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Essays on Continuous-Time Portfolio Optimization and Credit Risk

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St. Wendel, im Januar 2012 *Björn Bick*

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Part I

Zusammenfassung

Zusammenfassung

Björn Bick

1 Thematische Einordnung

Die vorliegende Arbeit beschäftigt sich mit der zeitstetigen Portfoliooptimierung sowie mit Themen aus dem Bereich des Kreditrisikos. Das Ziel der Portfoliooptimierung ist es, zu einem gegebenen Anfangskapital die bestmöglichen Konsum- und Investmentstrategien zu finden. Dies bedeutet, dass ein Investor zu jedem Zeitpunkt festlegen muss, wie viel Vermögen er konsumiert und wie viele Anteile er von welchem Wertpapier hält. In dieser Arbeit wird dabei vor allem der Einfluss von Einkommen auf diese Entscheidungen untersucht. Da einerseits jedoch der zukünftige Einkommensstrom vom Zufall bestimmt ist und es andererseits keine Finanzprodukte gibt, die diesen replizieren können, stellt die Einbindung von Einkommen in die Portfoliooptimierung ein großes Problem dar. Es führt dazu, dass die Annahmen eines vollständigen Marktes nicht weiter gelten, so dass die Standardmethoden zur Lösung nicht angewendet werden können. Diese Arbeit analysiert mehrere Ausprägungen dieses Problems und geht auf verschiedene Verfahren zur Lösung ein. Weiterhin untersucht diese Studie den Einfluss des Kreditrisikos einer Firma auf die jeweilige Firmenrendite. Dabei wird vor allem auf eine Anomalie, die bereits umfassend in der Literatur diskutiert wurde, Bezug genommen. Diese Anomalie besagt, dass Firmen mit hohen Ausfallwahrscheinlichkeiten geringere Renditen erwirtschaften als Firmen mit kleineren Ausfallwahrscheinlichkeiten. Eine weitere Frage, die in den Bereich des Kreditrisikos fällt, ist die Frage, inwieweit Modelle dazu in der Lage sind, strukturierte Produkte zu bewerten und abzusichern. Diese Arbeit versucht Antworten darauf zu geben.

Mehrere Literaturstränge bilden die Grundlage für diese Arbeit. Im Bereich der zeitstetigen Portfoliooptimierung unterscheidet man im Wesentlichen zwei mögliche Methoden zur Lösung des Problems. Ein Lösungsansatz geht auf die bahnbrechende Arbeit von Merton (1969) zurück, in der Methoden der stochastischen Steuerung verwendet werden. Einen anderen Ansatz liefert die sogenannte Martingalmethode, die von Pliska (1986) und Cox und Huang (1989) eingeführt wurde. Viele weitere Aufsätze wie Samuelson (1969), Kim und Omberg (1996), Sørensen (1999), Campbell und Viceira (2001), Brennan und Xia (2002) und Liu (2007) betrachten ebenfalls Portfolioprobleme. Im Gegenstaz zu den bereits erwähnten Arbeiten berücksichtigen Hakansson

(1970) und Bodie, Merton und Samuelson (1992) Arbeitseinkommen, wobei sie entweder annehmen, dass das Einkommen deterministisch ist oder durch die im Markt gehandelten Wertpapiere repliziert werden kann. Einige kürzlich erschienenen Arbeiten, darunter die Studien von Cocco, Gomes und Maenhout (2005), Van Hemert (2010) und Koijen, Nijman und Werker (2010), modellieren nicht-replizierbares Einkommen und erhalten dadurch einen unvollständigen Markt. Die beiden bereits erwähnten Verfahren der stochastischen Steuerung und der Martingalmethode liefern in diesem Fall keine Lösung, so dass sich diese Arbeiten numerischer Methoden bedienen, um das Problem zu lösen. Diese numerische Verfahren sind dabei meistens Anwendungen von Finite-Differenzen-Methoden, die auch in Brennan, Schwartz und Lagnado (1997), Munk (2000), Yao und Zhang (2005) und Munk und Soerensen (2010) verwendet werden. Neben den Finite-Differenzen-Methoden werden in den Arbeiten von Kogan und Uppal (2000), Viceira (2001) und Das und Sundaram (2002) noch weitere Verfahren vorgestellt, um Portfolioprobleme in einem unvollständigen Markt zu lösen. Ein weiterer wichtiger Beitrag zur Portfolioliteratur sind die beiden Artikel von Karatzas, Lehoczky, Shreve und Xu (1991) und Cvitanić und Karatzas (1992), die ein theoretisches Konzept vorschlagen, mit dem Probleme aus unvollständigen Märkten in Probleme in künstlich vervollständigten Märkten transformiert und gelöst werden. Die Arbeit von Liu (2007) betrachtet ein Portfolioproblem ohne Einkommen, nimmt aber im Gegenzug stochastische Marktvariablen an. Kraft (2005) analysiert eine Optimierungsaufgabe mit stochastischer Volatilität, wobei er für den Fall, dass der Investor ausschließlich sein Endvermögen maximieren will, geschlossene Lösungen berechnen kann. Neben dem Aktienmarkt und dem Geldmarktkonto beziehen Liu und Pan (2003) Derivate in ihre Überlegungen mit ein. Sie zeigen, dass der Investor durch diese zusätzliche Investmentmöglichkeit einen höheren Nutzen hat.

Weitere für diese Dissertation relevante Literatur befasst sich mit der Frage, welche Faktoren einen Einfluss auf die Renditen von Firmen haben. Im Capital-Asset-Pricing-Model (CAPM), das von Sharpe (1964), Lintner (1965) und Mossin (1966) unabhängig voneinander entwickelt wurde, wird unter anderem gezeigt, dass allein das systematische Risiko gepreist ist und dass das unsystematische Risiko keinen Einfluss auf die Renditen haben sollte, da angenommen wird, dass die Investoren sehr gut diversifiziert sind. In der Realität sind jedoch viele Anleger nicht genügend diversifiziert, so dass unsystematisches Risiko durchaus in deren Investmententscheidungen einfließt. Merton (1987) und Malkiel und Xu (2004) zeigen dabei wie unsystematisches Risiko gepreist werden kann. Die empirischen Erkenntnisse, inwieweit sich unsystematisches Risiko auf die Renditen auswirkt, sind dabei jedoch unterschiedlich. Während die Mehrheit der Studien (u.a. Spiegel und Wang (2005) und Fu (2009)) einen positive Abhängigkeit feststellen, belegen Ang, Hodrick, Xing und Zhang (2006) und Ang und Zhang (2009) eine negative Abhängigkeit. Wie bereits in der Studie von Fama und French (1993) gezeigt, kann das systematische Risiko alleine nicht die Renditen einer Firma erklären. So zeigen Chan und Chen

(1991), dass das Ausfallrisiko einer Firma nicht diversifizierbar ist, so dass Investoren eine Risikoprämie für Ausfallrisiko verlangen. Einige Forschungsarbeiten (Dichev (1998), Griffin und Lemmon (2002) und Campbell, Hilscher und Szilagyi (2008)) finden dabei einen negativen Zusammenhang zwischen Firmenrenditen und den relevanten Ausfallrisiken, wohingegen Vassalou und Xing (2004) und Da und Gao (2008) einen positiven Zusammenhang dokumentieren.

Auch existieren mehrere Arbeiten, die sich mit der Bewertung und Absicherung von strukturierten Produkten befassen und somit die vorliegende Studie beeinflusst haben. Zur Modellierung der zur Bewertung relevanten Portfolioausfallwahrscheinlichkeiten verwendet Li (2000) eine Copula. Dieses Modell gehört zur Klasse der sogenannten Bottom-up Modelle und wird in abgewandelter Form auch in den Beiträgen von Andersen, Sidenius und Basu (2003), Hull und White (2004) und Schloegl und O'Kane (2005) genutzt. Die umfassende Studie von Burtschell, Gregory und Laurent (2009) vergleicht mehrere Copula-Modelle und führt die jeweiligen Eigenschaften auf. Neben diesen Verfahren, können auch Top-down Modelle angewendet werden, um die Verlustverteilung des Portfolios zu erzeugen. Diese Methode konzentriert sich dabei auf die Portfolioebene und schenkt den zugrundeliegenden Einzelkrediten keine Beachtung. Vertreter dieses Verfahrens sind Brigo, Pallavicini und Torresetti (2006), Longstaff und Rajan (2008) und Errais, Giesecke und Goldberg (2010). Die Studien von Cont und Kan (2011) und Cousin, Crépey und Kan (2011) untersuchen, inwieweit Vertreter dieser beiden Modellklassen in der Lage sind, Strategien zu liefern, welche zur Risikoabsicherung dienen.

2 Struktur und Inhalt der Arbeit

Die vorliegende kumulative Dissertation besteht aus insgesamt vier Forschungspapieren:

- *Solving Constrained Consumption-Investment Problems by Simulation of Artificial Market Strategies* von Björn Bick, Holger Kraft und Claus Munk,
- *Consumption-Portfolio Choice with Unspanned Labor Income and Stochastic Volatility* von Björn Bick,
- *Default and Idiosyncratic Risk Anomalies Revisited* von Björn Bick, Holger Kraft, Christian Hirsch und Yildiray Yildirim,
- *Hedging Structured Credit Products During the Credit Crunch* von Björn Bick und Holger Kraft.

In dem Forschungspapier *Solving Constrained Consumption-Investment Problems by Simulation of Artificial Market Strategies* wird ein zeitstetiges Konsum-Investment Problem eines Investors untersucht, der seinen Nutzen über den gesamten Lebenszyklus maximieren will. Es scheint, dass die entsprechenden optimalen Konsum- und Investmentstrategien nicht in geschlossener Form berechnet werden können, sobald man ein realistisches Problem mit Investmentbeschränkungen und unvollständigen Märkten betrachtet. Gewöhnliche numerische Verfahren, die oft zur Lösung solcher Probleme verwendet werden, sind in diesem Fall sehr schwer oder unmöglich zu implementieren. In unserer Studie wird eine relativ einfache numerische Methode vorgeschlagen, die die abstrakte Idee eines künstlich vervollständigten und unbeschränkten Marktes mit Monte-Carlo Simulationen und einer Optimierungsroutine kombiniert. Diese Methode liefert eine Obergrenze für den Vermögensverlust im Vergleich zu der unbekannten optimalen Strategie.

Der Ausgangspunkt für diese Analyse bildet ein Investor, der die Möglichkeit hat sein Vermögen in einen Aktienindex oder in ein Geldmarktkonto zu investieren. Der Investor muss dann festlegen, wie viele Anteile er von dem jeweiligen Wertpapier wann halten will. Weiterhin muss er darüber entscheiden, welchen Anteil seines Vermögens er wann konsumieren will. Des Weiteren erhält die Person einen exogen vorgegebenen stochastischen Einkommensstrom während seiner Arbeitskarriere. Da dieses Arbeitseinkommen, wie der Aktienmarkt, zufälligen Schwankungen unterworfen ist und nicht perfekt mit dem Aktienmarkt korreliert ist, entsteht ein unvollständiges Optimierungsproblem. Weiterhin wird angenommen, dass der Investor keine Leerverkäufe tätigen und keine Kredite aufnehmen darf, so dass der prozentuale Anteil des Vermögens, das in die Aktie investiert wird, immer zwischen 0 und 100% liegen muss. Diese Annahme sowie die Annahme eines nicht replizierbaren Einkommensstroms sorgen dafür, dass das Problem im Allgemeinen nicht geschlossen gelöst werden kann. Unsere Herangehensweise an das Problem basiert auf der Idee von künstlichen Finanzmärkten. Dabei kann ein

Konsum-Investment Problem mit Investmentbeschränkungen im unvollständigen Markt in eine Familie von künstlich vervollständigten Konsum-Investment Problemen ohne Investmentbeschränkungen eingebunden werden. In unserem speziellen Fall bestimmen das Sharpe-Ratio eines künstlichen Wertpapiers sowie eine kontrollierte Störung des risikofreien Zinses und der erwarteten Aktienrendite den künstlichen Markt. Dabei wird das künstliche Wertpapier so konzipiert, dass der Einkommensstrom mit einer geeigneten Handelsstrategie repliziert werden kann. Im Allgemeinen können sowohl das Sharpe-Ratio als auch die kontrollierte Störung stochastische Prozesse sein. Für eine solche Wahl zeigen Karatzas, Lehoczky, Shreve und Xu (1991) und Cvitanić und Karatzas (1992) sogar, dass die optimalen Konsum- und Investmententscheidungen im wahren unvollständigen Markt identisch mit den Strategien sind, die man im schlechtesten aller künstlich vervollständigten Märkten erhält. Dabei handelt es sich allerdings nur um ein theoretisches Ergebnis und es wird nicht gezeigt wie man die entsprechenden Strategien im schlechtesten dieser künstlichen Märkten erhält. In unserer Analyse konzentrieren wir uns hingegen auf eine Unterfamilie aus "einfachen" künstlichen Märkten. In dieser Unterfamilie sind sowohl das Sharpe-Ratio als auch die kontrollierte Störung deterministische Funktionen, die von wenigen Parametern abhängen. Nach Liu (2007) können nun geschlossene Lösungen in dieser Unterfamilie berechnet werden. Indem man die Wertfunktion über die Parameter minimiert, sind wir in der Lage, den schlechtesten aller einfachen künstlichen Märkte zu finden. Dadurch erhalten wir auch eine Obergrenze des Nutzens, der in dieser Unterfamilie erreicht werden kann. Die aus diesen künstlichen Märkten resultierenden Strategien dürfen jedoch im Allgemeinen nicht im wahren unvollständigen Markt verwendet werden. Dies liegt zum einen daran, dass ein Teil des Vermögens in das künstliche Wertpapier investiert wird und zum anderen daran, dass die Strategien die Investmentbeschränkungen verletzen könnten. Aus diesem Grund passen wir die Strategien an, indem wir die Investition in das künstliche Wertpapier vernachlässigen und die Strategien entsprechend ändern, wenn Schranken verletzt werden. Auf diesem Weg erhält man nun zulässige Strategien, die jeweils durch eine geringe Anzahl von Konstanten parametrisiert werden. Mit Hilfe von Monte-Carlo Simulationen können wir den erwarteten Nutzen jeder Strategie berechnen. Durch die Anwendung einer Optimierungsroutine können wir dann eine zulässige Strategie im wahren, unvollständigen Markt finden. Diese Strategie ist dann die beste aller Strategien, die aus der angesprochenen Unterfamilie abgeleitet wurden. In einem letzten Schritt werden der erwartete Nutzen dieser Strategie und die Obergrenze des Nutzens verglichen, so dass wir eine Obergrenze für den Wohlfahrtsverlust erhalten.

Wir zeigen, dass die Obergrenzen der Wohlfahrtsverluste sowohl von der Korrelation zwischen Einkommen und Aktienrendite sowie vom Verhältnis aus anfänglichen finanziellem Vermögen und anfänglichem Einkommen abhängen. Für unsere Referenzparameter und einen Zeithorizont von 50 Jahren finden wir Obergrenzen, die in den meisten Fällen unter 0.5% des Gesamtvermögens liegen und im schlimmsten Fall 1.1% beträgt. Das Gesamtvermögen ist dabei die Summe aus finanziellem Vermögen und dem erwarteten diskontierten Wert aller zukünftigen Einkommenszahlungen. Wir zeigen, dass die Strategien aus unserer Methode genauso gut sind wie ein standardisiertes Finite-Differenzen Verfahren, das auf einem sehr feinen Gitter implementiert wurde. Weiterhin übertrifft unsere Methode das Finite-Differenzen Verfahren, sobald dies auf einem gröberen Gitter implementiert wird, was für alle höher-dimensionalen Probleme nötig wäre. Dies liegt vor allem daran, dass alle numerischen Methoden am sogenannten "Fluch der Dimensionen" leiden. Im Gegensatz zu numerischen Finite-Differenzen Methoden liefert unsere Herangehensweise weiterhin geschlossene Lösungen, die leicht zu interpretieren sind. Zum Schluss belegen wir die Effizienz des vorgeschlagenen Verfahrens, indem wir es bezüglich der Robustheit unterschiedlicher Parameter sowie einer Erweiterung auf stochastische Zinsen testen.

In dem Forschungspapier *Consumption-Portfolio Choice with Unspanned Labor Income and Stochastic Volatility* betrachten wir ebenfalls ein Konsum-Investment Problem mit exogenem Arbeitseinkommen. Während die vorige Arbeit von einer konstanten Volatilität im Aktienmarkt ausging, wird hier nun die realistischere Annahme der stochastischen Volatilität gemacht. Wir untersuchen, inwieweit und in welchem Ausmaß sich Arbeitseinkommen und stochastische Volatilität auf die Konsum- und Investmententscheidungen des Investors über die komplette Lebensdauer auswirken. Im Falle eines vollständigen Marktes, ein Markt, in dem das Einkommen durch die vorliegenden Wertpapiere repliziert werden kann, stellen wir geschlossene Lösungen zur Verfügung. Weiterhin diskutieren und vergleichen wir Lösungsmöglichkeiten in unvollständigen Märkten.

In unserer Studie untersuchen wir wiederrum eine Person, die ihre Entscheidungen in stetiger Zeit macht und den erwarteten Nutzen aus Konsum und Endvermögen maximiert. Der stochastische Volatilitätsprozess und der Aktienmarkt werden nach der Arbeit von Heston (1993) modelliert. Im Gegensatz zur vorherigen Studie, kann der Investor nicht nur in den Aktienmarkt und das Geldmarktkonto investieren, sondern darf zusätzlich sein Vermögen in ein Derivat auf den Aktienmarkt anlegen. Dieses Derivat ermöglicht es dem Investor, sich gegen das vorliegende Volatilitätsrisiko abzusichern (siehe auch Liu und Pan (2003)). Dennoch resultiert die Einbindung des Derivats nicht automatisch in einem vollständigen Markt, da das risikobehaftete Einkommen im Allgemeinen nicht repliziert werden kann. Deshalb werden in unserer Analyse zwei Szenarien untersucht. Im ersten Szenario gehen wir davon aus, dass der Einkommensstrom vollständig durch eine entsprechende Handelsstrategie repliziert werden kann und dass keine Investmentbeschränkungen vorliegen. Nach Merton (1969) können wir dann geschlossene Lösungen für den optimalen Konsum und die optimale Derivate- und Aktienmarktinvestition berechnen. Die gefundenen Strategien können jeweils in einen spekulativen Anteil und in Anteile, die zur Absicherung des Volatilitäts- und Einkommensrisikos dienen, aufgeteilt werden. Wir zeigen den großen Einfluss des menschlichen Vermögens auf die entsprechenden Investment- und Konsumstrategien. Die expliziten Lösungen zeigen weiterhin, dass die Position im Derivat vor allem von der Sensitivität des Derivats bezüglich der Volatilität abhängt. Dies macht vor dem Hintergrund Sinn, dass das Derivat vor allem deshalb eingeführt wurde, um die gewünschte Position bezüglich Volatilitätsrisiko einnehmen zu können. Unter der Annahme realistischer Parameter werden die gewonnenen Ergebnisse anhand einer Lebenszeitstudie illustriert. Dabei sehen wir, dass unser Model in der Lage ist, verschiedene Muster zu replizieren, die auch in anderen Lebenszyklusstudien gefunden wurden. Im zweiten Szenario nehmen wir nicht-replizierbares Einkommen an, was einen unvollständigen Markt nach sich zieht. Des Weiteren ist es dem Investor nicht mehr erlaubt, zukünftiges Arbeitseinkommen als Sicherheit für eine mögliche Kreditaufnahme einzusetzen. Um dieses Problem zu lösen, wenden wir sowohl die Methode, die im ersten Forschungspapier entwickelt wurde, als auch ein Finite-Differenzen Verfahren an. Die aus der Methodik des künstlichen Vervollständigen stammenden Obergrenzen der Wohlfahrtsverluste sind über einen Zeitraum von 40 Jahren im schlimmsten Fall 6.7%. Man beachte, dass es sich hierbei allerdings nur um Obergrenzen handelt. Unsere Resultate deuten darauf hin, dass die tatsächlichen Verluste, die durch diese Strategie erzielt werden, geringer sind. Diese Vermutung wird dadurch bestärkt, dass die Strategien aus der Finite-Differenzen Methode für sämtliche Tests weitaus schlechter abschneiden. Waren in unserer ersten Forschungsarbeit, die Resultate beider Methoden noch annähernd gleich, wird in dem zweiten Forschungspapier der bereits angesprochene "Fluch der Dimension" deutlich, da wir nun durch die stochastische Volatilität eine zusätzliche Dimension erhalten. Die Ergebnisse scheinen darauf hinzuweisen, dass für dieses Problem die Grenze der numerischen Verfahren, die auf der Diskretisierung des Problems beruhen, bereits erreicht ist, wohingegen das Verfahren der künstlich vervollständigten Märkte auch auf höhere Dimensionen angewendet werden kann.

Im Gegensatz zu den beiden ersten Arbeiten befassen wir uns in den restlichen Studien mit Themen aus dem Bereich des Kreditrisikos. Im Forschungspapier *Default and Idiosyncratic Risk Anomalies Revisited* greifen wir zwei Anomalien auf, die bereits ausführlich in der Finanzliteratur dokumentiert und diskutiert wurden. Die erste Anomalie, die wir untersuchen, bezieht sich auf folgendes Phänomen: Firmen mit hohen Ausfallwahrscheinlichkeiten erwirtschaften geringere Renditen als Firmen mit entsprechenden kleineren Wahrscheinlichkeiten. Diese Erkenntnis widerspricht der Intuition, dass höheres Risiko durch eine höhere Rendite honoriert wird. Im weiteren Verlauf werden wir diese Anomalie als Ausfallanomalie bezeichnen. Als Beispiel dient eine Studie von Campbell, Hilscher und Szilagyi (2008), die diese Anomalie empirisch belegen. Neben der bereits erwähnten Studie gibt es noch ein Vielzahl anderer Untersuchungen, die diese Anomalie bestätigen können. Zusätzlich betrachten wir eine weitere Anomalie bezüglich des unsystematischen Risikos. Unter den Voraussetzungen des Capital-Asset-Pricing-Model (CAPM) sollte nur systematisches Risiko gepreist werden, was hauptsächlich darauf zurückzuführen ist, dass alle Investoren laut Annahmen des CAPM diversifizierte Portfolios halten. Mehrere Studien wie Merton (1987) und Malkiel und Xu (2004) zeigen jedoch, dass unsystematisches Risiko sehr wohl gepreist werden kann.

In unserem Forschungspapier benutzen wir drei verschiedene Maße für Ausfallrisiko, um die oben aufgeführten Anomalien im Zeitraum von 2001 bis 2010 zu untersuchen. Unsere Analyse bezieht sich dabei auf mehr als 700 Firmen. Zum einen berechnen wir reale Ausfallwahrscheinlichkeiten mit der Methode von Campbell, Hilscher und Szilagyi (2008) und analysieren, inwieweit dieser Faktor im betrachteten Zeitraum gepreist wurde. Als weiteren Faktor berücksichtigen wir risikoneutrale Ausfallwahrscheinlichkeiten, die wir aus Credit Default Swap Preisen ableiten. Diese Faktoren könnten auch aus den Risikozuschlägen von Bondpreisen extrahiert werden. Nach Longstaff, Mithal und Neis (2005) führen jedoch CDS Preise zu einer besseren Abschätzung von risikoneutralen Ausfallwahrscheinlichkeiten. Als dritten Faktor schlagen wir die Ausfallrisikoprämie vor, die als Quotient aus risikoneutraler Ausfallwahrscheinlichkeit und realer Ausfallwahrscheinlichkeit berechnet werden kann. Auf der Basis eines dieser Ausfallrisikomaße sortieren wir unsere Beobachtungen in Portfolios. Um genauer zu sein, bilden wir in jedem Quartal fünf Portfolios, die auf den Median des jeweiligen Faktors des vorherigen Quartals sortiert wurden. Das erste Portfolio setzt sich dann aus den Firmen mit den kleinsten Ausfallmaßen, das fünfte aus den Firmen mit den größten Ausfallmaßen zusammen. Danach berechnen wir die nach Marktwerten gewichteten Renditen der Portfolios und führen Fama-French Regressionen auf die entsprechenden täglichen Renditen durch. Um statistische Aussagen zu treffen, dokumentieren wir die Durchschnitte der Zeitreihen sowie die t-Statistiken aller Koeffizienten.

Wie in Campbell, Hilscher und Szilagyi (2008) beobachten wir ebenfalls die Ausfallanomalie, sobald wir die realen Ausfallwahrscheinlichkeiten als zusätzlichen Faktor verwenden. Der Effekt dieser Anomalie wird etwas kleiner, verschwindet jedoch nicht komplett, wenn wir bezüglich der risikoneutralen Ausfallwahrscheinlichkeiten sortieren. Wird jedoch die Ausfallrisikoprämie als Approximation des Ausfallrisikos verwendet, verschwindet die Anomalie. In einem letzten Schritt setzen wir diese Erkenntnisse mit der weiteren Anomalie bezüglich des unsystematischen Risikos in Verbindung. Dabei dokumentieren wir zuerst, dass diese Anomalie in unserem Datensatz präsent ist, solange wir keinen der oben genannten Faktoren berücksichtigen. Des Weiteren zeigen wir, dass die Anomalie nicht mehr signifikant ist, wenn die Ausfallrisikoprämien mit einbezogen werden. Sortiert man hingegen auf die realen und risikoneutralen Ausfallwahrscheinlichkeiten, ist die Anomalie weiterhin signifikant.

Das vierte Forschungspapier *Hedging Structured Credit Products During the Credit Crunch* analysiert nun nicht mehr die Ausfallrisikoprämien von einzelnen Firmen, sondern befasst sich mit der Bewertung und der Absicherung (engl. Hedging) von strukturierten Kreditprodukten, die auf einem Portfolio mehrerer Firmen beruhen und somit verschiedene Ausfallrisiken gleichzeitig verbriefen. In unserer Analyse konzentrieren wir uns dabei auf Collateralized Debt Obligations (CDO).

Die andauernde Finanzkrise hat den Bedarf eines guten Risikomanagement von strukturierten Kreditprodukten deutlich gemacht. Daher nehmen wir in dieser Studie die Position eines Risikomanagers ein und stellen uns eine entscheidende Frage: War es in der Finanzkrise 2008 möglich, Positionen in unterschiedlichen CDO Tranchen mit den existierenden Modellen abzusichern? Wir analysieren mehrere Bottom-up und Top-down Kreditportfoliomodelle und berechnen die entsprechenden Delta-Hedging-Strategien, indem wir entweder einen Indexkontrakt oder ein Portfolio aus einzelnen Credit Default Swaps (CDS) als Hedging-Instrument verwenden. Wir berechnen die resultierenden Gewinn- und Verlustprofile und beurteilen dadurch die Effizienz der Strategien.

In der Summe untersuchen wir insgesamt 10 Modelle, die entweder zur Klasse der Bottom-up oder Top-down Modelle gehören. Die Gemeinsamkeit aller Bottom-up Modelle besteht darin, dass die Verlustverteilung des Kreditportfolios durch die Kombination der einzelnen Ausfallverteilungen mit einer bestimmten Korrelationsstruktur erzeugt wird. Wir konzentrieren uns dabei auf Ein-Faktor-Copula-Modelle, die vor allem in der Praxis oft verwendet werden. Wie der Name bereits vermuten lässt, wird die komplette Korrelationsstruktur durch einen gemeinsamen Faktor bestimmt. Die auf diesen gemeinsamen Faktor bedingten marginalen Ausfallwahrscheinlichkeiten sind dabei der Hauptinput des Verfahrens. Das prominenteste Beispiel einer Copula ist die Gauß-Copula, die auf Li (2000) zurückgeht. Da diese Copula allerdings mehrere Nachteile hat, berücksichtigen wir auch die Double-t Copula, Student-t Copula und Clayton Copula. Auch wenn diese Modelle einige Defizite der Gauß-Copula beseitigen, sind sie immer noch statische Verfahren. Dies bedeutet, dass sich die Korrelationsstruktur während der Laufzeit nicht ändert. Infolgedessen sind diese Modelle nicht fähig, eine von der Vergangenheit abhängige und dynamische Korrelation zu generieren. Diese als Ansteckungseffekt bezeichnete Eigenschaft wurde in empirischen Studien (siehe z.B. Longstaff und Rajan (2008)) belegt. Um die genannten Copula-Modelle an Marktdaten zu kalibrieren, folgen wir der Marktpraxis, so dass jede Tranche durch einen eigenen Korrelationsparameter repräsentiert wird. Da wir ein heterogenes Portfolio von Firmen annehmen, können wir die Sensitivitäten der Tranchen bezüglich der einzelnen CDS Kontrakte berechnen. Im Gegensatz zu den Bottom-up Verfahren, betrachten Top-down Modelle zuerst die Verlustverteilung des Gesamtportfolios ohne diese mit den einzelnen Firmen in Verbindung zu setzen. Die Top-down Modelle liefern also keinerlei Informationen, welche Firma nun eigentlich einen Verlust im Portfolio ausgelöst hat. Solange wir lediglich strukturierte Produkte wie CDO's bewerten wollen, stellt dies einen Vorteil dar, da die fehlenden Informationen über die Einzelwahrscheinlichkeiten nicht benötigt werden. Die fehlende Verbindung zwischen dem Verlustprozess auf der Portfolioebene und den einzelnen Ausfallwahrscheinlichkeiten wird allerdings zum großen Problem, sobald CDO Tranchen durch Produkte abgesichert werden sollen, die nur das Ausfallrisiko einer bestimmten Firma verbriefen (wie z.B. CDS Papiere). Um dieses Problem zu bewältigen, wenden wir das von Giesecke, Goldberg und Ding (2011) eingeführte

“Random Thinning“ an. Durch dieses Verfahren sind wir in der Lage, die Ausfallwahrscheinlichkeiten auf Portfolioebene den jeweiligen Firmen zuzuordnen. In unserer Studie fokussieren wir uns auf drei solcher Modelle, die bereits vorher von Brigo, Pallavicini und Torresetti (2006), Longstaff und Rajan (2008) und Errais, Giesecke und Goldberg (2010) untersucht wurden.

Wir finden heraus, dass die Student-t Copula die besten Ergebnisse liefert. Die betrachteten Top-down Modelle sind ebenfalls in der Lage, effektive Hedgingstrategien zu liefern, haben jedoch erhebliche Probleme, die Equity Tranche abzusichern. Indem wir die absoluten Hedgefehler der einzelnen Modelle in der Finanzkrise betrachten, können wir jedoch folgern, dass die einzelnen Strategien bei weitem keine zufriedenstellenden Ergebnisse liefern. Weiterhin vergleichen wir die Leistung der Copula-Modelle, die entweder mit dem Konzept der impliziten Korrelationen oder dem Konzept der Basiskorrelationen kalibriert wurden. Wir schlussfolgern, dass die mit Basiskorrelationen kalibrierten Copulas stabilere und bessere Resultate liefern. Zu dieser Erkenntnis gelangen auch Ammann und Brommundt (2009) und Cousin, Crépey und Kan (2011), die sich allerdings nur auf die Gauß-Copula konzentrieren. Unsere Haupterkenntnis besteht darin, dass sowohl Index- als auch CDS Kontrakte keine geeigneten Instrument zum Absichern von CDO Tranchen sind. Wir zeigen, dass dies vor allem daran liegt, dass das Verhalten von CDO Tranchen und den zugrundeliegenden Indexprodukten in unserem Datensatz sehr unterschiedlich war.

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Part II

Research Papers

Solving Constrained Consumption-Investment Problems by Simulation of Artificial Market Strategies

Björn Bick, Holger Kraft, Claus Munk

This paper was presented at

- 16th Annual Meeting of the German Finance Association (DGF), 2009, Frankfurt am Main (9.10. - 10.10.),
- 13th Conference of the Swiss Society for Financial Market Research (SGF), 2010, Zürich (19.3. - 19.3.),
- 6th World Congress of the Bachelier Finance Society, 2010, Toronto (22.6. - 26.6.),
- European Finance Association Annual Meeting, 2010, Frankfurt am Main, Germany (25.8. - 28.8.).

Summary. Utility-maximizing consumption and investment strategies seem impossible to find in closed form in realistic settings involving portfolio constraints, incomplete markets, and potentially a high number of state variables. Standard numerical methods are hard or impossible to implement for such cases. We propose a relatively simple numerical procedure that combines the abstract idea of artificial unconstrained complete markets, well-known closed-form solutions in affine or quadratic return models, straightforward Monte Carlo simulation, and a standard iterative optimization routine. Our method provides an upper bound on the wealth-equivalent loss compared to the unknown optimal strategy, and it facilitates our understanding of the economic forces at play by building on closed-form expressions for the strategies considered. We illustrate and test our method on the life-cycle problem of an individual who receives an unspanned labor income rate and faces a no borrowing and a no short-selling constraint. The upper loss bound is small and our method has a precision comparable to a standard finite difference type solution of the problem, but the latter solution method does not generalize to higher dimensions as our method does.

1 Introduction

Utility-maximizing consumption and investment strategies are notoriously difficult to compute when markets are incomplete and strategies are constrained. Closed-form solutions are only known in unrealistic special cases. Grid-based numerical dynamic programming is frequently used but suffers from the curse of dimensionality. The existing alternative numerical methods are complex. Little is known about the precision of any of these numerical methods. This paper introduces a simple numerical approach combining (i) the idea of artificially unconstrained and complete markets, (ii) well-known closed-form solutions for unconstrained consumption/portfolio problems in affine or quadratic settings, (iii) straightforward Monte Carlo simulation to evaluate various simple consumption and investment strategies, and (iv) a standard iterative optimization routine. We will henceforth refer to our approach as SAMS, short for *Simulation of Artificial Markets Strategies*.

In addition to its relative simplicity, SAMS has a number of attractive features. First, SAMS applies to high-dimensional models as long as the relevant state variables have affine or quadratic dynamics which is assumed in most existing models. Second, the consumption and investment strategy produced by SAMS is given in closed form (involving some parameters that we optimize over as a part of the approach) and is thus easy to interpret. Third, in contrast to the mainstream numerical methods, SAMS also delivers an upper bound on the welfare loss the individual incurs by using the strategy suggested by our procedure instead of the unknown optimal strategy.

We document the performance of SAMS for the classical life-cycle problem where a power-utility individual receives an unspanned labor income stream, has access to trade in a risk-free asset and a stock, but faces a strict borrowing constraint so that he always has to invest between 0 and 100% of current financial wealth in each asset. This is a prime example of a problem with no closed-form solution, but with incomplete markets and a portfolio constraint that for realistic parameter values is binding in a substantial part of the (time,state)-space.[1] SAMS produces a relatively simple closed-form near-optimal consumption and investment strategy. The upper bound on the welfare loss from following this strategy depends on the assumed income-stock correlation and the ratio of initial financial wealth to initial annual income. In our benchmark parametrization of the model with a 50-year time horizon, the upper bound is below 0.5% of total wealth for most of the combinations of the correlation and the wealth-income ratio and the highest upper bound is 1.1%. A standard solution method for life-cycle utility maximization problems is grid-based numerical dynamic programming; in fact, several papers formulate such problems directly in discrete time and then discretize the relevant state variables when implementing the dynamic programming solution. For our specific problem, we show that the consumption and investment strategy derived by our method is as good as the strategy found by a standard dynamic programming approach implemented on a very fine grid. Moreover, our method outperforms the dynamic programming algorithm implemented on a coarser grid which would be necessary to use for a problem with one or two additional state variables.

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[1] Optimal unconstrained strategies have been derived in closed form for some settings with negative exponential utility and normally distributed income (Svensson and Werner (1993), Henderson (2005), Christensen, Larsen and Munk (2011)) and for settings with deterministic or spanned labor income (Hakansson (1970), Bodie, Merton and Samuelson (1992)).

Concerning the economic properties of the solution, we demonstrate that the optimal fraction of financial wealth invested in the stock *during* retirement, where income is assumed risk-free, depends on the correlation between labor income and stock returns *before* retirement. This observation appears to be new. If this correlation is high, the individual invests less in the stock market before retirement, so at retirement financial wealth will often be small compared to the present value of the risk-free retirement income. To obtain the desired risk-return balance, the individual will therefore invest a relatively large fraction of financial wealth in the stock after retirement. Furthermore, before retirement the human wealth depends on financial wealth because of the unspanned income risk and the portfolio constraints. We show that the human wealth is increasing in the ratio of financial wealth to initial income and is decreasing in the stock-income correlation.

Finally, we document that the excellent performance of SAMS is robust to variations in key parameter values and to an extension of the model to stochastic interest rates.

Our method applies the idea of artificial financial markets. A constrained, incomplete-markets consumption-investment problem can be embedded in a family of consumption-investment problems in artificially unconstrained, complete-markets problems. For our specific problem each artificial market corresponds to a given choice of (i) the Sharpe ratio of an artificial asset allowing perfect hedging of income risk and (ii) a certain perturbation of the risk-free rate and stock drift. Both the Sharpe ratio and the perturbation are generally stochastic processes. Karatzas, Lehoczky, Shreve and Xu (1991) and Cvitanić and Karatzas (1992) have shown that the optimal consumption and investment strategy under incomplete markets and portfolio constraints is identical to the strategy which is optimal in the worst of all the artificial markets, but their analysis provides no practical procedure for finding the worst artificial market and thus the optimal strategy. We focus on the subfamily of "simple" artificial markets where both the Sharpe ratio and the perturbation are simple functions of time characterized by a low number of constant parameters, since the optimal strategies in those markets are then known in closed form due to Liu (2007) and others. By minimizing the value function over these parameters we find the worst of the simple artificial markets, which gives an upper bound on the utility that can be obtained in the true market.

The optimal strategy in any of the simple artificial markets is generally not a feasible strategy in the true market as it involves the artificial asset and may violate the portfolio constraints, but we can transform it into a feasible strategy by ignoring the investment in the artificial asset and by "pruning" the remaining part of the strategy to make sure constraints are respected. In this way we obtain a family of feasible strategies parameterized by a low number of constants. We can compute the expected utility associated with each strategy by straightforward Monte Carlo simulation and we embed this in a standard optimization routine, leading to a feasible and near-optimal consumption and investment strategy in the true constrained and incomplete market.

Comparing the expected utility of this strategy with the upper utility bound, we obtain an upper bound on the welfare loss—the utility loss stated in terms of total wealth—associated with our strategy. As explained above, we find small upper bounds in our quantitative examples, and the comparison with a well-established alternative numerical method indicates that the actual welfare loss is significantly smaller than the upper bound suggests.

Let us compare our method to the existing alternative methods. Grid-based dynamic programming, finite difference solution of the HJB equation, and Markov chain approximations are closely related and frequently applied methods for numerically solving low-dimensional consumption-investment problems related to the one we study; see Brennan, Schwartz and Lagnado (1997), Munk (2000), Cocco, Gomes and Maenhout (2005), Yao and Zhang (2005), Van Hemert (2010), and Munk and Sørensen (2010) for examples. However, these methods seem impossible to implement with more than three state variables and are computationally intensive even in lower dimensions. Hence, coarse grids have to be used despite the implied reduced precision. Moreover, relevant state variables such as wealth, income, or the wealth-income ratio tend to fluctuate considerably over the life-cycle so that an age-dependent scaling must be implemented to keep the state variables within the grid with high probability. The appropriate scaling has to be determined experimentally and depends on the specific setting and parameter values. Our SAMS approach is computationally less intensive, needs no scaling, can handle higher dimensions, and provides an upper bound on the error.

Various Monte Carlo simulation based approaches that can potentially handle higher-dimensional problems have been proposed. The approach of Detemple, Garcia and Rindisbacher (2003) is based on Malliavin calculus and requires complete markets and unconstrained portfolios. Cvitanić, Goukasian and Zapatero (2003) suggest another (simpler and slower) method for complete markets and unconstrained portfolios. Brandt, Goyal, Santa-Clara and Stroud (2005) adapt the least-squares Monte Carlo technique developed by Longstaff and Schwartz (2001) for American option pricing to discrete-time consumption and investment choice problems. Their method involves a grid with imposed boundaries that have to be determined experimentally (as for the finite difference type approaches). To handle portfolio constraints, the method must rely on high-dimensional *constrained* optimization algorithms whereas simpler *unconstrained* optimization techniques are sufficient for our approach. Detemple, Garcia and Rindisbacher (2005) provide experimental evidence that the least-squares approach of Brandt *et al.* is outperformed by the Malliavin-based approach. The least-squares approach has been further studied by van Binsbergen and Brandt (2007) and extended by Koijen, Nijman and Werker (2007) and Koijen, Nijman and Werker (2010). Garlappi and Skoulakis (2010) introduce a method for discrete-time problems based on Monte Carlo simulation, a Taylor expansion of the value function, and a certain decomposition of the state variables.

A variety of other methods have been proposed. Extending the log-linearization technique of Campbell and Viceira (1999), Viceira (2001) obtains a closed-form approximate solution for an infinite-horizon problem with unspanned income, constant investment opportunities, and no constraints on portfolios, but adapting the approximation to a finite time horizon, time-varying investment opportunities, and relevant portfolio constraints seems impossible. Kogan and Uppal (2000) and Das and Sundaram (2002) consider perturbation techniques.

By applying the idea of artificially unconstrained and complete markets, as we do, and the associated duality technique, Haugh, Kogan and Wang (2006) explain how to compute an upper bound on the expected utility from any given feasible consumption and investment strategy. A comparison of the expected utility delivered by the given strategy and the upper bound—both computed by Monte Carlo simulation—provides a measure of the performance of the strategy, an idea that we also apply. In contrast to their work, we search for the best possible strategy among a parameterized family of promising candidates motivated by simple artificial markets, and we also search over a parameterized family of upper utility bounds to find the tightest possible bound. We exploit the fact that the optimal strategies in many artificial markets are known in closed form.

Section 2 formulates the consumption and investment choice problem we focus on. The associated artificial markets are described in Section 3, which also presents the closed-form solutions for a parameterized family of artificial markets. Section 4 explains how we transform the solutions from the artificial markets into feasible strategies in the true market and how we determine the best of these feasible strategies. The performance of our method on the given problem is extensively studied in Section 5. We illustrate some economic properties of the solution in Section 6. Section 7 studies the sensitivity of our method to key parameters. The extension to stock/bond/cash-allocation with stochastic interest rates is discussed in Section 8. Finally, Section 9 concludes. The appendices contains proofs and some supplementary results.

2 The Problem

The computational approach we suggest applies to many highly relevant life-cycle consumption and portfolio problems and could be explained at a very general level. Instead, we choose to implement the approach for a specific problem which has been frequently studied in the literature and allows us to illustrate the power of our approach in a transparent way. Furthermore, this problem can also be solved by a standard numerical dynamic programming technique offering a yardstick for the performance of our approach.

The individual we consider has access throughout his life-time to trade in two financial assets, an instantaneously risk-free asset (a bank account) and a risky asset (a stock or stock index).

We assume a constant annualized risk-free rate given by r using continuous compounding; see Section 8 for an extension to stochastic interest rates. S_t denotes the price of the stock at time t and the price dynamics is assumed to be

$$dS_t = S_t \left[(r + \sigma_S \lambda_S)\, dt + \sigma_S \, dW_t \right],$$

where $W = (W_t)$ is a standard Brownian motion. Hence, σ_S is the volatility of the stock and λ_S is the Sharpe ratio of the stock, both assumed constant and positive.

The individual earns an exogenously given stochastic labor income until a predetermined retirement date $\tilde{T}$ after which the individual lives on until time $T > \tilde{T}$. The labor income rate at time t is denoted by Y_t and we assume that

$$dY_t = Y_t \left[\alpha \, dt + \beta \left(\rho \, dW_t + \sqrt{1-\rho^2}\, dW_{Yt} \right) \right], \quad 0 \le t \le \tilde{T}, \tag{1}$$

where $W_Y = (W_{Yt})$ is another standard Brownian motion, independent of W. The parameter α is the expected growth rate of labor income, β is the income volatility, and ρ is the instantaneous correlation between stock returns and income growth. We assume that α, β, and ρ are all constants, but our analysis goes through with the deterministic age-related variations in α and β documented by Cocco, Gomes and Maenhout (2005). Note that, unless $\beta = 0$ or $|\rho| = 1$, the investor faces an incomplete market, since he is not able to fully hedge against unfavorable income shocks. In retirement, income is risk-free and given by a fraction of income just prior to retirement,

$$Y_t = \Upsilon Y_{\tilde{T}}, \quad t \in (\tilde{T}, T], \tag{2}$$

for some constant non-negative constant Υ, the socalled replacement ratio.

The individual chooses a consumption strategy represented by a non-negative stochastic process $c = (c_t)$ and an investment strategy represented by a stochastic process $\pi_S = (\pi_{St})$, where π_{St} is the fraction of financial wealth invested in the stock at time t so that the fraction $1 - \pi_{St}$ of financial wealth is invested in the bank account. Let X_t denote the financial wealth at time t. For a given consumption and portfolio strategy (c, π_S), the wealth dynamics is

$$dX_t = X_t \left[(r + \pi_{St}\sigma_S \lambda_S)\, dt + \pi_{St}\sigma_S \, dW_t \right] + (Y_t - c_t)\, dt. \tag{3}$$

We impose the standard portfolio constraint $\pi_{St} \in [0,1]$ for all $t \in [0,T]$ ruling out short-selling of the stock as well as borrowing. We will say that a strategy (c, π) is admissible, if it is adapted, satisfies the portfolio constraint, and implies that financial wealth stays non-negative at all times, i.e., $X_t \ge 0$ (almost surely) for all $t \in [0,T]$.[2] We denote the set of admissible strategies from time t and onwards by $\mathcal{A}_t$.

[2] At any time before retirement, assuming $\beta > 0$, future income is only bounded from below by zero. Therefore, if financial wealth was negative at any time t, the individual cannot make sure that he ends up with non-negative terminal wealth, i.e., that he can pay back debts. In a more realistic setting with mortality risk, human wealth would also be risky in retirement, justifying the constraint on financial wealth throughout life.

The individual has time-additive expected utility of consumption and terminal wealth. An admissible consumption and investment strategy (c, π_S) generates the expected utility

$$J^{c,\pi_S}(t,x,y) = \mathrm{E}_t\left[\int_t^T e^{-\delta(s-t)}U(c_s)\,ds + \varepsilon e^{-\delta(T-t)}U(X_T)\right],$$

where the expectation is conditional on $X_t = x$ and $Y_t = y$. Here $\delta \geq 0$ is the subjective time preference rate, and $\varepsilon \geq 0$ models the relative weight of bequests and consumption. In retirement, we do not need y as a state variable. The indirect utility function is

$$J(t,x,y) = \max_{(c,\pi_S)\in\mathcal{A}_t} J^{c,\pi_S}(t,x,y). \tag{4}$$

We assume constant relative risk aversion $\gamma > 1$, i.e., $U(c) = c^{1-\gamma}/(1-\gamma)$.

If (a) portfolios are unconstrained and (b) income is risk-free ($\beta = 0$) or spanned ($|\rho| = 1$), the problem has the following known closed-form solution. The indirect utility function is

$$J^{\mathrm{com}}(t,x,y) = \frac{1}{1-\gamma}(g^{\mathrm{com}}(t))^{\gamma}(x + yF^{\mathrm{com}}(t))^{1-\gamma}, \tag{5}$$

and the optimal consumption and investment strategy is given by

$$c_t = \frac{X_t + Y_t F^{\mathrm{com}}(t)}{g^{\mathrm{com}}(t)}, \qquad \pi_{St} = \frac{\lambda_S}{\gamma\sigma_S} + \frac{Y_t F^{\mathrm{com}}(t)}{X_t}\left[\frac{\lambda_S}{\gamma\sigma_S} - \mathbf{1}_{\{t\leq\tilde{T}\}}\frac{\beta\rho}{\sigma_S}\right], \tag{6}$$

where it is understood that y is replaced by $Y_{\tilde{T}}$ in retirement and[3]

$$\begin{aligned}
g^{\mathrm{com}}(t) &= \frac{1}{r_g}\left(1 - e^{-r_g(T-t)}\right) + \varepsilon^{1/\gamma}e^{-r_g(T-t)}, \qquad (7)\\
F^{\mathrm{com}}(t) &= \begin{cases} \frac{\Upsilon}{r}\left(1 - e^{-r(T-t)}\right), & \text{for } t \geq \tilde{T}, \\ \frac{1}{r_F}\left(1 - e^{-r_F(\tilde{T}-t)}\right) + F^{\mathrm{com}}(\tilde{T})e^{-r_F(\tilde{T}-t)}, & \text{for } t < \tilde{T}, \end{cases}\\
r_g &= \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}r + \frac{1}{2}\frac{\gamma-1}{\gamma^2}\lambda_S^2,\\
r_F &= r - \alpha + \rho\beta\lambda_S.
\end{aligned}$$

As income is assumed risk-free in retirement, this solution will apply in retirement provided that it does not violate the short-selling constraint.

For young individuals, human wealth $Y_t F^{\mathrm{com}}(t)$ can easily be much higher than financial wealth X_t. If $\rho = +1$ and $\beta > \lambda_S/\gamma$, Eq. (6) shows that the unconstrained stock investment can then be negative. The labor income is equivalent to a large positive investment in the stock so, to obtain the desired overall risk exposure, the actual stock investment has to be negative. Conversely, if $\rho = -1$, the unconstrained stock investment will often be far above 100% since labor income then constitutes a negative implicit position in the stock.

For the more reasonable situation of risky and unspanned labor income as well as portfolio constraints, a closed-form solution for the optimal consumption and investment strategy and the

[3] If $r_g = 0$, interpret $\frac{1}{r_g}(1 - e^{-r_g(T-t)})$ as its limit as $r_g \to 0$, which is $T - t$. Similarly for r_F.

investor's indirect utility seems impossible to find. A separation like (5) does not hold as the valuation of future income will depend on wealth and risk aversion. Below we introduce a specific consumption and investment strategy, which is relatively simple to compute and implement, and we demonstrate that this strategy is close to optimal in a certain, very reasonable metric. The consumption and investment strategy we suggest will be motivated by the optimal solutions in various artificial markets to which we turn now.

3 The Artificial Markets

Karatzas, Lehoczky, Shreve and Xu (1991) and Cvitanić and Karatzas (1992) show how to construct the relevant artificial markets for a number of different portfolio constraints, including the constraint that the investor cannot trade a specific risk, i.e., that the market is incomplete. In the artificial markets associated with our problem, the drift of the stock and the risk-free rate are adjusted because of the constraint $\pi_{St} \in [0, 1]$. Let $\nu = (\nu_t)$ denote a stochastic process with values in $\mathbb{R}$ and define $\nu_t^- = \max(-\nu_t, 0)$ and $\nu_t^+ = \max(\nu_t, 0)$. Each ν defines a certain artificial market. In the artificially unconstrained market corresponding to any given ν, the risk-free rate is assumed to be $\tilde{r}_t = r + \nu_t^-$ (instead of just r as in the true market) and the drift of the stock is assumed to be $r + \sigma_S\lambda_S + \nu_t^+ = \tilde{r}_t + \sigma_S\lambda_S + \nu_t$. If the unconstrained π_{St} is above 1—which often happens with substantial human wealth and low or even negative ρ—we raise the risk-free rate and keep the drift of the stock fixed, which makes the stock less attractive relative to the bank account. This corresponds to a negative value of ν_t. If the unconstrained π_{St} is below 0—which may happen when ρ, β, γ are relatively high and λ_S relatively low—we increase the drift of the stock and keep the risk-free rate fixed, which makes the stock more attractive relative to the bank account. This corresponds to a positive value of ν_t.

Assuming $\beta > 0$ and $|\rho| < 1$, the individual faces unspanned income risk until retirement. The artificially unconstrained markets allow for investment in an "income contract" characterized by the market price of risk λ_{It} associated with the standard Brownian motion W_Y. The time t price is I_t and evolves according to

$$dI_t = I_t \left[(\tilde{r}_t + \lambda_{It}) \, dt + \, dW_{Yt} \right]. \tag{8}$$

Let π_{It} be the fraction of wealth invested in the income contract. λ_I can be positive or negative and is, in general, a stochastic process.

Every pair of processes (ν, λ_I), satisfying certain technical conditions, defines an artificial market. There are no constraints on the consumption and investment strategy in the artificial markets except for the standard integrability conditions and the constraint that consumption and terminal wealth have to be non-negative. As labor income is perfectly hedgeable in the artificial markets, we do not need $X_t \geq 0$ for all t to ensure $X_T \geq 0$, as we do in the true market.

Let $J(t, x, y; \nu, \lambda_I)$ denote the indirect utility in the artificial market corresponding to (ν, λ_I). A strategy (c, π_S) that is feasible in the true market will, together with a zero investment in the income contract, leads to at least the same expected utility in any of the artificial markets as in the true market. The reason is that the risk-free rate and the return on the risky investment is at least as big in the artificial markets and, hence, terminal wealth will also be at least as big. Many other strategies are feasible in the artificial markets, so the indirect utility in each artificial market is greater than or equal to the indirect utility in the true market, $J(t, x, y; \nu, \lambda_I) \geq J(t, x, y)$. Karatzas, Lehoczky, Shreve and Xu (1991) and Cvitanić and Karatzas (1992) show that the minimum of the indirect utility $J(t, x, y; \nu, \lambda_I)$ over *all* the processes (ν, λ_I) satisfying certain technical conditions is equal to the indirect utility in the true constrained market, i.e., the solution in the true constrained market is equal to the solution in the worst of all the artificially unconstrained markets. Alas, since we cannot compute $J(t, x, y; \nu, \lambda_I)$ for all (ν, λ_I), we cannot minimize over (ν, λ_I), so this result does not generally provide a way of finding the optimal constrained strategy.

However, we can compute $J(t, x, y; \nu, \lambda_I)$ in *some* artificial markets. To keep the solution simple, we focus on the artificial markets with deterministic (ν, λ_I) and use the notation $\nu(t)$ instead of ν_t and similarly for λ_I. The pair (ν, λ_I) representing the worst artificial market will presumably depend on financial wealth and income (and age) and will thus be stochastic processes.[4] Our method could be extended to certain exogenous stochastic processes ν and λ_I. As long as the price dynamics in the artificial market has an affine or quadratic structure (Liu (2007)), closed-form solutions exist (in some cases one or more simple ordinary differential equations have to be solved numerically), but the solutions will be more complex with stochastic (ν, λ). Apparently, we cannot allow ν or λ_I to depend explicitly on wealth and still obtain closed-form solutions. As we report below, the method is already very precise when restricted to simple deterministic (ν, λ_I).

At retirement, income becomes risk-free, which will presumably lead to a big shift in the allocation of the investment between the risk-free asset and the risky asset, so that a constraint which is binding just before retirement may not be binding immediately after and vice versa.

[4] In the HJB equation corresponding to the problem (4), π_S has to maximize $\pi_S \lambda_S J_x + \frac{1}{2}\pi_S^2 \sigma_S^2 x J_{xx} + \beta\rho\pi_S y J_{xy}$. If we impose the constraint $\pi_S \leq 1$ and let m denote the associated non-negative Lagrange multiplier, the Lagrangian consists of the terms listed before plus $m(1 - \pi_S)$. By maximizing with respect to π_S, we find

$$\pi_S = -\frac{J_x}{x J_{xx}} \frac{\lambda_S - \frac{m}{J_x}}{\sigma_S} - \frac{\beta\rho}{\sigma_S} \frac{y J_{xy}}{x J_{xx}}.$$

This shows that the appropriate reduction of the Sharpe ratio is closely related to the Lagrangian multiplier associated with the constraint. For $\pi_S = 1$, we get $m/J_x = \lambda_S + \beta\rho y J_{xy}/J_x + \sigma_S x J_{xy}/J_x$, which depends on x and y as well as the age and risk aversion of the individual.

Therefore, we allow for different $\nu(t)$ in retirement and in the active phase as represented by $\nu_R(t)$ and $\nu_A(t)$.

In retirement, there is no income risk so λ_I is irrelevant and the artificial markets are just characterized by ν_R. For any function $\nu_R(t)$, the solution to the utility maximization in the corresponding artificial unconstrained market is stated below; see Appendix A for proofs.

Theorem 1 (In retirement)**.** *The indirect utility during retirement in the artificial market characterized by $\nu_R(t)$ is given by*

$$J_R(t,x;\nu_R) = \frac{1}{1-\gamma} g_R(t;\nu_R)^\gamma \left(x + \Upsilon Y_{\tilde{T}} F_R(t;\nu_R)\right)^{1-\gamma},$$

where

$$\begin{aligned}
F_R(t;\nu_R) &= \int_t^T e^{-\int_t^u \left(r+\nu_R(\tau)^-\right) d\tau}\, du, \\
g_R(t;\nu_R) &= \varepsilon^{1/\gamma} e^{-\int_t^T h_R(\nu_R(\tau))\, d\tau} + \int_t^T e^{-\int_t^u h_R(\nu_R(\tau))\, d\tau}\, du, \\
h_R(\nu_R(\tau)) &= \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\left(r+\nu_R(\tau)^-\right) + \frac{\gamma-1}{2\gamma^2}\left(\lambda_S + \frac{\nu_R(\tau)}{\sigma_S}\right)^2.
\end{aligned}$$

The corresponding optimal consumption and investment strategy is

$$c_t = \frac{X_t + \Upsilon Y_{\tilde{T}} F_R(t;\nu_R)}{g_R(t;\nu_R)}, \tag{9}$$

$$\pi_{St} = \left[1 + \frac{\Upsilon Y_{\tilde{T}} F_R(t;\nu_R)}{X_t}\right] \frac{\sigma_S\lambda_S + \nu_R(t)}{\gamma\sigma_S^2}. \tag{10}$$

In retirement, income affects the optimal strategy only through the addition of human capital to current financial wealth and thus drives up consumption and the risky investment.

Note that during retirement, π_S will never be negative since λ_S is positive, so we can focus on the constraint $\pi_S \leq 1$. To avoid $\pi_S > 1$, we may need $\nu_R < 0$. Then F_R and the present value of future income is smaller, so π_S is indeed smaller. Since h_R can be smaller or bigger (than with $\nu_R = 0$), it is not clear whether g_R is smaller or bigger, so the effect on consumption is not obvious.

Let $J_A(t,x,y;\nu_A,\lambda_I)$ denote the indirect utility function in the active phase in this artificial market. We have the boundary condition

$$J_A(\tilde{T},x,y;\nu_A,\lambda_I) = J_R(\tilde{T},x;\nu_R) = \frac{1}{1-\gamma} g_R(\tilde{T};\nu_R)^\gamma \left(x + \Upsilon Y_{\tilde{T}} F_R(\tilde{T};\nu_R)\right)^{1-\gamma}.$$

Via this boundary condition, the indirect utility in the active phase will also depend on the perturbation $\nu_R(t)$ of the expected returns on the risk-free asset and the stock in retirement.

Theorem 2 (In active phase)**.** *The indirect utility before retirement in the artificial market characterized by $\nu_A(t)$ and $\lambda_I(t)$ is given by*

$$J_A(t,x,y;\nu_A,\nu_R,\lambda_I) = \frac{1}{1-\gamma} g_A(t;\nu_A,\nu_R,\lambda_I)^\gamma \left(x + yF_A(t;\nu_A,\nu_R,\lambda_I)\right)^{1-\gamma},$$

where

$$
\begin{aligned}
F_A(t;\nu_A,\nu_R,\lambda_I) &= e^{-\int_t^{\tilde{T}} r_A(\nu_A(u),\lambda_I(u))\,du}\Upsilon F_R(\tilde{T};\nu_R) + \int_t^{\tilde{T}} e^{-\int_t^u r_A(\nu_A(\tau),\lambda_I(\tau))\,d\tau}\,du,\\
g_A(t;\nu_A,\nu_R,\lambda_I) &= e^{-\int_t^{\tilde{T}} h_A(\nu_A(u),\lambda_I(u))\,du} g_R(\tilde{T};\nu_R) + \int_t^{\tilde{T}} e^{-\int_t^u h_A(\nu_A(\tau),\lambda_I(\tau))\,d\tau}\,du,\\
r_A(\nu_A(t),\lambda_I(t)) &= r + \nu_A(t)^- - \alpha + \beta\rho\left(\lambda_S + \frac{\nu_A(t)}{\sigma_S}\right) + \beta\sqrt{1-\rho^2}\lambda_I(t), \qquad (11)\\
h_A(\nu_A(t),\lambda_I(t)) &= \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\left(r+\nu_A(\tau)^-\right) + \frac{\gamma-1}{2\gamma^2}\left[\left(\lambda_S + \frac{\nu_A(t)}{\sigma_S}\right)^2 + \lambda_I(t)^2\right].
\end{aligned}
$$

The corresponding optimal consumption and investment strategy is

$$
c_t = \frac{X_t + Y_t F_A(t;\nu_A,\nu_R,\lambda_I)}{g_A(t;\nu_A,\nu_R,\lambda_I)}, \qquad (12)
$$

$$
\pi_{St} = \frac{\sigma_S\lambda_S + \nu_A(t)}{\gamma\sigma_S^2} + \frac{Y_t F_A(t;\nu_A,\nu_R,\lambda_I)}{X_t}\left[\frac{\sigma_S\lambda_S + \nu_A(t)}{\gamma\sigma_S^2} - \frac{\beta\rho}{\sigma_S}\right], \qquad (13)
$$

$$
\pi_{It} = \frac{\lambda_I(t)}{\gamma} + \frac{Y_t F_A(t;\nu_A,\nu_R,\lambda_I)}{X_t}\left[\frac{\lambda_I(t)}{\gamma} - \beta\sqrt{1-\rho^2}\right]. \qquad (14)
$$

Before retirement, labor income influences the optimal strategy via the addition of human wealth to financial wealth, just as in retirement. In addition, the optimal stock investment is affected via the term $-\frac{\beta\rho}{\sigma_S}\frac{Y_t F_A(t)}{X_t}$ which adjusts the explicit investment in the stock by the stock investment implicit in human wealth through the income-stock correlation ρ. This is in line with the intuition of Bodie, Merton and Samuelson (1992).

We want to minimize the indirect utility over the selected artificial markets since that provides an upper bound for the indirect utility in the true constrained market. To perform the minimization, we need to parameterize the functions $\nu_R(t)$, $\nu_A(t)$, and $\lambda_I(t)$.

First, consider $\nu_R(t)$. If $\lambda_S < \gamma\sigma_S$, the constraint will not be active without income, i.e., just before the terminal date. We can let $\nu_R(t) = -v_R(\hat{T}-t)^+$ for some $\hat{T} \leq T$ and some (presumably positive) constant v_R (the integrals is the expressions for F_R and g_R are then easily computed).

Next, consider the choice of $\nu_A(t)$ and $\lambda_I(t)$. How binding the constraints are, will depend on wealth. Since expected wealth in most models increases up to retirement, we try affine functions of time

$$
\lambda_I(t) = \Lambda_0 + \Lambda_1 t, \qquad \nu_A(t) = v_0 + v_1 t.
$$

The integrals in the expressions for F_A and g_A can then be computed by standard numerical integration techniques. With these specifications, the strategies and the indirect utility are parameterized by the six constants $\boldsymbol{\phi} = (v_0, v_1, v_R, \hat{T}, \Lambda_0, \Lambda_1)$, and we denote the associated indirect utility by $J_A(t,x,y;\boldsymbol{\phi})$ and the corresponding optimal strategy by $(c(\boldsymbol{\phi}), \pi_S(\boldsymbol{\phi}), \pi_I(\boldsymbol{\phi}))$.

We can now compute an upper bound on the indirect utility in the true constrained market by a minimization over the parameterized artificial markets,

$$\bar{J}(t,x,y) \equiv J_A(t,x,y;\bar{\boldsymbol{\phi}}) \equiv \min_{\bar{\phi}} J_A(t,x,y;\boldsymbol{\phi}),$$

which is implemented using a standard unconstrained numerical optimization algorithm.

4 A Near-Optimal Strategy in the True Market

We derive a promising candidate for a good consumption-investment strategy in the true constrained market from the optimal strategies in the parameterized family of artificial markets in the following way. For each $\boldsymbol{\phi} = (v_0, v_1, v_R, \hat{T}, \Lambda_0, \Lambda_1)$, we take the optimal strategy $(c(\boldsymbol{\phi}), \pi_S(\boldsymbol{\phi}))$ in the corresponding artificial market—the strategy given by (9)–(10) in retirement and (12)–(14) before retirement—and transform it into a strategy which is feasible in the true market.

We need to make sure that financial wealth stays non-negative. In particular, when financial wealth approaches zero (from above), we have to refrain from investing in the risky asset and to consume less than current income. Intuitively, this liquidity constraint implies that future income has a smaller present value when current financial wealth is small. A parsimonious way to capture this effect is by multiplying the human capital $Y_t F_A(t)$ by a factor $(1 - e^{-\eta X_t})$, where η is a positive constant to be determined. For large financial wealth, the factor is close to one so that human capital is not significantly reduced. When financial wealth approaches zero, the factor approaches zero so that human capital is reduced to zero.[5] Furthermore, we have to make sure $\pi_{St} \in [0,1]$, so an obvious candidate for a good strategy in retirement, i.e. for $t \in (\tilde{T}, T]$, is

$$c_t = \frac{X_t + \Upsilon Y_{\tilde{T}} F_R(t)(1 - e^{-\eta X_t})}{g_R(t)}, \tag{15}$$

$$\pi_{St} = \left(\min\left\{1, \left[1 + \frac{\Upsilon Y_{\tilde{T}} F_R(t)(1 - e^{-\eta X_t})}{X_t}\right] \frac{\sigma_S \lambda_S + \nu_R(t)}{\gamma \sigma_S^2}\right\}\right)^+, \tag{16}$$

where we have suppressed the dependence of F_R and g_R on ν_R.

In the active phase, we disregard the investment in the artificial income contract, and ensure that constraints are satisfied just as in retirement. The modified strategy for $t \in [0, \tilde{T}]$ is thus

$$c_t = \frac{X_t + Y_t F_A(t)(1 - e^{-\eta X_t})}{g_A(t)}, \tag{17}$$

$$\pi_{St} = \left(\min\left\{1, \frac{\sigma_S \lambda_S + \nu_A(t)}{\gamma \sigma_S^2} + \frac{Y_t F_A(t)(1 - e^{-\eta X_t})}{X_t} \left[\frac{\sigma_S \lambda_S + \nu_A(t)}{\gamma \sigma_S^2} - \frac{\beta \rho}{\sigma_S}\right]\right\}\right)^+, \tag{18}$$

where we have suppressed the dependence of F_A and g_A on ν_A, ν_R, λ_I.

For any set of the (seven) constants $\boldsymbol{\psi} = (\boldsymbol{\phi}, \eta)$, the Equations (15)–(18) define a feasible strategy $(c(\boldsymbol{\psi}), \pi_S(\boldsymbol{\psi}))$ in the true market. For any $\boldsymbol{\psi}$, we can approximate the expected utility

[5] In principle, consumption for near-zero financial wealth could exceed current income in which case we would have to reduce consumption to ensure financial wealth stays non-negative. However, this never happened in our simulations.

$J(t,x,y;\boldsymbol{\psi})$ generated with $(c(\boldsymbol{\psi}),\pi_S(\boldsymbol{\psi}))$ by Monte Carlo simulation of the income dynamics (1) and the wealth dynamics (3) substituting in $(c(\boldsymbol{\psi}),\pi_S(\boldsymbol{\psi}))$. Searching through different $\boldsymbol{\psi}$, we find the best of the feasible strategies $(c(\boldsymbol{\psi}^*),\pi_S(\boldsymbol{\psi}^*))$ with an associated expected utility of $J(t,x,y;\boldsymbol{\psi}^*)$. Again, this can be implemented by a standard unconstrained numerical optimization algorithm.

We can evaluate the performance of any admissible consumption and investment strategy (c,π_S)—including our candidate $(c(\boldsymbol{\psi}^*),\pi_S(\boldsymbol{\psi}^*))$ defined above—in the following way. We compare the expected utility generated by the strategy, $J^{c,\pi_S}(t,x,y)$, to the upper bound $\bar{J}(t,x,y)$ on the maximum utility. If the distance is close, the strategy is near-optimal. More precisely, we can compute an upper bound $L = L^{c,\pi_S}(t,x,y)$ on the welfare loss suffered when following the specific strategy (c,π_S) by solving the equation

$$J^{c,\pi_S}(t,x,y) = \bar{J}(t,x[1-L],y[1-L]).$$

$L^{c,\pi_S}(t,x,y)$ is interpreted as an upper bound on the fraction of total wealth (current wealth as well as current and future income) that the individual would be willing to throw away to get access to the unknown optimal strategy, instead of following the strategy (c,π_S). If we focus on the active phase, it follows from Theorem 2 that

$$\bar{J}(t,x[1-L],y[1-L]) = J_A(t,x[1-L],y[1-L];\bar{\boldsymbol{\phi}}) = (1-L)^{1-\gamma}J_A(t,x,y;\bar{\boldsymbol{\phi}}),$$

so the upper bound on the welfare loss becomes

$$L^{c,\pi_S}(t,x,y) = 1-\left(\frac{J^{c,\pi_S}(t,x,y)}{J_A(t,x,y;\bar{\boldsymbol{\phi}})}\right)^{\frac{1}{1-\gamma}}.$$

5 Numerical Results

The results presented below are based on Monte Carlo simulations using 10,000 paths. Along each path the consumption and investment strategy is reset with a frequency of $\Delta = 0.05$, i.e., 20 times a year (more frequent resetting of consumption and portfolio does not change results significantly). In order to reduce any simulation bias in the loss, we also compute the upper utility bound $J_A(t,x,y;\bar{\boldsymbol{\phi}})$ by Monte Carlo simulation—here it is the wealth dynamics in the artificial market which is simulated—applying the same set of random numbers as used in the computation of the utility $J(t,x,y;\boldsymbol{\psi}^*)$ for our best feasible strategy. The benchmark values for the parameters describing the characteristics of the individual, the income process, and the financial market are summarized in Table 1. The benchmark values are similar to those used in the existing literature; see, e.g., Cocco, Gomes and Maenhout (2005) and Kraft and Munk (2011) and the references therein. The individual has a relative risk aversion of 4, has 30 years until retirement, and subsequently lives for another 20 years. The initial time is $t = 0$, unless mentioned otherwise. Whenever we need to use levels of current wealth, labor income etc., we use a unit of USD 10,000 scaled by one plus the inflation rate in the perishable consumption good. As the benchmark we put $x = 2$ and $y = 2$ representing an initial financial wealth of USD 20,000 and an initial annual income of USD 20,000, which are in line with the median net worth and before-tax income statistics derived from the 2007 Survey of Consumer Finances for individuals of age 30-40, cf. (Bucks, Kennickell, Mach and Moore, 2009, pp. A5 and A11); the Survey also reveals a huge variation in wealth and income across individuals.

5.1 Main Results

First, we consider the size of the upper bound L on the welfare loss from following the strategy derived by our method instead of the unknown optimal strategy throughout the entire life. Consumption and investment strategies are known to depend on the ratio between financial wealth and income, as well as the correlation between stock returns and labor income, so we focus first on the sensitivity of the loss with respect to these quantities. Table 2 shows that for a wide range of values for the initial wealth/income ratio, the welfare loss bound is below 0.5% of current total wealth for an income-stock correlation of 0.2 or higher; in fact, in many cases the welfare loss bound is much lower than 0.5%. The welfare loss bound is somewhat higher for a zero income-stock correlation, but at most 1.1%. These results confirm that our proposed strategy is indeed near-optimal.

It is intuitively reasonable that the loss bound is largest for a zero correlation, because in this case the optimal unconstrained strategy will be a highly leveraged position in the stock for many years. Moreover, with zero correlation, the labor income is “far” from being spanned. Loosely speaking, the true market is very different from the artificial unconstrained markets. For intermediate values of the correlation, the portfolio constraint on the stock is rarely binding, and the loss bound is very small. For very high values of the correlation, the optimal unconstrained strategy would involve some shorting of the stock in the early years, so the loss bound is slightly higher than for intermediate correlations.

Table 3 shows how the upper bound on the welfare loss varies with the initial date and thus with the time to retirement assuming that the initial value of the wealth/income ratio is fixed at 1. As the investment horizon decreases, human wealth decreases. This reduces the wealth effect of labor income on the stock investment and it also reduces the adjustment for stock-like income risk. As the latter reduction depends on the correlation, the net effect of the decrease in the investment horizon also depends on correlation. For low correlations this implies that the no borrowing constraint is less tight and thus the loss bound tends to decrease. For higher correlations, constraints may become more frequently active over shorter horizons, and the loss bound may increase. As the initial date is moved close to the retirement date, the loss bound begins to increase for any correlation value.

5.2 A Comparison with a Standard Numerical Method

We have also solved the utility maximization problem (4) with the socalled Markov Chain Approximation (MCA) Method, which is a well-studied and frequently applied numerical approach (Kushner and Dupuis (2001); Munk (2000) and Munk (2003)). The indirect utility function is homogeneous of degree $1-\gamma$ and can therefore be written as $J(t,x,y) = y^{1-\gamma}H(t,x/y)$. From the HJB equation for J, a non-linear second-order PDE for H can be derived, and this PDE is the HJB equation for another stochastic control problem where the controls are simple scalings of the original consumption and portfolio plans. The MCA Method discretizes this control problem. The dynamics of the wealth/income ratio is approximated by a Markov chain on a grid defined by N equidistant time points and I equidistant values of the wealth/income ratio. In the continuous-time model, the wealth/income ratio is unbounded from above, but the MCA Method has to impose an upper bound. The optimization problem is solved by backwards recursion starting at the terminal date T. In each time step the value function for each state in the grid is maximized by policy iterations. The entire procedure is roughly equivalent to solving the HJB equation for H by a (specific) finite difference approach, similar to the one used by Brennan, Schwartz and Lagnado (1997) and others. The precision of the method depends heavily on the number of grid points and the size of the imposed upper bound. Ideally, the bound should be set so high that it is very unlikely that the wealth/income ratio would exceed that bound when the optimal strategies are followed. This can be checked by simulating the wealth/income ratio using the strategies obtained with the method for a given upper bound. If sufficiently many paths exceed the bound, the MCA method must be rerun with a higher imposed bound. This complicates the application of the MCA Method as well as other grid-based methods.

We have solved our problem with the MCA method both for a very fine grid ($N = 4000$, $I = 12000$) and a coarser—but still quite fine—grid ($N = 2000$, $I = 4000$). We evaluate the expected utility of the consumption and investment strategy derived with the MCA method by Monte Carlo simulation using the same random numbers as in the valuation of our suggested strategy, in order to avoid any bias stemming from the simulations. Table 4 presents the upper bounds on the percentage welfare loss for the MCA method for both grids and compares with the upper bound for our SAMS method. The loss difference is simply the loss bound for our method minus the loss bound for the MCA method, i.e., the increase in the upper bound by applying our SAMS method instead of the MCA method. The table assumes an initial wealth/income ratio of 1, but the results are similar for other values. The table shows that our method is roughly as good as the MCA method with the very fine grid in the sense that the loss from following our strategy instead of the MCA-based strategy is at most 0.115% of wealth. With the very fine grid the derived strategy can be expected to be very close to the truly optimal strategy, so the results that the strategy derived by our method is also very close to optimal and that the upper bound on the welfare loss is not very tight. Furthermore, our method beats the MCA method with the coarser grid.

For problems with a single state variable, like the one we consider, the MCA method can be implemented with a very fine grid. For problems with two or three state variables, however, fine grids are intractable. For example, a grid with two state variables and 12000 gridpoints per state variable, as we have used, would have 144 million gridpoints at each point in time considered, requiring a lot of computer memory and leading to long computation times. The above results suggest that our SAMS approach would outperform tractable implementations of the MCA method for problems with two or three state variables. For problems with more than three state variables (after any homogeneity is exploited), all grid-based methods seem computationally infeasible. In contrast, our SAMS method is still applicable.

5.3 Detailed Results from our Method

Next, we investigate the auxiliary parameters $\boldsymbol{\psi}^*$ underlying the best of the strategies of the form in (15)–(18). Table 5 shows the optimal auxiliary parameters, corresponding to $\boldsymbol{\psi}^*$, for different stock-income correlations and for an initial wealth/income ratio of 1. As for any multi-dimensional numerical optimization, some experimentation with starting values, possible sequential optimization over different subsets of parameters and so on is recommended. With the parameter values and initial state variables listed in Table 1, our experiments have shown that best results are obtained by fixing $v_R = 0$ (and then $\hat{T}$ is meaningless) and $\eta = 30$, and then run a simulated annealing optimization routine to find the optimal remaining parameters displayed in Table 5.[6] The values of Λ_0 and Λ_1 indicate that the risk premium which the individual associates with the unspanned income risk is positive and decreasing over life, and the risk premium is higher for low correlations. A high risk premium translates into a low value of the income multiplier $F_A(t)$, i.e., a low human wealth.

For a zero correlation, the optimized $\nu_{At} = v_0 + v_1 t$ is negative over most of the working life. This indicates that the risk-free rate is artificially increased to make the stock less attractive, because the unrestricted fraction of wealth invested in the stock would exceed one. For very high correlations, the optimized ν_{At} is positive, artificially increasing the expected return on the stock, which makes good sense early in life where the unrestricted stock investment would

[6] With our parameters, financial wealth will be sufficiently big at retirement that the portfolio constraints are not binding, so it is natural to have $\nu_R(t) = 0$. The large value of η indicates that downwards scaling of human wealth imposed in (17)–(18) is only significant for very small levels of financial wealth.

be negative in this case. However, Λ_0 and Λ_1 also affect the stock investment via the income multiplier $F_A(t)$, so that it is difficult to interpret the values of v_0 and v_1.

In general, we would optimize over the auxiliary parameters in $\boldsymbol{\psi}$ for each value of the wealth/income ratio x/y of interest. However, the optimal parameters vary only relatively little with x/y and, for a fixed x/y, the expected utility $J(t, x, y; \boldsymbol{\psi})$ is relatively insensitive to $\boldsymbol{\psi}$ around the optimal value $\boldsymbol{\psi}^*$. We have performed the following experiment. For any given correlation, we find the optimal parameters for $x/y = 1$ and then apply the same parameters for the other values of x/y considered. Obviously, applying the strategy based on the non-optimal parameters leads to a higher welfare loss bound. Table 6 documents that the increase in the percentage welfare loss caused by not re-optimizing over the parameters when x/y is different from 1 is very small. In particular, for a local sensitivity analysis of the near-optimal strategy with respect to changes in x or y, it is fair to keep the parameters fixed.

6 Some Properties of the Solution

Although the focus of this paper is on the solution technique, we take a brief look at some key properties of our solution. First, we have derived our near-optimal consumption and investment strategy as a function of age, financial wealth, and labor income. Then we have simulated 10,000 paths of financial wealth using this strategy and labor income over the life of the individual. Figure 1 shows how the average (over the 10,000 paths) fraction of wealth invested in the stock varies over the life-cycle for different values of the income-stock correlation ρ. For a zero or low stock-income correlation, the individual invests the entire financial wealth in the stock in the early years and then gradually, as human wealth decreases, replaces a fraction of the stock investment by a risk-free investment. For a high correlation, the individual would initially like to short the stock, so the optimal constrained investment strategy is to invest the entire financial wealth in the risk-free asset. Later, as human wealth decreases, the stock is included in the portfolio.

At retirement, the optimal asset allocation changes dramatically because the income risk is suddenly resolved. After retirement, both human wealth and the optimal fraction of total wealth invested in the stock are independent of what the stock-income correlation was before retirement. However, the financial wealth build up during the active phase will depend on the correlation. On average, individuals with a high stock-income correlation enter retirement with a low financial wealth, since they have been investing little in the stock compared to individuals with a low correlation. For an individual with a high correlation, financial wealth constitutes a lower fraction of total wealth at retirement, and to obtain the desired overall risk exposure of total wealth, this individual will have to invest a higher fraction of financial wealth in the stock, as shown in the graph.

Figure 2 depicts how the optimal fraction of financial wealth invested in the stock at time $t = 0$ depends on the initial ratio of financial wealth to annual income. As the wealth/income ratio approaches infinity, income becomes irrelevant so the investor should optimally invest the fraction $\lambda_S/(\gamma\sigma_S) = 0.3125$ of financial wealth in the stock, as in the no-income setting of Merton (1971). For a zero or low stock-income correlation, labor income mainly influences investments through the addition of human wealth to financial wealth. When the wealth/income ratio is small, the entire (but small) financial wealth is therefore invested in the stock in order to obtain the best possible overall risk exposure. As the wealth/income ratio is increased, the optimal fraction of financial wealth invested in the stock will eventually fall below 1 and thus decrease towards the asymptotic value of 0.3125. Conversely, for a high correlation labor income is much like an implicit stock investment. The optimal fraction of financial wealth invested in the stock will therefore be zero for low wealth/income ratios, but become positive for a high enough wealth/income ratio, and eventually approach the asymptotic 0.3125. These results demonstrate that the sensitivity of the optimal stock investment to the initial wealth and income is highly dependent on the risk characteristics of labor income.

Because of unspanned income risk (unless $|\rho| = 1$ or $\beta = 0$) and portfolio constraints, the future income stream cannot be valued like the dividends from a traded asset. The human wealth will depend on the stock-income correlation and on the risk aversion and wealth of the individual. We compute human wealth H as the minimum extra financial wealth that the individual would need as compensation if the entire income stream is taken away, i.e.,

$$J(t, x, y) = J(t, x + H, 0),$$

where the left-hand side is the indirect utility with the income stream and the right-hand side is the indirect utility without the income stream but a higher financial wealth. Given our benchmark parameter values, the portfolio constrains are not binding in the case without income, so the right-hand side equals

$$J(t, x + H, 0) = \frac{1}{1-\gamma}(g^{\text{com}}(t))^{\gamma}(x + H)^{1-\gamma},$$

where g^{com} is defined in (7). Hence, we can compute H as

$$H = [(1-\gamma)J(t, x, y)]^{\frac{1}{1-\gamma}} (g^{\text{com}}(t))^{\frac{\gamma}{\gamma-1}} - x.$$

We can interpret H/y as an income multiplier, since this is the factor that current income has to be multiplied by to get the human wealth. We replace the unkown indirect utility $J(t, x, y)$ with the expected utility generated by our near-optimal strategy.

Table 7 reports the income multiplier H/y for different combinations of the stock-income correlation and the initial wealth/income ratio. The income multiplier is increasing in the wealth/income ratio, because with relatively high financial wealth the individual is less concerned with unspanned income risk and the portfolio constraints will rarely bind, especially for low correlations. The income multiplier is decreasing in the stock-income correlation (except close to perfect correlation which is a very special case). The lower the correlation, the better the inherent income risk hedging properties of a positive investment in the risky asset and, thus, the more valuable the income stream. The differences in the displayed income multipliers are modest, however. For example, with zero correlation, the multiplier is 11.7% smaller starting with a low wealth ($x/y = 0.1$) than a high wealth ($x/y = 10$).

7 Comparative Statics

To check the robustness of our results, we now vary selected parameters of our benchmark case one by one. We focus on the relative risk aversion and the parameters driving the income process, and for each parameter we consider a value below and a value above the benchmark value. Table 8 reports both the upper loss bounds for our method and the increase in the loss bound relative to the MCA approach implemented with the fine grid. Overall, the welfare loss bound remains small for all the considered parameter values and the two methods provide very similar results. For 7 out of the 24 parameter combinations considered in the table, our method outperforms the MCA approach with the fine grid. Compared to the MCA with the coarser grid, our method does better for 16 out of the 24 parameter combinations considered in the table (results not shown, but are available upon request).

The loss bound is somewhat higher for a low risk aversion than for a high risk aversion. For a lower risk aversion, the unrestricted speculative stock demand will be more sensitive to the human capital and will stay above the imposed maximum of 100% for a longer period of time, so the imposed portfolio constraints are more binding for $\gamma = 3$ than for $\gamma = 5$ and a zero or moderate correlation. It is then not surprising that the welfare loss bound is higher for $\gamma = 3$ than for $\gamma = 5$.

The welfare loss bound tends to increase with the riskiness of the income stream measured by its volatility β, which makes sense as the unspanned income risk is then bigger. Note that for the case of high income volatility and zero income-stock correlation, in which the upper loss bound is highest (1.5%), our method performs significantly better than the MCA approach.

The expected income growth rate α enters the optimal strategies only via the F_A-function, which is increasing in α. As indicated by (11), variations in α that make constraints more or less binding are easily mitigated by varying ν_A and λ_I. Consequently, after optimizing over these parameters, the welfare loss bound is relatively insensitive to α.

Finally, the welfare loss is slightly increasing in the income replacement ratio Υ, as this increases human capital and thus tends to make portfolio constraints more binding early in life.

8 Extension to Stochastic Interest Rates

Until now we have assumed a simple Black-Scholes type financial market, but our approach applies to more general settings. As an example we consider the case where interest rates are stochastic as described by the Vasicek (1977) model so that the short-term interest rate r_t has dynamics

$$dr_t = \kappa[\bar{r} - r_t]\,dt - \sigma_r\,dW_{rt},$$

where $\bar{r}$, κ, and σ_r are constants, and W_r is a standard Brownian motion. The price B_t of any bond has dynamics of the form

$$dB_t = B_t\left[\big(r_t + \lambda_B \sigma_B(r_t,t)\big)\,dt + \sigma_B(r_t,t)\,dW_{rt}\right],$$

where λ_B is a constant market price of interest rate risk. For a zero-coupon bond with a time-to-maturity of τ, the price is of the form $B_t = \exp\{-\mathcal{A}(\tau) - \mathcal{B}_\kappa(\tau) r_t\}$, so that $\sigma_B(r_t,t) = \sigma_r \mathcal{B}_\kappa(\tau)$. Here $\mathcal{B}_m(\tau) = (1 - e^{-m\tau})/m$ for any constant m, and $\mathcal{A}$ is another deterministic function of minor importance for what follows. The dynamics of the stock price and the labor income is now assumed to be

$$\begin{aligned} dS_t &= S_t\left[(r_t + \lambda_S \sigma_S)\,dt + \sigma_S(\rho_{SB}\,dW_{rt} + \hat{\rho}_S\,dW_{St})\right], \\ dY_t &= Y_t\left[\alpha\,dt + \beta(\rho_{YB}\,dW_{rt} + \hat{\rho}_{YS}\,dW_{St} + \hat{\rho}_Y\,dW_{Yt})\right], \quad t < \tilde{T}, \end{aligned}$$

where W_r, W_S, W_Y are independent standard Brownian motions and

$$\hat{\rho}_S = \sqrt{1 - \rho_{SB}^2}, \quad \hat{\rho}_{YS} = \frac{\rho_{YS} - \rho_{SB}\rho_{YB}}{\sqrt{1 - \rho_{SB}^2}}, \quad \hat{\rho}_Y = \sqrt{1 - \rho_{YB}^2 - \hat{\rho}_{YS}^2}$$

where ρ_{SB}, ρ_{YB}, and ρ_{YS} are the pairwise stock-bond, income-bond, and income-stock correlations. In retirement, the income is again given by (2).[7]

The individual can trade in the stock, the instantaneously risk-free bank account, and a single bond index. The bond index is continuously rebalanced so that at any point in time it corresponds to a zero-coupon bond having a time-to-maturity of $\bar{\tau}$ and, consequently, a constant volatility of $\sigma_B = \sigma_r \mathcal{B}_\kappa(\bar{\tau})$. Let π_{St} and π_{Bt} denote the fractions of financial wealth invested in the stock and the bond index, respectively, at time t. The remaining financial wealth $W_t(1 - \pi_{St} - \pi_{Bt})$ is invested in the bank account. We impose the constraints that $\pi_{St}, \pi_{Bt} \in [0,1]$, $\pi_{St} + \pi_{Bt} \leq 1$ (borrowing prohibited), and before retirement we also have to make sure that financial wealth stays non-negative as in the problem studied in the preceding sections.

An artificial market corresponding to this constrained, incomplete market is characterized by a triple $(\nu_S, \nu_B, \lambda_I)$ of stochastic processes such that (see Cvitanić and Karatzas (1992))

[7] Dynamic portfolio choice with Vasicek-type interest rates has been studied by Sørensen (1999), Brennan and Xia (2000), and Campbell and Viceira (2001) without labor income and Koijen, Nijman and Werker (2010), Van Hemert (2010), Munk and Sørensen (2010), and Kraft and Munk (2011) with labor income.

(i) the short-term interest rate is $\tilde{r}_t = r_t + \max(\nu_{Bt}^-, \nu_{St}^-)$,

(ii) the drift of the stock price is $\tilde{r}_t + \sigma_S \lambda_S + \nu_{St}$,

(iii) the drift of the bond price is $\tilde{r}_t + \sigma_B \lambda_B + \nu_{Bt}$, and

(iv) until retirement the individual can trade in an income contract with price dynamics (8), that is with Sharpe ratio λ_{It}.

For artificial markets associated with *deterministic* processes $(\nu_S, \nu_B, \lambda_I)$, the unconstrained utility maximization problem can be solved in closed form by extending the results of Liu (2007) and Munk and Sørensen (2010). Before retirement, the indirect utility function is of the form

$$J_A(t, x, y, r; \nu_S, \nu_B, \lambda_I) = \frac{1}{1-\gamma} g_A(t,r)^\gamma \left(x + yF_A(t,r)\right)^{1-\gamma},$$

where the functions g_A and F_A depend on ν_S, ν_B, λ_I, and the optimal strategies are

$$
\begin{aligned}
c_t &= \frac{X_t + Y_t F_A(t,r_t)}{g_A(t,r_t)}, \\
\pi_{Bt} &= \left(\frac{1}{\gamma(1-\rho_{SB}^2)\sigma_B} \left[\lambda_B + \frac{\nu_B(t)}{\sigma_B} - \rho_{SB} \left(\lambda_S + \frac{\nu_S(t)}{\sigma_S} \right) \right] - \frac{\sigma_r}{\sigma_B} \frac{g_{Ar}(t,r_t)}{g_A(t,r_t)} \right) \frac{X_t + Y_t F_A(t,r_t)}{X_t} \\
&\quad - \left(\frac{\beta}{\sigma_B} \left[\rho_{YB} - \frac{\hat{\rho}_{YS}\rho_{SB}}{\hat{\rho}_S} \right] - \frac{\sigma_r}{\sigma_B} \frac{F_{Ar}(t,r_t)}{F_A(t,r_t)} \right) \frac{Y_t F_A(t,r_t)}{X_t} \\
\pi_{St} &= \frac{1}{\gamma(1-\rho_{SB}^2)\sigma_S} \left[\lambda_S + \frac{\nu_S(t)}{\sigma_S} - \rho_{SB} \left(\lambda_B + \frac{\nu_B(t)}{\sigma_B} \right) \right] \frac{X_t + Y_t F_A(t,r_t)}{X_t} - \frac{\beta}{\sigma_S} \frac{\hat{\rho}_{YS}}{\hat{\rho}_S} \frac{Y_t F_A(t,r_t)}{X_t} \\
\pi_{It} &= \frac{\lambda_I(t)}{\gamma} \frac{X_t + Y_t F_A(t,r_t)}{X_t} - \beta \hat{\rho}_Y \frac{Y_t F_A(t,r_t)}{X_t}.
\end{aligned}
$$

Here g_{Ar} and F_{Ar} denote the derivatives of g_A and F_A with respect to r. Similar expressions apply in retirement, however the income contract is not included and the terms with β in π_B and π_S vanish. Details can be found in Appendix B.

We specialize again to the simple deterministic functions

$$
\begin{aligned}
\nu_S(t) &= \begin{cases} \upsilon_{S0} + \upsilon_{S1} t & , \text{ for } t \in [0, \tilde{T}], \\ -\upsilon_{SR}(\hat{T}_S - t)^+ & , \text{ for } t \in [\tilde{T}, T], \end{cases} \\
\nu_B(t) &= \begin{cases} \upsilon_{B0} + \upsilon_{B1} t & , \text{ for } t \in [0, \tilde{T}], \\ -\upsilon_{BR}(\hat{T}_B - t)^+ & , \text{ for } t \in [\tilde{T}, T], \end{cases} \\
\lambda_I(t) &= \Lambda_0 + \Lambda_1 t,
\end{aligned}
$$

where $\hat{T}_S < T$ and $\hat{T}_B < T$, so that we only have to optimize over a small set of constants. We minimize $J_A(t, x, y, r; \nu_S, \nu_B, \lambda_I)$ over all the simple artificial markets to find the upper bound on the obtainable utility in the true market. The optimal strategy in any simple artificial market is transformed into a feasible strategy in the true market by disregarding the income contract, pruning the investments in the bond and the stock to comply with constraints,[8] and multiplying

[8] If π_S and π_B are both positive and their sum is above 1, we divide both of them by the sum.

human wealth by $(1 - e^{-\eta X_t})$ to ensure non-negative financial wealth before retirement. We can evaluate each of these feasible strategies by Monte Carlo simulation and build this evaluation into a standard optimization routine.

In the following numerical example we assume the values of the interest rate related parameters listed in Table 9, whereas the values of the other parameters are still the same as in Table 1. In particular, we consider a very long-duration bond index with a volatility of $\sigma_B \approx \sigma_r/\kappa = 5\%$ and an excess expected return of $\sigma_B \lambda_B \approx 0.5\%$. We assume that the initial value of the short-term interest rate equals the long-run average of 2%.

The upper bound on the percentage welfare loss associated with the strategy derived by our method is shown in Table 10 for different values of the stock-income correlation ρ_{YS} and the ratio x/y between initial financial wealth and initial annual income. The loss bound is at most 1.4% and often much lower (we find similar results for other parameter combinations). The results for the model with constant interest rates suggest that the bound is not very tight so that actual losses are much smaller.

To indicate that our solution makes economical sense, Figure 3 shows the average (over 10,000 paths) optimal allocation to the stock, the bond index, and the bank account over the life-cycle. Panel (a) is for a zero income-stock correlation, whereas Panel (b) is for a correlation of 0.8. With zero correlation, almost the entire financial wealth is invested in the stock early in life. The long-term bond would be useful to hedge interest rate risk but, on the other hand, the bond has a much smaller Sharpe ratio than the stock and the bond is positively correlated with income. In retirement, the portfolio is still dominated by the stock, but now a significant fraction of financial wealth should be invested in the bond because of its hedging property. When the income is highly correlated with the stock, all financial wealth is invested in the bond through most of working life. In retirement, income is risk-free so the stock becomes attractive. Note again that the income-stock correlation before retirement affects the optimal portfolio in retirement. With a high income-stock correlation, financial wealth at retirement tends to be lower so that a bigger share of that wealth has to be invested in the stock to obtain the desired overall risk-return balance in retirement.

9 Conclusion

This paper has suggested and tested an easy procedure for finding a simple, near-optimal consumption and investment strategy of an investor receiving an unspanned labor income stream. This procedure is valuable since it appears to be impossible to find the truly optimal solution in closed form and very difficult to approximate it precisely using standard numerical solution techniques. For illustrative purposes we have focused on standard models of the price dynamics of traded assets. However, we emphasize that the procedure can be generalized to models of the affine or quadratic classes considered in many recent papers on portfolio choice in the absence of labor income, since in those settings (i) we would still be able to find explicit solutions in the artificially completed markets and (ii) we can still evaluate the performance of a specific strategy by Monte Carlo simulations.

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A Proofs for Constant Interest Rates

Proof of Theorem 1. The dynamics of financial wealth in retirement in the artificial market characterized by $\nu_R(t)$ is

$$dX_t = X_t\left[(\tilde{r}(t) + \pi_{St}[\sigma_S\lambda_S + \nu_R(t)])\,dt + \pi_{St}\sigma_S\,dW_t\right] + (\Upsilon Y_{\tilde{T}} - c_t)\,dt,$$

where $\tilde{r}(t) = r + \nu_R(t)^-$. The Hamilton-Jacobi-Bellman (HJB) equation for the indirect utility function $J = J_R(t, x; \nu_R)$ is

$$\delta J = \sup_{c,\pi_S}\Big\{U(c) + J_t + (\Upsilon Y_{\tilde{T}} - c)J_x + (\tilde{r}(t) + \pi_S[\sigma_S\lambda_S + \nu_R(t)])\,xJ_x + \frac{1}{2}\pi_S^2\sigma_S^2x^2J_{xx}\Big\},$$

where subscripts on J denote partial derivatives, and the terminal condition is $J(T, x) = \varepsilon\frac{1}{1-\gamma}x^{1-\gamma}$. The first-order conditions for c and π_S lead to

$$c = J_x^{-1/\gamma}, \quad \pi_S = -\frac{\sigma_S\lambda_S + \nu_R(t)}{\sigma_S^2}\frac{J_x}{xJ_{xx}}. \tag{19}$$

After substitution of these controls, the HJB equation reduces to

$$\delta J = \frac{\gamma}{1-\gamma}J_x^{1-1/\gamma} + J_t + \Upsilon Y_{\tilde{T}}J_x + \tilde{r}(t)xJ_x - \frac{1}{2}\frac{(\sigma_S\lambda_S + \nu_R(t))^2}{\sigma_S^2}\frac{J_x^2}{J_{xx}}. \tag{20}$$

Conjecture a solution of the form

$$J(t, x) = \frac{1}{1-\gamma}g(t)^\gamma\left(x + \Upsilon Y_{\tilde{T}}F(t)\right)^{1-\gamma}, \tag{21}$$

where $g(T) = \varepsilon^{1/\gamma}$, $F(T) = 0$ to satisfy the terminal condition. After substituting (21) into (20), we collect terms involving $(x + \Upsilon Y_{\tilde{T}}F(t))^{1-\gamma}$ and the remaining terms that all involve $(x + \Upsilon Y_{\tilde{T}}F(t))^{-\gamma}$. This leads to the ordinary differential equations

$$F'(t) - \tilde{r}(t)F(t) + 1 = 0,$$
$$g'(t) - h(t)g(t) + 1 = 0,$$

where

$$h(\tau) = \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\left(r + \nu_R(\tau)^-\right) + \frac{\gamma-1}{2\gamma^2}\frac{(\sigma_S\lambda_S + \nu_R(\tau))^2}{\sigma_S^2}.$$

The solutions that satisfy the above-mentioned terminal values are

$$F(t) = \int_t^T e^{-\int_t^u \tilde{r}(\tau)\,d\tau}\,du,$$
$$g(t) = \varepsilon^{1/\gamma}e^{-\int_t^T h(\tau)\,d\tau} + \int_t^T e^{-\int_t^u h(\tau)\,d\tau}\,du.$$

Inserting the conjectured indirect utility function into (19), we obtain the optimal controls

$$c = \frac{x + \Upsilon Y_{\tilde{T}}F(t)}{g(t)}, \quad \pi_S = \frac{1}{\gamma}\frac{x + \Upsilon Y_{\tilde{T}}F(t)}{x}\frac{\sigma_S\lambda_S + \nu_R(t)}{\sigma_S^2},$$

which define an admissible strategy in this artificial market. □

Proof of Theorem 2. The dynamics of financial wealth before retirement in the artificial market characterized by $\nu_A(t), \nu_R(t), \lambda_I(t)$ is

$$dX_t = X_t\left[(\tilde{r}(t) + \pi_{St}[\sigma_S\lambda_S + \nu_A(t)] + \pi_{It}\lambda_I(t))\ dt + \pi_{St}\sigma_S\,dW_t + \pi_{It}\,dW_{Yt}\right] + (Y_t - c_t)\,dt,$$

where $\tilde{r}(t) = r + \nu_A(t)^-$. The Hamilton-Jacobi-Bellman (HJB) equation for the indirect utility function $J = J_A(t,x;\nu_A,\lambda_I)$ is

$$\delta J = \sup_{c,\pi_S,\pi_I}\Big\{U(c) + J_t + (y-c)J_x + (\tilde{r}(t) + \pi_S[\sigma_S\lambda_S + \nu_A(t)] + \pi_I\lambda_I(t))\,xJ_x$$
$$+ \frac{1}{2}(\pi_S^2\sigma_S^2 + \pi_I^2)x^2J_{xx} + \alpha yJ_y + \frac{1}{2}\beta^2y^2J_{yy} + (\rho\pi_S\sigma_S + \sqrt{1-\rho^2}\pi_I)\beta xyJ_{xy}\Big\},$$

where subscripts on J denote partial derivatives, and the terminal condition is

$$J(\tilde{T},x,y) = \frac{1}{1-\gamma}g_R(\tilde{T};\nu_R)\left(x + \Upsilon yF_R(\tilde{T};\nu_R)\right)^{1-\gamma}.$$

The first-order conditions for c, π_S, and π_I lead to

$$\begin{aligned} c &= J_x^{-1/\gamma},\\ \pi_S &= -\frac{\sigma_S\lambda_S + \nu_A(t)}{\sigma_S^2}\frac{J_x}{xJ_{xx}} - \frac{\beta\rho}{\sigma_S}\frac{yJ_{xy}}{xJ_{xx}},\\ \pi_I &= -\lambda_I(t)\frac{J_x}{xJ_{xx}} - \beta\sqrt{1-\rho^2}\frac{yJ_{xy}}{xJ_{xx}}. \end{aligned} \tag{22}$$

After substitution of these controls, the HJB equation becomes

$$\begin{aligned} \delta J =& \frac{\gamma}{1-\gamma}J_x^{1-1/\gamma} + J_t + yJ_x + \tilde{r}(t)xJ_x + \alpha yJ_y + \frac{1}{2}\beta^2y^2J_{yy}\\ &- \frac{1}{2}\left(\frac{(\sigma_S\lambda_S + \nu_A(t))^2}{\sigma_S^2} + \lambda_I(t)^2\right)\frac{J_x^2}{J_{xx}} - \frac{1}{2}\beta^2y^2\frac{J_{xy}^2}{J_{xx}}\\ &- \beta\left(\rho\frac{\sigma_S\lambda_S + \nu_A(t)}{\sigma_S} + \sqrt{1-\rho^2}\lambda_I(t)\right)y\frac{J_xJ_{xy}}{J_{xx}}. \end{aligned} \tag{23}$$

Conjecture a solution of the form

$$J(t,x,y) = \frac{1}{1-\gamma}g(t)^\gamma\left(x + yF(t)\right)^{1-\gamma}, \tag{24}$$

where $g(\tilde{T}) = g_R(\tilde{T};\nu_R)$, $F(\tilde{T}) = \Upsilon F_R(\tilde{T};\nu_R)$ to satisfy the terminal condition. After substituting (24) into (23), the terms involving $(x+yF(t))^{-\gamma-1}$ cancel. We collect terms involving $(x+yF(t))^{1-\gamma}$ and the remaining terms that all involve $(x+yF(t))^{-\gamma}$. This leads to the ordinary differential equations

$$F'(t) - r_A(t)(t)F(t) + 1 = 0,$$
$$g'(t) - h_A(t)g(t) + 1 = 0,$$

where

$$r_A(\tau) = \tilde{r}(\tau) - \alpha - \beta\left(\rho\frac{\sigma_S\lambda_S + \nu_A(\tau)}{\sigma_S} + \sqrt{1-\rho^2}\lambda_I(t)\right)$$

and

$$h_A(\tau) = \frac{\delta}{\gamma} + \frac{\gamma-1}{\gamma}\left(r + \nu_A(\tau)^-\right) + \frac{\gamma-1}{2\gamma^2}\left[\frac{(\sigma_S\lambda_S + \nu_A(\tau))^2}{\sigma_S^2} + \lambda_I(\tau)^2\right].$$

The solutions consistent with the above-mentioned values at time $\tilde{T}$ are

$$F(t) = e^{-\int_t^{\tilde{T}} r_A(u)\,du}\Upsilon F_R(\tilde{T};\nu_R) + \int_t^{\tilde{T}} e^{-\int_t^u r_A(\tau)\,d\tau}\,du,$$

$$g(t) = e^{-\int_t^{\tilde{T}} h_A(u)\,du} g_R(\tilde{T};\nu_R) + \int_t^{\tilde{T}} e^{-\int_t^u h_A(\tau)\,d\tau}\,du.$$

Inserting the conjectured indirect utility function into (22), we obtain the optimal controls

$$c = \frac{x + yF(t)}{g(t)},$$

$$\pi_S = \frac{1}{\gamma}\frac{x + yF(t)}{x}\frac{\sigma_S\lambda_S + \nu_A(t)}{\sigma_S^2} - \frac{\beta\rho}{\sigma_S}\frac{yF(t)}{x},$$

$$\pi_I = \frac{\lambda_I(t)}{\gamma}\frac{x + yF(t)}{x} - \beta\sqrt{1-\rho^2}\frac{yF(t)}{x},$$

which define an admissible strategy in this artificial market. □

B Detailed Results for Stochastic Interest Rates

Here we state results for the artificial markets in the model with stochastic interest rates considered in Section 8. The proofs are similar to the case of constant interest rates (see Appendix A), although many expressions and computations are more involved because of the additional state variable. Detailed proofs are available from the authors upon request.

In retirement, there is no income risk and thus no income contract, so λ_I is irrelevant and a simple artificial market is characterized by $\nu_S(t)$ and $\nu_B(t)$. The solution to the utility maximization problem in such an artificial market is as follows:

Theorem 3. *The indirect utility during retirement in the artificial market characterized by $\nu_S(t), \nu_B(t)$ is given by*

$$J_R(t,x,r;\nu_S,\nu_B) = \frac{1}{1-\gamma} g_R(t,r;\nu_S,\nu_B)^\gamma \left(x + \Upsilon Y_{\tilde{T}} F_R(t,r;\nu_S,\nu_B)\right)^{1-\gamma},$$

where (suppressing dependence on ν_S, ν_B for notational simplicity)

$$F_R(t,r) = \int_t^T e^{-\int_t^s \max\left[\nu_B^-(u),\nu_S^-(u)\right]du - A(t,s) - \mathcal{B}_\kappa(s-t)r}\,ds,$$

$$g_R(t,r) = \varepsilon^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(T-t) - \frac{\gamma-1}{\gamma}(D(t,T) + \mathcal{B}_\kappa(T-t)r)} + \int_t^T e^{-\frac{\delta}{\gamma}(s-t) - \frac{\gamma-1}{\gamma}(D(t,s) + \mathcal{B}_\kappa(s-t)r)}\,ds,$$

with auxiliary functions

$$A(t,s) = (\nu + \sigma_r \lambda_B)\frac{1}{\kappa}\big(s-t-\mathcal{B}_\kappa(s-t)\big) + \frac{\sigma_r}{\sigma_B}\int_t^s \nu_B(u)\mathcal{B}_\kappa(s-u)\,du$$
$$- \frac{\sigma_r^2}{2\kappa^2}\big(s-t-2\mathcal{B}_\kappa(s-t)+\mathcal{B}_{2\kappa}(s-t)\big),$$
$$D(t,s) = \int_t^s \max\left[\nu_B^-(u), \nu_S^-(u)\right]\,du + \frac{\gamma-1}{\gamma}A(t,s) + \frac{1}{\gamma}\int_t^s \nu_B(u)\mathcal{B}_\kappa(s-u)\,du + \frac{\tilde{D}(t,s)}{2\gamma(1-\rho_{SB}^2)},$$
$$\tilde{D}(t,s) = \left(\lambda_B^2 - 2\rho_S B\lambda_B\lambda_S + \lambda_S^2\right)(s-t) + \frac{1}{\sigma_B^2}\int_t^s \nu_B(u)^2\,du + \frac{2}{\sigma_B}(\lambda_B - \rho_{SB}\lambda_S)\int_t^s \nu_B(u)\,du$$
$$+ \frac{1}{\sigma_s^2}\int_t^s \nu_S(u)^2 du + \frac{2}{\sigma_S}(\lambda_S - \rho_{SB}\lambda_B)\int_t^s \nu_S(u)\,du - 2\frac{\rho_{SB}}{\sigma_B\sigma_S}\int_t^s \nu_S(u)\nu_B(u)\,du.$$

The optimal consumption and investment strategies are

$$c_t = \frac{X_t + \Upsilon Y_{\tilde{T}} F_R(t,r)}{g_R(t,r)},$$
$$\pi_{Bt} = \left(\frac{1}{\gamma(1-\rho_{SB}^2)\sigma_B}\left[\lambda_B + \frac{\nu_B(t)}{\sigma_B} - \rho_{SB}\left(\lambda_S + \frac{\nu_S(t)}{\sigma_S}\right)\right] - \frac{\sigma_r}{\sigma_B}\frac{g_{Rr}(t,r)}{g_R(t,r)}\right)\frac{X_t + \Upsilon Y_{\tilde{T}} F_R(t,r)}{X_t}$$
$$+ \frac{\sigma_r}{\sigma_B}\frac{\Upsilon Y_{\tilde{T}} F_R(t,r)}{X_t}\frac{F_{Rr}(t,r)}{F_R(t,r)},$$
$$\pi_{St} = \frac{1}{\gamma(1-\rho_{SB}^2)\sigma_S}\left[\lambda_S + \frac{\nu_S(t)}{\sigma_S} - \rho_{SB}\left(\lambda_B + \frac{\nu_B(t)}{\sigma_B}\right)\right]\frac{X_t + \Upsilon Y_{\tilde{T}} F_R(t,r)}{X_t}.$$

The indirect utility function J_A for the active phase must satisfy the boundary condition

$$J_A(\tilde{T}, x, y, r; \nu_S, nu_B, \lambda_I) = J_R(\tilde{T}, x, r; \nu_S, \nu_B)$$
$$= \frac{1}{1-\gamma} g_R(\tilde{T}, r; \nu_S, \nu_B)^\gamma \left(x + \Upsilon Y_{\tilde{T}} F_R(\tilde{T}, r; \nu_S), \nu_B\right)^{1-\gamma}.$$

The following theorem states the indirect utility and the optimal strategies before retirement.

Theorem 4. *The indirect utility before retirement in the artificial market characterized by $\nu_S(t), \nu_B(t), \lambda_I(t)$ is given by*

$$J_A(t, x, y, r; \nu_S, \nu_B, \lambda_I) = \frac{1}{1-\gamma} g_A(t, r; \nu_S, \nu_B, \lambda_I)^\gamma \left(x + y F_A(t, r; \nu_S, \nu_B, \lambda_I)\right)^{1-\gamma},$$

where (suppressing dependence on ν_S, ν_B, λ_I for notational simplicity)

$$F_A(t,r) = \Upsilon \int_{\tilde{T}}^T e^{-\int_t^s \left(\max\left[\nu_B^-(u), \nu_S^-(u)\right] + \mathbf{1}_{\{u \le \tilde{T}\}}\beta h(u)\right) du - A(t,s) + f(t,\tilde{T},s) - \mathcal{B}_\kappa(s-t) r}\,ds$$
$$+ \int_t^{\tilde{T}} e^{-\int_t^s \left(\max\left[\nu_B^-(u), \nu_S^-(u)\right] + \beta h(u)\right) du - A(t,s) + f(t,s,s) - \mathcal{B}_\kappa(s-t) r}\,ds,$$
$$g_A(t,r) = \varepsilon^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(T-t) - \frac{\gamma-1}{\gamma}\left(D(t,T) + \varLambda(t,\tilde{T}) + \mathcal{B}_\kappa(T-t) r\right)}$$
$$+ \int_t^T e^{-\frac{\delta}{\gamma}(s-t) - \frac{\gamma-1}{\gamma}\left(D(t,s) + \varLambda(t, s\wedge\tilde{T}) + \mathcal{B}_\kappa(s-t) r\right)}\,ds,$$

with additional auxiliary functions

$$h(u) = \left(\rho_{YB} - \frac{\hat{\rho}_{YS} - \rho_{SB}}{\hat{\rho}_S}\right)\left(\lambda_B + \frac{\nu_B(u)}{\sigma_B}\right) + \frac{\hat{\rho}_{YS}}{\hat{\rho}_S}\left(\lambda_S + \frac{\nu_S(u)}{\sigma_S}\right) + \hat{\rho}_Y\left(\varLambda_0 + \varLambda_1 u\right),$$
$$f(t,u,s) = (u-t)\alpha + \frac{\sigma_r \rho_{YB}\beta}{\kappa}\big(u - t - \mathcal{B}_\kappa(s-u) - \mathcal{B}_\kappa(s-t)\big),$$
$$\varLambda(t,s) = \frac{1}{2\gamma}\left[\varLambda_0^2(s-t) + \varLambda_0\varLambda_1(s^2-t^2) + \frac{1}{3}\varLambda_1^2(s^3-t^3)\right].$$

The optimal consumption and investment strategies are

$$
\begin{aligned}
c_t &= \frac{X_t + Y_t F_A(t, r_t)}{g_A(t, r_t)}, \\
\pi_{Bt} &= \left(\frac{1}{\gamma(1-\rho_{SB}^2)\sigma_B} \left[\lambda_B + \frac{\nu_B(t)}{\sigma_B} - \rho_{SB} \left(\lambda_S + \frac{\nu_S(t)}{\sigma_S} \right) \right] - \frac{\sigma_r}{\sigma_B} \frac{g_{Ar}(t, r_t)}{g_A(t, r_t)} \right) \frac{X_t + Y_t F_A(t, r_t)}{X_t} \\
&\quad - \left(\frac{\beta}{\sigma_B} \left[\rho_{YB} - \frac{\hat{\rho}_{YS} \rho_{SB}}{\hat{\rho}_S} \right] - \frac{\sigma_r}{\sigma_B} \frac{F_{Ar}(t, r_t)}{F_A(t, r_t)} \right) \frac{Y_t F_A(t, r_t)}{X_t} \\
\pi_{St} &= \frac{1}{\gamma(1-\rho_{SB}^2)\sigma_S} \left[\lambda_S + \frac{\nu_S(t)}{\sigma_S} - \rho_{SB} \left(\lambda_B + \frac{\nu_B(t)}{\sigma_B} \right) \right] \frac{X_t + Y_t F_A(t, r_t)}{X_t} - \frac{\beta}{\sigma_S} \frac{\hat{\rho}_{YS}}{\hat{\rho}_S} \frac{Y_t F_A(t, r_t)}{X_t} \\
\pi_{It} &= \frac{\lambda_I(t)}{\gamma} \frac{X_t + Y_t F_A(t, r_t)}{X_t} - \beta \hat{\rho}_Y \frac{Y_t F_A(t, r_t)}{X_t}.
\end{aligned}
$$

δ	time preference rate	0.03		λ_S	stock Sharpe ratio	0.25
γ	relative risk aversion	4		σ_S	stock volatility	0.2
$\tilde{T}$	retirement date	30		y	annual income	2
T	terminal date	50		α	expected income growth	0.01
x	financial wealth	2		β	income volatility	0.1
r	risk-free rate	0.02		Υ	replacement ratio	0.6

Table 1: Benchmark parameter values. The table shows the values of the model parameters used in the numerical computations unless mentioned otherwise. Time is measured in years. The initial wealth $x = 2$ and annual income $y = 2$ are interpreted as USD 20,000.

	stock-income correlation ρ					
wealth/income ratio x/y	0	0.2	0.4	0.6	0.8	1
0.1	0.686%	0.426%	0.327%	0.045%	0.084%	0.135%
0.25	0.714%	0.435%	0.305%	0.046%	0.078%	0.144%
1	0.842%	0.477%	0.182%	0.038%	0.082%	0.100%
4	1.110%	0.478%	0.153%	0.039%	0.035%	0.093%
10	0.815%	0.351%	0.149%	0.057%	0.019%	0.033%

Table 2: Upper bound on welfare loss. Upper bound on the welfare loss as a percentage of total wealth for different values of the stock-income correlation and the wealth/income ratio.

	stock-income correlation ρ					
initial time t	0	0.2	0.4	0.6	0.8	1
0	0.842%	0.477%	0.182%	0.038%	0.082%	0.100%
10	0.645%	0.415%	0.191%	0.029%	0.085%	0.120%
20	0.467%	0.377%	0.265%	0.228%	0.247%	0.289%
25	0.801%	0.795%	0.781%	0.749%	0.804%	0.844%

Table 3: Upper bound on welfare loss for different horizons. Upper bound on the percentage welfare loss for different time horizons and stock-income correlations. The wealth/income ratio at time t is fixed at 1.

	stock-income correlation ρ					
method	0	0.2	0.4	0.6	0.8	1
our SAMS method	0.842%	0.477%	0.182%	0.038%	0.082%	0.100%
MCA method, fine grid	0.728%	0.454%	0.185%	0.038%	0.080%	0.068%
loss difference	0.115%	0.023%	-0.003%	-0.000%	0.003%	0.032%
MCA method, coarse grid	0.879%	0.621%	0.212%	0.076%	0.105%	0.246%
loss difference	-0.037%	-0.144%	-0.030%	-0.038%	-0.023%	-0.147%

Table 4: Upper bounds on welfare loss for two different methods. Upper bounds on the percentage welfare loss for the two different methods and the increase in the upper bound (the loss difference) by applying our SAMS method instead of the MCA method. We assume the benchmark parameter values as well as $t = 0$ and $x/y = 1$.

parameter	stock-income correlation ρ 0	0.2	0.4	0.6	0.8	1
Λ_0	0.43578	0.40760	0.40764	0.28137	0.28139	0.00799
Λ_1	-0.00261	-0.00143	-0.00141	-0.00067	-0.00060	-0.00027
v_0	-0.01559	0.00070	0.00074	0.00008	0.00014	0.00009
v_1	0.00064	-0.00002	0.00006	0.00002	0.00005	0.00015

Table 5: Optimal parameters for the incomplete market. Optimal parameters for the incomplete market for $x/y = 1$ and $t = 0$. In addition, $v_R = 0$ (so that $\hat{T}$ is meaningless) and $\eta = 30$.

wealth/income ratio x/y	stock-income correlation ρ 0	0.2	0.4	0.6	0.8	1
0.1	0.08%	0.04%	0.00%	0.00%	0.00%	0.01%
0.25	0.04%	0.03%	0.01%	0.01%	0.00%	0.01%
1	0%	0%	0%	0%	0%	0%
4	0.15%	0.02%	0.00%	0.01%	0.00%	0.01%
10	0.37%	0.05%	0.01%	0.00%	0.00%	0.02%

Table 6: Increase in percentage welfare loss. Increase in percentage welfare loss bound from applying the strategy based on the parameter $\boldsymbol{\psi}^*$ optimal for $x/y = 1$ for other values of x/y. For instance, the 0.08% reported for $x/y = 0.1$ and $\rho = 0$ means that the upper bound on the welfare loss increases by 0.08 percentage points when using the auxiliary parameters found optimal for $x/y = 1$ in the consumption and investment strategy for $x/y = 0.1$ instead of the auxiliary parameters that are indeed optimal for $x/y = 0.1$.

wealth/income ratio x/y	stock-income correlation ρ 0	0.2	0.4	0.6	0.8	1
0.1	23.06	22.49	21.96	21.57	21.48	21.53
0.25	23.16	22.56	22.00	21.59	21.50	21.54
1	23.60	22.87	22.19	21.72	21.59	21.64
4	24.82	23.70	22.74	22.13	21.86	21.86
10	26.12	24.65	23.52	22.70	22.19	22.02

Table 7: Income multipliers. Income multipliers at time $t = 0$ for different wealth/income ratios x/y and different stock-income correlations ρ.

	stock-income correlation ρ					
	0		0.4		0.8	
parameter	bound	MCA diff.	bound	MCA diff.	bound	MCA diff.
$\gamma = 3$	0.938%	0.168%	0.414%	0.038%	0.102%	0.006%
$\gamma = 5$	0.813%	0.061%	0.044%	-0.022%	0.062%	0.020%
$\alpha = 0.005$	0.887%	0.097%	0.155%	-0.004%	0.098%	-0.000%
$\alpha = 0.015$	0.783%	0.125%	0.242%	-0.000%	0.067%	0.000%
$\beta = 0.05$	0.624%	0.156%	0.383%	0.068%	0.224%	0.027%
$\beta = 0.15$	1.500%	-0.357%	0.266%	-0.023%	0.457%	0.042%
$\Upsilon = 0.5$	0.809%	0.093%	0.151%	-0.005%	0.086%	0.001%
$\Upsilon = 0.7$	0.896%	0.139%	0.245%	0.006%	0.102%	0.003%

Table 8: Robustness checks. For each correlation value the left column shows the upper bound on percentage welfare losses when the value of the parameter is changed from its benchmark value as indicated. The right column shows the additional welfare loss from applying the consumption and investment strategy suggested by our method instead of the strategy suggested by the Markov Chain Approximation method implemented with a fine grid ($N = 4000$, $I = 12000$).

κ	mean reversion speed	0.2
$\bar{r}$	long-run short rate	0.02
σ_r	short rate volatility	0.01
λ_B	bond Sharpe ratio	0.1
$\bar{\tau}$	bond maturity	50
ρ_{SB}	stock-bond correlation	0.1
ρ_{YB}	income-bond correlation	0.25

Table 9: Additional parameter values with stochastic interest rates. The table shows the values of the additional parameters in the model with stochastic interest rates.

	stock-income correlation ρ_{YS}				
wealth/income ratio x/y	0	0.2	0.4	0.6	0.8
0.1	1.094%	0.958%	0.679%	0.400%	0.175%
0.25	1.113%	0.963%	0.629%	0.420%	0.190%
1	1.261%	1.151%	0.739%	0.350%	0.236%
4	1.431%	0.652%	0.428%	0.425%	0.451%
10	0.715%	0.327%	0.180%	0.149%	0.525%

Table 10: Upper bound on welfare loss for stochastic interest rates. Upper bound on the percentage welfare loss in the case with stochastic interest rates for different values of the wealth/income ratio and the stock-income correlation.

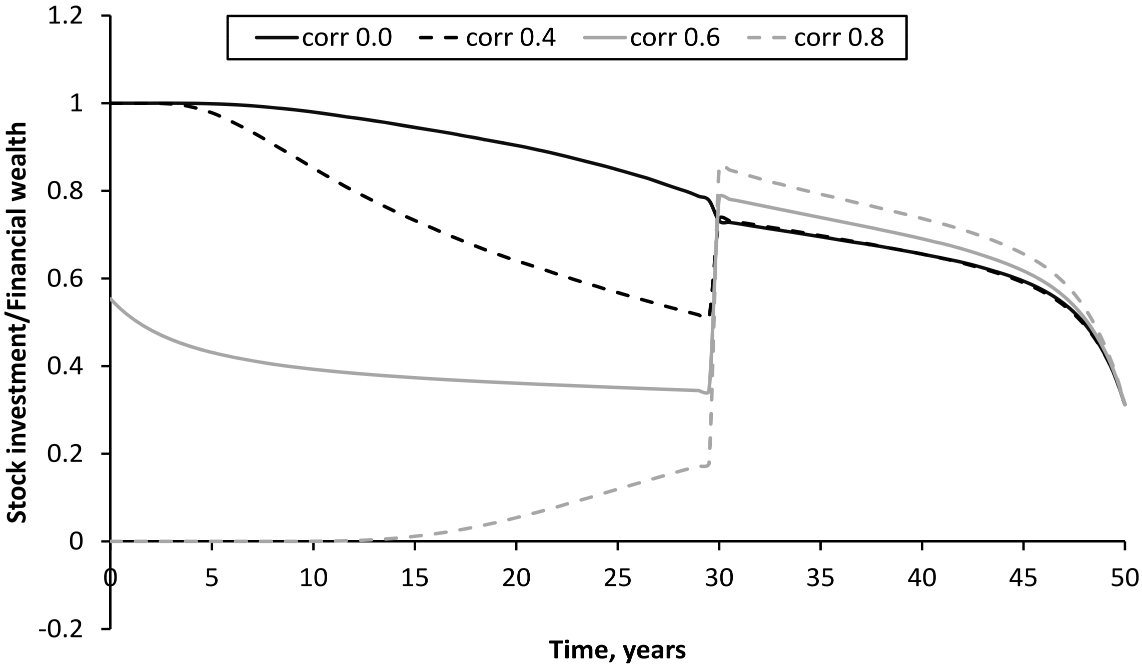


Figure 1: Stock weight over the life-cycle. For each point in time the graph shows the average fraction of financial wealth invested in the stock across all simulated paths. The initial wealth-income ratio is assumed to be $x/y = 1$.

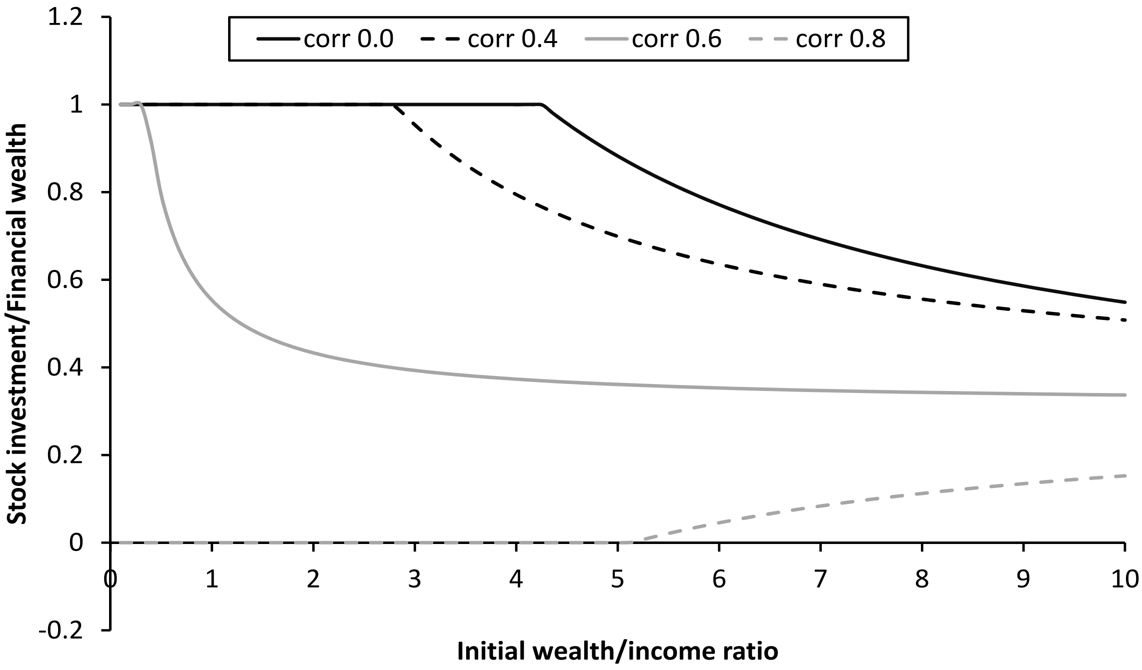
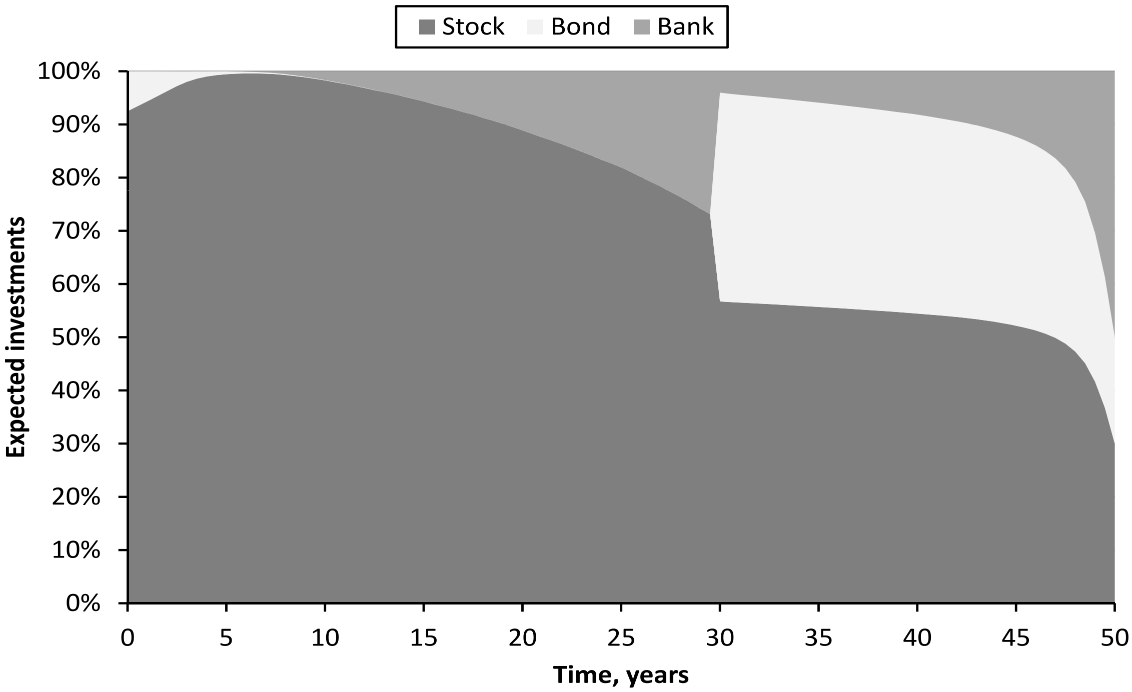
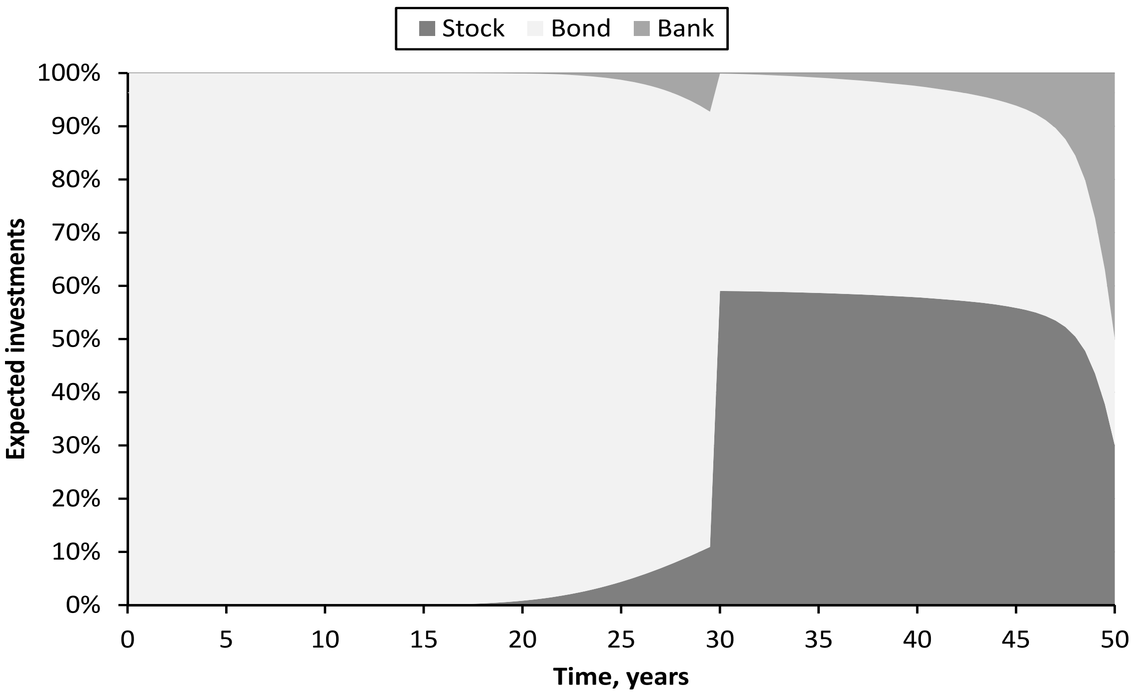


Figure 2: Financial wealth fractions. The initial fraction of financial wealth invested in the stock for different values of the stock-income correlation.



(a) Correlation $\rho_{YS} = 0.0$



(b) Correlation $\rho_{YS} = 0.8$

Figure 3: Optimal portfolios over the life-cycle. The graphs show how the average portfolio varies over the life-cycle when the income-stock correlation is 0 (Panel a) or 0.8 (Panel b). The optimal strategies are computed with our numerical method explained in the text. 10,000 paths of income, interest rates, and wealth (applying those strategies) are then simulated over the 50-year period considered. The graphs show averages over the paths.

Consumption-Portfolio Choice with Unspanned Labor Income and Stochastic Volatility

Björn Bick

Summary. We analyze a consumption-portfolio choice problem of an investor that receives labor income and faces a stock market with stochastic volatility. We show how labor income and stochastic volatility affect the decisions on optimal portfolio holdings and consumption over the lifetime. We provide closed-form solutions for a market where the income stream of the investor is spanned by traded assets. Furthermore, we discuss solution techniques for the case of unspanned income where explicit solutions do not exist. We compare a method suggested in Bick, Kraft and Munk (2011) with a standard finite-difference algorithm and show that the first method outperforms the numerical algorithm.

1 Introduction

Labor income is one of the dominant assets for most individuals. Since, at the beginning of an individuals life, his present value of all future income (also called human wealth) is worth much more than his current financial wealth, income has a large influence on his optimal portfolio and consumption decisions over the life-cycle. In addition, it is well-documented that the volatilities of asset returns are stochastic. Consequently, the investor has to take the additional volatility risk into account. In general, both labor income and stochastic volatility of returns involve additional idiosyncratic risk components. By including derivatives that are able to hedge the volatility shocks, we can handle this specific risk component. Nevertheless, we still face an incomplete market as the individuum is not able to hedge against unfavorable shocks to his income stream because the income stream is typically not spanned by traded assets. In contrast to the volatility risk that can be hedged by derivatives, there do not exist any markets where claims on income are traded.[1] Hence, it seems impossible to find closed-form expressions for the dynamic consumption and investment strategies maximizing the lifetime utility of an individual consumer-investor.

In our study, we examine an investor that makes his decisions in continuous-time and maximizes expected power utility from terminal wealth and intermediate consumption. During his

[1] Although there exist income insurance contracts offered by governments and insurance companies, they are far from being perfect due to moral hazard problems.

lifetime, the investor is allowed to invest in stocks and a money market account and has access to the derivative market to get the desired exposure to volatility risk. Moreover, he receives an exogenously given stochastic income stream until a predetermined retirement date. After retirement, income becomes risk-free and the individuum receives a constant fraction of his last salary. The stochastic volatility process and the asset returns are modeled using a Heston framework (Heston (1993)). The corresponding dynamic of the specific derivative follows by a replication argument and is unique once we have specified the market price of volatility risk (see Liu and Pan (2003)).

Nevertheless, we are still left with the risky and unspanned income risk. In our main analysis, we will investigate two scenarios. First, we assume that labor income is fully spanned by traded assets and that there are no constraints on investment decisions. In that case, human wealth, which is defined as the present value of all future labor income, can be computed by no-arbitrage arguments. We derive closed-form solutions for the optimal investment and consumption decisions, basically by replacing financial wealth by total wealth[2] and following the procedure of Merton (1969) and Merton (1971). Although the assumption of spanned income is debatable and not realistic, the explicit solutions provide some intuition behind the optimal strategies. In order to solve the involved optimization problem, it is formulated in terms of exposures to the different risk components. These exposures are then transformed to the optimal investment proportions in the stock and the option. The optimal strategies can be decomposed into speculative demands and hedging demands for volatility and stock-like income risk. The large influence of human wealth becomes evident when looking at the corresponding investment and consumption strategies. Regarding the investment in the derivative contract, we observe that the ability of the specific contract to hedge against volatility risk plays a crucial role. We illustrate our findings by several life-cycle studies where we apply a realistic calibrated parameter set. Our model is able to replicate patterns that are found in several life-cycle studies like hump-shaped financial wealth (see e.g Munk and Soerensen (2010) and Kraft and Munk (2011)). In the second scenario, we allow for unspanned income risk and impose liquidity constraints, i.e the investor must not borrow against future income during his working career. Consequently, we face an incomplete market and cannot solve the problem in closed-form. To obtain solutions, two approaches are applied. The first approach is based on a paper by Bick, Kraft and Munk (2011), who introduce a hypothetical income contract to create an artificial unconstrained market where they are able to compute closed-form solutions. The authors refer to their approach as SAMS, short for *Simulation of Artificial Markets Strategies.* Following this idea, we transform the strategies from these artificially markets to strategies that are admissible in the true incomplete market and measure their performances via upper bounds on welfare losses which represent the utility losses stated in terms of total wealth. Our upper bounds are

[2] Total wealth is the sum of financial wealth and human wealth.

at most about 6.7% over an investment horizon of 40 years and in many cases they are much lower. In the second approach, we use a standard grid-based numerical dynamic programming procedure. In our study, we choose the Markov chain approach, which is an application of a finite difference scheme and frequently applied to consumption-portfolio problems. Comparing both methods, we find that SAMS always outperforms the Markov chain technique although the latter is computationally much more intensive. We wish to stress that SAMS could also be applied to problems of higher dimensions whereas it seems impossible to solve grid-based schemes with more than three state variables.

Let us briefly review the related literature of this study. A common assumption in most of the literature about portfolio optimization is that investors live off financial wealth exclusively and disregard labor income completely (e.g. Merton (1969), Samuelson (1969), Campbell and Viceira (2001), Liu, Longstaff and Pan (2003), and Liu (2007)). There also exist several studies that include income risk in a setting with constant investment opportunities. Hakansson (1970) and Merton (1971) consider a deterministic income stream which is equivalent to an implicit investment in the risk-free asset. As a result, the optimal fraction of financial wealth invested into the stock market will increase if labor income is present. Bodie, Merton and Samuelson (1992) account for a risky income stream, but assume that human wealth is spanned by traded assets. They derive closed-form strategies and show that the optimal investments are strongly dependent on the stock-like income risk which can result in large positions. On the other hand, there are also several papers that allow for unspanned income risk (e.g. Munk (2000) and Cocco, Gomes and Maenhout (2005)). In all these papers, the authors have to use numerical solution techniques that are based on a finite difference scheme. The Markov chain approximation method that we use in this paper is adopted from Munk (2000). There are many studies that deal with the optimal portfolio decisions when the investor faces a stochastic opportunity set. More specific, when it is assumed that volatility is stochastic, Kraft (2005) and Liu (2007) show that there exist closed-form solutions for an investor with power utility that receives utility from terminal wealth only. Both studies can solve the problem in an incomplete market and do not need additional derivatives, which is only possible as they do not consider utility from intermediate consumption. Liu and Pan (2003) include derivatives to the investment set and show that there is significant benefit from using derivatives. In their paper, the investor only gets utility from terminal wealth. The work of Munk and Soerensen (2010) is similar to ours. They solve a model where interest rates are stochastic and the investor can additionally invest in the bond market to hedge the interest rate risk. The idea of artificially completing the market was developed in the work of Cvitanić and Karatzas (1992) who add penalty terms to the interest rate and drift of the stock to obtain unconstrained artificially markets.

The remainder of the paper is structured as follows. Section 2 describes the consumption and portfolio choice problem of the investor. Section 3 summarizes the solution for the case where labor income is spanned by traded assets. In Section 4, we describe the artificially completed markets and derive the corresponding optimal consumption and investment strategies. Moreover, we explain how to transform the optimal strategies in the artificial market into admissible strategies in the real market, how we find the best of such strategies, how we evaluate the performance of these strategies, and how we apply the Markov chain approximation technique to our problem. Section 5 discusses several numerical results, in both the complete and the incomplete market. Finally, Section 6 concludes. All proofs are summarized in the Appendix.

2 The Problem

We study the life-cycle consumption and portfolio decisions of an investor. We assume that the individual can invest his money in a single stock (e.g. representing the stock market index), an option on the stock, and a bank account offering a constant risk-free rate of r. Moreover, the investor receives a stochastic stream of income payments from non-financial sources, which we refer to as labor income. We assume that there is a single consumption good in the economy that serves as the numeraire. This means that all prices are expressed in units of this numeraire that we choose to be money.

2.1 Financial Assets

As we assume a constant short-term interest rate, the money market account becomes

$$M_t = e^{rt} \tag{1}$$

Additionally, the agent can put his money into a stock (e.g. representing the stock market index) that pays no dividends. We model the evolution of the stock by a stochastic volatility model. Such a model assumes that the volatility v_t of the stock is itself a stochastic process. The time t price of the stock is denoted by S_t and the price dynamics read

$$dS_t = S_t \left[(r + \eta_S v_t) dt + \sqrt{v_t} dW_{St} \right], \tag{2}$$

where $W_{St} = (W_{St})_{t \geq 0}$ is a standard Brownian motion and the excess return of the stock $\eta_S v_t$ is assumed to be the product of volatility and the constant parameter η_S. In order to obtain a tractable model, we choose the affine stochastic volatility model of Heston. Heston (1993) models the volatility by a square-root mean-reverting process

$$dv_t = (\nu - \kappa v_t) dt + \bar{\beta} \sqrt{v_t} \left(\hat{\rho}_v dW_{vt} + \rho_{sv} dW_{St} \right), \tag{3}$$

where $W_{vt} = (W_{vt})_{t \geq 0}$ is a standard Brownian motion independent of W_{St}. The parameters $\nu, \kappa, \bar{\beta}$ are assumed to be constant, ρ_{sv} is the instantaneous correlation between the stock and volatility, and

$$\hat{\rho}_v = \sqrt{1 - \rho_{sv}^2}.$$

It is well known that such a model is able to generate several empirical stylized facts such as heavy tails and high peaks of asset returns (e.g. Andersen, Benzoni and Lund (2002) and Eraker, Johannes and Polson (2003)). Further, we can create arbitrary correlations between the stock price and volatility which enables us to model the well-documented leverage effect.

In order to manage volatility risk, we introduce a derivative O_t that can be used to hedge this risk. If a derivative is traded, we assume that there exists a function h such that the price of the claim fulfills $O_t = h(t, S_t, v_t)$. Applying Ito's formula, we can deduce the following SDE

$$dO_t = O_t r\, dt + \left(h_s S_t + h_v \bar{\beta}\rho_{sv}\right)\left(\eta_S v_t\, dt + \sqrt{v_z}\, dW_{St}\right) + h_v \bar{\beta}\hat{\rho}_v \left(\eta_v v_t\, dt + \sqrt{v_t}\, dW_{vt}\right), \quad (4)$$

where

$$h_s = \frac{\partial h(t,s,v)}{\partial s}\Big|_{(S_t,v_t)}$$
$$h_v = \frac{\partial h(t,s,v)}{\partial v}\Big|_{(S_t,v_t)}$$

denote the corresponding derivatives. These derivatives express how sensitive the price of the option reacts to changes in stock price and volatility, respectively. Consequently, an option with $h_s \neq 0$ provides exposure to the price shock W_S; a derivative with non-zero h_v provides exposure to the volatility risk component. The functions h_s and h_v are also known as the delta and the vega of the option. The constant parameter η_v can be interpreted as the risk premium of volatility risk. To be more precise, $\eta_v\sqrt{v_t}$ is the risk premium for incurring one unit of W_{vt}.

2.2 Labor Income

In our analysis, the agent receives a continuous income stream of Y_t that is exogenously given. Until the predetermined retirement date $\tilde{T}$ the income stream is both stochastic and unspanned and follows the dynamics

$$dY_t = Y_t\left[\alpha dt + \hat{\beta}\sqrt{v_t}\left(\rho_{ys} dW_{St} + \hat{\rho}_{yv} dW_{vt} + \hat{\rho}_y dW_{Yt}\right)\right], \quad 0 \le t \le \tilde{T}, \quad (5)$$

where $W_{Yt} = (W_{Yt})_{t\geq 0}$ is a Brownian motion that is independent of W_{St} and W_{vt}. In the labor income dynamics, ρ_{ys} is the constant correlation between asset return and income growth, and

$$\hat{\rho}_{yv} = \frac{\rho_{yv} - \rho_{sv}\rho_{ys}}{\sqrt{1-\rho_{sv}^2}}, \quad \hat{\rho}_y = \sqrt{1 - \rho_{ys}^2 - \hat{\rho}_{yv}^2},$$

where ρ_{yv} is the constant correlation between income and volatility.[3]

The constant parameter α represents the expected income growth in the active working phase and the volatility $\hat{\beta}\sqrt{v_t}$ is represented by the constant parameter $\hat{\beta}$. Our analysis can

[3] To create such an interaction structure in a tractable form, we applied a Cholesky decomposition to the original correlation matrix.

easily be extended to account for a time dependent income growth rate in order to capture the fluctuations of labor income over the life-cycle (see e.g. Munk and Soerensen (2010), Cocco, Gomes and Maenhout (2005)).

In the retirement period $(\tilde{T}, T]$, the investor receives a retirement pension $\tilde{Y}_t$ which is assumed to be risk-free. The first income payment in retirement at $\tilde{T}$ equals a fraction Υ (called replacement rate) of his last observed income payment $Y_{\tilde{T}}$, so that we observe a downward jump at retirement. We also allow the income process to grow at a constant rate $\tilde{\alpha}$ in this period

$$d\tilde{Y}_t = \tilde{\alpha}\tilde{Y}_t dt, \quad \tilde{Y}_{\tilde{T}} = \Upsilon Y_{\tilde{T}}. \tag{6}$$

We wish to stress that after retirement the individual always faces a complete market. In contrast, when the investor is still working, we have in general to deal with an incomplete market setting. There are two special cases where the market is complete, even in the active phases. The first case emerges when the idiosyncratic risk component of income is assumed away, i.e $\hat{\rho}_y = 0$. This implies that as long as we do not impose any constraints on the investments, income shocks are spanned and we are always able to replicate the income stream. To derive closed-form strategies, we could also assume that income is risk-free by setting $\hat{\beta} = 0$ (see Hakansson (1970)). In the remainder, when considering complete markets, we focus primarily on the spanned income case. However, the results collapse into the solution to the problem with risk-free income stream by setting $\hat{\beta} = 0$.

2.3 Wealth Dynamics and Preferences

At every point in time, the individual can choose a consumption and an investment strategy. The consumption strategy is described by a stochastic process $c = (c_t)_{t\geq 0}$ with the requirement that $c_t \geq 0$ for all t. An investment strategy is characterized by the processes $\pi_S = (\pi_{St})_{t\geq 0}$ and $\pi_O = (\pi_{Ot})_{t\geq 0}$, which are the fractions of financial wealth invested in the stock and the option. Consequently, the remaining fraction $(1 - \pi_{St} - \pi_{Ot})$ is invested in the money market account. Let X_t denote the financial wealth of the individuum at time t. Then, for a specific consumption and investment choice (c, π)[4], the evolution of the wealth process is given by

$$\begin{aligned} dX_t = X_t \Bigg[& \left(r + \pi_S \eta_S v_t + \pi_O \frac{(h_s S_t + h_v \bar{\beta} \rho_{sv})\eta_S + h_v \bar{\beta} \hat{\rho}_v \eta_v}{O_t} v_t \right) dt \\ & + \left(\pi_S + \pi_O \frac{h_s S_t + h_v \bar{\beta} \rho_{sv}}{O_t} \right) \sqrt{v_t} dW_{St} + \pi_O \frac{h_v \bar{\beta} \hat{\rho}_v}{O_t} \sqrt{v_t} dW_{vt} \Bigg] \\ & + (\mathbf{1}_{t<\tilde{T}} Y_t + \mathbf{1}_{t\geq \tilde{T}} \tilde{Y}_t - c_t) dt. \end{aligned} \tag{7}$$

In the remainder of this study, we will examine two different sets of strategies. If we only demand positive terminal wealth $X_T \geq 0$ from a strategy, $\mathcal{A}_t^{unc}$ will denote the set of admissible

[4] The different strategies have to satisfy some technical conditions to have a well-defined wealth process X_t.

strategies over the time interval $[t, T]$ and we will use this set when we assume that income is spanned. For the case where we cannot replicate income, the individual is not allowed to have negative financial wealth during his working phase as he cannot make sure that he ends up with positive financial wealth, i.e. that he can liquidate all short positions until death. The set of admissible strategies with this liquidity constraint is denoted by $\mathcal{A}_t^{con}$.

The individual has time additive preferences with respect to consumption and terminal wealth. An admissible consumption and investment strategy $(c; \pi)$ generates the expected utility

$$J(t, x, v, y; c, \pi) = \mathbf{E}_t \left[\int_t^T e^{-\delta(s-t)} u(c_s) ds + \varepsilon e^{-\delta(T-t)} u(X_T) \right], \tag{8}$$

where the expectation is conditioned on $X_t = x$, $v_t = v$ and $Y_t = y$ [5], δ is the subjective discount rate, and ε models the relative weight of terminal wealth and intermediate consumption. The indirect utility function is given by

$$J(t, x, v, y) = \max_{(c;\pi) \in \mathcal{A}} J(t, x, v, y; c, \pi),$$

where $\mathcal{A}$ is either equal to $\mathcal{A}_t^{unc}$ or to $\mathcal{A}_t^{con}$. We will use both sets in the upcoming sections. Throughout the paper, we assume that the utility function exhibits a constant relative risk aversion, i.e. $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma > 1$.

In order to compute the optimal strategies, we solve the Hamilton-Jacobi-Bellmann equation (HJB), which is involved for the wealth equation (7). Therefore, we refrain from expressing the dynamics of the financial wealth process in terms of π and instead use risk exposures. As argued by Liu and Pan (2003) and Kraft (2003), working in terms of the optimal exposure is mathematically more convenient than working with the optimal portfolio weights. Under the assumption that the agent can always invest in a derivative with non-zero vega h_v and non-zero price $h \neq 0$, it is equivalent to decide on his optimal portfolio holdings or on his optimal exposure with respect to the risk factors. For given portfolio weights π_{St} and π_{Ot}, these factor exposures are defined as

$$\phi_{St} = \pi_{St} + \pi_{Ot} \frac{h_s S_t + h_v \bar{\beta} \rho_{sv}}{O_t}, \tag{9}$$

$$\phi_{vt} = \pi_{Ot} \frac{h_v \bar{\beta} \hat{\rho}_v}{O_t}. \tag{10}$$

In terms of factor exposures $\phi = (\phi_v, \phi_S)$, the dynamics of wealth are then given by

$$\begin{aligned} dX_t = {} & X_t[rdt + \phi_{St}(\eta_S v_t dt + \sqrt{v_t} dW_{St}) + \phi_{vt}(\eta_v v_t dt + \sqrt{v_t} dW_{vt})] \\ & + (\mathbf{1}_{t<\tilde{T}} Y_t + \mathbf{1}_{t \geq \tilde{T}} \tilde{Y}_t - c_t) dt, \end{aligned} \tag{11}$$

where ϕ_{St} can be interpreted as the the fraction of wealth invested into $\sqrt{v_t} W_{St}$. Equivalently, ϕ_{vt} is the fraction invested in $\sqrt{v_t} W_{vt}$.

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[5] If $t \geq \tilde{T}$, Y_t will be replaced by $\tilde{Y}_t$.

By investing π_{St} and π_{Ot} in the stock and the option, our agent holds the desired amounts ϕ_{St} in the diffusive part of the stock and ϕ_{vt} in the additional volatility risk. Taking a position of π_{St} in the stock market yields an exposure to the risk component in stock prices, which is quite intuitive. On the other side, taking a position π_{Ot} in the derivative market yields exposure to both the volatility diffusion W_v and the stock Brownian W_S as $h_v \neq 0$. Furthermore, it also provides exposure to the stock diffusion whenever the option-delta is different from 0. In all our calculations we will use the form (11) of the wealth process. To write the HJB equation in a compact and tractable form, we introduce some matrix notation. Let us define

$$\alpha_t = (\phi_v \sqrt{v_t}, \phi_S \sqrt{v_t}, 0)^T \,, \quad \lambda_t = (\eta_v \sqrt{v_t}, \eta_S \sqrt{v_t}, 0)^T \,,$$

$$\mu_{Zt} = (\nu - \kappa v_t, \alpha Y_t)^T \,, \quad \Sigma_{Zt} = \begin{pmatrix} \bar{\beta}\sqrt{v_t}\hat{\rho}_v & \bar{\beta}\sqrt{v_t}\rho_{sv} & 0 \\ \hat{\beta}\sqrt{v_t}\hat{\rho}_{yv} & \hat{\beta}\sqrt{v_t}\rho_{ys} & \hat{\beta}\sqrt{v_t}\hat{\rho}_y \end{pmatrix}, M_{Yt} = \begin{pmatrix} 1 & 0 \\ 0 & Y_t \end{pmatrix}.$$

Setting $Z = (v, Y)^T$,yields the following wealth and state dynamics

$$dX_t = \left(rX_t + X_t\alpha_t^T \lambda_t - c_t + Y_t\right) dt + X_t \alpha_t^T \, d \begin{pmatrix} W_{vt} \\ W_{St} \\ W_{Yt} \end{pmatrix}, \tag{12}$$

$$dZ_t = \mu_{Zt} \, dt + M_{Yt}\Sigma_{Zt} d \begin{pmatrix} W_{vt} \\ W_{St} \\ W_{Yt} \end{pmatrix}. \tag{13}$$

Now, we can rewrite the HJB equation as follows:[6]

$$0 = \max_{(c;\alpha)} \Bigg\{ J_t - \delta J + (rX + Y)J_X + \mu_Z^T J_Z + \frac{1}{2}\text{tr}\left[M_Y \Sigma_Z \Sigma_Z^T M_Y^T J_{ZZ}\right] + X\alpha^T \lambda J_X \tag{14}$$

$$+ \frac{1}{2} X^2 \alpha^T \Sigma^T \alpha J_{XX} + X\alpha^T \Sigma_Z^T M_Y J_{XZ} + \frac{1}{1-\gamma} c^{1-\gamma} - cJ_X \Bigg\},$$

where J_t, J_X, and J_Z denote the derivatives of the indirect utility function with respect to t, X, and Z, and equivalent definitions with respect to higher derivatives.

3 Solution for Spanned Income

In this section, we study the case when the income stream is spanned by traded assets. We thus set $\hat{\rho}_y = 0$ to get rid of the idiosyncratic shock to the income stream, which implies that

$$\hat{\rho}_{yv} =^{+}_{-} \sqrt{1 - \rho_{ys}^2}.$$

Therefore, we face a complete market[7] where the human capital, which we define as the expected discounted value of all future labor earnings, can be computed by risk-neutral pricing. Note that

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6 For shorter notations, we omit the time-dependency of variables and vectors.

7 We would also get a complete market if we had a risk-free income stream. We do not examine this case in detail. But, all results follow by just setting $\hat{\beta} = 0$.

we do not impose any constraints on the investor. In the notation of the previous section, we thus consider policies $(c;\pi) \in \mathcal{A}^{unc}$. In such a complete market with unique risk-neutral pricing measure $\mathbb{Q}$, the human capital is given by

$$L^{com}(t, v_t) = \mathrm{E}_t^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} Y_s \, ds\right], \tag{15}$$

which can, up to numerical integrals, be computed in closed-form. As we see in the following theorem, the time t value of human wealth can also be expressed as

$$L^{com}(t, v_t) = \begin{cases} Y_t F^{com}(t, v_t), & t \leq \tilde{T}, \\ \Upsilon Y_t F^{com}(t, v_t), & t \in (\tilde{T}, T]. \end{cases} \tag{16}$$

Next, following Hakansson (1970) and Bodie, Merton and Samuelson (1992), one can think of the investor selling his remaining income for the amount $L^{com}(t, v_t)$. Therefore, total wealth at time t becomes $X(t)+L^{com}(t, v_t)$. Notice that human capital depends on the volatility process v_t, which complicates all further computations.

We try to derive the optimal policies from the optimal strategies for the case without income, but with a financial wealth of $X_t + L^{com}(t, v_t)$ instead of just X_t. Due to the assumed completeness, the affine process dynamics, and the assumption of unconstrained controls, the problem can be solved explicitly. The following theorem summarizes the results.

Theorem 5. ***(Solution for Truly Complete Market)*** *Assume $\hat{\rho}_y = 0$. Then the indirect utility function is given by*

$$J^{com}(t, x, v, y) = \frac{1}{1-\gamma} g^{com}(t, v)^{\gamma} (x + y F^{com}(t, v))^{1-\gamma}, \tag{17}$$

where

$$g^{com}(t, v) = \begin{cases} \varepsilon^{\frac{1}{\gamma}} e^{-r_g(T-t) - A^r(\tilde{T}, T) - A_{\alpha(T)}(t, \tilde{T}) - B_{\alpha(T)}(t, \tilde{T}) v} + \int_{\tilde{T}}^T e^{-r_g(s-t) - A^r(\tilde{T}, s) - A_{\alpha(s)}(t, \tilde{T}) - B_{\alpha(s)}(t, \tilde{T}) v} \, ds \\ \qquad + \int_t^{\tilde{T}} e^{-r_g(s-t) - A_0(t,s) - B_0(t,s) v} ds, \quad t \leq \tilde{T}, \\ \varepsilon^{\frac{1}{\gamma}} e^{-r_g(T-t) - A^r(t,T) - B^r(t,T) v} \\ \qquad + \int_t^T e^{-r_g(s-t) - A^r(t,s) - B^r(t,s) v} ds, \quad t \in (\tilde{T}, T]. \end{cases}$$

$$F^{com}(t, v) = \begin{cases} \Upsilon F^{com}(\tilde{T}, v) e^{-(r-\alpha)(\tilde{T}-t) - A^F(t, \tilde{T}) - B^F(t, \tilde{T}) v} + \int_t^{\tilde{T}} e^{-(r-\alpha)(s-t) - A^F(t,s) - B^F(t,s) v} \, ds, & t \leq \tilde{T}, \\ \frac{1}{r-\tilde{\alpha}} \left[1 - e^{-(r-\tilde{\alpha})(T-t)}\right], & t \in (\tilde{T}, T]. \end{cases} \tag{18}$$

and we have introduced the constant

$$r_g = \frac{\delta}{\gamma} + \frac{\gamma - 1}{\gamma} r.$$

The deterministic functions A^F, B^F, $A_{\alpha(s)}$, $B_{\alpha(s)}$, A^r, and B^r are defined as

$$A^F(t,s) = -\frac{\nu(\tilde{\tilde{\kappa}} - a^F)}{\bar{\beta}^2}(s-t) + \frac{2\nu}{\bar{\beta}^2}\ln\left(\frac{1 - \frac{\tilde{\tilde{\kappa}}-a^F}{\tilde{\tilde{\kappa}}+a^F}e^{-a^F(T-t)}}{1 - \frac{\tilde{\tilde{\kappa}}-a^F}{\tilde{\tilde{\kappa}}+a^F}}\right),$$

$$B^F(t,s) = -\frac{(a^F - \tilde{\tilde{\kappa}})e^{-a^F(s-t)} + \tilde{\tilde{\kappa}} - a^F}{\bar{\beta}^2\left(\frac{a^F - \tilde{\tilde{\kappa}}}{\tilde{\tilde{\kappa}}+a^F}e^{-a^F(s-t)} + 1\right)},$$

$$A_{\alpha(s)}(t,s) = -\frac{\nu(\tilde{\kappa} - a^g)}{\bar{\beta}^2}(s-t) + \frac{2\nu}{\bar{\beta}^2}\ln\left(\frac{1 - k_\alpha e^{-a^g(s-t)}}{1 - k_\alpha}\right),$$

$$B_{\alpha(s)}(t,s) = -\frac{-k_\alpha(\tilde{\kappa} + a^g)e^{-a^g(s-t)} + \tilde{\kappa} - a^g}{\bar{\beta}^2\left(-k_\alpha e^{-a^g(s-t)} + 1\right)},$$

$$A^r = -\frac{\nu(\tilde{\kappa} - a)}{\bar{\beta}^2}(s-t) + \frac{2\nu}{\bar{\beta}^2}\ln\left(\frac{1 - \frac{\tilde{\kappa}-a}{\tilde{\kappa}+a}e^{-a(T-t)}}{1 - \frac{\tilde{\kappa}-a}{\tilde{\kappa}+a}}\right),$$

$$B^r = -\frac{(a - \tilde{\kappa})e^{-a(s-t)} + \tilde{\kappa} - a}{\bar{\beta}^2\left(\frac{a-\tilde{\kappa}}{\tilde{\kappa}+a}e^{-a(s-t)} + 1\right)}.$$

The corresponding parameters are given in the Appendix A. The optimal fractions of wealth invested in the specific risk exposures are given by

$$\begin{aligned}
\phi_{vt} =& \frac{1}{\gamma}\frac{X_t + Y_t F^{com}(t,v_t)}{X_t}\eta_v - \left(\hat{\beta}\hat{\rho}_{yv}\frac{Y_t F^{com}(t,v_t)}{X_t} + \bar{\beta}\hat{\rho}_v\frac{Y_t F^{com}(t,v_t)}{X_t}\frac{F_v^{com}(t,v_t)}{F^{com}(t,v_t)}\right) \\
&+ \bar{\beta}\hat{\rho}_v\frac{g_v^{com}(t,v_t)}{g^{com}(t,v_t)}\frac{X_t + Y_t F^{com}(t,v_t)}{X_t}, \\
\phi_{St} =& \frac{1}{\gamma}\frac{X_t + Y_t F^{com}(t,v_t)}{X_t}\eta_S - \left(\hat{\beta}\rho_{ys}\frac{Y_t F^{com}(t,v_t)}{X_t} + \bar{\beta}\rho_{sv}\frac{Y_t F^{com}(t,v_t)}{X_t}\frac{F_v^{com}(t,v_t)}{F^{com}(t,v_t)}\right) \\
&+ \bar{\beta}\rho_{sv}\frac{g_v^{com}(t,v_t)}{g^{com}(t,v_t)}\frac{X_t + Y_t F^{com}(t,v_t)}{X_t}.
\end{aligned}$$

The optimal consumption rate reads

$$c_t = \frac{X_t + Y_t F^{com}(t,v_t)}{g^{com}(t,v_t)}.$$

Up to numerical integrals, all expressions can be computed in closed-form. The specific integrals are calculated by a Romberg procedure.[8] Note that F^{com} does not depend on volatility v for $t \geq \tilde{T}$ since pension payments are not correlated with the volatility process at all. The function g^{com} captures the non-wealth dependent parts of the individual's indirect utility. Compared to a problem without labor income, the initial financial wealth is simply adjusted by adding the initial value of human wealth. The optimal consumption strategy is to consume a fraction of total wealth, $\frac{1}{g^{com}(t,v_t)}$.

3.1 Optimal Strategies

To obtain an attainable strategy, we have to transform the proportional risk exposures ϕ back to the optimal investment fractions π. This transformation is always possible due to our stand-

[8] Romberg's method computes an approximate solution to a definite integral. It is based on a Newton–Cotes formula and evaluates the integrand at equally-spaced points.

ing assumption of a non-zero option vega. The optimal positions in the risk securities follow immediately by applying formulas (9) and (10)

Lemma 1. *In a complete market setting, i.e* $\hat{\rho}_y = 0$*, and under the standing assumption that* $h_v \neq 0$ *and* $h \neq 0$*, the optimal portions of financial wealth invested in the corresponding assets are represented by*

$$\pi_{St} = \frac{1}{\gamma}\frac{X_t + Y_t F^{com}(t, v_t)}{X_t}\left[\eta_S - \eta_v \frac{\rho_{sv}}{\hat{\rho}_v}\right] + \hat{\beta}\left(\hat{\rho}_{yv}\frac{\rho_{sv}}{\hat{\rho}_v} - \rho_{ys}\right)\frac{Y_t F^{com}(t, v_t)}{X_t} - \pi_{Ot}\frac{h_s S_t}{O_t}, \tag{19}$$

$$\pi_{Ot} = \left\{\frac{1}{\gamma}\frac{X_t + Y_t F^{com}(t, v_t)}{X_t}\frac{\eta_v}{\bar{\beta}\hat{\rho}_v} - \left(\frac{\hat{\beta}}{\bar{\beta}}\frac{\hat{\rho}_{yv}}{\hat{\rho}_v}\frac{Y_t F^{com}(t, v_t)}{X_t} + \frac{Y_t F^{com}(t, v_t)}{X_t}\frac{F^{com}_v(t, v_t)}{F^{com}(t, v_t)}\right)\right. \tag{20}$$

$$\left. + \frac{g^{com}_v(t, v_t)}{g^{com}(t, v_t)}\frac{X_t + Y_t F^{com}(t, v_t)}{X_t}\right\}\frac{O_t}{h_v}.$$

The results of the previous lemma show that the optimal portfolio weights do not depend on financial wealth and human wealth separately, but only on the wealth-income ratio.

Regarding the demand for the derivative, recall that we added this security to be able to hedge volatility shocks. Hence, the derivative is mainly needed to get a specific exposure to the volatility risk. To understand how effective a specific claim is in creating exposure to volatility, one can use the ratio $\frac{h_v}{O_t}$. As in Liu and Pan (2003) this ratio represents the "volatility exposure for each dollar invested in the derivative security". This means that a contract with a high $\frac{h_v}{O_t}$ is more likely to create a desired exposure than a contract with a low ratio. Therefore, we can invest less money in the derivative when the ratio is high. This can be seen from (20) where the amount of financial wealth invested in the derivative is inversely proportional to the ratio $\frac{h_v}{O_t}$.

We also notice that the optimal option investment consists of three terms. The first term corresponds to the standard myopic optimal portfolio and depends on risk aversion γ and the risk premium η_v. As usual, the more risk-averse the investor is the smaller is the absolute position. The other decisive component in computing the speculative demand is the volatility risk premium. For example, for a positive volatility risk premium η_v, the investor wants to have a long position in volatility. This position can be achieved by buying the derivative whenever its vega is positive and by selling the security when its vega is negative. Note that the first component also highly depends on the wealth-income ratio and that both effects are amplified when human wealth is increasing. The second part represents an adjustment of the investments to the risk profile of human wealth and can be deduced in the following way. First, determine the optimal riskiness of total wealth with respect to the exogenous shock, which was originally determined by Merton (1969). Then we subtract the risk amount of human wealth to obtain the desired exposures towards the shocks. This term also reflects a correction for the option-like income risk and depends on the correlation structure. When the investor does not receive income, this part will vanish. Finally, the third term is a hedge against variations in future volatilities and crucially depends on the ratio $\frac{g_v^{com}}{g^{com}}$ that can be computed in closed-form (up to numerical integrals). The volatility hedge term is influenced by the wealth-income ratio, too.

Now, we examine the demand for the stock (19) that can be divided into three parts. The first term is the myopic or speculative demand. In contrast to the derivative, the myopic demand does not only depend on the stock risk premium η_S, but also on the volatility risk premium η_v and the imposed correlation structure. This is needed to account for the stock-like volatility risk. For example, by assuming a negative correlation ρ_{sv} between stock returns and volatility, a long position in volatility would imply a short position in the stock, and vice versa. The second component is again a correction term for income risk and is mainly driven by the correlation coefficients. In particular, the sign of the demand is determined by ρ_{sv}. Since the underlying of the option is the stock itself, there is always an implicit stock exposure when we invest in the option. The delta of the option h_S describes the magnitude of this implicit holding. Therefore, the stock holding is adjusted via the third term.

4 Solution for Unspanned Income

In the previous section, the indirect utility has satisfied a Merton-type separation

$$J^{com}(t,x,v,y) = \frac{1}{1-\gamma} g^{com}(t,v)^{\gamma}(x + yF^{com}(t,v))^{1-\gamma},$$

where financial wealth X_t was replaced by total wealth $X_t + Y_t F^{com}(t, v_t)$. For the more realistic situation of unspanned labor income risk and liquidity constraints[9], it is impossible to replicate human wealth by traded assets. Hence, such an approach and the associated intuitive derivation of the optimal strategy break down. One can even show that a separation like (23) does not hold in the true incomplete market. To summarize, it seems impossible to find closed-form solutions in this incomplete market.

In the remainder of this section, we will apply two different approaches to obtain controls in the incomplete market. We measure their performance by a simple metric, the so called welfare losses. The first method can be used to derive consumption and investment strategies in closed-form that are simple to compute and in some sense near-optimal. This method was developed in Bick, Kraft and Munk (2011) and we will refer to it as SAMS. In the second approach, we solve the involved dynamic optimization problem (14) with a Markov Chain Approximation (MCA) method that rests upon a finite difference scheme.

4.1 Artificially Completed Markets

Bick, Kraft and Munk (2011) use a hypothetical labor contract to complete the market. Following this idea, we consider an artificially completed market that consists of the original risk-free

[9] This means that we consider the set $\mathcal{A}_t^{con}$.

bank account and the stock, augmented by an asset making the market complete. We use their approach, which basically consists of three steps.

First, look at a subset of artificially completed markets. Varying, for example, the risk premium of the hypothetical asset, results in a whole family of such markets. Ignoring the investment in the artificial contract and after a small adjustment, we obtain strategies that can be applied in the true incomplete market. In the second step, we perform a utility maximization over this family of strategies. That will define a specific consumption and investment strategy in the incomplete market. The utility generated by the optimal incomplete market strategy is unknown, but by construction lower than the utility obtained in any of the artificially completed markets. Thus, we can derive an upper bound on the maximum obtainable utility in the incomplete market by minimizing over the expected utilities of the artificially completed markets.

To be more precise, we assume that in the artificially markets the agent can invest in a hypothetical income contract I until the retirement date $\tilde{T}$.[10] In the following, we will refer to this hypothetical asset as an *income contract*. This contract is characterized by the risk premium η_I. The time-t price I_t evolves according to

$$dI_t = I_t \left[(r + \eta_I v_t)\, dt + \sqrt{v_t} dW_{Yt} \right]. \tag{21}$$

Since the single purpose of this contract is to complete the market, we let him only depend on the idiosyncratic shock of income risk W_Y and assume that its volatility is given by $\sqrt{v_t}$.

Therefore, every artificial market is characterized by a choice of η_I. There are no constraints on the consumption and investment strategy in the artificial markets except for the standard integrability conditions and the constraint that terminal wealth X_T has to be non-negative. As labor income is perfectly hedgeable in the artificial markets, we do not have to impose any liquidity constraints to ensure $X_T \geq 0$, as we do in the true incomplete market.

Let π_{It} denote the fraction of financial wealth invested in the income contract I at time t. Due to the simplicity of the contract, we get $\pi_{It} = \phi_{It}$ with ϕ_{It} being the fraction invested in the diffusive component W_Y. In this artificially completed market, the wealth dynamics in terms of the factor exposures (ϕ_v, ϕ_S, ϕ_I) read

$$\begin{aligned} dX_t = X_t[rdt + \phi_{St}(\eta_S v_t dt + \sqrt{v_t} dW_{St}) + \phi_{vt}(\eta_v v_t dt + \sqrt{v_t} dW_{vt})] \\ + \mathbf{1}_{t<\tilde{T}} \phi_I(\eta_I v_t\, dt + \sqrt{v_t} dW_{Yt}) + (\mathbf{1}_{t<\tilde{T}} Y_t + \mathbf{1}_{t\geq\tilde{T}} \tilde{Y}_t - c_t)dt, \end{aligned} \tag{22}$$

and for a given market price of risk η_I, the indirect utility $J^{art}(t, x, v, y; \eta_I)$ is defined as

$$J^{art}(t, x, v, y; \eta_I) = \max_{(c;\phi_v,\phi_S,\phi_I)} \left\{ \mathbf{E}_t \left[\int_t^T e^{-\delta(s-t)} u(c_s) ds + \varepsilon e^{-\delta(T-t) u(X_T)} \right] \right\}$$

[10] Note that there is no demand for the income contract when $t \geq \tilde{T}$ as we already face a complete market setting in this period.

In general, the risk premium might be a stochastic process. As theoretically shown in Cvitanić and Karatzas (1992), the minimum of the indirect utility $J^{art}(t,x,v,y;\eta_I)$ over all (possible) stochastic processes η_I is equal to the indirect utility in the true constrained market. But this does not facilitate the actual computation of the optimal solution.

Although there do not exist closed-form solutions for a general stochastic risk premium η_I, we can still compute the optimal strategies for a constant risk premium and measure the performance of them.[11]. The following theorem gives us the desired solutions.

Theorem 6. ***(Solution with income contracts)*** *If the investor has access to income contracts until retirement date $\tilde{T}$, his indirect utility is given by*

$$J^{art}(t,x,v,y;\eta_I) = \frac{1}{1-\gamma} g^{art}(t,v;\eta_I)^{\gamma}(x + yF^{art}(t,v;\eta_I))^{1-\gamma}, \tag{23}$$

where

$$g^{art}(t,v;\eta_I) = \begin{cases} \varepsilon^{\frac{1}{\gamma}} e^{-r_g(T-t)-A^r(\tilde{T},T)-A_{\alpha(T)}(t,\tilde{T};\eta_I)-B_{\alpha(T)}(t,\tilde{T};\eta_I)v} \\ \qquad + \int_{\tilde{T}}^{T} e^{-r_g(s-t)-A^r(\tilde{T},s)-A_{\alpha(s)}(t,\tilde{T};\eta_I)-B_{\alpha(s)}(t,\tilde{T};\eta_I)v}\, ds \\ \qquad + \int_t^{\tilde{T}} e^{-r_g(s-t)-A_0(t,s;\eta_I)-B_0(t,s;\eta_I)v} ds, \quad t \le \tilde{T}, \\ g^{com}(t,v), \quad t \in (\tilde{T},T]. \end{cases}$$

$$F^{art}(t,v;\eta_I) = \begin{cases} \Upsilon\, F^{com}(\tilde{T},v) e^{-(r-\alpha)(\tilde{T}-t)-A^F(t,\tilde{T};\eta_I)-B^F(t,\tilde{T};\eta_I)v} \\ \qquad + \int_t^{\tilde{T}} e^{-(r-\alpha)(s-t)-A^F(t,s;\eta_I)-B^F(t,s;\eta_I)v}\, ds, \quad t \le \tilde{T}, \\ F^{com}(t,v), \quad t \in (\tilde{T},T]. \end{cases}$$

The deterministic functions $A^F(.,.;\eta_I)$, $B^F(.,.;\eta_I)$, $A_{\alpha(s)}(.,.;\eta_I)$, and $B_{\alpha(s)}(.,.;\eta_I)$ are defined as

$$A^F(t,s;\eta_I) = -\frac{\nu(\tilde{\tilde{\kappa}} - a^F(\eta_I))}{\bar{\beta}^2}(s-t) + \frac{2\nu}{\bar{\beta}^2} \ln\left(\frac{1 - \frac{\tilde{\tilde{\kappa}} - a^F(\eta_I)}{\tilde{\tilde{\kappa}} + a^F(\eta_I)} e^{-a^F(\eta_I)(T-t)}}{1 - \frac{\tilde{\tilde{\kappa}} - a^F(\eta_I)}{\tilde{\tilde{\kappa}} + a^F(\eta_I)}} \right),$$

$$B^F(t,s;\eta_I) = -\frac{(a^F(\eta_I) - \tilde{\tilde{\kappa}}) e^{-a^F(\eta_I)(s-t)} + \tilde{\tilde{\kappa}} - a^F(\eta_I)}{\bar{\beta}^2 \left(\frac{a^F(\eta_I) - \tilde{\tilde{\kappa}}}{\tilde{\tilde{\kappa}} + a^F(\eta_I)} e^{-a^F(\eta_I)(s-t)} + 1 \right)},$$

$$A_{\alpha(s)}(t,s;\eta_I) = -\frac{\nu(\tilde{\kappa} - a^g(\eta_I))}{\bar{\beta}^2}(s-t) + \frac{2\nu}{\bar{\beta}^2} \ln\left(\frac{1 - k_\alpha e^{-a^g(\eta_I)(s-t)}}{1 - k_\alpha} \right),$$

$$B_{\alpha(s)}(t,s;\eta_I) = -\frac{-k_\alpha(\tilde{\kappa} + a^g(\eta_I)) e^{-a^g(\eta_I)(s-t)} + \tilde{\kappa} - a^g(\eta_I)}{\bar{\beta}^2 \left(-k_\alpha e^{-a^g(\eta_I)(s-t)} + 1 \right)}.$$

The corresponding parameters are given in the Appendix A. The optimal fractions of wealth invested in the specific risk exposures are given by

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[11] Even for deterministic risk premia, we are not able to derive close-form solutions as the ODEs for the functions B and A have to be solved numerically.

$$
\begin{aligned}
\phi_{vt}(\eta_I) =& \frac{1}{\gamma}\frac{X_t + Y_t F^{art}(t, v_t; \eta_I)}{X_t}\eta_v - \left(\hat{\beta}\hat{\rho}_{yv}\frac{Y_t F^{art}(t, v_t; \eta_I)}{X_t} + \bar{\beta}\hat{\rho}_v \frac{Y_t F^{art}(t, v_t; \eta_I)}{X_t}\frac{F_v^{art}(t, v_t; \eta_I)}{F^{art}(t, v_t; \eta_I)}\right)\\
&+ \bar{\beta}\hat{\rho}_v \frac{g_v^{art}(t, v_t; \eta_I)}{g^{art}(t, v_t; \eta_I)}\frac{X_t + Y_t F^{art}(t, v_t; \eta_I)}{X_t},\\
\phi_{St}(\eta_I) =& \frac{1}{\gamma}\frac{X_t + Y_t F^{art}(t, v_t; \eta_I)}{X_t}\eta_S - \left(\hat{\beta}\rho_{ys}\frac{Y_t F^{art}(t, v_t; \eta_I)}{X_t} + \bar{\beta}\rho_{sv} \frac{Y_t F^{art}(t, v_t; \eta_I)}{X_t}\frac{F_v^{art}(t, v_t; \eta_I)}{F^{art}(t, v_t; \eta_I)}\right)\\
&+ \bar{\beta}\rho_{sv} \frac{g_v^{art}(t, v_t; \eta_I)}{g^{art}(t, v_t; \eta_I)}\frac{X_t + Y_t F^{art}(t, v_t; \eta_I)}{X_t},\\
\phi_{Yt}(\eta_I) =& \frac{1}{\gamma}\frac{X_t + Y_t F^{art}(t, v_t; \eta_I)}{X_t}\eta_I - \hat{\beta}\hat{\rho}_y \frac{Y_t F^{art}(t, v_t; \eta_I)}{X_t}.
\end{aligned}
$$

The optimal consumption rate reads

$$
c_t(\eta_I) = \frac{X_t + Y_t F^{art}(t, v_t; \eta_I)}{g^{art}(t, v_t; \eta_I)}.
$$

The corresponding investment fractions are equivalent to (19) and (20) and $\pi_{It} = \phi_{It}$.

Of course, the indirect utility in the artificially completed market for any choice of η_I will be at least as high as the unknown solution in the truly incomplete market. Given Theorem 6, one easily obtains $\bar{\eta}_I := \arg_{\eta_I} \min J^{art}(t, x, v, y; \eta_I)$ which gives us an upper bound on the truly incomplete market

$$
\bar{J}(t, x, v, y) = J^{art}(t, x, v, y; \bar{\eta}_I). \tag{24}
$$

Next, we take a strategy from an artificial market that is parameterized by η_I and transform it to a strategy that is feasible in the incomplete market. Recall that the individual has to comply with the liquidity constraint until retirement to make sure that he ends up with positive terminal wealth. In retirement, the income stream becomes deterministic and we allow the agent to borrow against his future income. Therefore, by ignoring the investment in the income contract, we obtain a family of strategies that is feasible in the retirement period. When the investor is still working, the problem becomes more involved. To satisfy the liquidity constraint in the active phase, we have to adapt the strategies in order to make them admissible. Following Bick, Kraft and Munk (2011), we modify the strategies as follows

$$
\begin{aligned}
\pi_{St}(\eta_I) =& \frac{1}{\gamma}\frac{X_t + \mathbf{1}_{\{X_t>k\}} Y_t F^{art}(t, v_t)}{X_t}\left[\eta_S - \eta_v \frac{\rho_{sv}}{\hat{\rho}_v}\right] + \hat{\beta}\hat{\rho}_{yv}\frac{\rho_{sv}}{\hat{\rho}_v}\frac{\mathbf{1}_{\{X_t>k\}} Y_t F^{art}(t, v_t)}{X_t} - \pi_{Ot}\frac{h_s S_t}{O_t},\\
\pi_{Ot}(\eta_I) =& \left\{\frac{1}{\gamma}\frac{X_t + \mathbf{1}_{\{X_t>k\}} Y_t F^{art}(t, v_t)}{X_t}\frac{\eta_v}{\bar{\beta}\hat{\rho}_v} - \mathbf{1}_{\{X_t>k\}}\left(\frac{\hat{\beta}}{\bar{\beta}}\frac{\hat{\rho}_{yv}}{\hat{\rho}_v}\frac{Y_t F^{art}(t, v_t)}{X_t} + \frac{Y_t F^{art}(t, v_t)}{X_t}\frac{F_v^{art}(t, v_t)}{F^{art}(t, v_t)}\right)\right.\\
&\left.+ \frac{g_v^{art}(t, v_t)}{g^{art}(t, v_t)}\frac{X_t + \mathbf{1}_{\{X_t>k\}} Y_t F^{art}(t, v_t)}{X_t}\right\}\frac{O_t}{h_v},
\end{aligned}
\tag{25}
$$

$$
c_t(\eta_I) = \begin{cases} \frac{X_t + Y_t F^{art}(t, v_t; \eta_I)}{g^{art}(t, v_t; \eta_I)}, & \text{if } \frac{X_t + Y_t F^{art}(t, v_t; \eta_I)}{g^{art}(t, v_t; \eta_I)} < Y_t \text{ OR } X_t > k,\\ \xi Y_t, & \text{otherwise,} \end{cases}
$$

where k and ξ are constants. After this transformation, we can ensure that financial wealth stays positive for $t < \tilde{T}$.[12]

While we are not able to derive the optimal consumption and investment strategy in the truly incomplete market, we can evaluate the performance of any admissible consumption and investment strategy in the following way. For any risk premium η_I, we get feasible strategies from (25) and approximate the corresponding expected utility $J(t,x,v,y;(c(\eta_I),\pi(\eta_I)))$ by Monte Carlo simulations. Then, we measure the performance of a strategy defined by η_I by comparing the expected utility $J(t,x,v,y;(c(\eta_I),\pi(\eta_I)))$ to the upper bound $\bar{J}(t,x,v,y)$. If this distance is small, Bick, Kraft and Munk (2011) call the strategy near-optimal. The metric that they use to assess whether the distance is close is an upper bound $L := L(t,x,v,y;(c,\pi))$ on the welfare loss which is defined as

$$J(t,x,v,y;(c(\eta_I),\pi(\eta_I))) = \bar{J}(t,[1-L]x,v,[1-L]y).$$

The value L can be interpreted as an upper bound on the fraction of total wealth that the investor would be willing to pay to get access to the unknown optimal strategy, instead of following $(c(\eta_I),\pi(\eta_I))$. Due to the specification of (23), we get a close-form expression for L

$$L = 1 - \left(\frac{J(t,x,v,y;(c(\eta_I),\pi(\eta_I)))}{\bar{J}(t,x,v,y)} \right)^{\frac{1}{1-\gamma}}. \tag{26}$$

Finally, we search through different η_I to find the best of feasible strategies $(c(\eta_I),\pi(\eta_I))$ which yields the smallest value L.

It is important to mention that this is only an upper bound and we do not have any information on how close $\bar{J}(t,x,v,y)$ is to the true incomplete indirect utility value. We will investigate this point in the later analysis.

4.2 Markov Chain Approximation

We have also solved the utility maximization problem (4) with the so called Markov Chain Approximation Method, which is a well-studied and frequently applied numerical approach (Kushner and Dupuis (1992), Munk (2000)). We solve a problem that is similar to the one studied in Munk and Soerensen (2010). The original problem consists of three state variables: wealth, income and volatility. Among others, Munk and Soerensen (2010) show that the indirect utility is homogeneous of degree $1-\gamma$ in wealth and income. The same holds true for our problem and we can reduce the problem to two dimensions by rewriting the indirect utility as $J(t,x,v,y) = y^{1-\gamma} H(t,\frac{x}{y},v)$, where the function H is defined as

$$H(t,z,v) = J(t,z,v,1). \tag{27}$$

[12] See Bick, Kraft and Munk (2011) for further details.

Substituting (27) into the original HJB (14) and rearranging yields a non-linear second-order PDE for H

$$
\begin{aligned}
0 &= H_t - \hat{\delta}(v)H + \frac{1}{1-\gamma}\hat{c}^{1-\gamma} + H_v\left[\nu - \kappa v + v\bar{\beta}\hat{\beta}\rho_{yv}(1-\gamma)\right] + \frac{1}{2}\bar{\beta}^2 v H_{vv} \qquad (28)\\
&+ H_z\left\{1 - \hat{c} + z\left[r - \alpha + \gamma\hat{\beta}^2 v + \alpha^T\lambda - \gamma\hat{\beta}\sqrt{v}\alpha^T(\hat{\rho}_{yv}, \rho_{ys})^T\right]\right\}\\
&+ \frac{1}{2}z^2 H_{zz}\left[\alpha^T\alpha + \hat{\beta}^2 v - 2\hat{\beta}\sqrt{v}\alpha^T(\hat{\rho}_{yv}, \rho_{ys})^T\right] + zH_{vz}\bar{\beta}\sqrt{v}\left[\alpha^T(\hat{\rho}_v, \rho_{sv})^T - \hat{\beta}\sqrt{v}\rho_{yv}\right],
\end{aligned}
$$

where we define $\hat{\delta}(v) = \delta - \alpha(1-\gamma) + 0.5\gamma(1-\gamma)\hat{\beta}^2 v$ and $\hat{c} = \frac{c}{y}$ is the ratio between consumption and income.

The MCA Method discretizes this control problem. The dynamics of the wealth-income ratio and the volatility process are approximated by a Markov chain on a grid defined by N equidistant time points, I equidistant values of the wealth/income ratio, and J equidistant values of the volatility process. In the continuous-time model, both the wealth/income ratio and the volatility process are unbounded from above, but the MCA Method has to impose upper bounds. The optimization problem is solved by backwards recursion starting at the terminal date T. In each time step the value function for each state in the grid is maximized by policy iterations. The entire procedure is roughly equivalent to solving the HJB equation for H by a (specific) finite difference approach, similar to the one used by Brennan, Schwartz and Lagnado (1997) and others. The precision of the method strongly depends on the number of grid points and the size of the imposed upper bound.

5 Numerical Results

This section contains a numerical study of the proposed investment and consumption strategies both for the complete market and the incomplete market. To examine these strategies, we fix a set of several parameters that describe the investors preferences, the income process, the volatility process, and the stock process. Many papers deal with the calibration of the Heston model (see, e.g., Pan (2002)) and provide parameter estimates for a stochastic volatility model. The parameters we use in this study resemble the ones found in the literature. The benchmark values for the income process are similar to those in the existing literature (Cocco, Gomes and Maenhout (2005), Munk and Soerensen (2010)). We assume a risk aversion of $\gamma = 4$, a time-preference rate of $\delta = 0.03$, and a weight between consumption and terminal wealth of $\varepsilon = 5$. This means that the utility that is derived from a terminal wealth of a is approximately the same as the utility one gets from consuming the value a in each of the last 5 years. Further, we assume the investor works for 20 years and subsequently receives a retirement pension of 40% of his last salary for the next 20 years. The benchmark parameters are summarized in Table 1. The fraction $\frac{\nu}{\kappa}$ gives us the mean reversion level of the volatility. The constant parameter

η_S in combination with the average volatility yields an average equity excess return of 4.95%. Similarly, we have an expected income volatility of 9.75%.

There are several empirical studies that support the choice of a negative volatility risk premium, i.e. we get a premium for short positions in volatility.[13] Therefore, we chose $\eta_v = -1$.

At this point, we do not specify the correlation parameter between stock and income ρ_{ys} and between volatility and income ρ_{yv}. Although, there is empirical evidence that ρ_{ys} is close to zero, we will vary it in the next part of this section to achieve several combinations of spanned income.[14] As the benchmark, we choose a financial wealth at time zero of $x = 2$ and an initial income of $y = 2$. These values can be interpreted as USD $20,000$. In the remainder, we will often vary the wealth income ratio to show different effects.

5.1 Results for Spanned Income

In this part, a complete market setup is analyzed. Hence, we have to assume that the income stream is either spanned by traded assets or risk-free and that the investor does not face any liquidity constraints. We will focus on the spanned income case, where the income stream can be replicated at every point in time. We thus have to consider combinations of ρ_{ys} and ρ_{yv} that result in $\hat{\rho}_y = 0$, i.e. for fixed ρ_{sv}, we must choose values of ρ_{ys} and ρ_{yv} such that

$$\rho_{yv} - \rho_{sv}\rho_{ys} =^{+}_{-} \sqrt{1-\rho_{ys}^2}\sqrt{1-\rho_{sv}^2}.$$

We are going to study the optimal investments given in Lemma 1 for three stock-income correlations ($\rho_{ys} = -1$, 0, and 1). We still have to specify the security h because we need to compute the price, the delta, and the vega of the option in Formulas (19) and (20). In order to get a clearer interpretation of the specific stock investment components, we get rid of the "delta-hedging" part in (19) and choose a derivative with a delta of zero. This results in an optimal stock investment of

$$\pi_{St} = \underbrace{\frac{1}{\gamma}\left(1 + \frac{L^{com}(t,v_t)}{X_t}\right)\left[\eta_S - \eta_v \frac{\rho_{sv}}{\hat{\rho}_v}\right]}_{\text{myopic demand}} + \underbrace{\hat{\beta}\left(\hat{\rho}_{yv}\frac{\rho_{sv}}{\hat{\rho}_v} - \rho_{ys}\right)\frac{L^{com}(t,v_t)}{X_t}}_{\text{income hedge}}. \tag{29}$$

As in Liu and Pan (2003), delta neutral straddles are the prime security in our research. A straddle is a long position in a call and a put option with the same strike and expiration date.[15] For a specific option maturity τ and spot price, one can always compute a strike price K such that the straddle becomes delta neutral. We choose a maturity of $\tau = 0.3$.[16] Table 2 shows the

[13] For a discussion of the empirical studies, we refer to Liu and Pan (2003).

[14] We cannot freely choose the different correlation parameters since we have to ensure that the correlation matrix stays positive definite.

[15] All options are assumed to be European options.

[16] We assume that the agent is able to invest in such a contract at every point in time.

dependency of π_S on different wealth-income ratios and four [17] combinations of the correlations at time 0. The optimal fractions of total wealth invested in the stock market are decomposed into its specific components. First, note that the ratio of wealth to income and thus the ratio of financial wealth to human wealth has a huge impact on the optimal decisions (no matter which correlation scenario we consider). This partially results in huge fractions of financial wealth (e.g. 297) that are invested in stock whenever human wealth dominates financial wealth, i.e. when we deal with a low $\frac{x}{y}$. However, the portions of total wealth seem quite moderate. Moreover, the results indicate that the optimal π_S converges to the case without income when human wealth is decreasing. We also notice that the optimal investments depend largely upon the correlation structure. For example when the idiosyncratic shocks to labor income and the stock market are perfectly positively correlated, the investor wants to take a large short position in the stock. In contrast when they are perfectly negatively correlated, he takes huge long positions. When both risk shocks are not correlated at all, we can examine two scenarios (two different ρ_{yv} are possible). In both cases, the agent takes long positions.

To get a better understanding of the previous results, the different components of the stock investment in (29), which consist of the myopic (also called speculative demand) and a hedge demand for income risk, are analyzed. Remember that we do not consider the "delta-hedging" effects here as we use delta-neutral straddles. As already mentioned, the correlation structure has a huge influence on the optimal investments. For instance, when $\rho_{ys} = 1$, income can be interpreted as an implicit investment in stock and hence, we should reduce the actual investment in stock. This already indicates that human wealth is driven by the specific correlations. Table 3 summarizes the human wealth values for an initial labor income of $y = 2$, a retirement period of $\tilde{T} = 20$ and a terminal date of $T = 40$. The results show that human wealth has a higher value when the income stream is negatively correlated with stock returns. This fact is emphasized when regarding the two extreme values of perfect negatively correlated streams and perfect positively correlated streams. The reason is that income payments that are high when stock returns are low are very valuable. Therefore, investing in the stock will provide a good hedge against unfavorable income shocks. Similarly, income payments that are positively correlated with volatility shocks have a high value. Regarding the respective hedging terms and the absolute values of the income hedges, we see that both terms are mainly affected by the fraction of human wealth to financial wealth $\frac{L^{com}(t,v)}{x}$. The sign of the myopic part is computed by $(\eta_S - \eta_v \frac{\rho_{sv}}{\hat{\rho}_v})$ which is always positive for our benchmark parameters. Due to the term $(\hat{\rho}_{yv} \frac{\rho_{sv}}{\hat{\rho}_v} - \rho_{ys})$, the sign of the income hedge term is influenced by the imposed correlation parameters and can change in the examined scenarios.

[17] Note that for a correlation of $\rho_{ys} = 0$, there exist two values of ρ_{yv} that yield a $\hat{\rho}_{yv}$ of zero.

Next, we have a look at the investments in the delta-neutral straddle which are displayed in Table 4. Like the optimal stock weights, the investments in the straddle are influenced by the impact of human wealth and the specific correlation structure, which was discussed in the previous paragraph. The speculative demand is always negative due to the negative volatility risk premium η_v. The sign of the income hedge term is mainly determined by $\hat{\rho}_{yv}$. In addition to these terms, the investor also hedges against volatility risk. The magnitude of this component also depends on human wealth and the sign is defined by g_v, which was negative in all our computations. The table also indicates that the assumed straddle has a high quality to create the desired exposure to volatility risk.

Of course, the specific portfolio weights for both the stock and the derivative would change when we used other derivatives to hedge volatility shocks. Therefore, we do the same analysis when the agent is allowed to invest in Call and Put options which are both traded at the money. The corresponding results are displayed in Table 5

After setting $\rho_{ys} = 0.2$ and $\hat{\rho}_{yv} = -0.948$, we conduct some robustness checks with respect to different parameters. Figure 1 depicts the results.

Unfortunately, we are not able to compute the expected value of total wealth in closed-form which is necessary for a study of portfolio and investment decisions of an investor over his life-cycle. However, we can simulate the process via a Monte Carlo simulation. In our study, we use 10,000 simulations and 100 discretization steps per year and simulate the expected values of total wealth, human wealth, and financial wealth over a period of 40 years. Within this period, the investor retires after 20 years. We simulate an investor with an initial wealth of $x = 2$ and income of $y = 2$ who can invest in delta-neutral straddle. Again we set $\rho_{ys} = 0.2$ $\hat{\rho}_{yv} = -0.948$. Figure 2 illustrates how the dollar amounts invested in the different assets change over the life-cycle. The expected amount invested in stock are stable at approximately 25 (250,000 USD) until 7 years before the retirement date and slowly decreases afterwards. At the retirement date $\tilde{T} = 20$, the amount drops from 21 to 16 and then decreases to zero at $T = 40$. This profile is in line with the advice most economists would give you (to invest more in stock at the beginning of the working career).

The straddle investment starts at roughly 6 and slowly decreases until retirement where the investor reallocates his portfolio and takes a short position of -4.5. Then the option investment increases to zero until the terminal date.

Figure 2 also illustrates the huge short positions of the investor in the bank account at time zero. He needs to borrow much money to mainly finance his stock investments and his consumption. But the magnitude of this position is fast declining when the investor approaches retirement. At the retirement date, he even deposits 10 in his bank account.

Figure 3 displays the expected values of financial, human, and total wealth that arise when following the optimal investments and consumption strategies discussed in the earlier sections. It becomes evident that human wealth is clearly dominating financial wealth until shortly before the retirement date where they overlap. Due to the replacement ratio, we observe a small hump at the retirement date. Afterwards human wealth monotonically decreases until the terminal date.

Moreover, we see that financial wealth is hump-shaped, a fact that is often found in life-cylce studies (e.g. Munk and Soerensen (2010), Kraft and Munk (2011)). We see that financial wealth is accumulated until 3 years after retirement. The investor does so to finance consumption in the retirement period where he only receives a fraction of his last salary. We also notice that financial wealth and total wealth are approaching with time and reach an expected terminal wealth of 5.5 that the individual passes on to his heirs.

5.2 Results for Unspanned Income

In this section, we examine the two methods proposed in part 4 to solve the involved problem. To provide results for the artificially completed market method (SAMS), we use Monte Carlo simulations to compute the bound on the true incomplete market $\bar{J}(t, x, v, y)$ and to evaluate the strategies $(c(\eta_I), \pi(\eta_I))$ that result in values $J(t, x, v, y; (c(\eta_I), \pi(\eta_I)))$. Note that, although we have $\bar{J}(t, x, v, y)$ in closed-form, we evaluate it by Monte Carlo simulations, too. This is to avoid any bias in the welfare losses due to simulation. Moreover, a derivative does not have to be specified since we simulate all dynamics with respect to the risk exposures (see (11)).

First, we want to consider the welfare losses L defined by (26) that we attain by following our strategies $(c(\eta_I), \pi(\eta_I))$. We only report the losses for the best of our feasible strategies.[18] The results presented below are based on our benchmark parameters that are tabulated in Table 1. We use $10,000$ simulations and 100 time steps per year to discretize the wealth and income process and fix the volatility-income correlation at $\rho_{yv} = -0.2$. Since we have to adapt the strategies to become admissible, we choose $k = 0.2$ and $\zeta = 0.95$.[19] Motivated by the results from the previous section, where we see that the involved strategies can change dramatically with the wealth-income ratios and volatilities, we focus on the sensitivity of the welfare loss with respect to these quantities. Table 6 reports the upper bounds on the welfare losses for different correlations ρ_{ys} and several wealth-income ratios. The welfare losses are at most 6.7% and in many cases the upper bounds on losses are much lower. The impact of the correlation between income and stock is evident. The loss bounds are lowest for high correlations (except

[18] This strategy is easily obtained by a standard line search algorithm.

[19] With this specification, we did not observe one single simulation path with a negative financial wealth, in all our simulations.

for $\rho_{ys} = 0.8$) and highest for correlations near zero. This is not surprising as we implement strategies in the true incomplete market that are derived in a complete market. It is also striking that the utility losses are decreasing with increasing wealth-income ratio. The effects of variations in this ratio stem from the fact that decreasing the wealth-income ratio leads to a higher value of the individual's labor income and thus makes the unspanned labor income relatively more important.

We wish to stress that all numbers represent upper bounds on welfare losses and that those bounds might be very weak. The upper bounds that we use are computed in an artificially completed market where the investor is allowed to borrow against expected future income which often results in negative financial wealth.[20] In contrast, we have to impose this liquidity constraint in the true incomplete market because we have to make sure that financial wealth stays positive during the working phase.[21] This indicates that a large part of the loss might be due to the liquidity constraint that is not captured in the artificially completed markets. Hence, we argue that the upper bound that was derived in the artificially completed market $\bar{J}(t, x, v, y)$ is far away from the true unknown utility in the incomplete market. The numbers in Table 6 support this fact as losses are higher for lower wealth-income ratios, i.e. when human wealth compared to financial wealth is large. When human wealth is a large part of financial wealth, the investor wants to borrow a large amount against his future human wealth. This behavior is supported by the results in the previous subsection.

To test SAMS further, it is compared with a well-established alternative numerical method. We choose the Markov chain approach that was explained in the previous chapter. In order to have a fair comparison, we evaluate the expected utility generated by the investment and consumption strategies derived with the Markov chain method by Monte Carlo simulations using the same random numbers as in the valuation of the near-optimal strategy.[22] Comparing this expected utility with the upper bound in the artificial completed market (see (24)), we obtain a welfare loss on the numerical policies. In spite of the dimension reduction, we still have to deal with two state variables and one time variable. The numerical PDE problem is solved with a very fine grid where the number of points approximating the wealth-income ratio are set to $I = 2000$ and the number of points for the volatility process are set to $J = 40$. This already results in $80,000$ grid points at each point time which requires a lot of computer memory and computation time. Further, we use 100 steps per year to discretize the time dimension. Table 7 summarizes the welfare losses for different wealth-income ratios and stock-income correlations and compares the Markov chain method with SAMS. We see that SAMS outperforms the

[20] Indeed, we can observe nearly 70% of paths on average that contain at least one negative realization of financial wealth.

[21] Recall that we cannot hedge the unspanned income process in the incomplete market.

[22] We do this to avoid any bias stemming from simulations.

grid search algorithm for all combinations of wealth-income ratio and stock-income correlation which is a strong argument for SAMS. In the analysis of Bick, Kraft and Munk (2011), the authors show that their method has roughly the same precision as a standard finite difference solution. However, they only have to deal with one state variable. Instead, we have to consider an additional state variable and thus the approximation suffers from the well-known curse of dimensionality. It is also known that numerical solution techniques have a poor performance at the boundaries of the grid (e.g see Munk (2000)). However, the investor in the true incomplete market often wants to take fairly extreme positions (see previous results) and thus frequently realizes values of financial wealth that are near the lower boundary. Therefore, the imprecise policies of the numerical algorithm near the boundary are frequently attained which results in high welfare losses compared to the analysis of Bick, Kraft and Munk (2011). Furthermore, the grid-based method does not give any intuition behind the optimal strategies and little is known about the precision. In contrast, SAMS gives us closed-form expressions that are easy to interpret. Moreover, it provides an upper bound on possible welfare losses.

As in the previous part, we can now use the near-optimal policies from SAMS to conduct a life-cycle study with an initial wealth of $x = 2$ and initial income of $y = 2$. The investor is assumed to retire after 20 years and the investment horizon is set to 40 years. We analyze a situation where the stock-income correlation is $\rho_{ys} = 0.2$ and the income-volatility correlation is $\rho_{yv} = -0.2$. We illustrate the expected amounts invested in the risky assets in Figure 4. We see that the investor takes more moderate positions in risky assets compared to the complete market setting. One reason for this behavior is the imposed liquidity constraint that the investor has to obey. It also becomes evident that the agent takes a short position in the straddle over his whole life-cycle since the volatility risk premium is chosen to be negative. The optimal investment proportions after retirement are by definition identical to the ones attained in the complete market setting. Following these strategies, the investor realizes expected values of financial wealth, human wealth, and total wealth that are depicted in Figure 5. The graphs show that human wealth is much lower compared to the analysis in the complete market, which also explains the more moderate positions in assets.

6 Conclusion

This paper has analyzed a consumption-portfolio choice problem of an investor that receives labor income and faces stochastic volatility. In our analysis, we have studied two scenarios. First, we have assumed a spanned income stream and we have provided closed-form solutions and investigated their components. Moreover, we have illustrated our findings by a life-cycle analysis where we could replicate the patterns found in similar studies. In the second part of this paper, we have allowed for unspanned income and have additionally imposed liquidity constraints on the portfolio strategies which results in a problem where closed-form solutions seem impossible to find. We have compared two methods that try to find a solution to this problem. Our results clearly indicate that SAMS performs better than a standard Markov chain approach when having more than one state variable. We wish to emphasize that SAMS could also be applied to higher dimensions whereas numerical solution techniques do not seem to be tractable when facing more than two state variables.

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A Proofs

Proof of Theorem 1. In the retirement phase, where the investor receives a risk-free income stream, the general Hamilton-Jacobi-Bellman (HJB) equation in (14) can be simplified to

$$0 = \max_{(c;\alpha)} \Bigg\{ J_t - \delta J + (rx + \tilde{Y})J_x + (\nu - \kappa v)J_v + \frac{1}{2}\bar{\beta}^2 v J_{vv} + xv(\pi_S \eta_S + \phi_V \eta_v)J_x \qquad (30)$$
$$+ \frac{1}{2}x^2 v(\phi_S^2 + \phi_v^2)J_{xx} + Xv\bar{\beta}(\hat{\rho}_v\phi_v + \rho_{sv}\phi_S)J_{xv} + \tilde{\alpha}\tilde{y}J_y + \frac{1}{1-\gamma}c^{1-\gamma} - cJ_x \Bigg\},$$

where subscripts denote partial derivatives and the terminal condition is given by $J(T, x, v) = \varepsilon \frac{1}{1-\gamma} x^{1-\gamma}$. From this HJB, we can derive the first order conditions (FOC) for ϕ_S, ϕ_v, and c

$$\phi_S = -\eta_S \frac{J_x}{xJ_{xx}} - \bar{\beta}\rho_{sv}\frac{J_{xv}}{xJ_{xx}},$$
$$\phi_v = -\eta_v \frac{J_x}{xJ_{xx}} - \bar{\beta}\hat{\rho}_v\frac{J_{xv}}{xJ_{xx}},$$
$$c = J_x^{\frac{-1}{\gamma}}.$$

Moreover, we conjecture a solution of the form

$$J(t, x, y, v) = \frac{1}{1-\gamma} g^{ret}(t, v)^{\gamma}(x + yF^{ret}(t))^{1-\gamma}, \qquad (31)$$

where $g^{ret}(T, v) = \varepsilon^{\frac{1}{\gamma}}$ and $F^{ret}(T) = 0$ satisfy the terminal condition. Now, after substituting (31) into (30) and using the FOC's, we arrive at

$$0 = g_t^{ret} - r_g^{ret}(v)g^{ret} + (\nu - \tilde{\kappa}v)g_v^{ret} + 0.5\bar{\beta}^2 v g_{vv}^{ret} + 1,$$
$$0 = F_t^{ret} - (r - \tilde{\alpha})F^{ret} + 1,$$

and we have introduced the function $r_g^{ret}(v) = r_g + \frac{\gamma-1}{2\gamma^2}(\eta_v^2 + \eta_S^2)v$ and the constant $\tilde{\kappa} = \kappa + \frac{\gamma-1}{\gamma}\bar{\beta}(\hat{\rho}_v\eta_v + \rho_{sv}\eta_S)$. The solutions that satisfy the above-mentioned equations are

$$F^{ret}(t) = \frac{1}{r-\tilde{\alpha}}\left[1 - e^{-(r-\tilde{\alpha})(T-t)}\right],$$
$$g^{ret}(t, v) = \varepsilon^{\frac{1}{\gamma}}\tilde{E}_t\left[e^{-\int_t^T r_g^{ret}(v_u)\,du}\right] + \int_t^T \tilde{E}_t\left[e^{-\int_t^s r_g^{ret}(v_u)\,du}\right].$$

Following a theorem from Kraft (2005), we can solve the expectation and get

$$g^{ret}(t, v) = \varepsilon^{\frac{1}{\gamma}} e^{-r_g(T-t)-A^r(t,T)-B^r(t,T)v}$$
$$+ \int_t^T e^{-r_g(s-t)-A^r(t,s)-B^r(t,s)v}ds,$$

with

$$A^r = -\frac{\nu(\tilde{\kappa}-a)}{\bar{\beta}^2}(s-t) + \frac{2\nu}{\bar{\beta}^2}\ln\left(\frac{1 - \frac{\tilde{\kappa}-a}{\tilde{\kappa}+a}e^{-a(T-t)}}{1 - \frac{\tilde{\kappa}-a}{\tilde{\kappa}+a}}\right),$$
$$B^r = -\frac{(a-\tilde{\kappa})e^{-a(s-t)} + \tilde{\kappa} - a}{\bar{\beta}^2\left(\frac{a-\tilde{\kappa}}{\tilde{\kappa}+a}e^{-a(s-t)} + 1\right)}.$$

The parameter a is a constat given by $a = \sqrt{\tilde{\kappa} + \frac{\gamma-1}{\gamma^2}(\eta_v^2 + \eta_S^2)\bar{\beta}^2}$.

Before retirement, we apply the law of iterated expectations and write the indirect utility as

$$J(t,x,y,v) = \mathrm{E}_t \left[\int_t^{\tilde{T}} e^{-\delta(s-t)} u(c_s)\, ds + e^{-\delta(\tilde{T}-t)} \frac{1}{1-\gamma} g^{ret}(\tilde{T},v)^{\gamma} (x_{\tilde{T}} + \Upsilon y_{\tilde{T}} F^{ret}(\tilde{T}))^{1-\gamma} \right].$$

The HJB equation associated with this problem is

$$0 = \mathcal{L}_1 J + \mathcal{L}_2 J + \mathcal{L}_3 J, \tag{32}$$

where[23]

$$\mathcal{L}_1 = \max_c \left\{ \frac{1}{1-\gamma} c^{1-\gamma} - c J_x \right\}, \tag{33}$$

$$\mathcal{L}_2 = \max_\alpha \left\{ \alpha^T \lambda J_x + 0.5 \alpha^T \Sigma^T \alpha J_{xx} + \alpha^T \Sigma_Z^T J_{xz} \right\}, \tag{34}$$

$$\mathcal{L}_3 = J_t - \delta J + (rx + y) J_x + \mu_Z^T J_z + 0.5 tr \left[\Sigma_Z \Sigma_Z^T J_{zz} \right] \tag{35}$$

The FOC's involved by $\mathcal{L}_1$ and $\mathcal{L}_2$ are

$$\alpha = \frac{1}{J_{xx}} \left(\Sigma^T \right)^{-1} \left[-\lambda J_x - \Sigma_Z^T J_{xz} \right] \text{ and} \tag{36}$$

$$c = J_x^{\frac{-1}{\gamma}}. \tag{37}$$

We handle each of these terms separately and then combine them afterwards.[24] This leads to two PDE's given by

$$0 = g_t - r_g^{ret}(v) g + (\nu - \tilde{\kappa} v) g_v^{ret} + 0.5 \bar{\beta}^2 v g_{vv}^{ret} + 1, \tag{38}$$

$$0 = F_t - r_F(v) F + \mu_F(v) + 0.5 \bar{\beta}^2 v F_{vv} + 1 = 0, \tag{39}$$

with terminal conditions $g(\tilde{T},v) = g^{ret}(\tilde{T},v)$ and $F(\tilde{T},v) = \Upsilon F^{ret}(\tilde{T})$. As above, we solve these PDE's by an application of Feynman Kac and the theorem of Kraft (2005). The solutions lead to the formulas in Theorem 5. The missing constants are defined as

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[23] Here, we use the matrix notation that was introduced in Section 2.

[24] The expressions for the operators become quite lengthy. Therefore, we do not report them. The formulas are available upon request.

$$\begin{aligned}
\hat{\lambda} &= \hat{\rho}_{yv}\eta_v + \rho_{ys}\eta_S, \\
b^F &= \hat{\beta}\hat{\lambda}, \\
\tilde{\tilde{\kappa}} &= \kappa + \bar{\beta}(\hat{\rho}_v\eta_v + \rho_{sv}\eta_S), \\
a^F &= \sqrt{\tilde{\tilde{\kappa}}^2 + 2b^F\bar{\beta}^2}, \\
b^g &= \frac{\gamma - 1}{2\gamma^2}\lambda, \\
a^g &= \sqrt{\tilde{\kappa}^2 + 2b^g\bar{\beta}^2}, \\
\alpha(s) &= B^r(\tilde{T}, s) \quad \forall s \geq \tilde{T}, \\
k_{\alpha(s)} &= \frac{\alpha(s)\bar{\beta}^2 + \tilde{\kappa} - a^g}{\alpha(s)\bar{\beta}^2 + \tilde{\kappa} + a^g}.
\end{aligned}$$

The optimal strategies follow by solving the FOC's.

Proof of Theorem 3. Clearly, the solution after retirement is equal to the first theorem since the investor is not allowed to invest in the income contract after his retirement.

In the active phase, we have to account for the additional hypothetical income contract. Therefore, we have to modify the matrix notation introduced in Section 2. Then, the proof is equivalent to the proof of Theorem 1. The general formulas will remain the same. We only have to adapt the specific parameters since they will depend on the risk premium of the income contract. The parameters are listed below

$$\begin{aligned}
\hat{\lambda}(\eta_I) &= \hat{\rho}_{yv}\eta_v + \rho_{ys}\eta_S + \hat{\rho}_y\eta_I, \\
b^F(\eta_I) &= \hat{\beta}\hat{\lambda}, \\
a^F(\eta_I) &= \sqrt{\tilde{\tilde{\kappa}}^2 + 2b^F(\eta_I)\bar{\beta}^2}, \\
b^g(\eta_I) &= \frac{\gamma - 1}{2\gamma^2}(\eta_I^2 + \lambda), \\
a^g(\eta_I) &= \sqrt{\tilde{\kappa}^2 + 2b^g(\eta_I)\bar{\beta}^2}, \\
k_{\alpha(s)}(\eta_I) &= \frac{\alpha(s)\bar{\beta}^2 + \tilde{\kappa} - a^g(\eta_I)}{\alpha(s)\bar{\beta}^2 + \tilde{\kappa} + a^g(\eta_I)}.
\end{aligned}$$

δ	time preference rate	0.03	ν	first param vola	0.12375
γ	risk aversion	4	κ	mean reversion speed	5.5
ε	weight	5	$\bar{\beta}$	vol of vol	0.25
$\tilde{T}$	retirement date	20	ρ_{sv}	correl. vola and stock risk	-0.5
T	terminal date	40	α	expected active income growth	0.015
r	interest rate	0.02	$\tilde{\alpha}$	expected retirement growth	0
η_S	stock excess return	2.2	$\hat{\beta}$	income volatility	0.65
η_v	volatility risk premium	-1	Υ	replacement ratio	0.4

Table 1: Benchmark parameter values. The table shows the values of the model parameters in our main analysis unless otherwise stated. Time is measured in years.

$\frac{x}{y}$	myopic demand	income hedge	delta hedge	total	over total wealth
		$\rho_{ys} = 0$ and $\rho_{yv} = 0.866$			
0.25	50.365	-46.218	0	4.148	0.033
0.5	25.386	-23.109	0	2.277	0.036
1	12.896	-11.554	0	1.341	0.042
2	6.651	-5.777	0	0.873	0.053
4	3.528	-2.889	0	0.64	0.074
no income	0.406	0	0	0.406	0.406
		$\rho_{ys} = 0$ and $\rho_{yv} = -0.866$			
0.25	34.093	31.164	0	65.257	0.776
0.5	17.249	15.582	0	32.832	0.772
1	8.828	7.791	0	16.619	0.764
2	4.617	3.896	0	8.512	0.748
4	2.511	1.948	0	4.459	0.72
no income	0.406	0	0	0.406	0.406
		$\rho_{ys} = 1$ and $\rho_{yv} = -0.5$			
0.25	27.65	-43.654	0	-16.004	-0.235
0.5	14.028	-21.827	0	-7.799	-0.226
1	7.217	-10.913	0	-3.697	-0.208
2	3.811	-5.457	0	-1.646	-0.175
4	2.108	-2.728	0	-0.62	-0.119
no income	0.406	0	0	0.406	0.406
		$\rho_{ys} = -1$ and $\rho_{yv} = 0.5$			
0.25	65.187	103.8	0	168.987	1.052
0.5	32.796	51.9	0	84.696	1.048
1	16.601	25.95	0	42.551	1.04
2	8.503	12.975	0	21.478	1.025
4	4.454	6.488	0	10.942	0.996
no income	0.406	0	0	0.406	0.406

Table 2: Stock investments for different correlation combinations in the complete market. This table shows how the optimal fraction of financial wealth invested in the stock market depends on different income-stock correlations and wealth/income ratios (total). We also consider the case for no income. We further show the specific components and the investment over total wealth (over total wealth). We set the volatility at time zero equal to its long run mean 0.0225.

ρ_{ys}	ρ_{yv}	$L^{com}(t,v)$
0	0.8666	94.959
0	-0.8666	53.642
1	-0.5	40.091
-1	0.5	140.615

Table 3: Human wealth for different correlation combinations in the complete market. This table shows the value of human wealth for different correlation combinations that imply a complete market. We set the volatility at time zero equal to its long run mean 0.0225. Initial financial wealth and income are set to 2.

$\frac{x}{y}$	myopic demand	income hedge	vola hedge	$\frac{O_t}{h_v}$	total	over total wealth
			$\rho_{ys}=0$ and $\rho_{yv}=0.866$			
0.25	-143.363	-386.1	-13.132	0.09	-48.654	-0.392
0.5	-72.259	-193.05	-6.619	0.09	-24.383	-0.39
1	-36.707	-96.525	-3.362	0.09	-12.248	-0.385
2	-18.931	-48.262	-1.734	0.09	-6.181	-0.377
4	-10.043	-24.131	-0.92	0.09	-3.147	-0.362
no income	-1.155	0	-0.106	0.09	-0.113	-0.113
			$\rho_{ys}=0$ and $\rho_{yv}=-0.866$			
0.25	-97.045	259.701	-8.889	0.09	13.788	0.164
0.5	-49.1	129.85	-4.497	0.09	6.838	0.161
1	-25.127	64.925	-2.302	0.09	3.362	0.155
2	-13.141	32.463	-1.204	0.09	1.625	0.143
4	-7.148	16.231	-0.655	0.09	0.756	0.122
no income	-1.155	0	-0.106	0.09	-0.113	-0.113
			$\rho_{ys}=1$ and $\rho_{yv}=-0.5$			
0.25	-78.704	18.639	-7.209	0.09	-6.032	-0.089
0.5	-39.929	9.32	-3.657	0.09	-3.073	-0.089
1	-20.542	4.66	-1.882	0.09	-1.593	-0.09
2	-10.848	2.33	-0.994	0.09	-0.853	-0.091
4	-6.002	1.165	-0.55	0.09	-0.483	-0.093
no income	-1.155	0	-0.106	0.09	-0.113	-0.113
			$\rho_{ys}=-1$ and $\rho_{yv}=0.5$			
0.25	-185.552	-46.213	-16.996	0.09	-22.306	-0.139
0.5	-93.353	-23.106	-8.551	0.09	-11.21	-0.139
1	-47.254	-11.553	-4.328	0.09	-5.661	-0.138
2	-24.204	-5.777	-2.217	0.09	-2.887	-0.138
4	-12.68	-2.888	-1.161	0.09	-1.5	-0.137
no income	-1.155	0	-0.106	0.09	-0.113	-0.113

Table 4: Investments in straddle for different correlation combinations in the complete market. This table shows how the optimal fraction of financial wealth invested in the straddle depends on different income-stock correlations and wealth/income ratios (total). We also consider the case of zero income (no income). We further show the specific components and the investment over total wealth (over total wealth). We set the volatility at time zero equal to its long run mean 0.0225.

$\frac{x}{y}$	$\frac{O_t}{h_v}$(C)	$\frac{O_t}{h_v}$(P)	total (C)	total (P)	over total wealth (C)	over total wealth (P)
$\rho_{ys} = 0$ and $\rho_{yv} = 0.866$						
0.25	0.099	0.083	-53.749	-44.857	-0.433	-0.361
0.5	0.099	0.083	-26.937	-22.481	-0.43	-0.359
1	0.099	0.083	-13.531	-11.292	-0.426	-0.355
2	0.099	0.083	-6.828	-5.698	-0.416	-0.348
4	0.099	0.083	-3.476	-2.901	-0.4	-0.334
no income	0.099	0.083	-0.125	-0.104	-0.125	-0.104
$\rho_{ys} = 0$ and $\rho_{yv} = -0.866$						
0.25	0.099	0.083	15.232	12.712	0.181	0.151
0.5	0.099	0.083	7.554	6.304	0.178	0.148
1	0.099	0.083	3.714	3.1	0.171	0.142
2	0.099	0.083	1.795	1.498	0.158	0.132
4	0.099	0.083	0.835	0.697	0.135	0.113
no income	0.099	0.083	-0.125	-0.104	-0.125	-0.104
$\rho_{ys} = 1$ and $\rho_{yv} = -0.5$						
0.25	0.099	0.083	-6.664	-5.562	-0.098	-0.082
0.5	0.099	0.083	-3.394	-2.833	-0.098	-0.082
1	0.099	0.083	-1.76	-1.469	-0.099	-0.083
2	0.099	0.083	-0.942	-0.786	-0.1	-0.084
4	0.099	0.083	-0.534	-0.445	-0.103	-0.086
no income	0.099	0.083	-0.125	-0.104	-0.125	-0.104
$\rho_{ys} = -1$ and $\rho_{yv} = 0.5$						
0.25	0.099	0.083	-24.642	-20.565	-0.153	-0.128
0.5	0.099	0.083	-12.383	-10.335	-0.153	-0.128
1	0.099	0.083	-6.254	-5.219	-0.153	-0.128
2	0.099	0.083	-3.189	-2.662	-0.152	-0.127
4	0.099	0.083	-1.657	-1.383	-0.151	-0.126
no income	0.099	0.083	-0.125	-0.104	-0.125	-0.104

Table 5: Investments in Call (C) and Put (P) for different correlation combinations in the complete market. This table shows how the optimal fraction of financial wealth invested in the derivatives depends on different income-stock correlations and wealth/income ratios (total). We also consider the case of zero income (no income). We further show the specific components and the investment over total wealth (over total wealth). We set the volatility at time zero equal to its long run mean 0.0225. Both options are at the money and have a maturity of 3 months.

	stock income correlation ρ_{ys}				
wealth-income ratio $\frac{x}{y}$	0	0.2	0.4	0.6	0.8
0.25	6.686	3.159	1.826	1.627	2.09
0.5	6.031	2.942	1.617	1.283	1.708
1	4.489	2.163	1.248	0.837	1.203
2	3.468	1.654	0.83	0.571	0.688
4	2.227	1.008	0.5	0.321	0.252

Table 6: Percentage welfare losses. This table shows the welfare losses for different stock-income correlations and several wealth-income ratios when using SAMS. We set the volatility at time zero equal to its long run mean 0.0225. The stock-volatility correlation reads $\rho_{yv} = -0.2$. All numbers represent percentage values.

	stock income correlation ρ_{ys}				
wealth-income ratio $\frac{x}{y}$	0	0.2	0.4	0.6	0.8
0.25	14.46	10.95	9.46	9.28	9.86
difference in losses	7.77	7.79	7.63	7.65	7.77
0.5	13.57	10.48	8.99	8.66	9.26
difference in losses	7.54	7.54	7.37	7.38	7.55
1	11.79	9.44	8.36	7.95	8.53
difference in losses	7.3	7.28	7.11	7.11	7.33
2	10.46	8.36	7.33	7.02	7.28
difference in losses	6.99	6.71	6.5	6.45	6.59
4	8.9	7.15	6.39	6.12	6.11
difference in losses	6.67	6.14	5.89	5.8	5.86

Table 7: Percentage welfare losses for two methods. This table shows the welfare losses for different stock-income correlations and several wealth-income ratios for the Markov chain method. The second row of each wealth-income specification represents the difference in welfare losses compared with SAMS (see Table 6). We set the volatility at time zero equal to its long run mean 0.0225. The stock-volatility correlation reads $\rho_{yv} = -0.2$. All numbers represent percentage values.

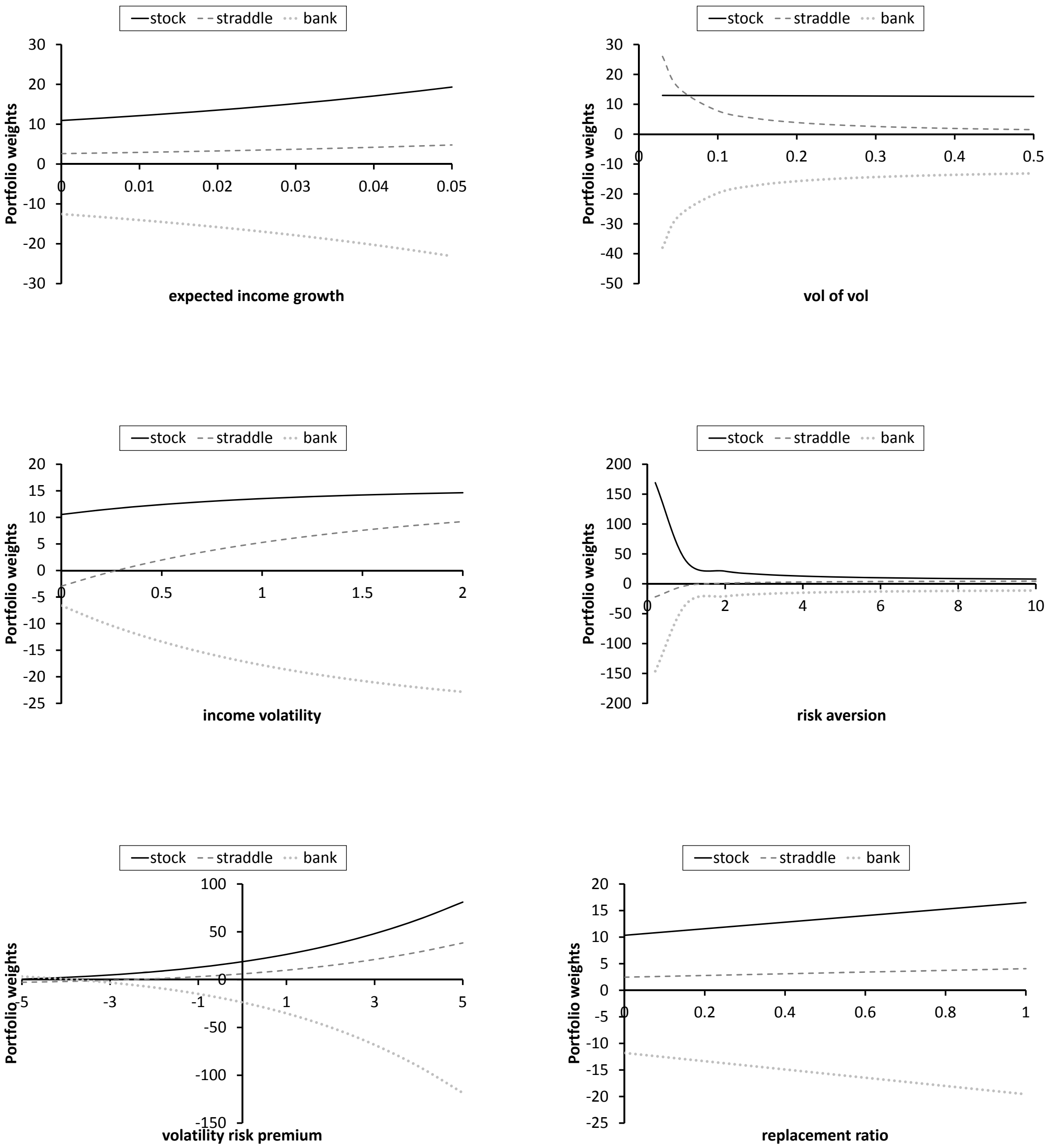

Figure 1: Optimal portfolio weights at time 0. The graphs display the optimal portfolio fractions on the risky stock (solid line), on the delta-neutral straddle straddle (dashed line), and on the risk-free bank account (dotted line) in a complete market. The base-case parameters are as described in Table 1. The volatility v_0 is set to its long run mean. Initial wealth and initial income are assumed to be 2. The stock-income correlation reads $\rho_{ys} = 0.2$ which implies $\hat{\rho}_{yv} = -0.948$.

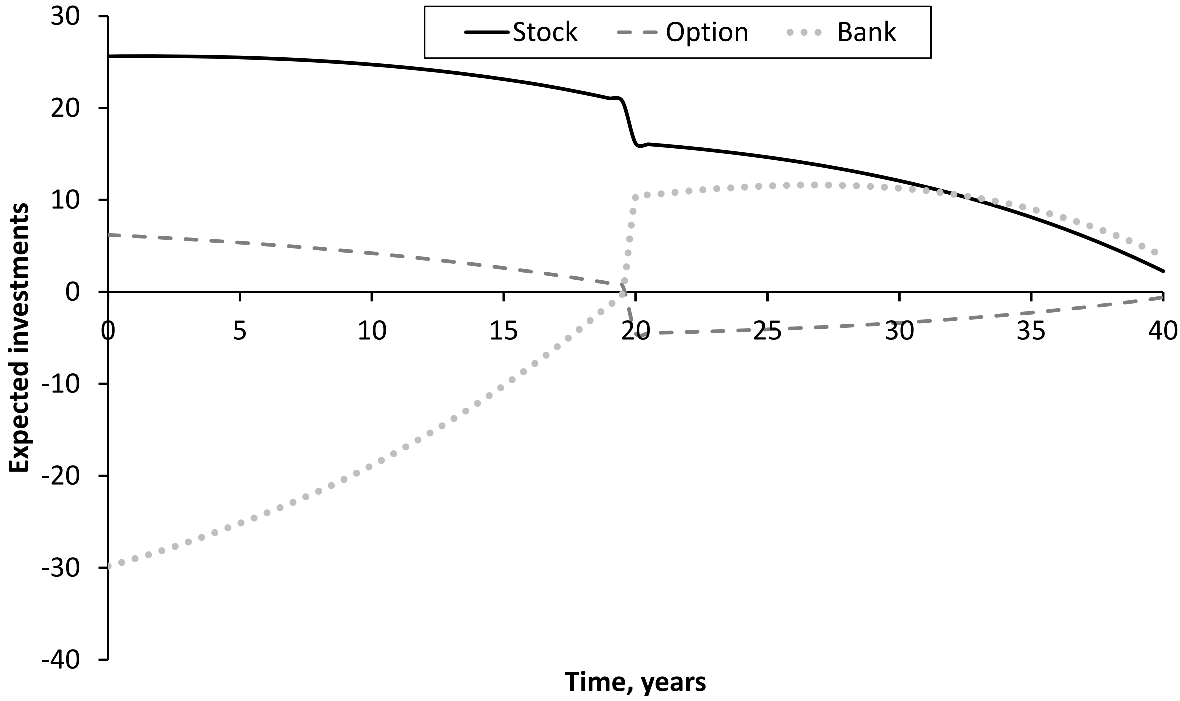

Figure 2: Expected investments over the life-cycle. The curves show the expected spending in thousands of US dollars on the risky stock (solid line), on the delta-neutral straddle (dashed line), and on the risk-free bank account (dotted line) in a complete market. The results are generated with the benchmark parameters in Table 1. The current volatility is set to its long run mean. The current financial wealth is x = 20,000 USD, and the current income is y = 20,000 USD per year. The stock-income correlation reads $\rho_{ys} = 0.2$ which implies $\hat{\rho}_{yv} = -0.948$.

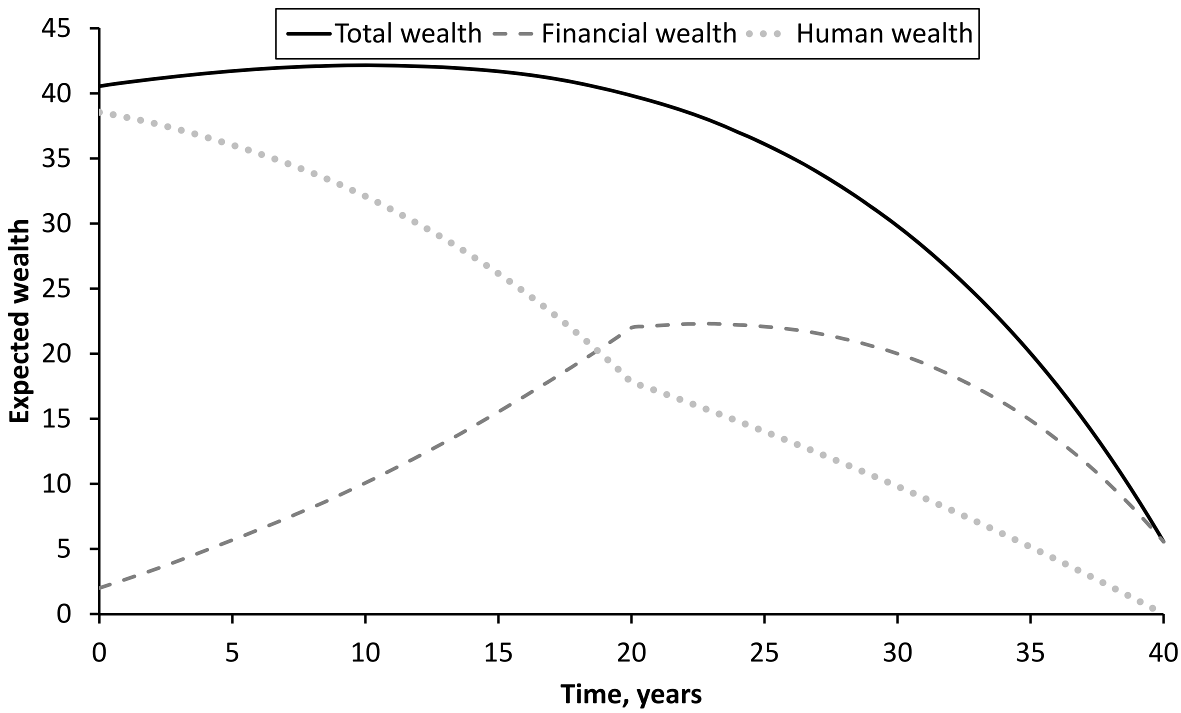

Figure 3: Expected wealth over the life-cycle. The curves show the expectations of the total wealth (solid line), financial wealth (dashed line), and human wealth (dotted line) over the life-cycle in a complete market. The results are generated with the benchmark parameters in Table 1. The current volatility is set to its long run mean. The current financial wealth is x = 20,000 USD, and the current income is y = 20,000 USD per year. The stock-income correlation reads $\rho_{ys} = 0.2$ which implies $\hat{\rho}_{yv} = -0.948$.

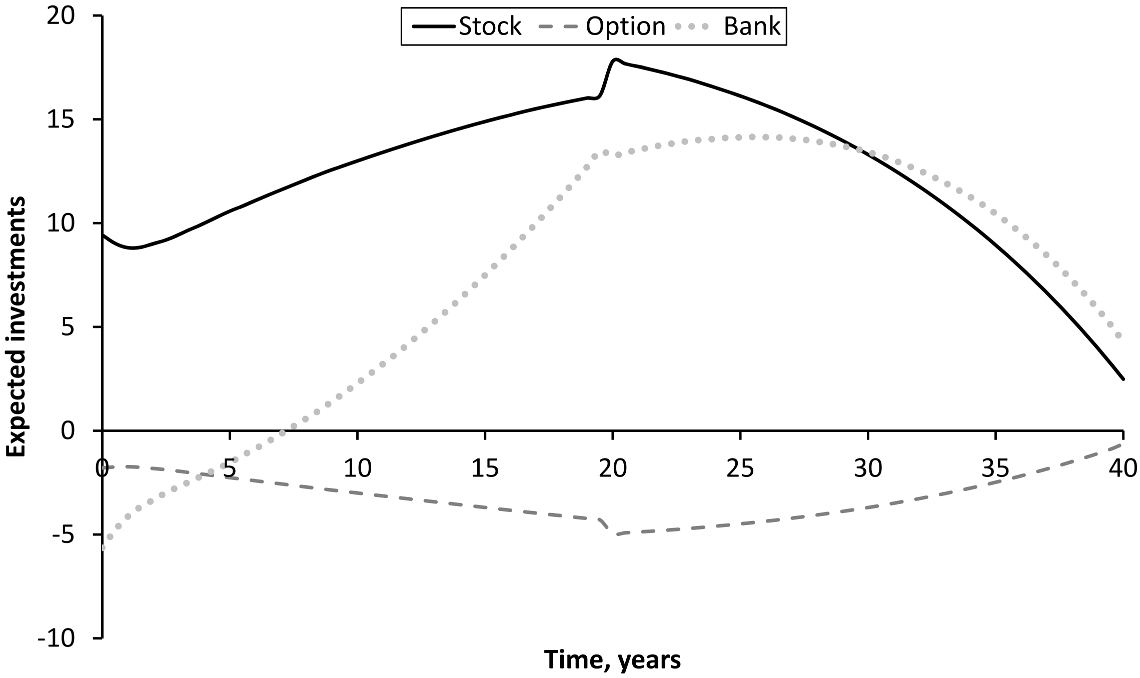

Figure 4: Expected investments over the life-cycle. The curves show the expected spending in thousands of US dollars on the risky stock (solid line), on the delta-neutral straddle (dashed line), and on the risk-free bank account (dotted line) in an incomplete market (We use SAMS to obtain the strategies). The results are generated with the benchmark parameters in Table 1. The current volatility is set to its long run mean. The current financial wealth is x = 20,000 USD, and the current income is y = 20,000 USD per year. The stock-income correlation reads $\rho_{ys} = 0.2$ and the income-volatility correlation is set to $\rho_{yv} = -0.2$.

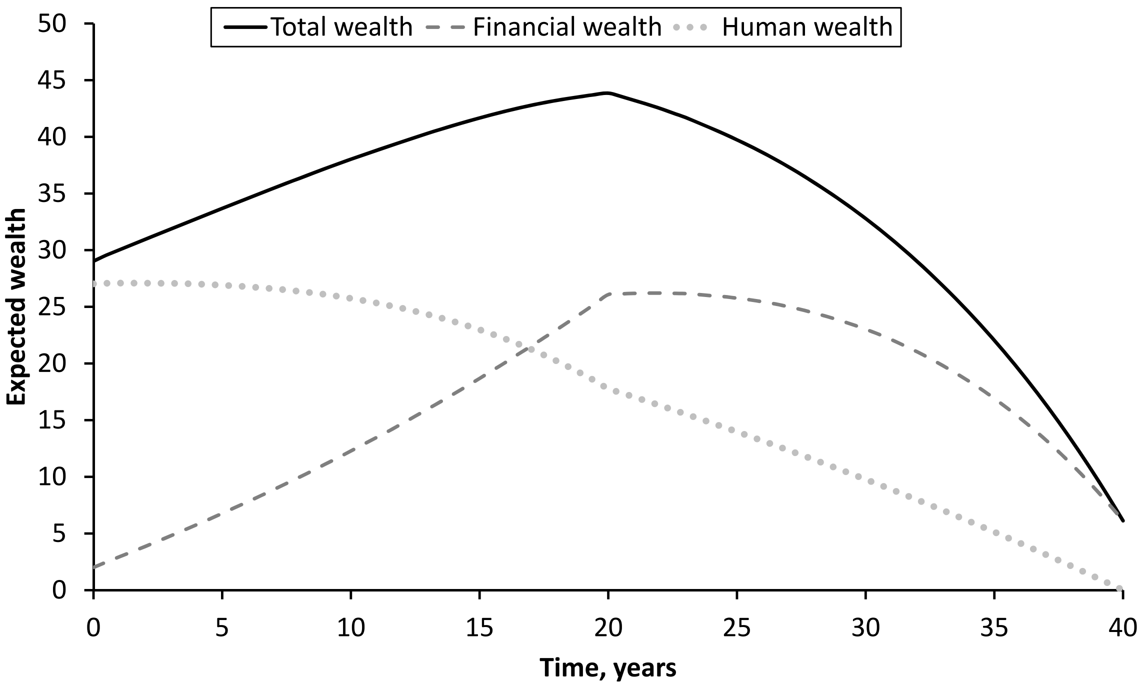

Figure 5: Expected wealth over the life-cycle. The curves show the expectations of the total wealth (solid line), financial wealth (dashed line), and human wealth (dotted line) over the life-cycle in an incomplete market (We use SAMS to obtain the strategies). The results are generated with the benchmark parameters in Table 1. The current volatility is set to its long run mean. The current financial wealth is x = 20,000 USD, and the current income is y = 20,000 USD per year. The stock-income correlation reads $\rho_{ys} = 0.2$ and the income-volatility correlation is set to $\rho_{yv} = -0.2$.

Default and Idiosyncratic Risk Anomalies Revisited

Björn Bick, Christian Hirsch, Holger Kraft, Yildiray Yildirim

Summary. This paper studies the default anomaly that has been documented in the literature. We show that after controlling for the default-risk premium the default anomaly disappears. In contrast, controlling for credit spreads does not fully eliminate the anomaly. We also relate our results to the IVOL anomaly and find evidence that this anomaly disappears when one controls for default risk via the default-risk premia.

1 Introduction

In this paper, we revisit two anomalies studied in the finance literature. The first anomaly is the idiosyncratic volatility (IVOL) anomaly. Under the assumptions of the Capital Asset Pricing Model (CAPM) only systematic risk is priced[1] suggesting that high expected returns are associated with high levels of systematic risk. This is because investors are assumed to hold well diversified portfolios. However, many investors are not well diversified[2], so both systematic and idiosyncratic risk might matter. Merton (1987) and Malkiel and Xu (2004) show that idiosyncratic risk can be priced. Less diversified investors care about the total risk and not only about the systematic risk. As a result, a risk premium for idiosyncratic risk might emerge. Empirical evidence of a cross-sectional relationship between idiosyncratic risk and stock returns is mixed. While the majority of studies such as Malkiel and Xu (2004), Spiegel and Wang (2005), Fu (2009), and others suggest a positive relationship, Ang et al. (2006, 2009) document a negative relationship.

The second anomaly we are going to address is the distress anomaly. The literature suggests financial distress as another missing factor of CAPM. In earlier studies, value and size effects are used as proxies for financial distress. Chan and Chen (1991) document that financial distress can not be diversified away and hence investors require a premium for bearing such risk. Fama and French (1996) show that book-to-market (BTM) and high-minus-low (HML) portfolios are

[1] If the CAPM holds, then the market portfolio is the tangency portfolio, and the estimated alpha should be zero and total risk is only the systematic (e.g. market) risk.

[2] Goetzmann and Kumar (2007) show that more than 25% of the investors hold only one stock in their portfolio.

proxies for distress. Ferguson and Shockley (2003) document that distress risk is priced in equity returns. Although distress risk seems to be a missing factor, the empirical evidence on the relation between distress risk and equity returns is mixed. For example, after controlling for the Fama-French factors, Campbell, Hilscher and Szilagyi (2008) show that distressed firms deliver lower returns than non-distressed firms. This result is interesting for two reasons: First, there are significant abnormal returns. Secondly, the result contradicts the intuition that higher risk should be rewarded by higher returns. These findings suggest that the size and value factors of the model by Fama and French (1993) cannot fully capture default risk and point in the direction of a missing factor that is related to default risk. While Dichev (1998), Griffin and Lemmon (2002) and Campbell, Hilscher and Szilagyi (2008) find a negative relationship between distress risk and equity returns, Vassalou and Xing (2004) and Da and Gao (2008) document a positive relationship.

Dichev (1998) documents that non-distressed firms outperform distressed firms when using both Altman's (1968) Z-score and Ohlson's (1980) O-score as proxies for financial distress. Griffin and Lemmon (2002) and George and Hwang (2010) also find evidence in support of the distress anomaly using the O-score. Campbell, Hilscher and Szilagyi (2008) use a discrete time hazard model similar to those used by Shumway (2001) and Chava and Jarrow (2004) and find that there is a strong negative distress premium. Avramov, Chordia, Gergana and Philipov (2009) use credit ratings from S&P and find a negative distress premium due to the poor performance of low rated firms during periods of distress. Garlappi, Shu and Yan (2008) use Expected Default Frequencies (EDF) from Moody's KMV and find that stock returns are not positively related to distress risk. However, they do not find evidence of a negative premium. Vassalou and Xing (2004) provide one of the few studies documenting a positive relationship between default risk and future returns. They use Merton's (1974) option pricing model and firm equity characteristics to form default likelihood indicators (DLI) as proxies for distress. However, Da and Gao (2010) find that the results of Vassalou an Xing are driven by high returns earned by high distance-to-default firms that recently experienced large negative returns. All of these studies calculate default probabilities under the physical measure (P measure) and not under the risk-neutral measure (Q measure).

There is evidence that the two anomalies are related. Campbell and Taksler (2003) show that idiosyncratic firm-level volatility can explain cross-sectional variation in corporate yields because it increases the probability of default. They document a link between the idiosyncratic risk and distress risk. In this paper, we first revisit the anomalies, and then address the link between them.

We argue that the distress anomaly should disappear when we correct for an appropriate measure of default risk. Our paper addresses this point and shows that the default-risk premium can successfully be used as a control to eliminate the default anomaly. The default-risk premium plays a central role in credit-risk theory and can be viewed as the Sharpe ratio of credit risk. It is defined as the ratio of the credit spread of a firm over its actual physical default probability.[3] Since the credit spread is the excess return that an investor receives for bearing a firm's default risk, the default-risk premium is the excess return per unit of default risk, i.e. it is a relative risk premium. Put differently, a small default risk premium indicates that a firm can cheaply issue debt relative to its default risk. Analogous to the ordinary Sharpe ratio, which is defined as the excess return of a stock over its volatility, the default risk premium is relevant for several reasons. First, Merton (1971) shows that an investor's optimal demand for a corporate bond is driven by the default risk premium. Secondly, the default risk premium characterizes the risk-neutral probability measure used to price credit-risky assets (see, e.g. Duffie (2001)).

The previous discussion suggests that apart from the default-risk premium there are two other default measures that can be used to control for credit risk. These are the credit spread (numerator of the default-risk premium) and the actual default probability (denominator of the default-risk premium). There are different ways of determining both variables. To estimate credit spreads, data for traded securities are needed and one can use quotes from either corporate bonds or credit default swaps (CDS). Longstaff, Mithal and Neis (2005) document that corporate bond prices are affected by several factors including liquidity and tax issues and suggest that CDS are a cleaner measure of default risk. To estimate actual default probabilities, a database with actual defaults is needed so that one can calibrate an econometric model to these data. Then, mainly two approaches are used: logit-type regressions[4] or methods based on the firm-value approach by Merton (1974). Campbell, Hilscher and Szilagyi (2008) document the default anomaly by using a logit specification to extract actual default probabilities and sorting on these probabilities. Running Fama-French regressions, they find positive alphas for portfolios with low default probabilities and negative alphas for portfolios with high default probabilities. They also show that their method of estimating actual default probabilities is superior over alternative methods. Anginer and Yildizhan (2010) compute credit spreads as the difference between a firm's bond yield and the corresponding maturity matched treasury rate. Using this particular measure of the credit spread they find that the default anomaly disappears. Berndt, Lookman and Obreja (2007) construct a default-risk related factor from the prices of default contingent claims. In contrast to the Fama-French factors, their factor is not the return of a trading strategy. Berndt, Douglas, Duffie, Ferguson and Schranz (2008) extract default risk premia of several firms. They obtain the firm's actual default probabilities using expected

[3] More precisely, it is the risk-neutral default probability and not the credit spread, but both concepts are closely related. We will come back to that point later on.

[4] See, e.g., Shumway (2001), Chava and Jarrow (2004), Bever, McNichols and Rhie (2005).

default frequencies as provided by Moody's KMV[5] and estimate the credit spreads from CDS data. Their focus differs from ours since they concentrate on the time variation of the risk premia. Vassalou and Xing (2004) analyze the effect of default risk on equity returns using a distance-to-default measure. They find some evidence that distressed stocks, mainly in the small value group, earn higher returns. Value and size effects have also been attributed to be proxies for financial distress (Chan and Chen (1991) and Fama and French (1996)).

In this paper, we use all three measures as controls. We extract the actual default probabilities using the approach by Campbell, Hilscher and Szilagyi (2008). We obtained CDS data from Markit to determine credit spreads for more than 700 firms from 2001 to 2010. We then calculate the risk premium by calculating the ratio of credit spread to the pyhsical default probability. Like Campbell, Hilscher and Szilagyi (2008), we find a default anomaly when sorting on actual default probabilities. The anomaly becomes smaller, but is not fully eliminated when we sort with respect to credit spreads extracted from CDS data. However, when we use the default risk premium to control for default risk, the anomaly completely disappears. This suggests that the default risk premium is a superior measure of default risk. Finally, we relate these results to the idiosyncratic volatility (IVOL) anomaly. We first document that the IVOL anomaly is present in our data. Then we provide evidence that this anomaly only disappears if default risk premia are used as controls. This is not true for physical default probabilities or credit spreads.

Apart from the previously mentioned papers, our study is related to several other papers. First, a number of existing papers study risk-neutral default probabilities using CDS data. Das, Hanouna and Sarin (2009) examine how accounting-based and market-based variables perform in pricing the risks of default by using a large sample of CDS data. They find that models using accounting data explain CDS spreads at least as well as structural models that use market data. Tang and Yan (2010) empirically investigate the explanatory power of macroeconomic conditions and firm characteristics and the effect of their interactions. They identify implied volatility as the most significant determinant of default risk among firm-level characteristics. Furthermore, Zhang, Zhou and Zhu (2009) try to explain CDS spreads by using volatility risks and jump risks of individual firms. They incorporate these risks in a Merton-type structural model in order to price CDS. Cesare and Guazzarotti (2010) analyze how the recent financial crisis has changed the way credit risk is priced in CDS markets. In their regressions, they include the theoretical spread that follows from a simple Merton model in order to deal with non-linear relationships. They find that including this theoretical spread improves the explanatory power of fundamental variables. Moreover, their model is able to explain more than one half of the variation in CDS spreads before and during the crisis. They also find that leverage affected

[5] Moody's KMV offer a proprietary database. Although KMV has made it public that their method is based on Merton's approach, the exact procedure of how the actual probabilities are calculated is not disclosed so that their estimates cannot easily be replicated. See, e.g. Duffie and Singleton (2003) for more details.

CDS spreads much more during the crisis than before. At the same time, the impact of equity volatility substantially decreased.

Secondly, there is a strand of literature that deals with bond risk premia including Cieslak and Povala (2010) , Driessen (2005), Ludvigson and Ng (2009), and Westerkamp and Zechner (2010), among others. Westerkamp and Zechner (2010) examine whether corporate bond risk premia vary systematically across bond specific characteristics and across time. Cieslak and Povala (2010) decompose long-term yields into a persistent component and maturity-related cycles to study the predictability of bond excess returns. Ludvigson and Ng (2009) study whether bond market risk premia are directly related to cyclical macroeconomic behavior and identify macroeconomic variables that drive bond risk premia. They find a strong predictable variation in excess bond returns that is related to macroeconomic activity. Their results suggest that investors are compensated for risks that are related to recessions. Furthermore, several authors study the components of corporate yield spreads including Eom, Helwege and Huang (2004), Huang and Huang (2003), Collin-Dufresne, Goldstein and Helwege (2003), and Longstaff, Mithal and Neis (2005).

The remainder of the paper is structured as follows: Section 2 explains how the default risk premium is defined and Section 3 explains the logit approach to estimate physical default probabilities. Section 4 discusses how the risk-neutral default probabilities are extracted and Section 5 reports the main results of the paper and shows that the default anomaly is eliminated once one controls for the default risk premim of a firm. The paper concludes in Section 6.

2 Default Risk Premium

Default risk premia are central for the understanding of this paper. Therefore, we briefly summarize some relevant properties. Following Jarrow and Turnbull (1995) and Duffie and Singleton (1999) we assume that default occurs as a sudden surprise. More formally, this can be modeled via the first jump of some exogenously given counting process jumps. For a short time interval, dt, the actual default probability is then approximately equal to $\lambda_t^P dt$, where λ_t^P denotes the physical default intensity. For small default probabilities, the intensity is roughly equal to the annual default probability. For instance, if $\lambda^P = 0.01$, then the annual default intensity is about 1%. This is because for a constant intensity the default probability over the time period $[t, T]$ is given by

$$\mathrm{PD}(t, T) = 1 - e^{\lambda^P (T-t)} \approx \lambda^P (T - t). \tag{1}$$

For the pricing of credit-risky securities, the risk-neutral default intensity, λ^Q, is relevant. The ratio of both intensities is the *default risk premium* which is defined by:

$$\chi = \frac{\lambda^Q}{\lambda^P},$$

where λ^Q is the risk-neutral default intensity. If default risk has a positive risk premium, then χ is greater than one.[6] In this paper, we are going to use all three possible measures of default risk: the physical default intensity λ^P, the risk-neutral default intensity λ^Q, and the default risk premium χ.

To estimate the physical default intensity, we use the approach by Campbell, Hilscher and Szilagyi (2008). The risk-neutral default intensities can be extracted from CDS quotes. In particular, under the assumption of a constant default intensity a firm's CDS quote is related to its risk-neutral intensity in the following way:[7]

$$\text{CDS} = \ell\lambda^Q,$$

where ℓ denotes the expected loss. Throughout this paper we assume that the expected loss is 60%. Therefore, we can extract implied risk-neutral default intensities from a time-series of CDS quotes given that intensities are constant which we assume in our benchmark analysis. Physical default intensity directly follow from (1). As a robustness check, we have also studied a model with stochastic intensities, but the results are very similar.

3 Estimating Physical Default Probabilities

To estimate actual default probabilities, we apply a dynamic logit approach. Following Campbell, Hilscher and Szilagyi (2008), among others, we assume that the marginal probability of default over the next period is given by a logistic distribution

$$\text{PD}(t-1,t) = P_{t-1}(D_{it}=1) = \frac{1}{1+\exp(-\alpha-\beta x_{i,t-1})}$$

where D_{it} is a dummy variable equal to one if the firm defaults within month t and zero otherwise. The term $x_{i,t-1}$ represents a vector of explanatory variables. We use the "best-model variables" as identified by Campbell, Hilscher and Szilagyi (2008).[8]

3.1 Data

Our default information ranges from January 1986 to December 2009 and comes from Moodys's default risk service database. All firms in our CDS sample have a rating. Faulkender and Petersen (2006) show that firms with access to the public bond market (i.e. rated firms) differ from firms without access to bond markets. Therefore, physical default probabilities are estimated

[6] Sometimes the default risk premium is defined as $\eta \equiv \chi - 1$ so that there is a premium if η greater than zero. Since this shifts all default risk premia at the same time and does not effect their rankings, we use χ.

[7] See, e.g., Longstaff, Mithal and Neis (2005).

[8] Model 2 in Table III of Campbell, Hilscher and Szilagyi (2008).

from rated firms only. To obtain information on firm characteristics we merge quarterly accounting data from Compustat and daily and monthly stock price information from CRSP. We assume that the accounting information is available two months after the end of the accounting quarter. We follow Faulkender and Petersen (2006) and classify firms in a given month as rated if the firm has a S&P long-term or S&P short-term rating outstanding in Compustat. Our sample period begins in January of 1986 because this is when rating information becomes available in Compustat. We exclude financial firms from our analysis.

Table 1 reports the number and percentage of defaulted firms in our sample. The second column presents the average number of active firms in the sample for a given year. The average number for firms is calculated in two steps. We first calculate the average number of firms in a given month. To arrive at the average over a given year reported in the table we average over the monthly averages. Compared to other studies that use monthly observations to predict default probabilities (e.g. Chava and Jarrow (2004) or Campbell, Hilscher and Szilagyi (2008)) we have less observations because these studies do not restrict their sample to include rated firms only. However, the ratio of defaulted firms to active firms is comparable to other studies.

Following Campbell, Hilscher and Szilagyi (2008) the variables for the prediction of physical default probabilities are constructed as follows: *TLMTA* is a measure of the firm's leverage. It is defined as the book value of total liabilities over the market value of total assets. The book value of total liabilities is the sum of total liabilities (Compustat item: LTQ) and minority interests (MIBQ). The market value of total assets is defined as the market value of equity (*ME*) obtained from CRSP and the book value of total liabilities (LTQ +MIBQ). The monthly CRSP stock file is used to calculate market value of equity. We expect leverage to be positively related to default. *CASHMTA* is a liquidity measure. It is constructed as cash and short-term investments (CHEQ) over market value of total assets. We expect that the more liquidity the firm has at hand, the lower is the likelihood of default. *PRICE* is the firm‘s log price per share truncated above 15 dollars. Firms that are close to default trade at very low stock prices. *RSIZE* is constructed as the log ratio of the market value of equity of the firm in a given month to the total market value of the constituents of the S&P 500 index in that month. We follow Davis, Fama and French (2000) and Cohen, Polk and Vuolteenaho (2003) in constructing the book value of equity for our *MB* variable. We further adjust the book value of equity (*BE*) to account for very small values that would lead to very large values of *MB* using the following equation:

$$BE_{adjusted,i,t} = BE_{i,t} + 0.1(ME_{i,t} - BE_{i,t}).$$

Campbell, Hilscher and Szilagyi (2008) argue that the market-to-book variable may reflect overvaluation of distressed firms that have recently experienced heavy losses. *SIGMA* is our measure of equity volatility. It is computed using the annualized 3-month standard deviation of the firms‘s daily stock return. If less than 5 non-missing return observations are available *SIGMA* is set to missing. We expect distressed firms to have very high volatility levels. Our measure of the profitability of the firm (*NIMTA*) is constructed as the ratio of net income (NIQ) to market value of total assets. Campbell, Hilscher and Szilagyi (2008) find that the history of profitability is a more informative measure of default than just the most recent observation since firms about to default tend to incur losses over longer time periods. To account for this, we construct the variable *NIMTAAVG* which is computed as the geometric weighted average level of profitability where the weight is halved over each month:

$$NIMTAAVG_{t-1,t-12} = \frac{1-\phi}{1-\phi^{12}}(NIMTA_{t-1} + ... + NIMTA_{t-12}) \tag{2}$$

with $\phi = 2^{-\frac{1}{3}}$.

Lastly, we include a measure of excess return (*EXRET*) which is the stock return of the firm minus the return of the S&P 500 index. Like profitability we also expect firms near default to experience negative returns for an extended period of time. We therefore construct *EXRETAVG* as

$$EXTRETAVG_{t-1,t-12} = \frac{1-\phi}{1-\phi^{12}}(EXRET_{t-1} + ... + EXRET_{t-12}) \tag{3}$$

with $\phi = 2^{-\frac{1}{3}}$.

We winzorize *TLMTA*, *NIMTA*, *EXRET*, *MB*, *RSIZE*, and *CASHMTA* at the 5/95 % level to account for outliers and other data problems. A firm-month is included in our default prediction sample if we observe *TLMTA*, *NIMTA*, *EXRET*, and *RSIZE*. We replace missing values of *SIGMA* and *CASHMTA* with its cross-sectional mean. When constructing *NIMTAAVG* and *EXRETAVG* we also replace lagged missing values of *NIMTA* and *EXRET* with their cross-sectional mean.

Table 2 provides descriptive statistics for the explanatory variables used in the dynamic logit default prediction model. In Panel A descriptive statistics for the entire sample are reported. Compared to Campbell, Hilscher and Szilagyi (2008) the firms in our sample have lower variations in returns as measured by *SIGMA*. E.g. Campbell, Hilscher and Szilagyi (2008) report a mean (median) SIGMA of 0.562 (0.471). In contrast, the mean (median) *SIGMA* in our data set is 0.418 (0.339). This is in line with rated firms using more leverage and therefore having lower volatility (Faulkender and Petersen (2006)). The firms in our data set are generally larger irrespective of size or price. Again this is consistent with the findings of Faulkender and Petersen (2006), among others, that rated firms tend to be large.

Table 2, Panel B reports descriptive statistics for the sample of defaulted firms. The explanatory variables are reported in the month immediately preceding default. Comparing Panel A and B shows that the two subgroups differ as expected in most of the explanatory variables.

Exceptions are *MB* and *CASHMTA*. Note that the difference in the mean *PRICE* variable between the two groups is higher (2.47 - 0.235) than reported in Campbell, Hilscher and Szilagyi (2008) (2.01 - 0.432). The same is true for the median. This means that in our sample the *PRICE* variable better discriminates between default and non-default firms. On the other hand *MB* and *CASHMTA* do not differ by much in the two subgroups. We expect these variables not to be valuable when it comes to prediction.

3.2 Results

The results of our logit analysis are presented in Table 3. All explanatory variables have the expected sign and are in line with the findings of Campbell, Hilscher and Szilagyi (2008). An exception is *RSIZE* which is negative and significant at the 10% level. Larger firms are less likely to go bankrupt.[9] Furthermore, we find *MB* not to be significant (although it shows the right sign). Note that the pseudo-R^2 of our regression is higher than in the paper of Campbell, Hilscher and Szilagyi (2008). One possible explanation for this could be that explanatory variables related to firm size (*RSIZE* and *PRICE*) better identify distressed firms in our sample. This is because in restricting our sample to rated firms we effectively select the largest firms in the population, while Campbell, Hilscher, and Szilagyi (2008) use the entire universe of firms from the merged Compustat-CRSP database.

4 Estimating Risk-neutral Default Intensities

Our CDS data comes from Markit. To calculate default risk premia, we have to restrict our analysis to CDS contracts written on firms for which data is available in Compustat and CRSP. We merge daily CDS spread data to our monthly predicted physical default probabilities using ticker and company names.[10] Table 4 presents summary statistics for the default risk premia sample. The number of observations per year is reported in Column (2), while Column (3) shows the average number of active firms in the sample. In total we have over 650 000 daily observations. Table 5 presents descriptive statistics for default risk premia calculated under the assumption that default intensities are constant. Compared to Berndt, Douglas, Duffie, Ferguson and Schranz (2008) our estimates are generally higher. However, their time series ends before the financial crisis.

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[9] Notice that for model 1 in Campbell, Hilscher and Szilagyi (2008) the loading is also negative.

[10] One reason why we are not able to find a match in Compustat and CRSP is that these firms are subsidaries (e.g. Ford Motor Credit Company or General Motors Acceptance Corp.) with no stock data available in CRSP.

5 Asset Pricing Tests

In this section, we sort our observations into portfolios according to three different measures for a firm's default risk. These are the physical default intensity λ^P, the risk-neutral default intensity λ^Q, and the default risk premium $\chi = \lambda^Q/\lambda^P$. More precisely, each quarter we form five portfolios sorted on each firm's median λ^P, λ^Q, or χ for the previous quarter. The first (fifth) portfolio consists of firms with the smallest (biggest) λ^P, λ^Q, or χ, respectively. We have decided to sort the firms into five rather than ten portfolios to ensure that we have on average 50 firms in every portfolio.[11] Sorting into five instead of ten portfolios should however make it harder to find anomalies since the first (fifth) portfolio is a blend of portfolios one and two (nine and ten) if one sorts into ten portfolios.

We then compute value-weighted returns for every day of the quarter and run the following Fama-French regressions on the daily returns of the portfolios:

$$r_t^i - r_t^f = \alpha_t^i + \beta_t^{M,i}(r^M - r^f)_t + \beta_t^{SMB,i} SMB_t + \beta_t^{HML,i} HML_t + \beta_t^{UMD,i} UMD_t + \varepsilon_t^i, \quad (4)$$

$i = 1, \ldots, 5$, where r_t^i is the return of the i-th portfolio, r_t^f is the Fama riskfree rate, and $(r^M - r^f)_t$, SMB_t, HML_t, and UMD_t denote the returns on the three Fama-French factor portfolios (market, size, book) and momentum. Finally, we calculate the time series averages and t-statistics of all coefficients. The alphas reported in the tables are annualized. Additionally, we also compute the average alpha of the difference portfolios ("five minus one").

Table 6 reports the regression results when we sort on the physical default intensities. In line with the findings of Campbell, Hilscher and Szilagyi (2008), we find significantly negative alphas for the portfolio consisting of firms with the highest physical default probabilities.[12] Furthermore, the alpha of the difference portfolio is also significantly negative, which documents the default anomaly in our dataset. Although the size of the firms decreases over the portfolios, all firms are still large compared to the firms in data set of Campbell, Hilscher and Szilagyi (2008). This shows that the default anomaly can also be found in samples consisting of bigger firms only.

Next we sort on the risk-neutral default intensities of the firms that we have calculated from CDS quotes. These intensities are relevant for the pricing of credit-risky securities issued by the firms. As explained above, such an intensity can be viewed as the credit spread of the particular firm. Anginer and Yildizhan (2010) perform a similar sort, but use yields of corporate bonds as proxies for the credit spread. However, Elton, Gruber, Agrawal and Mann

[11] Although we have a rich data set with many CDS quotes, the number of firms is smaller than in other papers using the entire Compustat-CRSP database.

[12] Notice that Campbell, Hilscher and Szilagyi (2008) find significant alphas for the firms with the lowest physical default probabilities only if they apply a three-factor Fama-French model.

(2001) and Longstaff, Mithal and Neis (2005), among others, document that significant parts of the yields cannot be attributed to default risk. Therefore, sorting by yields might not fully capture the pricing relevant default risk. In this case, it might be so that non-significant results can have two reasons: Either the yield is indeed the appropriate control for default risk or it is too much contaminated by other effects such as taxes or liquidity. Anginer and Yildizhan (2010) report that after controlling for corporate bond yields the default anomaly disappears. The abnormal return of their difference portfolio[13] is negative with a t-statistic of -0.68. Our results are reported in Table 7. The abnormal return of the difference portfolio is borderline significant at the 10% level (p-value: 10.5%). Therefore, the default anomaly is mitigated by controlling via risk-neutral default intensities, but it does not fully disappear. In particular, the point estimate of the abnormal return of the difference portfolio is still negative and comparable in magnitude to the result for the physical default intensity. Our result might be attributed to the fact that credit default swaps are a cleaner measure for default risk than corporate bond spreads.

Third, we sort on the default risk premia of the firms that are given by the ratios of the risk-neutral and the physical default intensities. This measure might be an appropriate control for default risk since it captures the relative costs of a firm's debt. Put differently, the relative costs for issuing corporate debt are low if the default risk premium is low as well. Everything else equal, equity holders should benefit from a low default risk premium. Table 8 reports our regression results which show that when using the default risk premium as a measure of default risk, the default anomaly is eliminated. None of the portfolios has a significant alpha and the alpha of the difference portfolio is also insignificant (p-value: 81.39 %). Besides, the average point estimates of the alphas are about 10 times smaller than when sorting on the physical or risk-neutral default intensities. Albeit not significant, the alpha of the difference portfolio is positive as opposed to the previous sorts.

Figure 1 confirms the previous results. It depicts the cumulative alphas of the difference portfolios from 2001 to 2010. It can be seen that the cumulative alphas become the largest when sorting on the physical default intensities (dotted line). This is in line with the default anomaly documented in Table 6. The anomaly becomes less pronounced, but does not fully disappear when sorting on the risk-neutral default intensities (grey line). When we sort on the default risk premium, the cumulative alpha is very small and even changes its sign over time (black line).

[13] They sort into 10 portfolios. Therefore, the return of the difference portfolio is the return of portfolio 10 minus the return of portfolio 1.

5.1 Number of Portfolios

First, we sort our observations into three instead of five portfolios. This could have two competing effects. If the sizes of the portfolios remained the same, then reducing the number of the portfolios should make it more difficult to find significant results. This is because the extreme portfolios (one and three) are then mixtures of two portfolios when we sort into five portfolios. However, the size of the portfolios increases, which is in fact one of the reasons why we perform this check. Therefore, it might be so that the effects observed for five portfolios are becoming more pronounced. Consequently, it is not obvious which results to expect. Table 9 shows our results that confirm the findings for the sorts into five portfolios and in the case of risk-neutral default intensities even become stronger. For physical default intensities, we find a pronounced default anomaly with significantly negative alphas for firms with high physical default intensity and for the difference portfolio. For the risk-neutral default intensity, we now obtain significantly positive alpha for firms with low risk-neutral intensity. The return of the difference portfolio is still borderline significant. When using the default risk premium, the alphas for the default anomaly are insignificant. To summarize, using the default risk premium as a control makes the distress anomaly disappear.

5.2 Stochastic Intensities: Black-Karasinski Model

As an additional robustness check, we use a one-factor stochastic intensity model for the physical default intensity. Following Berndt, Douglas, Duffie, Ferguson and Schranz (2008) the dynamics of the intensity are given by

$$d\lambda_t = \lambda_t \left[\kappa(\theta - \log \lambda_t)\, dt + \sigma dW_t \right], \tag{5}$$

where κ, θ, and σ are constants and W_t is a one-dimensional Brownian motion under the physical measure P. This implies that the logarithm of the intensity is mean-reverting and normally distributed. Such a model is called a Black-Karasinski (BK) model. By assuming that the default risk premium also follows a Black-Karasinski model, one can show that this is also true for the risk-neutral intensity. Further details can be found in Appendix B. Our main results however hardly change and are thus not reported here.

5.3 Idiosyncratic Volatility

Recent research has documented another anomaly that is related to the idiosyncratic volatility of a firm's stock (see, e.g., Ang, Hodrick, Xing and Zhang (2006)). The idiosyncratic volatility (short: IVOL) can be extracted using the Fama-French regression (4). We follow Ang, Hodrick, Xing and Zhang (2006) and disregard the momentum factor when computing the IVOL. The

IVOL is then defined as the standard deviation of the residuals, $\sqrt{\mathrm{Var}(\varepsilon_t^i)}$. There is evidence that this anomaly is related to the default anomaly (e.g., Chen, Jing, Lorán Chollete, and Rina Ray, 2010).

First, we check whether we can document the IVOL anomaly in our dataset. Therefore, we sort on idiosyncratic volatilities of the stocks in our sample and form five portfolios, where the first (fifth) portfolio consists of firms with the smallest (biggest) IVOL. Then we run Fama-French regressions (4) to obtain estimates for the average alphas of the portfolios.[14] Table 10 reports our results. The alphas across portfolios are decreasing (except for the third portfolio) starting with a significantly positive alpha for the first portfolio. The point estimates for portfolio four and five are negative. The return of the difference portfolio is negative, but not significant (p-value: 20.35%). In Table 11, we report results when sorting firms into three portfolios instead of five. For three portfolios, the IVOL anomaly becomes more pronounced and the abnormal return of the difference portfolio is significantly negative. We thus conclude that the IVOL anomaly is present in our data.

We now study whether the IVOL anomaly is related to the distress risk anomaly. First, we construct additional factors from the sorts into five portfolios on the default risk premia and the default intensities. More precisely, we use the returns of every difference portfolio as an additional factor in the Fama-French model. Then we sort our observations into three portfolios according to a firm's IVOL at the beginning of every quarter and run Fama-French regressions on the returns of every quarter adding one of the factors to the Fama-French model. The goal is to check whether these factors are able to eliminate the abnormal returns that we observe when sorting on the IVOL only (see Table 11). The results are presented in Tables 12, 13 and 14. It can be seen that all factors eliminate the significance of the alpha of the difference portfolio. However, the factor stemming from the sort on the risk-neutral default intensity is not able to eliminate the significance of the first portfolio's alpha. The results for the default risk premium and the physical default intensity are similar.

To get a clearer picture, we perform three double sorts. In all three cases, we first form three portfolios sorted according to the IVOLs of the firms over the previous quarter. Then, within each IVOL portfolio, we sort firms into two portfolios according to their median (i) physical default intensities, (ii) risk-neutral default intensities, or (iii) default risk premia. Table 15 reports the results when we sort quarterly and run four-factor Fama-French regressions on daily portfolio returns. When we double sort on the default risk premium, the alphas are insignificant. This means that the IVOL anomaly disappears if we control for the default risk premium of a firm. This is not the case for both intensities. For the risk-neutral default intensity, firms

[14] Although the momentum factor is disregarded when extracting the IVOL, Ang, Hodrick, Xing and Zhang (2006) include this factor in their the Fama-French regressions.

with low IVOL and low risk-neutral default intensity have significantly positive alphas. On the other hand, firms with high IVOL and high risk-neutral default intensity have significantly negative alphas. This is in line with the findings of Ang, Hodrick, Xing and Zhang (2006) on the IVOL anomaly and our results concerning the risk-neutral default intensities (see Table 7). For the physical default intensity, we obtain a surprising result: If we consider firms with low IVOLs only, then firms with high physical default intensity exhibit significantly *positive* alphas. Therefore, for these firms the default anomaly is reversed. To summarize, only for the default risk premia the IVOL anomaly is completely eliminated.

6 Conclusion

The capital asset pricing model predicts a positive relationship between expected stock returns and its risk. On the contrary, empirical evidence points in the direction of an opposite relationship if in addition to the Fama-French factors distress risk is incorporated as a missing factor. Several recent papers however quantify distress risk based on physical or risk-neutral default probabilities. In this paper, we show that distress risk premia can resolve the distress risk and idiosyncratic risk puzzles, whereas distress risk probabilities (physical or risk-neutral) are not able to eliminate the corresponding abnormal returns.

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A CDS Pricing

This appendix briefly summarizes how CDS contracts are prized in reduced-form models. A CDS consists of two legs, the fee and protection leg. During the lifetime of a CDS the buyer of a CDS (called the protection buyer) pays a fee (called the CDS premium) for a protection against default risk to the protection seller. This fee is paid quarterly or semiannually in arrear and is fixed at the time when the CDS is issued. For pricing purposes, it will be chosen such that the initial value of the CDS is zero. This payment stream of premium payments stops at the maturity of the CDS or at default, whichever occurs first. In case of a default before maturity of the CDS, the protection buyer is compensated for the loss that a typical bondholder would suffer. This compensation scheme is called recovery of treasury. We do not consider any counterparty risks in our setup.
In the remainder of this paper, we will work with the following setup. We consider a CDS which starts at time t and has a maturity of T. The time-t CDS spread is denoted by $S_t = S_t(T)$. During the lifetime of the CDS fee payments are made at times $t_j, j = 1, \ldots, n$ if default has not occurred before t_j. Note that the last payment date should coincide with the maturity of the contract, i.e. $T = t_n$. Moreover, payments are made at equidistant points in time, i.e. $\delta = t_j - t_{j-1}$ for all $j = 1, \ldots, n$. Since we look at spot CDS contracts, we have $t_0 = t$. For the sake of simplicity, we normalize the notional to one. As already described, the default time is modeled via the stopping time

$$\tau = \inf\{t \in \mathbb{R}_+ | N_t > 0\} \tag{6}$$

Employing these notations, the fee payment at time t_j is given by

$$S_t \delta \mathbf{1}_{\{\tau > t_j\}}. \tag{7}$$

If a default happens between t_{j-1} and t_j , then an accrued fee payment is due at default:

$$S_t \delta_d^j \mathbf{1}_{\{t_{j-1} \leq \tau \leq t_j\}}, \tag{8}$$

where $\delta_d^j := \tau - t_{j-1}$ is the length of the last interval.

Under the absence of arbitrage, we can apply risk-neutral pricing methods to evaluate noth legs. The value of the fee leg per 1bp of fee payments reads

$$V_t^{fee} = \sum_{j=1}^{n} \left(\delta \mathrm{E}_t^Q \left[\frac{\mathbf{1}_{\{\tau > t_j\}}}{B(t, t_j)} \right] + \mathrm{E}_t^Q \left[\frac{\delta_d^j \mathbf{1}_{\{t_{j-1} \leq \tau \leq t_j\}}}{B(t, \tau)} \right] \right), \tag{9}$$

where $B(t, s)$ is the value of the money market account.

The amount that the protection seller has to pay upon default reads

$$V_t^{prot} = \ell \, \mathrm{E}_t^Q \left[\frac{\mathbf{1}_{\{t_0 \leq \tau \leq T\}}}{B(t, \tau)} \right], \tag{10}$$

where ℓ is the expected loss given default. Assume that the short rate r and λ^Q are independent. After using integration by parts, we get

$$V_t^{fee} = \delta \sum_{j=1}^{n} P(t,t_j)q(t,t_j) + \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} (s - t_{j-1})P(t,s)\partial_s(1 - q(t,s))\, ds$$

$$V_t^{prot} = \ell \int_t^T P(t,s)\partial_s(1 - q(t,s))\, ds,$$

where $P(t,s)$ is the time-t price of a zero bond maturing at time s.

As it can be quite time-consuming to evaluate the integrals, we use an approximation proposed by Hull and White (2006). They assume that defaults occur in the middle of a period and get

$$V_t^{fee} \approx \delta \sum_{j=1}^{n} P(t,t_j)q(t,t_j) + 0.5 \sum_{j=1}^{n} \delta\left(q(t,t_{j-1}) - q(t,t_j)\right)\sqrt{P(t,t_i)P(t,t_{i-1})}$$

$$V_t^{prot} \approx \ell \sum_{j=1}^{n} \delta\left(q(t,t_{j-1}) - q(t,t_j)\right)\sqrt{P(t,t_i)P(t,t_{i-1})}.$$

Form now on, we use these approximations to compute CDS spreads. We also implemented the correct formulas of both legs and observed that the error of the approximating formulas are negligible. Therefore, we prefer the discrete version of both legs.

B Stochastic Intensity Model

Ito's lemma yields

$$d\log\lambda_t = \kappa(\theta - \frac{\sigma^2}{2\kappa} - \log\lambda_t)\, dt + \sigma dW_t.$$

In this model, the intensity is mean-reverting and positive. In contrast to the class of affine processes, the BK model does not have a closed-form solution for the default probablities. Therefore, we rely on numerical procedures to calculate the survival probabilities. We use a two-stage procedure for constructing trinomial trees (see, e.g., Hull and White (1994)).

In contrast to Berndt, Douglas, Duffie, Ferguson and Schranz (2008), we now assume that the risk premium satisfies a BK model as well:

$$d\phi_t = \phi_t\left[\kappa(\hat{\theta} - \log\phi_t)\, dt + \hat{\sigma} dW_t\right], \tag{11}$$

where κ, $\hat{\theta}$, and $\hat{\sigma}$ are constants. For simplicity, we assume that both the risk premium and the intensity under the physical measure have the same speed of mean reversion κ. Applying Ito's lemma again, we see that the risk neutral intensities under the physical measure fulfill the following SDE

$$d\lambda_t^Q = \lambda_t^Q\left[\kappa(\theta + \hat{\theta} + \frac{\sigma\hat{\sigma}}{\kappa} - \log\lambda_t^Q)\, dt + (\sigma + \hat{\sigma})dW_t\right].$$

To price CDS contracts, we have to work with risk-neutral survival probabilities. Hence, we need the dynamics of the risk-neutral intensities under the risk-neutral measure Q and specify the change of measure via

$$dW = dW^Q - \frac{\eta}{\sigma + \hat{\sigma}},$$

where η is the market price of risk (Girsanov kernel) that is assumed to be constant. Notice that there are two risk premia. One is due to the default event risk (ϕ) and one due to the market price of risk of the factor driving the changes in the risk-neutral intensity (η). Applying the change of measure, we arrive at

$$\begin{aligned} d\lambda_t^Q &= \lambda_t^Q \left[\kappa(\theta + \hat{\theta} - \frac{\eta}{\kappa} + \frac{\sigma\hat{\sigma}}{\kappa} - \log \lambda_t^Q)\, dt + (\sigma + \hat{\sigma}) dW_t^Q \right] \\ &= \lambda_t^Q \left[\kappa(\theta^Q - \log \lambda_t^Q)\, dt + \sigma^Q dW_t^Q \right]. \end{aligned} \tag{12}$$

with $\theta^Q = \theta + \hat{\theta} - \frac{\eta}{\kappa} + \frac{\sigma\hat{\sigma}}{\kappa}$ and $\sigma^Q = \sigma + \hat{\sigma}$. Consequently, the risk-neutral intensity has BK dynamics as well. Therefore, we can use the same numerical procedure to calculate the risk neutral probabilities. In contrast to the constant intensity model, we need to estimate this parametric model to be able to calculate risk premiums. This procedure is described in Appendix C.

C Maximum Likelihood Estimation of Default Intensities

In a first step, we want estimate the risk-neutral default intensities. For this purpose, our data consists of daily 5-year credit default swap rates. Therefore, for every firm, we observe several CDS spreads S_t at consecutive dates $t, t{+}d, t{+}2d, \ldots$, where d is a day. Given theses observations, we will estimate a time-series model of the Black-Karasinski process. This leads us with the parameter vector $\hat{\Theta} = (\kappa, \hat{\theta}, \hat{\sigma}, \eta)$. The estimation has to be done for every firm i. We assume that we have D days of data in total and let us denote the 5-year CDS spread at day d for firm i is given by C_d^i. The maximum likelihood estimator is obtained firm by firm. Therefore, we have to find a parameter vector $\tilde{\Theta}$ that solves

$$\sup_{\hat{\Theta}} h_P(C_0^i, C_1^i, \ldots, C_D^i | \hat{\Theta}), \tag{13}$$

with h_P being the joint P-density of the observed probabilities.

Applying the law of total probability and using the fact that the default intensity follows a Markov process, function (13) becomes

$$h(C_D^i | C_{D-1}^i, \hat{\Theta}) \cdot \ldots \cdot h(C_1^i | C_0^i, \hat{\Theta}) \times h(C_0^i | \hat{\Theta}),$$

where h denotes the marginal probability of the CDS spread. The log-likelihood function of the observations for a given parameter set $\hat{\Theta}$ now becomes

$$l(\hat{\Theta}; C_0^i, ...C_D^i) = \sum_{d=1}^{D} \ln(h(C_d^i | C_{d-1}^i, \hat{\Theta})) + \ln(h(C_0^i | \hat{\Theta})). \tag{14}$$

In our computations, we do not consider the las term $\ln(h(C_0^i|\hat{\Theta}))$. The MLE $\tilde{\Theta}$ of the unknown parameter vector $\hat{\Theta}$ is then given by $\sup_{\hat{\Theta}} l(\hat{\Theta}; C_0^i, ...C_D^i)$.

Now, we have to deduce the marginal distribution of CDS spreads. Since the CDS spread S_t is computed such that the initial value of the CDS is zero, it is obtained by

$$S_t = \frac{V_t^{prot}}{V_t^{fee}} \tag{15}$$

and both the fee leg and the protection leg depend on default probabilities. Hence, there exists a function g with $S_t = g(\lambda_t^Q; \hat{\Theta})$ and $\hat{\Theta} = (\kappa, \hat{\theta}, \hat{\sigma}, \eta)$ (Note that we can determine the default probabilities by a trinomial-tree scheme!). Here, the parameter η is included in the estimation procedure as we need the dynamics of the risk-neutral intensity process under the physical measure. The reason for this is that the Maximum-Likelihood procedure uses P-densities. Under mild technical conditions, the conditional P-density of C_d^i given C_{d-1}^i and $\hat{\Theta}$ is given by

$$h(c|C_{d-1}^i, \hat{\Theta}) = \frac{d\left(g^{-1}(c; \hat{\Theta}) | g^{-1}(C_{d-1}^i; \hat{\Theta}), \hat{\Theta}\right)}{g'(g^{-1}(c; \hat{\Theta}); \hat{\Theta})},$$

with $d(.|\lambda_{d-1}^{Q^i})$ being the conditional P-density of $\lambda_d^{Q^i}$. As the dynamics of the intensity process are of the Black-Karasisnki type, we know that the conditional P-density is lognormal.

Now, we can conduct the second step where we estimate our physical default intensities. At first, we ignore misspecification of our default probabilities itself and assume that $1 - p(t, t+1)$ is indeed the current one-month default probability. For every firm, we observe several default probabilities $p(t, t+1)$ at successive dates t, $t+m$, $t+2m$, where m is now a month. Note that the mean reversion speed κ will be the same for both intensities under the risk-neutral measure and for intensities under the real-world measure. So in terms of estimation, we only have to deal with the parameter vector $\Theta = (\theta, \sigma)$ and do not need to include κ that has already been calculated in the first step. Again, by applying numerics, we can compute the physical default probabilities and get a function f such that $p(t, t+1) = f(\lambda_t; \Theta)$. Now,we can almost apply the same procedure as in the preceding part by replacing the function g with f to determine the physical default intensities.

Year	Active Firms	Defaults	(%)
1986	868	9	1.03
1987	912	10	1.09
1988	856	10	1.16
1989	799	10	1.25
1990	745	19	2.54
1991	705	12	1.70
1992	747	7	0.93
1993	820	6	0.73
1994	906	4	0.44
1995	950	5	0.52
1996	1050	7	0.66
1997	1164	10	0.85
1998	1282	19	1.48
1999	1357	24	1.76
2000	1361	25	1.83
2001	1332	44	3.30
2002	1301	27	2.07
2003	1268	17	1.33
2004	1293	7	0.54
2005	1289	10	0.77
2006	1265	2	0.15
2007	1223	2	0.16
2008	1182	11	0.92
2009	1286	27	2.09

Table 1: Number of defaults per year.

The table presents average number of active firms and the number of defaults for every year of the sample used to predict default. The sample consists of all rated non-financial firms in the intersection of Compustat and CRSP in the period January 1986 to December 2009. A firm is classified as rated in a given month if the firm has a S&P long-term or S&P short-term rating outstanding in Compustat. The default information comes from the Moody's default history database. The first column reports the year. The average number for firms in the second column is calculated in two steps. We first calculate the average number of firms in a given month. To arrive at the average over a given year reported in the table we average over the monthly averages. The third column reports the number of defaults. Finally the forth column presents the percent of defaults relative to the average number of active firm in the sample.

Variable	Mean	Median	Std. Dev.	p25	p75
Panel A: Entire sample					
NIMTAAVG	.0032	.0056	.0133	.0009	.0101
TLMTA	.5676	.5654	.2401	.3818	.7775
EXRETAVG	-.0029	.0001	.0384	-.0212	.0192
SIGMA	.4186	.3399	.2670	.2393	.5028
RSIZE	-9.2308	-9.0282	1.5105	-10.1686	-7.8763
CASHMTA	.0603	.0300	.07849	.0103	.0782
MB	2.0621	1.8068	1.4875	1.1084	2.2757
PRICE	2.4743	2.7080	.5535	2.6119	2.7080
Panel B: Default subgroup					
NIMTAAVG	-.02856	-.02370	.0243	-.0482	-.0091
TLMTA	.8863	.9216	.0813	.8872	.9216
EXRETAVG	-.0842	-.0871	.0567	-.1260	-.0427
SIGMA	1.2003	1.326	.3232	.9765	1.4661
RSIZE	-12.3357	-12.5299	1.4814	-13.6115	-11.4249
CASHMTA	.0570	.0311	.0701	.0127	.0728
MB	2.2531	1.0040	2.4860	.4483	2.2757
PRICE	.2359	.2271	1.0631	-.4259	.9530

Table 2: Summary statistics.

The table reports summary statistics for the explanatory variables used to predict default. The sample consists of all rated non-financial firms in the intersection of Compustat and CRSP in the period January 1986 to December 2009. A firm is classified as rated in a given month if the firm has a S&P long-term or S&P short-term rating outstanding in Compustat. The default information comes from the Moody's default history database. Panel A presents summary statistics for the entire sample and Panel B reports summary statistics for the subgroup of firms that default over the next month.

The variables are defined as follows: *NIMTAAVG* is computed as the geometric weighted average level of *NIMTA* over the most recent 12 months where the weight is halved over each month. *NIMTA* is constructed as the ratio of net income (NIQ) to market value of total assets; *TLMTA* is defined as the book value of total liabilities over the market value of total assets. The book value of total liabilities is the sum of total liabilities (Compustat item: LTQ) and minority interests (MIBQ). The market value of total assets is defined as the market value of equity (*ME*) obtained from the monthly CRSP files and the book value of total liabilities (LTQ +MIBQ); *EXRETAVG* computed as the geometric weighted average level of *EXRET* over the most recent 12 months where the weight is halved over each month. *EXRET*) is the stock return of the firm relative to the return of the S&P 500 index; *SIGMA* is computed using the annualized 3-month standard deviation of the firms's daily stock return. If less than 5 non-missing return observations are available *SIGMA* is set to missing; *RSIZE* is constructed as the log ratio of the market value of equity of the firm in a given month to the total market value of the constituents of the S&P 500 index in that month; *CASHMTA* is constructed as cash and short-term investments (CHEQ) over market value of total assets; *MB* is constructed using Davis, Fama and French (2000) and Cohen, Polk and Vuolteenaho (2003) in constructing the book value of equity; finally *PRICE* is the firm's log price per share truncated above $ 15.

Explanatory variable	Coefficient (z-value)
NIMTAAVG	-16.66***
	(-6.42)
TLMTA	5.89***
	(7.44)
EXRETAVG	-4.51***
	(-4.02)
SIGMA	1.97***
	(8.97)
RSIZE	-.099*
	(-1.95)
CASHMTA	-4.51***
	(-5.41)
MB	.001
	(0.05)
PRICE	-.431***
	(-5.80)
Constant	-13.28***
	(-15.41)
Observations	311,436
Defaults	324
Pseudo-R^2	0.3644

Table 3: Default prediction.

The table presents results from a dynamic logit regression of default dummy variable on explanatory variables. The sample consists of all rated non-financial firms in the intersection of Compustat and CRSP in the period January 1986 to December 2009. A firm is classified as rated in a given month if the firm has a S&P long-term or S&P short-term rating outstanding in Compustat. The default information comes from the Moody's default history database. The dependent variable is a dummy variable equal to one if the firm defaults within the next month and zero otherwise. The explanatory variables are defined as follows: *NIMTAAVG* is computed as the geometric weighted average level of *NIMTA* over the most recent 12 months where the weight is halved over each month. *NIMTA* is constructed as the ratio of net income (NIQ) to market value of total assets; *TLMTA* is defined as the book value of total liabilities over the market value of total assets. The book value of total liabilities is the sum of total liabilities (Compustat item: LTQ) and minority interests (MIBQ). The market value of total assets is defined as the market value of equity (*ME*) obtained from the monthly CRSP files and the book value of total liabilities (LTQ +MIBQ); *EXRETAVG* computed as the geometric weighted average level of *EXRET* over the most recent 12 months where the weight is halved over each month. *EXRET*) is the stock return of the firm relative to the return of the S&P 500 index; *SIGMA* is computed using the annualized 3-month standard deviation of the firms‘s daily stock return. If less than 5 non-missing return observations are available *SIGMA* is set to missing; *RSIZE* is constructed as the log ratio of the market value of equity of the firm in a given month to the total market value of the constituents of the S&P 500 index in that month; *CASHMTA* is constructed as cash and short-term investments (CHEQ) over market value of total assets; *MB* is constructed using Davis, Fama and French (2000) and Cohen, Polk and Vuolteenaho (2003) in constructing the book value of equity; finally *PRICE* is the firm‘s log price per share truncated above $ 15. *** ,** , and * indicate significance at the 1%, 5%, and 10% level. z-values are reported in parenthesis.

Year	Observations	Active Firms
2001	24957	121
2002	48456	203
2003	56204	241
2004	72297	304
2005	76639	318
2006	75102	308
2007	86142	351
2008	73764	302
2009	86582	350
2010	73377	326

Table 4: Characteristics of the default risk premia sample by year.
The table reports characteristics of the default risk premia sample. For a firm day to be included in the sample we must observe CDS spreads and predicted default probabilities from the dynamic logit model in Table 3. Column 2 presents the number of observations per year for the entire sample and Column 3 reports the average number of firms in the sample in a given year The average number of firms is calculated in two steps. We first calculate the average number of firms in a given month. Column (3) reports the average over the monthly averages.

Year	Mean	Std. Dev.	p25	p50	p75	Observations
2001	64.04	107.45	10.01	24.81	66.20	24957
2002	99.80	183.89	11.35	31.65	94.92	48456
2003	78.84	158.49	10.80	27.47	77.36	56204
2004	93.14	183.43	15.88	42.15	103.18	72297
2005	107.20	220.79	18.75	50.23	122.88	76639
2006	107.27	227.14	18.00	47.03	109.28	75102
2007	127.11	263.51	21.22	55.41	134.45	86142
2008	140.92	362.50	18.59	58.94	144.87	73764
2009	78.46	155.33	10.92	33.39	85.15	86582
2010	120.92	154.23	29.71	71.80	148.39	73377
Total	105.24	222.02	15.85	45.05	113.87	673520

Table 5: Summary statistics for default risk premia by year.
The table reports descriptive statistics for default risk premia calculated using a constant default intensity model. For a firm day to be included in the sample we must observe CDS spreads and predicted default probabilities from the dynamic logit model in Table 3.

	PF1	PF2	PF3	PF4	PF5	$PF5 - PF1$
mktrf	0.9863***	0.9552***	1.0026***	1.0195***	1.1004***	
	(77.9184)	(55.3096)	(43.5272)	(41.5917)	(47.3288)	
smb	-0.1627***	-0.0774***	-0.0582*	-0.1217***	0.1655***	
	(-8.8456)	(-3.9018)	(-1.7144)	(-4.3627)	(3.8725)	
hml	-0.3868***	0.0870	0.2958***	0.2263***	0.6994***	
	(-11.8191)	(1.5729)	(5.4369)	(3.6903)	(7.6093)	
umd	0.1263***	0.0426	0.1419***	-0.0773**	-0.3039***	
	(3.6763)	(1.0333)	(2.9310)	(-2.4275)	(-5.9120)	
alpha	0.0113	-0.03105	0.04025	-0.0220	-0.102**	-0.1133**
	(0.7918)	(-1.1371)	(1.6099)	(-0.6319)	(-2.1201)	(-2.1050)
size	16.69	16.10	15.85	15.53	14.61	
average return	0.0634	0.0581	0.1201	0.0871	0.0664	

Table 6: Fama-French regressions for portfolios sorted on LAMBDA P.
Each quarter we form 5 portfolios sorted on each firm‘s median LAMBDA P for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest median, and PF5 contains the firms with the highest median. We run 4 Factor Fama-French regressions on daily returns each quarter. The table reports the time-series average of the coefficients. t-values are reported below coefficients.*** ,** , and * indicate significance at the 1%, 5%, and 10% level. All alphas and the average return are annualized by multiplying the daily alpha with 250.

	PF1	PF2	PF3	PF4	PF5	PF5−PF1
mktrf	0.9874***	0.9715***	0.9829***	1.0581***	1.1215***	
	(65.9147)	(55.2394)	(49.0442)	(46.0548)	(36.8089)	
smb	-0.2033***	-0.0661**	-0.0551**	-0.0151	0.3205***	
	(-9.6540)	(-2.7097)	(-2.5332)	(-0.4732)	(7.5157)	
hml	-0.2347***	0.1508**	0.1239**	0.2373***	0.4505***	
	(-7.2052)	(2.4407)	(2.2487)	(3.3918)	(4.5039)	
umd	0.0958***	0.1070**	0.0102	0.0606	-0.1898	
	(2.9631)	(2.4140)	(0.2785)	(1.1041)	(-2.0973)**	
alpha	0.0215	0.0331	-0.050	-0.025	-0.067	-0.089
	(1.6002)	(1.5333)	(-2.2792)**	(-0.6979)	(-1.3712)	(-1.6607)
size	16.86	16.09	15.74	15.42	14.58	
average return	0.0631	0.1013	0.0538	0.0967	0.0902	

Table 7: Fama-French regressions for portfolios sorted on LAMBDA Q.
Each quarter we form 5 portfolios sorted on each firm‘s median LAMBDA Q for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest median, and PF5 contains the firms with the highest median. We run 4 Factor Fama-French regressions on daily returns each quarter. The table reports the time-series average of the coefficients. t-values are reported below coefficients.*** ,** , and * indicate significance at the 1%, 5%, and 10% level. All alphas and the average return are annualized by multiplying the daily alpha with 250.

	PF1	PF2	PF3	PF4	PF5	PF5−PF1
mktrf	1.0328***	0.9955***	0.9484***	0.9622***	1.0307***	
	(37.2629)	(48.1947)	(45.3426)	(53.6187)	(72.5798)	
smb	-0.1515***	-0.0590*	-0.0824***	-0.1326***	-0.0696***	
	(-3.7585)	(-1.9074)	(-3.0216)	(-5.6114)	(-3.0878)	
hml	0.2182***	0.3648***	0.1306*	-0.1283***	-0.4298***	
	(3.7419)	(5.8824)	(1.9020)	(-3.4115)	(-10.3487)	
umd	-0.1708***	0.0392	0.0888*	0.0636*	0.1379***	
	(-4.5436)	(1.0764)	(1.8371)	(1.7814)	(3.8665)	
alpha	-0.0083	-0.0353	0.0045	-0.0037	0	0.0107
	(-0.2241)	(-1.3105)	(0.1791)	(-0.1755)	(0.1114)	(0.2371)
size	15.16	15.60	15.80	16.08	16.20	
average return	0.0969	0.0779	0.1028	0.0681	0.0583	

Table 8: Fama-French regressions for portfolios sorted on default risk premium.
Each quarter we form 5 portfolios sorted on each firm‘s median default risk premia for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest median, and PF5 contains the firms with the highest median. We run 4 Factor Fama-French regressions on daily returns each quarter. The table reports the time-series average of the coefficients. t-values are reported below coefficients. *** ,** , and * indicate significance at the 1%, 5%, and 10% level. All alphas and the average return are annualized by multiplying the daily alpha with 250.

	PF1	PF2	PF3	PF3−PF1
lambda p	0.0043	0.0197	-0.0687*	-0.0730*
	(0.2647)	(0.9825)	(-1.9815)	(-2.0088)
lambda q	0.0215*	-0.0235	-0.0519	-0.0734
	(1.7319)	(-1.0991)	(-1.1834)	(-1.5828)
drp	-0.0053	-0.0134	0.0000	0.0039
	(-0.1734)	(-0.6407)	(0.2281)	(0.2536)

Table 9: Alphas of Fama-French regressions for three portfolios.
Each quarter we form 3 portfolios sorted on lambda p, lambda q, or drp for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest, and PF3 contains the firms with the highest. We run 4 Factor Fama-French regressions on daily returns each quarter. The table reports the time-series average of the constant. t-values are reported below coefficients.*** ,** , and * indicate significance at the 1%, 5%, and 10% level. All alphas are annualized by multiplying the daily alpha with 250.

	PF1	PF2	PF3	PF4	PF5	PF5−PF1
mktrf	0.9273***	0.9630***	1.0605***	1.1140***	1.1143***	
	(44.6771)	(48.1792)	(63.1858)	(46.0331)	(24.4993)	
smb	-0.2096***	-0.1219***	-0.0919***	-0.0223	0.2353***	
	(-9.4239)	(-4.3660)	(-3.8205)	(-0.6612)	(4.9104)	
hml	-0.0864**	-0.0679	0.0519	0.0907	0.2559***	
	(-2.1441)	(-1.4975)	(0.9233)	(1.2252)	(2.8005)	
umd	-0.0030	0.0576*	0.1036**	0.1768**	0.0672	
	(-0.0719)	(1.7557)	(2.7045)	(2.3233)	(0.6241)	
alpha	0.0372*	0.016	-0.0361	-0.0312	-0.0322	-.0695
	(1.9092)	(0.6423)	(-1.5267)	(-1.0463)	(-0.7274)	(-1.2940)
size	16.44	16.06	15.89	15.62	14.92	
average return	0.0746	0.0727	0.0789	0.1225	0.0918	

Table 10: Fama-French regressions for portfolios sorted on IVOL.
Each quarter we form five portfolios sorted on each firm's IVOL for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest median, and PF5 contains the firms with the highest median. We run 4 Factor Fama-French regressions on daily returns each quarter. The table reports the time-series average of the coefficients. t-values are reported below coefficients. *** ,** , and * indicate significance at the 1%, 5%, and 10% level. All alphas and the average return are annualized by multiplying the daily alpha with 250.

	PF1	PF2	PF3	PF3−PF1
mktrf	0.9351***	1.0589***	1.1090***	
	(51.6256)	(80.1309)	(37.7101)	
smb	-0.2140***	-0.0776***	0.1001***	
	(-9.1999)	(-3.9353)	(3.0792)	
hml	-0.1145***	0.0229	0.1847**	
	(-3.8149)	(0.5114)	(2.4354)	
umd	0.0257	0.0865***	0.1373	
	(0.6742)	(2.6129)	(1.4655)	
alpha	0.0409**	-0.0311	-0.0437	-0.0846*
	(2.2105)	(-1.5927)	(-1.1900)	(-1.9490)
size	16.32	15.88	15.15	
average return	0.0760	0.0735	0.1003	

Table 11: Fama-French Regressions for three portfolios sorted on IVOL.
Each quarter we form three portfolios sorted on IVOL for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest, and PF3 contains the firms with the highest. We run 4 Factor Fama-French regressions on daily returns each quarter. The table reports the time-series average of the constant. t-values are reported below coefficients.*** ,** , and * indicate significance at the 1%, 5%, and 10% level. All alphas and the average return are annualized by multiplying the daily alpha with 250.

	PF1	PF2	PF3	PF3−PF1
mktrf	0.9245***	1.0631***	1.1095***	
	(61.5787)	(89.1333)	(41.9066)	
smb	-0.1934***	-0.0844***	0.0814**	
	(-9.5039)	(-4.8212)	(2.5631)	
hml	-0.1688***	0.0969**	0.2802***	
	(-5.1384)	(2.0435)	(3.9828)	
umd	0.0468	0.0707**	0.1062	
	(1.3532)	(2.1366)	(1.2509)	
drp	-0.0916***	0.0683***	0.1379***	
	(-3.8177)	(2.9964)	(3.6304)	
alpha	0.0207	-0.0310	-0.0269	-0.0476
	(1.1949)	(-1.6642)	(-0.8891)	(-1.3157)
size	16.32	15.88	15.15	
average return	0.0760	0.0735	0.1003	

Table 12: Fama-French Regressions for portfolios sorted on IVOL: default risk factor.
Each quarter we form 3 portfolios sorted on each firm's IVOL for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest median, and PF3 contains the firms with the highest median. We run 4 Factor Fama-French regressions with ADDITIONAL FACTOR: DRP on daily returns each quarter. The table reports the time-series average of the coefficients. t-values are reported below coefficients. *** ,** , and * indicate significance at the 1%, 5%, and 10% level.All alphas and the average return are annualized by multiplying the daily alpha with 250.

	PF1	PF2	PF3	PF3−PF1
mktrf	0.9431***	1.0673***	1.1136***	
	(52.6984)	(85.0083)	(36.9761)	
smb	-0.1927***	-0.0726***	0.0778**	
	(-8.2531)	(-3.6114)	(2.1211)	
hml	-0.0451	0.0676	0.1576*	
	(-1.2421)	(1.3392)	(1.9473)	
umd	-0.0022	0.0738**	0.1755**	
	(-0.0595)	(2.3879)	(2.1947)	
lambda p	-0.0526***	-0.0327*	0.0280	
	(-3.0702)	(-1.9102)	(0.8392)	
alpha	0.0269	-0.0316	-0.0202	-0.0471
	(1.4821)	(-1.6310)	(-0.6620)	(-1.2546)
size	16.32	15.88	15.15	
average return	0.0760	0.0735	0.1003	

Table 13: Fama-French regressions for portfolios sorted on IVOL: LAMBDA P factor.

Each quarter we form 3 portfolios sorted on each firm‘s IVOL for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest median, and PF3 contains the firms with the highest median. We run 4 Factor Fama-French regressions with ADDITIONAL FACTOR: LAMBDA P on daily returns each quarter. The table reports the time-series average of the coefficients. t-values are reported below coefficients. *** ,** , and * indicate significance at the 1%, 5%, and 10% level. All alphas and the average return are annualized by multiplying the daily alpha with 250.

	PF1	PF2	PF3	PF3−PF1
mktrf	0.9480***	1.0617***	1.0786***	
	(51.0030)	(82.2115)	(38.9201)	
smb	-0.1508***	-0.0621***	-0.0152	
	(-7.7129)	(-3.4009)	(-0.4779)	
hml	-0.0267	0.0450	0.0347	
	(-0.8879)	(0.9734)	(0.5654)	
umd	-0.0257	0.0709**	0.2191***	
	(-0.9067)	(2.1348)	(2.8674)	
lambda q	-0.1157***	-0.0094	0.2203***	
	(-8.8256)	(-0.5609)	(10.7477)	
alpha	0.0311*	-0.0354*	-0.0111	-0.0422
	(1.8882)	(-1.7471)	(-0.3713)	(-1.1960)
size	16.32	15.88	15.15	
average return	0.0760	0.0735	0.1003	

Table 14: Fama-French Regressions for portfolios sorted on IVOL: LAMBDA Q factor.

Each quarter we form three portfolios sorted on each firm‘s IVOL for that quarter. We compute value-weighted returns every day for the following quarter. PF1 contains the firms with the lowest median, and PF3 contains the firms with the highest median. We run 4 Factor Fama-French regressions with ADDITIONAL FACTOR: LAMBDA Q on daily returns each quarter. The table reports the time-series average of the coefficients. t-values are reported below coefficients. *** ,** , and * indicate significance at the 1%, 5%, and 10% level. All alphas and the average return are annualized by multiplying the daily alpha with 250.

	IVOL PF1	IVOL PF2	IVOL PF3	IVOL $PF3 - PF1$
Panel A: DRP				
DRP PF1	0.0331	-0.0289	-0.0330	-0.0662
	(1.4384)	(-0.9274)	(-0.6530)	(-1.1765)
DRP PF2	0.0230	-0.0262	-0.0375	-0.0606
	(0.8155)	(-1.1400)	(-0.9958)	(-1.2487)
Panel B: LAMBDA P				
LAMBDA P PF1	0.0247	0.0342	-0.0163	-0.0410
	(1.0693)	(-1.5048)	(-0.4216)	(-0.8471)
LAMBDA P PF2	0.0442*	-0.0111	-0.0760	-0.1202*
	(1.7562)	(-0.3342)	(-1.2876)	(-1.8274)
Panel C: LAMBDA Q				
LAMBDA Q PF1	0.0510**	-0.0180	-0.0094	-0.0604
	(2.3990)	(-0.8885)	(-0.2386)	(-1.2834)
LAMBDA Q PF2	0.0072	-0.0431	-0.0994*	-0.1066
	(0.2502)	(-1.1314)	(-1.7055)	(-1.6411)

Table 15: Fama-French Regressions for double sorts on IVOL and a default measure.

The table reports the regression results of three double sorts, where we double sort on IVOL and (i) MEDIAN DRP, (ii) MEDIAN LAMBDA P, or (iii) MEDIAN LAMBDA Q. Each quarter we first form three portfolios sorted according to the IVOL of the firm over the last quarter (IVOL PF1, IVOL PF2, IVOL PF3). Then we split up each of the three IVOL portfolios into two portfolios, one with low DRP, LAMBDA P, or LAMBDA Q and one with high DRP, LAMBDA P, or LAMBDA Q. For instance, for DRP the portfolio IVOL-PF1-DR-PF1 contains the firms with the lowest ivol and the lowest default risk premium, whereas IVOL-PF3-DR-PF2 contains the firms with the highest ivol and the highest default risk premium. Each quarter we run 4 Factor Fama-French regressions. The table reports the time-series average of the constant. t-values are reported below coefficients. All alphas are annualized by multiplying the daily alpha with 250.

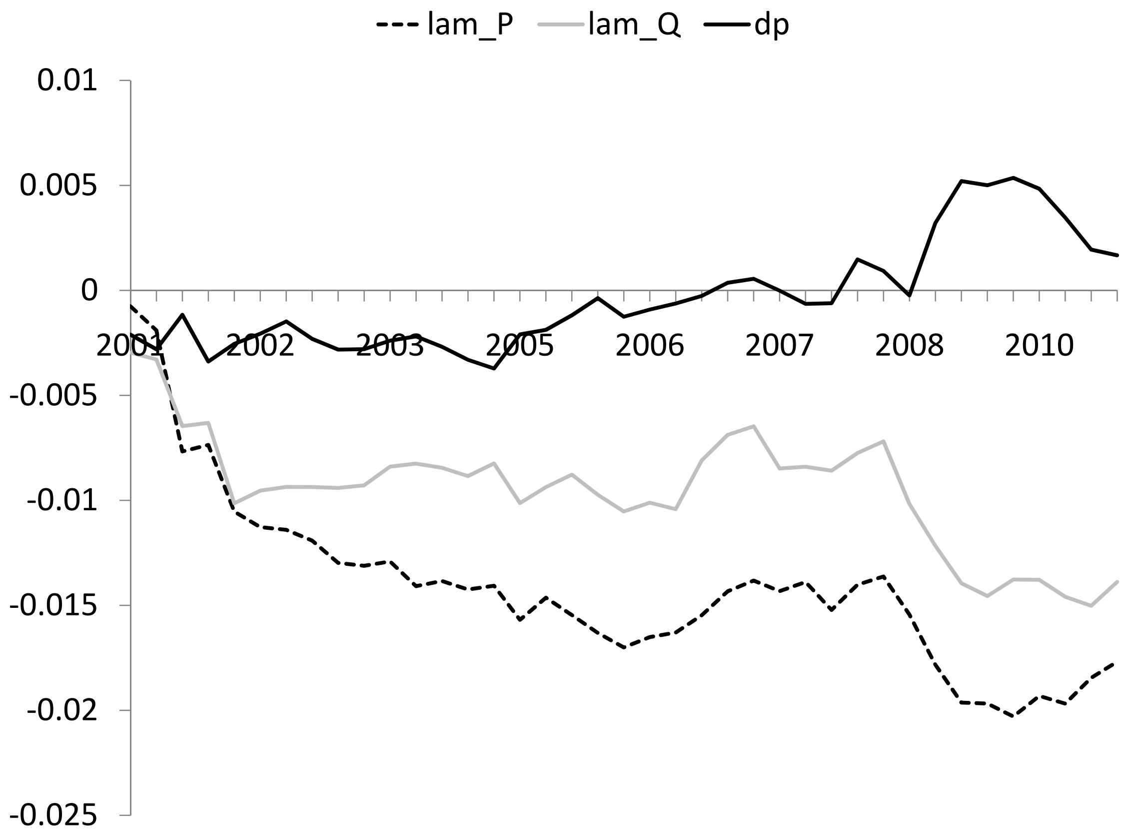

Figure 1: Cumulative Alphas. The figure depicts cumulative alphas of the three difference portfolios.

Hedging Structured Credit Products During the Credit Crunch

Björn Bick, Holger Kraft

Summary. Pricing and hedging structured credit products poses major challenges to financial institutions. This has become very clear during the recent credit crisis. This paper puts several valuation approaches through a crucial test: How did these models perform in one of the worst periods of economic history, September 2008, when Lehman Brothers went down? Did they produce reasonable hedging strategies? We study several bottom-up and top-down credit portfolio models and compute the resulting delta hedging strategies using either index contracts or a portfolio of single-name CDS contracts as hedging instruments. We compute the profit-and-loss profiles and assess the performances of these hedging strategies. Among all 10 pricing models that we consider the Student-t copula model performs the best. The dynamical generalized-Poisson loss model is the best top-down model, but this model class has in general problems to hedge equity tranches. Our major finding is however that single-name and index CDS contracts are not appropriate instruments to hedge CDO tranches.

1 Introduction

The ongoing credit crisis has documented the need of a proper risk-management of structured credit products. We put ourselves into the shoes of a risk manager in a financial institution and address a crucial question: Was it possible to hedge CDO tranches during the heat of the crisis using any of the existing credit portfolio approaches? To answer this question, we consider eleven[1] pricing models and compare them with respect to their abilities of fitting market data and providing accurate hedging strategies. We focus on delta hedging, an approach which is widely used in practice, and carry out a profit-and-loss analysis to assess the hedging strategies empirically. We use both index contracts and a portfolio of single-name CDS contracts as hedging instruments. The particular models are tested in realistic settings using market data from April and September 2008. Among all models that we consider the Student-t copula model performs the best. The dynamical generalized-Poisson loss model is the best top-down model, but this model class has in general problems to hedge equity tranches. On of our major findings is that single-name and index CDS contracts are no useful instruments to hedge CDO tranches.

[1] More precisely, we use eleven models to calibrate the data and ten to calculate deltas.

We document that this is because of the loose connection between the price dynamics of CDS contracts and CDO tranches. Therefore, the overall hedge performances of all models are poor. Even for the best models, the biggest daily losses can exceed 5% of the tranche notional.[2] Given that financial institutions are highly leveraged entities with leverage ratios of more than 90% these are severe numbers.

The first family of models that are studied are Copula models that are so-called bottom-up models. In bottom-up models, the loss distribution of a credit portfolio is constructed by combining marginal distributions of the portfolio constituents with a specific dependence structure. Copula models used to be the industry standard. In the one-factor version, default dependencies are modeled by a single common factor. Therefore, the conditional marginal default probabilities become the key input for this approach. A typical example of a copula is the Gaussian copula (see, e.g., Li (2000)) that is still important in practice. They are used to quote tranche prices via implied correlations, which is similar to the industry standard to quote option prices via implied volatilities. Unfortunately, Gaussian copulas have several limitations including a severe restriction on the amount of correlation that can be achieved in these models. Therefore, we will also consider other copulas like the Student-t or the Clayton copula. Although, these models overcome some of the problems inherent in the Gaussian copula, they are still static models in the sense that the correlation structure is static and thus does not change with time. Consequently, copula models are not able to generate correlation dynamics that depend on the default history. This stylized fact is documented in empirical studies (see, e.g., Longstaff and Rajan (2008)) and is referred to as contagion effects. To calibrate a copula pricing model we follow market practice and provide per tranche fits, i.e. every tranche is calibrated via a separate correlation parameter. We use both implied and base correlations and compare the respective hedging performances. Copula approaches typically make the so-called homogeneous-pool assumption (all firms are assumed to be identical), which simplifies the computation of the portfolio loss distribution (see, e.g., Cousin, Crépey and Kan (2011)). On the contrary, we allow for heterogeneous pools of firms (at least with respect to the default probabilities) to be able to compute single-name sensitivities of credit tranches. Papers applying copula models to price credit derivatives include Andersen, Sidenius and Basu (2003), Hull and White (2004), and Schloegl and O'Kane (2005), among others. In particular, Burtschell, Gregory and Laurent (2009) compare several copula models and list their properties. There are also studies that extend the standard copula models to random recoveries like the works of Andersen and Sidenius (2004) and Krekel (2008).

In contrast to bottom-up models, the top-down approach first puts some structure on the portfolio loss process without referring to specific constituents of the credit portfolio. This means

[2] We emphasize that the market value of the position is much smaller than the tranche notional.

that the difference between two top-down models only lies in the distributional assumptions about the loss distribution. They do not give any information about which firms have actually triggered these losses. For the pricing of structured credit tranches like CDOs, this might be an advantage since the latter information is not necessarily needed for this purpose. Moreover, it seems to be easier to include dynamic contagion effects in top-down models. The lacking connection between single defaults and the loss process however becomes a challenge when tranches shall be hedged via single-name credit products such as CDS contracts. To overcome this issue, we apply the so-called random thinning approach that was introduced by Giesecke, Goldberg and Ding (2011). More precisely, we use a variant of this method presented by Halperin and Tomecek (2009) that aims to find top-down matrices that allocate the defaults on portfolio level to their underlying single names. We focus on three top-down models that have previously been studied by Brigo, Pallavicini and Torresetti (2006), Longstaff and Rajan (2008), and Errais, Giesecke and Goldberg (2010).

To summarize, the contributions of our paper are the following: In a very comprehensive study involving more than 10 credit models, we compare top-down with bottom-up approaches. Applying random thinning we are able to compute index deltas and single-name deltas for all, but the Longstaff-Rajan model. In contrast, Cont and Kan (2011) apply a recalibration approach to derive index deltas in top-down models. We find that none of the top-down models is able to outperform the Student-t copula, which is the best bottom-up model in our study. In particular, top-down models fail to hedge equity tranches. Using average absolute hedging errors (instead of average hedging error as, e.g., in Cont and Kan (2011) and Cousin, Crépey and Kan (2011)), we also find that hedging performances of all models are unsatisfactory. Besides, we confirm that delta hedging with Gaussian copulas works very poorly, which is in line with the results by Giesecke, Goldberg and Ding (2011). We also compare the index and single-name deltas of several copula models that are either calibrated with compound or base correlations. We find evidence that the copulas calibrated with base correlations provide more stable profit-and-loss paths. This observation is consistent with the findings in Ammann and Brommundt (2009) and Cousin, Crépey and Kan (2011). In contrast to our study, both papers only analyze the Gaussian copula. Notice that Ammann and Brommundt (2009) do not use index- or single-name deltas. Instead they hedge CDO tranches via other tranches. As a robustness check, we have also performed a similar analysis which shows that hedging tranches with tranches leads to descent hedging results. The question however arises whether this is a tractable strategy in practice.

The remainder of the paper is structured as follows. Section 2 defines the general default processes and describes the payoffs of index default swaps, CDO tranches, and credit default swaps. Section 3 reviews the copula models studied in this paper and derives the relevant conditional default probabilities. In Section 4, we describe the top-down models and show how to derive the relevant portfolio probabilities. In Section 5, we present the methods used to fit the models to market data. Moreover, we provide numerical results on the model's calibration errors. Section 6 discusses the random thinning procedure and provides a detailed profit-and-loss analysis across all models. Finally, Section 7 concludes.

2 Portfolio Credit Derivatives

In our study, we consider a portfolio of I credit-risky securities such as loans or credit swaps that are issued by different entities. The main object of interest is the aggregate loss in this portfolio. The process N_t denotes the number of defaults up to time t.[3] The default stopping times of the single firms are denoted by $\tau_1, \tau_2, ..., \tau_I$. The default times of the portfolio are represented by an increasing sequence of stopping times $T_1 \leq T_2 \leq ... \leq T_I$ meaning that T_i is the i-th jump time of N_t. Therefore, the total number of defaults in the portfolio are given by

$$N_t = \sum_{i \geq 1}^{I} \mathbf{1}_{\{T_i \leq t\}}. \tag{1}$$

Equivalently, we can define

$$N_t = \sum_{i \geq 1}^{I} \mathbf{1}_{\{\tau_i \leq t\}}. \tag{2}$$

Since we are going to compare top-down and bottom-up models, we either use formula (1) or (2). The first formula is applied when analyzing top-down models, the second is needed to study bottom-up models.[4] To price credit derivatives, we have to specify the losses incurred by defaults in the portfolio and define the loss process L_t by

$$L_t = \sum_{i \geq 1}^{I} l_i \mathbf{1}_{\{\tau_i \leq t\}}, \tag{3}$$

where l_i is the loss associated with the default of firm i. Assuming a constant recovery rate R and an equally weighted portfolio with a common notional of $1/I$, the loss process can be rewritten as $L_t = \frac{1-R}{I} N_t$.

In the remainder, we will concentrate on two contracts: index credit default swaps (index CDS) and collateralized debt obligations (CDO). Both contracts involve two parties, a protection buyer and a protection seller. The protection buyer is compensated for all losses from defaults

[3] We assume a complete probability space $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t))$. The complete filtration $(\mathcal{F}_t)$ contains the information up to time t.

[4] Note that both methods generate different filtrations. The filtration generated by a top-down model is always fine enough to distinguish the arrival of defaults, but does not contain any information about the default identities. In contrast, the filtration implied by a bottom-up approach is much finer and gives us this additional information. This fact explains much of the structural differences of both methods. For a detailed discussion of bottom-up and top-down models, we refer to Giesecke (2008) and Bielecki, Crépey and Jeanblanc (2010).

in the pool. We refer to this payment stream as *default leg.* In exchange, the protection buyer pays a fixed fee to the protection seller at specific payment dates $t_1, ..., t_K$, where $T = t_K$ is the maturity date. These cash flows are said to be the *fee leg.* We assume that the payment dates are equidistant with $t_k - t_{k-1} = \Delta$, for some constant Δ. Moreover, we make the standing assumption that the risk-free interest (short) rate r and the default-counting process N are independent. We further assume that the distributions of the loss rates ($l_i, i = 1, ..., I$) are independent of each other and independent of r and N. Under this assumption, the distribution of the loss process can be derived from the default process (see Giesecke, Goldberg and Ding (2011)).

2.1 Index Credit Default Swaps

The underlying of an index CDS (e.g. CDX, iTraxx) is a portfolio that consists of single-name CDS contracts written on I different firms. All contracts have the same maturity T, equal premium payment dates (t_k) and are equally weighted (same notionals $1/I$). The buyer of such a contract receives full protection of this portfolio. Whenever a reference entity in the pool defaults, the protection seller will pay the corresponding losses. Hence, the default leg at time $t < T$ reads

$$Def_t^{index} = \mathrm{E}_t \left[\int_t^T e^{-\int_t^s r_u \, du} \, dL_s \right], \tag{4}$$

where $\mathrm{E}_t[\cdot]$ denotes the conditional expectation under a risk-neutral measure Q. Following Hull and White (2006) who assume that defaults can only occur in the middle of two consecutive payment dates, (4) simplifies to

$$Def_t^{index} \approx \sum_{t_k \geq t} B(t, \frac{t_{k-1} + t_k}{2}) \left(\mathrm{E}_t[L_{t_k}] - \mathrm{E}_t[L_{t_{k-1}}] \right), \tag{5}$$

where $B(t, s)$ is the time-t price of a zero-coupon Treasury bond maturing at time s. Since the approximation errors are small, we will use representation (5) throughout this paper.

The protection buyer pays a fee on the remaining notional in the portfolio. The remaining notional F_t at time t represents the total notional of all firms that have survived until time t and is given by

$$F_t = 1 - \frac{N_t}{I}. \tag{6}$$

The fee payment at time t_k is thus given by $S_t^{index} \Delta F_{t_k}$, where S_t^{index} denotes the annualized fixed index spread that was contracted at initiation t. Hence, the expected discounted value of all premium payments at time s, $s \geq t$, per one unit of fee payments is given by

$$Fee_s^{index} = \mathrm{E}_s \left[\sum_{t_k \geq s} e^{-\int_s^{t_k} r_u \, du} \Delta F_{t_k} \right] = \Delta \sum_{t_k \geq s} B(s, t_k) \left(1 - \frac{\mathrm{E}_s[N_{t_k}]}{I} \right), \tag{7}$$

such that the present value of total fee payments at time s reads $S_t^{index} Fee_s^{index}$.

Since the index spread S_t^{index} has to be computed such that the initial value of the contract at time t is zero, it is obtained by

$$S_t^{index} = \frac{Def_t^{index}}{Fee_t^{index}} = \frac{\sum_{t_k \geq t} B(t, \frac{t_{k-1}+t_k}{2}) \left(\mathrm{E}_t[L_{t_k}] - \mathrm{E}_t[L_{t_{k-1}}] \right)}{\Delta \sum_{t_k \geq t} B(t, t_k) \left(1 - \frac{\mathrm{E}_t[N_{t_k}]}{I} \right)}. \tag{8}$$

To obtain the fair spread, we thus need to compute expected loss $\mathrm{E}_t[L_s]$ and the expected number of defaults $\mathrm{E}_t[N_s]$. For the special case when $I = 1$, the above formulas can be used to calculate the fair spread of a single-name CDS with notional 1. The fair CDS spread of firm i becomes

$$S_t^i = \frac{l_i \sum_{t_k \geq t} B(t, \frac{t_{k-1}+t_k}{2}) \left(Q_t(\tau_i > t_{k-1}) - Q_t(\tau_i > t_k) \right)}{\Delta \sum_{t_k \geq t} B(t, t_k) Q_t(\tau_i > t_k)}. \tag{9}$$

2.2 Collateralized Debt Obligations

In general, CDOs are financial claims to the cash flows generated by a portfolio of I debt securities. To offer a variation in risk-return profiles, the total portfolio notional is sliced into M different tranches that are described by a sequence $0 = K_0 < K_1 ... < K_M = 1$ of attachment and detachment points. Thus, the m-th tranche is specified by its attachment point K_{m-1} and its detachment point K_m, $m = 1, ..., M$. It is assumed that the firm's notionals are the same and we normalize their sum to 1. Then, the face value of tranche m is given by $K_m - K_{m-1}$. The cash flows and losses generated by the portfolio are used to service the tranches according to their seniority. The percentage loss of tranche $[K_{m-1}, K_m]$ can be written as

$$L_t^m = \frac{1}{K_m - K_{m-1}} \left\{ \max\{L_t - K_{m-1}, 0\} - \max\{L_t - K_m, 0\} \right\}, \tag{10}$$

which is the scaled difference between two call payoffs. Now, the holder of a tranche receives interest payments on the remaining tranche notional, but has to take losses that are attributed to the tranche. Similar to (4), the default leg of a CDO contract with maturity T at time $t < T$ reads[5]

$$Def_t^m \approx \sum_{t_k \geq t} B(t, \frac{t_{k-1} + t_k}{2}) \left(\mathrm{E}_t[L_{t_k}^m] - \mathrm{E}_t[L_{t_{k-1}}^m] \right) \tag{11}$$

and the fee leg per one unit of fee payments at time s, $s \geq t$, is given by

$$Fee_s^m = \Delta \sum_{t_k \geq s} B(s, t_k) \left(1 - \mathrm{E}_s[L_{t_k}^m] \right). \tag{12}$$

Hence, the fair spread of tranche m is

$$S_t^m = \frac{\sum_{t_k \geq t} B(t, \frac{t_{k-1}+t_k}{2}) \left(\mathrm{E}_t[L_{t_k}^m] - \mathrm{E}_t[L_{t_{k-1}}^m] \right)}{\Delta \sum_{t_k \geq t} B(t, t_k) \left(1 - \mathrm{E}_t[L_{t_k}^m] \right)}. \tag{13}$$

[5] Here, we also assume that defaults can only occur in the middle of the period between two payment dates.

Some tranches (typically the equity tranche) are sometimes quoted in terms of an upfront payment u_t^m and a fixed running spread S^{fix}. Then, the present value of fee payments can be rewritten as

$$\widetilde{Fee}_t^m = u_t^m + S^{fix} \varDelta \sum_{t_k \geq t} B(t, t_k) \left(1 - \mathrm{E}_t[L_{t_k}^m]\right) \tag{14}$$

and we have to compute u_t^m. Similar to the computation of index swaps, the pricing formula (13) requires the computation of expressions of the form $\mathrm{E}_t[f(L_s)]$, where f is some deterministic function.

2.3 Marking to Market

As we want to study hedge ratios, we have to work with the mark-to-market values of the specific contracts. As in Masol and Schoutens (2008), Cont and Kan (2011) and Cousin, Crépey and Kan (2011) we concentrate on the mark-to-market value of a protection seller's position at time s who initiated the contract at time t. At time t the spread is chosen such that both legs have the same values. For an index CDS, this means that $S_t^{index} Fee_t^{index} - Def_t^{index} = 0$. At time $s > t$, the mark-to-market value of a protection seller's position in an index CDS is given by

$$MTM_s^{index} = \left(S_t^{index} - S_s^{index}\right) Fee_s^{index}. \tag{15}$$

The corresponding single-name CDS mark-to-market value follows by setting $I = 1$. This value is denoted by MTM_s^i. Similarly, the mark-to-market value of a protection seller's position in the m-th tranche of a CDO at time s is given by

$$MTM_s^m = \left(S_t^m - S_s^m\right) Fee_s^m. \tag{16}$$

An upfront payment in some tranche m will result in a mark-to-market value of

$$MTM_s^m = -u_s^m.$$

3 Bottom-Up Models: Copulas

In this section, we briefly summarize some results on copula models (see, e.g., Li (2000), Andersen and Sidenius (2004), Laurent and Gregory (2005)). In our analysis, we focus on one-factor models. As we deal with a bottom-up approach, the single default probabilities of all firms are explicitly modeled. We model each firm's default time as the first jump of an inhomogeneous Poisson process (see, e.g., Jarrow and Turnbull (1995)). Then the probability of firm i to default before time s conditioned on survival up to time t reads

$$Q_t(\tau_i < s) = 1 - e^{-\int_t^s \lambda_i(u)\, du}, \tag{17}$$

where $\lambda_i(t)$ is a deterministic intensity. To shorten notations, we disregard the subscript t in the remainder of this section and denote the distribution function of τ_i by $Q^i(\cdot)$. We assume that these default probabilities are known for each company.[6] In order to generate a factor model for the default times $(\tau_1, ...\tau_I)$, we link them to a vector of random variables $(X_1, ..., X_I)$, which we also refer to as latent variables. The distribution of this random vector depends on the specific copula. Details about the copulas and the involved portfolio probabilities used in our paper can be found in Appendix A.1.

4 Top-Down Models

In this section, we briefly summarize some facts about three top-down models that we will consider later on. These are the models by Brigo, Pallavicini and Torresetti (2006), Longstaff and Rajan (2008), and Errais, Giesecke and Goldberg (2010). Instead of focusing on the aggregation of single-name default probabilities, the portfolio loss process itself becomes the modeling primitive in every top-down approach. This means that the losses are not directly attributed to particular firms. The main difference between these models are their approaches to model contagion effects.

4.1 Longstaff-Rajan

Longstaff and Rajan (2008) model contagion effects via joint defaults. More precisely, the model allows for small losses (single firms), medium losses (sector), and large losses (economy). This means that, in contrast to other methods, the authors explicitly allow for multiple defaults at the same time. The loss process per \$1 notional is given by

$$L_t = 1 - e^{-\gamma_1 N_{1t}} e^{-\gamma_2 N_{2t}} e^{-\gamma_3 N_{3t}} \tag{18}$$

and the parameters γ_i are nonnegative constants that define the jump sizes. The portfolio loss is generated by three factors represented by three independent Cox processes N_{it}. The intensities of the three Cox processes are given by

$$\begin{aligned} d\lambda_{1t} &= \sigma_1 \sqrt{\lambda_{1t}} dW_{1t}, \\ d\lambda_{2t} &= \sigma_2 \sqrt{\lambda_{2t}} dW_{2t}, \\ d\lambda_{3t} &= \sigma_3 \sqrt{\lambda_{3t}} dW_{3t}, \end{aligned}$$

[6] In practice, we can usually bootstrap the default probabilities from single-name CDS quotes. As single-name CDS quotes are only available at discrete points in time, the intensity process λ_i will become a piecewise constant function.

where W_{it} are independent Brownian motions and σ_i are constant volatility parameters for $i = 1, 2, 3$. As the intensities are stochastic, default correlations vary over time. Longstaff and Rajan (2008) show that the expectation of an arbitrary function of the loss process $f(L_s)$ is given by

$$\mathrm{E}[f(L_s)] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{P_j^1(\lambda_{1\,0}, s)}{j!}\frac{P_k^2(\lambda_{2\,0}, s)}{k!}\frac{P_l^3(\lambda_{3\,0}, s)}{l!} f_{j,k,l}, \tag{19}$$

where we define $f_{j,k,l} = f\left(1 - e^{-\gamma_1 j} e^{-\gamma_2 k} e^{-\gamma_3 l}\right)$. The function $P_j^i(\lambda_{i\,0}, s)$ is equal to $j!$ times the probability that $N_{is} = j$. Using formula (19), all portfolio products can be priced.

4.2 Self-Exciting Framework

Errais, Giesecke and Goldberg (2010) present an affine point process framework to price portfolio credit derivatives. The portfolio default times are generated by an intensity that is driven by market factors which are assumed to follow an affine jump process. In their setup, the loss process itself becomes such a risk factor. This so-called self-exciting property captures the empirically documented effects of default clustering and contagion. Defaults arrive with intensity λ given by the dynamics

$$d\lambda_t = \kappa(\theta - \lambda_t)\, dt + \delta\, dL_t, \tag{20}$$

where κ, θ, and δ are positive constants. Ito's Lemma yields

$$\lambda_t = \theta + (\lambda_0 - \theta)e^{-\kappa t} + \delta \int_0^t e^{-\kappa(t-s)}\, dL_s.$$

Upon default the intensity increases by the realized loss scaled by the sensitivity parameter δ. The impact of an default event exponentially decays over time with rate κ such that the intensity reverts back to its long-run mean θ. The self-exciting effect is controlled via the constant δ: the bigger δ, the bigger the effect. In contrast to Errais, Giesecke and Goldberg (2010), who allow for a stochastic recovery distribution at the default times, we assume a constant loss given default of l.[7] We further assume an equally weighted portfolio with a common notional of $1/I$ such that the loss and default process are related via $L_t = \frac{l}{I} N_t$. Therefore, it is sufficient to consider the default distribution which can be calculated using the transform approach of Duffie, Pan and Singleton (2000):

$$\mathrm{E}_t\left[e^{uN_s}\right] = e^{a(t)+b(t)\lambda_s+uN_t}, \tag{21}$$

where $u \in \mathbb{C}$ and the coefficient functions $a(t) = a(u,t,s)$ and $b(t) = b(u,t,s)$ satisfy the following ODEs:

$$\partial_t b(t) = \kappa b(t) - e^{u+\delta l b(t)} + 1,$$
$$\partial_t a(t) = -\kappa\theta b(t),$$

[7] We also tested a self-exciting framework with stochastic recoveries, but our results are hardly affected. Therefore, we decided to use a constant recovery rate.

with boundary conditions $a(s) = b(s) = 0$. We use a FFT to invert (21), which yields the default distribution.

4.3 Dynamical Generalized-Poisson Loss Model

Finally, we consider the model by Brigo, Pallavicini and Torresetti (2006) that is based on I independent Poisson processes $M_1, ..., M_I$. We use homogeneous Poisson processes with constant intensities $\lambda_1, ..., \lambda_I$ and define the total number of defaults by the stochastic process

$$Z_t = \sum_{i=1}^{I} \alpha_i M_{it} \qquad (22)$$

where $\alpha_i \in \{1, ..., I\}$ can be interpreted as jump sizes. The default process is then defined by $N_t = \min(Z_t, I)$. When a jump occurs in some M_i, this event triggers α_i defaults to happen simultaneously. Brigo, Pallavicini and Torresetti (2006) call this process a generalized-Poisson process since Z can have multiple jumps at the same time which is also possible in the LR model, but different from the self-exciting approach.[8] The distribution of Z, which can easily be transformed into the distribution of N (see Brigo, Pallavicini and Torresetti (2006)), is given by the Laplace transform of Z

$$\mathrm{E}\left[e^{uZ_t}\right] = \exp\left(\sum_{i=1}^{I} \lambda_i \left(e^{iu} - 1\right)\right), \qquad (23)$$

with $u \in \mathbb{C}$. As for the previous model, we employ a FFT to invert (23) which gives us the desired distribution of the default process after a simple transformation. Assuming a constant recovery rate, the loss distribution can easily be calculated.

5 Calibration

To demonstrate how the different approaches can be calibrated to market data, we use mid quotes of market prices of CDOs, index CDS, and their corresponding constituent CDS spreads. Market quotes at time t are denoted by CDO_t^m, $Index_t$, and CDS_t^i, respectively. Here, m represents the particular tranche and i stands for the firm. For numerical illustration, we use the 5-year iTraxx.EU tranche and index data throughout the paper. The iTraxx.EU is a pool consisting of $I = 125$ obligors that represent the most liquid names in the European credit derivatives market. There are several CDO tranches written on the iTraxx: $[0-3\%]$,$[3-6\%]$,$[6-9\%]$,$[9-12\%]$, $[12-22\%]$, and $[22-100\%]$. We consider the series iTraxx.EU S9 that was on the run in 2008. In our analysis, we concentrate on market data from April and September 2008 and exclude the super senior tranche $[22-100\%]$ since data is sparse.

[8] Here, a jump of M_1 can be interpreted as idiosyncratic risk, a jump of M_{10} as a catastrophic event, and a jump of M_{100} as systemic risk.

All considered contracts have a maturity of $T = 5$ years. Index, tranche, and single-name data are quoted in basis points (bp) based on quarterly premium payments ($\Delta = 0.25$). An exception is the equity tranche ($[0 - 3\%]$ tranche) which according to market conventions is quoted in terms of an upfront payment and a running spread of 500 bps. Figure 1 depicts the tranche and index data in September 2008 and shows the single-name CDS spreads of three constituents that represent high risk, middle risk, and low risk firms.

In addition to this data, we collect 3-month, 6-month, 12-month Libor rates, and 2-year, 3-year, and 5-year swap rates that we get from Datastream. To approximate the term structure, we bootstrap the specific discount factors from these data by splines. Furthermore, all calibrations assume a fixed recovery rate of $R = 40\%$.[9] To shorten our exposition, we introduce abbreviations that are summarized in Table 1.

5.1 Basis Adjustment

Before calibrating the models, we perform a so-called basis adjustment that ensures that the market index spread is matched by its theoretical bottom-up equivalent. This theoretical counterpart is derived in the following way. Starting from the single default level, the index spread at time t can also be expressed as

$$Index_t = \frac{\sum_{i=1}^{I} CDS_t^i Fee_t^i}{\sum_{i=1}^{I} Fee_t^i},$$

where Fee_t^i is firm i's present value of all premium payments per 1 bp of fee payments and can be derived from (7) by setting $I = 1$. However, we typically observe a deviation between this bottom-up index spread and the market index spread because index and single-name CDS contracts might be written on different sets of credit events and have different levels of liquidity. This difference is called the CDS basis. As it is market practice,[10] we remove this CDS basis by commonly adjusting the single-name CDS spreads. In our applications, the basis turned out be at most 8 bp and was on average 3 bp.

5.2 Bottom-Up

As we consider an inhomogeneous pool of firms, all single default probabilities must be calibrated to single-name CDS spreads, which has to be done for each time point in the data set. By assuming constant intensities that trigger the firm defaults, a general approximation of the intensities is given by $\frac{CDS^i}{1-R}$ and the desired default probabilities follow by (17).

[9] In the Longstaff-Rajan model, we do not have to assume a recovery rate since recovery rates are implicitly computed in the model.

[10] See, e.g, Halperin and Tomecek (2009), Eckner (2009), and Giesecke, Goldberg and Ding (2011).

It is well known that one-factor copulas cannot price tranches consistently (see Burtschell, Gregory and Laurent (2009)). Consequently, we use different correlation parameters for each tranche and calibrate these parameters to the respective market data points.[11] The implied correlations are called *implied* or *compound* correlations. Figure 2 shows a calibrated curve of compound correlations if a Gaussian copula is applied. The curve does not exhibit the typical skew form that we saw in the market before the crisis.[12] Table 2 lists the implied correlations of the other copulas. In total, we study 8 copula models taking into account that we consider different specifications of the Student-t and Double-t copula. It can be seen that all models lead to similar forms of compound correlations as the one depicted in Figure 2. We further note that the Student-t copula is not able to fit the $[6-9\%]$ mezzanine tranche for this specific date. This is a known problem that especially became apparent during the financial crisis where copula models could not longer fit all tranches. To analyze this problem in detail, Table 3 tabulates the average calibration errors for the period of September. It becomes obvious that all copulas have problems to calibrate the $[6-9\%]$ tranche, especially the Student-t copula.

As there are several drawbacks of the compound correlation concept (e.g. two implied compound correlations for one tranche), we also consider another definition that is known as *base correlation*.[13] The concept of base correlations decomposes each tranche into combinations of first loss tranches. For example, a tranche with attachment point a and detachment point b is decomposed into the first loss tranches $[0, a]$ and $[0, b]$. The particular base correlations are then calculated via a bootstrapping procedure. One major advantage of this approach is that it delivers unique solutions. Moreover, one can apply the base correlation concept to determine the value of tranches that are not actively traded.[14] We list the base correlations in Table 4. By construction, base and compound correlations are equal for equity tranches and then increase with seniority. To compare the calibration performance of both concepts, Table 5 reports the calibration errors when we fit tranche prices via base correlations. The calibration errors (especially for the Student-t copula) are on average smaller and the mezzanine tranche is now fitted perfectly.

[11] In our study, we either have to fit the correlation parameter ρ or the frailty parameter θ. This can be done by a simple bisection method.

[12] De Servigny (2007) and Burtschell, Gregory and Laurent (2009) observe a correlation skew based on pre-crisis market data.

[13] For more details about both concepts, we refer the reader to O'Kane and Livesey (2004) and Kakodkar, Galiani and Shchetkovskiy (2004).

[14] This can be seen in the linear structure of base correlations in Figure 2.

5.3 Top-Down

For the top-down models, we must calibrate several parameters to the observed tranche and index spreads (instead of just one parameter for the single-factor bottom-up models). On each trading day in our data set the respective model is recalibrated by minimizing the root mean squared percentage calibration error (RMSPE), which for day t is defined as

$$\left(\sum_m \left(\frac{CDO_t^m - Mod_t^m(\Theta)}{CDO_t^m}\right)^2 + \left(\frac{Index_t - Mod_t^{ind}(\Theta)}{Index_t}\right)^2\right)^{1/2}, \tag{24}$$

where Θ is a vector of parameters and Mod represents the specific model. For the LR and the SE model, we solve this minimization problem by a numerical algorithm like Powell's method.[15] As Powell's algorithm only provides us with local minima, the optimization procedure is repeated several times and we choose the optimal parameters to be the solution to (24) with the smallest RMSPE amongst all runs. The starting values for Θ are drawn form a uniform distribution in every run. This procedure is very similar to the one used in Giesecke, Goldberg and Ding (2011). In the following, we summarize our calibration results for every model.

Longstaff-Rajan In the LR model, we have to estimate three jumps sizes, three volatilities, and three intensities so that the parameter vector becomes $\Theta = (\lambda_i, \sigma_i, \gamma_i; i = 1, 2, 3)$. In contrast to Longstaff and Rajan (2008), we daily recalibrate the jump and volatility parameters. Table 6 summarizes the calibrated parameters for a particular day in our data set. The estimated jump sizes are in line with the intuition provided by Longstaff and Rajan (2008) that the three factors represent small, medium, and large losses. For instance, the jump size associated with first Cox process is 0.0084 which is exactly the weight of one firm in the portfolio ($1/125 = 0.008$) and can thus be interpreted as idiosyncratic risk. Furthermore, the intensities decrease with the corresponding jump size which shows that unsystematic shocks happen more often than systemic shocks.

Self-Exciting For the self-exciting model, we need to calibrate four parameters summarized in the vector $\Theta = (\lambda_0, \kappa, \theta, \delta)$. In September 2008, the average value of the parameter δ turned out to be about 7, which reflects a high risk of contagion effects in this period.

DGLP In the DGLP model, the parameter vector Θ becomes high-dimensional (in fact, the dimension is 250) since both the vector of jump sizes α_i and the respective constant intensities λ_i have to be calibrated. Hence, ordinary numerical optimization algorithms would be slow and become intractable. To fit the parameters, we apply an approach suggested in Brigo, Pallavicini

[15] See Powell (1964). Powell's method is a conjugate gradient algorithm to find local minima of a function. We have also used a simulated annealing procedure with similar results. However, we decided to apply Powell's method since it turned out to be much faster.

and Torresetti (2006) that adds one jump process in every step. Details on the procedure can be found in Appendix A.2.

Table 7 shows the relative calibration errors for all tranches across all three top-down models. It can be seen that the LR model yields the best fit across all tranches. This can be attributed to number of parameters that are used to calibrate this model. On the contrary, the SE model provides a poor fit of the $[3-6\%]$ and the $[12-22\%]$ tranche although calibration errors of 10% might still be seen as reasonable (see Cont and Kan (2011)).

6 Hedging

In this section, we compare the performances of the hedging strategies that can be derived from the different models. We consider an agent who has sold protection on a CDO tranche and is thus exposed to changes in the default risk of the constituents. As it is market practice (see, e.g., Petrelli, Zhang, Jobst and Kapoor (2007), Masol and Schoutens (2008), and Cont and Kan (2011)), we will use single-name or index CDS contracts as hedging instrument. To hedge the mark-to-market (MTM) dynamics of the tranche position, the agent thus buys protection on the single-name or index CDS. The tranche sensitivities with respect to these contracts are said to be the single-name delta and index delta that are defined as follows[16]

$$\psi_t^{m,i} = \frac{\partial MTM_t^m}{\partial MTM_t^i}, \tag{25}$$

$$\psi_t^{m,index} = \frac{\partial MTM_t^m}{\partial MTM_t^{index}}. \tag{26}$$

In all models, the hedge ratios (25) and (26) have to be computed numerically. As we have seen in Section 3, the tranche spreads of bottom-up models exclusively depend on the single default probabilities and some model parameters Θ. Therefore, (25) is approximated by

$$\psi_t^{m,i} \approx \frac{MTM_t^m((S_t^1, ..., S_t^i + \varepsilon, ..., S_t^I); \Theta) - MTM_t^m((S_t^1, ..., S_t^i, ..., S_t^I); \Theta)}{MTM_t^i(S_t^i + \varepsilon) - MTM_t^i(S_t^i)}, \tag{27}$$

so that the single-name delta is defined as the ratio of the change of the MTM tranche value to the change of the MTM single-name CDS value when we disturb the i-th CDS spread by ε. To compute (27), we follow the approach explained in Section 5.2 and calibrate both the marginal default probabilities and the correlation/frailty parameter to market prices. We then shift the single-name CDS spread of firm i by $\varepsilon = 1$ bp and recalibrate the marginal default probability of firm i (the default probabilities of all other firms and the correlation parameter are not changed).[17] In the last step, the new values for the tranche spread and single-name CDS spread of obligor i are computed.

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[16] Since the fee legs given in (7) and (12) are expressed in terms of normalized notional values, the MTM values are also normalized and we do not have to adjust the corresponding deltas.

[17] We also tested shifts of 5 bp and 10 bp, but the results are hardly affected.

Since a change in the index spread can be interpreted as a parallel shift in all individual spreads, (26) is approximated by

$$\psi_t^{m,index} \approx \frac{MTM_t^m((S_t^1+\varepsilon,...,S_t^i+\varepsilon,...,S_t^I+\varepsilon);\Theta) - MTM_t^m((S_t^1,...,S_t^i,...,S_t^I);\Theta)}{MTM_t^{index}((S_t^1+\varepsilon,...,S_t^i+\varepsilon,...,S_t^I+\varepsilon);\Theta) - MTM_t^{index}((S_t^1,...,S_t^i,...,S_t^I);\Theta)}. \quad (28)$$

To compute this formula, we apply a similar approach as for the single-name spreads except that now all single-name CDS spreads are shifted by $\varepsilon = 1$ bp. As a robustness check, we will also use CDO tranches as hedging instrument, i.e. we use tranches to hedge other tranches. As in Ammann and Brommundt (2009), the delta of tranche m with respect to tranche k is defined as

$$\psi_t^{m,k} = \frac{\partial MTM_t^m}{\partial MTM_t^k} = \frac{\frac{\partial MTM_t^m}{\partial MTM_t^{index}}}{\frac{\partial MTM_t^k}{\partial MTM_t^{index}}} \quad (29)$$

and (28) can be applied to compute this hedge ratio.

Pricing a tranche with either the tranche's compound or base correlation in general produces the same result. In fact, both correlations are calibrated from market data to match traded prices (see Section 5). This is not true for the sensitivity parameters that both concepts deliver, which has been documented in the literature. For example, Cont and Kan (2011) use compound correlations to compute deltas in a Gaussian copula model, whereas Cousin, Crépey and Kan (2011) use base correlations to compute deltas.

6.1 Random Thinning

In top-down models hedging becomes more involved. This is because top-down models deliberately abstract from the identities of the defaulted firms. At first sight, this might complicate risk management considerably because a calculation of single-name sensitivities as in (25) seems impossible. This problem can be resolved by an approach called random thinning (RT) that is discussed in Halperin and Tomecek (2009) and Giesecke, Goldberg and Ding (2011). It allows us to attribute risk to the portfolios constituents. In our study, we follow the approach of Halperin and Tomecek (2009) who take a probabilistic view of RT. Central to their approach is the computation of the following matrix

$$TD_{ij}^{(l)} = Q\left(\{\tau_i = T_j\} \cap \{t_{l-1} \le \tau_i \le t_l\}\right), \quad i,j = 1,...,I, \quad (30)$$

where $TD_{ij}^{(l)}$ represents the probability that the i-th name is the j-th defaulter and that the corresponding default event happens between t_{l-1} and t_l. We refer to this matrix as the thinning matrix. Details on this procedure can be found in Appendix A.3. Having calibrated this matrix, we are able to compute the CDS deltas by (27) and (28). Figure 3 shows an example of a calibrated RT matrix where the portfolio probabilities were generated by the DGPL model.

6.2 Hedging Results

The values of single-name and index deltas are model dependent. Figure 4 depicts the index deltas of different models for the equity and the $[9-12\%]$ tranche when using compound correlations. It shows that there can be significant differences across models. In this example, the index deltas computed with the Gaussian, Clayton, and Student-t copula are of the same order and are quite stable. On the contrary, the implied deltas of Double-t copulas are much more volatile and different. Figure 5 depicts the corresponding deltas induced by the base correlation method which significantly differ across copulas. As already discussed, it becomes obvious that both concepts provide different deltas. We even observe negative ones, which has also been documented by Morgan and Mortensen (2007) and Halperin and Tomecek (2009). To compare these results with the results for top-down models, we also compute index deltas of the SE and DGPL models (see Figure 6). Notice that, although the LR model almost yields a perfect fit to market data, we cannot apply the random thinning approach to this model since it is not possible to derive the distribution of the number of defaults in the LR model. Consequently, it cannot be included in this analysis. In comparison to the Gaussian index deltas, the top-down deltas are higher for the equity tranche and lower for the mezzanine tranche. The deltas computed in the SE model are always higher than the corresponding ones from the DGPL model. These results suggest that there is a significant model risk involved when CDO tranches are hedged.

To measure the performances of the particular hedging strategies, we run a profit-and-loss analysis (P&L). The P&L of the delta at time t is defined as

$$PL_t = \Delta MTM_t^m - \psi_{t-1}^{m,index/CDS} \Delta MTM_t^{CDS/Index},$$

where ΔMTM_t^m and $\Delta MTM_t^{CDS/Index}$ represent the changes in the mark-to-market values of the corresponding contracts between times $t-1$ and t. In our analysis, we rebalance the hedging portfolio every day and report the daily profit/loss, PL_t. More precisely, it is assumed that our position in the single-name/index CDS that we entered the day before is liquidated at the end of the following day. Then we set up a new hedge portfolio such that the total position is delta neutral. Disregarding transaction costs, entering a new position at time t does not involve a payment because the spread at time t is set in such a way that the MTM value of the single-name/index CDS is zero (see Section 2). To assess the hedging strategies, we use a metric proposed by Ederington (1979) who measures the performance of a hedging strategy as the percentage reduction in standard deviations of the hedged and unhedged portfolio. We further report a hedging error that is defined as the ratio between the mean absolute values of the hedged and unhedged portfolio.[18] Formally, hedging error and volatility reduction are

[18] Cont and Kan (2011) and Cousin, Crépey and Kan (2011) define the hedging error as the ratio of average P&L of the hedged and unhedged portfolio and do not take absolute values. Therefore, in their analysis a gain of 1m dollars cancels out a loss of 1m dollars, although at both days the hedging strategy was off 1m dollars.

defined as

$$\text{hedging error} = \frac{\text{Average absolute P\&L of the hedged position}}{\text{Average absolute P\&L of the unhedged position}}, \tag{31}$$

$$\text{vola residual} = \frac{\text{Volatiltiy of the P\&L of the hedged position}}{\text{Volatiltiy of the P\&L of the unhedged position}}. \tag{32}$$

An efficient hedging strategy should reduce both numbers.

6.2.1 Index Hedging

First, we analyze the hedging errors (31) and volatility residuals (32) when using the index contract as hedging instrument. We consider two different months, April and September 2008. To implement a hedging strategy, we recalibrate the models on every day (see Section 5) and rebalance the portfolio. In this section, we will focus on four bottom-up models (Gauss, Clay, DT(4/4), and T(4)) and the two top-down models (SE and DGPL)).[19] Figure 7 depicts the performances of the bottom-up copula models in April and September when using compound correlations. In both periods, index delta hedging only provides useful strategies for the tranches [0−3%] and [3−6%] where both the hedging errors and volatilities are reduced. In both periods, index delta hedging reduces hedging errors and volatilities only for the tranches [0 − 3%] and [3 − 6%]. Nevertheless, the hedging errors are still larger than 70% which might result in huge losses. For the [6 − 9%] tranche the models fail to work which can be attributed to poor calibrations of this tranche (see Table 3). We further conclude that bottom-up models do not provide effective strategies when it comes to hedging the more senior tranches (especially in September 2008).

For the Gaussian copula, our results are comparable to the ones of Cont and Kan (2011) who study hedge ratios of tranches on the CDX index in the same period. Figure 8 depicts the results for the concept of base correlations. By definition, the results for the equity tranche are the same as in Figure 7. Although the hedging strategies for base correlations are still not satisfying, they seem to be more stable and to perform better than the ones we get for compound correlation calibration.[20] Although a similar result has been documented for the Gaussian copula,[21] we thus show that this result carries over to other copula models. Consequently, we will always use the base correlation concept to compare top-down models with bottom-up models.

[19] The results for the omitted models are available upon request. We do not include the Student-t copulas with 8 and 12 degrees of freedom in this analysis since the results are comparable to the T(4) model. Moreover, the results for the other Double-t copulas are not reported here as they turned out to be worse than the ones for the DT(4/4) model.

[20] In particular, for the T(4) model hedging errors and volatilities are reduced across all tranches.

[21] See, e.g., Cousin, Crépey and Kan (2011). Ammann and Brommundt (2009) also find evidence that base correlations provide better hedging strategies. However, they do not study index deltas, but examine deltas of tranches with respect to other tranches.

Now, we consider the performance of hedging strategies following from top-down models. Figure 9 depicts our findings and compares them to two bottom-up models. Obviously, both top-down strategies do not provide a reasonable hedge of the equity tranche. However, with increasing seniority the performance of the hedging strategies improves. The SE model performs worse than the DGPL model (especially in April) which can be attributed to the inferior calibration of some tranches (see Table 7). Furthermore, we do not find clear evidence that top-down hedging strategies dominate bottom-up strategies. Although the hedging strategies implied by both top-down models are far away from being perfect, we remark that combining top-down models with random thinning might overcome the problem of hedging the senior tranches that is mentioned by Cousin, Crépey and Kan (2011) and Cont and Kan (2011). Both studies use a local intensity model, which also belongs to the class of top-down approaches, and carry out a similar analysis in the same period. In contrast to our findings, their model does not result in effective hedging strategies for senior tranches. In fact, they realize hedging errors of more than 200% for the senior tranches. However, they do not apply random thinning, but use martingale representation techniques (see Cousin, Crépey and Kan (2011)) to derive index deltas.

6.2.2 Single-Name Hedging

In the next step, we hedge CDO tranches by a portfolio of several single-name CDS contracts. We consider a case where the single-name CDS portfolio consists of three contracts.[22] We only report the results for the bottom-up models that are calibrated using base correlations. The corresponding results for the compound correlations can be found in the Appendix. Since the weight of a single CDS is typically really small in the portfolio, the delta that is calculated due to (27) will be small, too. Therefore, building a hedging strategy with just one contract and neglecting the others will virtually have no effect on the profit-and-loss dynamics. Hence, we divide the pool in three groups ordered with respect to the sizes of the single-name CDS spreads and search a firm in each group that has a spread close to the average single-name CDS spread in this group. For instance, the three constituents depicted in Figure 1 can be interpreted as representatives of small-risk, middle-risk, and high-risk names.

Table 8 reports the tranche sensitivities for these three firms, Danone, France Telecom, and Tate and Lyle. Comparing the deltas across firms shows that changes in credit risk of higher risk names have a higher effect on the equity tranche, which is true for all three models. This is because the equity tranche is primarily exposed to idiosyncratic risks of single high-risk firms. For both top-down models, we find that the low-risk firms have a large influence on the

[22] As robustness check, we also studied portfolios with five CDS contracts. However, results are hardly affected. Therefore, we will not report them.

senior tranche.[23] This is reasonable since defaults of low-risk firms are usually triggered by systematic events. To summarize, the hedge ratios of top-down models with respect to single names correctly attribute their loadings to the tranches: Senior tranches load on low-risk firms, whereas equity and mezzanine tranches load on high-risk firms.

In contrast, the Gaussian copula model performs poorly for senior tranches since deltas can even become negative.[24] This can be attributed to the static nature of copulas, which makes it harder to capture dynamic contagion effects. Table 9 illustrates the hedging errors and volatility residuals for single-name and index hedging in September 2008.[25] We see that delta hedging with a portfolio of three single-name CDS contracts does not improve the hedge performance significantly. Moreover, we find that a standard Gaussian copula model fails to hedge the $[9 - 12\%]$ and $[12 - 22\%]$ tranches with a portfolio of single-name CDS contracts, whereas the Student-t copula and both top-down models provide at least efficient strategies for these tranches although hedging errors and volatility residuals are still above 80%.

6.3 What Complicates Hedging?

In the previous analysis, we have seen that delta hedging both with index and single-name CDS contracts can lead to substantial hedging errors. The smallest hedging errors and residual volatilities across all models and tranches were still around 65%. The sizes of the biggest losses and expected shortfalls as reported in Table 10 suggest that the proposed strategies can even worsen the portfolio performance compared to an unhedged position.[26]

These numbers indicate that the models failed to provide useful hedges during the heat of the credit crisis. Although we allow for daily rebalancing of the portfolio, which is unrealistic in practice due to transaction costs, losses and expected shortfalls for the hedged portfolios can be significantly higher than for the unhedged position. There are two possible explanations for the poor performance of the hedging strategies: Either CDS contracts are not the appropriate hedging instruments to hedge CDO tranches or the models are not able to capture the underlying dynamics. To address the first point, we examine the correlation between the realized returns of the tranches and the index (Table 11). As expected, the correlations between back-to-back tranches (tranches that share attachment/detachment points) are the highest (up to 98%). On the other hand, the correlations between the tranches and the index are quite low, which suggests that the index was not an appropriate hedging instrument in the credit crisis. The reason for

[23] Eckner (2009) who applies an affine jump diffusion model to compute sensitivities, finds a similar behavior of single-name deltas.

[24] This problem has already been discussed in Section 5.

[25] The corresponding results for the period of April 2008 can be found in the Appendix.

[26] The results for April 2008 are shifted to the Appendix.

these low correlations can be seen in Figure 10 that depicts the comovements of index and tranche returns. Obviously, there were several days where index and tranches returns moved in opposite directions.[27] For example, 9 out of 22 returns of the equity tranche and the index have opposite signs. Since the model deltas are in general positive, frequent opposite movements lead to serious problems. This explains much of the bad performance of our hedging strategies. The patterns for single-name CDS returns are similar and thus single-name/index CDS contracts were apparently not an appropriate instrument to hedge CDO tranches during the credit crisis.

To address the question of whether the models were wrong, we study the hedge performances when tranches are hedged by other tranches. Notice that we observed high correlations between tranche returns and that returns move in tandem (see Figure 11). We focus on the T(4) and the DGPL model who turned out to provide the most efficient strategies in the previous analysis. Table 12 illustrates the performance of both models. We use all tranches as hedging instruments and find that both hedging errors and volatility residuals are much smaller than before. For example, disregarding the equity tranche, the DGPL model provides hedging errors that are, in the worst case, about 58% and in all other cases between 19% and 30% when using back-to-back tranches as hedging instrument. Moreover, residual volatilities are significantly reduced (about 30%). It also becomes evident that the DGPL model still fails to hedge equity tranches. On the other hand, DGPL outperforms the T(4) model when we hedge senior tranches. The results for April 2008 are similar and can be find in the Appendix. For the sake of completeness, Table 13 reports the biggest loss and expected shortfall when we use the $[12 - 22\%]$ tranche as hedging instrument. Apart from the equity tranche, we see that the corresponding numbers are much lower than in Table 10.

7 Conclusion

This paper has analyzed bottom-up and top-down credit portfolio models. First, we have calibrated all models to market data from 2008 and studied the specific calibration errors. Then we have computed both index deltas and single-name deltas and backtested their hedging performance by a profit-and-loss analysis. Our results indicate that the base correlation concept provides better results than the compound correlation approach across all copula models, a result that was previously found for Gaussian copulas. Using a random thinning method, we are also able to calculate deltas in top-down models. We find that the market practice of delta hedging CDO tranches using CDS contracts failed during the crisis. Even for the best models (Student-t and dynamical generalized-Poisson model) daily losses can easily exceed 5% of the tranche notional. This is a huge number given that banks are highly leveraged with equity

[27] See also Cont and Kan (2011) who study the CDX index in the same period.

ratios of around 10% or less. Neither single-name nor index CDS contracts were appropriate instruments to hedge CDO tranches.

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A Technical Part

A.1 Copula Models

Gaussian Copula In the Gaussian copula model by Li (2000), the random variables X_i are defined as

$$X_i = \rho M + \sqrt{1-\rho^2} Z_i, \tag{33}$$

where ρ is a constant parameter between 0 and 1. The random variables Z_i and M are standard normally distributed are assumed to be independent of each other. This implies that the latent variables X_i are standard normal distributed for $i = 1, ..., I$, too. In this scenario, Z_i can be interpreted as the individual risk factor whereas M represents a common risk factor. This setup defines a common correlation structure between the X_i that is exclusively implied by the single parameter ρ and we have $Corr(X_i, X_j) = \rho^2$. We will thus refer to ρ as the correlation parameter. As outlined in Hull and White (2004) and Burtschell, Gregory and Laurent (2009), we can map the latent variable X_i to the default time τ_i via the relation $\tau_i = {Q^i}^{-1}(\Phi(X_i))$, where $\Phi(.)$ denotes the cumulative distribution function of a standard-normal random variable. Hence, we can express the conditional marginal default probability of firm i as

$$Q^{i|M}(s) := Q(\tau_i < s|M) = \Phi\left\{\frac{\Phi^{-1}(Q^i(s)) - \rho M}{\sqrt{1-\rho^2}}\right\}. \tag{34}$$

We see that a correlation parameter of $\rho = 0$ implies independent default times and that an increase in ρ leads to an increase in the default dependence. In other words, this meas that the probability mass is shifted to both ends of the probability distribution.

Student-t Copula To overcome some of the shortcomings of the Gaussian copula model, a simple extension of the Gaussian copula is the Student-t copula which has also been applied in the finance literature (see Andersen, Sidenius and Basu (2003) and Schloegl and O'Kane (2005)). A detailed description of this type of copula can be found in Demarta and McNeil (2005) who show for example that the Student-t copula can generate fat-tailed distributions. In contrast, the Gaussian copula cannot produce fat-tails which has often been criticized in literature. Similar to the Gaussian framework, we set up a factor model and define

$$X_i = \frac{\rho M + \sqrt{1-\rho^2} Z_i}{\sqrt{\frac{P}{\nu}}}, \tag{35}$$

where M and Z_i are independent normal distributed random variables and P follows a χ^2_ν distribution with ν degrees of freedom such that X_i is Student-t distributed with ν degrees of freedom.[28] Then, we have $\tau_i = {Q^i}^{-1}(t_\nu(X_i))$, where $t_\nu(.)$ denotes the cumulative distribution function of a Student-t random variable with ν degrees of freedom. In contrast to the previous

[28] A random variable of the form $\frac{X}{\sqrt{\frac{Y}{\nu}}}$ with X standard normal and Y χ^2_ν distributed, is Student-t distributed.

approach, we now have to condition on both factors M and P to get independence of default times which results in the following conditional marginal default probabilities of firm i:

$$Q^{i|M,P}(s) := Q(\tau_i < s|M,P) = \Phi\left\{\frac{\sqrt{\frac{P}{\nu}}t_\nu^{-1}(Q^i(s)) - \rho M}{\sqrt{1-\rho^2}}\right\}. \tag{36}$$

Furthermore, we want to remark that setting $\rho = 0$ does not result in independent default times which is induced by the tail dependence of the Student-t copula.

Double-t Copula Like the already discussed copulas, the Double-t copula also bases on a factor model and belongs to the class of elliptical copulas. Hull and White (2004) and Burtschell, Gregory and Laurent (2009) discuss the properties of the Double-t copula and apply it to price CDO contracts. Here, the latent variables X_i are a mixture of Student-t variables and are defined as

$$X_i = \rho\sqrt{\frac{\nu-2}{\nu}}M + \sqrt{1-\rho^2}\sqrt{\frac{\tilde{\nu}-2}{\tilde{\nu}}}Z_i, \tag{37}$$

where M and Z_i are independent random variables that follow a Student-t distribution with ν and $\tilde{\nu}$ degrees of freedom, respectively. Both random variables are scaled by a factor to ensure unit variance. Unfortunately, the sum of two Student-t distributed variables does not give a Student-t variable. The cumulative distribution function of X_i is denoted by $H(.)$.[29] In a work of Vrins (2009), the author claims that there does not exist an analytical solution of this distribution function such that it has to be computed numerically. In our paper, we adapt the semi-analytical approach of Vrins (2009) to solve this problem. Having solved this problem, the default times are given by $\tau_i = {Q^i}^{-1}(H(X_i))$ and the conditional marginal default probabilities read

$$Q^{i|M}(s) := Q(\tau_i < s|M) = \Phi\left\{\sqrt{\frac{\tilde{\nu}}{\tilde{\nu}-2}}\frac{H^{-1}(Q^i(s)) - \rho\sqrt{\frac{\nu}{\nu-2}M}}{\sqrt{1-\rho^2}}\right\}. \tag{38}$$

Clayton Copula As last approach, we present the Clayton copula which belongs to the class of Archimedean copulas. Here, the latent variable follows

$$X_i = \psi\left(\frac{-\ln(Z_i)}{M}\right), \tag{39}$$

where Z_i are independent uniform random variables which are also independent of M. The common factor M, which is also called frailty, is assumed to be standard Gamma distributed with parameter $1/\theta$ and $\psi(s) = (1+s)^{-1/\theta}$ is a deterministic function. Eventually, the conditional marginal default probabilities read

$$Q^{i|M}(s) := Q(\tau_i < s|M) = \exp\left(V(1 - Q^i(s)^{-\theta})\right). \tag{40}$$

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[29] Note that the distribution function $H(.)$ does not depend on the specific firm i since the random variables Z_i are identically distributed. Moreover, $H(.)$ will depend on the parameter ρ.

For a general derivation of this method and for further information we refer the reader to Laurent and Gregory (2005) and Schönbucher (2003).

Number of Defaults To price portfolio products, the distribution of the number of defaults is required. In the previous analysis, the conditional marginal default probabilities of the different copula models were derived. With the formulas given in (34), (36), (38), and (40), it becomes straightforward to compute the conditional distribution of the number of defaults because the single default times are independent when conditioned on the common factor. For instance, when dealing with a homogeneous pool of assets, one only has to evaluate a standard binomial distribution. However, when dealing with a heterogenous pool of assets, i.e. the firms have different default probabilities, it can become computationally expensive to get the portfolio loss distribution when I becomes large. This is the case since we have to consider $\binom{I}{k}$ combinations of getting k defaults in a pool of I obligors and every combination involves further computations (see Hull and White (2004)). In the literature, there are basically two semi-analytical solution techniques to tackle this problem: Fast Fourier techniques (FFT) (see Laurent and Gregory (2005)) and the procedure of probability bucketing (see Andersen, Sidenius and Basu (2003) and Hull and White (2004)). We tested these methods in several experiments and it turned out that they both outperform the combinatorial technique in terms od CPU time. Furthermore, the probability bucketing approach is always faster than the FFT and the differences in the resulting probabilities are negligible. We thus decided to use the bucketing approach that bases on a recursion algorithm. This algorithm provides us with the probability distribution of the number of defaults conditioned on the common factor. We denote the conditional probability of having k defaults at time s by $q_k^{i|M}(s)$ where $k = 0, ..., I$.

In the last step, the unconditional probability distribution can be determined by numerically integrating $q_s^{i|M}(k)$ over the distribution of M.[30] This integration is accomplished in a fast way by using quadrature techniques. As the common factor is distributed differently depending on the copula, we use several quadrature techniques that are summarized in table (17)

A.2 Calibration of the DGPL model

Following Brigo, Pallavicini and Torresetti (2006), we calibrate the DGPL model as follows:

(i) Set $\alpha_1 = 1$ and calibrate λ_1. All other α_i are assumed to be zero.

(ii) Add the jump size α_2 and find its best positive integer value between 1 and 125. This is done by calibrating the intensities λ_1 and λ_2 (we take the calibrated value of λ_1 from step 1 as an initial guess) for each value of α_2 in $[1, 125]$.

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[30] Recall that we have to perform a two-dimensional integration when we deal with the Student-t copula.

(iii) Repeat the second step for α_i, $i = 3, ..., I$, and calibrate the intensities $\lambda_1, \lambda_2, ..., \lambda_i$ by minimizing (24). As in step 2, use $\lambda_1, \lambda_2, ..., \lambda_{i-1}$ from the previous step as initial guess for the intensities.

In all cases, we have needed less than 7 iterations to achieve a perfect fit (defined as the RMSPE being smaller than 1bp). Nevertheless, repeating this calibration exercise every day would be time-consuming. Hence, we decided to calibrate the full DGPL model on the first day only and keep the size parameters α_i for the calibrations on the next days. Therefore, we only have to estimate the respective intensities which still provides remarkable good fits. The estimated parameters for a particular day in the data set are shown in Table 18.

A.3 Random Thinning Procedure

The aim of this method is to choose the elements $TD_{ij}^{(l)}$ such that all single-name CDS spreads are fitted. To this end, we have to introduce further notation and define $w^{(l)}(j)$ as the probability that the number of defaults in the portfolio at time t_l is at least j. Furthermore, $Q_i^{(l)}$ denotes the probability of firm i to default before time t_l. The probabilities $w^{(l)}(j), j = 0, ..., I$ are known from the respective top-down model calibrated to market data and $Q_i^{(l)}, i = 1, ..., I$ can be calibrated using CDS quotes. By construction, these probabilities and the elements of the RT matrix are related via

$$\sum_{i=1}^{I} TD_{ij}^{(l)} = w^{(l)}(j) - w^{(l-1)}(j), \text{ for all } j = 1, ..., I, \tag{41}$$

$$\sum_{j=1}^{I} TD_{ij}^{(l)} = Q_i^{(l)} - Q_i^{(l-1)}, \text{ for all } j = 1, ..., I. \tag{42}$$

In our study, we apply the iterative scaling algorithm proposed by Halperin and Tomecek (2009) to back out the probabilities $TD_{ij}^{(l)}$. Having found the thinning matrix, we proceed as follows:

(i) Calibrate the relevant top-down model and the marginal default probabilities to market data, which gives us $w^{(l)}(j), j = 0, ..., I$ and $Q_i^{(l)}, i = 1, ..., I$.

(ii) Shift the single-name CDS spread of firm i by 1 bp, recalibrate the firms default probability, and store the proportional change ξ of this probability.

(iii) Change the i-th row of the RT matrix such that $TD_{ij}^{(l)} = TD_{ij}^{(l)}(1+\xi)$ for all $j = 1, ..., I$. This adjustment results in the desired shift of the default probability of firm i (see (42)). In addition, the change of the matrix yields new top down probabilities $w^{(l)}(j) = w^{(l)}(j) + TD_{ij}^{(l)}\xi$ for all $j = 1, ..., I$ (see (41)).

(iv) Use the adjusted top-down probabilities of step (iii) to compute the change in the MTM value of a given tranche.

A similar approach can be used to calculate the index spread. Here, we only have to shift all CDS spreads in step (ii) by 1 bp. The resulting RT matrix adjustment in step (iii) will then affect all rows.[31]

[31] Note that our approach is slightly different form the approach in Halperin and Tomecek (2009) who use the same proportional change ξ for every firm. Additionally, they only use approximations of the MTM values of the specific securities.

Model	Abbreviation
Gaussian copula	Gau
Clayton copula	Clay
Student-t copula with ν degrees of freedom	T(ν)
Double-t copula with ν and $\tilde{\nu}$ degrees of freedom	DT($\nu/\tilde{\nu}$)
Longstaff-Rajan	LR
Self-exciting framework	SE
Dynamical generalized-Poisson Loss model	DGPL

Table 1: Abbreviations of the relevant bottom-up and top-down models.

Model	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
Gau	0.49	0.94	0.03	0.12	0.23
Clay	0.41	1.36	0.02	0.07	0.15
DT(4/4)	0.6	0.82	0.04	0.16	0.23
DT(6/4)	0.59	0.85	0.03	0.13	0.2
DT(6/6)	0.57	0.85	0.03	0.15	0.22
T(4)	0.38	0.89	1	0.01	0.04
T(8)	0.44	0.9	1	0.01	0.14
T(12)	0.46	0.9	1	0.05	0.17

Table 2: Compound correlations on 09/05/08. This table shows the compound correlations for the copula models under study.

Model	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
Gau	0	0.57	4.69	0	0
Clay	0	0	0.82	0	0
DT (4/4)	0	0	0.25	0	0
DT (6/4)	0	0	0.4	0	0
DT (6/6)	0	0	0.39	0	0
T (4)	0	0	13.12	10.93	0.57
T (8)	0	0	13.44	8.79	0
T (12)	0	0	13.85	1.86	0

Table 3: Average calibration error in percent implied by compound correlations. This table shows the average calibration errors as percentages of market spreads of the particular bottom-up models when we use compound correlations. The time period is 09/01/08 to 09/22/08.

Model	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
Gau	0.49	0.62	0.69	0.75	0.94
Clay	0.41	0.59	0.71	0.86	1.52
DT(4/4)	0.6	0.66	0.68	0.74	0.85
DT(6/4)	0.59	0.66	0.69	0.74	0.9
DT(6/6)	0.57	0.64	0.68	0.73	0.89
T(4)	0.38	0.53	0.61	0.69	0.91
T(8)	0.44	0.57	0.65	0.72	0.92
T(12)	0.46	0.59	0.66	0.73	0.92

Table 4: Base correlations on 09/05/08. This table shows the base correlations for the copula models under study.

Model	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
Gau	0	0	0	0.01	0.26
Clay	0	0	0	0	0
DT (4/4)	0	0	0	0.01	2.49
DT (6/4)	0	0	0	0.01	2.03
DT (6/6)	0	0	0	0.01	1.85
T (4)	0	0	0	0	0.72
T (8)	0	0	0	0.01	0.85
T (12)	0	0	0	0.01	0.73

Table 5: Average calibration error in percent implied by base correlations. This table shows the average calibration errors as percentages of market spreads of the particular bottom-up models when we use base correlations. The time period is 09/01/08 to 09/22/08.

	vola σ_i	jump γ_i	intensity λ_i
first process	0.3114	0.0084	0.4718
second process	0.2625	0.0825	0.0257
third process	0.2211	0.1584	0.0082

Table 6: Calibrated parameters in the LR model on 09/05/08.

Model	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
SE	0	16.16	2.83	0.94	13.06
DGPL	0.01	0.85	1.47	6.46	3.49
LR	0.06	0.08	0.06	0.12	0.07

Table 7: Average calibration error in percent across all top-down models. This table shows the average calibration errors as a percentage of market spreads of the top-down models under study. The time period is 09/01/08 to 09/22/08.

Firm	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
	Gaussian copula with base correlation				
Danone	0.0289	0.0206	0.0077	0.019	-0.0149
France Telecom	0.0413	0.0318	0.014	0.0301	-0.0115
Tate and Lyle	0.0674	0.0719	0.0517	0.0245	0.0409
	Self-exciting				
Danone	0.1231	0.061	0.0349	0.0228	0.0114
France Telecom	0.1403	0.0626	0.0327	0.0199	0.0089
Tate and Lyle	0.1578	0.0635	0.0299	0.0162	0.0059
	DGPL				
Danone	0.079	0.0321	0.0222	0.0157	0.0088
France Telecom	0.0986	0.0361	0.0228	0.015	0.0075
Tate and Lyle	0.1244	0.0412	0.0232	0.0138	0.0054

Table 8: Tranche deltas with respect to single-name CDS on 09/05/08. This table shows single-name deltas with respect to the three firms Danone (small-risk), France Telecom (middle-risk), and Tate and Lyle (high-risk) computed with three different models.

Model	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
	Index				
Gau	83.7 (75.27)	82.01 (76.35)	84.08 (77.28)	108.65 (95.31)	113.48 (118.23)
T(4)	82.5 (76.66)	83.39 (77.76)	79.64 (74.83)	96.83 (86.67)	94.02 (85.07)
SE	230.32 (223.69)	140.09 (131.43)	108.26 (102.53)	126.61 (117.22)	118.8 (108.53)
DGPL	165.74 (153.33)	91.66 (86.22)	84.5 (82.84)	104.42 (96.45)	106.5 (96.09)
	Portfolio of three single-name CDS				
Gau	75.08 (70.61)	69.89 (68.63)	71.92 (66.31)	114.87 (112.78)	110.4 (101.96)
T(4)	77.81 (74.37)	72.18 (69.55)	70.04 (68.99)	90.96 (78.94)	91.09 (89.14)
SE	194.11 (198.56)	111.04 (105.29)	83.65 (81.07)	96.61 (93.29)	94.19 (89.13)
DGPL	142.47 (138.93)	75.63 (72.53)	72.63 (70.32)	88.51 (81.71)	89.75 (83.74)

Table 9: Comparison of index CDS and a portfolio of three single-name CDS contracts in September 2008. This table compares the hedging errors and residual volatilities (in brackets) that are either computed with the index as hedging instrument or with a portfolio of three single-name CDS contracts. The period is September 2008.

	0–3%	3–6%	6–9%	9–12%	12–22%	0–3%	3–6%	6–9%	9–12%	12–22%
	Biggest loss					Expected shortfall				
	Unhedged position									
	7.71	5.89	3.62	2.08	1.02	2.03	1.29	0.81	0.61	0.23
	Index hedging									
Gau	4.5	3.87	2.29	1.77	1.43	1.66	1.34	1.07	0.45	0.26
T(4)	4.63	3.84	2.6	1.24	0.68	1.61	1.26	0.92	0.65	0.28
SE	13.03	5.09	2.99	1.84	0.9	4.48	2.3	1.2	0.85	0.38
DGPL	7.73	4.28	2.67	1.58	0.83	3.29	1.41	0.98	0.71	0.3
	Single-name hedging									
Gau	4.9	4.21	2.47	2.04	0.89	1.53	1.1	0.87	0.54	0.39
T(4)	5.22	4.21	2.85	1.35	0.85	1.38	1.15	0.77	0.63	0.25
SE	11.19	5.16	2.98	1.66	0.77	3.98	1.99	0.84	0.7	0.26
DGPL	7.31	4.39	2.7	1.54	0.74	2.97	1.26	0.82	0.57	0.25

Table 10: Biggest losses and expected shortfall in September 2008. This table shows the biggest losses and the expected shortfall (expected value of P&L path conditioned on losses) for different hedging strategies and different hedging instruments (index hedging and single-name hedging). All numbers are given in percentage of the tranche notional. We also report the corresponding values if the position is not hedged. The period is September 2008.

	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]	Index
[0 − 3%]	100					
[3 − 6%]	96.63	100				
[6 − 9%]	94.06	98.52	100			
[9 − 12%]	89.9	95.91	97.05	100		
[12 − 22%]	88.94	92.78	94.85	97.92	100	
Index	63.16	64.24	66.38	59.55	59.88	100

Table 11: Correlations between tranches and index CDS. This table plots the correlation of tranche and index CDS returns that were realized in September 2008.

T(4)

Hedging instrument	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
[0 − 3%]	0 (0)	26.35 (27.21)	29.92 (33.42)	60.3 (61.34)	52.99 (52.99)
[3 − 6%]	26.3 (25.74)	0 (0)	22.27 (20.46)	48.85 (45.68)	46.18 (43.26)
[6 − 9%]	29.86 (33.86)	21.91 (20.88)	0 (0)	49.69 (52.03)	45.49 (41.74)
[9 − 12%]	98.46 (104.55)	80.89 (81.94)	87.18 (93.41)	0 (0)	46.65 (46.74)
[12 − 22%]	82.23 (85.82)	67.11 (67.04)	71.54 (71.37)	42.06 (40.13)	0 (0)

DGPL

Hedging instrument	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
[0 − 3%]	0 (0)	57.31 (53.01)	59.2 (56.64)	59.3 (56.42)	61.07 (57.93)
[3 − 6%]	111.47 (105.3)	0 (0)	18.75 (16.6)	36.71 (31.04)	48.03 (41.64)
[6 − 9%]	116.59 (113.83)	19.24 (17.13)	0 (0)	29.51 (29.29)	40.95 (37.59)
[9 − 12%]	106.17 (100.43)	34.23 (28.32)	26.86 (25.52)	0 (0)	20.92 (21.03)
[12 − 22%]	108.31 (109.24)	42.41 (37)	34.94 (31.88)	19.66 (20.52)	0 (0)

Table 12: Hedging tranches with tranches in September 2008. This table shows the hedging errors and residual volatilities (numbers in brackets) when we hedge tranches with other tranches. We plot the results for the models T(4) and DGPL. The tranches in the column labeled "Hedging instrument" represent the tranches that are used for hedging. The period is September 2008.

	0 – 3%	3 – 6%	6 – 9%	9 – 12%	12 – 22%	0 – 3%	3 – 6%	6 – 9%	9 – 12%	12 – 22%
	Biggest loss					Expected shortfall				
	Unhedged position									
	7.71	5.89	3.62	2.08	1.02	2.03	1.29	0.81	0.61	0.23
	Hedging with the [12 – 22%] tranche									
DGPL	3.78	1.82	0.78	0.16	0	2.06	0.51	0.27	0.09	0

Table 13: Biggest losses and expected shortfall in September 2008. This table shows the biggest losses and the expected shortfall (expected value of P&L path conditioned on losses) for the DGPL model. We use the [12−22%] as hedging instrument. All numbers are given in percentage of the tranche notional. For comparison, we also plot the corresponding values when we do not hedge (Unhedged position). The period is September 2008.

Model	[0 – 3%]	[3 – 6%]	[6 – 9%]	[9 – 12%]	[12 – 22%]
	Index				
Gau	84.37 (81.03)	68.78 (72.25)	78.38 (83.72)	68.75 (71.71)	103.24 (105.91)
T(4)	87.14 (83.38)	71.85 (74.66)	66.81 (73.11)	75.5 (81.04)	67.99 (72.11)
SE	202.21 (174.34)	125.02 (121.11)	78.58 (82.16)	64.78 (72.52)	64.34 (67.39)
DGPL	128.38 (114.24)	68.86 (72.15)	67.31 (73.1)	64.77 (70.27)	64.11 (68.14)
	Portfolio of three single-name CDS				
Gau	109.53 (99.15)	92.81 (96.11)	92.67 (100.17)	101.68 (104)	100.16 (119.92)
T(4)	129.79 (116.98)	95.32 (97.99)	82.06 (84.74)	84.45 (94.92)	77.9 (78.04)
SE	262.03 (223.07)	152.46 (148.17)	91.63 (93.15)	76.8 (81.25)	68.5 (73.18)
DGPL	175.89 (153.01)	81.04 (84.26)	75.8 (79.66)	72.32 (78.17)	69.11 (74.46)

Table 14: Comparison of index CDS and a portfolio of three single-name CDS contracts in April 2008. This table compares the hedging errors and residual volatilities (in brackets) that are either computed with the index as hedging instrument or with a portfolio of three single-name CDS contracts. The period is April 2008.

	0 – 3%	3 – 6%	6 – 9%	9 – 12%	12 – 22%	0 – 3%	3 – 6%	6 – 9%	9 – 12%	12 – 22%
	Biggest loss					Expected shortfall				
	Unhedged position									
	3.28	1.96	1.49	0.99	0.73	1.55	0.71	0.76	0.48	0.23
	Index hedging									
Gau	2.53	1.01	1.33	0.63	0.76	1.07	0.61	0.54	0.43	0.28
T(4)	2.66	1.07	1.26	0.86	0.46	1.15	0.64	0.5	0.36	0.18
SE	6.33	2.41	1.14	0.78	0.42	3.07	1.22	0.59	0.31	0.15
DGPL	4	1.08	1.22	0.72	0.44	2.19	0.6	0.47	0.3	0.15
	Single-name hedging									
Gau	3.52	2.02	1.94	1.4	1.07	1.65	0.91	0.74	0.68	0.28
T(4)	4.21	2.11	1.35	1.51	0.45	2.04	0.93	0.65	0.55	0.23
SE	8.78	3.35	1.6	1.06	0.51	4.01	1.34	0.72	0.47	0.16
DGPL	5.69	1.8	1.27	0.99	0.53	2.86	0.74	0.57	0.38	0.17

Table 15: Biggest losses and expected shortfall in April 2008. This table shows the biggest losses and the expected shortfall (expected value of P&L path conditioned on losses) for different hedging strategies and different hedging instruments (Index hedging and single-name hedging). All numbers are given in percentage of the tranche notional. For comparison, we also plot the corresponding values when we do not hedge (Unhedged position). The period is April 2008.

T(4)

Hedging instrument	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
[0 − 3%]	0 (0)	27.43 (32.82)	42.22 (43.98)	46.11 (48.98)	61.37 (62.28)
[3 − 6%]	28.7 (31.41)	0 (0)	25.93 (25.63)	35.14 (37.58)	48.29 (49.29)
[6 − 9%]	54.55 (52.8)	31.52 (31.03)	0 (0)	36.9 (38.67)	44.78 (47.8)
[9 − 12%]	69.09 (67.3)	58.08 (67.39)	50.25 (59.36)	0 (0)	67.94 (74.71)
[12 − 22%]	376.5 (825.28)	319.83 (774.3)	217 (520.11)	337.87 (890.85)	0 (0)

DGPL

Hedging instrument	[0 − 3%]	[3 − 6%]	[6 − 9%]	[9 − 12%]	[12 − 22%]
[0 − 3%]	0 (0)	47.45 (46.9)	58.35 (57.18)	59.14 (58.59)	59.99 (60)
[3 − 6%]	87.68 (81.23)	0 (0)	24.44 (24.67)	26.35 (29.99)	28.34 (31.82)
[6 − 9%]	130.01 (119.43)	29.25 (29.6)	0 (0)	22.47 (25.6)	27.16 (30.18)
[9 − 12%]	131.06 (116.55)	30.67 (33.18)	22.31 (24.49)	0 (0)	27.84 (27.33)
[12 − 22%]	134.76 (129.72)	33.59 (38.15)	27.41 (31.05)	28.53 (29.8)	0 (0)

Table 16: Hedging tranches with tranches in April 2008. This table shows the hedging errors and residual volatilities (numbers in brackets) when we hedge tranches with other tranches. We plot the results for the models T(4) and DGPL. The tranches in column "Hedging instrument" represent the tranches that are used for hedging. The period is April 2008.

Copula	Method
Gaussian	Gauss-Hermite
Student-t	Gauss-Hermite (M)
	Gauss-Laguerre (P)
Double-t	Gauss-Legendre
Clayton	Gauss-Laguerre

Table 17: *Methods for numerical integration.* *This table shows the numerical methods that are used to approximate the specific integrals. The Student-t copula requires a two-dimensional integration.*

	N_1	N_2	N_3	N_4	N_5	N_6
α_i	1	19	3	125	4	20
λ_i	0.1309	0.0178	0.0566	0.0064	0.1221	0.0016

Table 18: Calibrated parameters in the DGPL model on 09/05/08.

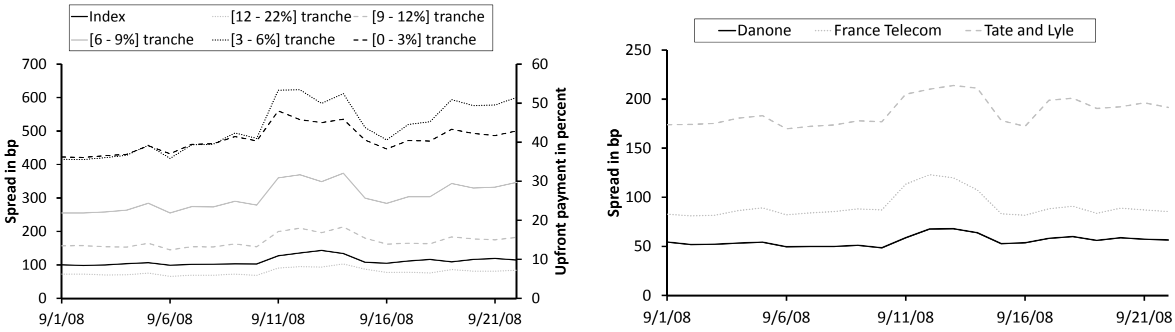

Figure 1: Tranche, index, and constituent spreads on 5-year iTraxx.EU Series 9 from 09/01/08 to 09/22/08.

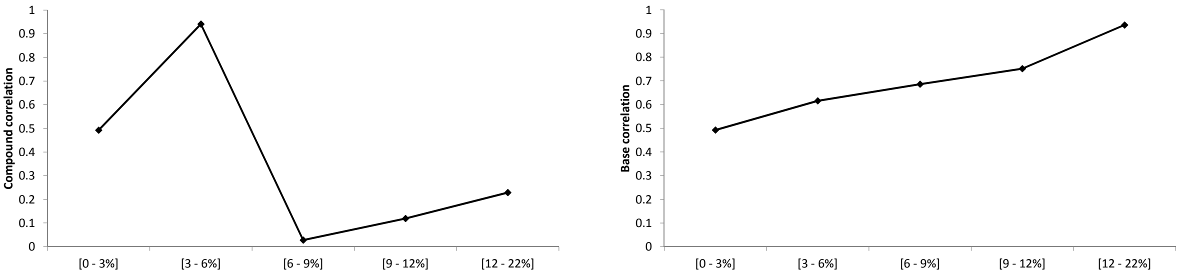

Figure 2: Compound and base correlations for Gaussian copula on 09/05/08.

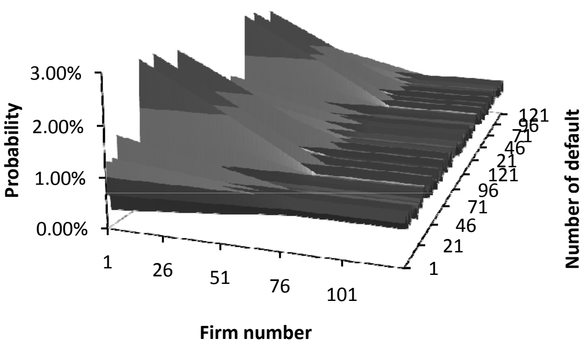

Figure 3: Random thinning. This figure shows the random thinning matrix generated by the DGPL model on 09/05/08.

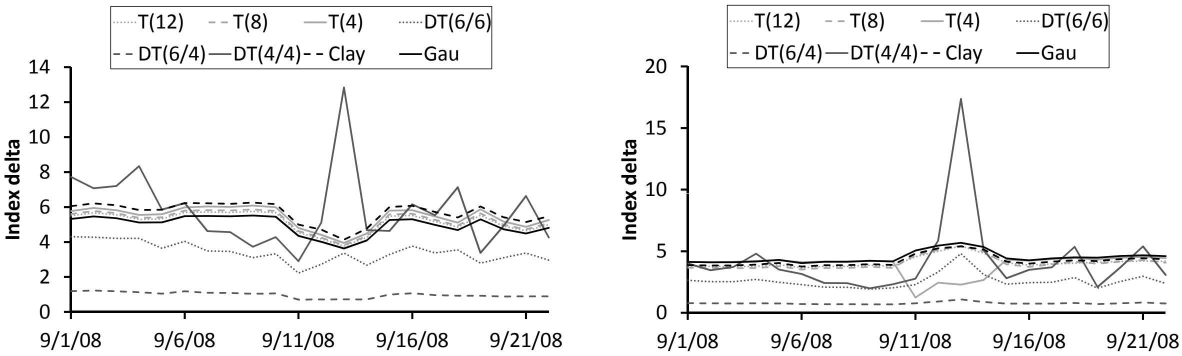

Figure 4: Index deltas across copula models calibrated with compound correlations. The left panel shows the index deltas for the equity tranche in September 2008, the right panel depicts the corresponding deltas for the $[9-12\%]$ tranche.

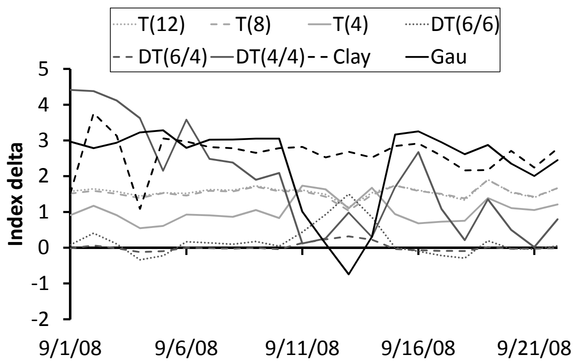

Figure 5: Index deltas across copula models calibrated with base correlations. The figure shows the index deltas for the $[9-12\%]$ tranche in September 2008.

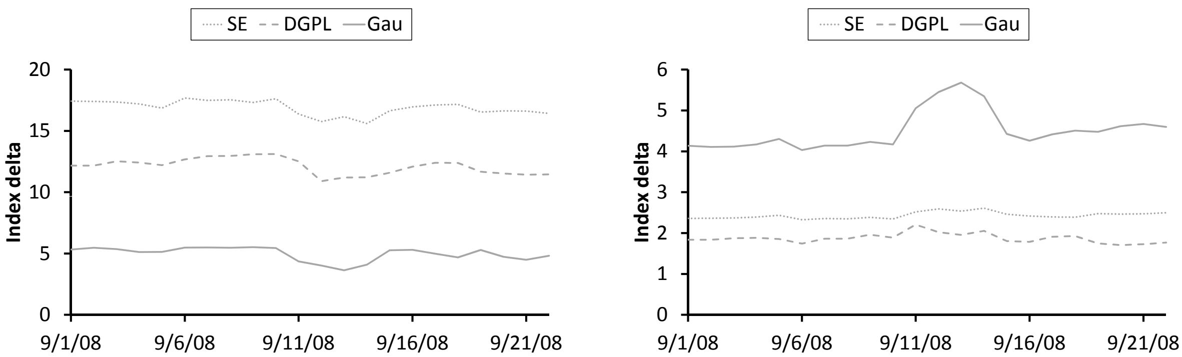

Figure 6: Index deltas across top-down models. The left panel shows the index deltas for the equity tranche in September 2008, the right panel depicts the corresponding deltas for the $[9-12\%]$ tranche. For comparison, the respective deltas for the Gaussian copula (calibrated with compound correlations) are plotted.

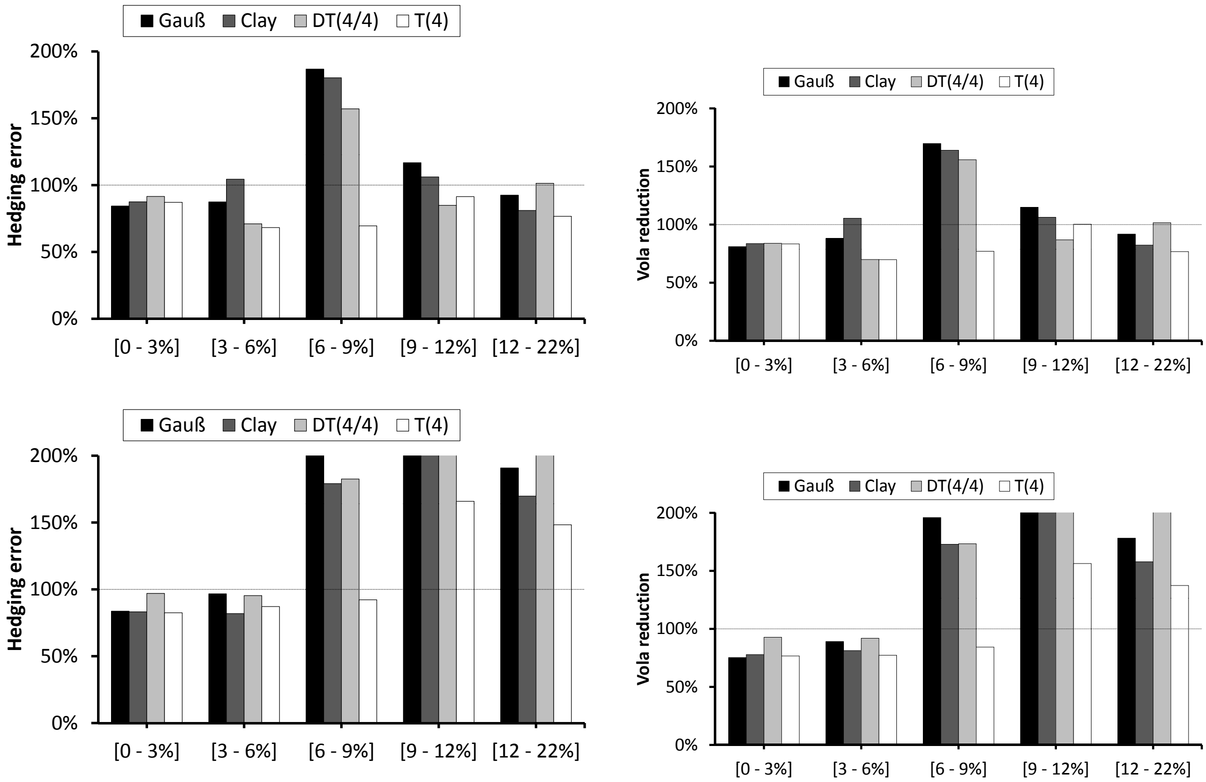

Figure 7: Bottom-up P&L analysis with delta hedging index and compound correlations. The upper pictures show the hedging errors and volatility reductions of hedging strategies implied by several bottom-up models in April 2008. The lower pictures show the corresponding results for the period of September. The deltas are computed using the compound correlation method.

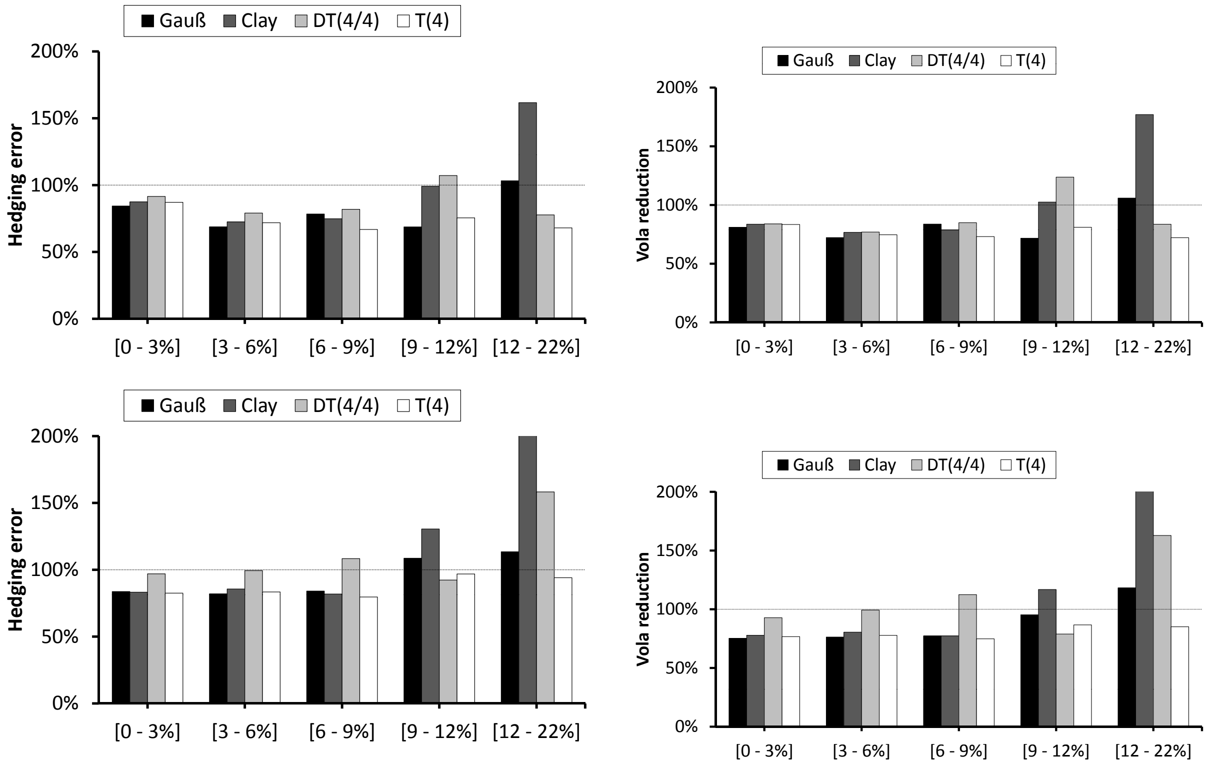

Figure 8: Bottom-up P&L analysis with index hedging and base correlations. The upper figures show the hedging errors and volatility reductions of hedging strategies implied by several bottom-up models in April 2008. The lower pictures show the corresponding results for September 2008. The deltas are computed using the base correlation method.

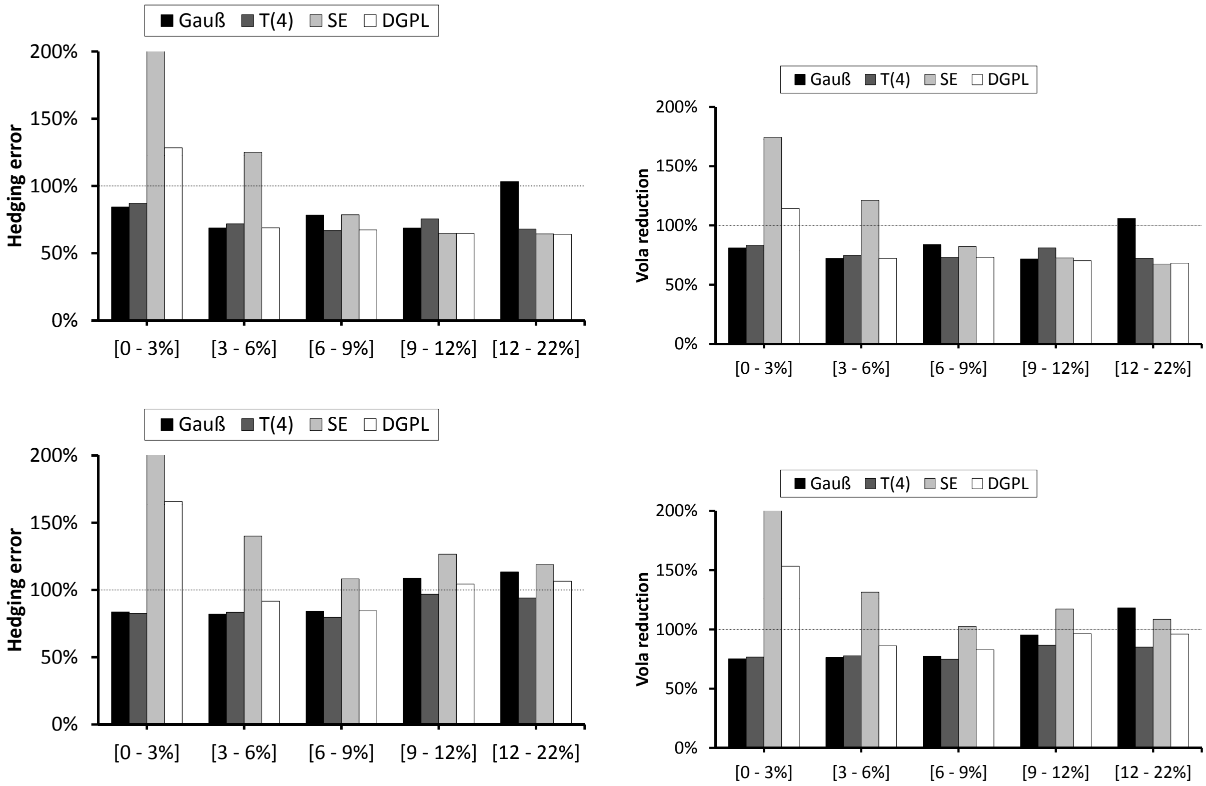

Figure 9: Top-down P&L analysis with index hedging. The upper pictures show the hedging errors and volatility reductions of hedging strategies implied by the SE and DGPL models in April 2008. To make the results comparable, the numbers for the Gaussian and Clayton copula are also plotted. The lower pictures show the corresponding results for the period of September.

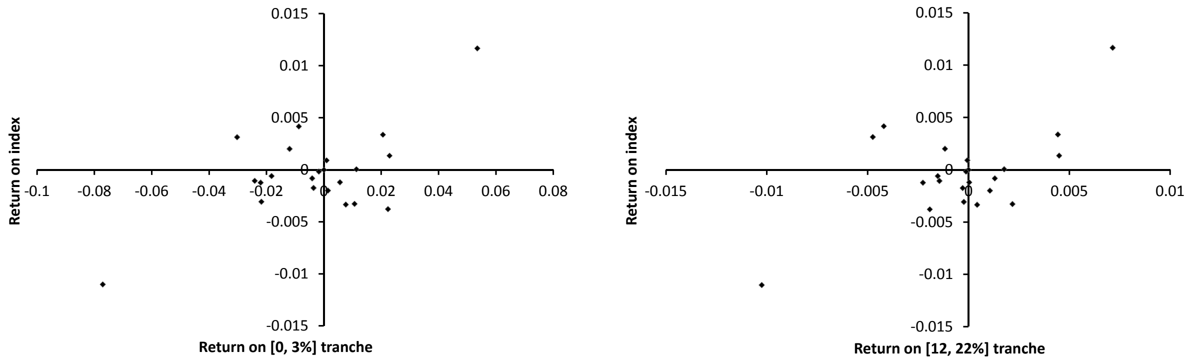

Figure 10: Daily index returns versus daily tranche returns. The pictures display scatter plots of realized index returns and tranche returns in September 2008. The left panel contains equity tranche returns, the right panel contains senior tranche returns.

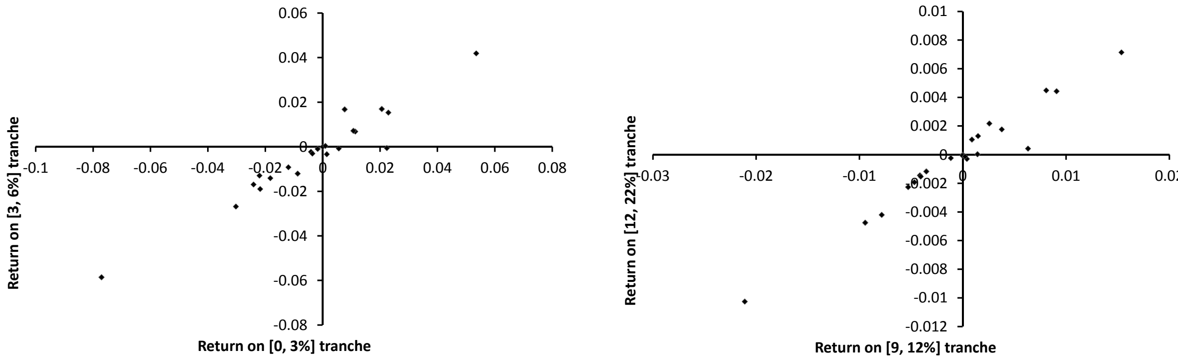

Figure 11: Daily tranche returns versus daily tranche returns. The pictures display scatter plots of realized index returns and tranche returns in September 2008. The left panel contains equity tranche returns, the right panel contains senior tranche returns.

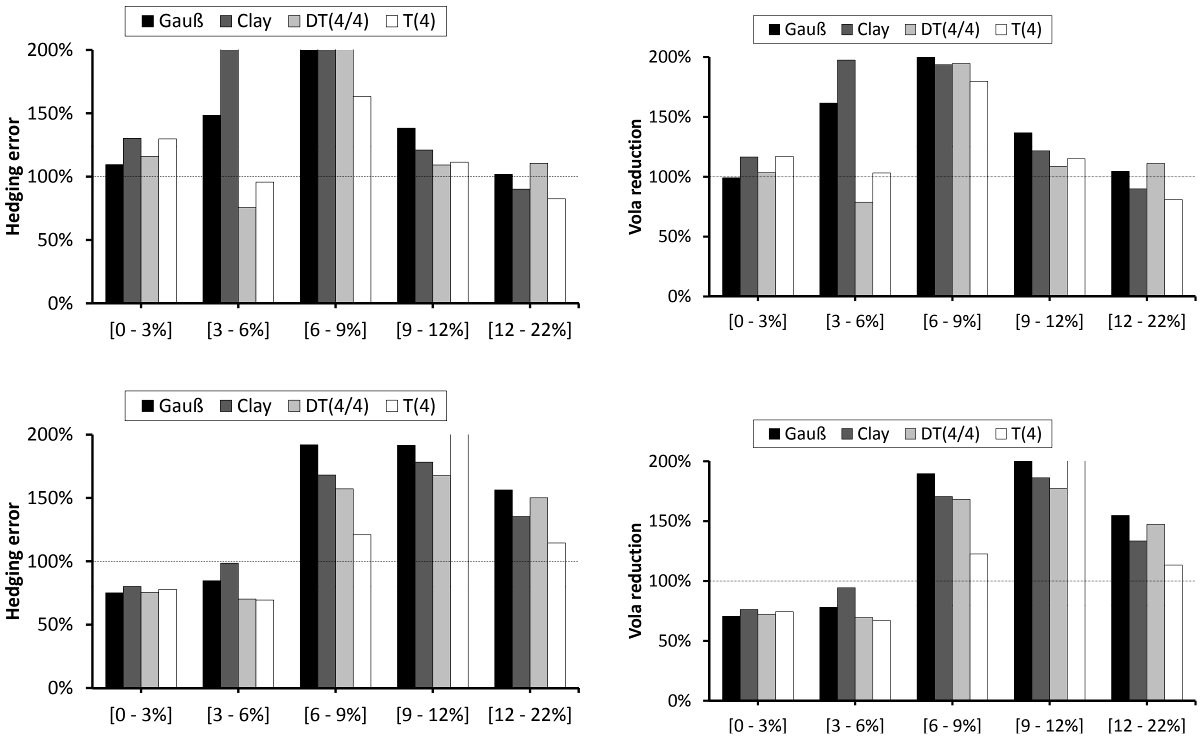

Figure 12: Bottom-up P&L analysis with three single-name CDS contracts as hedging instruments and compound correlations. The upper pictures show the hedging errors and volatility reductions of hedging strategies implied by several bottom-up models in April 2008. The lower pictures show the corresponding results for the period of September. The deltas are computed by the concept of compound correlations.

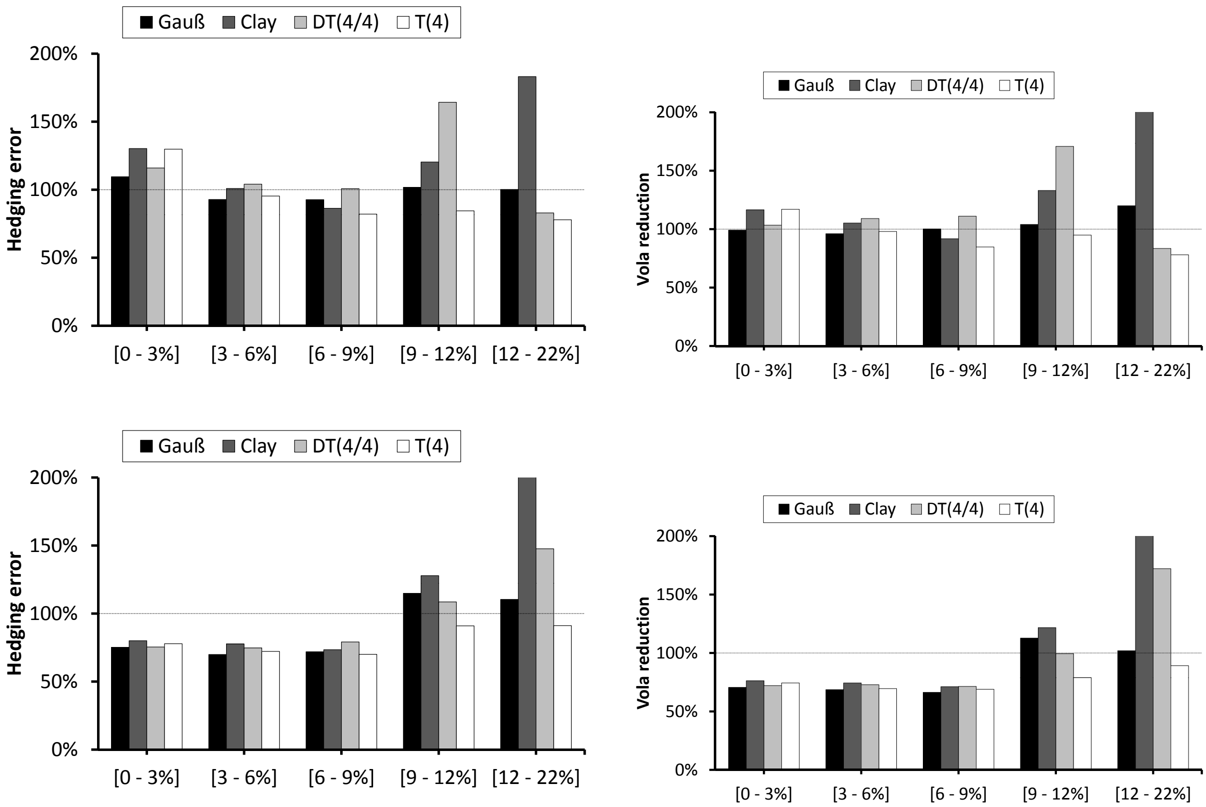

Figure 13: Bottom-up P&L analysis with three single-name CDS contracts as hedging instruments and base correlations. The upper pictures show the hedging errors and volatility reductions of hedging strategies implied by several bottom-up models in April 2008. The lower pictures show the corresponding results for the period of September. The deltas are computed by the concept of base correlations.

Part III

Appendix

Curriculum Vitae

BJÖRN BICK
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EDUCATION

Since 04/2008	**Ph.D. Student in Finance** Goethe University Frankfurt, Germany Supervisor: Prof. Dr. Holger Kraft
10/2002 - 03/2007	**Diplom-Mathematiker** (Master of Mathematics) University of Kaiserslautern, Germany
08/1992 - 06/2001	**General A-levels** Arnold-Janssen-Gymnasium, St. Wendel, Germany

WORK EXPERIENCE

Since 03/2008	**Research Assistant** at the UBS Endowed Chair of Asset Pricing Goethe University Frankfurt, Germany
07/2007 - 03/2008	**Research Assistant** at the Department of Financial Mathematics Fraunhofer ITWM, Kaiserslautern, Germany
04/2007 - 06/2007	**Student Assistant** at the Department of Financial Mathematics Fraunhofer ITWM, Kaiserslautern, Germany

PUBLICATIONS

Solving Constrained Consumption-Investment Problems by Simulation of Artificial Market Strategies
with Holger Kraft and Claus Munk, Working Paper

Consumption-Portfolio Choice with Unspanned Labor Income and Stochastic Volatility
Working Paper

Default and Idiosyncratic Risk Anomalies Revisited
with Holger Kraft, Christian Hirsch, and Yildiray Yildirim, Working Paper

Hedging Structured Credit Products During the Credit Crunch
with Holger Kraft, Working Paper