Dirac particles in Rindler space

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We show that a uniformly accelerated observer experiences a "thermal" flux of Dirac particles in the ordinary Minkowski vacuum.

It has been known for several years\(^1\) that the particle interpretation of quantum field theory in a general Riemannian manifold is not unique, but depends on the observer's coordinate frame. The particle concept, endowed with a nonlocal nature by the uncertainty relation, depends on the global topological structure of that submanifold which is naturally connected with the observer's state of motion.

In this article we show that a uniformly accelerated observer in Minkowski space experiences a flux of Dirac particles in the ordinary Minkowski vacuum. To do this we utilize a method of Rumpf\(^4\) allowing one to define particle and antiparticle states in quite general circumstances.

Owing to the dynamics of the uniformly accelerated observer, two-dimensional Minkowski space is divided into four sectors, as can be seen from Fig. 1: right (I), left (II), future (F), and past (P) with respect to the origin \(x=t=0\). Minkowski coordinates \((t,x)\) may be transformed into Rindler coordinates \((v,u)\) according to [the following conventions are used: \(R=c=1\), \((x^0=t,x^3=x)\) in \(\mathbb{M}\), \((x^0=v,x^3=u)\) in I II, and \((x^0=u,x^3=v)\) in F P]

\[
\begin{align*}
& t = u \sinh v, \quad x = u \cosh v, \quad \text{in I II} \quad (1a) \\
& v = \tanh(t/x), \quad u = \text{sgn}(x^2 - t^2)^{1/2}, \\
& t = u \cosh v, \quad x = u \sinh v, \quad \text{in F P} \quad (1b) \\
& v = \tanh(x/t), \quad u = \text{sgn}(t^2 - x^2)^{1/2}.
\end{align*}
\]

This leads to the line element

\[
\begin{align*}
& ds^2 = u^2 dv^2 - du^2 \quad \text{in I II}, \quad (2a) \\
& ds^2 = du^2 - u^2 dv^2 \quad \text{in F P}.
\end{align*}
\]

It is well known that a world line \(u = \text{const}\) in I corresponds to the trajectory of a uniformly accelerated observer. In I the timelike coordinate \(v\) is connected with the observer's proper time via \(v = \gamma t\) where \(\gamma\) is the observer's acceleration. We start calculating Dirac wave functions in the four sectors of Minkowski space. The covariant Dirac equation reads\(^5\)

\[
[i
\gamma^\alpha (\partial_\alpha + \Gamma_\alpha) - m] \psi = 0.
\]

Let us first concentrate on sector I. \(\partial_\alpha\) being a (timelike) Killing vector, we may solve for stationary states \(\psi(v,u) = \varphi(u) \exp(-i\omega u)\):

\[
\omega \varphi = \left[-i\alpha \gamma^0 \frac{\partial}{\partial u} \gamma^1 + \beta \gamma^0 u \right] \varphi = \hat{H}_R \varphi.
\] (4)

For spin-up states the only normalizable solution reads\(^6\)

\[
\varphi(u) = \begin{cases} 
\phi_+^\omega + \phi_-^\omega & \\
0 & \\
\phi_-^\omega - \phi_+^\omega & \\
0 &
\end{cases} \quad \phi_-^\omega = H^{1/2}_{\text{ant}} / (imu)
\] (5)

with \(\phi_+^\omega\) obeying the differential equation

\[
\left( u \frac{d}{du} + \frac{d}{du} \right) \phi_+^\omega = \left[ m^2 u^2 - \left( \omega \mp \frac{i}{2} \right)^2 \right] \phi_+^\omega \quad (\omega \in \mathbb{R}).
\] (6)

It is interesting to note that the difference in the boson case is the term \(\frac{1}{2}i\) in Eq. (6) which does not...
appear in the Klein–Gordon Equation. This term arises due to the nonvanishing derivative of $\hat{H}_R$ [Eq. (4)]. Let us denote the solution of the Dirac equation in sector I as

$$\begin{bmatrix} \Phi^- \\Phi^+ \end{bmatrix} = e^{-i\omega u} \begin{bmatrix} \Phi^+ \\Phi^- \\theta(t+x)\delta(x-t) \\end{bmatrix},$$

indicating that $I \subset \Re$ is the support of $\psi_\omega$. The norm of $\psi_\omega$ is given by

$$\langle \psi_\omega, \psi_\omega \rangle = \frac{8\epsilon_{\omega}}{m\cosh(\pi \omega)} \delta(\omega - \omega'),$$

where the Dirac scalar product $\langle , \rangle$ is defined as

$$\langle \psi_1, \psi_2 \rangle = \int d\sigma \psi_1^\dagger \psi_2 \\text{on some spacelike hypersurface } \Sigma.$$  

We obtain the following two independent solutions for Eq. (10):

$$F_\psi^{-}(\omega) = e^{-i\omega u} \begin{bmatrix} \Phi^- - i\Phi^+ \\Phi^- + i\Phi^+ \\end{bmatrix}, \quad \epsilon_\omega = H_0^{(1)}(mu),$$

$$F_\psi^{+}(\omega) = e^{-i\omega u} \begin{bmatrix} \Phi^- + i\Phi^+ \\Phi^- - i\Phi^+ \\end{bmatrix}, \quad \epsilon_\omega = H_0^{(2)}(mu),$$

with

$$\langle F_\psi^{(i)}, F_\psi^{(k)} \rangle = \frac{16\epsilon_{i\omega}}{m} \delta(\omega - \omega') \delta_{ik} \quad (i, k = \pm).$$

We obtain analogous wave functions in II and P through

$$\begin{bmatrix} \Phi^+ \\Phi^+ \end{bmatrix} = e^{-i\omega u} \begin{bmatrix} \Phi^+ \\Phi^- \\theta(t-x)\delta(x-t) \\end{bmatrix},$$

$$\begin{bmatrix} \Phi^+ \\Phi^+ \end{bmatrix} = e^{-i\omega u} \begin{bmatrix} \Phi^+ \\Phi^- \\theta(t+x)\delta(x-t) \\end{bmatrix}.$$
scribed by the unique normalized wave functions of the homogeneous equation in $B = I \cup F \cup P$:

$$\psi_\omega = \frac{m^{1/2} e^{-\omega/2}}{[96 \cosh(\pi \omega)]^{1/2}} \left( \begin{array}{c} 2 \cosh(\pi \omega)^2 \psi_\omega + e^{-\omega/2} F \psi_\omega \\
- e^{\omega/2} F \psi_\omega \end{array} \right) + e^{\omega/2} P \psi_\omega,$$

(17)

Note that $\psi_\omega$ does not contain $\phi_\omega$ because the observer's information is decoupled from $\Pi \subset \Re$. It is at this point where the global aspect of the submanifold $B$ plays a crucial role for the particle interpretation of $\beta$.

The observer modes are related to the Minkowski modes according to

$$\psi_\omega = \alpha_\omega \psi_\omega + \beta_\omega \psi'_\omega,$$

(18)

with

$$\alpha_\omega = \frac{e^{\omega/2}}{[2 \cosh(\pi \omega)]^{1/2}}, \quad \beta_\omega = \frac{e^{-\omega/2}}{[2 \cosh(\pi \omega)]^{1/2}}.$$

Now Unruh$^3$ and others have shown that the accelerated observer experiences a spectral representation of $\beta$ according to positive and negative values of $\omega$:

$$\beta^B = \sum_n \int_0^\infty d\omega (\hat{c}_{\omega,a} \psi_{\omega,a} + \hat{c}_{\omega,a}^\dagger \psi_{\omega,a}^*),$$

(19a)

defining the $B$ vacuum

$$\hat{c}_{\omega} |0_B\rangle = \hat{d}_{\omega} |0_B\rangle = 0.$$  

(19b)

We may calculate the number of $B$ particles in the ordinary Minkowski vacuum as $\langle \omega, \omega' | 0 \rangle > 0$

$$\langle 0 | \hat{c}_{\omega,a}^\dagger \hat{c}_{\omega',a}^\dagger | 0 \rangle = \langle 0 | \delta(\omega - \omega') \delta_{aa'}.$$  

(20)

This can be interpreted that the observer measures a number of created particles per unit interval of proper time:

$$\frac{dn}{d\tau} = 2 \int_0^\infty \frac{d\omega/2\pi}{e^{\omega/\tau} + 1},$$

(21)

where the factor of 2 is introduced by the two spin projections. This means that he measures a "thermal" flux of Dirac particles where the effective "temperature" is the Fulling-Unruh temperature

$$T_B = \frac{\hbar k}{2 \pi c k} = 10^{-22} \text{ K sec}^2/\text{cm}.$$  

(22)

Thus one expects that the action of the Dirac field onto a particle detector corresponds to the action of an isotropic temperature bath of temperature $T_B$. Note that the change from the boson to fermion statistics resulted from the details of the Dirac equation, i.e., the additional term $\frac{1}{2}i$ in the Dirac equation changes the behavior of the wave functions as $\omega = 0$ as compared to the Klein-Gordon equation. This leads to $\beta_\omega = (e^{2\omega} - 1)^{-1/2}$ instead of $\beta_\omega = (e^{2\omega} - 1)^{-1/2}$ as was found by Fulling$^1$ and Rumpf$^4$ for scalar particles.

We finally note that a pure quantum state defined on the whole Minkowski manifold may appear as a mixed state when measured on a submanifold for the following reason: Since the observer modes $\psi_\omega$ vanish on sector $\Pi \subset \Re$, $\beta^B$ generates only a subalgebra of the whole Minkowski field algebra. Let $\hat{P}_B$ be the projector onto the set of all states generated by $|0_B\rangle$. Then $\hat{P}_B |0_B\rangle$ will be a mixed state containing particle antiparticle excitations.

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