos^{p-j}\phi\, \sin^j\phi\, \bar{y}^{p-j} e_n^j, \tag{4.6}$$

where $\bar{y} = \frac{1}{\sum_{i=1}^{n-1} y_i^2}(y_1, \dots, y_{n-1})$ is the normalised projection of y to S^{n-1}. As $H^*dx^n = 0$, we may now compute:

$$\begin{aligned} H^*\Phi^n_{(k,0,q)} &= H^* \sum_{\pi(1)=n} \operatorname{sgn}\pi y_n\, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes e_{i_1\dots i_q} \cdot e_{\pi(2)\dots\pi(q+1)} \\ &\quad + (n-k-1) H^* \sum_{\pi(k+2)=n} \operatorname{sgn}\pi y_{\pi(1)}\, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{n\pi(k+3)\dots\pi(n)} \otimes e_{i_1\dots i_q} \cdot e_{\pi(2)\dots\pi(q+1)} \\ &\quad + q\, H^* \sum_{\pi(2)=n} \operatorname{sgn}\pi y_{\pi(1)}\, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes e_{i_1\dots i_q} \cdot e_{n\pi(3)\dots\pi(q+1)} \\ &=: I + (n-k-1)\cdot II + q \cdot III, \end{aligned}$$

where the sums are over $\pi \in S_n$ and $i_1, \dots, i_q = 1, \dots, n$. Furthermore, if $n \notin \{\pi(a), \pi(a+1), \dots, \pi(b)\}$, then it is straight-forward to show:

$$\begin{aligned} H^* \sum_{\pi\in S_\ell} \operatorname{sgn}(\pi) dy^{\pi(a),\dots,\pi(b)} &= \sum_{\pi\in S_\ell} \operatorname{sgn}(\pi) \bigwedge_{i=a}^{b} (-\sin\phi\, y_{\pi(i)}\, d\phi + \cos\phi\, dy^{\pi(i)}) \\ &= \sum_{\pi\in S_\ell} \operatorname{sgn}(\pi) \left(\cos^{b-a+1}\phi\, dy^{\pi(a)\dots\pi(b)} - (b-a+1) y_{\pi(a)} \sin\phi \cos^{b-a}\phi\, d\phi \wedge dy^{\pi(a+1)\dots\pi(b)} \right). \end{aligned}$$

In the terms I, II, and III, we only keep multiples of $d\phi$, since the fiber integration of the other terms would yield 0. The equality modulo some non-multiple of $d\phi$ is then denoted by $\equiv$. Then the following holds:

$$\begin{aligned} I &= \sum_{\pi(1)=n} \operatorname{sgn}\pi \sin\phi\, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge (\cos^{n-k-1}\phi\, dy^{\pi(k+2)\dots\pi(n)} \\ &\qquad - (n-k-1)\, y_{\pi(k+2)} \sin\phi \cos^{n-k-2}\phi\, d\phi \wedge dy^{\pi(k+3)\dots\pi(n)}) \otimes e_{i_1\dots i_q} \cdot e_{\pi(2)\dots\pi(q+1)} \\ &\equiv -(n-k-1) \sum_{\pi(1)=n} \operatorname{sgn}\pi\, y_{\pi(k+2)} \sin^2\phi \cos^{n-k-2}\phi\, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge d\phi \\ &\qquad \wedge dy^{\pi(k+3)\dots\pi(n)} \otimes e_{i_1\dots i_q} \cdot e_{\pi(2)\dots\pi(q+1)}. \end{aligned}$$

The above sums are over $\pi \in S_n$ and $i_1, \dots, i_q = 1, \dots n-1$, since $H^*dx^n = 0$. By applying the permutation $\pi' := (1 \dots k+2)$ and taking the term $d\phi$ to the last position, we obtain

$$\begin{aligned} I &= (n-k-1) \sum_{\pi(1)=n} \operatorname{sgn}\pi\, \sin^2\phi \cos^{n-k-2}\phi\, y_{\pi(1)}\, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \wedge d\phi \\ &\qquad \otimes e_{i_1\dots i_q} \cdot e_{\pi(2)\dots\pi(q+1)} \\ &= (n-k-1) \sin^2\phi \cos^{n-k-2}\phi\, \Phi^{n-1}_{(k,0,q)} \wedge d\phi. \end{aligned}$$

Similar computation yields

$$
\begin{aligned}
II &= \sum_{\pi(k+2)=n} \operatorname{sgn}\pi\ \cos\phi\, y_{\pi(1)} dx^{i_1\ldots i_q \pi(q+2)\ldots\pi(k+1)} \wedge (\cos^{n-k-1}\phi\, d\phi \wedge dy^{\pi(k+3)\ldots\pi(n)}) \\
&\qquad\qquad\qquad\qquad \otimes e_{i_1\ldots i_q}\cdot e_{\pi(2)\ldots\pi(q+1)} \\
&= \sum_{\pi(k+2)=n} \operatorname{sgn}\pi\ \cos^{n-k}\phi\, y_{\pi(1)}\, dx^{i_1\ldots i_q \pi(q+2)\ldots\pi(k+1)} \wedge dy^{\pi(k+2)\ldots\pi(n-1)} \wedge d\phi \\
&\qquad\qquad\qquad\qquad \otimes e_{i_1\ldots i_q}\cdot e_{\pi(2)\ldots\pi(q+1)} \\
&= \cos^{n-k}\phi\, \Phi^{n-1}_{(k,0,q)} \wedge d\phi
\end{aligned}
$$

and

$$
\begin{aligned}
III &= \sum_{\pi(1)=n} \operatorname{sgn}\pi \cos\phi\, y_{\pi(1)} dx^{i_1\ldots i_q \pi(q+2)\ldots\pi(k+1)} \wedge (\cos^{n-k-1}\phi\, dy^{\pi(k+2)\ldots\pi(n)} \\
&\quad -(n-k-1)\, y_{\pi(k+2)} \sin\phi \cos^{n-k-2}\phi\, d\phi \wedge dy^{\pi(k+3)\ldots\pi(n)}) \otimes\ e_{i_1\ldots i_q}\cdot e_{\pi(2)\ldots\pi(q+1)} \\
&\equiv -(n-k-1) \sum_{\pi(1)=n} \operatorname{sgn}\pi \cos^{n-k-1}\phi\, \sin\phi\, y_{\pi(1)} y_{\pi(k+2)}\, dx^{i_1\ldots i_q \pi(q+2)\ldots\pi(k+1)} \wedge d\phi \\
&\qquad\qquad\qquad\qquad \wedge dy^{\pi(k+3)\ldots\pi(n)} \otimes\ e_{i_1\ldots i_q}\cdot e_{\pi(2)\ldots\pi(q+1)} \\
&= 0.
\end{aligned}
$$

The last identity can be seen, as we may permute all elements by the 2-cycle $\pi' := (1, k+2)$, which changes the sign while preserving the form, thus yielding $III = -III$.

By summing I and II, one obtains:

$$
H^*\Phi^n_{(k,0,q)} = I + (n-k-1)II = (n-k-1)\cos^{n-k-2}\phi\, \Phi^{n-1}_{(k,0,q)} \wedge d\phi.
$$

As $\Phi^n_{(k,p,q)} = \Phi^n_{(k,0,q)} \cdot y^p$ and H^* is a linear map, we obtain by (4.6):

$$
H^*\Phi^n_{(k,p,q)} = (n-k-1)\sum_{j=0}^{p} \binom{p}{j} \sin^j\phi \cos^{n-k+p-j-2}\phi\, (\Phi^{n-1}_{(k,p-j,q)} \cdot e_n^j) \wedge d\phi.
$$

For $\Xi^n_{(k,p,q)} = \Phi^n_{(k,p,q)} \wedge y$, we have in a similar fashion:

$$
\begin{aligned}
H^*\Xi^n_{(k,p,q)} &= H^*\Phi^n_{(k,p,q)} \wedge H^*y = H^*\Phi^n_{(k,p,q)} \wedge H^*(\cos\phi\, \bar{y} + \sin\phi\, e_n) \\
&= (n-k-1)\sum_{j=0}^{p} \binom{p}{j} \Big[\sin^j\phi \cos^{n-k+p-j-1}\phi\, (\Phi^{n-1}_{(k,p-j,q)} \wedge (\bar{y} \otimes 1 \otimes e_n^j)) \wedge d\phi \\
&\qquad + \sin^{j+1}\phi \cos^{n-k+p-j-2}\phi\, (\Phi^{n-1}_{(k,p-j,q)} \wedge (e_n \otimes 1 \otimes e_n^j)) \wedge d\phi\Big] \\
&= (n-k-1)\sum_{j=0}^{p} \binom{p}{j} \Big[\sin^j\phi \cos^{n-k+p-j-1}\phi\, (\Xi^{n-1}_{(k,p-j,q)} \cdot e_n^j) \wedge d\phi \\
&\qquad + \sin^{j+1}\phi \cos^{n-k+p-j-2}\phi\, (\Phi^{n-1}_{(k,p-j,q)} \wedge (e_n \otimes 1 \otimes e_n^j)) \wedge d\phi\Big],
\end{aligned}
$$

where $(v \otimes 1 \otimes e_n^j) \in W_{(p;q;1)-(p-j;q;0)}V$, $v \in \{\bar{y}, e_n\}$. A similar but slightly more intricate computation yields for $\Psi^n_{(k,p,q)}$:

$$\begin{aligned} H^*\Psi^n_{(k,p,q)} &= H^*\Phi^n_{(k,p,q-1)} \wedge H^* y^2 \\ &= (n-k-1)\sum_{j=0}^{p}\binom{p}{j}\sin^j\phi\cos^{n-k+p-j-2}\phi\,\Big[\cos^2\phi\,\Psi^{n-1}_{(k,p-j,q)}\cdot e_n^j \qquad (4.7) \\ &\quad + 2\sin\phi\cos\phi\,\Xi^{n-1}_{(k,p-j,q-1)}\wedge(1\otimes e_n\otimes e_n^j) + \sin^2\phi\,\Phi^{n-1}_{(k,p-j,q-1)}\wedge(e_n^2\otimes e_n^j)\Big]\wedge d\phi. \end{aligned}$$

With the above identities at hand, we may now compute the restrictions of the differential forms in question. Recall that $\int_0^{\frac{\pi}{2}}\cos^k\phi\,\sin^\ell\phi\,d\phi = \frac{s_{k+\ell+1}}{s_k s_\ell}$ and thus:

$$\int_{-\pi/2}^{\pi/2}\cos^k\phi\,\sin^\ell\phi\,d\phi = \begin{cases} 2\frac{s_{k+\ell+1}}{s_k s_\ell} = \frac{s_{k+2j+1}}{s_k}\frac{(2j)!}{(4\pi)^j j!} & \text{if } \ell = 2j \text{ is even;} \\ 0 & \text{if } \ell \text{ is odd,} \end{cases}$$

where we have used the identity $s_{2j} = \frac{2^{2j+1}\pi^j j!}{(2j)!}$, $\forall j\in\mathbb{N}$, in the even case. Since $\sin^{2j+1}\phi$ is odd for all $j\in N$, it follows:

$$\begin{aligned} \iota^*\Phi^n_{(k,p,q)} &= \int_{-\pi/2}^{\pi/2} H^*\Phi^n_{(k,p,q)} \\ &= (n-k-1)\sum_{j=0}^{\lfloor p/2\rfloor}\int_{-\pi/2}^{\pi/2}\binom{p}{2j}\sin^{2j}\phi\cos^{n-k+p-2j-2}\phi\,(\Phi^{n-1}_{(k,p-2j,q)}\cdot e_n^{2j})\,d\phi \\ &= (n-k-1)\sum_{j=0}^{\lfloor p/2\rfloor}\frac{p!}{(p-2j)!(4\pi)^j j!}\,\frac{s_{n-k+p-1}}{s_{n-k+p-2j-2}}\,\Phi^{n-1}_{(k,p-2j,q)}\cdot e_n^{2j}, \end{aligned}$$

It is easy to see that the above normalisation of $\Phi^n_{(k,p,q)}$ fulfils the Theorem's conditions. The cases $\iota^*\Xi^n_{k,p,q}$ and $\iota^*\Psi^n_{k,p,q}$ may be solved in the same way as $\iota^*\Phi^n_{k,p,q}$. We omit the tedious computations here. ☐

Motivated by curvature measures $\Psi_{k,q}$ from Theorem 3.40, we now turn to the question whether one can obtain a "purely horizontal" basis for all tensor-valued curvature measures. To that end, we will prove the following intermediate result.

Lemma 4.25. *Let $Q_n = \sum_{s=1}^n e_s^2 \in \mathrm{Sym}^2(\mathbb{R}^n)$ and T_p^n, $n\in\mathbb{N}$, $p\in\mathbb{N}$ be a family of non-trivial tensor-valued $SO(n)$-covariant curvature measures satisfying the restriction behaviour:*

$$\iota^* T_p^n = \sum_{r=0}^{\lfloor p/2\rfloor} a_r\, T^{n-1}_{p-2r}\cdot e_n^{2r},$$

where $a_r\in\mathbb{R}$ depends only on r for $r = 1,\ldots,\lfloor p/2\rfloor$. Then, the two-parameter family of $SO(n)$-covariant curvature measures

$$\mathrm{Hor}\,T_p^n := \sum_{j=0}^{\lfloor p/2\rfloor} b_j\, T^n_{p-2j}\cdot Q_n^j$$

is horizontal if and only if $b_j = (-1)^j a_j$ and $a_j = \frac{c^j}{j!}$ for some $c\neq 0$.

Proof. Since $\iota^* Q_n = Q_{n-1} + e_n^2$, we first compute

$$\iota^* \tilde{T}_p^n \quad = \sum_{\substack{j=0,\ldots,\lfloor p/2\rfloor \\ r=0,\ldots,\lfloor p/2\rfloor - j \\ s=0,\ldots,j}} \binom{j}{s} b_j a_r T_{p-2(j+r)}^{n-1} \cdot Q_{n-1}^{j-s} e^{2(r+s)}.$$

The next steps are just re-orderings of the above sum::

$$\begin{aligned} \iota^* \tilde{T}_p^n \quad &= \sum_{\substack{\ell=0,\ldots,\lfloor p/2\rfloor \\ r=0,\ldots,\ell \\ s=0,\ldots,\ell-r}} \binom{\ell - r}{s} b_{\ell-r} a_r T_{p-2\ell}^{n-1} \cdot Q_{n-1}^{\ell-(r+s)} e^{2(r+s)} = \sum_{\substack{\ell=0,\ldots,\lfloor p/2\rfloor \\ m=0,\ldots,\ell \\ r=0,\ldots,m}} \binom{\ell - r}{m - r} b_{\ell-r} a_r T_{p-2\ell}^{n-1} \cdot Q_{n-1}^{\ell-m} e^{2m} \\ &= \sum_{\ell=0}^{\lfloor p/2\rfloor} b_\ell a_0 \, T_{p-2\ell}^{n-1} \cdot Q^\ell + \sum_{\substack{\ell=0,\ldots\lfloor p/2\rfloor \\ m=1,\ldots,\ell}} \left(\sum_{r=0}^{m} b_{\ell-r} a_r \frac{(\ell - r)!}{(m-r)!} \right) \frac{m!}{(\ell - m)! m!} T_{p-2\ell}^{n-1} \cdot Q_{n-1}^{\ell-m} e^{2m}. \end{aligned}$$

It follows that $\tilde{T}_p^n$ is horizontal if and only if $a_0 = 1$ and the second term identically vanishes for all m and ℓ in the sum. Since the terms that are independent from r are not zero, the second terms vanishes if and only if:

$$\sum_{r=0}^{m} b_{\ell-r} a_r \frac{(\ell - r)!}{(m-r)!} = 0 = (f_{\ell,m} - f_{\ell,m})^m = f_{\ell,m} \sum_{r=0}^{m} \frac{m!}{r!(m-r)!} (-1)^r,$$

where $f_{\ell,m} \neq 0$ are any constants dependent only on ℓ and m. Component-wise comparison yields $b_{\ell-r} = (-1)^r \frac{f_{\ell,m} m!}{r!(\ell-r)! a_r}$. Since $b_{\ell-r}$ is independent from m, one has $f_{\ell,m} := \frac{1}{m!} f'_\ell$, where $f'_\ell \neq 0$ is again any non-zero constant dependent only on ℓ, and:

$$b_{\ell-r} = (-1)^r \frac{f'_\ell}{r!(\ell - r)! a_r}.$$

As $b_{\ell-r} = b_{\ell+1-(r+1)}$, one has:

$$(-1)^r \frac{f_\ell}{r!(\ell - r)! a_r} = (-1)^{r+1} \frac{f_{\ell+1}}{(r+1)!(\ell - r)! a_{r+1}},$$

whence follows $\frac{f_{\ell+1}}{f_\ell} = -\frac{(r+1) a_{r+1}}{a_r}$. Obviously, the left-hand side does not depend on r, so $\frac{f_{\ell+1}}{f_\ell} = c \neq 0$ and $\frac{a_{r+1}}{a_r} = \frac{c}{r+1}$ and we recursively obtain

$$a_\ell = (-1)^\ell \frac{c^\ell}{\ell!} \qquad \text{and} \qquad b_\ell = b_{\ell-0} = \frac{c^\ell}{\ell! a_0} = \frac{c^\ell}{\ell!} = (-1)^\ell a_\ell,$$

which proves the claim. □

Writing out the computation steps in the above proof for these special values of a_j and b_j yields the following identity.

Corollary 4.26. *If a_j, b_j are as in Lemma 4.25, then the family T_p^n satisfies the following identity:*

$$\sum_{\substack{j=0,\ldots,\lfloor p/2\rfloor \\ s=0,\ldots,\lfloor p/2\rfloor - j}} \frac{(-1)^j c^{j+s}}{j! s!} T_{p-2j-2s}^{n-1} \cdot e_n^{2s} \cdot (Q_{n-1} + e_n^2)^j = \sum_{j=0}^{\lfloor p/2\rfloor} \frac{(-1)^j c^j}{j!} T_{p-2j}^{n-1} \cdot Q_{n-1}^j = \operatorname{Hor} T_p^{n-1}$$

Theorem 4.27 (Horizontality Theorem)**.** *Let $T^n_{k,p,q}$ be the McMullen-basis.*

a) We may canonically assign to $\Phi^n_{k,p,q}$ exactly one horizontal curvature measure:

$$\tilde{\Phi}^n_{k,p,q} := \operatorname{Hor}\Phi^n_{k,p,q} = \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{(-1)^j}{(4\pi)^j j!} \Phi^n_{k,p-2j,q} \cdot Q^j.$$

In particular, there are canonical horizontal curvature measures $\tilde{\Psi}^n_{k,p,1} := \tilde{\Phi}^n_{k,p+2,0}$ as well as $\tilde{\Xi}^n_{k,p,0} := \tilde{\Phi}^n_{k,p+1,0}$.

b) If $q > 0$, then $\Xi^n_{k,p,q}$ may be uniquely horizontalised to:

$$\begin{aligned} \tilde{\Xi}^n_{k,p,q} &= \operatorname{Hor}\Xi^n_{k,p,q} - \frac{1}{2\pi}\tilde{\Phi}^n_{k,p-1,q} \cdot Q'_n \\ &= \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{(-1)^j}{(4\pi)^j j!} \Xi^n_{k,p-2j,q} \cdot Q^j - \frac{1}{2\pi} \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} \frac{(-1)^j}{(4\pi)^j j!} \Phi^n_{k,p-2j-1,q} \cdot Q^j \cdot Q'_n, \end{aligned}$$

where $Q'_n = \sum_{r=1}^n (e_r \otimes 1^{q-1} \otimes e_r) \in W_{[q;p;1]-[q;p-1;0]}V$.

c) If $q > 1$, the unique horizontalisation of $\Psi^n_{k,p,q}$ is:

$$\tilde{\Psi}^n_{k,p,q} := \operatorname{Hor}\Psi^n_{k,p,q} - \frac{1}{2\pi}\tilde{\Phi}^n_{k,p,q-1} \wedge Q - \frac{1}{\pi}\tilde{\Xi}^n_{k,p-1,q-1} \cdot Q'_n.$$

Proof. The existence and uniqueness of the horizontal curvature measures $\tilde{\Phi}^n_{k,p,q} := \operatorname{Hor}\Phi^n_{k,p,q}$ follow directly from the Lemma 4.25. In particular, $\operatorname{span}(\operatorname{Hor}\Phi^n_{k,p,q})_{p\geq 0} = \operatorname{span}(\Phi^n_{k,p,q})_{p\geq 0}$ for all n, k, q. The cases $\tilde{\Psi}^n_{k,p,1}$ and $\tilde{\Xi}^n_{k,p,0}$ now follow from (4.5).

Recall from McMullen's Theorem 4.22 for $\Xi^n_{k,p,q}$ that:

$$\iota^*\Xi^n_{k,p,q} \equiv \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{1}{(4\pi)^j j!} \Xi^{n-1}_{k,p-2j,q} \cdot e_n^{2j} \qquad \operatorname{mod} \Phi^n_{k,\cdot,q}.$$

By Lemma 4.25, the only way to horizontalise $\Xi^n_{k,p,q}$ modulo Φ is to apply the Hor-operator. Writing out $\iota^* \operatorname{Hor}\Xi^n_{k,p,q}$ and simplifying the resulting terms by means of Corollary 4.26, we obtain:

$$\iota^* \operatorname{Hor}\Xi^n_{k,p,q} = \operatorname{Hor}\Xi^{n-1}_{k,p,q} + \frac{1}{2\pi}\operatorname{Hor}\Phi^{n-1}_{k,p-1,q} \cdot (e_n \otimes 1^{q-1} \otimes e_n),$$

where $e_n \otimes 1^{q-1} \otimes e_n \in W_{[q;p;1]-[q;p-1;0]}V$. We now need to compensate for the Φ-terms neglected so far. Setting $\tilde{\Xi}^{n-1}_{k,p,q} := \operatorname{Hor}\Xi^{n-1}_{k,p,q} - \frac{1}{2\pi}\operatorname{Hor}\Phi^{n-1}_{k,p-1,q} \cdot Q'_n$, we obtain a unique horizontal curvature measure corresponding to $\Xi^n_{k,p,q}$. This correspondence is one-to-one and $\tilde{\Xi}^n_{k,p,q}$, $q > 0$, as well as $\tilde{\Phi}^n_{k,p,q}$, $q \geq 0$, are all linearly independent by construction. Hence, they form a unique horizontal basis of $\Phi^n_{k,q} \oplus \Xi^n_{k,q}$ for all n, k, q.

A similar, but slightly more involved, argument may be applied to horizontalise $\Psi^n_{k,p,q}$. By McMullen's Theorem,

$$\iota^*\Psi^n_{k,p,q} \equiv \sum_{j=0}^{\lfloor p/2 \rfloor} \frac{1}{(4\pi)^j j!} \tilde{\Psi}^{n-1}_{k,p-2j,q} \cdot e_n^{2j} \quad \operatorname{mod} \Phi^n_{k,\cdot,q-1}, \Xi^n_{k,\cdot,q-1}$$

and, again, by Lemma 4.25, the only way to horizontalise $\Psi^n_{k,p,q}$ modulo Φ, Ξ is to apply the Hor-operator. We may re-write the Φ^{n-1}-part of $\iota^*\Psi^n_{k,p,q}$ in a more suitable way:

$$\iota^*\Psi^n_{k,p,q} \overset{\text{mod } \Psi,\Xi}{\equiv} \frac{1}{2\pi}\sum_{j=0}^{\lfloor p/2\rfloor} \frac{2j+1}{(4\pi)^j j!}\,\Phi^{n-1}_{k,p-2j,q-1}\wedge e_n^2\cdot e_n^{2j}$$

$$= \frac{1}{2\pi}\sum_{j=0}^{\lfloor p/2\rfloor}\frac{1}{(4\pi)^j j!}\,\Phi^{n-1}_{k,p-2j,q-1}\wedge e_n^2\cdot e_n^{2j} + \frac{1}{4\pi^2}\sum_{j=0}^{\lfloor \frac{p-2}{2}\rfloor}\frac{1}{(4\pi)^j j!}\,\Phi^{n-1}_{k,p-2j-2,q-1}\cdot e_n^{2j}\cdot e_n^2\wedge e_n^2.$$

so as to be able to apply Corollary 4.26 in all cases. Setting $e'^2_n := (e_n\otimes 1^{q-2}\otimes e_n)$, we have:

$$\begin{aligned}\iota^*\operatorname{Hor}\Psi^n_{k,p,q} \quad &= \quad \operatorname{Hor}\Psi^{n-1}_{k,p,q} + \frac{1}{\pi}\operatorname{Hor}\Xi^{n-1}_{k,p-1,q-1}\cdot e'^2_n \\ &\quad + \frac{1}{2\pi}\operatorname{Hor}\Phi^{n-1}_{k,p,q}\wedge e_n^2 + \frac{1}{4\pi^2}\operatorname{Hor}\Phi^{n-1}_{k,p-1,q-1}\cdot e_n^2\wedge e_n^2.\end{aligned}$$

Correcting the behaviour of terms $\operatorname{Hor}\Xi$ and the $\operatorname{Hor}\Phi^n_{k,p,q}$ is now obvious. We just subtract $\frac{1}{\pi}\operatorname{Hor}\Xi^n_{k,p-1,q-1}\cdot Q'_n$ and $\frac{1}{2\pi}\operatorname{Hor}\Phi^n_{k,p,q}\wedge Q$. Setting

$$A^n_{k,p,q} := \operatorname{Hor}\Psi^n_{k,p,q} - \frac{1}{\pi}\operatorname{Hor}\Xi^n_{k,p-1,q-1}\cdot Q'_n - \frac{1}{2\pi}\operatorname{Hor}\Phi^n_{k,p,q}\wedge Q$$

and taking into account the restriction behaviour of $\operatorname{Hor}\Xi^n_{k,p-1,q-1}$ computed above, we obtain the following restriction behaviour:

$$\begin{aligned}\iota^*A^n_{k,p,q} \quad &= \quad \operatorname{Hor}\Psi^{n-1}_{k,p,q} - \frac{1}{\pi}\operatorname{Hor}\Xi^{n-1}_{k,p-1,q-1}\cdot Q'_{n-1} - \frac{1}{2\pi}\Phi^{n-1}_{k,p,q}\wedge Q_{n-1} \\ &\quad + \frac{1}{4\pi^2}\operatorname{Hor}\Phi^{n-1}_{k,p-1,q-1}\cdot e_n^2\wedge e_n^2 - \frac{1}{2\pi^2}\operatorname{Hor}\Phi^{n-1}_{k,p-2,q-1}\cdot e'^2_n\cdot(e'^2_n + Q'_{n-1}),\end{aligned}$$

where the last term comes from the correction term for $\operatorname{Hor}\Xi$. Since

$$\operatorname{Hor}\Phi^{n-1}_{k,p-2,q-1}\cdot e'^2_n\cdot e'^2_n = \operatorname{Hor}\Phi^{n-1}_{k,p-2,q-1}\cdot e_n^2\wedge e_n^2,$$

we may re-write the above identity as follows:

$$\begin{aligned}\iota^*A^n_{k,p,q} \quad &= \quad \operatorname{Hor}\Psi^{n-1}_{k,p,q} - \frac{1}{\pi}\operatorname{Hor}\Xi^{n-1}_{k,p-1,q-1}\cdot Q'_{n-1} - \frac{1}{2\pi}\Phi^{n-1}_{k,p,q}\wedge Q_{n-1} \\ &\quad - \frac{1}{4\pi^2}\operatorname{Hor}\Phi^{n-1}_{k,p-1,q-1}\cdot e'^2_n\cdot e'^2_n - \frac{1}{2\pi^2}\operatorname{Hor}\Phi^{n-1}_{k,p-2,q-1}\cdot e'^2_n\cdot Q'_{n-1}.\end{aligned}$$

Adding another correction term $B^n_{k,p,q} := \frac{1}{4\pi^2}\operatorname{Hor}\Phi^n_{k,p-2,q-1}\cdot Q'^2_n$ with the restriction behaviour

$$\begin{aligned}\iota^*B^n_{k,p,q} \quad &= \quad \frac{1}{4\pi^2}\operatorname{Hor}\Phi^{n-1}_{k,p-1,q-1}\cdot e'^2_n\cdot e'^2_n + \frac{1}{2\pi^2}\operatorname{Hor}\Phi^{n-1}_{k,p-2,q-1}\cdot e'^2_n\cdot Q'_{n-1} \\ &\quad + \frac{1}{4\pi^2}\operatorname{Hor}\Phi^{n-1}_{k,p-2,q-1}\cdot Q'^2_{n-1}\end{aligned}$$

yields the horizontal curvature measure $\tilde{\Psi}^n_{k,p,q} := A^n_{k,p,q} + B^n_{k,p,q}$. □

Definition 4.28. We call the basis given by $\tilde{\Phi}^n_{k,p,q}$, $\tilde{\Xi}^n_{k,p,q}$, $\tilde{\Psi}^n_{k,p,q}$ and $\tilde{\Theta}^n_p := \Theta^n_{(p)}$ the *horizontal basis.*

4.3 Globalisation

The space $\mathrm{TVal}_{\Gamma_\lambda}$ is a quotient space of $\mathrm{TCurv}_{\Gamma_\lambda}$, Therefore, the map glob : $\mathrm{TCurv}_{\Gamma_\lambda} \to \mathrm{TVal}_{\Gamma_\lambda}$ has a kernel, which we now completely describe in the following Theorem.

Theorem 4.29. *For all n, k, p, q, one has:*

$$\operatorname{glob} \Xi_{(k,p,q)} = 0. \tag{4.8}$$

For all n, p, and $0 \leq k \leq \frac{n-1}{2}$ one has:

$$\operatorname{glob} \Psi_{(k,p,k+1)} = 0. \tag{4.9}$$

For all n, k, p, one has:

$$\operatorname{glob} \Theta^n_{(p)} = 0. \tag{4.10}$$

For all n, k, p, and $q \geq 1$, such that $\dim \mathrm{TCurv}^{SO(n)}_{k,\Gamma_0^{q,p}} \geq 2$, *one has:*

$$q(n-k+p) \operatorname{glob} \Psi_{(k,p,q)} + (k-q+1)(c_{q-1}p+1) \operatorname{glob} \Phi_{(k,p,q)} = qd_q \operatorname{glob} \Phi_{(k,p,q-1)} \wedge Q, \tag{4.11}$$

where

$$c_m = \begin{cases} 1 & \text{if } m \text{ is even,} \\ \frac{1}{2} & \text{if } m \text{ is odd,} \end{cases} \qquad \text{and} \qquad d_q = \begin{cases} 1 & \text{if } q > 1, \\ p+1 & \text{if } q = 1. \end{cases}$$

Proof. The cases (4.8) and (4.9) follow immediately from 3.36 and 4.4, as the corresponding curvature measures $\Xi_{(k,p,q)}$ and $\Psi_{(k,p,k')}$ assume values in $SO(n)$-modules which occur in Curv^{sm}_k but are missing in Val_k.

The case (4.10) follows, as $\Theta^n_{(p)}$ is $SO(n)$- but not $O(n)$-covariant by Remark 4.16 and there are no valuations with such covariance properties.

The last case (4.11) is a direct consequence of the below Lemma: $dX_{k,p,q}$ is obviously exact and, thus, always globalises to 0. If $q > 1$, then $\operatorname{glob} \Xi_{(k,p-1,q-1)} = 0$ by (4.8). If $q = 1$, then $\Xi_{(k,p-1,0)} \wedge (1 \otimes Q_{p-1}) = \Phi_{(k,p,0)} \cdot Q$ by (4.4), and the result follows. □

Corollary 4.30. *For all n, k, p and $q \geq 1$, such that* $\dim \mathrm{TCurv}^{SO(n)}_{k,\Gamma_0^{q,p}} \geq 2$, *one has:*

$$q(n-k+p) \operatorname{glob} \Psi_{[k,p,q]} + (k-q+1)(c_{q-1}p+1) \operatorname{glob} \Phi_{[k,p,q]} = 0 \tag{4.12}$$

Proof. As $\Phi_{(k,p,q-1)}$ assumes values in the $SL(n)$-module Γ_μ of type $[q; p-1; r]$ embedded $SO(n)$-invariantly into the $SL(n)$-module Γ_λ of type $[q; p; r]$, the claim follows immediately from the above result and Lemma 2.63. □

Lemma 4.31. *For any k, p and $q \geq 1$, such that* $\dim \mathrm{TCurv}^{SO(n)}_{k,\Gamma_0^{q,p}} \geq 2$, *one has:*

$$\begin{aligned} &(n-k-1)\, dX_{k,p,q-1} - pq\, \Xi_{(k,p-1,q-1)} \cdot Q'_n \\ &\quad \overset{\text{mod } \alpha}{\equiv} q\, \Phi_{(k,p,q-1)} \wedge Q - q(n-k+p)\Psi_{(k,p,q)} - (k-q+1)(c_{q-1}p+1)\Phi_{(k,p,q)}, \end{aligned}$$

where $dX_{k,p,q}$ is some exact form which will be explicitly constructed in the proof.

Proof. Consider the function $h : V \to \mathrm{Sym}^2 V$ given by $y \mapsto y^2$ whose gradient is

$$\operatorname{grad} h = 2 \sum_{j=1}^{n} \frac{\partial}{\partial y^j} \otimes y e_j.$$

Contracting $\frac{\partial}{\partial y^j}$ with the differential-form part of $\Phi_{(k,p,q)}$, wedging ye_j with its tensor-part, and using the shorthand notation $e_{iy} := e_i \wedge y$ yields the following form:

$$\iota_{\operatorname{grad} h}\Phi_{(k,p,q)} = C_{n,k} \sum \operatorname{sgn}\pi \, y_{\pi(1)} \, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+3)\dots\pi(n)} \otimes e_{i_1\dots i_q y} \cdot e_{\pi(2)\dots\pi(q+1)\pi(k+2)} \cdot y^p,$$

where $C_{n,k} = 2(n-k-1)$, the sum is over $i_1, \dots, i_q = 1, \dots, n$ and $\pi \in S_n$. If we denote $X_{k,p,q} := \frac{(-1)^k}{C_{n,k}} \iota_{\operatorname{grad} h}\Phi_{(k,p,q)}$, then:

$$\begin{aligned} dX_{k,p,q} = & \sum \operatorname{sgn}\pi \, dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(1)\pi(k+3)\dots\pi(n)} \otimes e_{i_1\dots i_q y} \cdot e_{\pi(2)\dots\pi(q+1)\pi(k+2)} \cdot y^p \\ & + p \sum_{j=1}^{n} \operatorname{sgn}\pi \, y_{\pi(1)} \wedge dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{j\pi(k+3)\dots\pi(n)} \otimes e_{i_1\dots i_q y} \cdot e_{\pi(2)\dots\pi(q+1)\pi(k+2)} \cdot y^{p-1} e_j \\ & + \sum_{j=1}^{n} \operatorname{sgn}\pi \, y_{\pi(1)} dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{j\pi(k+3)\dots\pi(n)} \otimes e_{i_1\dots i_q j} \cdot e_{\pi(2)\dots\pi(q+1)\pi(k+2)} \cdot y^p. \end{aligned}$$

We denote each respective term by I, II, and III and analyse them separately.

Consider term I. First, we multiply it with $1 = \sum_{j=1}^{n} y_j^2 = \sum_{j=\pi(1)}^{\pi(n)} y_j^2$ and regard it at the point $(x, y) := (0, e_1)$. As $y_j = 1$ if and only if $j = 1$ and 0 otherwise, we have $y_j dx^j = \alpha = dx^1$ if and only if $j = 1$ and 0 otherwise as well as $y_j dy^j = 0$ for all j. Hence, we may write at point $(0, e_1)$:

$$I \overset{\text{mod } \alpha}{\equiv} \sum_{j} \operatorname{sgn}\pi \, y_j dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(1)\pi(k+3)\dots\pi(n)} \otimes e_{i_1\dots i_q y} \cdot e_{\pi(2)\dots\pi(q+1) y_j \pi(k+2)} \cdot y^p,$$

where the sum is over $j = \pi(2), \dots, \pi(q+1), \pi(k+2)$. Let $j_1, j_2 = 2, \dots, q+1, k+2$, set $\tau := (j_1 j_2)$ the permutation exchanging j_1 and j_2, and assume, without loss of generality, $j_1 < j_2$. Then

$$\begin{aligned} \sum_{j_1,j_2} \operatorname{sgn}\pi \, e_{\pi(j_1)} \wedge y_{\pi(j_2)} e_{\pi(j_2)} &= \sum_{j_1,j_2} \operatorname{sgn}(\pi\circ\tau) \, e_{\pi\circ\tau(j_1)} \wedge y_{\pi\circ\tau(j_2)} e_{\pi\circ\tau(j_2)} \\ &= \operatorname{sgn}\tau \sum_{j_1,j_2} \operatorname{sgn}(\pi) e_{\pi(j_2)} \wedge y_{\pi(j_1)} e_{\pi(j_1)} = -\operatorname{sgn}\tau \sum_{j_1,j_2} \operatorname{sgn}(\pi) \, y_{\pi(j_1)} e_{\pi(j_1)} \wedge e_{\pi(j_2)} \\ &= \sum_{j_1,j_2} \operatorname{sgn}(\pi) y_{\pi(j_1)} e_{\pi(j_1)} \wedge e_{\pi(j_2)}. \end{aligned}$$

Hence, term I may be re-written as

$$\begin{aligned} I \overset{\text{mod } \alpha}{\equiv} & (q+1) \sum_{j} \operatorname{sgn}\pi \, y_{\pi(k+2)} dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(1)\pi(k+3)\dots\pi(n)} \\ & \otimes e_{i_1\dots i_q y} \cdot e_{\pi(2)\dots\pi(q+1) y_{\pi(k+2)} \pi(k+2)} \cdot y^p. \end{aligned}$$

and the only summands that are not zero are those with $\pi(k+2) = 1$. Then one has:

$$I \overset{\text{mod } \alpha}{\equiv} (q+1) \sum_{j} \operatorname{sgn}\pi \, y_{\pi(k+2)} dx^{i_1\dots i_q \pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(1)\pi(k+3)\dots\pi(n)} \otimes e_{i_1\dots i_q y} \cdot e_{\pi(2)\dots\pi(q+1) y} \cdot y^p.$$

Applying the same permutation-based argument as before – but this time for $\tau = ((k+2)\ 1)$ – we obtain:

$$I \overset{\text{mod } \alpha}{\equiv} -(q+1)\Psi_{(k,p,q+1)}. \tag{4.13}$$

In what follows, we will make frequent use of the above permutation-based argument combined with the anti-commutativity of the wedge-product – sometimes with different cycles τ without writing them out in detail. For example, the sum over j in term III at $(0, e_1)$ may be decomposed into two sums over $j = \pi(2), \ldots, \pi(q+1), \pi(k+2)$ and $j = \pi(q+2), \ldots, \pi(k+1)$, respectively, with the sum over $j = \pi(1), \pi(k+2), \ldots, \pi(n)$ being 0 for the same reason as for term I. We exclude $j = \pi(1)$, as $\pi(1) = 1$ and $dy^1 = 0$ at $(0, e_1)$. Now, by applying the permutation argument again, we obtain:

$$\begin{aligned} III \overset{\text{mod } \alpha}{\equiv} & (q+1)\sum \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k+1)} \wedge dy^{\pi(k+2)\ldots\pi(n)} \\ & \qquad \otimes e_{i_1\ldots i_q\pi(k+2)} \cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+2)} \cdot y^p \\ & + (k-q)\sum \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k+1)} \wedge dy^{\pi(k+1)\pi(k+3)\ldots\pi(n)} \\ & \qquad \otimes e_{i_1\ldots i_q\pi(k+1)} \cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+2)} \cdot y^p. \end{aligned}$$

We denote the terms A and B, respectively, and re-write B as follows:

$$\begin{aligned} B &= (k-q)\sum_{j=q+2}^{k+1} \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k)\pi(j)} \wedge dy^{\pi(k+1)\pi(k+3)\ldots\pi(n)} \\ & \qquad \otimes e_{i_1\ldots i_q\pi(j)} \cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+2)} \cdot y^p \\ &= \frac{k-q}{n-k-1}\sum_{\substack{j=q+2\\ j\neq k+2}}^{n} \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k)\pi(j)} \wedge dy^{\pi(k+1)\pi(k+3)\ldots\pi(n)} \\ & \qquad \otimes e_{i_1\ldots i_q\pi(j)} \cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+2)} \cdot y^p, \end{aligned}$$

where the second identity again follows from the permutation argument. Define a form

$$\begin{aligned} C &:= \frac{k-q}{n-k-1}\sum_{j} \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k)\pi(j)} \wedge dy^{\pi(k+1)\pi(k+3)\ldots\pi(n)} \\ & \qquad \otimes e_{i_1\ldots i_q\pi(j)} \cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+2)} \cdot y^p, \end{aligned}$$

where $j = 2, \ldots, q+1, k+2$. Then $A + B = (B + C) + (A - C)$ and

$$\begin{aligned} B + C &= \frac{k-q}{n-k-1}\sum_{j=2}^{n} \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k)\pi(j)} \wedge dy^{\pi(k+1)\pi(k+3)\ldots\pi(n)} \\ & \qquad \otimes e_{i_1\ldots i_q\pi(j)} \cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+2)} \cdot y^p \\ &= -\frac{k-q}{n-k-1}\sum_{j=2}^{n} \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\ldots i_q j\pi(q+3)\ldots\pi(k+1)} \wedge dy^{\pi(k+2)\ldots\pi(n)} \\ & \qquad \otimes e_{i_1\ldots i_q\pi(j)} \cdot e_{\pi(2)\ldots\pi(q+2)} \cdot y^p \\ &= -\frac{k-q}{n-k-1}\Phi_{(k,p,q+1)}. \end{aligned}$$

To compute $A - C$, we first use the permutation argument to re-write A as follows:

$$A = \frac{q+1}{n-k-1}\sum_{j=k+2}^{n} \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k+1)} \wedge dy^{\pi(k+2)\ldots\pi(n)} \otimes e_{i_1\ldots i_q\pi(j)} \cdot e_{\pi(2)\ldots\pi(q+1)\pi(j)} \cdot y^p$$

Furthermore, by multiple permutation arguments, we obtain the following identity for $-C$:

$$\begin{aligned}
-C &= -\frac{(k-q)(q+1)}{n-k-1}\sum \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\dots i_q\pi(q+2)\dots\pi(k)\pi(k+2)} \wedge dy^{\pi(k+1)\pi(k+3)\dots\pi(n)} \\
&\qquad\qquad \otimes e_{i_1\dots i_q\pi(k+2)}\cdot e_{\pi(2)\dots\pi(q+1)\pi(k+2)}\cdot y^p \\
&= \frac{(k-q)(q+1)}{n-k-1}\sum \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \\
&\qquad\qquad \otimes e_{i_1\dots i_q\pi(k+1)}\cdot e_{\pi(2)\dots\pi(q+1)\pi(k+1)}\cdot y^p \\
&= \frac{q+1}{n-k-1}\sum_{j=2}^{k+1} \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \\
&\qquad\qquad \otimes e_{i_1\dots i_q\pi(j)}\cdot e_{\pi(2)\dots\pi(q+1)\pi(j)}\cdot y^p.
\end{aligned}$$

It follows:

$$\begin{aligned}
A-C &= \frac{q+1}{n-k-1}\sum_{j=2}^{n} \operatorname{sgn}\pi\, y_{\pi(1)} dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \otimes e_{i_1\dots i_q\pi(j)}\cdot e_{\pi(2)\dots\pi(q+1)\pi(j)}\cdot y^p \\
&= \frac{q+1}{n-k-1}\Phi_{(k,p,q)}\wedge (Q-y^2).
\end{aligned}$$

As $\Phi_{(k,p,q)}\wedge y^2 = \Psi_{(k,p,q+1)}$ by definition, we obtain:

$$III \overset{\text{mod }\alpha}{\equiv} \frac{q+1}{n-k-1}\Phi_{(k,p,q)}\wedge Q - \frac{q+1}{n-k-1}\Psi_{(k,p,q+1)} - \frac{k-q}{n-k-1}\Phi_{(k,p,q+1)}. \tag{4.14}$$

The computation scheme for term II is similar to that of term III. As before, we regard the form at point $(0,e_1)$ and first observe that only terms with $\pi(1)=1$ are horizontal and not equal to zero. Then we decompose III in two sums, the first one being over $j=\pi(2),\dots,\pi(q+1),\pi(k+2)$ and the second one – over $j=\pi(q+2),\dots,\pi(k+1)$. The same permutation argument as for term III yields:

$$\begin{aligned}
II \overset{\text{mod }\alpha}{\equiv}\; & p(q+1)\sum \operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+2)\dots\pi(n)} \\
&\qquad \otimes e_{i_1\dots i_q y}\cdot e_{\pi(2)\dots\pi(q+1)\pi(k+2)}\cdot y^{p-1}e_{\pi(k+2)} \\
&+ p(k-q)\sum \operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)} \wedge dy^{\pi(k+1)\pi(k+3)\dots\pi(n)} \\
&\qquad \otimes e_{i_1\dots i_q y}\cdot e_{\pi(2)\dots\pi(q+1)\pi(k+2)}\cdot y^{p-1}e_{\pi(k+1)}.
\end{aligned}$$

Denoting the above terms by D and E, respectively, we obtain:

$$D = \frac{p(q+1)}{n-k-1}\sum_j \operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)}\wedge dy^{\pi(k+2)\dots\pi(n)}\otimes e_{i_1\dots i_q y}\cdot e_{\pi(2)\dots\pi(q+1)\pi(j)}\otimes y^{p-1}e_{\pi(j)},$$

where the sum is over $j=2,\dots q+1, k+2,\dots,n$. As above, we may write $II=(D+F)+(E-F)$ with

$$F := \frac{p(q+1)}{n-k-1}\sum_{j=q+2}^{k+1} \operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\dots i_q\pi(q+2)\dots\pi(k+1)}\wedge dy^{\pi(k+2)\dots\pi(n)}\otimes$$
$$e_{i_1\dots i_q y}\cdot e_{\pi(2)\dots\pi(q+1)\pi(j)}\cdot y^{p-1}e_{\pi(j)}.$$

We immediately see:

$$\begin{aligned} D+F &= \frac{p(q+1)}{n-k-1}\sum_{j=2}^{n}\operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k+1)}\wedge dy^{\pi(k+2)\ldots\pi(n)} \\ &\qquad\qquad \otimes e_{i_1\ldots i_q y}\cdot e_{\pi(2)\ldots\pi(q+1)\pi(j)}\cdot y^{p-1}e_{\pi(j)} \\ &= \frac{p(q+1)}{n-k-1}\Xi_{(k,p-1,q)}\cdot(Q'_n - y\otimes 1^{q-1}\otimes y)). \end{aligned}$$

As $\Xi_{(k,p-1,q)}\wedge(1\otimes y\otimes 1^{p-1}\otimes y)=\Psi_{(k,p,q+1)}$, we obtain:

$$D+F=\frac{p(q+1)}{n-k-1}\Xi_{(k,p-1,q)}\cdot Q'_n-\frac{p(q+1)}{n-k-1}\Psi_{(k,p,q+1)}.$$

Similarly to term C, we re-write:

$$\begin{aligned} F &= -\frac{p(k-q)}{n-k-1}\sum_{j}\operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k)\pi(j)}\wedge dy^{\pi(k+2)\ldots\pi(n)} \\ &\qquad\qquad \otimes e_{i_1\ldots i_q y}\cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+1)}\cdot y^{p-1}e_{\pi(j)}, \end{aligned}$$

where $j=2,\ldots,q+1,k+2$. Observing that

$$\begin{aligned} E &= -p(k-q)\sum\operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k)\pi(k+2)}\wedge dy^{\pi(k+2)\ldots\pi(n)} \\ &\qquad\qquad \otimes e_{i_1\ldots i_q y}\cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+1)}\cdot y^{p-1}e_{\pi(k+2)} \\ &= -\frac{p(k-q)}{n-k-1}\sum_{j=k+2}^{n}\operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k)\pi(j)}\wedge dy^{\pi(k+2)\ldots\pi(n)} \\ &\qquad\qquad \otimes e_{i_1\ldots i_q y}\cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+1)}\cdot y^{p-1}e_{\pi(j)}, \end{aligned}$$

we conclude

$$\begin{aligned} E-F &= -\frac{p(k-q)}{n-k-1}\sum_{j=2}^{n}\operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\ldots i_q\pi(q+2)\ldots\pi(k)j}\wedge dy^{\pi(k+2)\ldots\pi(n)} \\ &\qquad\qquad \otimes e_{i_1\ldots i_q y}\cdot e_{\pi(2)\ldots\pi(q+1)\pi(k+1)}\cdot y^{p-1}e_j \\ &= -\frac{p(k-q)}{n-k-1}\sum_{j=2}^{n}\operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\ldots i_q j\pi(q+3)\ldots\pi(k+1)}\wedge dy^{\pi(k+3)\ldots\pi(n)} \\ &\qquad\qquad \otimes e_{i_1\ldots i_q y}\cdot e_{\pi(2)\ldots\pi(q+2)}\cdot y^{p-1}e_j. \end{aligned}$$

We now apply the Bianchi-identity from Lemma 2.28 to the column $e_{i_1\ldots i_q y}$ and the box filled by e_j to obtain:

$$\begin{aligned} E-F &= -\frac{p(k-q)}{n-k-1}\sum_{j,m}\operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\ldots i_q j\pi(q+3)\ldots\pi(k+1)}\wedge dy^{\pi(k+3)\ldots\pi(n)} \\ &\qquad\qquad \otimes e_{i_1\ldots i_q j}\cdot e_{\pi(2)\ldots\pi(q+2)}\cdot y^p \\ &\quad -\frac{p(k-q)}{n-k-1}\sum_{j,m}\operatorname{sgn}\pi\, y_{\pi(1)}\wedge dx^{i_1\ldots i_q j\pi(q+3)\ldots\pi(k+1)}\wedge dy^{\pi(k+3)\ldots\pi(n)} \\ &\qquad\qquad \otimes e_{\underbrace{i_1\ldots j\ldots i_q y}_{j \text{ instead of } i_m}}\cdot e_{\pi(2)\ldots\pi(q+2)}\cdot y^{p-1}e_{i_m}, \end{aligned}$$

where $m = 1, \dots, q$. The first term is obviously equal to $-\frac{p(k-q)}{n-k-1}\Phi_{(k,p,q+1)}$ while the second one can directly be seen to be equal to $\sum_{m=1}^{q}(-1)^{q+1-m}(E-F)$ by renaming j to i_m and vice versa and using anti-commutativity of the wedge-product. Hence, the second term vanishes for even q and $E-F = -\frac{p(k-q)}{n-k-1}\Phi_{(k,p,q+1)}$. For odd q, the second term evaluates to $-(E-F)$ and $E-F = -\frac{1}{2}\frac{p(k-q)}{n-k-1}\Phi_{(k,p,q+1)}$. Altogether, we find for term II:

$$II \overset{\text{mod }\alpha}{\equiv} \frac{p(q+1)}{n-k-1}\Xi_{(k,p-1,q)} \cdot Q'_n - \frac{p(q+1)}{n-k-1}\Psi_{(k,p,q+1)} - c_q\frac{p(k-q)}{n-k-1}\Phi_{(k,p,q+1)}, \tag{4.15}$$

where $c_q = 1$ for even q and $1/2$ for odd q.

If we multiply (4.13), (4.15), (4.14) by $(n-k-1)$, sum the results up, and map $q \mapsto q-1$, we obtain the claimed identity. □

4.4 Comparisons, Corollaries, and Conclusions

We conclude our work with a range of Corollaries which we hope will prove useful in the future research.

Let us begin by comparing the different families $T^n_{k,p,q}$ with the known families of curvature measures. For example, the curvature measures $\Psi_{k,q}$ from Theorem 3.40 are multiples of $\tilde{\Phi}^n_{k,0,q} = \Phi^n_{k,0,q}$:

$$\Psi_{k,q} = -\binom{k}{q}\Phi^n_{k,0,q} \qquad \text{mod } d\alpha. \tag{4.16}$$

We may also recover the basis $Q^m\Phi_k^{0,s,j}$, $j \in \{0,1\}$, $2m+2j+s = p$, of $\operatorname{Sym}^p V$-valued valuations and curvature measures from Theorems 3.41 and 3.43. As $\operatorname{Sym}^p V = \bigoplus_{j=0}^{\lfloor p/2 \rfloor}\Gamma_{[p-2j]}$ is multiplicity-free, the embedding $\iota_m : \Gamma_{[p-2m]} \to \operatorname{Sym}^p V$, $\tau \mapsto \tau \cdot Q^m$ is canonical. Then one sees by comparing the constants:

$$Q^m\Phi_k^{0,s,0} = \iota_m\Phi^n_{k,s,0} = \Phi^n_{k,s,0}\cdot Q^m, \qquad Q^m\Phi_k^{0,s,1} = k\iota_m\Phi_{k,s,1} = k\Phi_{k,s,1}\cdot Q^m. \tag{4.17}$$

In particular, Remark 4.1 from [49] is a special case of (4.11) for $q = 1$. Taking into account the globalisation Theorem 4.29 and the relations (4.5) between curvature measures for $q \le 1$, McMullen's Theorem 3.48 for $\operatorname{Sym}^p V$-valued valuations is a special case of Theorem 4.22 for $\Phi^n_{k,p,1}$. To our knowledge, no McMullen's Theorem for $\operatorname{Sym}^p V$-valued *curvature measures* has been published. In this sense, Theorem 4.22 is a genuinely new result.

The explicit characterisation of $\mathrm{TCurv}^{SO(n)}$ also yields an explicit basis of Curv^{sm}.

Corollary 4.32. *The coefficients of $\Phi^n_{[k,p,q]}$, $*\Phi^n_{[k,p,q]}$ for $n = 2j$ even and $k = j-1, j$, $\Xi^n_{[k,p,q]}$ for $q > 0$, $\Psi^n_{[k,p,q]}$ for $q > 1$, and $\Theta^n_{[p]}$ form a basis of Curv^{sm}_k.*

Proof. The coefficients of all linearly independent Γ_λ-valued curvature measures (as Theorem 4.17 shows, there may be more than one, in contrast to the case of valuations) span the isotypical component Γ_λ in the space Curv^f_k of the so-called *$SO(n)$-finite vectors* in Curv^f_k. The claim now follows, since Curv^f_k lies dense in Curv^{sm}_k. We refer to [80, Section 3.2] for the details on G-finite vectors in infinite-dimensional representations. □

The relations between globalised curvature measures give us a means of carrying over several results to valuations as well. First, let us show the following fact.

Proposition 4.33. *Any continuous tensor-valued $SO(n)$-covariant translation-invariant valuation is smooth.*

Proof. Let ϕ be a Γ-valued $SO(n)$-covariant translation-invariant valuation, where Γ an $SO(n)$-module. Since smooth translation-invariant valuations lie dense in Val, we may find a sequence ϕ_i of smooth Γ-valued translation-invariant valuations which converges to ϕ.

Consider the following map A for any translation-invariant tensor-valued valuation τ:

$$(A\tau)(K) := \int_{SO(n)} g^{-1}\tau(gK)\,dg.$$

Obviously, if τ is smooth, then so is $A\tau(K)$. Furthermore, for any $h \in SO(n)$, one has

$$A\tau(hK) = \int_{SO(n)} g^{-1}\tau(ghK)\,dg \overset{\tilde{g}:=gh}{=} \int_{SO(n)} (\tilde{g}h^{-1})^{-1}\phi(\tilde{g}K)\,d\tilde{g} = h(A\tau(K)),$$

i.e. $A\tau$ is also $SO(n)$-covariant.

Applying A to both the sequence ϕ_i and ϕ, one obtains a new sequence $A\phi_i$ of smooth $SO(n)$-covariant translation-invariant valuations converging to $A\phi = \phi$. We have seen in the previous Sections that the space $\mathrm{TVal}_\Gamma^{sm}$ of smooth Γ-valued $SO(n)$-covariant translation-invariant valuations is finite-dimensional and, thus, closed. Hence, $\phi = \lim_i A\phi_i$ lies in $\mathrm{TVal}_\Gamma^{sm}$ and the result now follows. □

We are now able to give an explicit basis of the space $\mathrm{TVal}^{SO(n)}$ of tensor-valued $SO(n)$-covariant tensor-invariant valuations.

Theorem 4.34. *The elements from the family $T^n_{[k,p,q]}$ span $\mathrm{TVal}^{SO(n)}$. In particular, the elements $\Phi^n_{[k,p,q]}$, $*\Phi^n_{[k,p,q]}$ for $n = 2j$ even and $k = j-1, j$ as well as $\Psi^n_{[k,p,k+1]}$ for $\frac{n-1}{2} < k \leq n-1$ form a basis of $\mathrm{TVal}^{SO(n)}$.*

Definition 4.35. We refer to these elements as the $\mathrm{TVal}^{SO(n)}$-basis $T^n_{[k,p,q]}$.

Proof. The forms $\Xi^n_{[k,p,q]}$ and $\Theta^n_{[p]}$ are excluded, since they globalise to trivial valuations by (4.8) and (4.10). The forms $\Psi^n_{[k,p,q]}$ need to be excluded, since they either globalise to trivial valuations by (4.9) or $\mathrm{glob}\,\Psi^n_{[k,p,q]} = C_{n,k,p,q}\,\mathrm{glob}\,\Phi^n_{[k,p',q']}$ for some constant $C_{n,k,p,q}$ and some p' and q' by (4.4) or (4.12). The only exception is $\Psi^n_{[k,p,k+1]}$ for $\frac{n-1}{2} < k \leq n-1$, as none of the relations apply to them. □

Corollary 4.36. *The coefficients of $\mathrm{TVal}^{SO(n)}$-basis $T^n_{[k,p,q]}$ form a Schauder-basis of* Val.

Proof. By the same arguments as for Corollary 4.32, we see that the coefficients of the $\mathrm{TVal}^{SO(n)}$-basis $T^n_{k,p,q}$ form a basis for Val^f. Since Val^f lies dense in Val, we obtain the claim. □

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