Euclidean decompositions of hyperbolic manifolds and their duals

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Abstract

Epstein and Penner constructed in [EP88] the Euclidean decomposition of a non-compact hyperbolic n-manifold of finite volume for a choice of cusps, $n \geq 2$. The manifold is cut along geodesic hyperplanes into hyperbolic ideal convex polyhedra. The intersection of the cusps with the Euclidean decomposition determined by them turns out to be rather simple as stated in Theorem 2.2. A dual decomposition resulting from the expansion of the cusps was already mentioned in [EP88]. These two dual hyperbolic decompositions of the manifold induce two dual decompositions in the Euclidean structure of the cusp sections. This observation leads in Theorems 5.1 and 5.2 to easily computable, necessary conditions for an arbitrary ideal polyhedral decomposition of the manifold to be a Euclidean decomposition.

1 Introduction and notation

1.1 Minkowski space H^n

A good reference for the following is [Rat94].

The symmetric bilinear form $\_ \circ \_$ of type $(n, 1)$ turns $\mathbb{R}^{n+1}$ into Minkowski (or Lorentzian) space with norm $(x \circ x)^{\frac{1}{2}}$. The vectors with norm equal to zero are called *light-like.* If the norm is positive (resp. imaginary), the vector is called *space-like* (resp. *time-like*). If a light-like or a time-like vector has a positive last coordinate, the vector is called *positive.* The positive light-like vectors form the *positive light cone* L^+. Its convex hull $\overline{L}^+$ consists of the positive light-like and the positive time-like vectors.

The vectors of imaginary norm i form a two-sheeted hyperboloid, and the sheet with the positive time-like vectors forms the *Minkowski model* H^n of hyperbolic space. The distance $d(x, y)$ of any two points x and y of H^n is defined by $\cosh d(x, y) = -x \circ y$. The isometries of H^n are exactly those maps that are induced by linear maps of $\mathbb{R}^{n+1}$ that preserve the bilinear form and that do not exchange H^n with the other sheet of the hyperboloid. These linear maps are called *positive Minkowski transformations.*

A vector subspace V of $\mathbb{R}^{n+1}$ is called *time-like* if and only if V has a time-like vector, *space-like* if and only if every non-zero vector in V is space-like, or *light-like* otherwise. A plane in H^n of dimension m is the intersection of H^n with a time-like vector subspace of $\mathbb{R}^{n+1}$ of dimension $(m+1)$. The Minkowski-orthogonal complement of a vector subspace A is denoted by A^L. A and A^L do not span $\mathbb{R}^{n+1}$ in general. We embed the projective model D^n in $\overline{L}^+$ as the set $\{(x_1, \dots, x_n, 1) | x_1^2 + \dots + x_n^2 < 1\}$. Radial projection induces an isometry between D^n and H^n.

1.2 Convex hull construction

We consider a non-compact hyperbolic manifold D^n/Γ of finite volume, where Γ is a freely acting, discrete subgroup of the isometries of D^n. For a decomposition of D^n/Γ into a compact part and finitely many cusps, denote by S the preimage of the cusps in D^n. S consists of pairwise disjoint Γ-invariant horoballs. A horoball b of S is mapped by radial projection to a horoball of H^n. The horoballs in H^n are the sets $\{x \in H^n | x \circ v \geq -1\}$ for $v \in L^+$, and so b determines a unique point of L^+, called its representative. The set of representatives of horoballs of S will be denoted by B. The Euclidean closed convex hull of B will be denoted by C. Radial projection maps ∂C to the Euclidean decomposition of D^n corresponding to S. Each subpolyhedron of this decomposition is the hyperbolic convex hull of finitely many base points of horoballs of S. The decomposition of D^n is Γ-invariant, so it induces a decomposition of D^n/Γ. Because polyhedra in D^n look like Euclidean polyhedra, these decompositions are called Euclidean.

For a three-dimensional link space, the cusps are solid tori from which the cores have been removed. There are many possibilities to choose such topological neighbourhoods of the removed link components. But the definition of a cusp restricts these possibilities because the preimage of a cusp has to be a horoball. So there is only a one-parameter choice for each of the cusps that corresponds to parallel cuts producing different volumes of the cusp. It turns out that the Euclidean decompositions agree for two choices of cusps if there is a common factor α so that the volume of each cusp of the second choice is α times the volume of the corresponding cusp of the first choice. If the volumes of all the cusps are chosen to be equal, the decomposition is uniquely determined by the hyperbolic manifold, and it is called *canonical*. If the manifold has only one cusp, every Euclidean decomposition is canonical.

The decomposition is described by the isometry classes of the ideal polyhedra and the gluing isometries between their $(n-1)$-dimensional sides. Theorem 1.1 ensures that only finitely many polyhedra occur. If $n \geq 3$, a gluing isometry is determined by the combinatorial pairing of vertices. The case $n = 2$ is special because an isometry that leaves a line invariant is not necessarily the reflection in this line or the identity. Further, if $n \geq 3$, an ideal n-polyhedron is determined up to isometry by its combinatorial structure, i. e. by which vertices each of the sides is spanned, and by the angles between sides. This last result follows from Th. 14.1 of [Riv94] where it is proved for the essential case $n = 3$.

Theorem 1.1 *The operation of* Γ *on the Euclidean decomposition of* D^n *is free and it has only finitely many orbits.*

Proof: We denote by π the covering $D^n \to D^n/\Gamma$. Removing the open cusps of D^n/Γ, we obtain a compact set K. As the distance of any two points of D^n/Γ is the minimal distance between their preimages in D^n, we can find a compact set $U \subset D^n$ such that $K \subset \pi(U)$.

Every polyhedron of the Euclidean decomposition of D^n has at least two ideal vertices, and so it cannot lie completely in S. So each polyhedron of the decomposition intersects ΓU. The decomposition is locally finite (Proposition 3.5 of [EP88]), U is compact, and this implies the finiteness of the orbits.

Assume that $\gamma \in \Gamma$ leaves an ideal polyhedron σ invariant. A power of γ induces then the identity on the vertices of this polyhedron. Because Γ contains only fixed point free isometries, σ has to be a geodesic line. This implies that γ is a hyperbolic isometry that fixes both of the ideal points of σ. But the ideal points of the Euclidean decomposition are fixed points of parabolic isometries of Γ. Because parabolic and hyperbolic isometries cannot share a common fixed point in a discrete group, the assumed existence of γ leads us to a contradiction. ♣

Remark: Theorem 1.1 fills a small gap of [EP88]. The result on the finiteness of the orbits is obvious in dimension $n = 2$ because the area of each ideal polygon is a multiple of π, and the finiteness of the area of the surface implies the finiteness of the orbits. But the situation changes for $n \geq 3$ because in those dimensions there exist ideal polyhedra with arbitrarily small volumes.

1.3 Isometries of the manifold

The group of isometries of D^n/Γ is equal to $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normalizer of Γ in the group of isometries of D^n. An isometry γ of D^n/Γ maps a cusp to another cusp. If all cusps are chosen to have the same volume, i. e. the decomposition is canonical, the cusps are permuted by γ. The lifting $\tilde{\gamma}$ of γ to D^n permutes then the preimages of the cusps, the set S of horoballs. So the set B of light-like representatives of S is permuted, too. Therefore, the closed convex hull C of B is kept invariant by (the isometry of H^n corresponding to) $\tilde{\gamma}$, so $\tilde{\gamma}$ and γ preserve the canonical decomposition.

In particular, γ induces a combinatorial automorphism of the ideal polyhedra of the canonical decomposition. On the other hand side, if the dimension of the manifold is greater than two, any combinatorial automorphism is by Mostow's rigidity theorem homotopic to an isometry while leaving fixed the ideal points during the homotopy. So every combinatorial automorphism is indeed an isometry. This fact was used by Weeks' computer program SnapPea [Wee] to compute combinatorially the group of isometries of the manifold. If the manifold is a knot space, this is the symmetry group of the knot, too.

2 How cusps intersect the Euclidean decomposition

Lemma 2.1 *$z + \overline{L}^+$ is a subset of C for each $z \in B$.*

Proof: We set $\langle x \rangle_+ = \{\alpha x | \alpha \geq 0\}$ for $x \in \mathbb{R}^{n+1}$. It is easy to see that the closed convex hull of $a + \langle b \rangle_+$ and c contains $c + \langle b \rangle_+$ for any a, b and c of $\mathbb{R}^{n+1}$.

Let v and z be elements of B. By Lemma 3.3 of [EP88], the half-line $v + \langle v \rangle_+$ is contained in C. As $z \in C$, and C is closed and convex, we have $z + \langle v \rangle_+ \subset C$. The rays through points of B are dense in L^+. It follows $z + L^+ \subset C$ because C is closed and further $z + \overline{L}^+ \subset C$ because C is convex. ♣

Definition: An ideal polyhedron that is spanned by a subset of the ideal vertices of a polyhedron of the Euclidean decomposition of D^n is called a *generalized subpolyhedron* of the Euclidean decomposition.

Theorem 2.2 *A horoball h of S intersects a generalized subpolyhedron σ of the Euclidean decomposition of D^n determined by S if and only if the base point of h is an ideal vertex of σ.*

Proof: A horoball based at an ideal vertex of σ clearly intersects σ.

We consider in the following the case that the base point of h is not an ideal vertex of the generalized subpolyhedron σ. Denote by z_0 the element of B corresponding to h. Then σ corresponds via radial projection to the Euclidean convex hull σ' of a finite subset of $B - \{z_0\}$. Further, σ' lies in ∂C.

We denote for $v \in L^+$

$$\begin{aligned} W(v) &= \{x \in \mathbb{R}^{n+1} | x \circ v = -1\}, \\ W^+(v) &= \{x \in \mathbb{R}^{n+1} | x \circ v \geq -1\}, \\ W^-(v) &= \{x \in \mathbb{R}^{n+1} | x \circ v < -1\}. \end{aligned}$$

The minimal distance between two disjoint horoballs being represented in L^+ by u and v is $\ln(-(u \circ v)/2)$ (see e.g. Lemma 4.2.c in [Wee93]). This implies $-(z \circ z_0)/2 > 1$, and so $z \in W^-(z_0/2)$ for $z \in B - \{z_0\}$. As a half-space is convex, we have $\sigma' \subset W^-(z_0/2)$.

The horoball h is the projection of $W^+(z_0) \cap H^n$ to D^n by definition of z_0. Denote by $\langle x \rangle$ the ray from the origin through a point x of h. Let

$$x_1 = \langle x \rangle \cap H^n, \qquad x_2 = \langle x \rangle \cap \partial C, \qquad x_3 = \langle x \rangle \cap W(z_0/2).$$

There exists $\alpha > 0$ so that $x_3 = \alpha x_1$. It follows by $x_3 \circ (z_0/2) = -1$ and $x_1 \circ z_0 \geq -1$ that $\alpha \geq 2$. This implies $x_3 \circ x_3 \leq -4$ and so $(x_3 - z_0) \circ (x_3 - z_0) \leq 0$. Because $(x_3 - z_0) \circ x_3 < 0$, the last coordinate of $(x_3 - z_0)$ is positive, and therefore $(x_3 - z_0) \in \overline{L}^+$. By Lemma 2.1 we have $x_3 \in C$. Therefore, the point x_2 must lie on the segment from the origin to x_3, and so $x_2 \in W^+(z_0/2)$. We showed above that $\sigma' \subset W^-(z_0/2)$. Therefore $x_2 \notin \sigma'$. This means that $x \notin \sigma$, and we have thus proved that h and σ are disjoint. ♣

Remark: Sometimes it is useful to admit tangential points between the cusps. It is a consequence of Theorem 2.2 that in this case a horoball h of S intersects a generalized subpolyhedron not having the base point of h as an ideal vertex at most in a point of its bounding horosphere ∂h.

One can deduce easily from Theorem 2.2 the well-known result that for a cusped hyperbolic surface of finite area and any choice of cusps, the area of the cusps is at most $3/\pi$ times the area of the whole surface. One has only to verify that three disjoint (or at most tangential) horoballs cover at most the area 3 of the ideal triangle spanned by their base points.

3 Voronoi domains

We start with a description of the supporting planes for C that results from the first part of the proof of Proposition 3.5 in [EP88].

Lemma 3.1 *For a supporting plane T for C there exists either $z \in B$ so that*

$$T = \langle z \rangle^L \text{ and } C \subset \{x \in \mathbb{R}^{n+1} | x \circ z \leq 0\},$$

or there exists $w \in H^n$ and $c < 0$ so that

$$T = \{x \in \mathbb{R}^{n+1} | x \circ w = c\} \text{ and } C \subset \{x \in \mathbb{R}^{n+1} | x \circ w \leq c\}.$$

Further, the mapping of a supporting plane to its normal vector is a bijection between the supporting planes for C and the elements of B and H^n.

The *distance* of a point x to a horoball h is the signed hyperbolic distance from x to the bounding horosphere ∂h. The sign is defined to be negative if $x \in h$, otherwise positive.

Definition: The *Voronoi domain* of the horoball $h \in S$ is the set of points of H^n that are at most as far away from h as from any other horoball of S.

We shall now start to prove that Voronoi domains are polyhedra and that they form a locally finite covering of H^n. The Voronoi domains are Γ-invariant because S is Γ-invariant and Γ contains only isometries. So a decomposition of the hyperbolic manifold is induced. This Voronoi decomposition turns out to be dual to the Euclidean decomposition if both of the decompositions are constructed using the same choice of cusps.

The following Lemma is proved in [SW95b] as Lemma 3.2. It implies the following two Corollaries.

Lemma 3.2 *The distance of a point $x \in H^n$ to a horoball that is represented by $w \in L^+$ is $\ln(-x \circ w)$.*

Corollary 3.3 *Given two horoballs h_1 and h_2 that are represented by u_1 resp. u_2. The set of points that are at most as far away from h_1 as from h_2 is the half-space*

$$\{x \in \mathbb{R}^{n+1} | x \circ (u_2 - u_1) \leq 0\} \cap H^n.$$

The set of points that are equidistant from h_1 and h_2 is the hyperplane

$$\{x \in \mathbb{R}^{n+1} | x \circ (u_2 - u_1) = 0\} \cap H^n.$$

♣

Corollary 3.4 *Let h_0 be a horoball of S that is represented by z_0 in B. Then the Voronoi domain of h_0 is*

$$V(z_0) = \bigcap_{z \in B - \{z_0\}} \{x \in H^n | x \circ (z - z_0) \leq 0\}.$$

♣

Lemma 3.5 *Let z_0 be a point of B. Let w be a point of H^n and denote by $T(w)$ the supporting plane at C determined by Lemma 3.1 that has w as a normal vector. Then the point w lies in $V(z_0)$ if and only if $T(w)$ contains z_0. Further, there exists a neighbourhood $U_\varepsilon(w)$ of w in H^n such that $U_\varepsilon(w)$ intersects no other Voronoi domains but those which belong to horoballs that are represented by elements of $T(w) \cap B$.*

Proof: By Lemma 3.2, the point w of H^n lies in $V(z_0)$ if and only if there exists a $q < 0$ such that $w \circ z_0 = q$ and $w \circ z < q$ for any $z \in B - \{z_0\}$. This means that B is a subset of the Euclidean half-space $R = \{x \in \mathbb{R}^{n+1} | w \circ x \leq q\}$, and z_0 lies in the hyperplane $\partial R = \{x \in \mathbb{R}^{n+1} | w \circ x = q\}$. This is equivalent to the statement that C is a subset of R and z_0 lies in ∂R. As a supporting plane at C with normal vector w is unique, w lies in $V(z_0)$ if and only if z_0 lies in $T(w)$.

The intersection of $T(w)$ with B is a finite set, $\{z_1, \dots, z_s\}$. By the above calculation we have for some $q < 0$

$$\begin{aligned} w \circ z &= q \text{ for } z \in \{z_1, \dots, z_s\}, \\ w \circ z &< q \text{ for } z \in B - \{z_1, \dots, z_s\}. \end{aligned}$$

As the intersection of a space-like hyperplane with $\overline{L}^+$ is compact and B is discrete, we can slightly translate $T(w)$ along its normal vector without touching new points of B. So there exists an $\varepsilon_0 > 0$ such that

$$w \circ z \leq q - \varepsilon_0 \text{ for } z \in B - \{z_1, \dots, z_s\}.$$

Setting $\varepsilon = (\ln(\varepsilon_0 - q) - \ln(-q))/2$, we get by use of the triangle inequality a neighbourhood $U_\varepsilon(x)$ that is disjoint to the Voronoi domains corresponding to elements of $B - \{z_1, \dots, z_s\}$. ♣

Corollary 3.6 *The Voronoi domains $V(z)$, $z \in B$, form a locally finite covering of H^n.*

Proof: Let w be a point of H^n and denote by $T(w)$ the supporting plane assigned by Lemma 3.1. $T(w) \cap B$ is a finite, non empty subset of B. By Lemma 3.5, the set $U_\varepsilon(w)$ intersects only finitely many Voronoi domains but at least one Voronoi domain. ♣

Lemma 3.7 *For $z_0 \in B$, the interior of the Voronoi domain $V(z_0)$ is*

$$\mathring{V}(z_0) = \bigcap_{z \in B - \{z_0\}} \{x \in H^n | x \circ (z - z_0) < 0\}.$$

Proof: Denote $\bigcap_{z \in B - \{z_0\}} \{x \in H^n | x \circ (z - z_0) < 0\}$ by M. By Corollary 3.3, this is the set of points that lie in $V(z_0)$ but not in any other Voronoi domain $V(z)$, $z \in B - \{z_0\}$. Lemma 3.5 ensures a neighbourhood $U_\varepsilon(w)$ in H^n that is contained in M. So the points of M are interior points of $V(z_0)$.

For a point x of $V(z_0) - M$ there exists a z distinct from z_0 such that $x \circ (z - z_0) = 0$. The point x cannot be an interior point of $V(z_0)$ because even in $\{x \in H^n | x \circ (z - z_0) \leq 0\}$ the point x has no neighbourhood that is open in H^n. So M is the interior of $V(z_0)$. ♣

Lemma 3.8 *Denote by V the Voronoi domain in D^n for a horoball h. Then the closure $\overline{V}$ of V as a subset of the Euclidean space $\mathbb{R}^n$ intersects ∂D^n only in the base point of h.*

Proof: Denote by p the light-like representative of h in B. Radial projection to D^n maps p to the base point of h, say P. Let us suppose the existence of a point Q in $\overline{V} \cap \partial D^n$ different from P. Because $\overline{V}$ is closed and convex, the line g joining P and Q lies in $\overline{V}$. Choose any vector q in L^+ that maps to Q by radial projection.

We turn now to the Minkowski model H^n. The intersection of the vector subspace $< p, q >$ with H^n, a line g', is mapped by radial projection to g. Denote by r_w the distance of a point $w \in g'$ to h. As shown in the proof of the first part of Lemma 3.5, the Euclidean hyperplane

$$H_w = \{y \in \mathbb{R}^{n+1} | w \circ y = -e^{r_w}\}$$

is a supporting plane for C at the point p for any $w \in g'$. As w moves along g' to infinity, we obtain for each direction of movement a light-like supporting plane for C at the point p. One of the two is $< p >^L$, the other is parallel to $< q >^L$. But $< p >^L$ is by Lemma 3.1 the only supporting plane for C at p. So the assumed existence of a point $Q \in \overline{V} \cap \partial D^n$ different from P leads us to a contradiction. ♣

Lemma 3.9 *A Voronoi domain is an n-dimensional convex polyhedron. Its sides are compact.*

Proof: Denote by V the Voronoi domain corresponding to a horoball h that is represented by $z_0 \in B$. Let x be a point of the boundary of V. Corollary 3.4 and Lemma 3.7 imply that x lies in at least one Voronoi domain distinct from V. By Corollary 3.6 there is a neighbourhood $U(x)$ and a finite subset $\{z_0, z_1, \ldots, z_g\}$ of B, representing horoballs $h_0, \ldots, h_g$, $g \geq 1$, such that for any point y of $U(x)$

$$d(y, h_i) < d(y, h_j) \;\text{ for } i = 0, 1, \ldots, g \text{ and } j \geq g + 1.$$

It follows

$$U(x) \cap V = U(x) \cap \bigcap_{i=1}^{g} \{x \in H^n | x \circ (z_i - z_0) \leq 0\}.$$

We have thus shown that the set of half-spaces $\{x \in H^n | x \circ (z_i - z_0) \leq 0\}$ (whose intersection is V) is locally finite at the boundary of V. So V is a polyhedron. Its dimension is n because it contains the horoball h.

The sides of a polyhedron are closed. We have to show now that they are bounded. Let us suppose the existence of an unbounded side τ of V. In the projective model D^n, the closure of τ as a subset of $\mathbb{R}^n$ touches ∂D^n. By Lemma 3.8 this has to be the base point P of the horoball h. Now choose a point Q of τ. The half-line r connecting Q and the ideal point P lies in τ by convexity and closedness of τ. But r clearly intersects h which is in the interior of V. We thus derive the contradiction that a side of V intersects the interior of V. So the assumption about the unboundedness of τ is false. ♣

Remark: We showed that certain subsets of all the hyperplanes $\langle z_i - z_j \rangle^L$ for $z_i, z_j \in B$ are locally finite near the boundary of V. But the complete set of those hyperplanes is never locally finite in the whole of H^n. This can be seen in the upper half-space model U^n as follows.

First, we consider the case $n \geq 3$. The Euclidean $(n-1)$-manifold at a cusp section is compact, and it is therefore finitely covered by a Euclidean $(n-1)$-torus. Especially, the stabilizer of any parabolic fixed point in Γ contains the Poincaré extensions of two Euclidean translations with different translation axes. Choose a parabolic fixed point and denote two of the independent translations of its stabilizer in Γ by γ and δ. Bring this parabolic fixed point in the upper half-space model U^n to ∞ and choose any other parabolic fixed point $x \in \partial U^n$. The points $\gamma^i \delta^j(x)$, i and j integers, constitute a lattice that is contained in a Euclidean two-dimensional plane $E \subset \partial U^n$.

Now restrict attention to the hyperbolic 3-space F lying vertically above E. The intersection of F with the horoballs of S based at the lattice points in E is a set of horoballs of equal Euclidean radii. Choose two of these horoballs, say h and h'. Denote by g the mid perpendicular to their base points in E. The set of points in F that are equidistant from h and h' is the hyperbolic plane that lies vertically above g. Denote by g_i the mid perpendicular to the points $\gamma^{-i}\delta^{-1}(x)$ and $\gamma^i \delta(x)$. These lines are not locally finite, indeed they converge to the mid perpendicular to the points $\gamma^{-1}(x)$ and $\gamma(x)$. So, the planes F_i lying vertically above g_i are not locally finite, too. The set of all the considered hyperplanes contains all the F_i's and it is not locally finite, too.

The two-dimensional case is handled similarly. Again, we construct a not locally finite subset of the hyperplanes $\langle z_i - z_j \rangle^L$ for $z_i, z_j \in B$. As in the above, we position in U^2 a horodisk of S with its base point at ∞. The stabilizer of ∞ in Γ is generated by the Poincaré extension τ of a Euclidean translation of ∂U^2. Let h be any horodisk with a base point, say P, on ∂U^2. The points that are equidistant from h and $\tau(h)$ form a line that joins the Euclidean mid point of P and $\tau(P)$ with ∞.

Now let h_0 be a horodisk of S. The base points of horodisks are dense in $\partial U^2 \cup \{\infty\}$. Therefore, a sequence h_i of horodisks exists whose base points converge to the base point of h_0. The lines that are equidistant from h_i and $\tau(h_i)$, $i = 1, 2, \ldots$, are not locally finite. This had to be shown.

Definition: A k-dimensional polyhedron σ spans a k-dimensional subspace which we denote by $\langle\sigma\rangle$. Two polyhedra σ_1 and σ_2 are said to be *quasi-orthogonal* if $\langle\sigma_1\rangle$ and $\langle\sigma_2\rangle$ intersect orthogonally in exactly one point. If σ_2 is a point, quasi-orthogonality means just that σ_2 is an element of $\langle\sigma_1\rangle$.

Definition: Let σ be a subpolyhedron of the Euclidean decomposition of D^n with respect to Γ. The ideal vertices of σ correspond to elements $z_1, \ldots, z_m$ of B. The *dual polyhedron* σ^* is the intersection of the Voronoi domains $V(z_1), \ldots, V(z_m)$.

Theorem 3.10 *Let η and σ be subpolyhedra of a Euclidean decomposition of H^n with respect to Γ.*

- *$\eta \subset \sigma$ if and only if $\sigma^* \subset \eta^*$.*
- *If the dimension of σ is k, $1 \leq k \leq n$, then σ^* is a compact polyhedron of dimension $(n-k)$.*
- *The polyhedra σ and σ^* are quasi-orthogonal.*
- *For γ in Γ we have $(\gamma(\sigma))^* = \gamma(\sigma^*)$.*
- *If the decomposition is canonical, then for $\gamma \in N(\Gamma)$, the normalizer of Γ as a subgroup of the isometries of U^n, we have $(\gamma(\sigma))^* = \gamma(\sigma^*)$.*

Proof: The vertices of σ correspond to elements of B which may be numbered as $z_1, \ldots, z_m$, representing horoballs $h_1, \ldots, h_m$, $m \geq 2$.

To verify the first claim, we suppose $\eta \subset \sigma$. The vertices of η correspond to, say, $z_1, \ldots, z_j$ with $j \leq m$. A point that is equidistant from the horoballs $h_1, \ldots, h_m$ is equidistant from $h_1, \ldots, h_j$. So $\eta \subset \sigma$ implies $\sigma^* \subset \eta^*$.

Now assume $\sigma^* \subset \eta^*$. Because σ corresponds by radial projection to a side of C, there exists a supporting plane at C that intersects B only in $z_1, \ldots, z_m$. So, by Lemma 3.5, σ^* contains a point that is equidistant from $h_1, \ldots, h_m$ and farther away from any other horoball of S. But x is equidistant from the horoballs corresponding to the vertices of η, too, and farther away or equidistant from any other horoball of S. So the vertices of η have to be a subset of the vertices of σ, and therefore $\eta \subset \sigma$.

Now we prove the second claim. σ^* is closed because it is the intersection of Voronoi domains that are, as we know, closed. Voronoi domains V_1 and V_2 belong to the horoballs h_1 and h_2 based at the vertices of σ. Because $\sigma \subset V_1 \cap V_2$, σ^* lies in a side of V_1. We know that sides of Voronoi domains are compact, so σ^* is compact, too.

We compute the dimension of σ^*. The set of points that are equidistant from the horoballs $h_1, \ldots, h_m$ is the plane

$$\begin{aligned} T &= \bigcap_{i=2}^{m} \langle z_i - z_1 \rangle^L \cap H^n \\ &= \langle z_2 - z_1, \ldots, z_m - z_1 \rangle^L \cap H^n. \end{aligned}$$

We remark that $\langle z_2 - z_1, \ldots, z_m - z_1 \rangle$ is a proper subspace of $\langle z_1, \ldots, z_m \rangle$ because the first space is space-like, while the second one is time-like.

$$\begin{aligned} \mathrm{Dim}\, T &= \mathrm{Dim}(\langle z_2 - z_1, \ldots, z_m - z_1 \rangle^L) - 1 \\ &= (n + 1 - \mathrm{Dim}\langle z_2 - z_1, \ldots, z_m - z_1 \rangle) - 1 \\ &= n - \mathrm{Dim}\langle z_2 - z_1, \ldots, z_m - z_1 \rangle \\ &= n - (\mathrm{Dim}\langle z_1, \ldots, z_m \rangle - 1) \\ &= n - \mathrm{Dim}\langle \sigma \rangle. \end{aligned}$$

We showed in the above proof of the first claim that a point x exists in σ^* that is equidistant from $h_1, \ldots, h_m$ and farther away from any other horoball of S. Lemma 3.5 ensures even a neighbourhood of x with this property. Therefore, σ^* has a non empty interior in T. So the dimension of σ^* is the dimension of T.

We shall prove now the third claim. Remember that $m \geq 2$. The subspace $\langle z_1, \ldots, z_m \rangle$ is time-like, its dimension will be denoted by r. By the above proof of the second claim, we know that σ^* spans in H^n the plane

$$P_1 = \langle z_2 - z_1, \ldots, z_m - z_1 \rangle^L \cap H^n.$$

Because σ spans $P_2 = \langle z_1, \ldots, z_m \rangle \cap H^n$, we have to prove that P_1 and P_2 intersect orthogonally in one point.

By linear algebra one can show that there exists a positive Minkowski transformation A that maps $\langle z_1, \ldots, z_m \rangle$ to $U_1 = \langle e_1, \ldots, e_{r-1}, e_{n+1} \rangle$, and it maps $\langle z_2 - z_1, \ldots, z_m - z_1 \rangle$ to $\langle e_1, \ldots, e_{r-1} \rangle$. Because A preserves orthogonal complements, $\langle z_2 - z_1, \ldots, z_m - z_1 \rangle^L$ is mapped by A to $U_2 = \langle e_r, \ldots, e_{n+1} \rangle$. The planes $U_1 \cap H^n$ and $U_2 \cap H^n$ intersect orthogonally in the point e_{n+1}. Because A induces an isometry in H^n, P_1 and P_2 intersect orthogonally in one point, so σ and σ^* are quasi-orthogonal.

The fourth and the fifth claim follow immediately by the operations of the corresponding groups on the Euclidean (or canonical) decomposition and by the definition of Voronoi domains. ♣

Theorem 3.11 *The operation of Γ on the dual polyhedra has only finitely many orbits. The operation of Γ on the dual polyhedra of dimension smaller than n is free.*

Proof: This follows immediately from Theorem 1.1 and Theorem 3.10. Remember that the ideal points of the polyhedra are not part of the Euclidean decomposition, so the dual polyhedra of dimension n, i. e. the Voronoi domains, have to be excluded. ♣

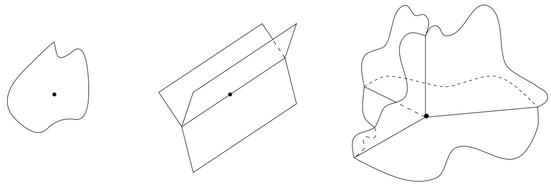

Figure 1: Neighbourhoods in a special spine.

4 Spines

A horoball collapses to its bounding horosphere by contracting each ray that starts at the base point of the horoball to its intersection with the horosphere. This is the nearest point retraction. We extend the rays based at the base points of the horoballs of S until they intersect the boundaries of the Voronoi domains. By contracting these rays, D^n collapses to the $(n-1)$-skeleton of the Voronoi decomposition of D^n. By the way, note that this is not the nearest point retraction. The fourth statement of Theorem 3.10 ensures that the collapsing is Γ-invariant. So the $(n-1)$-skeleton of the Voronoi decomposition is a spine of the hyperbolic manifold. By duality, this is a standard spine if and only if all the polyhedra of the Euclidean decomposition are simplices. For $n = 3$, the neighbourhoods in a special spine look like in figure 1.

5 Induced decompositions of the cusp sections

5.1 Duality in the cusp sections

We consider only cusp sections that are the boundaries of the chosen cusps. So, the preimage of a cusp section under the covering $H^n \to H^n/\Gamma$ is a horosphere that bounds a horoball of S. It is convenient to place this horosphere, say h, in the upper half-space model U^n with the base point at ∞. By this, h becomes a Euclidean hyperplane parallel to ∂U^n. In the following we fix a certain choice of cusps, and we consider the resulting Euclidean decomposition. We denote by Z_∞ all the polyhedra of the Euclidean decomposition with one vertex equal to ∞. The intersection of a polyhedron τ of Z_∞ with h is a compact polyhedron in the Euclidean structure of h because, by Theorem 2.2, h is disjoint to the side of τ opposite to ∞. If the dimension of τ is k, the dimension of the induced polyhedron in h is $(k-1)$. Z_∞ induces a decomposition on h which we call Z_E. It is invariant under Γ_∞, the stabilizer of ∞ in Γ.

By Lemma 3.8, the boundary of the Voronoi domain V of the horoball bounded by h is homeomorphic to h. Every line having the base point of h as an ideal vertex intersects both of ∂V and h in exactly one point. This bijection is in U^n the vertical projection along the n-th axis. A compact hyperbolic polyhedron of ∂V is mapped to a compact Euclidean polyhedron on h of the same dimension. So we get a decomposition on h which we call Z_V. By the fourth statement of Theorem 3.10, it is invariant under Γ_∞.

Z_E and Z_V induce decompositions of the cusp sections.

Definition: Let η be a polyhedron of Z_E of dimension k, $0 \leq k \leq n-1$. It is the intersection of h with an ideal polyhedron η_0 of dimension $(k+1)$ of the Euclidean decomposition. The projection of the dual polyhedron η_0^* to h is (by the above) an $(n-k-1)$-dimensional polyhedron of Z_V denoted by η^*. It is called the *dual polyhedron* of η in Z_V.

This duality is a bijection between the polyhedra of Z_E and of Z_V. We describe now explicitly this duality structure.

Theorem 5.1 *Let η and σ be subpolyhedra of Z_E for a specified cusp section.*

- *$\eta \subset \sigma$ if and only if $\sigma^* \subset \eta^*$.*
- *If the dimension of σ is k, the dimension of σ^* is $(n-k-1)$.*
- *The polyhedra σ and σ^* are quasi-orthogonal.*
- *For $\gamma \in \Gamma_\infty$ we have $(\gamma(\sigma))^* = \gamma(\sigma^*)$.*
- *If the decomposition is canonical, then for $\gamma \in N(\Gamma)_\infty$, the stabilizer of ∞ in the normalizer of Γ as a subgroup of the isometries of U^n, we have $(\gamma(\sigma))^* = \gamma(\sigma^*)$.*

Proof: Only the third claim does not follow immediately by Theorem 3.10 and the above considerations. We prove the quasi-orthogonality of σ and σ^*. Compare with figure 2.

Denote the dimension of σ by k, $0 \leq k \leq n-1$. Quasi-orthogonality is preserved by isometries, so we may assume that σ lies in a horosphere h that has ∞ as its base point. Let σ_0 be the polyhedron of Z_∞ whose intersection with h is σ.

The projection of σ_0^* to h is σ^*. σ_0^* spans an $(n-k-1)$-dimensional plane $\langle\sigma_0^*\rangle$ in U^n, and this is a Euclidean hemisphere which is orthogonal to ∂U^n. By Theorem 3.10, $\langle\sigma_0\rangle$ and $\langle\sigma_0^*\rangle$ intersect orthogonally in one point, say x. Denote by T the $(n-k-1)$-dimensional Euclidean tangency plane for $\langle\sigma_0^*\rangle$ at the point x. T and $\langle\sigma_0\rangle$ intersect orthogonally in the Euclidean and in the hyperbolic structure of U^n because U^n is a conformal model.

The projection of T to h is the Euclidean plane that is spanned by σ^*. And the intersection of $\langle\sigma_0\rangle$ with h is the Euclidean plane that is spanned by σ. These two planes intersect orthogonally in one point which is the projection of x to h. So σ and σ^* are quasi-orthogonal in the Euclidean structure of h. ♣

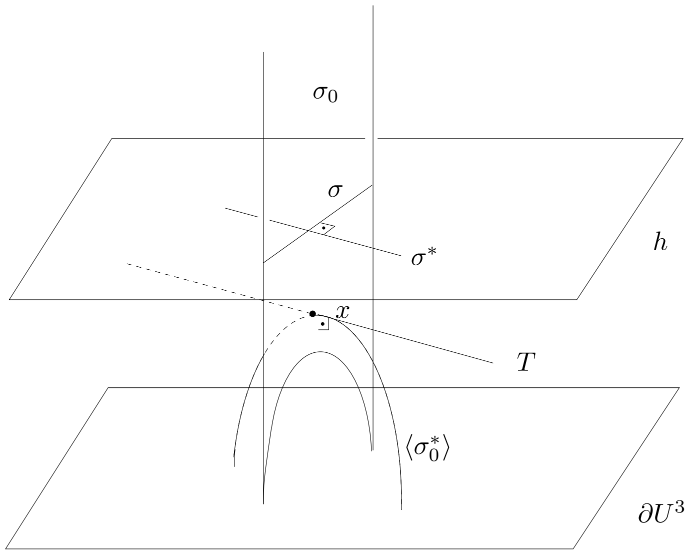

Figure 2: Dual decompositions on a horosphere.

5.2 Isometric dual polyhedra

Let $M = U^3/\Gamma$ be a manifold with exactly one cusp, and suppose that its Euclidean decomposition consists of ideal tetrahedra. Because there is only one cusp, the operation of Γ on the parabolic fixed points is transitive. So the Γ-orbit of each tetrahedron of the Euclidean decomposition of U^3 determines by its intersection with the horosphere h based at ∞ four Γ_∞-orbits of Euclidean triangles in Z_E. Because opposite angles in an ideal tetrahedron are equal, these triangles are similar.

A Γ-orbit of an edge of the Euclidean decomposition of U^3 contributes to Z_E two Γ_∞-orbits of points. We shall prove in the following Theorem that the dual polyhedra to these two points are isometric in the Euclidean structure of h. But, in general, this isometry does not lie in the normalizer of Γ_∞ as a subset of the Euclidean isometries of h. So, in general, this isometry will not induce an isometry in the Euclidean structure of h/Γ_∞, i. e. in the cusp section.

Theorem 5.2 *Let $M = U^n/\Gamma$ be a hyperbolic n-manifold of finite volume and exactly one cusp, $n \geq 2$. Let σ be a k-dimensional ideal polyhedron of the canonical decomposition of U^n determined by Γ. Let h be any horosphere that is the preimage of the boundary of the chosen cusp. As described above, every ideal vertex of σ determines on h a Γ_∞-orbit of $(k-1)$-dimensional polyhedra of Z_E. Let σ_1 and σ_2 be such polyhedra corresponding to different vertices of σ. Then there exists a Euclidean isometry f that maps σ_1^* to σ_2^*. If M is orientable, f can be chosen to be orientation reversing.*

Proof: Compare with figure 3. We bring the base point of h in U^n to ∞. By an isometry of Γ, σ is positioned so that $\sigma \cap h = \sigma_1$. There exists an ideal vertex P of σ and an isometry γ of Γ so that $\sigma_2 = \gamma(\sigma) \cap h$. P is different from ∞ because σ_1 and σ_2 belong to different vertices of σ.

A horoball of S is based at P, and we denote its bounding horosphere by h_P. Denote by E the hyperbolic hyperplane consisting of points that are equidistant from h and from h_P. σ^* lies in E. The reflection ι at E maps h to h_P. So $\gamma\iota$ leaves the horosphere h invariant. This implies that $\gamma\iota$ is the Poincaré extension of a Euclidean isometry of ∂U^n. This means that there exists a Euclidean isometry f of $\partial U^n = \mathbb{R}^{n-1}$ so that $\gamma\iota(x,t) = (f(x),t)$, $x \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$. Denote by p the orthogonal projection of U^n to h. This projection commutes with Poincaré extensions. For a point x of σ^* we have

$$\begin{aligned} p\gamma(x) &= p\gamma\iota(x) \\ &= \gamma\iota p(x) \\ &= fp(x). \end{aligned}$$

By definition of the dual decomposition of the cusp, $p(\sigma^*)$ is σ_1^*. Because $\gamma(\sigma^*) = (\gamma(\sigma))^*$ (see Theorem 3.10), we have $p\gamma(\sigma^*) = \sigma_2^*$. So $\sigma_2^* = f(\sigma_1^*)$.

M is orientable if and only if every element γ of Γ is orientation preserving. Because the reflection ι is orientation reversing, f is orientation reversing if and only if γ is orientation preserving. Remark that there might exist an orientation

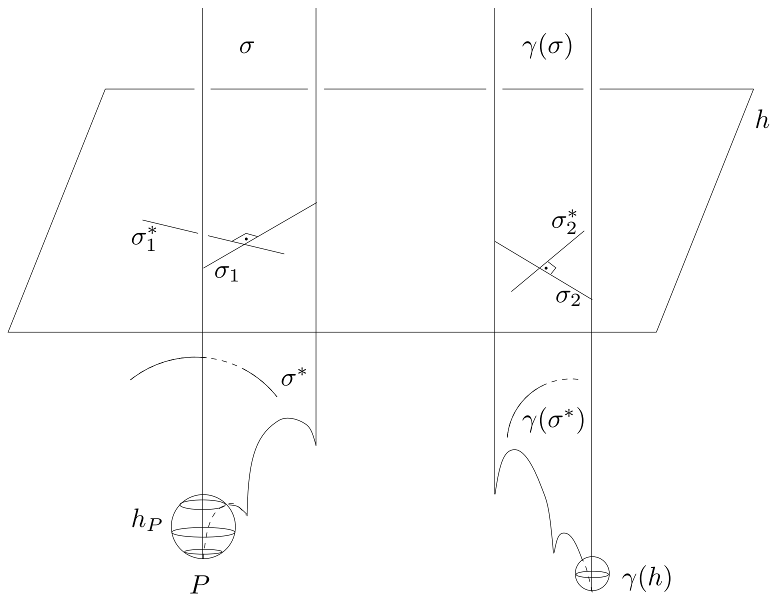

Figure 3: Isometric dual polyhedra.

reversing Euclidean isometry that leaves σ_1^* invariant. So σ_1^* and σ_2^* might be mapped onto each other by an orientation preserving isometry, too. ♣

Remark: Theorem 5.2 can be extended to manifolds with more than one cusp. To do this, a Euclidean structure can be given to any horosphere h as follows. Choose an isometry ζ so that $\zeta(h)$ is in U^n a Euclidean hyperplane parallel to ∂U^n of height 1. Then define the Euclidean distance of any two points of h as the Euclidean distance of their image points. This is well defined because an isometry of U^n that fixes $\zeta(h)$ is the Poincaré extension of a Euclidean isometry of ∂U^n.

Assume that in the above proof the horospheres h_P and h are not equivalent under the operation of Γ. Denote by q_1 the orthogonal projection of U^n to h_P, i. e. for $x \in U^n$ the point $q_1(x)$ is the intersection of h_P with the line passing through x and P. Correspondingly, denote by q_2 the orthogonal projection to h.

The reflection ι at E satisfies $\iota q_1(x) = q_2(x)$ for any $x \in E$. So, ι maps the polyhedron $q_1(\sigma^*)$ of the dual decomposition of h_P to the polyhedron $q_2(\sigma^*)$ by an isometry of the Euclidean structures of the two horospheres.

Whether this isometry is orientation reversing of preserving depends on the chosen orientations of the cusp sections.

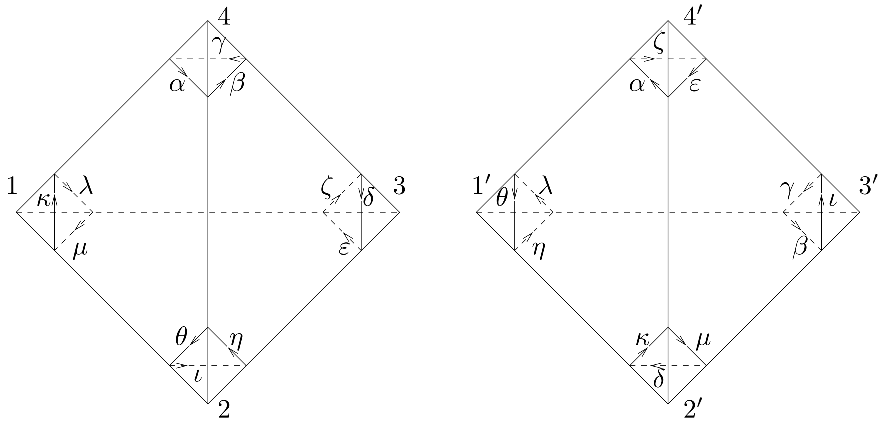

Figure 4: Two ideal regular tetrahedra.

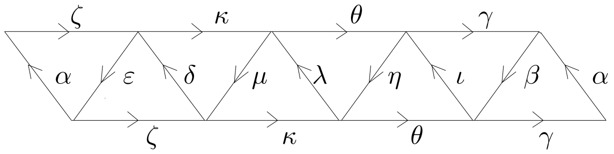

Figure 5: The cusp section is a torus.

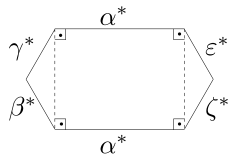

Figure 6: Dual hexagon.

6 Applications

6.1 The figure eight knot

We shall apply the preceding results on the complement of the figure eight knot. This space can be decomposed into two ideal regular tetrahedra. In figure 4, the edges of the small triangles with the same label are to be glued together (see e. g. [Rat94] or [Thu79]). Indeed, this is the canonical decomposition. One can verify this by computing the tilts as described in [SW95b]. The cusp of this manifold is homeomorphic to $S^1 \times S^1 \times [1, \infty)$. The induced Euclidean decomposition Z_E of the cusp section consists of Euclidean equilateral triangles with identifications as in figure 5.

We position the horosphere h with the decomposition Z_E in the upper half-space model U^n with the base point at ∞. The stabilizer Γ_∞ of ∞ in Γ is generated by two translations, say ξ_1 and ξ_2. To describe them, we set the height of an equilateral triangle equal to 1. The length, say a, of the edges is then $2/\sqrt{3}$. We choose a Euclidean coordinate system in h with an x-axis parallel to κ. With suitable orientations of the axes we have

$$\xi_1 : (x, y) \to \left(x - \frac{a}{2}, y + 1\right) \text{ and } \xi_2 : (x, y) \to (x + 4a, y).$$

We shall compute the dual decomposition of h. One can see easily that the dual point to each one of the triangles is its centre, but we would like to know

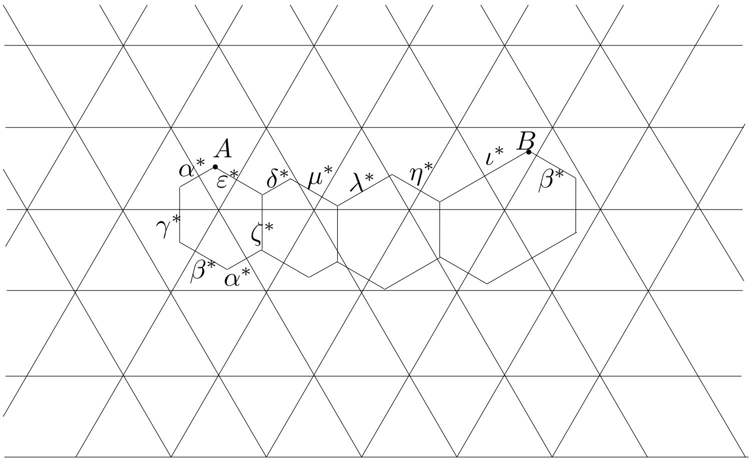

Figure 7: Dual edges in the continuation of figure 5.

how far the Theorems 5.1 and 5.2 determine the dual decomposition of the cusp section. We remark that if, as here, Γ_∞ contains only translations, these two Theorems determine the dual decomposition at most up to translation.

Let us denote by Z^* any decomposition of h that assigns to every polyhedron σ of Z_E a polyhedron σ^* of Z^* satisfying the conditions one to four of Theorem 5.1. An example for Z^* is Z_V.

There are four Γ_∞-orbits of vertices in Z_E. The dual two-dimensional polyhedra have to be hexagons because six edges come together at every vertex of Z_E. By quasi-orthogonality, every interior angle has to be 120°.

Consider in figure 7 the dual hexagon with sides α^*, γ^*, β^*, α^*, ζ^*, and ε^* (where we denoted $\xi_1^{-1}(\alpha^*)$ by α^*, too). We shall denote a side and its length by the same symbol. The two sides denoted by α^* are orthogonal to the translation direction of ξ_1. So the hexagon has to have the shape of figure 6.

The translation length a of ξ_1 is $2/\sqrt{3}$. It follows that the sides γ^*, β^*, ζ^* and ε^* all have the same length $a/(2\cos 30°)$, i. e. 2/3. Considering the other dual hexagons, we have

$$\gamma^* = \beta^* = \zeta^* = \varepsilon^* = \mu^* = \kappa^* = \eta^* = \theta^* = \frac{2}{3}. \tag{1}$$

The decomposition Z^* has to be invariant with respect to ξ_2, too. In figure 7, the point A has to be mapped to B by ξ_2. This is satisfied if and only if A and B have the same y-component and the distance of A and B is $4a$, i. e.

$$d_1 = d_2 \text{ and} \tag{2}$$
$$(4a)^2 = d_1^2 + d_2^2 - 2d_1 d_2 \cos 120°, \tag{3}$$

where

$$d_1 = \alpha^* + \delta^* + \lambda^* + \iota^* \text{ and } d_2 = \varepsilon^* + \mu^* + \eta^* + \beta^*.$$

If (1) is satisfied, then (2) and (3) reduce to

$$\alpha^* + \delta^* + \lambda^* + \iota^* = \frac{8}{3}. \tag{4}$$

The decomposition Z_V satisfies

$$\alpha^* = \delta^* = \lambda^* = \iota^* = \frac{2}{3}.$$

So, besides Z_V, there exist many dual decompositions satisfying the statements one to four of Theorem 5.1. For example, we can shrink the edge ι^* a little bit in figure 7 so that A is mapped to B by ξ_2.

Now, which constraints on Z^* are imposed by Theorem 5.2? There are four Γ-orbits of ideal triangles in the Euclidean decomposition of U^3 because the eight sides are identified pairwise. Every triangle determines three Γ_∞-orbits of edges in the decomposition Z_E of h. For example, the triangle with the ideal vertices 1, 2 and 4 determines the edges α, κ and θ. Theorem 5.2 implies that the lengths of the edges are related as follows

$$\begin{aligned} \alpha^* &= \kappa^* = \theta^* \\ \gamma^* &= \zeta^* = \lambda^* \\ \eta^* &= \delta^* = \beta^* \\ \iota^* &= \mu^* = \varepsilon^*. \end{aligned}$$

Equation (1) implies that each of the dual edges has the length 2/3. So, in this example, the Theorems 5.1 and 5.2 determine the dual decomposition of h up to translation.

One can compute by hand the combinatorial automorphisms of the decomposition in figure 4. It is a dihedral group of order 8 generated by two automorphisms which are described by the permutation of the ideal vertices:

$$\tau_1 = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,4\,1\,3 \end{pmatrix}, \tau_2 = \begin{pmatrix} 1'\,2'\,3'\,4' \\ 3'\,4'\,4'\,2' \end{pmatrix} \text{ and } \omega_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1' & 2' & 3' & 4' \end{pmatrix}, \omega_2 = \begin{pmatrix} 1' & 2' & 3' & 4' \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Let us see how far we get if we do not use the information that the ideal tetrahedra are regular. In figure 4, the identification maps the 12 edges onto two edges. The combinatorial automorphism determined by τ_1 and τ_2 permutes these two edges. Each edge induces two points in the Euclidean decomposition Z_E of the cusp section torus. So there are two $N(\Gamma)_\infty$-orbits of points in Z_E. The fifth statement of Theorem 5.1 implies that there are only two isometry classes of dual hexagons in Z_V. Theorem 5.2 even implies that all the dual hexagons are isometric. Though this does not imply immediately that the dual hexagons are regular, this is quite a strong relation.

If the cusp section is a torus, any translation commutes with Γ_∞, so there exist uncountably many Euclidean isometries for the cusp section. Therefore, the isometries of the manifold do not induce all the isometries of the cusp sections. It is surprising that there exist isometries of the cusp sections that preserve both of the dual decompositions of the cusp sections but that are not induced by isometries of the manifold. An example can be seen in figure 5. It is the reflection in the line that is orthogonal to the edge λ and that passes through the centre of λ.

6.2 Two-bridge links

Sakuma and Weeks described in [SW95a] a topological decomposition of the complements of hyperbolic two-bridge links. The decomposition consists of ideal tetrahedra. In section II.4 of their paper they considered the induced combinatorial decomposition of the cusp sections. Then they calculated how the tetrahedra have to be realized as hyperbolic ideal tetrahedra so that the gluing produces a complete hyperbolic structure. In Theorem II.5.5, they obtained an equation whose solutions with positive imaginary part produce a complete hyperbolic structure. It is the conjecture of Sakuma and Weeks that one of these solutions produces the canonical decomposition.

The Theorems 5.1 and 5.2 provide necessary conditions for the decomposition of the torus to be induced by the canonical decomposition. The combinatorial automorphisms of the decomposition have been computed by Sakuma and Weeks, so the fifth statement of Theorem 5.1 can be applied immediately, too.

These additional constraints might be strong enough so that only one solution of the above mentioned equation satisfies them. For this candidate, one could compute the tilts (according [SW95b]) and maybe verify that the decomposition is the canonical one.

7 Degenerate ideal triangulations

It is a conjecture that every non-compact hyperbolic 3-manifold of finite volume can be decomposed into hyperbolic ideal tetrahedra. A Euclidean decomposition solves only part of this problem because it uses ideal polyhedra. Now decompose each of the polyhedra arbitrarily into ideal tetrahedra without introducing new vertices. This induces a decomposition of the two-dimensional sides into triangles. In general, the identifications of the polyhedra's two dimensional sides do not respect this new decomposition. But the concept of 'flat' ideal tetrahedra will resolve this difficulty.

It helps, if we regard the gluing of polyhedra in a two-stage process. First, we have finitely many combinatorial polyhedra with combinatorial gluing maps. Then we realize the polyhedra as hyperbolic ideal polyhedra with gluing isometries that are still determined by the combinatorial mapping of the ideal vertices.

We start with two pyramids, say C_1 and C_2, having a triangulation as in figure 8. We want to glue the sides as indicated there for the vertices. This is not compatible with the triangulation, but, for the moment, we do not care. This situation can be easily realized in hyperbolic 3-space with ideal vertices. By the gluing, we obtain an octahedron. In figure 9, we change the gluing a little

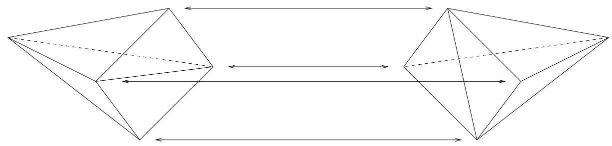

Figure 8: Two pyramids.

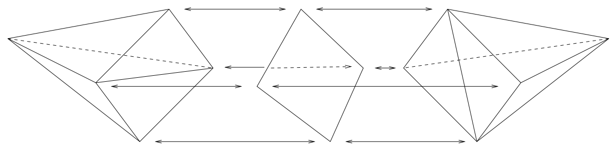

Figure 9: An additional square.

bit by inserting an additional square between those sides of C_1 and C_2 that are glued onto each other. The result is the same. But the description changed: We start with two pyramids, C_1 and C_2, and a tetrahedron T. The combinatorial gluing is as in figure 10. This gluing respects now the triangulation. Then we choose realizations of C_1 and C_2 as before, but for T we choose a degenerate realization as a square. This insertion of degenerate tetrahedra opens the way to glue sides with different decompositions.

Let us call the exchange of the triangles in figure 11 a 2-process. To handle the general case, one has to verify that any two triangulations of a k-gon can be transformed into each other by finitely many 2-processes. This can be done by induction.

The attempt to produce degenerate ideal triangulations in higher dimensions fails in general. In any dimension it is possible to triangulate the polyhedra of the Euclidean decomposition without introducing new vertices. But for $n \geq 4$, the induced decompositions on the $(n-2)$-dimensional sides are non-trivial. Because the insertion of flat n-simplices does not change the $(n-2)$-skeleton of the triangulation, different triangulations cannot be made equal.

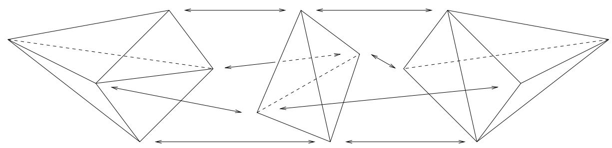

Figure 10: Adding a tetrahedron.

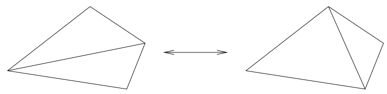

Figure 11: A 2-process.

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