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Abstract

Stocks are exposed to the risk of sudden downward jumps. Additionally, a crash in one stock (or index) can increase the risk of crashes in other stocks (or indices). Our paper explicitly takes this contagion risk into account and studies its impact on the portfolio decision of a CRRA investor both in complete and in incomplete market settings. We find that the investor significantly adjusts his portfolio when contagion is more likely to occur. Capturing the time dimension of contagion, i.e. the time span between jumps in two stocks or stock indices, is thus of first-order importance when analyzing portfolio decisions. Investors ignoring contagion completely or accounting for contagion while ignoring its time dimension suffer large and economically significant utility losses. These losses are larger in complete than in incomplete markets, and the investor might be better off if he does not trade derivatives. Furthermore, we emphasize that the risk of contagion has a crucial impact on investors' security demands, since it reduces their ability to diversify their portfolios.

Keywords: Asset Allocation, Jumps, Contagion, Model Risk

JEL-Classification: G12, G13

1 Introduction and Motivation

The notion of contagion in financial markets refers to a phenomenon where losses in one asset, one asset class, or one country increase the risk of subsequent losses in other assets, other asset classes, or other countries. Contagion may arise due to firm-specific relations, e.g. dependency on a main customer, due to the exposure to common macroeconomic risk factors, e.g. interest rates, or due to psychological reasons, e.g. bank runs.[1] One example for an event inducing contagion is the recent subprime crisis that has been threatening the financial markets all over the world: When real estate prices in the US started to decrease, homeowners who had borrowed heavily against the equity in their homes were suddenly realizing that they could no longer afford to keep up their mortgage payments. An estimate from December 2007 states that "subprime borrowers will probably default on 220 billion – 450 billion of mortgages".[2] Initially, this threat has had a significant effect on the markets for structured credit contracts like Collateralized Debt Obligations (CDOs) leading to huge losses that the banks have started to report. All along the way, the fear has extended into equity markets:

> "Fears about an end to the leveraged buy-out boom triggered heavy selling of global equities yesterday, leading to the FTSE 100's worst one-day slide for more than four years. [...] The FTSE 100 fell more than 200 points, or 3.2%, to 6.251,2; its biggest drop since March 2003 in the run-up to the Iraq war. [...] By early afternoon in New York, the Dow Jones Industrial Average was down more than 300 points, or 2.4%." (FT, July 27, 2007)

> "'In this sort of climate it is all about sentiment, not about the numbers at all, and sentiment at present is all about fear and nervousness,' said Kevin Gardiner, head of global equity strategy at HSBC." (WSJ, July 27, 2007)

or as catchily summarized:

> "The grievous experience of two centuries of financial busts is that when the banking system is in difficulties the mess spreads." (Economist, Dec 19, 2007)

These examples show how losses in one part of the economy or in one country can spread out into other parts of the economy or other countries.

[1] The relevance of contagion is empirically documented in Bae, Karolyi, and Stulz (2003) and Boyson, Stahel, and Stulz (2007).

[2] See Economist, Dec 19 2007.

Our paper analyzes the optimal portfolio choice of a CRRA investor in a stock market exposed to contagion risk. The stock prices in our economy follow jump-diffusion processes. Large losses in the stocks are captured by downward jumps. Additionally, we take the above-described empirical fact into account that large losses in one asset can increase the risk of subsequent large losses in the same or other assets. Therefore, in contrast to papers that model contagion by an increase in the correlation between the diffusion components, we concentrate on the dependence between these large downward jumps. To capture this dependence, we build in a Markov chain with two states, a calm state and a contagion state. In the calm state, the probability of downward jumps is rather low, while it increases when the economy enters the contagion state. Downward jumps in the calm state can (but need not) trigger a jump of the economy into the contagion state. On the other hand, a jump back into the calm state occurs without a jump in stock prices.

Our approach allows us to capture two stylized facts at the same time: Firstly, contagion is not a "one time event" in the sense that it occurs, leads to immediate losses in several stocks, but has no longer-lasting impact. Usually, the probability for subsequent crashes remains higher for some time. This time dimension of contagion implies that an investor can adjust his portfolio when the threat of contagion becomes apparent. Secondly, contagion is usually triggered by an initial crash in a particular market, i.e. the jump into the contagion state occurs when some asset prices drop. Put differently, our approach allows to correlate the jump processes of two stocks where correlation is induced by jumps themselves. This is not possible if stock dynamics depend on ordinary Poisson or Cox processes. Note that Cox processes are correlated, but the correlation results from diffusion processes that drive the corresponding intensities. Therefore, the probabilities for jumps change only gradually over time. This is in contrast to our approach where the probabilities for jumps in stock prices can jump themselves.

Our paper is related to the literature on continuous-time portfolio choice starting with Merton (1969, 1971). There are two approaches to deal with contagion effects in portfolio problems. One strand of the literature models contagion as joint Poisson jumps. Papers in this area include Das and Uppal (2004) and Kraft and Steffensen (2008), among others. Their approaches however disregard the time dimension of contagion. In particular, the probability of subsequent crashes remains the same after a joint jump has happened. This is because Poisson processes are memoryless (Markov property). Therefore, in this framework, one cannot study the investors' reactions on the advent of contagion. The second strand of the literature are so-called regime-switching models. Papers in this area include Ang and Bekaert (2002) and Guidolin and Timmermann (2007, 2008), among others. Although these models capture the time dimension of contagion, regime shifts are triggered by an exogenous process and do not occur as the result of crashes in certain assets.

Our paper generalizes Kraft and Steffensen (2009) to stock markets and addresses the following points:[3] Firstly, we solve for the optimal stock demands in the calm and in the contagion state both in a complete and in an incomplete market. We show that there is a hedging demand for those jumps that trigger the economy to switch the state. The sign of this hedging demand depends on the investment opportunities in both states and on the risk aversion of the investor relative to the log investor. Furthermore, we compare the optimal portfolios in the calm and in the contagion state. It turns out that the investor revises his portfolio significantly when the economy changes its state. The sizes of these portfolio revisions depend on the differences between the calm and the contagion state, while their signs depend on the market prices of risk.

Secondly, we analyze the utility loss an investor suffers from if he ignores contagion or if he ignores the time dimension of contagion. We show that the utility loss due to model mis-specification can be significant. This is particularly true when the market is completed by derivatives. In this case, an investor with a rather low risk aversion of 1.5 might annually lose more than 20% when he makes his decision based on an incorrect model. If the calm and contagion state differ significantly, then the utility loss is largest if the investor ignores contagion completely. For smaller differences, the utility losses are the largest if he only ignores the time dimension of contagion. Applying the latter model also results in the largest losses if the market is incomplete. These losses are however smaller than in a complete market, where the investor does not only suffer from basing his portfolio decision on an incorrect model, but also from implementing his (seemingly) optimal strategy using an incorrect pricing model for the derivatives. The utility loss from this second mistake can become so large that it more than offsets the utility gain from having access to derivatives. Therefore, the investor might be better off if he does not trade derivatives at all.

The remainder of the paper is structured as follows. In Section 2, we present the model and the portfolio planning problem. The optimal portfolios both in complete and incomplete markets are derived in Section 3. In Section 4, we analyze two benchmark models where the investor either completely ignores contagion or just its time dimension. Section 5 provides some numerical examples, discusses the impact of model mis-specification, and provides some robustness checks. Section 6 concludes. All proofs can be found in the Appendix.

[3]From a theoretical point of view, our paper extensively looks at incomplete markets. From an economical point of view, we analyze the economic value of derivatives. Both aspects are not considered by Kraft and Steffensen (2009).

2 Model Setup

2.1 The Economy

We consider an economy where uncertainty is described by the complete filtered probability space $(\Omega, \mathcal{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T^*]})$ and $\mathcal{F} = \mathcal{F}_{T^*}$. To rule out arbitrage, we assume that an equivalent martingale measure Q exists under which discounted asset prices are (local) martingales.[4] Our economy is characterized by eight states that will be specified below. Let $Z(t)$ denote the state at time $t \in [0, T^*]$ and let Z be a right-continuous process with left limits (RCLL). Then the associated 8-dimensional counting process $N = (N^k)_k$ is an RCLL process, where N^k counts the number of transitions into state k, i.e.

$$N^k(t) = \#\{s | s \in (0, t], Z(s-) \neq k, Z(s) = k\}.$$

Investors can borrow and lend using a money market account with dynamics

$$dM(t) = M(t) r dt, \quad M(0) = 1,$$

where, for simplicity, the interest rate r is assumed to be constant.[5] Besides, there are two stocks A and B with jump-diffusion dynamics ($i \in \{A, B\}$)

$$\frac{dS_i(t)}{S_i(t)} = \mu_i^{Z(t)} dt + \sigma_i^{Z(t)} dW_i(t) - \sum_{k\neq Z(t)} L_i^{Z(t),k} dN^k(t),$$

where W_A and W_B denote correlated Brownian motions. Their correlation is given by ρ^Z, i.e. we allow for a state dependent correlation of diffusive risk. The Brownian motions capture normal stock price movements. Additionally, there can be sudden large losses upon transition from one state into another state of economy. For instance, $L_i^{j,k}$ denotes the loss of stock i if the economy jumps from state j into state k. It is assumed that for fixed i, j, and k the loss sizes are constant, but this assumption can be relaxed.[6]

We interpret the states of the economy as calm and contagion states. In our model, these states mainly differ with respect to the jump intensities. While the jump intensities are low in a calm state, they increase when the economy enters a contagion state. Formally, contagion is modeled using a Markov chain that jumps from state j into state

[4]See Harrison and Kreps (1979) and Delbaen and Schachermayer (1994) for the essential equivalence of the existence of such a measure and the absence of arbitrage.

[5]Our analysis can easily be generalized to stochastic interest rates along the lines of Korn and Kraft (2001) and Munk and Soerensen (2004), among others.

[6]Note that in our notation $L_i^{j,k} > 0$ corresponds to a loss.

k with intensity $\lambda^{j,k}$, $j \neq k$. As mentioned above, we use a Markov chain with eight states $\{cont_{A1}, cont_{A2}, cont_{B1}, cont_{B2}, calm_{A1}, calm_{A2}, calm_{B1}, calm_{B2}\}$ that is illustrated in Figure 1. The first subscript of the state indicates the stock that has exhibited the most recent downward jump. The second subscript comes from the fact that we also wish to model stock price jumps not leading to regime shifts. For instance, if stock A jumps without leaving the calm state, then the Markov chain jumps from state $calm_{A1}$ to $calms_{A2}$, or vice versa.

The intensity of a jump in stock i that does not trigger contagion is $\lambda_i^{calm,calm}$, and the corresponding loss in stock i is $L_i^{calm,calm}$ (the loss in the other stock is zero). The intensity of a jump in stock i that does trigger contagion is $\lambda_i^{calm,cont}$ and the loss of stock i for such a jump is $L_i^{calm,cont}$. If the economy is in a contagion state, the intensity for a loss in stock i is $\lambda_i^{cont,cont}$, and the corresponding loss size is $L_i^{cont,cont}$. After spending some time in the contagion state, the economy will eventually jump back into the calm state. The intensity for this to happen is $\lambda^{cont,calm}$, and it is assumed that this event does not induce any losses in stocks, i.e. $L_i^{cont,calm} \equiv 0$, $i \in \{A, B\}$. The intensities for all other jumps are equal to zero.

To summarize, the Markov chain has four contagion states and four calm states. We assume that the model parameters coincide in all calm states and in all contagion states. This implies that all calm states and all contagion states are identical in the sense that optimal portfolios and indirect utilities are the same. As explained above, the use of four contagion and four calm states is for technical reasons only.

Finally, we specify the drift and the risk premia of the stocks. The drift of stock i is equal to

$$\mu_i^{Z(t)} = r + \phi_i^{Z(t)} + \sum_{k \neq Z(t)} L_i^{Z(t),k} \lambda^{Z(t),k}$$

where the last term is the compensator of the jump processes. The risk premium of the stock is thus given by

$$\phi_i^{Z(t)} = \sigma_i^{Z(t)} \eta_i^{Z(t)} + \sum_{k \neq Z(t)} L_i^{Z(t),k} \lambda^{Z(t),k} \eta^{Z(t),k}$$

where η_i^j is the premium for diffusive risk W_i when the economy is in state j, and $\eta^{j,k}$ is the premium for jumps from j into k. The intensity for a jump from j into k under the risk neutral measure is thus $(1 + \eta^{j,k})$ times the intensity under the physical measure.

With our definition of the Markov chain, the risk premium only depends on whether the economy is in one of the calm or in one of the contagion states. Consequently, the risk premia of stock i can be rewritten as

$$\begin{aligned}\phi_i^{calm} &= \sigma_i^{calm}\eta_i^{calm} + L_i^{calm,calm}\lambda_i^{calm,calm}\eta_i^{calm,calm} + L_i^{calm,cont}\lambda_i^{calm,cont}\eta_i^{calm,cont} \\ \phi_i^{cont} &= \sigma_i^{cont}\eta_i^{cont} + L_i^{cont,cont}\lambda_i^{cont,cont}\eta_i^{cont,cont}.\end{aligned}$$

Apart from stocks and the money market account, the investor might also have access to derivatives. We assume that there are either no derivatives at all, or enough derivatives to complete the market. The exposure of the derivatives to the risk factors can be calculated using Ito's lemma.

2.2 The Investor

We consider an investor with CRRA-utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$ denotes his relative risk aversion. The investor's planning horizon is denoted by $T < T^*$, and it is assumed that he maximizes expected utility from terminal wealth X_T only. Therefore, his time-t indirect utility in state j is defined as

$$G^j(t, X_t) = \max_{\pi \in \mathcal{A}^j(t, X_t)} \left\{ E\left[u(X_T) | Z(t) = j\right] \right\},$$

where $\mathcal{A}^j(t, X_t)$ denotes the set of all trading strategies π for a current wealth level of X_t that are admissible at time t in state j.

3 Asset Allocation

3.1 Complete Market

In a complete market, the investor can separate his decision upon the optimal exposures to the risk factors from finding the strategy that implements these exposures. Generalizing an idea of Liu and Pan (2003) to our Markov chain framework, the investor's budget restriction reads

$$\begin{aligned}\frac{dX(t)}{X(t)} &= r dt + \theta_A^{Z(t)}(t)\left[dW_A(t) + \eta_A^{Z(t)} dt\right] + \theta_B^{Z(t)}(t)\left[dW_B(t) + \eta_B^{Z(t)} dt\right] \\ &\quad + \sum_{k \neq Z(t), \lambda^{Z(t),k} \neq 0} \theta^{Z(t),k}(t)\left[dN^k(t) - \lambda^{Z(t),k} dt - \eta^{Z(t),k}\lambda^{Z(t),k} dt\right],\end{aligned} \tag{1}$$

where θ_i^j denotes the investor's state-j exposure to diffusive risk W_i and $\theta^{j,k}$ is his exposure to a jump from state j into state k. In a calm state, we have to choose the four exposures to jumps in stock A and stock B that (do not) induce contagion, and we denote these exposures by $\theta_i^{calm,cont}$ ($\theta_i^{calm,calm}$). In the contagion state, we have to choose the three exposures to jumps in stock A, jumps in stock B, and jumps back from the contagion into the calm state. These exposures are denoted by $\theta_i^{cont,cont}$ and $\theta^{cont,calm}$. The portfolio planning problem of the investor is given by

$$G^j(t, X_t) = \max_{\{\theta_A^j(s),\theta_B^j(s),\theta^{j,k}(s),t\leq s<T\}} E\left[u(X_T)|Z(t) = j\right]$$

subject to the budget restriction (1).

The following proposition shows how the optimal exposures to diffusion risk, $\theta^i_{A/B}$, and to jump risk, $\theta^{i,j}_{A/B}$, are linked to the model parameters.

Proposition 3.1 (Contagion, Complete Market) *In an economy with contagion, the optimal exposures to the risk factors are*

$$\begin{aligned}
\theta_A^j &= \frac{\eta_A^j - \rho^j\eta_B^j}{\gamma(1-(\rho^j)^2)} & \theta_B^j &= \frac{\eta_B^j - \rho^j\eta_A^j}{\gamma(1-(\rho^j)^2)} \\
\theta_A^{calm,calm} &= (1+\eta_A^{calm,calm})^{-\frac{1}{\gamma}} - 1 & \theta_B^{calm,calm} &= (1+\eta_B^{calm,calm})^{-\frac{1}{\gamma}} - 1 \\
\theta_A^{calm,cont} &= (1+\eta_A^{calm,cont})^{-\frac{1}{\gamma}} \frac{f^{cont}}{f^{calm}} - 1 & \theta_B^{calm,cont} &= (1+\eta_B^{calm,cont})^{-\frac{1}{\gamma}} \frac{f^{cont}}{f^{calm}} - 1 \\
\theta_A^{cont,cont} &= (1+\eta_A^{cont,cont})^{-\frac{1}{\gamma}} - 1 & \theta_B^{cont,cont} &= (1+\eta_B^{cont,cont})^{-\frac{1}{\gamma}} - 1 \\
\theta^{cont,calm} &= (1+\eta^{cont,calm})^{-\frac{1}{\gamma}} \frac{f^{calm}}{f^{cont}} - 1.
\end{aligned}$$

The indirect utility function of the investor is

$$G^j(t,x) \quad = \quad \frac{x^{1-\gamma}}{1-\gamma}\left(f^j(t)\right)^\gamma \tag{2}$$

where

$$\begin{pmatrix} f^{calm}(t) \\ f^{cont}(t) \end{pmatrix} = \exp\left\{\begin{pmatrix} C^{calm,calm} & C^{calm,cont} \\ C^{cont,calm} & C^{cont,cont} \end{pmatrix}(T-t)\right\}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with

$$\begin{aligned}
C^{calm,calm} &= \frac{1-\gamma}{\gamma}\Bigg[r + \frac{(\eta_A^{calm})^2 + (\eta_B^{calm})^2 - 2\rho^{calm}\eta_A^{calm}\eta_B^{calm}}{2\gamma(1-(\rho^{calm})^2)} \\
&\qquad + \left(1+\eta_A^{calm,calm} - \frac{1}{1-\gamma}\right)\lambda_A^{calm,calm} + \left(1+\eta_B^{calm,calm} - \frac{1}{1-\gamma}\right)\lambda_B^{calm,calm} \\
&\qquad + \left(1+\eta_A^{calm,cont} - \frac{1}{1-\gamma}\right)\lambda_A^{calm,cont} + \left(1+\eta_B^{calm,cont} - \frac{1}{1-\gamma}\right)\lambda_B^{calm,cont}\Bigg] \\
&\quad + \left(1+\eta_A^{calm,calm}\right)^{1-\frac{1}{\gamma}}\lambda_A^{calm,calm} + \left(1+\eta_B^{calm,calm}\right)^{1-\frac{1}{\gamma}}\lambda_B^{calm,calm} \\
C^{calm,cont} &= \left(1+\eta_A^{calm,cont}\right)^{1-\frac{1}{\gamma}}\lambda_A^{calm,cont} + \left(1+\eta_B^{calm,cont}\right)^{1-\frac{1}{\gamma}}\lambda_B^{calm,cont} \\
C^{cont,cont} &= \frac{1-\gamma}{\gamma}\Bigg[r + \frac{(\eta_A^{cont})^2 + (\eta_B^{cont})^2 - 2\rho^{cont}\eta_A^{cont}\eta_B^{cont}}{2\gamma(1-(\rho^{cont})^2)} \\
&\qquad + \left(1+\eta_A^{cont,cont} - \frac{1}{1-\gamma}\right)\lambda_A^{cont,cont} + \left(1+\eta_B^{cont,cont} - \frac{1}{1-\gamma}\right)\lambda_B^{cont,cont} \\
&\qquad + \left(1+\eta^{cont,calm} - \frac{1}{1-\gamma}\right)\lambda^{cont,calm}\Bigg] \\
&\quad + \left(1+\eta_A^{cont,cont}\right)^{1-\frac{1}{\gamma}}\lambda_A^{cont,cont} + \left(1+\eta_B^{cont,cont}\right)^{1-\frac{1}{\gamma}}\lambda_B^{cont,cont} \\
C^{cont,calm} &= \left(1+\eta^{cont,calm}\right)^{1-\frac{1}{\gamma}}\lambda^{cont,calm}.
\end{aligned}$$

The proof is given in Appendix A.1.

Following Merton (1971), the optimal exposures can be decomposed into a speculative demand and a hedging demand. The demand for diffusive risk is purely speculative, since diffusive risk does not have any impact on the investment opportunity set. It depends on the risk premia (and the correlations) only. The optimal exposure to jump risk is more involved. The speculative demand for a jump from state *old* to state *new* (where the two states might coincide) is given by

$$(1+\eta^{old,new})^{-\frac{1}{\gamma}} - 1.$$

If the market price of jump risk $\eta^{old,new}$ is positive, jumps are more likely under the risk-neutral measure than under the true measure, and the optimal exposure to this kind of jumps is negative. In line with intuition, it increases in absolute terms in the risk premium, and it decreases in absolute terms in risk aversion. The second part of the demand for jump risk is the hedging demand, which is given by

$$(1+\eta^{old,new})^{-\frac{1}{\gamma}}\left(\frac{f^{new}}{f^{old}} - 1\right).$$

It differs from zero only if the old and the new state are not equal, i.e. if the economy changes from calm to contagion or vice versa. In this case, the investor takes changes in the investment opportunity set into account, where his reaction to these changes depends on whether he is more or less risk-averse than the log-investor, as explained in Kim and Omberg (1996), Liu and Pan (2003) or Liu, Longstaff, and Pan (2003), among others. For $f^{new} > f^{old}$, the induced hedging demand is positive. If $\gamma > 1$, $f^{new} > f^{old}$ implies that investment opportunities are worse in the new state than in the old state (see Equation (2)). The investor is more risk-averse than the log investor, he cares about hedging, and he wants to have more wealth in those states of the world where investment opportunities are bad. This results in a positive hedging demand. If $\gamma < 1$, $f^{new} > f^{old}$ implies that investment opportunities are better in the new state than in the old state. The investor is less risk-averse than the log investor and he speculates on changes in the investment opportunity set. He thus wants to have more wealth in the good new state, and the induced 'hedging demand' is positive.

To assess how good the investment opportunities in state j are, we rely on the certainty equivalent return (CER). It is defined by

$$G^j(t,x) = \frac{\left(xe^{CER^j(t,x)(T-t)}\right)^{1-\gamma}}{1-\gamma}.$$

The CER gives the deterministic return on wealth that would result in the same indirect utility as the optimal investment in the risky assets.

When the economy changes from the calm state to the contagion state (or vice versa), the indirect utility of the investor changes due to two reasons. First, his wealth changes where the loss or gain depends on his exposure towards the jump. Second, the investment opportunity set and thus the CER changes. Consider, e.g., the case where the optimal exposure to a jump from the calm into the contagion state is negative. If the investment opportunities are worse in the contagion state, then the investor will be worse off after the jump has occurred. If, on the other hand, the investment opportunities are better in the contagion state, then the overall impact on the indirect utility depends on the trade-off between the lower wealth and the higher CER.

3.2 Incomplete Market

If the investor can only trade in the two stocks and in the money market account, the market is incomplete. The budget restriction then becomes

$$\frac{dX(t)}{X(t)} = \pi_A^{Z(t)}(t)\frac{dS_A(t)}{S_A(t)} + \pi_B^{Z(t)}(t)\frac{dS_B(t)}{S_B(t)} + \left(1 - \pi_A^{Z(t)}(t) - \pi_B^{Z(t)}(t)\right) rdt,$$

where $\pi_i^j(t)$ is the proportion of wealth invested in stock i ($i = A, B$) at time t and in state j. The optimal portfolio strategy is given in the following proposition.

Proposition 3.2 (Contagion, Incomplete Market) *In an economy with contagion where only the two stocks and the money market account are traded, the investor's indirect utility in state $j \in \{calm, cont\}$ is*

$$G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma} f^j(t)$$

where f^j solves the ordinary differential equations

$$\begin{aligned}
0 \;=\;& f_t^{calm} + (1-\gamma)\left[r + \pi_A^{calm}(\mu_A^{calm} - r) + \pi_B^{calm}(\mu_B^{calm} - r)\right] f^{calm} && (3)\\
& - 0.5\gamma(1-\gamma)\left[(\pi_A^{calm}\sigma_A^{calm})^2 + (\pi_B^{calm}\sigma_B^{calm})^2 + 2\pi_A^{calm}\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right] f^{calm} \\
& + \lambda_A^{calm,cont}\left[(1-\pi_A^{calm}L_A)^{1-\gamma} f^{cont} - f^{calm}\right] + \lambda_A^{calm,calm}\left[(1-\pi_A^{calm}L_A)^{1-\gamma} - 1\right] f^{calm} \\
& + \lambda_B^{calm,cont}\left[(1-\pi_B^{calm}L_B)^{1-\gamma} f^{cont} - f^{calm}\right] + \lambda_B^{calm,calm}\left[(1-\pi_B^{calm}L_B)^{1-\gamma} - 1\right] f^{calm} \\
0 \;=\;& f_t^{cont} + (1-\gamma)\left[r + \pi_A^{cont}(\mu_A^{cont} - r) + \pi_B^{cont}(\mu_B^{cont} - r)\right] f^{cont} && (4)\\
& - 0.5\gamma(1-\gamma)\left[(\pi_A^{cont}\sigma_A^{cont})^2 + (\pi_B^{cont}\sigma_B^{cont})^2 + 2\pi_A^{cont}\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right] f^{cont} \\
& + \lambda_A^{cont,cont}\left[(1-\pi_A^{cont}L_A)^{1-\gamma} - 1\right] f^{cont} + \lambda_B^{cont,cont}\left[(1-\pi_B^{cont}L_B)^{1-\gamma} - 1\right] f^{cont} \\
& + \lambda^{cont,calm}(f^{calm} - f^{cont}).
\end{aligned}$$

and where the optimal portfolio weights solve

$$\begin{aligned}
\mu_A^{calm} - r - \gamma(\sigma_A^{calm})^2\pi_A^{calm} - \gamma\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm} \qquad & \\
-L_A\lambda_A^{calm,cont}(1-\pi_A^{calm}L_A)^{-\gamma}\frac{f^{cont}}{f^{calm}} - L_A\lambda_A^{calm,calm}(1-\pi_A^{calm}L_A)^{-\gamma} &= 0 \quad (5)\\
\mu_B^{calm} - r - \gamma(\sigma_B^{calm})^2\pi_B^{calm} - \gamma\pi_A^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm} \qquad & \\
-L_B\lambda_B^{calm,cont}(1-\pi_B^{calm}L_B)^{-\gamma}\frac{f^{cont}}{f^{calm}} - L_B\lambda_B^{calm,calm}(1-\pi_B^{calm}L_B)^{-\gamma} &= 0 \quad (6)\\
\mu_A^{cont} - r - \gamma(\sigma_A^{cont})^2\pi_A^{cont} - \gamma\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont} - L_A\lambda_A^{cont,cont}(1-\pi_A^{cont}L_A)^{-\gamma} &= 0 \quad (7)\\
\mu_B^{cont} - r - \gamma(\sigma_B^{cont})^2\pi_B^{cont} - \gamma\pi_A^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont} - L_B\lambda_B^{cont,cont}(1-\pi_B^{cont}L_B)^{-\gamma} &= 0. \quad (8)
\end{aligned}$$

The proof is given in Appendix A.2.

Equations (3), (4), (5), and (6) form a system of so-called differential-algebraic equations which can only be solved numerically.

As compared to the complete market, the investor can in general no longer achieve the optimal exposures, since he is restricted to the package of exposures offered by the two stocks, as e.g. pointed out in Liu and Pan (2003). As we will show in some numerical examples in Section 5, his exposure to some risk factors will thus be too high, while the exposure to some other risk factors will be too low compared to the complete market case. The exposure to jumps from the contagion to the calm state plays a special role. Since the exposure of both stocks to this jump is assumed to be zero, the investor has no exposure to this jump at all, and he cannot even approximately implement his hedging demand.

4 Simpler Models: Benchmark Cases

We consider two benchmark cases. In the first case ('no contagion'), the investor ignores contagion completely. The stocks jump independently of each other, and the jump intensities are constant over time. In the second case ('joint jumps'), studied e.g. by Das and Uppal (2004), the investor takes contagion into account by assuming that stock price jumps can only happen simultaneously.

Our model is in between these extreme cases in two respects. First, we assume that some jumps do not trigger contagion, while other jumps induce contagion. Second, we allow for a time dimension of contagion. If the economy enters into the contagion state, then the investor can adjust his portfolio and take a smaller (or larger) position in the risky assets. In the benchmark model with joint jumps, the jumps happen simultaneously, and the investor cannot react to the event of contagion any more.

4.1 No Contagion: Independent Downward Jumps

In the first benchmark case, there is no contagion at all, and downward jumps in the stocks happen independently of each other. The dynamics of stock i are

$$\frac{dS_i(t)}{S_i(t-)} = \Big[\underbrace{r + \phi_i + L_i\lambda_i}_{\mu_i}\Big]dt + \sigma_i dW_i(t) - L_i dN_i(t).$$

The Wiener processes W_A and W_B are correlated with correlation ρ. N_i is a Poisson process with intensity λ_i. The risk premium on the stock is

$$\phi_i = \sigma_i \eta_i^{diff} + L_i \lambda_i \eta_i^{jump}$$

where η_i^{diff} is the premium for diffusion risk and η_i^{jump} is the premium for jumps. In a complete market, the investor can again choose the exposures to the risk factors. The budget restriction becomes

$$\begin{aligned}\frac{dX(t)}{X(t)} &= rdt + \theta_A^{diff}(t)\left[dW_A(t) + \eta_A^{diff}dt\right] + \theta_B^{diff}(t)\left[dW_B(t) + \eta_B^{diff}dt\right] \\ &\quad + \theta_A^{jump}(t)\left[dN_A(t) - \lambda_A dt - \eta_A^{jump}\lambda_A dt\right] \\ &\quad + \theta_B^{jump}(t)\left[dN_B(t) - \lambda_B dt - \eta_B^{jump}\lambda_B dt\right]\end{aligned}$$

where θ_i^{diff} is the exposure to diffusive risk W_i, and θ_i^{jump} is the exposure to jumps in stock i. The optimal portfolio exposures are given in the following proposition.

Proposition 4.1 (No Contagion, Complete Market) *If there are no contagion effects in the market, the optimal exposures to the risk factors are*

$$\theta_A^{diff} = \frac{\eta_A^{diff} - \rho\eta_B^{diff}}{\gamma(1-\rho^2)} \qquad \theta_B^{diff} = \frac{\eta_B^{diff} - \rho\eta_A^{diff}}{\gamma(1-\rho^2)}$$
$$\theta_A^{jump} = (1+\eta_A^{jump})^{-\frac{1}{\gamma}} - 1 \qquad \theta_B^{jump} = (1+\eta_B^{jump})^{-\frac{1}{\gamma}} - 1.$$

The indirect utility function of the investor is

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\exp\left\{\gamma\, C^{nc,c}\cdot(T-t)\right\},$$

where

$$\begin{aligned}C^{nc,c} &= \frac{1-\gamma}{\gamma}\Bigg[r + \frac{(\eta_A^{diff})^2 + (\eta_B^{diff})^2 - 2\rho\eta_A^{diff}\eta_B^{diff}}{2\gamma(1-\rho^2)} \\ &\qquad + \left(1+\eta_A^{jump}\right)\lambda_A + \left(1+\eta_B^{jump}\right)\lambda_B - \frac{1}{1-\gamma}(\lambda_A+\lambda_B)\Bigg] \\ &\quad + \left(1+\eta_A^{jump}\right)^{1-\frac{1}{\gamma}}\lambda_A + \left(1+\eta_B^{jump}\right)^{1-\frac{1}{\gamma}}\lambda_B.\end{aligned}$$

The proof is given in Appendix B.1.

The investment opportunity set is constant. There is thus speculative demand only. Both for diffusion risk and for jump risk, this speculative demand has the same structure as in the contagion model discussed in Section 3 and is driven by the risk premia (and the diffusion correlation) only.

The certainty equivalent return is given by $\frac{\gamma}{1-\gamma}\, C^{nc,c}$. It captures how good the investment opportunities are. In a complete market, it does not depend on asset specific parameters like stock price volatilities and loss sizes, but only on economy-wide variables like the risk premia and the jump intensities. Obviously, the certainty equivalent return is increasing in the risk premia. Furthermore, it is increasing in the jump intensities λ_A and λ_B, which is formally shown in Appendix B.2. To get the intuition, notice that the risk premium the investor earns on his optimal portfolio is increasing in the optimal exposure to jumps (i.e. the loss in case of a jump), the market prices of jump risk, and the jump intensities (i.e. the overall amount of jump risk in the market). The CER is thus increasing in these three variables, too.

In the incomplete market, the investor chooses the optimal weights of the two stocks, which are given in the next proposition.

Proposition 4.2 (No Contagion, Incomplete Market) *If there are no contagion effects in the market and only the money market account and the two stocks are traded, then the indirect utility of the investor is given by*

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\exp(C^{nc,ic}\cdot(T-t)),$$

where

$$\begin{aligned} C^{nc,ic} &= (1-\gamma)\left[r + \pi_A(\mu_A - r) + \pi_B(\mu_B - r) - \frac{\gamma}{2}(\pi_A^2\sigma_A^2 + \pi_B^2\sigma_B^2 + 2\pi_A\pi_B\sigma_A\sigma_B\rho)\right] \\ &\quad +\lambda_A\left[(1-\pi_A L_A)^{1-\gamma} - 1\right] + \lambda_B\left[(1-\pi_B L_B)^{1-\gamma} - 1\right]. \end{aligned}$$

The optimal portfolio weights are given as the unique solution of

$$\begin{aligned} \mu_A - r - \gamma\sigma_A^2\pi_A - \gamma\pi_B\sigma_A\sigma_B\rho - L_A\lambda_A(1-\pi_A L_A)^{-\gamma} &= 0 \\ \mu_B - r - \gamma\sigma_B^2\pi_B - \gamma\pi_A\sigma_B\sigma_A\rho - L_B\lambda_B(1-\pi_B L_B)^{-\gamma} &= 0. \end{aligned}$$

The proof is given in Appendix B.3.

4.2 Joint Downward Jumps

In the second benchmark case, the investor takes contagion into account by assuming that stock price jumps happen simultaneously. The dynamics for stock i are

$$\frac{dS_i(t)}{S_i(t-)} = \Big[\underbrace{r + \phi_i + L_i\lambda_{joint}}_{\mu_i}\Big]dt + \sigma_i dW_i(t) - L_i dN_{joint}(t).$$

The risk premium on the stock is

$$\phi_i = \sigma_i\eta_i^{diff} + L_i\lambda_{joint}\eta_{joint}^{jump}.$$

We want the behavior of the individual stocks to be the same in both benchmark cases, so that only the joint behavior differs. Consequently, we assume that the parameters for the individual stocks are the same as in Section 4.1, and we set $\lambda_{joint} = \lambda_A = \lambda_B$ and $\eta^{jump}_{joint} = \eta^{jump}_A = \eta^{jump}_B$.

In the complete market, the solution to the portfolio planning problem is given in the next proposition.

Proposition 4.3 (Joint Downward Jumps, Complete Market) *If there are joint downward jumps, the optimal exposures to the risk factors are*

$$\theta_A^{diff} = \frac{\eta_A^{diff} - \rho\eta_B^{diff}}{\gamma(1-\rho^2)}, \qquad \theta_B^{diff} = \frac{\eta_B^{diff} - \rho\eta_A^{diff}}{\gamma(1-\rho^2)},$$
$$\theta_{joint}^{jump} = (1+\eta_{joint}^{jump})^{-\frac{1}{\gamma}} - 1.$$

The indirect utility function of the investor is

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma} \exp\left\{\gamma\, C^{jj,c} \cdot (T-t)\right\},$$

where

$$C^{jj,c} = \frac{1-\gamma}{\gamma}\Bigg[r + \frac{(\eta_A^{diff})^2 + (\eta_B^{diff})^2 - 2\rho\eta_A^{diff}\eta_B^{diff}}{2\gamma(1-\rho^2)} + \left(1+\eta_{joint}^{jump}\right)\lambda_{joint} - \frac{1}{1-\gamma}\lambda_{joint}\Bigg] + \left(1+\eta_{joint}^{jump}\right)^{1-\frac{1}{\gamma}}\lambda_{joint}.$$

The optimal exposures depend on the market prices of risk (and on the correlation) only. With identical parameters for the behavior of the individual stocks, they are thus the same as in the case of independent jumps. If a jump happens, the investor loses exactly the same amount of money, no matter whether he assumes independent jumps or joint jumps. What differs, however, is the optimal portfolio held by the investor. If there are joint jumps, the portfolio that is optimal with independent jumps would have a jump risk exposure that is twice as high as optimal. With joint jumps, the investor is thus more conservative.

The CER is lower with joint jumps than with independent jumps. To get the intuition, note that the market prices of risk are identical, while the average number of jumps is twice as large in the case of independent jumps as in the case of joint jumps. Since the CER increases in the jump intensity and thus in the average number of jumps, it is indeed smaller with joint jumps.

In the incomplete market, the investor is again restricted to the package of exposures offered by the stocks. The optimal portfolio is given in the next proposition.

Proposition 4.4 (Joint Downward Jumps, Incomplete Market) *If there are joint downward jumps and only the money market account and the two stocks are traded, then the indirect utility of the investor is given by*

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma} \exp\{C^{jj,ic} \cdot (T-t)\},$$

where

$$\begin{aligned} C^{jj,ic} &= (1-\gamma)\left[r + \pi_A(\mu_A - r) + \pi_B(\mu_B - r) - \frac{\gamma}{2}\left(\pi_A^2\sigma_A^2 + \pi_B^2\sigma_B^2 + 2\pi_A\pi_B\sigma_A\sigma_B\rho\right)\right] \\ &\quad + \lambda_{joint}\left[(1 - \pi_A L_A - \pi_B L_B)^{1-\gamma} - 1\right]. \end{aligned}$$

The optimal portfolio weights are given as the unique solutions of

$$\begin{aligned} \mu_A - r - \gamma\sigma_A^2\pi_A - \gamma\pi_B\sigma_A\sigma_B\rho - L_A\lambda_{joint}(1 - \pi_A L_A - \pi_B L_B)^{-\gamma} &= 0 \\ \mu_B - r - \gamma\sigma_B^2\pi_B - \gamma\pi_A\sigma_B\sigma_A\rho - L_B\lambda_{joint}(1 - \pi_A L_A - \pi_B L_B)^{-\gamma} &= 0. \end{aligned}$$

Just as in the model setup without contagion, the investment opportunity set is constant and the investor does not have a hedging demand in the incomplete market, either.

5 Numerical Results

5.1 Parameter Choice and Model Calibration

We consider a CRRA-investor with a relative risk aversion of $\gamma = 3$ and a planning horizon of 20 years. The interest rate is set to $r = 0.01$. The two stocks are assumed to follow identical processes. We rely on the parameter estimates of Eraker, Johannes, and Polson (2003) and Broadie, Chernov, and Johannes (2007). Since we want to focus on the impact of contagion, which is reflected in the difference between the jump intensities in the calm and in the contagion state, all other parameters are assumed to be equal in both states.[7]

The diffusion volatility σ is set to 0.15, and the Wiener processes driving the stock price dynamics are correlated with $\rho = 0.5$. The jump intensity in the benchmark models is set to 1.5, and we calibrate the jump intensities in our contagion model such that the average long-run jump intensity is equal to 1.5, too. More details on this step of the calibration will be given below. The loss in case of a jump in one of the stocks is assumed to be

[7]We also assume a constant riskless rate here although recent experience of the US subprime crisis suggests something different. To keep the numerical results clear and simple, however, we mainly focus on the impact of jump risk and do not consider market liquidity or related issues.

constant and set equal to 0.05, which is slightly higher than the estimate provided in models that also include stochastic volatility. Remember that the loss for a jump back from the contagion into the calm state equals zero.

The market price of diffusion risk is assumed to equal 0.35 in both states. Jumps from the contagion state back into the calm state are not priced. For the other market prices of jump risk, we consider two extreme cases. In the first case (parametrization 1), we assume that they are identical in all states. This implies a rather high stock price drift in the contagion state. In the second case (parametrization 2), we assume that the expected excess stock returns are equal in both states, which results in larger market prices of risk in the calm state and lower ones in the contagion state. We calibrate the market prices of jump risk such that the average expected excess return of the stocks is equal to 8.25% for both parametrizations, which is in line with Broadie, Chernov, and Johannes (2007).

The two benchmark models without contagion and with joint jumps are calibrated such that the stock price behavior in the benchmark models is as similar as possible to the behavior in our model. Therefore, we set the local moments in the benchmark models equal to the long run averages of the local moments in our model. Details of the calibration can be found in appendix C.

The different jump intensities in our model are chosen such that the average number of jumps per year, which follows from Equation (15), is equal to the benchmark value of 1.5. Since we want to focus on contagion, we explicitly control for its severeness and thus for the wedge driven between the two states. The difference between the jump intensities in the calm and contagion state is captured by $\xi \geq 1$:

$$\lambda_i^{cont,cont} = \xi_i \left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right) \qquad i \in \{A, B\}.$$

The conditional probability that a loss in a stock actually triggers contagion is given by the parameter α:

$$\lambda_i^{calm,cont} = \alpha_i \left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right) \qquad i \in \{A, B\},$$

and the average time the economy stays in the contagion state depends on ψ:

$$\lambda^{cont,calm} = \psi \left(\lambda_A^{cont,cont} + \lambda_B^{cont,cont}\right).$$

Given ξ, α, and ψ and the average jump intensity of 1.5, all other jump intensities can be calculated. In the base case calibration, we set $\xi = 4$, $\alpha = 0.5$ and $\psi = 0.25$. The resulting parameters are given in Table 1. Table 2 shows the resulting conditional equity risk premia and variances of stock returns for both parameterizations and in the benchmark models as well as their decomposition into diffusion and jump components. Table 3 gives some other combinations of parameters used in robustness checks, where we choose $\xi \in [1, 10]$, $\alpha \in [0.2, 0.5]$ and $\psi \in [0.2, 2/3]$.

5.2 Optimal Exposures and Optimal Portfolios

Table 4 gives the solution to the portfolio planning problem for the base-case parameters from Table 1 both for the complete and the incomplete market. We discuss the case of complete markets first, where the investor can achieve any desired payoff profile.

The demand for diffusion risk is driven by the speculative component only. It is identical in the calm and in the contagion state and for both parametrizations, because the market prices of diffusion risk are identical by assumption.

The demand for jump risk can be decomposed into a speculative component and – for those jumps that change the state – a hedging component. The speculative demand is an increasing function of the market prices of jump risk $\eta^{i,j}_{A/B}$. If the market prices of risk are identical in all states (parametrization 1), the speculative demand does not depend on the state and coincides with the speculative demand in the two benchmark models. If equity risk premia are constant (parametrization 2), on the other hand, the market price of risk is lower in the contagion state than in the calm state, and consequently, the speculative demand is lower in absolute terms in the contagion state, too. Since jumps from the contagion state back to the calm state are not priced by assumption, this speculative demand is zero.

The sign of the hedging demand depends on which of the two states is the better one. The right panel of Figure 2 shows the certainty equivalent returns in both states. If the market prices of risk are constant (parametrization 1), the investment opportunity set is better in the contagion state where jumps happen more often than in the calm state. Given that $\gamma > 1$, the hedging demand for jumps from the calm to the (better) contagion state is negative, which implies that the investor takes a more aggressive position in jump risk in the calm state. In the contagion state, on the other hand, his optimal exposure to jumps back to the (worse) calm state is positive. If the expected returns are equal (parametrization 2), the calm state is better than the contagion state which switches the sign of the hedging demands.

The optimal exposures are different in the calm and in the contagion state, and the investor will adjust his portfolio when the state of the economy changes. He thus profits from the time dimension of contagion captured in our model. His trading desire due to contagion is much more pronounced for the case of equal equity risk premia (parametrization 2), where trading is induced by changes in the market prices of risk and in the hedging demand, than for the case of identical market prices of risk, where trading is induced by changes in the hedging demand only.

If the market is incomplete, the investor cannot implement the overall optimal exposures. As can be seen in Table 4, the realized exposures will be somewhere in between the optimal exposures from the complete case. The position in risky assets is larger in the state in which investment opportunities are better, that is in the calm state in case of equal equity risk premia and in the contagion state in case of equal market prices of risk.

In the benchmark models, the investor does not distinguish between calm and contagion states. If he ignores contagion completely, the optimal position in stocks is somewhere in between the optimal positions in the calm and in the contagion state. If the investor assumes that there are joint jumps, he is more conservative and reduces his optimal position in stocks significantly.

The certainty equivalent returns in our model and in the two benchmark models are shown in the left panel of Figure 2. As expected, the utility loss due to market incompleteness is largest in our contagion model since the investor fails to implement the optimal myopic demand as well as the intertemporal hedging demand, whereas a hedging demand does not exist in both benchmark models. In absolute numbers, the joint jumps model gives the lowest utility both in an incomplete and in a complete market, since the average number of jumps is cut in half compared to the other models.

Robustness checks show that the results do not change qualitatively when we vary ξ, α and ψ, i.e. the overall size of contagion, the risk of entering the contagion state, and the (reciprocal of the) average duration of the contagion state. In line with intuition, a larger difference between the calm and contagion state, i.e. a larger value of ξ, leads to larger trading incentives due to changes of the state and to larger utility losses due to market incompleteness. The probability α of entering the contagion state does not have much impact on the results. On the other hand, the smaller ψ, i.e. the longer the economy stays in the contagion state once it has entered this state, the more extreme the portfolio weights, exposures and utility functions.

5.3 Model Mis-Specification

If the investor relies on a benchmark model instead of the true model from Section 2.1, he will not hold the optimal portfolio. In this section, we analyze the utility loss he suffers from due to this suboptimal behavior.

5.3.1 Incomplete Market

In the incomplete market, the investor can only invest into the two stocks and into the money market account. In case of model mis-specification, he (incorrectly) uses one of the benchmark models to determine the optimal portfolio. For both these models, the optimal portfolio weights are constant over time. The indirect utility derived from this strategy is given in the next proposition.

Proposition 5.1 (Model Mis-Specification, Incomplete Market) *In an economy with contagion where only the two stocks and the money market account are traded and for an investor who uses the portfolio weights $\widehat{\pi}_A, \widehat{\pi}_B$, the indirect utility in state $j \in \{calm, cont\}$ is*

$$G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\widehat{f^j}(t)$$

where $\widehat{f^j}$ is given by

$$\begin{pmatrix} \widehat{f}^{calm}(t) \\ \widehat{f}^{cont}(t) \end{pmatrix} = \exp\left\{\begin{pmatrix} \widehat{C}^{calm,calm} & \widehat{C}^{calm,cont} \\ \widehat{C}^{cont,calm} & \widehat{C}^{cont,cont} \end{pmatrix}(T-t)\right\}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where

$$\begin{aligned}
\widehat{C}^{calm,calm} &= (1-\gamma)\left[r + \widehat{\pi}_A(\mu_A^{calm} - r) + \widehat{\pi}_B(\mu_B^{calm} - r)\right] \\
&\quad - 0.5\gamma(1-\gamma)\left[(\widehat{\pi}_A\sigma_A^{calm})^2 + (\widehat{\pi}_B\sigma_B^{calm})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right] \\
&\quad - \lambda_A^{calm,cont} + \lambda_A^{calm,calm}\left[(1-\widehat{\pi}_A L_A)^{1-\gamma} - 1\right] \\
&\quad - \lambda_B^{calm,cont} + \lambda_B^{calm,calm}\left[(1-\widehat{\pi}_B L_B)^{1-\gamma} - 1\right] \\
\widehat{C}^{calm,cont} &= \lambda_A^{calm,cont}(1-\widehat{\pi}_A L_A)^{1-\gamma} + \lambda_B^{calm,cont}(1-\widehat{\pi}_B L_B)^{1-\gamma} \\
\widehat{C}^{cont,calm} &= \lambda^{cont,calm} \\
\widehat{C}^{cont,cont} &= (1-\gamma)\left[r + \widehat{\pi}_A(\mu_A^{cont} - r) + \widehat{\pi}_B(\mu_B^{cont} - r)\right] \\
&\quad - 0.5\gamma(1-\gamma)\left[(\widehat{\pi}_A\sigma_A^{cont})^2 + (\widehat{\pi}_B\sigma_B^{cont})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{cont}\sigma_B^{calm}\rho^{cont}\right] \\
&\quad - \lambda^{cont,calm} + \lambda_A^{cont,cont}\left[(1-\widehat{\pi}_A L_A)^{1-\gamma} - 1\right] + \lambda_B^{cont,cont}\left[(1-\widehat{\pi}_B L_B)^{1-\gamma} - 1\right].
\end{aligned}$$

The proof is given in Appendix D.1.

The upper panels of Figure 3 and 4 show the certainty equivalent returns in case of model mis-specification for equal market prices of risk and equal equity risk premia, respectively. For the base case parametrization, the investor loses up to 20 basis points a year if he relies on an incorrect model. The losses are larger for equal market prices of risk (parametrization 1) than for equal equity risk premia (parametrization 2), since the differences in the optimal portfolios between the states which the investor fails to pick up are larger in the first case. Surprisingly, the investor is (slightly) worse off if he assumes joint jumps and thus only ignores the time dimension of contagion than if he ignores contagion completely. And again, the results, i.e. the utility losses, increase in the difference between the calm and contagion state as measured by ξ.

5.3.2 Complete Market

Next, we analyze the impact of model mis-specification if the market is complete. To determine whether enough derivatives are traded for market completeness, the investor relies on the benchmark model. In the case of independent jumps, four risky assets are needed, while in the case of joint jumps, three risky assets are enough. We assume that the investor uses the two stocks, an ATM-call on stock A with a time to maturity of 3 months, and – if needed – an identical ATM-call on stock B. These short-term ATM-options are usually among the most liquid contracts. Note however that the choice of contracts will have an impact on the utility loss due to model mis-specification.

The analysis of model mis-specification is more complicated than in case of an incomplete market. In the first step, the investor determines the seemingly optimal exposures in the benchmark model. In the second step, he uses the risky assets and their risk exposure to implement these seemingly optimal exposures, where he (incorrectly) determines the sensitivities of the derivatives in the benchmark model. Given the seemingly optimal portfolio, we (but not the investor) can then use the sensitivities from the true model to determine the realized exposure. Given these realized exposures $\widehat{\theta}$, which are again constant over time, we can then finally calculate the realized indirect utility.

Proposition 5.2 (Model Mis-Specification, Complete Market) *In a complete market with contagion effects, the utility obtained by an investor who uses the incorrect risk factor exposures $\widehat{\theta}$ is given by*

$$\widehat{G}^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\,\widehat{f}^j(t)$$

where $j \in \{calm, cont\}$ and

$$\begin{pmatrix} \widehat{f}^{calm}(t) \\ \widehat{f}^{cont}(t) \end{pmatrix} = \exp\left\{ \begin{pmatrix} \widehat{C}^{calm,calm} & \widehat{C}^{calm,cont} \\ \widehat{C}^{cont,calm} & \widehat{C}^{cont,cont} \end{pmatrix} (T-t) \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

with

$$
\begin{aligned}
\widehat{C}^{calm,calm} &= (1-\gamma)\Big[r + \widehat{\theta}_A^{calm}\eta_A^{calm} + \widehat{\theta}_B^{calm}\eta_B^{calm} \\
&\qquad - \widehat{\theta}_A^{calm,calm}\lambda_A^{calm,calm}(1+\eta_A^{calm,calm}) - \widehat{\theta}_A^{calm,cont}\lambda_A^{calm,cont}(1+\eta_A^{calm,cont}) \\
&\qquad - \widehat{\theta}_B^{calm,calm}\lambda_B^{calm,calm}(1+\eta_B^{calm,calm}) - \widehat{\theta}_B^{calm,cont}\lambda_B^{calm,cont}(1+\eta_B^{calm,cont})\Big] \\
&\quad - 0.5\gamma(1-\gamma)\Big[(\widehat{\theta}_A^{calm})^2 + (\widehat{\theta}_B^{calm})^2 + 2\rho^{calm}\widehat{\theta}_A^{calm}\widehat{\theta}_B^{calm}\Big] \\
&\quad + \lambda_A^{calm,calm}\Big[(1+\widehat{\theta}_A^{calm,calm})^{1-\gamma} - 1\Big] + \lambda_B^{calm,calm}\Big[(1+\widehat{\theta}_B^{calm,calm})^{1-\gamma} - 1\Big] \\
&\quad - \lambda_A^{calm,cont} - \lambda_B^{calm,cont} \\
\widehat{C}^{calm,cont} &= \lambda_A^{calm,cont}(1+\widehat{\theta}_A^{calm,cont})^{1-\gamma} + \lambda_B^{calm,cont}(1+\widehat{\theta}_B^{calm,cont})^{1-\gamma} \\
\widehat{C}^{cont,calm} &= \lambda^{cont,calm}(1+\widehat{\theta}^{cont,calm})^{1-\gamma} \\
\widehat{C}^{cont,cont} &= (1-\gamma)\Big[r + \widehat{\theta}_A^{cont}\eta_A^{cont} + \widehat{\theta}_B^{cont}\eta_B^{cont} \\
&\qquad - \widehat{\theta}_A^{cont,cont}\lambda_A^{cont,cont}(1+\eta_A^{cont,cont}) - \widehat{\theta}_B^{cont,cont}\lambda_B^{cont,cont}(1+\eta_B^{cont,cont}) \\
&\qquad - \widehat{\theta}^{cont,calm}\lambda^{cont,calm}(1+\eta^{cont,calm})\Big] \\
&\quad - 0.5\gamma(1-\gamma)\Big[(\widehat{\theta}_A^{cont})^2 + (\widehat{\theta}_B^{cont})^2 + 2\rho^{cont}\widehat{\theta}_A^{cont}\widehat{\theta}_B^{cont}\Big] \\
&\quad + \lambda_A^{cont,cont}\Big[(1+\widehat{\theta}_A^{cont,cont})^{1-\gamma} - 1\Big] + \lambda_B^{cont,cont}\Big[(1+\widehat{\theta}_B^{cont,cont})^{1-\gamma} - 1\Big] \\
&\quad - \lambda^{cont,calm}
\end{aligned}
$$

The proof is given in Appendix D.2.

The lower panels of Figure 3 and 4 show the certainty equivalent returns when the correct model is used and when one of the benchmark models is used to determine the (seemingly) optimal portfolio. The CER losses are highly economically significant, and they are much higher than in the incomplete market, since the investor now makes an additional mistake. To set up the optimal portfolio, he has to convert the optimal exposures into portfolio weights. While the exposures of the stocks are model independent, the exposures of the derivatives depend on the model, and an investor using an incorrect model for portfolio planning will use the same incorrect model for pricing derivatives, too. As can be seen from the figures, the mistakes in calculating the exposures and in pricing the derivatives do not cancel each other, but rather add up.

Figure 5 compares the utility losses for different values of ξ, where we assume equal equity risk premia in both states. The results for equal market prices of risk (not shown here) are qualitatively similar. As can be seen from the graphs, the difference between the calm and contagion state has a very large impact on the utility losses. They are already far from negligible for a rather low value of $\xi = 2$, and increase to around 10%-15% a year for $\xi = 10$. For this high level of ξ, the CER can even become negative, and the investor would be better off if he just invested his wealth at the risk-free rate only, ignoring all risky assets.

Different from the incomplete market, it now depends on ξ, i.e. on the severeness of contagion, which of the two benchmark models leads to the smaller utility loss. For low values of ξ, the investor is still better off if he ignores contagion completely. For higher values of ξ, however, he is significantly better off if he just ignores the time dimension of contagion. In a model with joint jumps, the investor holds less derivatives which lowers his utility loss due to derivatives mispricing, but increases his utility loss due to a too conservative portfolio strategy. The exact trade-off between these two arguments depends on ξ.

An investor who relies on the correct model is obviously better off in the complete market. In case of model mis-specification, this may no longer be true, as can be seen in Figure 3 and 4. While an investor who incorrectly bases his decisions on a model with joint jumps is still better off in the complete market, an investor ignoring any contagion might be better off in the incomplete market. In this case, the utility gain from having access to derivatives (and thus more payoff patterns) is more than offset by the utility loss from using the incorrect sensitivities and implementing the seemingly optimal strategy in the wrong way.

We also did a robustness check with respect to α and ψ, which govern the risk of entering the contagion state and the average time the economy stays in the contagion state. As already seen above, the impact of the exact size of these two parameters is rather small, and the qualitative results do not change.

5.4 Robustness Checks

In the preceding sections, we have shown that contagion has a substantial effect on optimal exposures, optimal portfolio weights, and the investor's expected utility. Furthermore, an investor who uses an incorrect model might suffer large utility losses in particular in a complete market where he also uses derivatives. While we have already discussed the sensitivity of our results with respect to the severeness of contagion, we now do some additional robustness checks with respect to the risk aversion, the size of the losses, and the diffusion correlation between the stocks.

5.4.1 Relative Risk Aversion

The results up to now have been based on a relative risk aversion of $\gamma = 3$. We have redone the analysis for values of γ between 1.5 and 10. In line with intuition, the results become less extreme the higher the risk aversion and the less the investor therefore invests in risky assets. The qualitative results, however, do not change.

While the utility losses due to model mis-specification decrease in γ, they are still highly economically significant even for a high risk aversion of $\gamma = 10$. The investor is much more conservative in this case. Nevertheless, the loss in CER can well exceed 8% in the complete market and is thus far from negligible.

5.4.2 Loss Size

In a second step, we have changed the loss size from $L = 0.05$ to the more moderate value of $L = 0.03$. This has no impact on the results in the complete market, which are independent of the exact losses in the stocks, but depend only on the intensity of jumps and their market prices of risk. In the incomplete market, however, the smaller loss size decreases the utility of the investor, since the package offered by stocks now fits the optimal exposure even worse. Consequently, the utility loss due to market incompleteness increases.

The impact of the loss size on the losses due to model mis-specification is mixed. While the utility loss in the incomplete market and in case the joint jumps model is used decreases with the lower loss size, the opposite is true in a complete market and in case the investor relies on a model with no contagion at all. Overall, however, the results do not change qualitatively when we change the loss size.

5.4.3 Diffusion Correlation

As an additional robustness check, we consider different values for the diffusion correlation parameter ρ, which was set to $\rho = 0.5$ in our base case. We redo the analysis for $\rho = 0$ and $\rho = -0.5$.

The utility loss due to market incompleteness is smallest for $\rho = -0.5$ in our contagion model. This can be explained by the fact that the package offered by the stocks is closest to the overall optimal exposure in this case. The result is specific to the parameters used and will not hold in general.

Concerning our model mis-specification analysis, it depends on ρ whether the investor is better off if he ignores contagion completely or if he just ignores the time dimension of contagion. To get the intuition, remember that the model with joint jumps leads to a portfolio that is too conservative, but reduces the impact of calculating the incorrect sensitivities. For $\rho = -0.5$, the optimal portfolio includes only a small position in derivatives, so that the model with joint jumps performs worse than the model with no contagion at all. For $\rho = 0.5$, on the other hand, the investor is better off if he uses the model with joint jumps, since the position in derivatives is now significantly larger. Again, the utility loss due to model mis-specification may exceed the utility gain due to market completeness if the differences between the calm and the contagion state are large enough. This again suggests that the investor may be better off if he does not use derivatives at all instead of using them in the wrong way.

6 Conclusion

The paper analyzes the optimal portfolio in case of contagion risk. Instead of capturing contagion by joint jumps in the stocks, we assume that some large loss in stocks can increase the jump intensities significantly. This adds a time dimension to contagion. The investor is thus able to adjust his portfolio when the economy switches its state, and our results document that he indeed uses this possibility. The direction of the portfolio adjustment depends on his relative risk aversion and on the market prices of risk.

If the investor incorrectly uses a simpler model, then he suffers a utility loss: Surprisingly, in an incomplete market the investor's utility loss is larger if he assumes joint jumps (and thus ignores only the time dimension of contagion) than if he ignores contagion completely. On the other hand, if the investor has also access to derivatives, then his utility loss is larger if he disregards all aspects of contagion and if the calm and contagion state are rather distinct. Furthermore, an investor worrying about model mis-specification might be better off if he does not use derivatives at all, since the utility gain from having access to derivatives can be more than offset by the utility loss due to using an incorrect model.

There are several directions for future research. First, one can drop the assumption that the investor can observe the true state of the economy. In this case, he needs to learn about the current state by observing stock prices over time. He will use a filtering approach to continuously update the probabilities of being in the two states. Second, we have shown that the assumptions about the market prices of risk have a significant impact on the optimal portfolios. It would thus be interesting to consider a general equilibrium setup in which market prices of risk are determined endogenously. This would allow us to analyze how investors price contagion risk.

A Contagion

A.1 Complete Market - Proof

We solve the portfolio problem in a complete market for a general Markov chain with states $k \in \{1, \dots, K\}$. The indirect utility function in state j at time t and for a current wealth level of x is denoted by $G^j(t,x)$. The functions G must solve the system of Hamilton-Jacobi-Bellman equations, where we have one equation for each state j:

$$\begin{aligned} 0 &= \max\Bigg\{ G_t^j + G_x^j x \Big[r + \theta_A^j(t)\eta_A^j + \theta_B^j(t)\eta_B^j - \sum_{k\neq j} \theta^{j,k}(t)\lambda^{j,k}\left(1+\eta^{j,k}\right) \Big] \\ &\quad + 0.5 G_{xx}^j x^2 \left[\theta_A^j(t)^2 + \theta_B^j(t)^2 + 2\rho^j \theta_A^j(t)\theta_B^j(t)\right] \\ &\quad + \sum_{k\neq j} \left[G^k(t, x(1+\theta^{j,k}(t))) - G^j(t,x)\right]\lambda^{j,k} \Bigg\}. \end{aligned}$$

Subscripts of G denote partial derivatives. We assume constant relative risk aversion, and rely on the usual guess for the indirect utility function

$$G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\left(f^j(t)\right)^\gamma.$$

The partial derivatives are

$$\begin{aligned} G_t^j(t,x) &= \frac{x^{1-\gamma}}{1-\gamma}\gamma\left(f^j(t)\right)^{\gamma-1} f_t^j(t) \\ G_x^j(t,x) &= x^{-\gamma}\left(f^j(t)\right)^\gamma \\ G_{xx}^j(t,x) &= -\gamma x^{-\gamma-1}\left(f^j(t)\right)^\gamma, \end{aligned}$$

and the change in the indirect utility due to a jump is

$$G^k(t, x(1+\theta^{j,k}(t))) - G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\left[\left(f^k(t)\right)^\gamma (1+\theta^{j,k}(t))^{1-\gamma} - \left(f^j(t)\right)^\gamma\right].$$

Plugging these expressions into the HJB-equations and simplifying gives

$$\begin{aligned} 0 &= \max\Bigg\{ \gamma\frac{f_t^j}{f^j} + (1-\gamma)\Big[r + \theta_A^j(t)\eta_A^j + \theta_B^j(t)\eta_B^j - \sum_{k\neq j} \theta^{j,k}(t)\lambda^{j,k}\left(1+\eta^{j,k}\right) \Big] \\ &\quad - 0.5\gamma(1-\gamma)\left[\theta_A^j(t)^2 + \theta_B^j(t)^2 + 2\rho^j\,\theta_A^j(t)\theta_B^j(t)\right] \\ &\quad + \sum_{k\neq j} \left[\left(\frac{f^k}{f^j}\right)^\gamma (1+\theta^{j,k}(t))^{1-\gamma} - 1\right]\lambda^{j,k} \Bigg\}. \end{aligned}$$

Solving the first order conditions for the optimal exposures gives

$$\theta^j_A = \frac{\eta^j_A - \rho^j \eta^j_B}{\gamma(1-(\rho^j)^2)} \qquad\qquad \theta^{j,k} = (1+\eta^{j,k})^{-\frac{1}{\gamma}} \frac{f^k}{f^j} - 1$$
$$\theta^j_B = \frac{\eta^j_B - \rho^j \eta^j_A}{\gamma(1-(\rho^j)^2)}.$$

We then plug the optimal exposures back into the HJB-equations to get

$$\begin{aligned} 0 &= \gamma\frac{f^j_t}{f^j} + (1-\gamma)\left[r + \frac{(\eta^j_A)^2 + (\eta^j_B)^2 - 2\rho^j \eta^j_A \eta^j_B}{\gamma(1-(\rho^j)^2)}\right] \\ &\quad - (1-\gamma)\sum_{k\neq j}\left[\left(1+\eta^{j,k}\right)^{1-\frac{1}{\gamma}} \lambda^{j,k}\frac{f^k}{f^j} - \lambda^{j,k}\left(1+\eta^{j,k}\right)\right] \\ &\quad - 0.5(1-\gamma)\frac{(\eta^j_A)^2 + (\eta^j_B)^2 - 2\rho^j \eta^j_A \eta^j_B}{\gamma(1-(\rho^j)^2)} \\ &\quad + \sum_{k\neq j}\left[\frac{f^k}{f^j}\left(1+\eta^{j,k}\right)^{1-\frac{1}{\gamma}} - 1\right]\lambda^{j,k}. \end{aligned}$$

The resulting linear system of homogeneous ordinary differential equations for $f^j(t)$ ($j = 0, 1, 2$) with boundary condition $f^j(T) = 1$ is

$$\begin{aligned} 0 &= f^j_t + \frac{1-\gamma}{\gamma}\left[r + \frac{(\eta^j_A)^2 + (\eta^j_B)^2 - 2\rho^j \eta^j_A \eta^j_B}{2\gamma(1-(\rho^j)^2)}\right] f^j \\ &\quad + \frac{1-\gamma}{\gamma}\sum_{k\neq j}\left[\left(1+\eta^{j,k}\right) - \frac{1}{1-\gamma}\right]\lambda^{j,k} f^j + \sum_{k\neq j}\left(1+\eta^{j,k}\right)^{1-\frac{1}{\gamma}}\lambda^{j,k} f^k. \end{aligned}$$

This is equivalent to

$$0 = f^j_t + C^{(j,j)} f^j + \sum_{k\neq j} C^{(j,k)} f^k$$

where the coefficients C depend on the parameters only

$$\begin{aligned} C^{(j,j)} &= \frac{1-\gamma}{\gamma}\left[r + \frac{(\eta^j_A)^2 + (\eta^j_B)^2 - 2\rho^j \eta^j_A \eta^j_B}{2\gamma(1-(\rho^j)^2)}\right] + \frac{1-\gamma}{\gamma}\sum_{k\neq j}\left[\left(1+\eta^{j,k}\right) - \frac{1}{1-\gamma}\right]\lambda^{j,k} \\ C^{(j,k)} &= \left(1+\eta^{j,k}\right)^{1-\frac{1}{\gamma}}\lambda^{j,k}. \end{aligned}$$

The system of ordinary differential equations can thus be written as

$$\begin{pmatrix} f^1 \\ \vdots \\ f^K \end{pmatrix}_t = -\begin{pmatrix} C^{1,1} & C^{1,2} & \cdots & C^{1,K} \\ C^{2,1} & C^{2,2} & \cdots & C^{2,K} \\ \vdots & \vdots & \ddots & \vdots \\ C^{K,1} & C^{K,2} & \cdots & C^{K,K} \end{pmatrix}\begin{pmatrix} f^1 \\ \vdots \\ f^K \end{pmatrix},$$

and its solution is

$$\begin{pmatrix} f^1 \\ \vdots \\ f^K \end{pmatrix} = e^{C\cdot(T-t)} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Proposition 3.1 then follows by applying this result to our Markov chain and further aggregating the formally eight states to the two economic states 'calm' and 'contagion' as described in section 2.1.

A.2 Incomplete Market - Proof

In the incomplete market, the investor decides on the portfolio weights π_A^{calm} and π_B^{calm} of the two stocks. The HJB-equation in the calm state is

$$\begin{aligned} 0 \;=\; \max_{\pi_A^{calm},\pi_B^{calm}} & \{G_t^{calm} + x(r + \pi_A^{calm}(\mu_A^{calm} - r) + \pi_B^{calm}(\mu_B^{calm} - r))G_x^{calm} \\ & + 0.5x^2 \left[(\pi_A^{calm}\sigma_A^{calm})^2 + (\pi_B^{calm}\sigma_B^{calm})^2 + 2\pi_A^{calm}\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right] G_{xx}^{calm} \\ & + \lambda_A^{calm,cont}[G^{cont}(t, x(1 - \pi_A^{calm}L_A)) - G^{calm}(t,x)] \\ & + \lambda_B^{calm,cont}[G^{cont}(t, x(1 - \pi_B^{calm}L_B)) - G^{calm}(t,x)] \\ & + \lambda_A^{calm,calm}[G^{calm}(t, x(1 - \pi_A^{calm}L_A)) - G^{calm}(t,x)] \\ & + \lambda_B^{calm,calm}[G^{calm}(t, x(1 - \pi_B^{calm}L_B)) - G^{calm}(t,x)]\} \end{aligned}$$

and the HJB-equation in the contagion state is

$$\begin{aligned} 0 \;=\; \max_{\pi_A^{cont},\pi_B^{cont}} & \{G_t^{cont} + x(r + \pi_A^{cont}(\mu_A^{cont} - r) + \pi_B^{cont}(\mu_B^{cont} - r))G_x^{cont} \\ & + 0.5x^2 \left[(\pi_A^{cont}\sigma_A^{cont})^2 + (\pi_B^{cont}\sigma_B^{cont})^2 + 2\pi_A^{cont}\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right] G_{xx}^{cont} \\ & + \lambda_A^{cont,cont}[G^{cont}(t, x(1 - \pi_A^{cont}L_A)) - G^{cont}(t,x)] \\ & + \lambda_B^{cont,cont}[G^{cont}(t, x(1 - \pi_B^{cont}L_B)) - G^{cont}(t,x)] \\ & + \lambda^{cont,calm}[G^{calm}(t,x) - G^{cont}(t,x)]\}. \end{aligned}$$

With the guess $G^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma} f^j(t)$, the HJB-equation in the calm state becomes

$$\begin{aligned} 0 \;=\; \max_{\pi_A^{calm},\pi_B^{calm}} & \{f_t^{calm} + (1-\gamma)\left(r + \pi_A^{calm}(\mu_A^{calm} - r) + \pi_B^{calm}(\mu_B^{calm} - r)\right) f^{calm} \\ & - 0.5\gamma(1-\gamma)\left((\pi_A^{calm}\sigma_A^{calm})^2 + (\pi_B^{calm}\sigma_B^{calm})^2 + 2\pi_A^{calm}\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right) f^{calm} \\ & + \lambda_A^{calm,cont}\left((1 - \pi_A^{calm}L_A)^{1-\gamma} f^{cont} - f^{calm}\right) \\ & + \lambda_B^{calm,cont}\left((1 - \pi_B^{calm}L_B)^{1-\gamma} f^{cont} - f^{calm}\right) \\ & + \lambda_A^{calm,calm}\left((1 - \pi_A^{calm}L_A)^{1-\gamma} f^{calm} - f^{calm}\right) \\ & + \lambda_B^{calm,calm}\left((1 - \pi_B^{calm}L_B)^{1-\gamma} f^{calm} - f^{calm}\right)\} \end{aligned}$$

and the HJB-equation in the contagion state becomes

$$\begin{aligned}
0 &= \max_{\pi_A^{calm},\pi_B^{calm}} \{ f_t^{cont} + (1-\gamma)\left(r + \pi_A^{cont}(\mu_A^{cont} - r) + \pi_B^{cont}(\mu_B^{cont} - r)\right) f^{cont} \\
&\quad -0.5\gamma(1-\gamma)\left((\pi_A^{cont}\sigma_A^{cont})^2 + (\pi_B^{cont}\sigma_B^{cont})^2 + 2\pi_A^{cont}\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right) f^{cont} \\
&\quad + \lambda_A^{cont,cont}\left((1-\pi_A^{cont}L_A)^{1-\gamma} - 1\right) f^{cont} \\
&\quad + \lambda_B^{cont,cont}\left((1-\pi_B^{cont}L_B)^{1-\gamma} - 1\right) f^{cont} \\
&\quad + \lambda^{cont,calm}(f^{calm} - f^{cont})\}.
\end{aligned}$$

The first order conditions for the portfolio weights are

$$\begin{aligned}
\mu_A^{calm} - r - \gamma(\sigma_A^{calm})^2\pi_A^{calm} - \gamma\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm} & \\
-L_A\lambda_A^{calm,cont}(1-\pi_A^{calm}L_A)^{-\gamma}\frac{f^{cont}}{f^{calm}} - L_A\lambda_A^{calm,calm}(1-\pi_A^{calm}L_A)^{-\gamma} &= 0 \quad (9) \\
\mu_B^{calm} - r - \gamma(\sigma_B^{calm})^2\pi_B^{calm} - \gamma\pi_A^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm} & \\
-L_B\lambda_B^{calm,cont}(1-\pi_B^{calm}L_B)^{-\gamma}\frac{f^{cont}}{f^{calm}} - L_B\lambda_B^{calm,calm}(1-\pi_B^{calm}L_B)^{-\gamma} &= 0 \quad (10) \\
\mu_A^{cont} - r - \gamma(\sigma_A^{cont})^2\pi_A^{cont} - \gamma\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont} - L_A\lambda_A^{cont,cont}(1-\pi_A^{cont}L_A)^{-\gamma} &= 0 \quad (11) \\
\mu_B^{cont} - r - \gamma(\sigma_B^{cont})^2\pi_B^{cont} - \gamma\pi_A^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont} - L_B\lambda_B^{cont,cont}(1-\pi_B^{cont}L_B)^{-\gamma} &= 0. \quad (12)
\end{aligned}$$

With the optimal portfolio weights, the differential equations become

$$\begin{aligned}
0 &= f_t^{calm} + (1-\gamma)\left(r + \pi_A^{calm}(\mu_A^{calm} - r) + \pi_B^{calm}(\mu_B^{calm} - r)\right) f^{calm} \qquad (13) \\
&\quad - 0.5\gamma(1-\gamma)\left((\pi_A^{calm}\sigma_A^{calm})^2 + (\pi_B^{calm}\sigma_B^{calm})^2 + 2\pi_A^{calm}\pi_B^{calm}\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right) f^{calm} \\
&\quad + \lambda_A^{calm,cont}\left((1-\pi_A^{calm}L_A)^{1-\gamma} f^{cont} - f^{calm}\right) \\
&\quad + \lambda_B^{calm,cont}\left((1-\pi_B^{calm}L_B)^{1-\gamma} f^{cont} - f^{calm}\right) \\
&\quad + \lambda_A^{calm,calm}\left((1-\pi_A^{calm}L_A)^{1-\gamma} - 1\right) f^{calm} \\
&\quad + \lambda_B^{calm,calm}\left((1-\pi_B^{calm}L_B)^{1-\gamma} - 1\right) f^{calm} \\
0 &= f_t^{cont} + (1-\gamma)\left(r + \pi_A^{cont}(\mu_A^{cont} - r) + \pi_B^{cont}(\mu_B^{cont} - r)\right) f^{cont} \qquad (14) \\
&\quad - 0.5\gamma(1-\gamma)\left((\pi_A^{cont}\sigma_A^{cont})^2 + (\pi_B^{cont}\sigma_B^{cont})^2 + 2\pi_A^{cont}\pi_B^{cont}\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right) f^{cont} \\
&\quad + \lambda_A^{cont,cont}\left((1-\pi_A^{cont}L_A)^{1-\gamma} - 1\right) f^{cont} + \lambda_B^{cont,cont}\left((1-\pi_B^{cont}L_B)^{1-\gamma} - 1\right) f^{cont} \\
&\quad + \lambda^{cont,calm}(f^{calm} - f^{cont}).
\end{aligned}$$

Conditions (11) and (12) can be solved numerically for the optimal portfolio weights in the contagion state. Conditions (13),(14),(9) and (10) form a so-called differential-algebraic system for the functions f^{calm}, f^{cont}, π_A^{calm} and π_B^{calm}. This system can be solved numerically using a Runge-Kutta method of order 3, namely the implicit Radau form of order 3, which is for example studied in Hairer, Lubich, and Roche (1989).

B Benchmark Models: Independent Jumps

B.1 Complete Market - Proof

The model with independent jumps can be interpreted as a special case of the model with contagion where the parameters are identical in all states. The indirect utility function is then no longer state dependent. The optimal exposures are

$$\theta_A^{diff} = \frac{\eta_A^{diff} - \rho\eta_B^{diff}}{\gamma(1-\rho^2)} \qquad \theta_A^{jump} = (1+\eta_A^{jump})^{-\frac{1}{\gamma}} - 1$$
$$\theta_B^{diff} = \frac{\eta_B^{diff} - \rho\eta_A^{diff}}{\gamma(1-\rho^2)} \qquad \theta_B^{jump} = (1+\eta_B^{jump})^{-\frac{1}{\gamma}} - 1.$$

The ordinary differential equation for f becomes

$$0 = f_t + C^{nc,c} f$$

where

$$\begin{aligned} C^{nc,c} = \frac{1-\gamma}{\gamma}\Bigg[& r + \frac{(\eta_A^{diff})^2 + (\eta_B^{diff})^2 - 2\rho\eta_A^{diff}\eta_B^{diff}}{2\gamma(1-\rho^2)} \\ & + \left(1+\eta_A^{jump}\right)\lambda^A + \left(1+\eta_B^{jump}\right)\lambda^B - \frac{1}{1-\gamma}(\lambda^A+\lambda^B)\Bigg] \\ & + \left(1+\eta_A^{jump}\right)^{1-\frac{1}{\gamma}}\lambda^A + \left(1+\eta_B^{jump}\right)^{1-\frac{1}{\gamma}}\lambda^B. \end{aligned}$$

The function f can be solved for in closed form:

$$f(t) = \exp\left\{C^{nc,c}\cdot(T-t)\right\}.$$

The indirect utility is

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\exp\left\{\gamma\, C^{nc,c}\cdot(T-t)\right\}.$$

B.2 Complete Market: Impact of Jump Intensity

Lemma B.1 (Independent Jumps, Complete Market: Impact of Jump Intensity) *If there are no contagion effects and if the market is complete, the indirect utility is increasing in λ_A and λ_B.*

Proof: The partial derivative of G w.r.t. λ_i is

$$\frac{\partial G}{\partial \lambda_i} \quad = \quad \frac{w^{1-\gamma}}{1-\gamma}\gamma e^{\gamma C^{nc,c}\cdot(T-t)}\left[(1+\eta_i^{jump})^{1-\frac{1}{\gamma}} - 1 - \left(1-\frac{1}{\gamma}\right)\eta_i^{jump}\right](T-t).$$

The term in square brackets is positive (negative) if $(1+\eta_i^{jump})^{1-\frac{1}{\gamma}}$ is a convex (concave) function of η_i^{jump}, i.e. if $\gamma < 1$ ($\gamma > 1$), since $1+\left(1-\frac{1}{\gamma}\right)\eta_i^{jump}$ is just the first-order Taylor expansion of $(1+\eta_i^{jump})^{1-\frac{1}{\gamma}}$ around 0. The other terms are positive (negative) if $\gamma < 1$ ($\gamma > 1$). Put together, the partial derivative of the indirect utility function with respect to λ_i is positive, and the indirect utility is increasing in the jump intensity λ_i.

B.3 Incomplete Market - Proof

Again, the model can be interpreted as a special case of the model with contagion. The guess for the indirect utility function is

$$G(t,x) = \frac{x^{1-\gamma}}{1-\gamma}f(t)$$

where G does not depend on the state any more. The optimal portfolio weights π_A and π_B satisfy

$$\begin{aligned} \mu_A - r - \gamma\sigma_A^2\pi_A - \gamma\pi_B\sigma_A\sigma_B\rho - L_A\lambda_A(1-\pi_A L_A)^{-\gamma} &= 0 \\ \mu_B - r - \gamma\sigma_B^2\pi_B - \gamma\pi_A\sigma_B\sigma_A\rho - L_B\lambda_B(1-\pi_B L_B)^{-\gamma} &= 0 \end{aligned}$$

which can be solved numerically. The HJB-equation simplifies dramatically, and with the optimal portfolio weights, the differential equation for f is

$$f_t = -C^{nc,ic}f$$

with boundary condition $f(T) = 1$ and

$$\begin{aligned} C^{nc,ic} \quad = \quad & (1-\gamma)\left[r + \pi_A(\mu_A - r) + \pi_B(\mu_B - r) - 0.5\gamma(\pi_A^2\sigma_A^2 + \pi_B^2\sigma_B^2 + 2\pi_A\pi_B\sigma_A\sigma_B\rho)\right] \\ & +\lambda_A\left[(1-\pi_A L_A)^{1-\gamma} - 1\right] + \lambda_B\left[(1-\pi_B L_B)^{1-\gamma} - 1\right]. \end{aligned}$$

The solution is given by $f(t) = \exp\{C^{nc,ic}\cdot(T-t)\}$.

C Benchmark Models: Calibration

The stationary probability of the calm and contagion state is

$$\begin{aligned} p^{calm} \quad &= \frac{\lambda^{cont,calm}}{\lambda^{cont,calm}+\lambda_A^{calm,cont}+\lambda_B^{calm,cont}} \\ p^{cont} \quad &= \frac{\lambda_A^{calm,cont}+\lambda_B^{calm,cont}}{\lambda^{cont,calm}+\lambda_A^{calm,cont}+\lambda_B^{calm,cont}}, \end{aligned}$$

and we know from the ergodic theorem for Markov chains[8] that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t g(Z(s))ds = g(calm)p^{calm} + g(cont)p^{cont}$$

where g is some state-dependent function.

Firstly, we want the stocks to have the same risk in the contagion model and in the benchmark models. We thus equate the variance of the stock, which gives

$$\begin{aligned}(\sigma_i)^2 + L_i^2\lambda_i \quad = \quad & p^{calm}\left[\left(\sigma_i^{calm}\right)^2 + \left(L_i^{calm,calm}\right)^2\lambda_i^{calm,calm} + \left(L_i^{calm,cont}\right)^2\lambda_i^{calm,cont}\right] \\ & + p^{cont}\left[\left(\sigma_i^{cont}\right)^2 + \left(L_i^{cont,cont}\right)^2\lambda_i^{cont,cont}\right] \\ & + p^{calm}p^{cont}\Big[\sigma_i^{calm}\eta_i^{calm} + L_i^{calm,calm}\lambda_i^{calm,calm}\eta_i^{calm,calm} \\ & \qquad + L_i^{calm,cont}\lambda_i^{calm,cont}\eta_i^{calm,cont} \\ & \qquad - \sigma_i^{cont}\eta_i^{cont} - L_i^{cont,cont}\lambda_i^{cont,cont}\eta_i^{cont,cont}\Big]^2.\end{aligned}$$

We also equate the jump intensity (for those jumps that result in a loss) and the average jump size

$$\lambda_i \quad = \quad p^{calm}\left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right) + p^{cont}\lambda_i^{cont,cont} \tag{15}$$

$$\begin{aligned}L_i \quad = \quad & p^{calm}\left[\frac{\lambda_i^{calm,calm}}{\lambda_i^{calm,calm}+\lambda_i^{calm,cont}}\cdot L_i^{calm,calm} + \frac{\lambda_i^{calm,cont}}{\lambda_i^{calm,calm}+\lambda_i^{calm,cont}}\cdot L_i^{calm,cont}\right] \\ & + p^{cont}\cdot L_i^{cont,cont}\end{aligned}$$

Secondly, we want the stocks to have the same expected excess returns. Since the investor might deal differently with jump and diffusion risk, we also equate the risk premia earned on stock diffusion risk and stock jump risk. This gives two additional restrictions

$$\begin{aligned}\sigma_i\eta_i^{diff} \quad &= \quad p^{calm}\sigma_i^{calm}\eta_i^{calm} + p^{cont}\sigma_i^{cont}\eta_i^{cont} \\ L_i\lambda_i\eta_i^{jump} \quad &= \quad p^{calm}\left(L_i^{calm,calm}\lambda_i^{calm,calm}\eta_i^{calm,calm} + L_i^{calm,cont}\lambda_i^{calm,cont}\eta_i^{calm,cont}\right) \\ & \qquad + p^{cont}L_i^{cont,cont}\lambda_i^{cont,cont}\eta_i^{cont,cont}.\end{aligned}$$

D Model Mis-Specification

D.1 Incomplete Market - Model Mis-Specification

In case of model mis-specification, the optimal portfolios are determined in the benchmark model. With independent jumps, the weights of the stocks are constant over time. The

[8]See, e.g., Brémaud (2001).

indirect utility functions in the two states are then given by

$$\widehat{G}^j(t,x) = E_t\left[\frac{X_T^{1-\gamma}}{1-\gamma} \mid X_t = x\right]$$

subject to the budget restriction

$$\frac{dX(t)}{X(t)} = \widehat{\pi}_A(t)\frac{dS_A(t)}{S_A(t)} + \widehat{\pi}_B(t)\frac{dS_B(t)}{S_B(t)} + (1 - \widehat{\pi}_A(t) - \widehat{\pi}_B(t))\, rdt$$

where $\widehat{\pi}_A$ and $\widehat{\pi}_B$ denote the seemingly optimal portfolio weights. Since the indirect utility $\widehat{G}$ is a martingale, it holds that

$$\begin{aligned}
0 \;=\; & \widehat{G}_t^{calm} + x(r + \widehat{\pi}_A(\mu_A^{calm} - r) + \widehat{\pi}_B(\mu_B^{calm} - r))\widehat{G}_x^{calm} \\
& + 0.5x^2\left[(\widehat{\pi}_A\sigma_A^{calm})^2 + (\widehat{\pi}_B\sigma_B^{calm})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right]\widehat{G}_{xx}^{calm} \\
& + \lambda_A^{calm,cont}[\widehat{G}^{cont}(t, x(1 - \widehat{\pi}_A L_A)) - \widehat{G}^{calm}(t,x)] \\
& + \lambda_B^{calm,cont}[\widehat{G}^{cont}(t, x(1 - \widehat{\pi}_B L_B)) - \widehat{G}^{calm}(t,x)] \\
& + \lambda_A^{calm,calm}[\widehat{G}^{calm}(t, x(1 - \widehat{\pi}_A L_A)) - \widehat{G}^{calm}(t,x)] \\
& + \lambda_B^{calm,calm}[\widehat{G}^{calm}(t, x(1 - \widehat{\pi}_B L_B)) - \widehat{G}^{calm}(t,x)]
\end{aligned}$$

and

$$\begin{aligned}
0 \;=\; & \widehat{G}_t^{cont} + x(r + \widehat{\pi}_A(\mu_A^{cont} - r) + \widehat{\pi}_B(\mu_B^{cont} - r))\widehat{G}_x^{cont} \\
& + 0.5x^2\left[(\widehat{\pi}_A\sigma_A^{cont})^2 + (\widehat{\pi}_B\sigma_B^{cont})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{cont}\sigma_B^{cont}\rho^{cont}\right]\widehat{G}_{xx}^{cont} \\
& + \lambda^{cont,calm}[\widehat{G}^{calm}(t,x) - \widehat{G}^{cont}(t,x)] \\
& + \lambda_A^{cont,cont}[\widehat{G}^{cont}(t, x(1 - \widehat{\pi}_A L_A)) - \widehat{G}^{cont}(t,x)] \\
& + \lambda_B^{cont,cont}[\widehat{G}^{cont}(t, x(1 - \widehat{\pi}_B L_B)) - \widehat{G}^{cont}(t,x)].
\end{aligned}$$

Since the investor has constant relative risk aversion, we can use a separation approach and set

$$\widehat{G}^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\,\widehat{f}^j(t).$$

Plugging in and simplifying gives

$$\begin{aligned}
0 \;=\; & \widehat{f}_t^{calm} + (1-\gamma)\left(r + \widehat{\pi}_A(\mu_A^{calm} - r) + \widehat{\pi}_B(\mu_B^{calm} - r)\right)\widehat{f}^{calm} \\
& - 0.5\gamma(1-\gamma)\left((\widehat{\pi}_A\sigma_A^{calm})^2 + (\widehat{\pi}_B\sigma_B^{calm})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right)\widehat{f}^{calm} \\
& + \lambda_A^{calm,cont}\left((1 - \widehat{\pi}_A L_A)^{1-\gamma}\widehat{f}^{cont} - \widehat{f}^{calm}\right) \\
& + \lambda_B^{calm,cont}\left((1 - \widehat{\pi}_B L_B)^{1-\gamma}\widehat{f}^{cont} - \widehat{f}^{calm}\right) \\
& + \lambda_A^{calm,calm}\left((1 - \widehat{\pi}_A L_A)^{1-\gamma}\widehat{f}^{calm} - \widehat{f}^{calm}\right) \\
& + \lambda_B^{calm,calm}\left((1 - \widehat{\pi}_B L_B)^{1-\gamma}\widehat{f}^{calm} - \widehat{f}^{calm}\right)
\end{aligned}$$

and

$$
\begin{aligned}
0 \;=\;& \widehat{f}_t^{cont} + (1-\gamma)\left(r + \widehat{\pi}_A(\mu_A^{cont} - r) + \widehat{\pi}_B(\mu_B^{cont} - r)\right)\widehat{f}^{cont} \\
& - 0.5\gamma(1-\gamma)\left((\widehat{\pi}_A\sigma_A^{cont})^2 + (\widehat{\pi}_B\sigma_B^{cont})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{cont}\sigma_B^{calm}\rho^{cont}\right)\widehat{f}^{cont} \\
& + \lambda^{cont,calm}\left(\widehat{f}^{calm} - \widehat{f}^{cont}\right) \\
& + \lambda_A^{cont,cont}\left((1-\widehat{\pi}_A L_A)^{1-\gamma}\widehat{f}^{cont} - \widehat{f}^{cont}\right) \\
& + \lambda_B^{cont,cont}\left((1-\widehat{\pi}_B L_B)^{1-\gamma}\widehat{f}^{cont} - \widehat{f}^{cont}\right).
\end{aligned}
$$

This results in a system of two linear ordinary differential equations

$$
\begin{pmatrix} \widehat{f}_t^{calm} \\ \widehat{f}_t^{cont} \end{pmatrix} = - \begin{pmatrix} \widehat{C}^{1,1} & \widehat{C}^{1,2} \\ \widehat{C}^{2,1} & \widehat{C}^{2,2} \end{pmatrix} \begin{pmatrix} \widehat{f}^{calm} \\ \widehat{f}^{cont} \end{pmatrix}
$$

where

$$
\begin{aligned}
\widehat{C}^{1,1} \;=\;& (1-\gamma)\left(r + \widehat{\pi}_A(\mu_A^{calm} - r) + \widehat{\pi}_B(\mu_B^{calm} - r)\right) \\
& - 0.5\gamma(1-\gamma)\left((\widehat{\pi}_A\sigma_A^{calm})^2 + (\widehat{\pi}_B\sigma_B^{calm})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{calm}\sigma_B^{calm}\rho^{calm}\right) \\
& - \lambda_A^{calm,cont} - \lambda_B^{calm,cont} \\
& + \lambda_A^{calm,calm}\left((1-\widehat{\pi}_A L_A)^{1-\gamma} - 1\right) + \lambda_B^{calm,calm}\left((1-\widehat{\pi}_B L_B)^{1-\gamma} - 1\right) \\
\widehat{C}^{1,2} \quad & \lambda_A^{calm,cont}(1-\widehat{\pi}_A L_A)^{1-\gamma} + \lambda_B^{calm,cont}(1-\widehat{\pi}_B L_B)^{1-\gamma} \\
\widehat{C}^{2,1} \;=\;& \lambda^{cont,calm} \\
\widehat{C}^{2,2} \;=\;& (1-\gamma)\left(r + \widehat{\pi}_A(\mu_A^{cont} - r) + \widehat{\pi}_B(\mu_B^{cont} - r)\right) \\
& - 0.5\gamma(1-\gamma)\left((\widehat{\pi}_A\sigma_A^{cont})^2 + (\widehat{\pi}_B\sigma_B^{cont})^2 + 2\widehat{\pi}_A\widehat{\pi}_B\sigma_A^{cont}\sigma_B^{calm}\rho^{cont}\right) \\
& - \lambda^{cont,calm} \\
& + \lambda_A^{cont,cont}\left((1-\widehat{\pi}_A L_A)^{1-\gamma} - 1\right) + \lambda_B^{cont,cont}\left((1-\widehat{\pi}_B L_B)^{1-\gamma} - 1\right).
\end{aligned}
$$

The solution for $\widehat{f}$ is

$$
\begin{pmatrix} \widehat{f}^{calm}(t) \\ \widehat{f}^{cont}(t) \end{pmatrix} = \exp\left\{ \begin{pmatrix} \widehat{C}^{1,1} & \widehat{C}^{1,2} \\ \widehat{C}^{2,1} & \widehat{C}^{2,2} \end{pmatrix} (T-t) \right\} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

D.2 Complete Market - Model Mis-Specification

In case of model mis-specification in a complete market setup, the investor does not implement his optimal risk factor exposures $\theta(t)$, but sub-optimal exposures $\widehat{\theta}$ which are constant over time. As in A.1, we solve for the indirect utility function for a general

Markov chain with states $k \in \{1, \ldots, K\}$. The indirect utility functions in the K states are then given by

$$\widehat{G}^j(t,x) = E_t\left[\frac{X_T^{1-\gamma}}{1-\gamma} \mid X_t = x\right]$$

subject to the budget restriction

$$\begin{aligned}
\frac{dX(t)}{X(t)} &= rdt + \widehat{\theta}_A^{Z(t)}\left[dW_A(t) + \eta_A^{Z(t)}dt\right] + \widehat{\theta}_B^{Z(t)}\left[dW_B(t) + \eta_B^{Z(t)}dt\right] \\
&\quad + \sum_{k \neq Z(t), \lambda^{Z(t),k} \neq 0} \widehat{\theta}^{Z(t),k}\left[dN^k(t) - \lambda^{Z(t),k}dt - \eta^{Z(t),k}\lambda^{Z(t),k}dt\right].
\end{aligned}$$

Since the indirect utility $\widehat{G}$ is a martingale, it holds that

$$\begin{aligned}
0 &= \widehat{G}_t^j + \widehat{G}_x^j x\left[r + \widehat{\theta}_A^j \eta_A^j + \widehat{\theta}_B^j \eta_B^j - \sum_{k \neq j} \widehat{\theta}^{j,k}\lambda^{j,k}\left(1 + \eta^{j,k}\right)\right] \\
&\quad + 0.5G_{xx}^j x^2\left[(\widehat{\theta}_A^j)^2 + (\widehat{\theta}_B^j)^2 + 2\rho^j \widehat{\theta}_A^j \widehat{\theta}_B^j\right] \\
&\quad + \sum_{k \neq j}\left[G^k(t, x(1 + \widehat{\theta}^{j,k})) - G^j(t,x)\right]\lambda^{j,k}.
\end{aligned}$$

Since the investor has constant relative risk aversion, we can use a separation approach and set

$$\widehat{G}^j(t,x) = \frac{x^{1-\gamma}}{1-\gamma}\widehat{f}^j(t).$$

Plugging in and simplifying gives a system of linear ordinary differential equations

$$\begin{aligned}
0 &= \widehat{f}_t^j + (1-\gamma)\left[r + \widehat{\theta}_A^j \eta_A^j + \widehat{\theta_B}^j \eta_B^j - \sum_{k \neq j} \widehat{\theta}^{j,k}\lambda^{j,k}\left(1 + \eta^{j,k}\right)\right]\widehat{f}^j \\
&\quad - 0.5\gamma(1-\gamma)\left[(\widehat{\theta}_A^j)^2 + (\widehat{\theta}_B^j)^2 + 2\rho^j\,\widehat{\theta}_A^j \widehat{\theta}_B^j\right]\widehat{f}^j \\
&\quad + \sum_{k \neq j}\left[\widehat{f}^k(1 + \widehat{\theta}^{j,k})^{1-\gamma} - \widehat{f}^j\right]\lambda^{j,k}\Bigg\}
\end{aligned}$$

whose solution with respect to the boundary conditions $f^j(T) = 1$ becomes in our case

$$\begin{pmatrix} \widehat{f}^{calm}(t) \\ \widehat{f}^{cont}(t) \end{pmatrix} = \exp\left\{\begin{pmatrix} \widehat{C}^{calm,calm} & \widehat{C}^{calm,cont} \\ \widehat{C}^{cont,calm} & \widehat{C}^{cont,cont} \end{pmatrix}(T-t)\right\}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with

$$
\begin{aligned}
\widehat{C}^{calm,calm} \;=\;& (1-\gamma)\Big[r + \widehat{\theta}_A^{calm}\eta_A^{calm} + \widehat{\theta}_B^{calm}\eta_B^{calm} \\
&\quad - \widehat{\theta}_A^{calm,calm}\lambda_A^{calm,calm}(1+\eta_A^{calm,calm}) - \widehat{\theta}_A^{calm,cont}\lambda_A^{calm,cont}(1+\eta_A^{calm,cont}) \\
&\quad - \widehat{\theta}_B^{calm,calm}\lambda_B^{calm,calm}(1+\eta_B^{calm,calm}) - \widehat{\theta}_B^{calm,cont}\lambda_B^{calm,cont}(1+\eta_B^{calm,cont})\Big] \\
&- 0.5\gamma(1-\gamma)\Big[(\widehat{\theta}_A^{calm})^2 + (\widehat{\theta}_B^{calm})^2 + 2\rho^{calm}\widehat{\theta}_A^{calm}\widehat{\theta}_B^{calm}\Big] \\
&+ \lambda_A^{calm,calm}\Big[(1+\widehat{\theta}_A^{calm,calm})^{1-\gamma} - 1\Big] + \lambda_B^{calm,calm}\Big[(1+\widehat{\theta}_B^{calm,calm})^{1-\gamma} - 1\Big] \\
&- \lambda_A^{calm,cont} - \lambda_B^{calm,cont} \\
\widehat{C}^{calm,cont} \;=\;& \lambda_A^{calm,cont}(1+\widehat{\theta}_A^{calm,cont})^{1-\gamma} + \lambda_B^{calm,cont}(1+\widehat{\theta}_B^{calm,cont})^{1-\gamma} \\
\widehat{C}^{cont,calm} \;=\;& \lambda^{cont,calm}(1+\widehat{\theta}^{cont,calm})^{1-\gamma} \\
\widehat{C}^{cont,cont} \;=\;& (1-\gamma)\Big[r + \widehat{\theta}_A^{cont}\eta_A^{cont} + \widehat{\theta}_B^{cont}\eta_B^{cont} \\
&\quad - \widehat{\theta}_A^{cont,cont}\lambda_A^{cont,cont}(1+\eta_A^{cont,cont}) - \widehat{\theta}_B^{cont,cont}\lambda_B^{cont,cont}(1+\eta_B^{cont,cont}) \\
&\quad - \widehat{\theta}^{cont,calm}\lambda^{cont,calm}(1+\eta^{cont,calm})\Big] \\
&- 0.5\gamma(1-\gamma)\Big[(\widehat{\theta}_A^{cont})^2 + (\widehat{\theta}_B^{cont})^2 + 2\rho^{cont}\widehat{\theta}_A^{cont}\widehat{\theta}_B^{cont}\Big] \\
&+ \lambda_A^{cont,cont}\Big[(1+\widehat{\theta}_A^{cont,cont})^{1-\gamma} - 1\Big] + \lambda_B^{cont,cont}\Big[(1+\widehat{\theta}_B^{cont,cont})^{1-\gamma} - 1\Big] \\
&- \lambda^{cont,calm}.
\end{aligned}
$$

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	Our model			Benchmark model		
		Parametrization 1	Parametrization 2		Parametrization 1	Parametrization 2
Data-generating process	σ_A^{calm}	**0.15**		σ_A	0.1515	0.15
	σ_A^{cont}	**0.15**				
	ρ^{calm}	**0.50**		ρ	0.50	0.50
	ρ^{cont}	**0.50**				
	$\lambda_A^{calm,calm}$	*0.375*		$\lambda_A/\lambda^{joint}$	**1.50**	**1.50**
	$\lambda_A^{calm,cont}$	*0.375*				
	$\lambda_A^{cont,cont}$	*3.000*				
	$\lambda^{cont,calm}$	*1.500*				
	$L_A^{calm,calm}$	**0.05**		L_A	0.05	0.05
	$L_A^{calm,cont}$	**0.05**				
	$L_A^{cont,cont}$	**0.05**				
	$L_A^{cont,calm}$	**0.00**				
	ξ_A	**4.00**				
	α_A	**0.50**				
	ψ	**0.25**				
Market prices of risk	η_A^{calm}	**0.35**	**0.35**	η_A^{diff}	0.3466	0.35
	η_A^{cont}	**0.35**	**0.35**			
	$\eta_A^{calm,calm}$	**0.40**	0.80	η_A^{jump}	0.40	0.40
	$\eta_A^{calm,cont}$	**0.40**	0.80			
	$\eta_A^{cont,cont}$	**0.40**	0.20			
	$\eta^{cont,calm}$	**0.00**	0.00			

Table 1: Parameters

The table gives the parameters for the stocks under the physical measure (upper part) and the market prices of risk (lower part) for our base case as explained in Section 5.1. The two stocks are assumed to follow identical processes, so that we only give the parameters for stock A. The market prices of risk in our model are chosen such that either the market prices of jump risk are identical in the calm and the contagion state (parametrization 1) or such that the expected excess return on the stock is identical in both states (parametrization 2). The parameters written in bold numbers have been set in line with recent empirical studies. The jump intensities in our model (written in italic numbers) have been set in the second step. All other numbers have been calibrated in a third step such that the average equity risk premium is identical for both parametrizations (market prices of risk) or such that the benchmark models are as close as possible to our model.

		our model		benchmark
		calm	contagion	
Parametrization 1 (identical market prices of risk)	Excess Return	0.0675	0.1125	0.0825
	... from diffusion	0.0525	0.0525	0.0525
	... from jumps	0.0150	0.0600	0.0300
	Variance	0.0244	0.0300	0.0267
	... from diffusion	0.0225	0.0225	0.0230
	... from jumps	0.0019	0.0075	0.0038
Parametrization 2 (identical equity risk premium)	Excess Return	0.0825	0.0825	0.0825
	... from diffusion	0.0525	0.0525	0.0525
	... from jumps	0.0300	0.0300	0.0300
	Variance	0.0244	0.0300	0.0263
	... from diffusion	0.0225	0.0225	0.0225
	... from jumps	0.0019	0.0075	0.0038

Table 2: Conditional Moments

The table gives the conditional expected excess returns and the conditional variances of stock returns in the calm and in the contagion state as well as in the benchmark models for the parameter set from Table 1. Furthermore, we show the contribution of diffusion risk and jump risk to the local moments. For parametrization 1, the market prices of risk are assumed to be equal in the calm and in the contagion state, while for parametrization 2, the expected excess returns are equal across states.

	$\lambda_A/\lambda^{joint}$	contagion parameters			calibrated jump parameters			
		ξ_A	α_A	ψ	$\lambda_A^{calm,calm}$	$\lambda_A^{calm,cont}$	$\lambda_A^{cont,cont}$	$\lambda^{cont,calm}$
no contagion	1.50	1	0.50	0.25	0.75000	0.75000	1.50	0.75
	1.50	2	0.50	0.25	0.50000	0.50000	2.00	1.00
base case	1.50	4	0.50	0.25	0.37500	0.37500	3.00	1.50
	1.50	10	0.50	0.25	0.30000	0.30000	6.00	3.00
	1.50	4	1/3	1/3	0.62500	0.31250	3.75	2.50
	1.50	10	1/3	1/3	0.55000	0.27500	8.25	5.50
	1.50	4	0.20	0.20	0.75000	0.18750	3.75	1.50
	1.50	10	0.20	0.20	0.66000	0.16500	8.25	3.30
	1.50	4	0.50	0.50	0.46875	0.46875	3.75	3.75
	1.50	10	0.50	0.50	0.41250	0.41250	8.25	8.25
	1.50	4	1/3	2/3	0.75000	0.37500	4.50	6.00
	1.50	10	1/3	2/3	0.70000	0.35000	10.50	14.00

Table 3: Selection of calibrated jump parameters

Each line of the table shows one possible combination of contagion and jump parameters leading to an 'average' (i.e. benchmark) jump intensity of 1.5. The line marked 'no contagion' describes a situation where the calm and the contagion state equal (since ξ equals 1). The line marked 'base case' shows the parameters for our base case parameter set also described in Table 1.

		our model		benchmark models	
		calm	cont	no cont	joint
Parametrization 1: identical market prices of risk					
complete	Diff-Exposure	0.0778	0.0778	0.0770	0.0770
	Jump-Exposure				
	no change of state	-0.1061	-0.1061	-0.1061	-0.1061
	change of state	-0.1307	0.0284		
	hedging demand	< 0	> 0	0	0
incomplete	π_A	0.6388	0.8934	0.7150	0.6396
	Diff-Exposure	0.0958	0.1340	0.1083	0.0968
	Jump-Exposure				
	no change of state	-0.0319	-0.0447	-0.0358	-0.0640
	change of state	-0.0319	0		
Parametrization 2: identical expected excess returns					
complete	Diff-Exposure	0.0778	0.0778	0.0778	0.0778
	Jump-Exposure				
	no change of state	-0.1779	-0.0590	-0.1061	-0.1061
	change of state	-0.1601	-0.0212		
	hedging demand	> 0	< 0	0	0
incomplete	π_A	0.7671	0.6583	0.7277	0.6495
	Diff-Exposure	0.1151	0.0988	0.1092	0.0974
	Jump-Exposure				
	no change of state	-0.0384	-0.0329	-0.0364	-0.0649
	change of state	-0.0384	0		

Table 4: Optimal Portfolios/Exposures

The table shows the optimal portfolios for our model and for the two benchmark models in a complete and in an incomplete market for a planning horizon of 20 years and for the benchmark parameters of Table 1. For the complete market, we give the optimal exposures to diffusion risk and the optimal exposure to jumps that (do not) induce a change from calm to contagion or vice versa. For the incomplete market, we give the optimal weight of stock A, as well as the induced exposures to the risk factors. Since the weights of stock B and the exposures to risk factors related to stock B coincide with those for stock A, we only show the results for A.

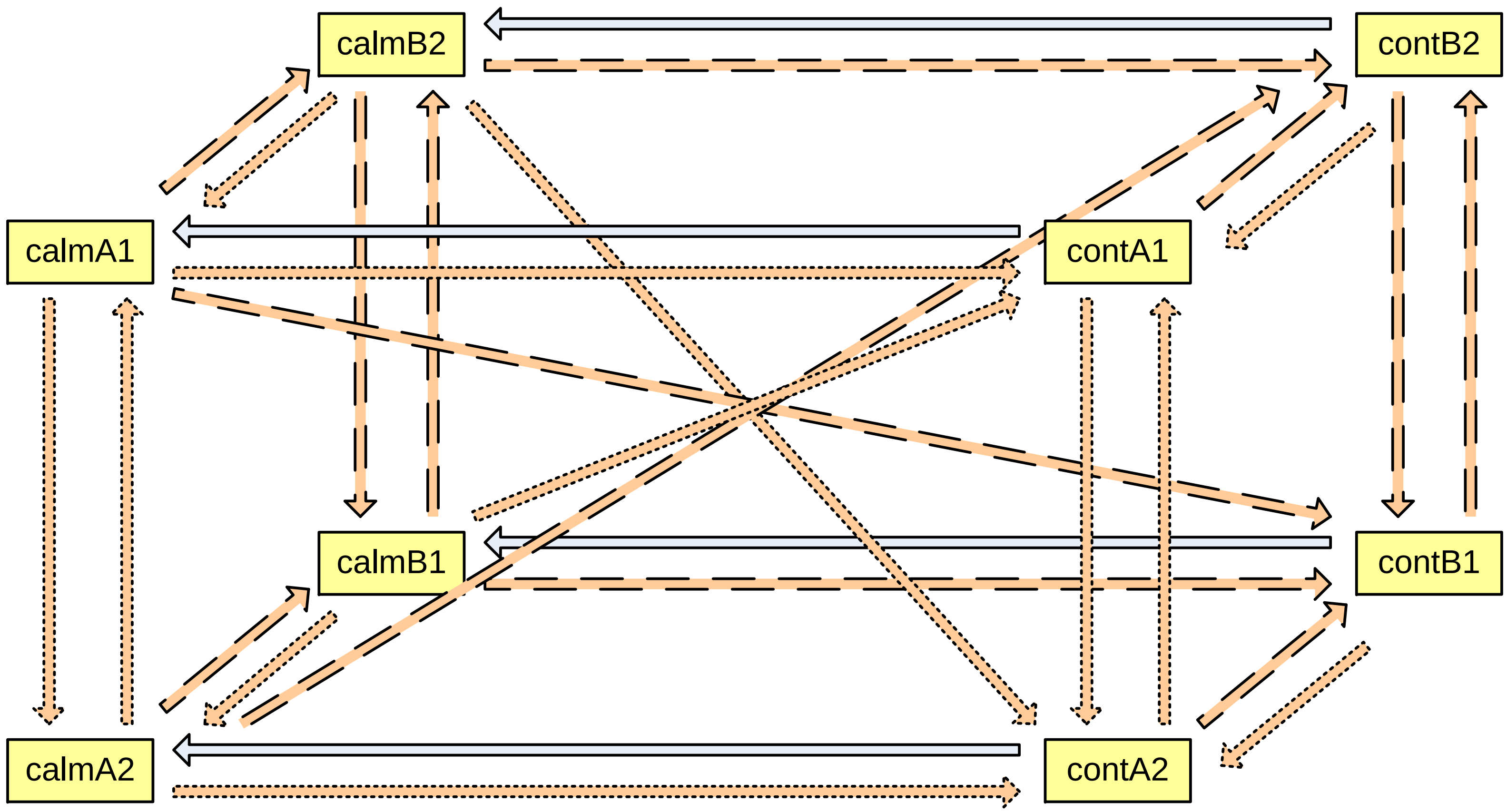


Figure 1: Our Markov Chain

This figure shows the Markov chain used in our model setup. The left states denote the calm states of the economy, the right ones denote those states in which the stocks are affected by contagion. The dotted (orange) arrows indicate a jump event leading to a loss in stock A, the dashed (orange) arrows a jump event leading to a loss in stock B. The solid (blue) arrows denote a change of state without any impact on the stock prices, i.e. a jump from the contagion state back to the calm state.

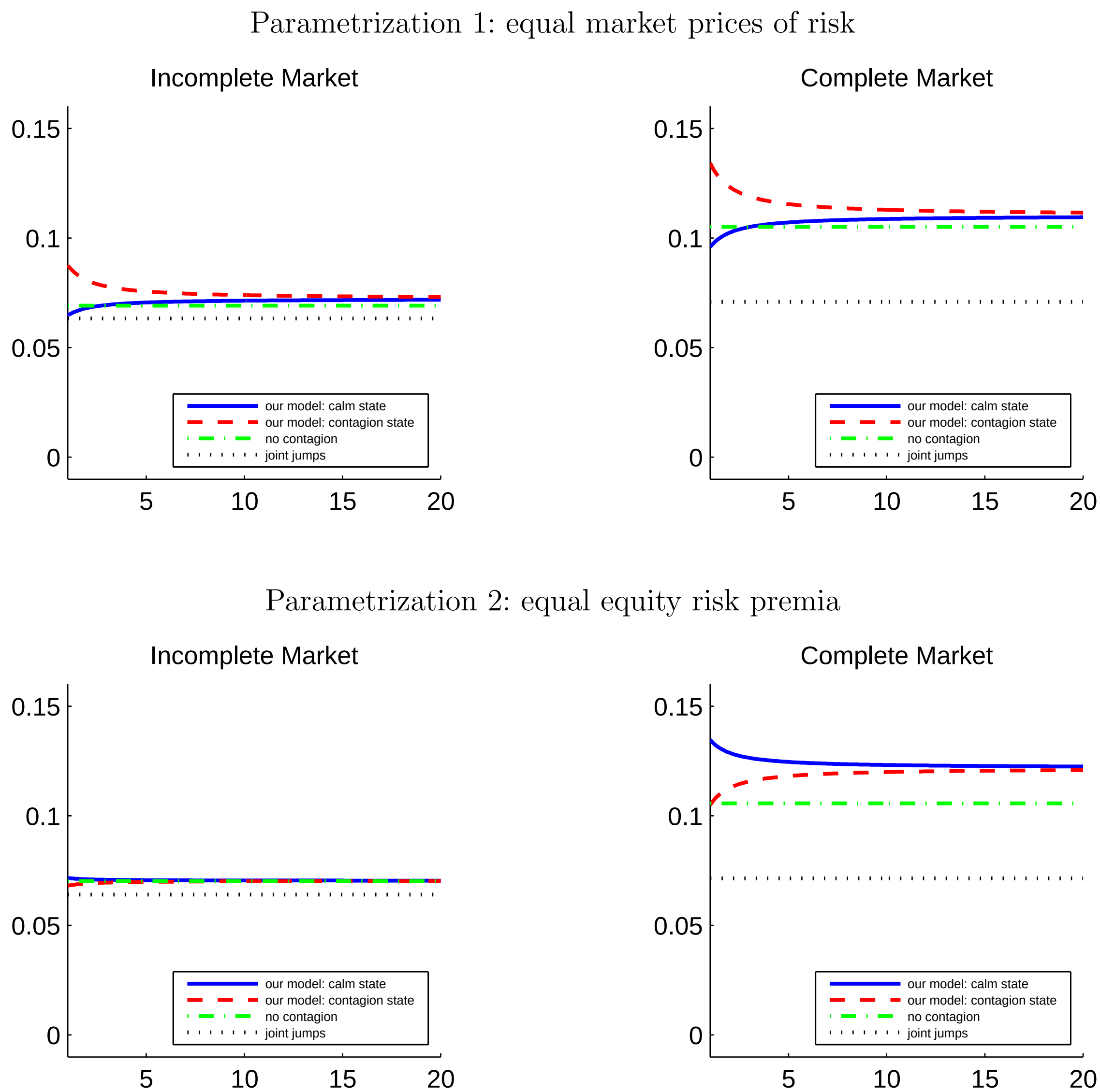


Figure 2: Certainty Equivalent Returns

The figures show the certainty equivalent returns as a function of the planning horizon for the case of equal market prices of risk (upper row) and equal equity risk premia (lower row) in the calm and in the contagion state as well as in the benchmark cases. The results for the incomplete market are given in the left column, the results for the complete one in the right column. The solid blue lines give the certainty equivalent returns in the calm state, the dashed red lines the certainty equivalent returns in the contagion state. The dash-dotted green lines denote the certainty equivalent returns in the benchmark case with no contagion, the dotted black lines the certainty equivalent returns in the model with joint jumps. The results are based on the parameters given in Table 1.

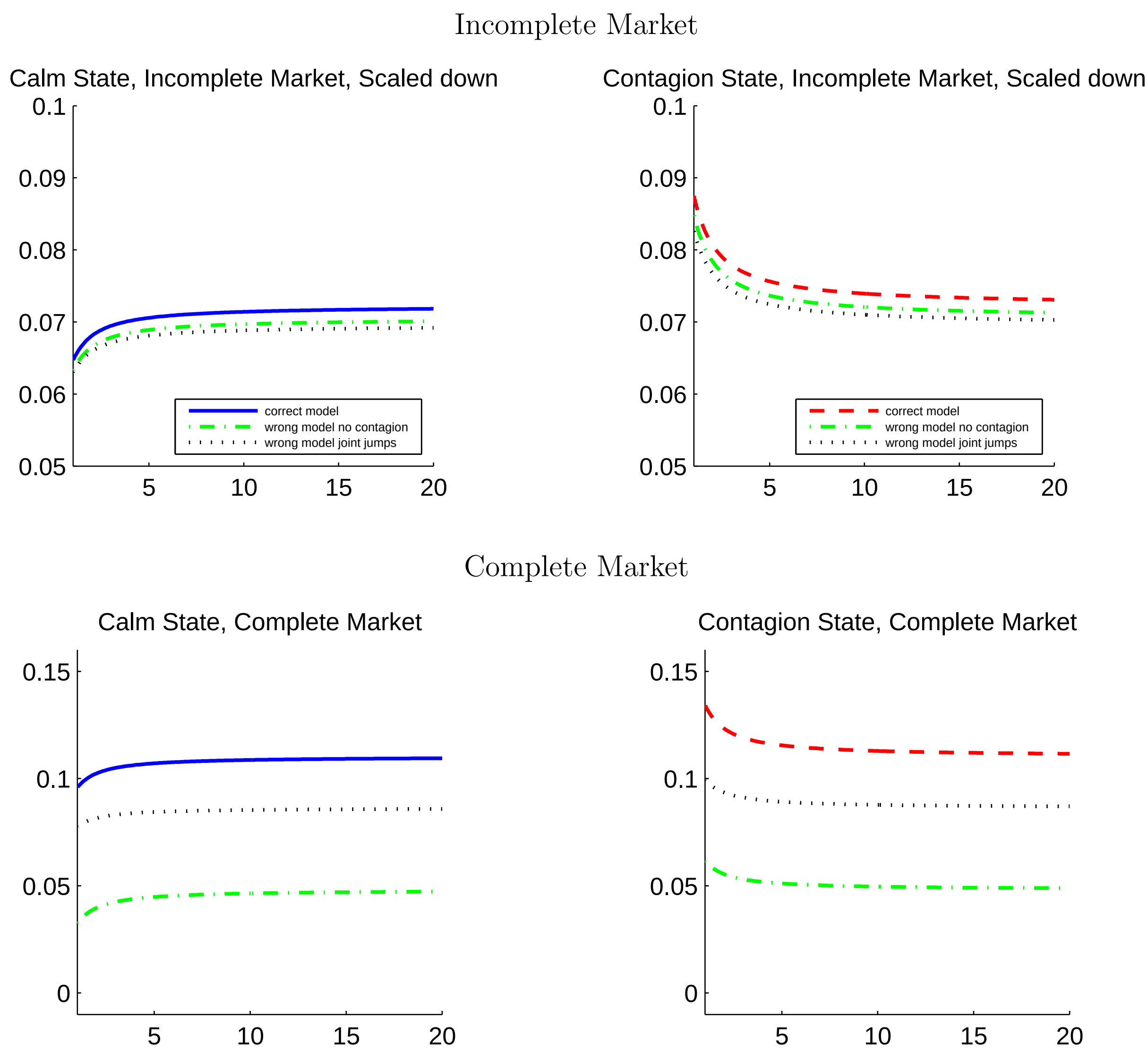
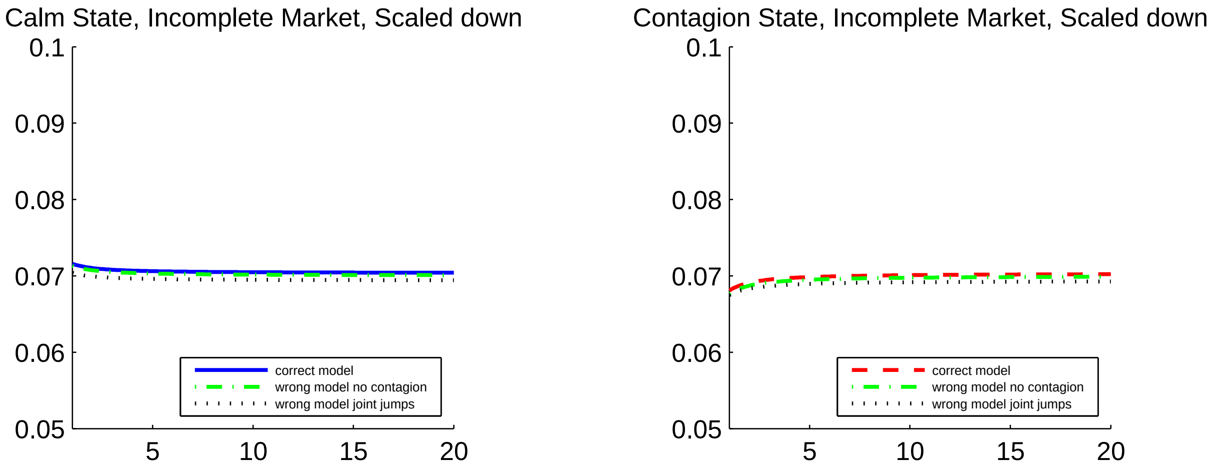


Figure 3: Model Mis-Specification: certainty equivalent returns for equal market prices of risk

The figures show the certainty equivalent returns as a function of the planning horizon for the incomplete (upper panel) and complete market (lower panel) if the economy is in the calm state (left column) and in the contagion state (right column), depending on which model is used for portfolio planning. The solid blue lines and the dashed red lines give the certainty equivalent returns n the calm and contagion state, respectively, if the correct model is used. The dash-dotted green lines indicate the CERs if a model with no contagion is used, the dotted black lines are the CERs if a model with joint jumps is used. The results are based on parametrization 1 from Table 1 for which the market prices of risk are equal in both states.

Incomplete Market



Complete Market

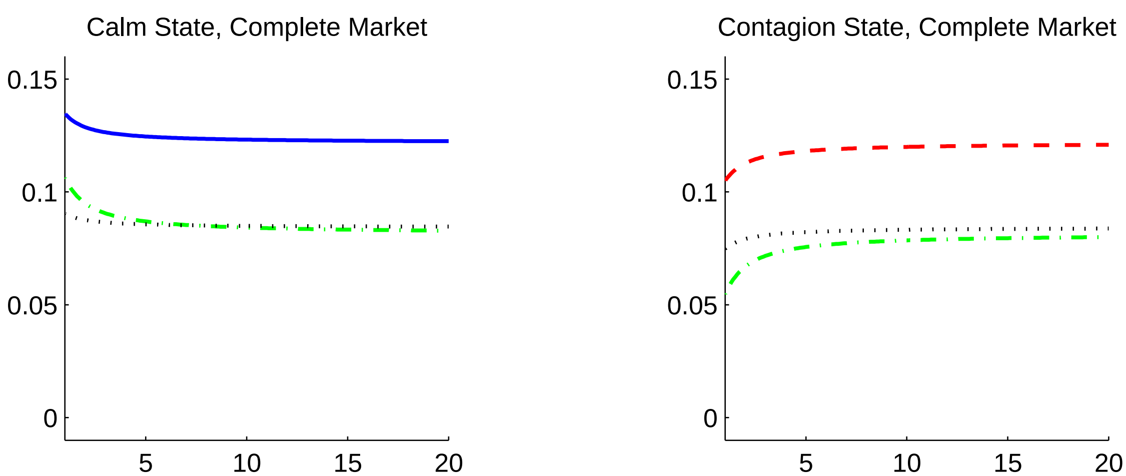


Figure 4: Model Mis-Specification: certainty equivalent returns for equal equity risk premia

The figures show the certainty equivalent returns as a function of the planning horizon for the incomplete (upper panel) and complete market (lower panel) if the economy is in the calm state (left column) and in the contagion state (right column), depending on which model is used for portfolio planning. The solid blue lines and the dashed red lines give the certainty equivalent returns in the calm and contagion state, respectively, if the correct model is used. The dash-dotted green lines indicate the CERs if a model with no contagion is used, the dotted black lines are the CERs if a model with joint jumps is used. The results are based on parametrization 2 from Table 1 for which the equity risk premium is equal in both states.

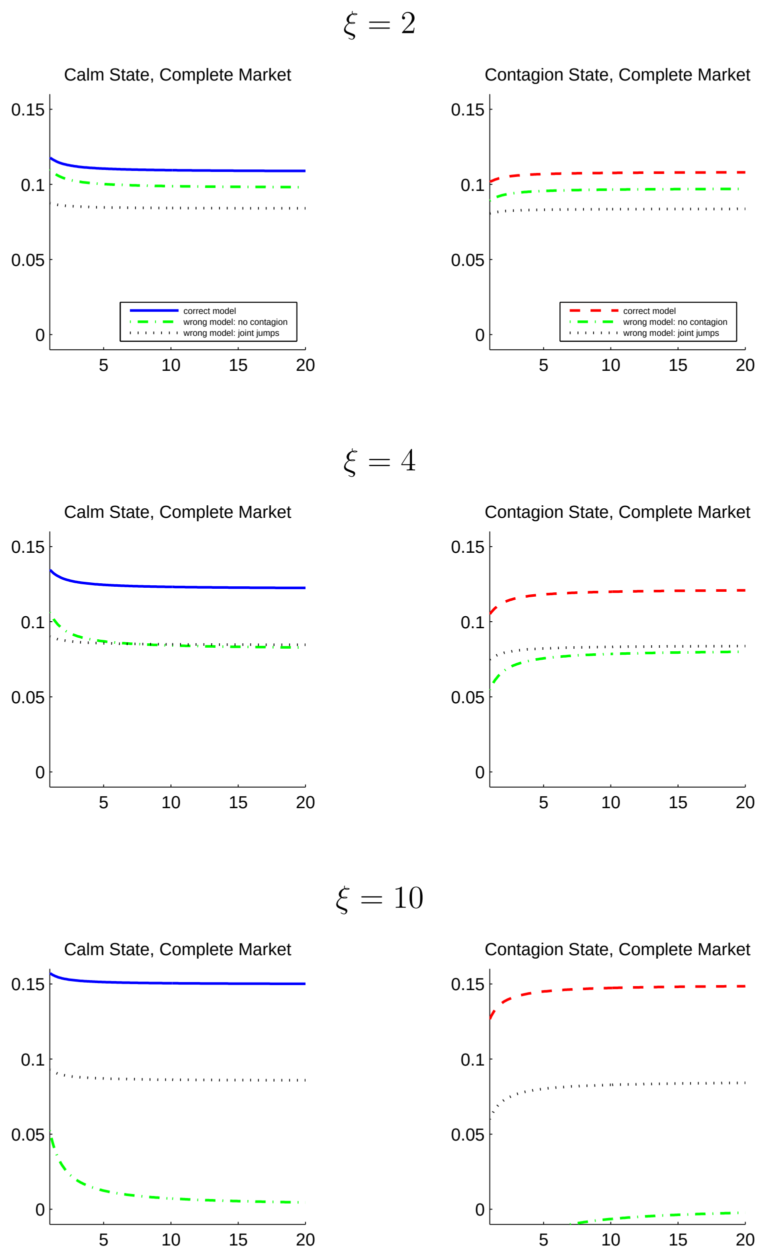


Figure 5: Model Mis-Specification: certainty equivalent returns for different values of ξ

The figures show the certainty equivalent returns for the complete market in case of model mis-specification as a function of the planning horizon for different values of $\xi_A = \xi_B = \xi$. The solid blue lines and the dashed red lines give the certainty equivalent returns in the calm and contagion state, respectively, if the correct model is used. The dash-dotted green lines indicate the CERs if a model with no contagion is used, the dotted black lines are the CERs if a model with joint jumps is used. The results are based on parametrization 2 (equal equity risk premia) from Table 1 where we have chosen $\xi = 2, 4, 10$ and thus changed the jump intensities according to Table 3.

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