APS/123-QED

Gravitational radiation from elastic particle scattering in models with extra dimensions

B. Koch* and M. Bleicher

Institut für Theoretische Physik,

J.W. Goethe Universität,

60054 Frankfurt am Main, Germany

Frankfurt Institute for Advanced Studies (FIAS),

60054 Frankfurt am Main, Germany

(Dated: April 15, 2006)

Abstract

In this paper we derive a formula for the energy loss due to elastic N to N particle scattering in models with extra dimensions that are compactified on a radius R. In contrast to a previous derivation we also calculate additional terms that are suppressed by factors of frequency over compactification radius. In the limit of a large compactification radius R those terms vanish and the standard result for the non compactified case is recovered.

*Electronic address: koch@th.physik.uni-frankfurt.de

Contents

I. MOTIVATION

Finding a unified theory of gravity and of the standard model of particle physics remains an elusive goal in quantum field theory. A crucial ingredient of superstringtheory is that it needs more than 3 spatial dimensions for their consistency. Also supergravity which is recognised as the low energy effective description of an 3+d=10 dimensional M-theory [1, 2]. On the one hand there are several attempts to incorporate extra dimensions into low energy field theory [3, 4, 5, 6, 7]. Most of these models have in common, that only gravity is allowed to propagate into the extra dimensions.

On the other hand, classical gravitational waves are being looked for since a long time and in the forthcoming years one finally expects to detect this important probe for general relativity [8]. Therefore also classical gravitational waves within models with extra dimensions do provide a good framework to study new physics.

In the following chapters we derive the cross section for gravitational radiation in models with an even number of extra dimensions. Although all equations are formulated for asymptotically flat space, we keep in mind that some spatial dimensions might be compactified. Therefore we do not immediately drop terms that are suppressed by higher powers of the observer distance, as this distance is limited in some directions by the chosen compactification radius R.

Then we show that only in the limit of large compactification radius (or no compactification at all) certain terms of this cross section can be neglected, which leads to the cross section which was already given [9].

II. EINSTEIN'S EQUATIONS WITH MORE DIMENSIONS

Einstein's field equations with $3+d$ spatial dimensions are a straight forward generalisation of the 3 dimensional case, however all the indices N,M run from $0\ldots 3+d$ instead of $0\ldots 3$, i.e.

$$R_{MN} - \frac{1}{2} g_{MN} R = -8\pi G T_{MN}. \tag{1}$$

The trace of this gives the Ricci scalar R

$$R(1 - \frac{4+d}{2}) = -8\pi G T_N^N. \tag{2}$$

From this one finds the $(3+d)$ dimensional Ricci-tensor R_{MN} as

$$R_{MN} = -8\pi G (T_{MN} - \frac{1}{2+d} g_{MN} T_L^L), \tag{3}$$

and therefore the $3+d$ dimensional gravitational source term S_{MN} is defined as

$$S_{NM} := (T_{NM} - \frac{1}{2+d} g_{MN} T_L^L). \tag{4}$$

A. Gravitational waves in $3 + d$ spatial dimensions

Assuming small perturbations from the $3 + d$ dimensional Minkowski metric η_{NM} with the signature $(+, -, -, -, -, ...)$ we take the following ansatz for the metric tensor

$$g_{NM} = \eta_{MN} + h_{MN}. \tag{5}$$

Inserting this ansatz into equation (3), Einstein's field equations to first order in the perturbation h read

$$\partial_L \partial^L h_{MN} - \partial_L \partial_N h^L_M - \partial_L \partial_M h^L_N + \partial_M \partial_N h^L_L = -8\pi G S_{MN}. \tag{6}$$

Here the definition of the $(3 + d)$ dimensional Riemann tensor

$$\begin{aligned} R_{MNOP} &= \tfrac{1}{2}\left[\partial_N \partial_P g_{MO} - \partial_M \partial_P g_{NO} - \partial_N \partial_O g_{MP} + \partial_M \partial_O g_{NP}\right] + g_{AB}\left[\Gamma^A_{OM}\Gamma^B_{NP} - \Gamma^A_{PM}\Gamma^B_{N0}\right] \\ &:= \tfrac{1}{2}\left[\partial_N \partial_P g_{MO} - \partial_M \partial_P g_{NO} - \partial_N \partial_O g_{MP} + \partial_M \partial_O g_{NP}\right] + X_{MNOP} \end{aligned} \tag{7}$$

is used. Notice that X_{MN} contributes only with quadratic or higher order terms in h. Now we make use of the gauge invariance of Einstein's field equations. We choose the so called harmonic coordinate system, for which

$$g^{KL}\Gamma^N_{KL} = 0. \tag{8}$$

Remembering the definition of the Christoffel symbol

$$\Gamma^A_{BC} = \frac{1}{2} g^{AD}\left[\partial_C g_{BD} + \partial_B g_{CD} - \partial_D g_{BC}\right] \tag{9}$$

and expanding (8) to first order in h gives

$$\partial_L h^L_N = \frac{1}{2}\partial_L h^N_N. \tag{10}$$

Using (10) in (6) we find

$$\partial^L \partial_L h_{MN} = -16\pi G S_{MN}. \tag{11}$$

The retarded ($\tau = t - t_0 - |x - y| > 0$) solution for (11) can be found with the $3 + d$ dimensional Greens function $G^{(3+d)}(|x - y|)$

$$h_{MN}(t,x) = N \int dt_0 \int d^{3+d}\underline{y} G^{(3+d)}_{ret}(t - t_0, |\underline{x} - \underline{y}|) S_{MN}(t_0, \underline{y}), \tag{12}$$

where N is a normalisation factor given by

$$N = -16\pi G. \tag{13}$$

The $3 + d$-dimensional retarded Greens function [9, 10, 11] is

$$G^{3+d}_{ret}(t,x) = -\frac{1}{(2\pi)^{4+d}} \int d^{3+d}\underline{k} e^{i\underline{kx}} \int dk_0 \frac{e^{-ik_0(t-T_0)}}{k_0^2 - \underline{k}^2}. \tag{14}$$

For an even number of flat extra dimensions this can be integrated to [11]

$$G^{3+d}_{ret}(t,x) = \frac{1}{4\pi} \left[\frac{-1}{2\pi r}\frac{\partial}{\partial r}\right]^{d/2} \left[\frac{\delta((t - t_0) - r)}{r}\right] \text{ , } d \text{ even.} \tag{15}$$

As the cases with an even number of extra dimensions are easier to discuss, we will postpone the cases with an odd number of extra dimensions. It is convenient to bring all derivatives in (15) to the right hand side. Therefore we define the commutator brackets

$$\begin{aligned} \left[\partial_r, \tfrac{1}{r}\right]_{-1} &:= 1, \\ \left[\partial_r, \tfrac{1}{r}\right]_{0} &:= \tfrac{1}{r}, \\ \left[\partial_r, \tfrac{1}{r}\right]_{1} &:= \tfrac{-1}{r^2}, \\ &\ldots \\ \left[\partial_r, \tfrac{1}{r}\right]_{n} &:= (-1)^n n! \tfrac{1}{r^{n+1}}. \end{aligned} \tag{16}$$

Now we decompose $(\partial_r \frac{1}{r})^n \delta$ into a number (A(k,n)) times the k^{th} derivative of δ with respect to its argument.

$$(\partial_r \frac{1}{r})^n \delta := \sum_{k=0}^{n} A(k,n) \delta^{(k)} \tag{17}$$

Using the definitions (16, 17) we find a recursive formula for (15). Knowing the Greens function for $d-2$ extra dimensions the Greens function for d extra dimensions is given by

$$\begin{aligned} G_{ret}^{3+d}(t,x) &= \tfrac{1}{4\pi r}(\tfrac{1}{2\pi})^{d/2}\sum_{i=0}^{d/2}\left(\sum_{l=0}^{d/2-i}\left|\left[\partial_r,\tfrac{1}{r}\right]_l\right|\mathrm{A}(l+i-1,d/2-1)\tfrac{(l+i)!}{l!i!}\right)\delta^{(i)}((t-t_0)-r) \\ &:= \tfrac{1}{4\pi r}(\tfrac{1}{2\pi})^{d/2}\sum_{i=0}^{d/2}K(r,i)\delta^{(i)}((t-t_0)-r). \end{aligned} \tag{18}$$

For the cases of $d=0,2,4,6$ the Greens functions are:

$$\begin{aligned} G_{ret}^{3}(t,x) &= \tfrac{\delta((t-t_0)-r)}{4\pi r}, \\ G_{ret}^{3+2}(t,x) &= \tfrac{(\delta((t-t_0)-r)+r\delta^{(1)}((t-t_0)-r))}{8\pi^2 r^3}, \\ G_{ret}^{3+4}(t,x) &= \tfrac{\delta^{(2)}((t-t_0)-r)r^2+3\delta^{(1)}((t-t_0)-r)r+3\delta((t-t_0)-r)}{16\pi^3 r^5}, \\ G_{ret}^{3+6}(t,x) &= \tfrac{\delta^{(3)}r^3+6\delta^{(2)}((t-t_0)-r)r^2+15\delta^{(1)}((t-t_0)-r)r+15\delta((t-t_0)-r)}{32\pi^4 r^7}. \end{aligned} \tag{19}$$

Lets assume that the observer ($|\underline{x}|$) is sitting far away in comparison with the extension of the source ($|\underline{y}|$). This means for $|\underline{x}| \gg |\underline{y}|$ that

$$\tau = t - t_0 - |\underline{x}-\underline{y}| \approx t - t_0 - |\underline{x}| + \underline{y}\frac{\underline{x}}{|\underline{x}|}. \tag{20}$$

Keeping this in mind (12) gives

$$\begin{aligned} h_{MN}(x) &= N\int dt_0\int d^3y_{\|}d^dy_{\perp}G_{ret}^{(3+d)}(t-t_0,|\underline{x}-\underline{y}|)S_{MN}(t_0,\underline{y}) \\ &= \int dt_0\int d^3y_{\|}d^dy_{\perp}\tfrac{N}{4\pi|\underline{x}-\underline{y}|}(\tfrac{1}{2\pi})^{d/2}\sum_{i=0}^{d/2}K(|\underline{x}-\underline{y}|,i)\delta^{(i)}(t-t_0-|\underline{x}-\underline{y}|)S_{MN}(t_0,\underline{y}). \end{aligned} \tag{21}$$

Partial integration with respect to t_0 shuffles the derivatives from the δ function to the source S

$$\begin{aligned} &h_{MN}(x) = \\ &\int dt_0\int d^3y_{\|}d^dy_{\perp}\tfrac{N}{4\pi|\underline{x}-\underline{y}|}(\tfrac{1}{2\pi})^{d/2}\sum_{i=0}^{d/2}K(|\underline{x}-\underline{y}|,i)\delta(t-t_0-|\underline{x}-\underline{y}|)(\tfrac{\partial}{\partial t_0})^i S_{MN}(t_0,\underline{y}). \end{aligned} \tag{22}$$

The delta function tells us at which time we have to evaluate $S_{MN}(t_0,\underline{y})$. The source term S is positive definite and can be expressed by its Fourier integral

$$S_{MN}(\tau,\underline{y}) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty}d\omega S_{MN}(\omega,\underline{y})e^{-i\omega\tau} + c.c. \tag{23}$$

Every derivative with respect to the time brings down a factor $-i\omega$ from (23). After using (20) and integrating out the δ functions this leads to

$$\begin{aligned} h_{MN}(x) &= N\frac{1}{\sqrt{2\pi}4\pi}\frac{1}{(2\pi)^{d/2}}\int d\omega \exp\left(-i\omega(t-|\underline{x}|)\right) \\ &\sum_{j=0}^{d/2} K(|\underline{x}-\underline{y}|,j)\int d^3y_{\parallel}d^dy_{\perp}\frac{1}{|\underline{x}|}(i\omega)^j S_{MN}(\omega,\underline{y})\exp\left(-i\omega\underline{y}\frac{\underline{x}}{|\underline{x}|}\right) \end{aligned} \tag{24}$$

The mono pole part of this gravitational wave is found by taking $\frac{|\underline{y}|}{|\underline{x}|} \ll 1$ and therefore to lowest order $\frac{1}{|\underline{x}-\underline{y}|^j} \approx \frac{1}{|\underline{x}|^j}$ and $K(|\underline{x}-\underline{y}|,j) \approx K(|\underline{x}|,j)$.

$$\begin{aligned} h^{(0)}_{MN}(x) &= N\frac{1}{\sqrt{2\pi}4\pi}\frac{1}{(2\pi)^{d/2}}\int d\omega \exp\left(-i\omega(t-|\underline{x}|)\right) \\ &\sum_{j=0}^{d/2} K(|\underline{x}|,j)\int d^3y_{\parallel}d^dy_{\perp}\frac{1}{|\underline{x}|}(i\omega)^j S_{MN}(\omega,\underline{y})\exp\left(-i\omega\underline{y}\frac{\underline{x}}{|\underline{x}|}\right) \\ &= \int d\omega \exp\left(-i\omega(t-|\underline{x}|)\right)e_{MN}(\underline{x},\omega). \end{aligned} \tag{25}$$

This looks like a plane wave solution. As the final result of this section it is shown that the polarisation tensor of the induced gravitational wave is given by

$$\begin{aligned} e_{MN}(\underline{x},\omega) &= N\frac{1}{4\pi}\frac{1}{(2\pi)^{d/2}\sqrt{2\pi}}\sum_{j=0}^{d/2} K(|\underline{x}|,j)\frac{1}{|\underline{x}|}(i\omega)^j\int d^3y_{\parallel}d^dy_{\perp}S_{MN}(\omega,\underline{y})\exp\left(-i\omega\underline{y}\frac{\underline{x}}{|\underline{x}|}\right)+c.c. \\ &= N\frac{1}{4\pi}\frac{1}{(2\pi)^{d/2}\sqrt{2\pi}}\sum_{j=0}^{d/2} K(|\underline{x}|,j)\frac{1}{|\underline{x}|}(i\omega)^j\hat{S}_{MN}(\omega)+c.c. \end{aligned} \tag{26}$$

The charge conjugated part is not shown explicitly, to keep the formula to a readable size, but of course it contributes to the polarisation tensor as well.
In section (III) we will explicitly calculate the source term $S_{MN}(\omega,\underline{y})$. When doing so it is useful to remember that the "time" coordinate corresponding to ω is τ from equation (20) and not t. One would also get this result by doing two more Fourier transformations on S and then performing a stationary phase analysis on the exponential functions in the integrand.

B. The energy and momentum of a gravitational wave

In this subsection we derive the energy momentum tensor t_{MN} of the gravitational wave given in (25). When we derived (6) we just took the first order of h in R_{MN}. Considering the approximate solution (12) in the complete field equations (1) we will find the energy momentum tensor t_{MN} of the gravitational wave (12). Expanding and rearranging (1) with

$R^{(1)}_{MN} - \frac{1}{2}\eta_{MN} R^{(1)}$ gives

$$R^{(1)}_{MN} - \frac{1}{2}\eta_{MN} R^{(1)} = -8\pi G \left[T_{MN} + \frac{1}{8\pi G}(R_{MN} - \frac{1}{2}\eta_{MN} R - R^{(1)}_{MN} \frac{1}{2}\eta_{MN} R^{(1)}) \right]. \tag{27}$$

Now we can define the energy momentum tensor of (21)

$$t_{MN} := \frac{1}{8\pi G}(R_{MN} - \frac{1}{2}\eta_{MN} R - R^{(1)}_{MN} \frac{1}{2}\eta_{MN} R^{(1)}) \tag{28}$$

and the total energy momentum tensor

$$\tau_{MN} = T_{MN} + t_{MN} \tag{29}$$

of the gravitational wave. The total energy momentum tensor in (29) has now two parts, the energy momentum tensor of the source T_{MN} and the energy momentum tensor t_{MN} of the propagating wave itself. In order to calculate (28) we need to expand the $(3+d)$ dimensional Riemann tensor (7) to 2^{nd} order in h. Therefore we note

$$\begin{aligned} R_{AB} &= R^{(1)}_{AB} + R^{(2)}_{AB} + \mathcal{O}(h^3), \\ R &= g^{AB} R_{AB} \\ &= \eta^{AB} R^{(1)}_{AB} + \eta^{AB} R^{(2)}_{AB} + h^{AB} R^{(1)}_{AB} + \mathcal{O}(h^3), \\ R^{A(1)}_{A} &= \eta^{AB} R^{(1)}_{AB}. \end{aligned} \tag{30}$$

Using these relations, (28) takes the form

$$t_{MN} = \frac{1}{8\pi G} \left[R^{(2)}_{MN} - \frac{1}{2} h_{MN} R^{(1)} - \frac{1}{2}\eta_{MN}\eta^{AB} R^{(2)}_{AB} - \frac{1}{2}\eta_{MN} h^{CD} R^{(1)}_{CD} \right] + \mathcal{O}(h^3). \tag{31}$$

For the freely propagating gravitational wave, the metric $g_{MN} = \eta_{MN} + h_{MN}$ satisfies the first-order Einstein equation $R^{(1)}_{MN} = 0$. The first order terms in (31) drop out and (31) simplifies to

$$t_{MN} = \frac{1}{8\pi G} \left[R^{(2)}_{MN} - \frac{1}{2}\eta_{MN}\eta^{AB} R^{(2)}_{AB} \right] + \mathcal{O}(h^3). \tag{32}$$

The challenge is now to derive the h dependence of $R^{(2)}_{MN}$. First we check the h^2 dependence of X_{MN} in (7). As the Christoffel symbols (Γ) in X_{MN} (see (7)) contain derivatives of the metric G_{MN} and X_{MN} is proportional to Γ^2, the second order part of X_{MN} contains only terms of the form $(\partial h)(\partial h)$ in particular

$$\begin{aligned} X^{(2)}_{MN} = \tfrac{1}{4}\ \Big[&(2\partial_L h^L_S - \partial_S h^{\grave{L}}_L)(\partial_M h^S_P + \partial_M h^S_P - \partial^S h_{MN}) \\ &-(\partial_N h^L_S + \partial^L h_{NP} - \partial_S h^L_N)(\partial_M h^S_L + \partial_L h^S_M - \partial^S h_{ML})\Big] \end{aligned} \tag{33}$$

The first part of (7) contributes terms proportional to h times second derivatives of h, in particular

$$\begin{aligned} R^{(2)}_{MN}\Big|_{\text{first part}} &= h^{LS} R^{(1)}_{MLSN} \\ &= \tfrac{1}{2} h^{LS}(\partial_M\partial_N h_{LS} - \partial_L\partial_M h_{NS} - \partial_S\partial_N h_{ML} + \partial_L\partial_S h_{MN}) \end{aligned} \tag{34}$$

Putting (33) and (34) together we find the second order of R in h

$$\begin{aligned} R^{(2)}_{MN} \ &=\ R^{(2)}_{MN}\Big|_{\text{first part}} + X^{(2)}_{MN} \\ &=\ \tfrac{1}{2} h^{LS}(\partial_M\partial_N h_{LS} - \partial_L\partial_M h_{NS} - \partial_S\partial_N h_{ML} + \partial_L\partial_S h_{MN}) + \\ &\quad \Big[(2\partial_L h^L_S - \partial_S h^{\grave{L}}_L)(\partial_M h^S_P + \partial_M h^S_P - \partial^S h_{MN}) \\ &\quad -(\partial_N h^L_S + \partial^L h_{NP} - \partial_S h^L_N)(\partial_M h^S_L + \partial_L h^S_M - \partial^S h_{ML})\Big]\,. \end{aligned} \tag{35}$$

Now we can use the plane wave solution (24) and plug it respectively into (35) and (32). The result will be quite lengthy and depends on some phase factors from (32). By averaging over a spatial region, large compared to $1/|k|$ we can integrate out these phase factors and simplify the result. The average is indicated by the $\langle\rangle$ brackets. If we also remember that $k_L k^L = 0$ and that we are free to choose the harmonic coordinate system condition (10) we find that the trace part of (32) vanishes. Finally, we obtain the averaged energy momentum tensor of a plane gravitational wave

$$\begin{aligned} \langle t_{MN}\rangle\ &= \langle R^{(2)}_{MN}\rangle \\ &= \tfrac{k_M k_N}{16\pi G}(\langle e^{SL*}(\underline{x},\tau) e_{SL}(\underline{x},\tau)\rangle - \tfrac{1}{2}|\langle e^L_L\rangle|^2), \end{aligned} \tag{36}$$

in dependence of the polarisation tensor e_{MN}. This polarisation tensor again depends on the energy momentum tensor of a given source. Such a source tensor for elastic collisions is derived in the next section.

III. ENERGY MOMENTUM TENSOR OF AN ELASTIC COLLISION

In this section we focus on the energy momentum tensor of colliding standard model particles. This tensor is needed, because it enters the source term for the gravitational wave (23). In the ADD model [12] all standard model particles are confined to the brane and the total energy momentum tensor for one of these particles can be defined using a delta function [13, 14]

$$T_{MN}(x) = \eta^{\mu}_{M}\eta^{\nu}_{N}T_{\mu\nu}(x)\delta^{d}(x_{\perp}). \tag{37}$$

In other models like those with universal extra dimensions [15, 16] this delta function restriction is not needed, but the discussion made in this section is easily translated to models of this type as well. One can decompose the energy momentum tensor into an incoming and outgoing part

$$T_{MN} = T^{(in)}_{MN} + T^{(out)}_{MN}. \tag{38}$$

The incoming and outgoing energy momentum tensors are given in terms of the 4 momenta of C colliding particles

$$\begin{aligned} T^{(in)}_{MN} &= \delta^{(d)}(x_{\perp})\eta_{M\mu}\eta_{N\nu}\sum_{j=1}^{C}\frac{P^{\mu}_{(j)}P^{\nu}_{(j)}}{P^{0}_{(j)}}\delta^{(3)}(x_{\parallel}-v_{(j)\parallel}t)\theta(-t) \\ &=: \delta^{(d)}(x_{\perp})\eta_{M\mu}\eta_{N\nu}T^{\mu\nu}_{(in)} \\ T^{(out)}_{MN} &= \delta^{(d)}(x_{\perp})\eta_{M\mu}\eta_{N\nu}\sum_{j=1}^{C}\frac{P^{\mu}_{(j)}P^{\nu}_{(j)}}{P^{0}_{(j)}}\delta^{(3)}(x_{\parallel}-v_{(j)\parallel}t)\theta(t) \\ &=: \delta^{(d)}(x_{\perp})\eta_{M\mu}\eta_{N\nu}T^{\mu\nu}_{(out)}. \end{aligned} \tag{39}$$

The source term (4) for incoming states gives

$$\begin{aligned} S^{(in)}_{MN}(t,x) &= T^{(in)}_{MN} - \frac{1}{2+d}\eta_{MN}T^{(in)L}_{L} \\ &= \delta^{(d)}(x_{\perp})(\eta_{M\mu}\eta_{N\nu} - \frac{1}{2+d}\eta_{MN}\eta_{\mu\nu})T^{\mu\nu}_{(in)}(t,x_{\parallel}). \end{aligned} \tag{40}$$

The incoming and the outgoing S_{MN} will now be the used as a source term for the induced gravitational wave (25). In order to know the polarisation tensor of the wave (25) we have to perform this $3+d$ dimensional y integral

$$\begin{aligned} \hat{S}_{MN}(\omega) &= \int d^3y_{\parallel}d^dy_{\perp}S_{MN}(\omega,y)\exp\left(-i\omega \underline{y}\frac{\underline{x}}{|\underline{x}|}\right) \\ &= \frac{1}{\sqrt{2\pi}}\int d\tilde{t}\int d^3y_{\parallel}d^dy_{\perp}S_{MN}(\tilde{t},y)\exp\left(-i\omega(\tilde{t}+\underline{y}\frac{\underline{x}}{|\underline{x}|})\right). \end{aligned} \tag{41}$$

For the incoming particles (as well as for the outgoing particles) the delta function in (40) helps us to do this integral and the last part of (41) reads

$$\begin{aligned}\int d^3y_\| d^dy_\perp \;\; & S_{MN}(\tilde{t},y)\exp\left(-i\omega \underline{y}\tfrac{\underline{x}}{|\underline{x}|}\right) = \int d^3y_\| d^dy_\perp S^{(in)}_{MN}(\tau,\underline{y})\exp\left(-i\omega \underline{y}\tfrac{\underline{x}}{|\underline{x}|}\right) \\ &= \left(\eta_{M\mu}\eta_{N\nu}-\tfrac{1}{2+d}\eta_{MN}\eta_{\mu\nu}\right)\sum_{j=1}^{C}\frac{P^\mu_{(j)}P^\nu_{(j)}}{P^0_{(j)}}\int d^3y_\|\delta^{(3)}((y_\|-v_{(j)\|}\tau)\theta(-\tau))\exp\left(-i\omega \underline{y}\tfrac{\underline{x}}{|\underline{x}|}\right) \\ &=: \left(\eta_{M\mu}\eta_{N\nu}-\tfrac{1}{2+d}\eta_{MN}\eta_{\mu\nu}\right)\sum_{j=1}^{C}\frac{P^\mu_{(j)}P^\nu_{(j)}}{P^0_{(j)}}J^{(in)}.\end{aligned} \tag{42}$$

After some transformations, $J^{(in)}$ can be brought to a form compatible with the Fourier decomposition of h_{MN}:

$$\begin{aligned}J^{(in)} &= \int d^3y_\|\delta^{(3)}((y_\|-v_{(j)\|}\tau)\theta(-\tau))\exp\left(-i\omega \underline{y}\tfrac{\underline{x}}{|\underline{x}|}\right) \\ &= \int d^3y_\| \int \frac{d^3k_\|}{(2\pi)^3}\exp\left(ik_\|(y_\|-v_{(j)}\tau)\right)\int\frac{d\omega_0}{-2\pi i}\frac{e^{-i\omega_0\tau}e^{-i\omega \underline{y}\frac{\underline{x}}{|\underline{x}|}}}{\omega_0-i\epsilon} \\ &= \int d^3y_\| \int \frac{d^3k_\|}{(2\pi)^3}\exp\left(ik_\|(y_\|)\right)\int\frac{d\omega_0}{-2\pi i}\frac{e^{-i(\omega_0+v_{(j)}k_\|)\tau}e^{-i\omega \underline{y}\frac{\underline{x}}{|\underline{x}|}}}{\omega_0-i\epsilon} \\ &= \int d^3y_\| \int \frac{d^3k_\|}{(2\pi)^3}\exp\left(ik_\|(y_\|)\right)\int\frac{d\tilde{\omega}}{-2\pi i}\frac{e^{-i(\tilde{\omega})\tau}e^{-i\omega \underline{y}\frac{\underline{x}}{|\underline{x}|}}}{\tilde{\omega}-k_\|v_{(j)}-i\epsilon} \\ &= \int d\tilde{\omega}e^{-i\tilde{\omega}\tau}\int\frac{d^3k_\|}{-i(2\pi)^4}\int d^3y_\|\frac{e^{ik_\|y_\|}e^{-i\omega \underline{y}\frac{\underline{x}}{|\underline{x}|}}}{\tilde{\omega}-k_\|v_{(j)}-i\epsilon}.\end{aligned} \tag{43}$$

Here, first the Fourier transform of the δ and θ function is used, then the terms under the integrals are rearranged and the substitution $\tilde{\omega} := \omega_0 + k_\| v_{(j)}$ is made. Now the Fourier definition of the δ function is used in order to get rid of the two three dimensional integrals

$$\begin{aligned}J^{(in)} &= \int d\tilde{\omega}e^{-i\tilde{\omega}\tau}\int\frac{d^3k_\|}{-i(2\pi)^4}\frac{1}{\tilde{\omega}-k_\|v_{(j)}-i\epsilon}\int d^3y_\|\exp\left(-i(\omega\tfrac{x_\|}{|x_\||}-k_\|)y_\|\right) \\ &= \int d\tilde{\omega}e^{-i\tilde{\omega}\tau}\frac{1}{-i2\pi}\frac{1}{\tilde{\omega}-\underline{k}v_{(j)}-i\epsilon}.\end{aligned} \tag{44}$$

From $\underline{k}v_{(j)} = k_\| v_{(j)}$ we see that $k_\|$ can be replaced by $\underline{k}$. For outgoing particles the procedure is the same, one just has to use the Fourier transform of $\theta(-t)$

$$J^{(out)} = -\int d\tilde{\omega}e^{-i\tilde{\omega}\tau}\frac{1}{-i2\pi}\frac{1}{\tilde{\omega}-\underline{k}v_{(j)}+i\epsilon}. \tag{45}$$

We see that the difference between the incoming and outgoing J can be expressed by a change of the sign of J and ϵ. These results can be plugged back into (42). For high energetic particles the denominator is $P^0_{(j)}(\omega-\underline{k}v_{(j)}) = k\cdot P_{(j)}$. As this is > 0 we can drop the ϵ. Using (41, 42, 44, 45) one sees that the source terms are given by:

$$\hat{S}^{(in)}_{MN}(\omega) =: (\hat{T}^{(in)}_{MN} - \eta_{MN}\hat{T}^{(in)L}_{L}) = (\eta_{M\mu}\eta_{N\nu} - \frac{1}{2+d}\eta_{MN}\eta_{\mu\nu}) \sum_{j=1}^{C} \frac{P^{\mu}_{(j)}P^{\nu}_{(j)}}{P_{(j)}k} \tag{46}$$

For the outgoing particles this reads

$$\hat{S}^{(out)}_{MN}(\omega) =: (\hat{T}^{(out)}_{MN} - \eta_{MN}\hat{T}^{(out)L}_{L}) = -(\eta_{M\mu}\eta_{N\nu} - \frac{1}{2+d}\eta_{MN}\eta_{\mu\nu}) \sum_{j=1}^{C} \frac{P^{\mu}_{(j)}P^{\nu}_{(j)}}{P_{(j)}k}. \tag{47}$$

IV. GRAVITATIONAL RADIATION FROM ELASTIC SCATTERING

Based on the discussion in (II) and (III) we will now calculate the classically radiated energy into gravitational waves from an elastic scattering.

A. Radiated energy and the energy momentum tensor

The momentum P^i of an extended object is defined as the volume integral over the density of the t^{0i} component of the energy momentum tensor. In $3+d$ dimensions this is

$$P^i = \int_V d^{3+d}x t^{0i}. \tag{48}$$

The energy change in time $\frac{dE}{d\tau}$ of a system can be rewritten by using the conservation of the energy momentum tensor

$$\frac{dE}{d\tau} = \int_V d^{3+d}x\, \partial_0 t^{00} = \int_V d^{3+d}x\, \partial_i t^{0i} = \partial_i P^i. \tag{49}$$

Applying Gauss law to $\partial_i P^i$ and using (48) gives

$$\partial_i P^i = \int_V d^{3+d}x\, \partial_i P^i = \int_{\mathcal{O}(V)} dS\, n_i t^{0i} = \int_{\mathcal{O}(V_E)} d\Omega\, |x|^{2+d} n_i t^{0i}. \tag{50}$$

By differentiating (49) after $d\Omega$, averaging over the space and integrating over $d\tau$ we get from (50) the average energy radiated into the space-segment $d\Omega$

$$\frac{d\langle E\rangle}{d\Omega} = \int d\tau \frac{\langle \partial_i P^i\rangle}{d\Omega} = \int d\tau |x|^{2+d} n_i \langle t^{0i}\rangle. \tag{51}$$

B. Radiated gravitational energy

Using the general relation between radiated energy and the energy momentum tensor t^{MN} (see (IV A)) we will now quantify how much energy is radiated away by a gravitational wave. Therefore one has to plug the energy momentum tensor of this wave (36) into equation (51). In the Fourier formulation of (51) we use (36) and $k_0^2 = k_i^2 = \omega$

$$\begin{aligned} \frac{dE}{d\Omega} &= \frac{1}{2\pi} \int\int\int d\tau d\tilde{\omega} d\omega |\underline{x}|^{2+d} \frac{\tilde{\omega}\omega}{16\pi G} (\langle e^{SL*}(\underline{x},\omega) e_{SL}(\underline{x},\tilde{\omega})\rangle - \tfrac{1}{2}\langle e_L^{L*}(\underline{x},\omega)\rangle\langle e_L^L(\underline{x},\tilde{\omega})\rangle) e^{i\tau(\tilde{\omega}-\omega)} \\ &= \int d\omega |\underline{x}|^{2+d} \frac{\omega^2}{16\pi G} (\langle e^{SL*}(\underline{x},\omega) e_{SL}(\underline{x},\omega)\rangle - \tfrac{1}{2}|\langle e_L^L(\underline{x},\omega)\rangle|^2). \end{aligned} \tag{52}$$

Now we can bring the $d\omega$ to the left side and get

$$\begin{aligned} \frac{dE}{d\Omega d\omega} &= |x|^{2+d} n_i \langle t^{0i}\rangle \\ &= |x|^{2+d} \frac{\omega^2}{16\pi} (\langle e^{SL*}(\underline{x},\omega) e_{SL}(\underline{x},\omega)\rangle - \tfrac{1}{2}|\langle e_L^L\rangle|^2). \end{aligned} \tag{53}$$

We use the relation $\omega = |k^0| = |n_i k^i|$. From (26 we get the polarisation tensors e_{MN} of the radiated gravitational wave,

$$\langle e_{MN}(\underline{x},\omega)\rangle = N \frac{1}{4\pi} \frac{1}{(2\pi)^{d/2}\sqrt{2\pi}} \hat{S}_{MN}(\omega) \langle \textstyle\sum_{j=0}^{d/2} K(|\underline{x}|, j) \frac{1}{|\underline{x}|} (i\omega)^j \rangle. \tag{54}$$

Here we define $\hat{S}_{MN}(\omega) := (\hat{T}_{MN}(\omega) - \frac{1}{2+d}\eta_{MN}\hat{T}_L^L(\omega))$, which is the Fourier transform of the $(\hat{S}_{MN}(\tau)^{(in)} + \hat{S}_{MN}(\tau)^{(out)})$ we know from equation (42). Let us first calculate the $e^{MN}e^*_{MN}$ part of (53) by using (54)

$$\begin{aligned} e^{SL*}(\underline{x},\omega) e_{SL}(\underline{x},\omega) &= \frac{N^2}{32\pi(2\pi)^d} \sum_{j,k=0}^{d/2} \langle K(|\underline{x}|, j) K(|\underline{x}|, k) \frac{1}{|\underline{x}|^2} (i\omega)^{j+k}\rangle\, \hat{S}^{SL}(\omega) \hat{S}^*_{SL}(\omega) \\ &= \frac{8G^2}{\pi(2\pi)^d} \sum_{j,k=0}^{d/2} \langle K(|\underline{x}|, j) K(|\underline{x}|, k) \frac{1}{|\underline{x}|^2} (i\omega)^{j+k}\rangle \\ &\qquad (\hat{T}^{SL}(\omega)\hat{T}^*_{SL}(\omega) - \frac{d}{(2+d)^2}|T_K^K|^2). \end{aligned} \tag{55}$$

Proceeding the same way with $|e_L^L|^2$ we find

$$\begin{aligned} |e_L^L|^2 &= \frac{8G^2}{\pi(2\pi)^d} \sum_{j,k=0}^{d/2} \langle K(|\underline{x}|, j) K(|\underline{x}|, k) \frac{1}{|\underline{x}|^2} (i\omega)^{j+k}\rangle\, \hat{S}_N^N \hat{S}_L^{L*} \\ &= \frac{8G^2}{\pi(2\pi)^d} \sum_{j,k=0}^{d/2} \langle K(|\underline{x}|, j) K(|\underline{x}|, k) \frac{1}{|\underline{x}|^2} (i\omega)^{j+k}\rangle\, |T_L^L|^2 (\tfrac{2}{2+d})^2. \end{aligned} \tag{56}$$

Evaluating the T terms in (56, 55) separately leads to

$$\hat{T}^{SL}\hat{T}^*_{SL} = (\hat{T}^{(in)SL} + \hat{T}^{(out)SL})(\hat{T}^{(in)*}_{SL} + \hat{T}^{(out)*}_{SL}). \tag{57}$$

In the notation of (46, 47) this will be rather lengthy. But we can take Sums ($\sum_I ..$) over all involved states instead of initial and final states separately ($\sum_i .. + \sum_j ..$) and use that every outgoing state brings one $-$ sign. After defining

$$\eta_I = \begin{cases} +1 \ I \ \text{ in initial state} \\ -1 \ I \ \text{ in final state,} \end{cases} \tag{58}$$

we have

$$\hat{T}^{MN} = (\eta^{M\mu}\eta^{N\nu}\sum_I \frac{P_{(I)\mu}P_{(I)\nu}\eta_I}{kP_{(I)}}). \tag{59}$$

In this notation we find that

$$\hat{T}^{SL}\hat{T}^*_{SL} = \sum_{I,J} \frac{(P^\mu_{(I)}P_{(J)\mu})^2\eta_I\eta_J}{(P_{(I)}k)(P_{(J)}k)}, \tag{60}$$

and that

$$\hat{T}^L_L\hat{T}^{S*}_S = \sum_{I,J} \frac{P^2_{(I)}\,P^2_{(J)}\eta_I\eta_J}{(P_{(I)}k)(P_{(J)}k)}. \tag{61}$$

The last two equations can be put into (55 and 53) to derive the energy carried away by induced gravitational radiation

$$\begin{array}{rl} \frac{dE}{d\Omega d\omega} = & \frac{G\left|\underline{x}^{2+d}\right|}{2\pi^2(2\pi)^d}\sum_{j,k=0}^{d/2}\langle K(|\underline{x}|,j)K(|\underline{x}|,k)\frac{1}{|\underline{x}|^2}(i\omega)^{j+k}\rangle \\ & \sum_{I,J}\frac{\eta_I\eta_J}{(P_{(I)}k)(P_{(J)}k)}\left[(P^\mu_{(I)}P_{(J)\mu})^2 - \frac{1}{2+d}P^2_{(I)}\,P^2_{(J)}\right] \end{array}. \tag{62}$$

In the second step all the simplifying definitions (56, 55, 18) are used . The $(3+d)$ dimensional gravitational constant G has a (d) dependent mass dimension. This becomes more obvious by the definition of the coupling G through a mass scale M_f

$$G = \frac{1}{M_f^{2+d}}. \tag{63}$$

In the case of $d = 0$ this gives $G = \frac{1}{M_f^2} = \frac{1}{M_P^2}$ which is the definition of the Planck mass. For the cases with even extra dimensions $d = 0, 2, 4, 6$ equation (62) gives

$$\begin{aligned}
\frac{dE(d=0)}{d\Omega d\omega} &= \frac{1}{M_P^2}\frac{1}{2\pi^2}\omega^2 \sum_{I,J} \frac{\eta_I \eta_J}{(P_{(I)}k)(P_{(J)}k)} \left[(P^{\mu}_{(I)}P_{(J)\mu})^2 - \frac{1}{2}P^2_{(I)}\,P^2_{(J)}\right] \\
\frac{dE(d=2)}{d\Omega d\omega} &= \frac{1}{M_f^4}\frac{1}{8\pi^4}\left(\omega^4 + 2\frac{\omega^3}{|\underline{x}|} + \frac{\omega^2}{|\underline{x}|^2}\right) \sum_{I,J} \frac{\eta_I \eta_J}{(P_{(I)}k)(P_{(J)}k)} \left[(P^{\mu}_{(I)}P_{(J)\mu})^2 - \frac{1}{4}P^2_{(I)}\,P^2_{(J)}\right] \\
\frac{dE(d=4)}{d\Omega d\omega} &= \frac{1}{M_f^6}\frac{1}{32\pi^6}\left(\omega^6 + 6\frac{\omega^5}{|\underline{x}|} + 15\frac{\omega^4}{|\underline{x}|^2} + 18\frac{\omega^3}{|\underline{x}|^3} + 9\frac{\omega^2}{|\underline{x}|^4}\right) \\
&\quad \sum_{I,J} \frac{\eta_I \eta_J}{(P_{(I)}k)(P_{(J)}k)} \left[(P^{\mu}_{(I)}P_{(J)\mu})^2 - \frac{1}{6}P^2_{(I)}\,P^2_{(J)}\right] \\
\frac{dE(d=6)}{d\Omega d\omega} &= \frac{1}{M_f^8}\frac{1}{128\pi^8}\left(\omega^8 + 12\frac{\omega^7}{|\underline{x}|} + 66\frac{\omega^6}{|\underline{x}|^2} + 210\frac{\omega^5}{|\underline{x}|^3} + 405\frac{\omega^4}{|\underline{x}|^4} + 450\frac{\omega^3}{|\underline{x}|^5} + 225\frac{\omega^2}{|\underline{x}|^6}\right) \\
&\quad \sum_{I,J} \frac{\eta_I \eta_J \left[(P^{\mu}_{(I)}P_{(J)\mu})^2 - \frac{1}{8}P^2_{(I)}\,P^2_{(J)}\right]}{(P_{(I)}k)(P_{(J)}k)}.
\end{aligned} \tag{64}$$

C. Interpretation and physical relevance of the obtained cross sections

In the limit of no extra dimensions, (64) agrees with [17]. For $d \neq 0$ there are several terms contributing: There is always one with a ω^{d+2} dependence and there are terms with the same mass-dimension, but containing a conspicuous looking $|\underline{x}|$ dependence $\frac{\omega^{d+2-i}}{|\underline{x}|^i}$. For a uncompactified $4+d$ dimensional space-time for a distant observer those terms vanish and only the ω^{2+d} term survives.

For compactified Large Extra Dimensions we start from the following setup: The collision region for massive particles or black holes is smaller than the compactification radius R. For $\underline{x} < R$ equation (64) holds and the $|\underline{x}|$ terms get weaker and weaker with distance. But when the distance $|\underline{x}|$ reaches R, the attenuation of those terms stops as the world starts to look again four dimensional. So for a given frequency ω they can be replaced by $\frac{\omega^{d+2-i}}{R^i}$. In the ADD model [12] the radius is related to the Planck-mass M_P and the new fundamental mass scale M_f by

$$M_P^2 = M_f^{2+d} R^d. \tag{65}$$

Using this relation we can estimate for which kind of scenarios the new terms become relevant. As the radiated energy is increasing rapidly with ω some cut off has to be used to estimate the amount of gravitationally radiated energy. In a 2 to 2 particle process emitting gravitational radiation this cut is at least reached as soon as the gravitational radiation takes away the invariant energy $\sqrt{s}/2$ of one of the participants. Strongest suppression of the $1/R$ terms is reached when we take this extreme value for ω. Limits on the compactification radius down to the μm range (depending on d) have been derived from a large number of physical observations [18, 19, 20, 21]. Under the condition of

$$\omega \gg \frac{1}{R} \text{ or } \sqrt{s} \gg 2M_f \left(\frac{M_f}{M_P}\right)^{2/d}, \tag{66}$$

equation (64) gives the original result from [9]. This shows that the additional terms only play a role for small $\sqrt{s}$ or very large M_f. On the one hand for particle scattering with invariant energy in the TeV range, M_f would have to be up to 1000 TeV, for the new terms to be relevant. On the other hand the whole cross-section is suppressed by a factor $1/M_f^{2+d}$ and would be negligible then. Summarising one can say that for elastic high energy N to N particle collisions in models with large extra dimensions the energy loss into gravitational radiation stays as described in [9]

$$\frac{dE}{d\Omega d\omega} = \frac{1}{M_f^{2+d}} \frac{\omega^{2+d}}{2(\pi)^2(2\pi)^d} \sum_{I,J} \frac{\eta_I \eta_J}{(P_{(I)}k)(P_{(J)}k)} \left[(P^{\mu}_{(I)} P_{(J)\mu})^2 - \frac{1}{2+d} P^2_{(I)} P^2_{(J)} \right]. \tag{67}$$

This result is valid for elastic N→N particle scattering with high particle velocities so that the interaction can be approximated to be instantaneous. Equation (64) was derived from classical general relativity and gives an quantitative idea for the gravitationally radiated energy. A quantum calculation for example in the ADD model was not yet performed, but is considered to be the next step to do.

V. SUMMARY

The main concern of this paper was to derive the general energy loss formula due to gravitational radiation in models with extra dimensions that are compactified on a radius R

$$\begin{aligned}
\frac{dE(d=0)}{d\Omega d\omega} &= \frac{1}{M_P^2}\frac{1}{2\pi^2}\omega^2 \sum_{I,J} \frac{\eta_I\eta_J}{(P_{(I)}k)(P_{(J)}k)}\left[(P^\mu_{(I)}P_{(J)\mu})^2 - \frac{1}{2}P^2_{(I)}\,P^2_{(J)}\right] \\
\frac{dE(d=2)}{d\Omega d\omega} &= \frac{1}{M_f^4}\frac{1}{8\pi^4}\left(\omega^4 + 2\frac{\omega^3}{|R|} + \frac{\omega^2}{|R|^2}\right)\sum_{I,J} \frac{\eta_I\eta_J}{(P_{(I)}k)(P_{(J)}k)}\left[(P^\mu_{(I)}P_{(J)\mu})^2 - \frac{1}{4}P^2_{(I)}\,P^2_{(J)}\right] \\
\frac{dE(d=4)}{d\Omega d\omega} &= \frac{1}{M_f^6}\frac{1}{32\pi^6}\left(\omega^6 + 6\frac{\omega^5}{|R|} + 15\frac{\omega^4}{|R|^2} + 18\frac{\omega^3}{|R|^3} + 9\frac{\omega^2}{|R|^4}\right)\sum_{I,J} \frac{\eta_I\eta_J}{(P_{(I)}k)(P_{(J)}k)}\left[(P^\mu_{(I)}P_{(J)\mu})^2 - \frac{1}{6}P^2_{(I)}\,P^2_{(J)}\right] \\
\frac{dE(d=6)}{d\Omega d\omega} &= \frac{1}{M_f^8}\frac{1}{128\pi^8}\left(\omega^8 + 12\frac{\omega^7}{|R|} + 66\frac{\omega^6}{|R|^2} + 210\frac{\omega^5}{|R|^3} + 405\frac{\omega^4}{|R|^4} + 450\frac{\omega^3}{|R|^5} + 225\frac{\omega^2}{|R|^6}\right) \\
&\quad \sum_{I,J} \frac{\eta_I\eta_J\left[(P^\mu_{(I)}P_{(J)\mu})^2 - \frac{1}{8}P^2_{(I)}\,P^2_{(J)}\right]}{(P_{(I)}k)(P_{(J)}k)}.
\end{aligned} \tag{68}$$

Then we showed that for models with large compactification radii (compared to the wave length of the gravitational radiation) this goes into

$$\frac{dE}{d\Omega d\omega} = \frac{1}{M_f^{2+d}}\frac{\omega^{2+d}}{2(\pi)^2(2\pi)^d}\sum_{I,J}\frac{\eta_I\eta_J}{(P_{(I)}k)(P_{(J)}k)}\left[(P^\mu_{(I)}P_{(J)\mu})^2 - \frac{1}{2+d}P^2_{(I)}\,P^2_{(J)}\right], \tag{69}$$

in line with Ref. [9]. For small compactification radii (and therefore large M_f) the overall $\frac{1}{M_f^{2+d}}$ factor strongly suppresses all terms.

Acknowledgments

The authors thank S. Hofmann, U. Harbach and S. Hossenfelder for fruitful discussions and the Frankfurt International Graduate School of Science (FIGSS) for financial support through a PhD fellowship.

---

[1] E. Witten, Nucl. Phys. **B443**, 85 (1995), hep-th/9503124.

[2] P. Horava and E. Witten, Nucl. Phys. **B460**, 506 (1996), hep-th/9510209.

[3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999), hep-th/9906064.

[4] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999), hep-ph/9905221.

[5] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Rev. **D59**, 086004 (1999), hep-ph/9807344.

[6] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. **B436**, 257 (1998), hep-ph/9804398.

[7] H. Kodama and K. Uzawa (2005), hep-th/0512104.

[8] A. Abramovici et al., Science **256**, 325 (1992).

[9] V. Cardoso, O. J. C. Dias, and J. P. S. Lemos, Phys. Rev. **D67**, 064026 (2003), hep-th/0212168.

[10] S. Hassani, Springer-Verlag, New York (1972).

[11] D. V. Galtsov, Phys. Rev. **D66**, 025016 (2002), hep-th/0112110.

[12] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, Phys. Lett. **B429**, 263 (1998), hep-ph/9803315.

[13] G. F. Giudice, R. Rattazzi, and J. D. Wells, Nucl. Phys. **B544**, 3 (1999), hep-ph/9811291.

[14] T. Han, J. D. Lykken, and R.-J. Zhang, Phys. Rev. **D59**, 105006 (1999), hep-ph/9811350.

[15] I. Antoniadis, Phys. Lett. **B246**, 377 (1990).

[16] J. D. Lykken, Phys. Rev. **D54**, 3693 (1996), hep-th/9603133.

[17] S. Weinberg, ISBN 0-471-92567-5 (1972).

[18] L. Anchordoqui, T. Paul, S. Reucroft, and J. Swain, Int. J. Mod. Phys. **A18**, 2229 (2003), hep-ph/0206072.

[19] S. Hannestad and G. G. Raffelt, Phys. Rev. Lett. **88**, 071301 (2002), hep-ph/0110067.

[20] J. D. Barrow, Phys. Rev. **D35**, 1805 (1987).

[21] S. Hossenfelder et al., Phys. Lett. **B575**, 85 (2003), hep-th/0305262.