LYAPUNOV FUNCTIONS FOR LINEAR NONAUTONOMOUS DYNAMICAL EQUATIONS ON TIME SCALES

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The existence of a Lyapunov function is established following a method of Yoshizawa for the uniform exponential asymptotic stability of the zero solution of a nonautonomous linear dynamical equation on a time scale with uniformly bounded graininess.

1. Introduction

Lyapunov functions are a very useful tool for investigating the behaviour of dynamical equations. They have been used now for over a century for differential equations of many types [15] as well as difference equations [1]. They were first used in the context of time scales in [12]. See [16] for a more recent and an extensive investigation of Lyapunov functions on time scales.

Much of the literature on Lyapunov functions, especially applications oriented, deals with sufficient conditions, assuming that a Lyapunov function is known. An important theoretical issue with practical implications is whether or not a Lyapunov function characterizing a particular dynamical property actually exists—such results are known as necessary conditions.

In this paper, we establish the existence of a Lyapunov function characterising the uniform exponential asymptotic stability of the zero solution of a nonautonomous linear dynamic equation

$$x^{\Delta} = A(t)x \tag{1.1}$$

on a time scale $\mathbb{T}$ with a bounded graininess, where the matrix-valued mapping $t \mapsto A(t)$ is right dense continuous (rd-continuous) on $\mathbb{T}$, that is, $A \in \mathscr{C}^{\mathrm{rd}}\mathscr{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Such linear dynamical equations and their inhomogeneous variants play an important role in investigations of the dynamical behaviour, both in themselves and also as

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linearizations of nonlinear systems, see, for example, [3, 7, 11]. We note that Pötzsche et al. [13] investigated various necessary and sufficient conditions for the uniform exponential stability of the zero solution of systems (1.1) in terms of spectral properties rather than Lyapunov functions.

In the next section, we show the invertibility of the cylinder transformation on the Hilger complex plane. This is required in our main result, which is presented and proved in the final section.

Background concepts and results on time scales are taken from Bohner and Peterson [4] (see also [2, 5, 8, 9]) and, for brevity, will not be stated explicitly here in general.

2. Invertibility of the cylinder transformation

A classical result from function theory (see, e.g., [6, Pages 38–45]) says that the main branch of the complex logarithm $\operatorname{Log}(z) := \log(|z|) + \imath\phi$, where $z = |z|e^{\imath\phi}$ with $-\pi < \phi \leq \pi$, is well defined for $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, but is not continuous for $z \in \mathbb{C}^*$ with $\operatorname{Im}(z) = \pi$. Moreover, $\operatorname{Log}(z)$ is a holomorphic function for $z \in \mathbb{C}^- := \mathbb{C} \setminus \{z \in \mathbb{R} : z \leq 0\}$.

Define $\mathbb{C}_h$ and $\mathbb{Z}_h$ for $h > 0$ by

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \qquad \mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \operatorname{Im}(z) < \frac{\pi}{h} \right\}. \tag{2.1}$$

$\mathbb{C}_h$ is called the Hilger complex plane. Hilger [10] also introduced the *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$, which is defined by

$$\xi_h(z) := \begin{cases} \dfrac{1}{h}\operatorname{Log}(1 + zh), & h > 0, \\ z, & h = 0. \end{cases} \tag{2.2}$$

Proposition 2.1 (Hilger [10]). *Let $\mathbb{T}$ be a time scale with graininess μ and let $p \in \mathscr{C}^{rd}(\mathbb{T}, \mathbb{C})$ be regressive on $\mathbb{T}$. Then the composition $\xi_\mu(p)$ of the cylinder transformation with μ and p is well defined, rd-continuous on $\mathbb{T}$, and thus Cauchy integrable.*

The cylinder transformation is useful for investigating, among other things, linear dynamical equations. It is invertible when suitably restricted.

Lemma 2.2. *Let $\mathbb{A}_h := \{z \in \mathbb{C}_h : z < -1/h\}$ for $h > 0$ and define $\xi_h : \mathbb{C}_h \setminus \mathbb{A}_h \to \mathbb{Z}_h$ by*

$$\xi_h(z) := \frac{1}{h}\operatorname{Log}(1 + zh). \tag{2.3}$$

Then ξ_h is a biholomorphism of $\mathbb{C}_h \setminus \mathbb{A}_h$ onto $\mathbb{Z}_h$ and its inverse $\xi_h^{-1} : \mathbb{Z}_h \to \mathbb{C}_h \setminus \mathbb{A}_h$ is given by

$$\xi_h^{-1}(z) := \frac{\exp(zh) - 1}{h}. \tag{2.4}$$

Proof. Since ξ_h and ξ_h^{-1} are well defined and continuous, it remains to prove that

(1) $\xi_h(\mathbb{C}_h \setminus \mathbb{A}_h) \subset \mathbb{Z}_h$ and $\xi_h^{-1}(\xi_h(z)) = z$ for all $z \in \mathbb{C}_h \setminus \mathbb{A}_h$.

Let $z \in \mathbb{C}_h \setminus \mathbb{A}_h$. Then $1 + zh \in \mathbb{C}^-$ and

$$\xi_h(z) = \frac{1}{h}\operatorname{Log}(1+zh) = \frac{\log|1+zh|}{h} + i \cdot \frac{\operatorname{Arg}(1+zh)}{h} \tag{2.5}$$

with $-\pi < \operatorname{Arg}(1+zh) < \pi$, from which it follows that

$$-\frac{\pi}{h} < \frac{\operatorname{Arg}(1+zh)}{h} < \frac{\pi}{h}, \tag{2.6}$$

hence $\xi_h(\mathbb{C}_h \setminus \mathbb{A}_h) \subset \mathbb{Z}_h$. Furthermore, a classical result from function theory says that $\exp(\operatorname{Log}(1+zh)) = 1 + zh$ for $z \in \mathbb{C}_h \setminus \mathbb{A}_h$ and $h > 0$ (see, e.g., [14, Pages 120 and 126]). Hence

$$\xi_h^{-1}(\xi_h(z)) = \frac{\exp\left(h \cdot (1/h)\operatorname{Log}(1+zh)\right) - 1}{h} = \frac{1+zh-1}{h} = z. \tag{2.7}$$

(2) $\xi_h^{-1}(\mathbb{Z}_h) \subset \mathbb{C}_h \setminus \mathbb{A}_h$ and $\xi_h(\xi_h^{-1}(z)) = z$ for all $z \in \mathbb{Z}_h$.

We assume that there exists a $z \in \mathbb{Z}_h$ such that for $h > 0$

$$\exp(zh) \leq -\frac{1}{h}. \tag{2.8}$$

Applying Euler's formula we see that (2.8) holds for $z \in \mathbb{Z}_h$ if and only if $\operatorname{Im}(z) = 0$, that is, $z \in \mathbb{R}$. But then a contradiction follows directly from (2.8), since the exponential function is strictly positive on $\mathbb{R}$. Hence $\xi_h^{-1}(\mathbb{Z}_h) \subset \mathbb{C}_h \setminus \mathbb{A}_h$. We have $-\pi/h < \operatorname{Im}(z) < \pi/h$ whenever $z \in \mathbb{Z}_h$. It then follows from [14, Pages 120 and 126] that $\operatorname{Log}(\exp(zh)) = zh$ and hence

$$\frac{1}{h}\operatorname{Log}\left(1 + h \cdot \frac{\exp(zh) - 1}{h}\right) = \frac{1}{h}\operatorname{Log}\left(\exp(zh)\right) = z \tag{2.9}$$

for $z \in \mathbb{Z}_h$ and $h > 0$.

We have shown that $\xi_h : \mathbb{C}_h \setminus \mathbb{A}_h \to \mathbb{Z}_h'$ is bijective with inverse function $\xi_h^{-1} : \mathbb{Z}_h \setminus \mathbb{A}_h$. From function theory, it is known that exp is a holomorphism on $\mathbb{C}$ and that Log is a holomorphism on $\mathbb{C}^-$ (see, e.g., [14, Page 64]). Hence ξ_h is a holomorphism on $\mathbb{C}_h \setminus \mathbb{A}_h$ and ξ_h^- is a holomorphism on $\mathbb{Z}_h$ for $h > 0$. □

Remark 2.3. Write $\mathbb{C}_0 := \mathbb{C}$ for $h = 0$. Then we define $\xi_0 :=$ id on $\mathbb{C}$ and $\xi_0^{-1} :=$ id on all of $\mathbb{C}$, where id is the identity function on $\mathbb{C}$.

For fixed $c \in \mathbb{R}$, the composition of the inverse cylinder transformation with the graininess μ of a time scale $\mathbb{T}$ will be needed in considering Lyapunov functions. Let $c \in \mathbb{R}$ be fixed and $\overline{\xi}_c : \mathbb{T} \to \mathbb{R}$ be defined by

$$\overline{\xi}_c(t) := \begin{cases} \dfrac{\exp\left(\mu(t) \cdot c\right) - 1}{\mu(t)}, & t < \sigma(t), \\ c, & t = \sigma(t). \end{cases} \tag{2.10}$$

Then

(i) the function $t \mapsto \overline{\xi}_c(t)$ is rd-continuous in all $t \in \mathbb{T}$ for arbitrary $c \in \mathbb{R}$ as the composition of the graininess function $t \mapsto \mu(t)$, which is always rd-continuous (see [4, Example 1.59 and Theorem 1.60, Page 22]) and the continuity of $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) := \begin{cases} \dfrac{\exp(x) - 1}{x}, & x \neq 0, \\ 1, & x = 0; \end{cases} \tag{2.11}$$

(ii) since $f(x) > 0$ for all $x \in \mathbb{R}$, for $c > 0$, it follows that $\overline{\xi}_c(t) > 0$ for all $t \in \mathbb{T}$ and for $c < 0$, it follows that $\overline{\xi}_c(t) < 0$ for all $t \in \mathbb{T}$;

(iii) moreover, $\overline{\xi}_c(t) = c$ for all right dense $t \in \mathbb{T}$;

(iv) for $h > 0$, let $\mathbb{T} := h\mathbb{Z}$, then $\mu(t) = h$ and

$$\overline{\xi}_c(t) = \frac{\exp(h \cdot c)}{h} \tag{2.12}$$

for all $t \in \mathbb{T}$, that is, $t \mapsto \overline{\xi}_c(t)$ is a constant function.

3. Existence of a Lyapunov function

Our main result, the following theorem, is a necessary condition for the existence of a Lyapunov function characterizing the uniform exponential asymptotic stability of the zero solution of a nonautonomous linear dynamical equation. It is adapted from Yoshizawa [15, Theorem 19.1] for linear ordinary differential equations.

Theorem 3.1. *Let $\mathbb{T}$ be an unbounded time scale with bounded graininess and suppose that for each $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$, there exists a classical solution $y : \mathbb{T} \to \mathbb{R}^n$, $y(t) = y(t, t_0, x_0)$ of the initial value problem (IVP)*

$$\begin{gathered} y^\Delta = A(t)y, \quad A \in \mathscr{C}^{rd}\mathscr{R}(\mathbb{T}, \mathbb{R}^{n\times n}),\ t \in \mathbb{T}, \\ y(t_0) = x_0, \quad x_0 \in \mathbb{R}^n,\ t_0 \in \mathbb{T}, \end{gathered} \tag{3.1}$$

for all $t \in \mathbb{T}$, $t \geq t_0$. Furthermore, suppose that there exist constants $K > 0$ and $c > 0$, which can be chosen independently of t_0 such that

$$||y(t, t_0, x_0)|| \leq K e^{-c(t-t_0)} ||x_0|| \tag{3.2}$$

holds for any $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$, and all $t \geq t_0$, that is, the zero solution is uniformly exponentially stable.

Then there exists a function $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ such that

(i) *$\|x\| \leq V(t, x) \leq K\|x\|$ for all $x \in \mathbb{R}^n$, $t \in \mathbb{T}$;*

(ii) *$|V(t, x) - V(t, \tilde{x})| \leq K\|x - \tilde{x}\|$ for any fixed $t \in \mathbb{T}$ and all $x, \tilde{x} \in \mathbb{R}^n$;*

(iii) *for arbitrary fixed* $(t_0,x_0) \in \mathbb{T} \times \mathbb{R}^n$, *define function* $V_* : \mathbb{T} \to \mathbb{R}$, $V_*(t) := V(t,y(t))$, *where* $y(t) = y(t,t_0,x_0)$, *then the upper right Dini derivative of* V_* *defined by*

$$V_*^{\Delta}(t) := \begin{cases} \overline{\lim}_{h\searrow 0,\, h+t\in\mathbb{T}} \dfrac{V_*(t+h) - V_*(t)}{h}, & \text{if } t = \sigma(t), \\ \dfrac{V_*(\sigma(t)) - V_*(t)}{\mu(t)}, & \text{if } t < \sigma(t), \end{cases} \tag{3.3}$$

exists and the estimate

$$V_*^{\Delta}(t) \le \overline{\xi}_{-c}(t) V_*(t) < 0 \tag{3.4}$$

holds for all $t \ge t_0$;

(iv) V *is continuous from the right in* $(t,x) \in \mathbb{T} \times \mathbb{R}^n$, *that is,*

$$\lim_{(\tilde{t},\tilde{x})\to(t,x),\, \tilde{t}\ge t} \left| V(\tilde{t},\tilde{x}) - V(t,x) \right| = 0. \tag{3.5}$$

Proof. Fix $t \in \mathbb{T}$. Then we define

$$A_t := \{\tau \in [0,\infty) : t+\tau \in \mathbb{T}\}, \tag{3.6}$$

since $0 \in A_t$, we note that A_t is nonempty. Further, we define $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}$ by

$$V(t,x) := \sup_{\tau\in A_t} ||y(t+\tau,t,x)|| e^{c\tau}, \tag{3.7}$$

where $y : \mathbb{T} \to \mathbb{R}^n$ is the unique solution of (3.1) with initial value $y(t) = x$. Equation (3.7) is well defined since y exists for all initial values $(t,x) \in \mathbb{T} \times \mathbb{R}^n$ and all $t+\tau$, $\tau \in A_t$ by assumption and, as we will see in detail, it also follows from the assumptions that V is bounded.

It will be shown that V satisfies the properties (i)–(iv) asserted in the theorem.

It is obvious from the definition of V that $\|x\| \le V(t,x)$, while from (3.2) it follows for all $\tau \in A_t$ that

$$||y(t+\tau,t,x)|| \le K e^{-c(t+\tau-t)} \|x\| = K e^{-c\tau} \|x\|, \tag{3.8}$$

so

$$V(t,x) = \sup_{\tau\in A_t} ||y(t+\tau,t,x)|| e^{c\tau} \le \sup_{\tau\in A_t} K e^{-c\tau} \|x\| e^{c\tau} = K\|x\|. \tag{3.9}$$

Thus V satisfies property (i).

Let $t \in \mathbb{T}$ be fixed and let $x, \tilde{x} \in \mathbb{R}^n$ be arbitrary. By the triangle inequality and the superposition of solutions of a linear system, we have

$$
\begin{aligned}
|V(t,x) - V(t,\tilde{x})| &= \left| \sup_{\tau \in A_t} ||y(t+\tau,t,x)|| e^{c\tau} - \sup_{\tau \in A_t} ||y(t+\tau,t,\tilde{x})|| e^{c\tau} \right| \\
&= \left| \sup_{\tau \in A_t} e^{c\tau} \left(||y(t+\tau,t,x)|| - ||y(t+\tau,t,\tilde{x})|| \right) \right| \\
&\le \left| \sup_{\tau \in A_t} e^{c\tau} \left(||y(t+\tau,t,x) - y(t+\tau,t,\tilde{x})|| \right) \right| \\
&\le \sup_{\tau \in A_t} e^{c\tau} ||y(t+\tau,t,x-\tilde{x})|| = V(t,x-\tilde{x}) \le K||x-\tilde{x}||,
\end{aligned}
\tag{3.10}
$$

which shows that V is globally Lipschitz—continuous in x for fixed $t \in \mathbb{T}$, that is, satisfies property (ii).

We will prove property (iii) next. Let $(t_0, x_0) \in \mathbb{T} \times \mathbb{R}^n$ be arbitrary. We will distinguish two cases, $\sigma(t) = t$ and $\sigma(t) > t$ in the proof.

Suppose that $\sigma(t) = t$ and let $h \in A_t$. Then

$$
V_*^{\Delta}(t) = \overline{\lim_{h \searrow 0, h \in A_t}} \frac{V_*(t+h) - V_*(t)}{h},
\tag{3.11}
$$

where, by the uniqueness of solutions of (3.1) (see [4, Theorem 8.20, Page 324]), it follows that

$$
\begin{aligned}
V_*(t+h) &= \sup_{\tau \in A_{t+h}} ||y(t+h+\tau, t+h, y(t+h,t_0,x_0))|| e^{c\tau} \\
&= \sup_{\tau \in A_{t+h}} ||y(t+h+\tau, t, y(t,t_0,x_0))|| e^{c\tau} \\
&= \sup_{\tau \in \{\tau \in [h,\infty): t+\tau \in \mathbb{T}\}} ||y(t+\tau, t, y(t,t_0,x_0))|| e^{c\tau} e^{-ch} \\
&\le \sup_{\tau \in \{\tau \in [0,\infty): t+\tau \in \mathbb{T}\}} ||y(t+\tau, t, y(t,t_0,x_0))|| e^{c\tau} e^{-ch} \\
&= V_*(t) e^{-ch},
\end{aligned}
\tag{3.12}
$$

hence we have

$$
V_*^{\Delta}(t) \le \overline{\lim_{h \searrow 0}} V_*(t) \cdot \frac{e^{-ch} - 1}{h} = V_*(t) \cdot (-c) = V_*(t) \cdot \overline{\xi}_{-c}(t),
\tag{3.13}
$$

since $\mu(t) = 0$ by assumption.

Now suppose that $\sigma(t) > t$, in which case

$$
V_*^{\Delta}(t) = \frac{V_*(\sigma(t)) - V_*(t)}{\mu(t)}.
\tag{3.14}
$$

But

$$\begin{aligned} V_*(\sigma(t)) &= \sup_{\tau \in A_{\sigma(t)}} \|y(\sigma(t)+\tau, \sigma(t), y(\sigma(t), t_0, x_0))\| e^{c\tau} \\ &= \sup_{\tau \in A_{\mu(t)+t}} \|y(\mu(t)+t+\tau, \mu(t)+t, y(\mu(t)+t, t_0, x_0))\| e^{c\tau} \\ &= \sup_{\tau \in \{\tau \in [\mu(t),\infty): t+\tau \in \mathbb{T}\}} \|y(t+\tau, t, y(t, t_0, x_0))\| e^{c\tau} e^{-\mu(t)c} \\ &\le V_*(t) e^{-c\mu(t)}, \end{aligned} \tag{3.15}$$

so

$$V_*^{\Delta}(t) \le V_*(t) \cdot \frac{e^{-c\mu(t)} - 1}{\mu(t)} = V_*(t) \cdot \overline{\xi}_{-c}(t), \tag{3.16}$$

thus property (iii) follows from (3.13) and (3.16).

In order to show that V is continuous in the sense of (iv), let $t \in \mathbb{T}$, $x \in \mathbb{R}^n$ be fixed and choose $\varepsilon > 0$ arbitrary. Then $\delta_1 > 0$ and $\delta_2 > 0$ must be found such that

$$|V(\tilde{t}, \tilde{x}) - V(t,x)| < \varepsilon \tag{3.17}$$

holds for all

$$\tilde{t} = t + \nu, \quad \nu \in A_t,\ 0 \le \nu < \delta_1 \tag{3.18}$$

and all $\tilde{x} \in B_{\delta_2}(x)$, where $B_{\delta_2}(x)$ is the open ball centered on x of radius δ_2.

If $t \in \mathbb{T}$ is right scattered, we can always choose a suitable $\delta_1 > 0$ such that $\tilde{t} = t$ is the only point satisfying condition (3.18). Then the assertion follows since V is globally Lipschitz continuous in x for fixed $t \in \mathbb{T}$.

Let $t \in \mathbb{T}$ be right dense and $\tilde{t} = t + \nu$ for $\nu \in A_t$. Then (3.17) will follow from the inequalities

$$\begin{aligned} |V(\tilde{t}, \tilde{x}) - V(t,x)| = |V(t+\nu, \tilde{x}) - V(t,x)| &\le |V(t+\nu, \tilde{x}) - V(t+\nu, x)| \\ &\quad + |V(t+\nu, x) - V(t+\nu, y(t+\nu, t, x))| \\ &\quad + |V(t+\nu, y(t+\nu, t, x)) - V(t,x)|. \end{aligned} \tag{3.19}$$

Now we can use the Lipschitz continuity of V in x to estimate

$$|V(t+\nu, \tilde{x}) - V(t+\nu, x)| \le K\|x - \tilde{x}\|. \tag{3.20}$$

Hence

$$|V(t+\nu, x') - V(t+\nu, x)| < \frac{\varepsilon}{3} \tag{3.21}$$

for all $x' \in B_{\delta_2}(x)$ when $\delta_2 < \varepsilon/3K$.

In addition, the Lipschitz continuity of V in x also implies that

$$|V(t+\nu,x) - V(t+\nu,y(t+\nu,t,x))| \leq K\|x - y(t+\nu,t,x)\|, \tag{3.22}$$

where, by the right continuity of y, we have

$$\lim_{\nu\to 0,\,\nu\in A_t} y(t+\nu,t,x) = x, \tag{3.23}$$

hence there exists a $\delta_1' > 0$, such that

$$\|x - y(t+\nu,t,x)\| < \frac{\varepsilon}{3}\cdot\frac{1}{K} \tag{3.24}$$

for all $\nu \in A_t$ with $0 \leq \nu < \delta_1'$, which combines with (3.22) to give

$$|V(t+\nu,x) - V(t+\nu,y(t+\nu,t,x))| \leq \frac{\varepsilon}{3} \tag{3.25}$$

for all $\nu \in A_t$ with $0 \leq \nu < \delta_1'$.

In order to estimate the last term in (3.19), we use the uniqueness of solutions of our initial value problem (3.1) (see [4, Theorem 8.20, Page 324]) to obtain

$$y(t+\nu+\tau,t+\nu,y(t+\nu,t,x)) = y(t+\nu+\tau,t,x). \tag{3.26}$$

Define

$$\Delta V := |V(t+\nu,y(t+\nu,t,x)) - V(t,x)|, \tag{3.27}$$

that is,

$$\Delta V = \left| \sup_{\tau\in A_{t+\nu}} \|y(t+\nu+\tau,t,x)\|e^{c\tau} - \sup_{\tau\in A_t}\|y(t+\tau,t,x)\|e^{c\tau}\right|. \tag{3.28}$$

Rearrangement gives

$$\sup_{\tau\in A_{t+\nu}} \|y(t+\nu+\tau,t,x)\|e^{c\tau} = \sup_{\tau\in\{\tau\in[\nu,\infty):t+\tau\in\mathbb{T}\}} \|y(t+\tau,t,x)\|e^{c\tau}e^{-c\nu}, \tag{3.29}$$

so (3.28) can be rewritten as

$$\Delta V = \left| \sup_{\tau\in\{\tau\in[\nu,\infty):\tau+y\in\mathbb{T}\}} \|y(t+\tau,t,x)\|e^{c\tau}e^{-c\nu} - \sup_{\tau\in\{\tau\in[0,\infty):t+\tau\in\mathbb{T}\}} \|y(t+\tau,t,x)\|e^{c\tau}\right|. \tag{3.30}$$

Define $\alpha(\nu) := \sup_{\tau\in\{\tau\in[\nu,\infty):t+\tau\in\mathbb{T}\}} \|y(t+\tau,t,x)\|e^{c\tau}$ for $\nu \in A_t$. Then

$$\Delta V \leq |\alpha(\nu)e^{-c\nu} - \alpha(0)|. \tag{3.31}$$

Now $\alpha(\nu) \leq \alpha(0)$ for all $\nu \in A_t$, where $\nu \geq 0$ by definition. Moreover, $\alpha(\nu)$ is nonincreasing function in ν with

$$\lim_{\nu \to 0, \nu \in A_t} \alpha(\nu) = \alpha(0). \tag{3.32}$$

Hence, there exists a $\delta_1'' > 0$ such that $\Delta V < \varepsilon/3$ for all $\nu \in A_t$ with $0 \leq \nu \leq \delta_1''$.

Now choose $\delta_1 := \min\{\delta_1', \delta_1''\}$. For $\tilde{t} = t + \nu$ with $\nu \in A_t$, where $0 \leq \nu < \delta_1$ and $\tilde{x} \in B_{\delta_2}(x)$, combining all of the above estimates of the terms in (3.19), then gives

$$|V(\tilde{t}, \tilde{x}) - V(t, x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \tag{3.33}$$

which proves property (iv) and completes the proof of Theorem 3.1. □

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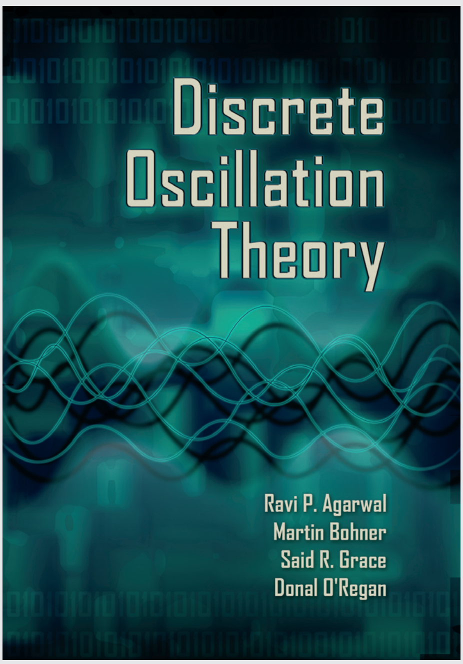
DISCRETE OSCILLATION THEORY

Ravi P. Agarwal, Martin Bohner, Said R. Grace, and Donal O'Regan



This book is devoted to a rapidly developing branch of the qualitative theory of difference equations with or without delays. It presents the theory of oscillation of difference equations, exhibiting classical as well as very recent results in that area. While there are several books on difference equations and also on oscillation theory for ordinary differential equations, there is until now no book devoted solely to oscillation theory for difference equations. This book is filling the gap, and it can easily be used as an encyclopedia and reference tool for discrete oscillation theory.

In nine chapters, the book covers a wide range of subjects, including oscillation theory for second-order linear difference equations, systems of difference equations, half-linear difference equations, nonlinear difference equations, neutral difference equations, delay difference equations, and differential equations with piecewise constant arguments. This book summarizes almost 300 recent research papers and hence covers all aspects of discrete oscillation theory that have been discussed in recent journal articles. The presented theory is illustrated with 121 examples throughout the book. Each chapter concludes with a section that is devoted to notes and bibliographical and historical remarks.

The book is addressed to a wide audience of specialists such as mathematicians, engineers, biologists, and physicists. Besides serving as a reference tool for researchers in difference equations, this book can also be easily used as a textbook for undergraduate or graduate classes. It is written at a level easy to understand for college students who have had courses in calculus.