I. INTRODUCTION

In a recent paper\(^1\) we have developed a formalism for the treatment of narrow resonances by shell-model methods. In this paper we shall generalize the treatment to the case of arbitrary continuum states. A central role here again is played by the natural boundary conditions. However, we abandon the previously used R-matrix theory and use, instead, a representation in which the S matrix is diagonal. In this representation all formal relations are very simple and transparent. We shall call the eigenstates of the S matrix the "eigenchannels" and the phase shifts associated with the eigenvalues of the S matrix the "eigenphases." As will be seen below the eigenchannels and the eigenphases can be obtained directly from the shell model. To this end, the configuration space is separated into an inside and an outside region. The nuclear problem is solved in the inside region by using properly defined natural boundary conditions for the wave function at the matching radius, \( r = a \), which separates the inside and the outside region; the obtained solutions are already without further transformations the eigenchannel solutions.

The previously treated case of a narrow resonance\(^1\) corresponds to the situation in which the reaction is dominated by a single compound state, and which can be described by a single-level Breit-Wigner formula. In the eigenchannel language this corresponds to the domination of the reaction by a single eigenchannel, which has at the resonance energy an eigenphase of \( \pi/2 \). Then all the other channels can be relegated to the "background."

In general all eigenchannels contribute comparably to the reaction amplitude. It is thus necessary to find all eigenchannels and all eigenphases. Once this has been done one actually has obtained the complete solution of the problem. This can be seen formally very simply. Writing the eigenvalue equation as

\[ SV'(\omega) = \epsilon_n V'(\omega), \]

Then the eigenvalues can be written in the form

\[ \epsilon_n = e^{i 2\pi \alpha}, \]

where, because of the unitarity of the S matrix, the eigenphases \( \delta_n \) are real. Combining the column vectors \( V'(\omega) \) into a "solution matrix" \( W \) and constructing with the eigenvalues \( e^{i 2\pi \alpha} \) the eigenvalue matrix \( \epsilon \) which is a diagonal matrix whose diagonal elements are the eigenvalues \( \epsilon_n \), we have the well-known representation\(^2\) of the S matrix

\[ S = WEW^{-1}, \]

which, then, is the complete formal solution of the problem.

Evidently, the formal aspects of the reaction theory are so trivial in the present treatment that it even may not deserve the name "reaction theory." However, for exactly the same reason we believe that it is eminently suited as a method for the computation of reaction cross sections. Actual numerical calculations using this method are underway at this time. They will be reported on in a separate publication.

There exist several procedures in the literature for treating the problem of this paper.\(^2-7\) In fact, the simplest cases, e.g., \( 0^+ \) in the one-particle-one-hole approximation, can be treated by every method. Which of the methods is easier to apply, and which of the methods is easier to generalize to more complicated systems and to higher approximations, will have to be proven out by practical experience.

In the present paper we still limit ourselves to the energy region below the two-particle threshold, i.e., all considered channels are assumed to have only a single incoming or outgoing particle. Also, the center-of-mass problem has been left unresolved. However, this is a separate problem, and it would disappear simply by using such wave functions in which the spurious components have been eliminated by means of some arbitrary, although as yet unknown, prescription. Without this refinement, the results will contain uncertainties of the order \( 1/A \).

In Sec. II we give the precise definition of the eigenchannels and discuss their physical meaning; we also

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\(^1\) Michael Danos and Walter Greiner, Phys. Rev. 138, B93 (1965).
\(^6\) A. D. Hill and B. Buck, Proceedings of the International Conference on Nuclear Structure, Antwerp, 1965 (to be published); B. Buck (private communication).
\(^7\) F. S. Levin, Phys. Rev. 140, B1099 (1965).
describe some special cases. Some formal relations are discussed in Sec. III. The procedure for computing the $S$ matrix is given in Sec. IV. Many details of this procedure can be taken over completely from Ref. 1. In Sec. V we show in which way resonating and nonresonating processes appear in the eigenchannel method and discuss the relations of this method with the $R$-matrix formalism.

II. THE EIGENCHANNELS

In this section we give the precise definitions of the eignchannels and discuss their physical meaning. The $S$ matrix is defined, as usually, by the asymptotic form of the wave function, i.e., by the wave function at such distances where the short-range nuclear forces have a negligible magnitude. There then holds

$$\Psi = \sum_i [A_i I_e + B_i 0_e] \psi_i.$$  \hspace{1cm} (4)

Here $I_e$ and $0_e$ are the radial parts of the incoming and outgoing particle in the channel $e$, $\psi_i$ is the wave function of the $(A-1)$ system together with the angular parts of the projectile particle, and the summation is over all open channels. To distinguish these channels $e$ from the eigenchannels of the $S$ matrix, we shall call them the "experimental channels." The $S$ matrix then is defined by

$$B_e = -\sum e' S_{ee'} A_{e'}.$$  \hspace{1cm} (5)

Considering (1) and (2), and assuming neutral particles, it then holds that for an eigenchannel

$$\Psi^{(a)} = \sum_i [V_e^{(a)} I_e - e^{i\delta_e} V_e^{(0)}] \psi_i$$
$$= \sum_i V_e^{(a)} e^{i\delta_e} [e^{-i\delta_e} I_e - e^{i\delta_e} 0_e] \psi_i$$
$$= \sum_i V_e^{(a)} e^{i\delta_e} e^{i\delta_e} [e^{i\delta_e} I_e - e^{i\delta_e} 0_e] \psi_i$$
$$= -2i e^{i\delta_e} \sum_i V_e^{(a)} \sin(\delta_e + \delta_e - i\pi/2) \psi_i.$$  \hspace{1cm} (6)

The eigenchannel thus corresponds to standing waves in all open experimental channels with a common phase shift, the eigenphase $\delta_e$. This is true also for channels in which the outgoing particle is charged.

We now turn to the discussion of (3), i.e., to the representation of the $S$ matrix in terms of the eigenchannels and eigenphases. The solution matrix $W$ which has the elements

$$W_{ik} = V^{(k)}$$  \hspace{1cm} (7)

can always be inverted since the solutions $V^{(a)}$ form a complete set. In particular, there are as many eigenchannels as there are open experimental channels at the considered energy, say $N$. The vectors $V^{(a)}$ are orthogonal as long as the eigenphases are nondegenerate. Otherwise they can be orthogonalized. Furthermore, they can be normalized. From now on we shall assume that the ortho-normalization has indeed been carried out. Then

$$W^{-1} = \bar{W},$$  \hspace{1cm} (8)

and together with the definition of the eigenvalue matrix elements

$$\epsilon_{ek} = \epsilon_{ik} \delta_{ek},$$  \hspace{1cm} (9)

Eq. (3) becomes

$$S_{cc'} = \sum_a V_c^{(a)} e^{i\delta_a} V_{c'}^{*(a)}.$$  \hspace{1cm} (10)

Equation (10) shows that in the end no matrix inversions are necessary if the complete set of eigenchannels has been obtained. It remains to be shown that in fact it is possible to obtain such solutions. Before turning to that question we discuss some special cases of the $S$ matrix.

In potential scattering the different experimental channels do not mix and the $S$ matrix is thus diagonal in the experimental channel representation. Here the experimental channels are already the eigenchannels. The different eigenphases have no relation to each other.

Somewhat more tricky is the case of degeneracy of eigenphases. Let us consider the case where a certain number, say $n$, of the eigenphases are equal. Then, writing $M$ for the inverse of the $W$ matrix (8), i.e., putting $M = W^{-1}$, and writing $1_n$ for the $n$-dimensional unit matrix, we have

$$S = \begin{pmatrix} W_{aa} & W_{ab} & 1_n & 0 & M_{aa} & M_{ab} \\ W_{ba} & W_{bb} & 0 & \Gamma_a & M_{ba} & M_{bb} \end{pmatrix},$$  \hspace{1cm} (11)

which is (3) expanded such that the degenerate states are explicitly separated from the nondegenerate states. Furthermore, we have written $\Delta = e^{i\pi} \delta$ for the degenerate eigenvalues and $\Gamma_a$ for the diagonal eigenvalue matrix of the other eigenchannels. Note that the first subscript of $W$ and the second subscript of $M$ designate the experimental channels; the eigenchannels are associated with the second subscript in $W$ and the first in $M$.

Equation (11) can be rewritten as

$$S = \begin{pmatrix} W_{aa} & W_{ab} & 1_n & 0 & M_{aa} & M_{ab} \\ W_{ba} & W_{bb} & 0 & \Gamma_a & M_{ba} & M_{bb} \end{pmatrix}.$$

Here $m = N - n$ is the number of nondegenerate eigenchannels.

The $S$ matrix thus splits into two parts, a diagonal part and a nondiagonal part. The latter depends only on the nondegenerate eigenvectors. This can be seen more clearly by writing out explicitly the matrices making up (12), for example,

$$(W_{ab}[\Gamma_a - 1_n \Delta] M_{ba} + W_{ba}[\Gamma_a - 1_n \Delta] M_{ab}).$$  \hspace{1cm} (12)

We thus see that a degeneracy leads in a certain sense to a "decoupling" of the channels, which is a manifestation of the freedom associated with the arbitrariness of the choice of the degenerate eigenvectors. Any linear combination of degenerate eigenvectors is itself
an eigenvector. Complete degeneracy leads to a complete decoupling of the channels. However, as long as even one eigenvalue is nondegenerate, a well-defined, in general finite, inelastic-scattering cross section does exist, in general connecting all channels. Since the eigenphases are functions of the energy, two eigenphases may coincide at some energy. If at that energy only very few channels are open then, under favorable conditions, the resulting “decoupling” could be observed, e.g., in the branching ratios.

III. DECOMPOSITION OF THE S MATRIX

In this section we discuss a formal decomposition of the S matrix in terms of “S matrices” defined within one eigenchannel.

In the asymptotic region, the most general wave function is given in terms of the eigenchannel wave functions (6) as

$$\Psi = \sum_a q_a(\alpha) \Psi^{(a)}$$

$$= \sum_a q_a(\alpha) \sum_c [V_c^{(a)} \phi_c + \tilde{V}_c^{(a)} \phi_c].$$

(14)

We have here introduced the abbreviations

$$\tilde{V}_c^{(a)} = -e^{-2i\delta_c} V_c^{(a)}; \quad \phi_c = I \phi_c; \quad \phi_c = 0 \phi_c.$$

(15)

A particular wave function describes the situation in which an incoming wave exists in only one channel, say $c$, and outgoing waves exist in all channels. This is the situation in which the S matrix can be defined in the most immediate manner. We denote the eigenchannel amplitudes describing this situation by $q_c^{(a)}$. Thus we have

$$\sum_a q_a^{(a)} V_c^{(a)} = \delta_{cc}.'$$

(16)

Recalling the definition of the solution matrix $W$ in (7) we find immediately

$$q_c^{(a)} = -W_{ac}.$$  

(17)

Inserting (17) in (14) we obtain indeed

$$\Psi = \phi_c - \sum_c W_{ac} e_a \phi_c \phi_c,$$

i.e., the defining equation of the S matrix.

We now define the “eigenchannel S matrix” by the equation

$$V_c^{(a)} = -\sum_c U_c^{(a)} V_c^{(a)}.$$  

(19)

Inserting (19) into (14) we find

$$S_{ac} = \sum_c U_c^{(a)} W_{ac} M_{ac},$$

which shows that a possible solution of (10) is

$$U_c^{(a)} = e_\delta \delta_c,$$

(21)

which is immediately obvious. However, (19) has an infinite set of solutions which is evident by counting the number of unknowns and the number of equations.

Another solution is obtained when making the ansatz

$$U_c^{(a)} = e_\delta \tau_c^{(a)} \tau_c^{(a)},$$

which when inserted in (19) shows that

$$\tau_c^{(a)} = V_c^{(a)} f_c^{(a)},$$

with the normalization constant

$$1/f_c^{(a)} = \sum_c \epsilon_c (V_c^{(a)})^2,$$

which is not necessarily real. Inserting (22) into (20) we again obtain (10). We see that even though (19) has no unique solution, such a definition still may be of use since one always can construct the S matrix using the $U$ matrices if one has a calculational procedure which gives a particular set of solutions.

IV. COMPUTATION OF THE EIGENCHANNELS AND EIGENPHASES

The amplitudes $V_c^{(a)}$ and the eigenphases $\delta_a$ have to be supplied for each energy of the compound system by the solution of the nuclear Schrödinger equation. This can be done by a method similar to one discussed in detail in an earlier publication. In short, the procedure is as follows. Fixing an energy $E$, the logarithmic derivatives of the reduced radial wave functions are computed for each open channel with an assumed common phase shift $\delta$, i.e.,

$$b_\delta = [r f_c^{(a)}(kr)/f_c(kr)]_{max}.$$

(25)

The matching radius $a$ is chosen as small as possible, consistent with the requirements of channel orthogonality, as discussed earlier. For neutral particles, $f_c$ is given by

$$f_c(x) = (x/k)[\cos j_l(x) - \sin b_l(x)],$$

where $x = kr$. For charged particles the spherical Bessel functions are replaced by Coulomb functions. With these boundary conditions the eigenstates of the nuclear Schrödinger equation are found for the inside region, i.e., for $r \leq a$. We call the eigenvalues $E_\delta(\delta)$. The eigenphases $\delta_a$ then are found from the condition

$$E_\delta(\delta_a) = E.$$

(27)

The roots of this equation can, for example, be found by a graphical method or an iteration procedure. Equation (27) has as many, in general, nondegenerate solutions as there are open channels, viz. $N$. The discussion of the problems which appear in the computation of the eigenchannels, in particular the choice of a complete orthonormal set of wave functions obeying the different boundary conditions $b_{\delta a}$. Eq. (25), in the different channels at a given phase $\delta$, can be taken over completely from Ref. 1.

In practice the solution of (27) is rather simple since it is only necessary to check whether one eigenvalue of the Hamiltonian matrix coincides with the energy $E$. This can be done simply by computing the determinant

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of the matrix $H-E$. If this determinant vanishes $H$ has
an eigenvalue $E$. Only then does one compute the
eigenvector to this eigenvalue, which then requires only
a matrix inversion and not a diagonalization. The main
computational effort thus lies in the determination of
the wave functions and the matrix elements.

This essentially completes the computation of the
eigenchannels. Namely, the nuclear wave function
corresponding to the eigenvalue $a$ can now be written
in the asymptotic region, i.e., $r=a$, in the form (6).
This involves a projection with $\psi_i^*$, at which point the
channel orthogonality requirement enters, viz.
the relation
\[
\int \psi_i^* \psi_{i'} \ dS = \delta_{ii'}
\]  
(28)
must hold with sufficient accuracy. This way the channel
wave function for the channel $c$ is split off and the amplitude $V_{c}(a)$ can be read from Eq. (6) by in-
spection. As a final step, the amplitudes $V_{c}(a)$ must be
normalized.

The generalized natural boundary conditions dis-
cussed in the Introduction now are seen to be given by
(25) when computed with an eigenphase $\phi_a$. Thus, at
each energy there exist $N$, in general, different natural
boundary conditions.

V. RESONATING AND NONRESONATING
PROCESSES

An isolated sharp resonance can be described by a
one-level Breit-Wigner formula. As a matter of fact, the
possibility of a description by such a Breit-Wigner
formula can even be used as the defining condition for
a sharp resonance. In terms of the eigenchannel de-
scription of the $S$ matrix one sees that the simplest,
nondegenerate situation which leads to such a resonance is
that in which the eigenphase in one eigenchannel changes rapidly by about $\pi/2$ over an energy interval
$\Delta E=\Gamma$ while all other parameters remain essentially
constant over this energy region. The contributions of
all other eigenchannels then can be collected into an
essentially constant "background" term. Then an $S$-
matrix element has the form
\[
S_{a}\sim V_{c}(a)^2 \delta_{aa} V_{c}(a) + B_{ee},
\]  
(29)
and, writing for the diverse amplitudes their absolute
values and phases, the cross section is proportional to
\[
\sigma_{ee} \propto |S_{ee}|^2 = |V_{c}(a)|^2 |V_{c}(a)|^2 + |B_{ee}|^2 + 2 |B_{ee}| |V_{c}(a)||V_{c}(a)| \times \cos(\phi_e - \phi_c + \varphi_{ee} - 2\theta_a),
\]  
(30)
which indeed shows a sharp peak for the discussed condi-
tions. Since $\varphi_e - \varphi_c = 0$ which can be seen from
(6), the peak will occur at $\theta_a = \pi/2$ if the phase $\phi_{ee} = \pi$.
For such a situation evidently it should not be necessary
to compute all eigenchannels. We shall return to this
point later.

In the general case no single resonance dominates the
cross section. The same holds for the eigenchannels.
Then all eigenchannels have to be computed and no
simplifications are possible in the description. Naturally,
it still is possible to perform a meromorphic expansion for
the different cross sections $\sigma_{ee}$ as a function of the
energy. However, the usefulness of such an expansion is
questionable for higher level densities. Firstly,
because the number of needed resonance parameters is
then very large, viz., positions and residua of all poles
for each partial cross section. Secondly, a nonresonating
background term, an entire function of the energy, also
has to be accounted for in addition. In any case, the
numerical methods proposed in this paper in practice
would not allow the performing of any "exact" mathe-
atical operations which would be needed to dis-
tinguish a maximum in the entire background function
from a maximum resulting from a pole since in our case
only the cross sections are known as a function of
energy, and only as a table of numbers of limited
accuracy at that. We do not want to claim, however,
that there might not exist cases where such an expansion
could be of interest. For example, some poles may
appear at the same position in all partial cross sections,
while others may be associated only with some channels.
In any case, our aim is limited to the actual computation
of cross sections, at least for the time being.

Finally, one or two words are in order concerning the
connection of the eigenchannel treatment with the ear-
eral formalisms. The nuclear Schrödinger equation
can be solved with arbitrary boundary conditions. We
are here limiting ourselves to standing-wave solutions.
Each solution of the inside problem in general will be a
linear combination of all eigenchannel wave functions.
The expansion in terms of the eigenchannel wave
functions will thus in general involve all eigenchannels
and will converge uniformly except at the matching
boundary. However, for the special case of natural
boundary conditions the expansion will contain only
those eigenchannel wave functions which are associated
with the employed natural boundary conditions, i.e.,
here one solution coincides with, in general, one eigen-
channel wave function. In other words, here the ob-
tained solution does not span the complete function
space, and therefore not all the information concerning
the nuclear system is contained in the wave function.
It thus seems that in the Wigner-Eisenbud procedure
either one has an incomplete solution or one is saddled
with the problem of non-uniform convergence at the
boundary.

However, if the obtained solution dominates the
reaction, as it is the case in a narrow isolated resonence,
than the information contained in the dominant eigen-
channel wave function should suffice for the description
of this resonance, i.e., it should yield position, width,
and branching ratios. That this is actually the case,
one sees by going to a neighboring energy where
the boundary conditions are not the natural boundary conditions any more. There the Wigner-Eisenbud procedure works since the wave function spans the complete function space. The obtained one-level Breit-Wigner function space. The obtained one-level Breit-Wigner solution of the nuclear problem seems to us to have been minimized.

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**I. INTRODUCTION**

The inelastic scattering of α particles is a useful method for studying the surface shapes of medium-mass nuclei. The levels most strongly excited are the 2+, 4+, and 3− collective states. It is therefore for the excitation of such levels that most of the angular distributions of scattered particles have been measured. The shapes of the angular distributions are determined by the angular momentum transfer L, while the absolute value of the differential cross section depends upon the collective strength of the level excited.

The light nuclei present many opportunities for studying inelastic scattering of α particles from targets for which the structure of the initial and final nuclear states is rather well understood. In many cases the transitions should be almost pure single-particle rather than collective. Large numbers of levels are sufficiently well separated in energy to permit resolution of the corresponding groups of scattered particles.

In the present survey experiment, elastic and inelastic...