Spin(9)-invariant valuations

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Dekan :	Prof. Dr. Thorsten Theobald
Gutachter :	Prof. Dr. Andreas Bernig Prof. Dr. Gil Solanes
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Abstract

The aim of this thesis is to compute the dimension of the space $\mathrm{Val}^{\mathrm{Spin}(9)}$ of Spin(9)-invariant valuations on $\mathbb{R}^{16}$.
McMullen's theorem allows us to decompose this space in its homogeneous components. We can then study them separately. It is done through the construction of the following exact sequence

$$0 \longrightarrow (\Lambda^k V)^{\mathrm{Spin}(9)} \xrightarrow{d_Q} \Omega_p^{k,0}(SV)^{\mathrm{Spin}(9)} \xrightarrow{\quad d_Q \quad} \dots$$

$$\dots \xrightarrow{\quad d_Q \quad} \Omega_p^{k,n-k-1}(SV)^{\mathrm{Spin}(9)} \xrightarrow{nc} Val_k^{\mathrm{Spin}(9)} \longrightarrow 0.$$

A large part of this work is then the computation of the dimensions of the spaces $(\Lambda^k V)^{\mathrm{Spin}(9)}$ and $\Omega_p^{k,l}(SV)^{\mathrm{Spin}(9)}$.
In order to do that, we present a description of the action of Spin(9) on $\mathbb{R}^{16}$. It uses the properties of the 8-dimensional division algebra $\mathbb{O}$ of the octonions to decompose the Lie algebra so(9) of Spin(9). The operation of the components on $\mathbb{O}^2$ is then explicitely given.

Using this description as well as representation-theoretic formulas, we can compute the dimensions of $(\Lambda^k V)^{\mathrm{Spin}(9)}$ and $\Omega_p^{k,l}(SV)^{\mathrm{Spin}(9)}$.
Hence we obtain the following dimensions of $\mathrm{Val}_k^{\mathrm{Spin}(9)}$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim \mathrm{Val}_k^{\mathrm{Spin}(9)}$	1	1	2	3	6	10	15	20	27	20	15	10	6	3	2	1	1

It is clear, that the k-th intrinsic volume μ_k belongs to $\mathrm{Val}_k^{\mathrm{Spin}(9)}$. The last chapter of this work present the construction of a new element of $\mathrm{Val}_2^{\mathrm{Spin}(9)}$, which is linearly independent of μ_2.
On a Riemannian manifold (M, g), we construct a valuation by integrating the curvature tensor on the disc bundle. We associate to this valuation on M a family of valuation on the tangent spaces. We show, that this valuations are even and homogeneous of degree 2. Moreover, since the valuation on M is invariant under the action of the isometry group $Isom(M, g)$ of M, the induced valuation on T_pM is invariant under the action of the stabilisator $Stab_p$ for all $p \in M$.
In the special case $M = \mathbb{O}P^2$, this construction yields an even, homogeneous of degree 2, Spin(9)-invariant valuation on $\mathbb{O}^2 \cong \mathbb{R}^{16}$, whose Klain function is not constant, i.e. which is linearly independent of the second intrinsic volume.

Deutsche Zusammenfassung

Das Ziel dieser Arbeit ist es, die Dimension des Raumes der Spin(9)-invarianten Bewertungen zu bestimmen.
Bevor wir das eigentliche Problem behandeln können, brauchen wir ein paar allgemeine Definitionen.
Auf einem n-dimensionalen Euklidischen Vektorraum V, betrachten wir den Raum $\mathcal{K}(V)$ der kompakten konvexen Teilmengen in V. Eine Bewertung ist ein reell- oder komplexwertiges Funktional μ auf $\mathcal{K}(V)$ mit folgender Additivitätseigenschaft :

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L),$$

für alle $K, L \in \mathcal{K}(V)$ mit $K \cup L \in \mathcal{K}(V)$.
Beispiele von Bewertungen sind das Volumen, der Flächeninhalt des Randes und die Euler Charakteristik.
Eine Bewertung μ heisst stetig, wenn μ stetig bezüglich der Hausdorff Topologie ist. μ heisst translationsinvariant, wenn für jedes $K \in \mathcal{K}(V)$ und $x \in V$,

$$\mu(K + x) = \mu(K).$$

Wir beschränken den Rahmen dieser Arbeit auf den Vektorraum der stetigen translationsinvarianten Bewertungen, der mit Val bezeichnet wird.
Sei μ eine Bewertung, und $k \in \mathbb{C}$. μ heisst homogen vom Grad k, wenn

$$\mu(tK) = t^k \mu(K),$$

für alle $t > 0$ und $K \in \mathcal{K}(V)$. Zum Beispiel ist das Volumen homogen vom Grad n, der Flächeninhalt des Randes vom Grad $n-1$ und die Euler Charakteristik vom Grad 0.
Der Raum Val lässt sich wie folgt zerlegen (Satz von McMullen) :

$$\mathrm{Val} = \bigoplus_{k=0}^{n} \mathrm{Val}_k.$$

Insbesondere ist der Homogenitätsgrad eine ganze Zahl zwischen 0 und n.
Eine Bewertung μ heisst gerade, bzw. ungerade, wenn

$$\mu(-K) = \mu(K), \qquad \text{bzw.} \qquad \mu(-K) = -\mu(K).$$

Jeder Raum Val_k lässt sich schreiben als

$$\mathrm{Val}_k = \mathrm{Val}_k^+ \oplus \mathrm{Val}_k^-,$$

wobei der oberer Index $+$, bzw. $-$, für gerade, bzw. ungerade, Bewertungen steht.
Gerade k-homogene Bewertungen können durch ihre Klainfunktion charakterisiert werden; sei $\mu \in \mathrm{Val}_k^+(V)$, und $E \in Gr_k(V)$ ein k-dimensionaler Unterraum von V. Dann ist $\mu\big|_E$ ein Vielfaches $c(E)$ des k-dimensionalen Lebesguemass auf E. Die Klainfunktion von μ ist dann durch

$$\mathrm{Kl}_\mu(E) = c(E)$$

definiert. Die induzierte Abbildung $\mathrm{Kl} : \mathrm{Val}_k^+ \to C(Gr_k(V))$ ist injektiv.

Die Gruppe $\mathrm{GL}(n, V)$ operiert auf Val durch

$$(g\mu)(K) = \mu(g^{-1}K),$$

für $g \in \mathrm{GL}(n, V)$, $\mu \in \mathrm{Val}$ und $K \in \mathcal{K}(V)$. Diese Operation ist stetig und der Homogenitätsgrad und die Parität bleiben unter dieser Operation erhalten. Ausserdem gilt nach dem Irreduzibilitätstheorem von Alesker ([1]):

Die Darstellung von $\mathrm{GL}(n,V)$ *auf* $\mathrm{Val}_k^{\pm}$ *ist irreduzibel für jedes* $k = 0, ..., n$.

Sei G eine kompakte Untergruppe von $\mathrm{SO}(n) = \mathrm{SO}(V)$. Wir betrachten den Untervektorraum $\mathrm{Val}^G \subset \mathrm{Val}$ von G-invarianten Bewertungen. Es wurde durch Alesker bewiesen [1], dass dieser Raum nur endliche Dimension hat, wenn G transitiv auf der Einheitssphäre $S(V)$ von V operiert. Die Bestimmung aller kompakten zusammenhängenden Lie Gruppen, die transitiv auf der Einheitssphäre operieren, wurde durch Montgomery–Samelson [36] und Borel [19] ausgeführt. Es sind

$$\mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1),$$

und die drei Ausnahmegruppen

$$\mathrm{G}_2, \mathrm{Spin}(7), \mathrm{Spin}(9).$$

Das Problem der Bestimmung der Dimension von Val^G für G aus dieser Liste ist nur teilweise gelöst.
Der Fall $G = \mathrm{SO}(n)$ wurde vollständig von Hadwiger [30] studiert : Die Dimension von $\mathrm{Val}^{\mathrm{SO}(n)}$ ist $n+1$; eine Basis dieses Raumes ist durch die sogenannten intrinsischen Volumina gegeben.
Für $G = \mathrm{U}(n)$, wurde das Thema von Park [37] für $n = 2, 3$ und von Alesker [2] für n beliebig behandelt :

$$\dim \mathrm{Val}^{\mathrm{U}(n)} = \binom{n+2}{2}.$$

Bernig hat dann in [14] die Dimensionen von $\mathrm{Val}_k^{\mathrm{SU}(n)}$ gegeben :

$$\begin{array}{rcll} \mathrm{Val}_k^{\mathrm{SU}(n)} & = & \mathrm{Val}_k^{\mathrm{U}(n)} & \text{for } k \neq n, \\ \dim \mathrm{Val}_n^{\mathrm{SU}(n)} & = & \mathrm{Val}_k^{\mathrm{U}(n)} + 4 & \text{if } n \cong 0 \mod 2, \\ \dim \mathrm{Val}_n^{\mathrm{SU}(n)} & = & \mathrm{Val}_k^{\mathrm{U}(n)} + 2 & \text{if } n \cong 1 \mod 2. \end{array}$$

Für die drei letzte Reihen von der oberen Liste, $\mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, hat Bernig in [16] kombinatorische Dimensionsformeln von Val^G, $G = \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, hergestellt. Die Fälle $G = \mathrm{G}_2$ und $G = \mathrm{Spin}(7)$ wurden vollständig von Bernig in [15] untersucht. Der Beweis benutzt die Inklusionen

$$\mathrm{SU}(3) < \mathrm{G}_2 < \mathrm{Spin}(7), \quad \mathrm{SU}(4) < \mathrm{Spin}(7),$$

und die für SU(3) und SU(4) bekannten Resultate.
Bis jetzt wurden für Spin(9) nur partielle Resultate von Alesker erhalten : er hat einige Beispiele von Spin(9)-invariante Bewertung in [9] konstruiert.
In der vorliegende Arbeit stellen wir eine Methode vor, um die Dimension von $\mathrm{Val}^{\mathrm{Spin}(9)}$ zu berechnen.

Das erste nützliche Hilfsmittel ist der Rumin-Operator [38]. Er wird folgendermassen definiert :
Eine Form ω auf der Kontaktmannigfaltigkeit $SV = V \times S(V)$ heisst vertikal, wenn $\alpha \wedge \omega = 0$, wobei α die Kontaktform auf SV ist.
Rumin hat in [38] gezeigt, dass es für $\omega \in \Omega^{k,n-k-1}(SV)$ eine einzige vertikale Form ω_v existiert, so dass $d(\omega + \omega_v)$ vertikal ist. Der Rumin-Operator ist dann definiert durch

$$D\omega = d(\omega + \omega_v).$$

Der Raum der Differentialformen auf $SV = V \times S(V)$ hat eine Bigraduierung

$$\Omega^*(SV) = \bigoplus_{k,l} \Omega^{k,l}(SV),$$

wobei $\Omega^{k,l}(SV)$ den Raum der Differentialformen vom Bigrad (k,l) auf SV bezeichnet.

Sei G wie vorher eine kompakte Untergruppe von $\mathrm{SO}(V)$, die transitiv auf $S(V)$ operiert.

Auf $\Omega^{k,l}(SV)$ definieren wir folgende Teilräume :

$$\begin{array}{lll} \mathcal{I}^{k,l}(SV) & := & \{\omega \in \Omega^{k,l}(SV) \,|\, \omega = \alpha \wedge \xi + d\alpha \wedge \psi,\, \xi \in \Omega^{k-1,l}(SV),\, \psi \in \Omega^{k-1,l-1}(SV)\}, \\ \Omega_v^{k,l}(SV)^G & := & \{\omega \in \Omega^{k,l}(SV)^G \,|\, \alpha \wedge \omega = 0\} \qquad \text{space of vertical forms,} \\ \Omega_h^{k,l}(SV)^G & := & \Omega^{k,l}(SV)^G \Big/ \Omega_v^{k,l}(SV)^G \qquad \text{space of horizontal forms,} \\ \Omega_p^{k,l}(SV)^G & := & \Omega^{k,l}(SV)^G \Big/ \mathcal{I}^{k,l}(SV)^G\,, \end{array}$$

wobei der obere Index G wie vorher den Raum der G-invarianten Formen bezeichnet.
Man betrachtet zusätzlich folgende Konstruktion :
Zu einem kompakten konvexen Körper $K \in \mathcal{K}(V)$, können wir eine $n-1$-dimensionale Lipschitz Untermannigfaltigkeit von SV assoziieren, durch

$$N(K) := \{(x, v) \in SV \,|\, x \in \partial K, v \text{ äusserer Einheitsnormalenvektor an } K \text{ im Punkt } x\}.$$

$N(K)$ heisst Normalenzyklus von K.
Man definiert den Operator

$$nc : \Omega_p^{k,n-k-1}(SV)^G \to \mathrm{Val}_k^G, \qquad \omega \mapsto \int_{N(\cdot)} \omega.$$

Dann gilt
Theorem 2.2.1.([16]) *Für $0 < k < n$, ist die Sequenz*

$$0 \longrightarrow (\Lambda^k V)^G \longrightarrow \Omega_p^{k,0}(SV)^G \xrightarrow{d_Q} \ldots \xrightarrow{d_Q} \Omega_p^{k,n-k-1}(SV)^G \xrightarrow{nc} Val_k^G \longrightarrow 0$$

exakt.
Es folgt also
Korollar 2.2.2.([16]) *Für $0 \leq k, l \leq n$, sei*

$$b_k := \dim(\Lambda^k V)^G,$$
$$b_{k,l} := \dim \Omega_h^{k,l}(SV)^G,$$

und $b_k = 0$, $b_{k,l} = 0$ für andere Werte von k und l.
Dann ist für $0 \leq k \leq n$:

$$\dim Val_k^G = \sum_{l=0}^{n-k-1} (-1)^{n-k-l-1}(b_{k,l} - b_{k-1,l-1}) + (-1)^{n-k} b_k.$$

Um diese Koeffizienten b_k und $b_{k,l}$ bestimmen zu können, müssen wir zuerst die Spin(9) Aktion auf $\mathbb{R}^{16}$ untersuchen. Dafür präsentieren wir eine Darstellung von Sudbery [41], die mit Hilfe von Eigenschaften der 8-dimensionalen Divisionalgebra $\mathbb{O}$ der Oktonionen gegeben ist. $\mathbb{R}^{16}$ wird als $\mathbb{O}^2$ betrachtet, und der Beweis folgendes Satzes wird vorgestellt :
Theorem 4.3.3.([41]) *Die Lie Algebra $so(9)$ von $Spin(9)$ kann durch*

$$\mathrm{so}(9) = A_2'(\mathbb{O}) \oplus \mathrm{so}(\mathbb{O}'),$$

dargestellt werden, und die Aktion ρ von $\mathrm{so}(9)$ *auf $\mathbb{O}^2 \cong \mathbb{R}^{16}$ (Spindarstellung) ist gegeben durch*

$$\begin{array}{llll} A \in A_2'(\mathbb{O}) & \Rightarrow & \rho(A)(x) := A \cdot x & \text{(Matrixmultiplikation)} \\ T \in \mathrm{so}(\mathbb{O}') & \Rightarrow & \rho(T)(x) := T^\sharp x & \text{(komponentenweise).} \end{array}$$

Wir können dann verifizieren, dass die Aktion vom Stabilisator Stab $\cong$ so(7) auf den Tangentialraum $T_{(1,0)}\mathbb{S}^{15} = \mathbb{O}' \oplus \mathbb{O}$ die Summe der Standarddarstellung von so(7) auf $\mathbb{O}' = \mathbb{R}^7$ und der

Spindarstellung von so(7) auf $\mathbb{O} = \mathbb{R}^8$ ist.

Damit können wir dann die Koeffizienten $b_{k,l}$ berechnen. Nach Umformung gilt nämlich

$$b_{k,l} = \dim(\Lambda^k(\mathbb{O} \oplus \mathbb{O}') \otimes \Lambda^l(\mathbb{O} \oplus \mathbb{O}'))^{\mathrm{Spin}(7)}.$$

Benutzt man die oben angegebene explizite Darstellung und Formeln aus der Theorie der Darstellungen und ihrer Charaktern, erhalten wir die Koeffizienten b_k und $b_{k,l}$.

Insgesamt erhalten wir folgende Dimensionen von $\mathrm{Val}_k^{\mathrm{Spin}(9)}$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim \mathrm{Val}_k^{\mathrm{Spin}(9)}$	1	1	2	3	6	10	15	20	27	20	15	10	6	3	2	1	1

Es ist klar, dass das k-te intrinsische Volumen in $\mathrm{Val}_k^{\mathrm{Spin}(9)}$ enthalten ist. Das letzte Kapitel dieser Arbeit präsentiert eine Konstruktion eines weiteren Elements von $\mathrm{Val}_2^{\mathrm{Spin}(9)}$, der linear unabhängig von μ_2 ist.
Wir betrachten eine glatte n-dimensionale orientierte Mannigfaltigkeit M. Da wir in diesem Zusammenhang von Konvexität nicht mehr sprechen können, muss man eine andere Klasse von Teilmengen von M betrachten. Sei $\mathcal{P}(M)$ die Menge der kompakten Untermannigfaltigkeiten mit Ecken. Ein Funktional $\mu : \mathcal{P}(M) \to \mathbb{C}$ ist dann eine Bewertung auf der Mannigfaltigkeit M, wenn die gleiche Additivitätsbedingung wie im linearen Fall erfüllt ist, für jedes Paar von Elementen in $\mathcal{P}(M)$ deren Schnitt und Vereinigung wieder in $\mathcal{P}(M)$ sind.
Für ein Element K aus $\mathcal{P}(M)$ kann man den normalen Zyklus $N(K) \subset S^*M$ definieren. $N(K)$ ist wie vorher eine $n-1$-dimensionale Lipschitz Untermannigfaltigkeit von S^*M. Zu $K \in \mathcal{P}(M)$ können wir zudem eine andere Teilmenge $N_1(K) \subset T^*M$ zuordnen, das Diskbündel. $N_1(K)$ wird durch Summieren von $K \times \{0\}$ und dem Bild von $[0,1] \times N(K)$ unter der Homothetie in dem 2. Faktor bekommen. Integration einer n-Form auf T^*M über das Diskbündel liefert eine glatte Bewertung auf M.
Für jeden Punkt $p \in M$, hat der Raum $\mathrm{Val}^\infty(T_pM)$ der translationsinvarianten glatten konvexen Bewertungen auf T_pM, nach dem Satz von McMullen, eine Graduierung nach Homogenitätsgrad:

$$\mathrm{Val}^\infty(T_pM) = \bigoplus_{i=0}^{n} \mathrm{Val}_i^\infty(T_pM).$$

Man bezeichnet mit $\mathrm{Val}^\infty(TM)$ das Bündel, dessen Faser über einem Punkt p der Raum $\mathrm{Val}^\infty(T_pM)$ ist. Dann haben wir folgende Graduierung

$$\mathrm{Val}^\infty(TM) = \bigoplus_{i=0}^{n} \mathrm{Val}_i^\infty(TM).$$

Dann gilt
Theorem 6.1.6 [10] *Es existiert eine kanonische Filtrierung des Raumes $\mathcal{V}^\infty(M)$ von glatten Bewertungen durch abgeschlossene Teilmengen*

$$\mathcal{V}^\infty(M) = \mathcal{W}_0 \supset \mathcal{W}_1 \supset ... \supset \mathcal{W}_n,$$

so dass der assoziierte graduierte Raum $gr_W\mathcal{V}^\infty(M) := \bigoplus_{i=0}^n \mathcal{W}_i/\mathcal{W}_{i+1}$ kanonisch isomorph zu $C^\infty(M, \mathrm{Val}^\infty(TM))$, dem Raum der glatten Schnitte von dem unendlich dimensionalen Vektorbündel $\mathrm{Val}^\infty(TM) \longrightarrow M$, ist.

Sei G eine Lie Gruppe, die isotropisch auf M operiert.
Dann beschränkt sich der oben angegebene Isomorphismus zu

$$gr_W\mathcal{V}^\infty(M)^G \cong C^\infty(M, \mathrm{Val}^\infty(TM))^G \cong \mathrm{Val}^H(T_pM)$$

wobei $H \subset G$ der Stabilisator des Punktes p ist; insbesondere gilt

$$(\mathcal{W}_i/\mathcal{W}_{i+1})^G \cong \mathrm{Val}_i^H(T_pM).$$

Wir können diesen Isomorphismus explizit angeben :
Sei $\mu \in \mathcal{W}_k$ und $p \in M$. Sei $\tau : U \to V \subset \mathbb{R}^n$ eine Karte um p. Für $K \in \mathcal{P}(T_pM)$ definieren wir

$$T_p^k\mu(K) := \frac{1}{k!}\frac{d^k}{dt^k}\bigg|_{t=0} (\tau^{-1})^*\mu\left(\tau(p) + t(d\tau_p(K) - \tau(p))\right).$$

Dann ist $T_p^k\mu$ unabhängig von der Wahl der Karte τ und gehört zu $\mathrm{Val}_k^\infty(T_pM)$.

Ist M eine Riemannsche Mannigfaltigkeit, so hat man eine zusätzliche Struktur, die man benutzt um eine kanonische Bewertung zu konstruieren.
Der Krümmungstensor R_p einer Riemannschen Mannigfaltigkeit ist ein Element von $Sym^2\Lambda^2T_p^*M$. Mit Hilfe des Hodge-$*$ Operators und einer Riemannschen Submersion, kann man R_p als Element R_p^* von $\Lambda^n(T^*_{(p,v)}(TM))$ ansehen. Man definiert dann eine n-Form auf T^*M durch

$$\omega_{(p,v)} = \frac{1}{C_{n-2}}R_p^*,$$

wobei C_{n-2} das Volumen des $n-2$-dimensionalen Einheitsballs ist. Dann ist die Bewertung μ auf M, die durch Integration von ω über das Diskbündel definiert ist, Element von $\mathcal{W}_2$. Also ist die Bewertung $T_p^2\mu$ auf T_pM vom Homogenitätsgrad 2. Zudem ist $T_p^2\mu$ invariant unter dem Stabilisator von p, da ω, und daher μ, invariant unter der Operation von $G = Isom(M,g)$ ist. Wir zeigen noch, dass $T_p^2\mu$ eine gerade Bewertung ist, deren Klainfunktion genau die Schnittkrümmung der Mannigfaltigkeit M ist.
Wir wenden dann diese Konstruktion auf vier Beispiele.
Das erste Beispiel ist die n-dimensionale Sphäre $M = \mathbb{S}^n$. Die konstruierte Bewertung $T_p^2\mu$ auf T_pM ist dann das zweite intrinsische Volumen.
Dann betrachten wir $M = \mathbb{C}P^n$, den komplex projektiven Raum. Die Bewertung $T_p^2\mu$ auf $T_pM = \mathbb{C}^n$ ist dann $\mathrm{U}(n)$-invariant; die Klainfunktion von $T_p^2\mu$ lautet

$$\mathrm{Kl}_{T_p^2\mu}(E_{x,y}) = 1 + 3\cos^2\varphi(x,y),$$

wobei $E_{x,y}$ die Ebene ist, die durch x und y erzeugt wird, und φ der Kählerwinkel, definiert durch

$$\cos^2\varphi(x,y) = \langle x, iy\rangle^2.$$

In der Basis der Tasaki Bewertungen (s. [18]) gilt also :

$$T_p^2\mu = \tau_{2,0} + 3\tau_{2,1}.$$

Im Fall $M = \mathbb{H}P^n$, der quaternionisch projektive Raum, ist die Bewertung $T_p^2\mu$ $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$-invariant. Die Klainfunktion von $T_p^2\mu$ wird durch

$$\mathrm{Kl}_{T_p^2\mu}(E_{x,y}) = 1 + 3\cos^2\alpha(x,y),$$

angegeben; α wird analog zum Kählerwinkel definiert.
Das letzte Beispiel ist für uns das interessanteste. Nehmen wir $M = \mathbb{O}P^2$, die oktonionisch projektive Ebene, so ist $T_pM = \mathbb{O}^2$ und $T_p^2\mu$ ist eine $\mathrm{Spin}(9)$-invariante Bewertung auf $\mathbb{O}^2 \cong \mathbb{R}^{16}$, deren Klainfunktion nicht konstant ist. $T_p^2\mu$ ist deshalb linear unabhängig von dem zweiten intrinsischen Volumen μ_2. Da $\mathrm{Val}_2^{\mathrm{Spin}(9)}$ Dimension 2 hat, kann man als Basis das zweite intrinsische Volumen μ_2 und das oktonionische Pseudo-Volumen τ, eingeführt in [9], nehmen. In dieser Basis ist die neu konstruierte Bewertung

$$T_p^2\mu = 4\mu_2 - 3\tau.$$

Contents

Introduction

The main object studied through this work are valuations. They are defined as follows:
On an n-dimensional Euclidean vector space V, we consider the space $\mathcal{K}(V)$ of compact convex subsets in V. A valuation is a real or complex valued functional μ on $\mathcal{K}(V)$ with the following additivity property :

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L),$$

for all $K, L \in \mathcal{K}(V)$ with $K \cup L \in \mathcal{K}(V)$.
First examples of valuations are given by the volume or the Euler characteristic.
In this work we consider only continuous (with respect to the Hausdorff topology on $\mathcal{K}(V)$) and translation invariant valuations; we denote by Val the space of these valuations.
Let G be a compact subgroup of the special linear group $\mathrm{SO}(n)$ of V, and consider the space $\mathrm{Val}^G \subset \mathrm{Val}$ of G-invariant valuations. It was shown by Alesker [1] that this vector space has finite dimension if and only if G acts transitively on the unit sphere $S(V)$ in V. The determination of compact connected Lie groups acting transitively on $S(V)$ was performed by Montgomery-Samelson [36] and Borel [19]. The compact connected Lie groups acting transitively on the sphere are

$$\mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1), \tag{1}$$

and the three exceptional groups

$$\mathrm{G}_2, \mathrm{Spin}(7), \mathrm{Spin}(9). \tag{2}$$

The space of $\mathrm{SO}(n)$-invariant valuations is described by Hadwiger's theorem, which states that $\mathrm{Val}^{\mathrm{SO}(n)}$ has dimension $n+1$. A basis of this space is formed by the so called intrinsic volumes.
The next case is $G = \mathrm{U}(n)$ acting on $\mathbb{C}^n$. Park in [37] gave the dimension of $\mathrm{Val}^{\mathrm{U}(n)}$ for $n = 2, 3$, and Alesker proved in [2] for general n :

$$\dim \mathrm{Val}^{\mathrm{U}(n)} = \binom{n+2}{2}.$$

Bernig proved in [14] a Hadwiger-type theorem for $\mathrm{SU}(n)$:

$$\begin{aligned}
\mathrm{Val}_k^{\mathrm{SU}(n)} &= \mathrm{Val}_k^{\mathrm{U}(n)} && \text{for } k \neq n, \\
\dim \mathrm{Val}_n^{\mathrm{SU}(n)} &= \mathrm{Val}_k^{\mathrm{U}(n)} + 4 && \text{if } n \cong 0 \mod 2, \\
\dim \mathrm{Val}_n^{\mathrm{SU}(n)} &= \mathrm{Val}_k^{\mathrm{U}(n)} + 2 && \text{if } n \cong 1 \mod 2.
\end{aligned}$$

For the three remaining series of the list (1), $\mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, Bernig recently found combinatorial dimension formulas for Val^G, $G = \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, using the Rumin-de Rahm complex and tools from representation theory [16].
The cases of G_2 and $\mathrm{Spin}(7)$ have been treated completely by Bernig in [15]. The proof uses the inclusions

$$\mathrm{SU}(3) < \mathrm{G}_2 < \mathrm{Spin}(7), \quad \mathrm{SU}(4) < \mathrm{Spin}(7),$$

and the known results for $\mathrm{SU}(3)$ and $\mathrm{SU}(4)$. Another important argument is the following :
$\mathrm{Spin}(7)$ acts transitively on the sphere $\mathbb{S}^7$, the stabilizer of this action is G_2; G_2 operates on the

tangent space of $\mathbb{S}^7$ which we identify with $\mathbb{R}^7$, and this action is transitive on the sphere $\mathbb{S}^6$. The stabilizer of this action is then SU(3), which operates on the tangent space of $\mathbb{S}^6$, identified with $\mathbb{R}^6$, and this action is again transitive on the sphere $\mathbb{S}^5$.
Until now, only some partial results have been obtained by Alesker in the case $G = \mathrm{Spin}(9)$: he constructed a few examples of Spin(9)-invariant valuation in [9]. The same argument as for G_2 and Spin(7) cannot apply in the case $G = \mathrm{Spin}(9)$, since the stabilizer of the action on $\mathbb{R}^{16}$ is Spin(7), which does not operate transitively on the 14-sphere in the tangent space of $\mathbb{S}^{15}$. Therefore, we have to develop other techniques to determine a Hadwiger-type theorem in this case. This is the subject of this thesis.

The present work is organized as follows :
In the first chapter we recall definitions and properties of certain classes of valuations. Since all G-invariant valuations are even, we remind of the tools used to describe them; in particular, the definitions of Klain function and Klain embedding are given. Moreover the different equivalent definitions of smoothness of a valuation are explained. In this context arise the notion of normal cycle.
In Chapter 2 we use the Rumin operator to derive a formula for the dimension of the space of translation invariant, continuous, homogeneous of degree k, G-invariant valuations in terms of dimensions of spaces of differential forms on the sphere bundle SV, namely :
Corollary 2.2.2.([16]) *For $0 \leq k, l \leq n$, set*

$$b_k := \dim(\Lambda^k V)^G,$$
$$b_{k,l} := \dim \Omega_h^{k,l}(SV)^G,$$

and $b_k = 0$, $b_{k,l} = 0$ for other values of k and l.
Then for $0 \leq k \leq n$:

$$\dim Val_k^G = \sum_{l=0}^{n-k-1} (-1)^{n-k-l-1}(b_{k,l} - b_{k-1,l-1}) + (-1)^{n-k} b_k.$$

In Chapter 3, we collect facts from representation theory of Lie groups. We discuss in particular the irreducible representations of $\mathrm{so}(2m+1)$, which are involved in the computations of the dimensions b_k and $b_{k,l}$ of Corollary 2.2.2. It will be useful to look at these representations at the level of their characters, computed using Weyl's character formula.
Chapter 4 is devoted to the better understanding of the spin groups and of the spin representations. First we describe the spin groups in terms of Clifford algebras. This description, although it gives us information on the structure of the spin groups, is not the right setting for our aim. Therefore, we present a work by Sudbery ([40],[41]) on division algebras and their relations to field theory. His approach applied to the division algebra $\mathbb{O}$ of the octonions yields an explicit description of the spin actions of Spin(7) on $\mathbb{R}^8 \cong \mathbb{O}$ and Spin(9) on $\mathbb{R}^{16} \cong \mathbb{O}^2$.
In Chapter 5, we use the results and tools established in the previous three chapters to compute the coefficients b_k and $b_{k,l}$ from Corollary 2.2.2. As a consequence, we get the following dimensions for the spaces $\mathrm{Val}_k^{\mathrm{Spin}(9)}$:

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim \mathrm{Val}_k^{\mathrm{Spin}(9)}$	1	1	2	3	6	10	15	20	27	20	15	10	6	3	2	1	1

It is clear that the intrinsic volume μ_k is element of $\mathrm{Val}_k^{\mathrm{Spin}(9)}$ for each $k = 0, 1, ..., 16$.
The goal of Chapter 6 is to construct a second, linearly independent of μ_2, element of $\mathrm{Val}_2^{\mathrm{Spin}(9)}$. We obtain it as a special case of a general procedure.
We extend the setting to valuations on a smooth oriented manifold M, and recall first definitions and properties. Then, restricting ourselves to a Riemannian manifold (M, g), we consider the

curvature tensor R, with which we derive an n-form ω on T^*M. Integrating this form on the disc bundle yields a smooth valuation on M, which is invariant with respect to the isometry group G of M. By theorems of Alesker ([6],[7]), this valuation induces a family of valuations on the tangent spaces, which are in fact invariant under the action of the stabilizer of G in p. This valuations are moreover of homogeneity degree 2 and even; therefore we can compute their Klain function, which turns out to be the sectional curvature of the manifold M. We illustrate then this result by a range of examples, one of them being $M = \mathbb{O}\mathrm{P}^2$; we get thereby a valuation on $T_pM \cong \mathbb{O}^2$ which is invariant under $\mathrm{Stab}_p = \mathrm{Spin}(9)$, the stabilizer of p. Moreover, since its Klain function is not constant, this valuation is linearly independent of the second intrinsic volume.

Chapter 1

Real valued valuations

1.1 Definition and basic properties

Let V be an n-dimensional real vector space. We denote by $\mathcal{K}(V)$ the space of compact convex subsets of V. With the choice of a scalar product on V, we can endow $\mathcal{K}(V)$ with the **Hausdorff metric**

$$d(A,B) := \inf\{r \geq 0 \mid A \subset B_r, B \subset A_r\},$$

where

$$A_r := \{x \in V \mid \operatorname{dist}(x,A) \leq r\}.$$

The induced **Hausdorff topology** does not depend on the choice of the scalar product.
With respect to **Minkowski addition**

$$K + L := \{x + y \mid x \in K, y \in L\},$$

$\mathcal{K}(V)$ is a semigroup.

Definition 1.1.1. *A* **valuation** *is a functional $\mu : \mathcal{K}(V) \to \mathbb{R}$ with the following additivity property :*

$$\mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L),$$

for every $K, L \in \mathcal{K}(V)$ with $K \cup L \in \mathcal{K}(V)$.

Examples of valuations are the volume of a convex body denoted by vol, its area and the constant valuation $\mu : \mathcal{K}(V) \to \mathbb{R}, K \mapsto 1$, called Euler characteristic and denoted by χ.

A particularly important class of valuations is that of continuous, translation invariant valuations.

Definition 1.1.2. *A valuation is called* **continuous** *if it is continuous with respect to the Hausdorff topology.*
A valuation is **translation invariant** *if*

$$\mu(K + v) = \mu(K),$$

for every $K \in \mathcal{K}(V)$, $v \in V$.

We denote by $\operatorname{Val}(V) = \operatorname{Val}$ the vector space of all continuous and translation invariant valuations on V. Val is an infinite dimensional vector space.

Definition 1.1.3. *A valuation μ is* **homogeneous of degree** *k if for every $K \in \mathcal{K}(V)$, $t \geq 0$,*

$$\mu(tK) = t^k \mu(K).$$

A valuation μ is **even** *if $\mu(-K) = \mu(K)$ and* **odd** *if $\mu(-K) = -\mu(K)$.*

We denote by $\mathrm{Val}_k^+, \mathrm{Val}_k^-$ the subspace of Val of even resp. odd valuations of homogeneity degree k. Each continuous translation invariant valuation on V can be uniquely decomposed in his even resp. odd components of degree $k = 0, 1, ..., n = \dim(V)$:

Theorem 1.1.4 (McMullen's decomposition theorem, [35])**.**

$$\mathrm{Val} = \bigoplus_{\substack{k=0 \\ \varepsilon=\pm}}^{n} \mathrm{Val}_k^{\varepsilon}.$$

Note in particular that this implies that the degree of homogeneity can only be an integer between 0 and n. It is known that $\mathrm{Val}_0(V)$ is one-dimensional and is spanned by the Euler characteristic χ, and $\mathrm{Val}_n(V)$ is also one-dimensional and is spanned by a Lebesgue-measure [30]. There is a natural operation of the general linear group $\mathrm{GL}(n, V)$ on Val given by

$$(g\mu)(K) = \mu(g^{-1}K).$$

Theorem 1.1.5 (Alesker's irreducibility theorem, [1])**.** *The natural representation of the group* $\mathrm{GL}(n, V)$ *on each space* $\mathrm{Val}_k^{\pm}$ *is irreducible for any* $k = 0, 1, ..., n$.

This means that any invariant closed subspace of $\mathrm{Val}_k^{\pm}$ is either $\{0\}$ or the entire space $\mathrm{Val}_k^{\pm}$. If the subspace is not closed, it is therefore either $\{0\}$ or dense in $\mathrm{Val}_k^{\pm}$.

Definition 1.1.6. *A valuation* $\mu \in \mathrm{Val}$ *is* **smooth** *if the map* $\mathrm{GL}(n, V) \longrightarrow \mathrm{Val},\ g \mapsto g\mu$ *is smooth.*
We denote by $Val^{sm}(V)$ *the space of smooth translation invariant continuous valuations on* V.

$\mathrm{Val}^{sm}(V)$ is a dense $\mathrm{GL}(n, V)$-invariant vector subspace of Val.
We recall the definition from [3] of the following operator on $\mathrm{Val}(V)$. Let B be the unit Euclidean ball in V. Then the operator

$$(\mathfrak{L}\mu)(K) := \frac{d}{dt}\Big|_{t=0} \mu(K + tB),$$

for $K \in \mathcal{K}(V)$, is called **derivation operator**; $\mathfrak{L}$ preserves the parity and decreases the homogeneity degree of μ by 1. We have

Theorem 1.1.7 (Hard Lefschetz theorem, [3], [17])**.** *For* $\frac{n}{2} < k \leq n$, *the operator*

$$\mathfrak{L}^{2k-n} : \mathrm{Val}_k^{sm} \to \mathrm{Val}_{n-k}^{sm}$$

is an isomorphism.

For a compact subgroup G of $SO(V) \cong SO(n)$, we let $\mathrm{Val}^G \subset \mathrm{Val}$ denote the subspace of G-**invariant valuations**, namely :

$$\mathrm{Val}^G := \{\mu \in \mathrm{Val} \,|\, \mu(gK) = \mu(K),\, \forall g \in G,\, K \in \mathcal{K}(V)\}.$$

Theorem 1.1.8 ([1])**.** *Let* G *be a compact subgroup of* $\mathrm{SO}(n)$ *for* $n \geq 2$. *Then* $\dim \mathrm{Val}^G < \infty$ *if and only if* G *acts transitively on the sphere* $S(V)$.

Proposition 1.1.9 ([4])**.** *Let* G *be a compact subgroup of* $\mathrm{SO}(n)$ *for* $n \geq 2$ *acting transitively on the sphere of* V. *Then*

$$\mathrm{Val}^G(V) \subset \mathrm{Val}^{sm}(V).$$

The condition that G acts transitively on the sphere was studied by Montgomery-Samelson [36] and Borel [19]. They gave a complete classification of the connected compact Lie groups satisfying this condition; they are namely

$$\mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1),$$

and the three exceptional groups

$$\mathrm{G}_2, \mathrm{Spin}(7), \mathrm{Spin}(9).$$

There are various natural inclusions among these groups:

$$\mathrm{U}(n), \mathrm{SU}(n) < \mathrm{SO}(2n),\ \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1) < \mathrm{SO}(4n),$$

$$\mathrm{SU}(4) < \mathrm{Spin}(7),\ \mathrm{G}_2 < \mathrm{SO}(7),\ \mathrm{SU}(3) < \mathrm{G}_2 < \mathrm{Spin}(7) < \mathrm{SO}(8),\ \mathrm{Spin}(9) < \mathrm{SO}(16).$$

The last two inclusions are the spin representations, which play an important role in this work and will be described further in Chapter 4.

Proposition 1.1.10. *If G is a compact subgroup of $\mathrm{SO}(n)$, $n \geq 2$, acting transitively on the sphere, then*

$$\mathrm{Val}^G \subset \mathrm{Val}^+.$$

This is clear if $-1 \in G$. The only groups for which this is not true are $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ for n odd and G_2; proofs for these three cases are given in [30], [14] and [15] respectively.

1.2 Hadwiger's theorem

Remarkable examples of continuous translation invariant valuations are the intrinsic volumes. We can define them by the

Theorem 1.2.1 (Steiner formula)**.**
Let B be the n-dimensional unit ball in V and ω_k the volume of the $k-$dimensional unit ball. For $t > 0$ and $K \in \mathcal{K}(V)$, let $K + tB$ be the t-tube around K. Then $\mathrm{vol}(K + tB)$ is a polynomial in t given by

$$\mathrm{vol}(K + tB) = \sum_{k=0}^{n} \mu_{n-k}(K)\omega_k t^k.$$

Definition 1.2.2. *The valuations μ_k, $k = 0, ..., n$, are called* **intrinsic volumes***.*

The k-th intrinsic volume μ_k has the following properties :

1. μ_k is a continuous translation invariant and $SO(n)$-invariant valuation
2. μ_k is homogeneous of degree k
3. the restriction of μ_k to a k-plane is the $k-$dimensional Lebesgue measure on that plane
4. μ_k is even.

In particular the μ_k are in $\mathrm{Val}^{SO(n)}$. Hadwiger's theorem states conversely that all valuations in $\mathrm{Val}^{SO(n)}$ are obtained by linear combinations of intrinsic volumes.

Theorem 1.2.3 (Hadwiger's theorem, [30])**.** *The vector space $Val^{SO(n)}$ is of finite dimension $n + 1$, and the valuations μ_0, $\mu_1, ..., \mu_n$ form a basis of it.*

This theorem was proved in 1957 by Hadwiger ([30]); more recently a shorter proof was given by Klain ([32]) using the following theorem :

Theorem 1.2.4 (Klain's theorem, characterization of the volume, [32])**.** *Let μ be an even, translation invariant, continuous valuation which is* **simple***, i.e. $\mu(K) = 0$ if $\dim(K) < n = \dim(V)$. Then there exists some constant $c \in \mathbb{R}$ such that*

$$\mu = c \cdot \mathrm{vol}\,.$$

1.3 Even valuations

By Proposition 1.1.10, the valuations of interest in this work are all even, so it makes sense to try to understand them better.

Let μ be an even valuation of homogeneity degree k. For $E \in Gr_k(V)$, the restriction $\mu|_E$ is a continuous, translation invariant simple valuation. By Klain's theorem 1.2.4, $\mu|_E = c(E) \cdot \mathrm{vol}_E$ with some constant $c(E) \in \mathbb{R}$.

Definition 1.3.1. *The map* $\mathrm{Kl}_\mu : Gr_k(V) \longrightarrow \mathbb{R}, \mathrm{Kl}_\mu(E) = c(E)$ *is called* **Klain function** *of* μ. *It is a continuous function on* $Gr_k(V)$.

Proposition 1.3.2. *The induced map* $\mathrm{Kl} : Val_k^+ \longrightarrow C(Gr_k(V)), \mu \mapsto \mathrm{Kl}_\mu$ *is injective.* Kl *is called* **Klain embedding**.

Proof. Suppose $\mathrm{Kl}_\mu = 0$ for some $\mu \in \mathrm{Val}_k^+$. Let F be a subspace of minimal dimension such that $\mu|_F \neq 0$, in particular $\dim F = j > k$. Then $\mu|_F$ is simple, hence by Klain's theorem $\mu|_F = c\,\mathrm{vol}_j$. Since μ is homogeneous of degree k, this is only possible if $\mu|_F = 0$. Hence there exists no subspace F of V with $\mu|_F \neq 0$, i.e. $\mu = 0$. □

Even smooth valuations of degree k can be characterized in the following way :
Let η be a smooth, translation invariant signed measure on the affine Grassmannian $\overline{Gr}_{n-k}(V)$ which is invariant under maps $E \mapsto -E$ and satisfying $(m_t)_*\eta = t^{-k}\eta$ with $m_t : \overline{Gr}_{n-k}(V) \longrightarrow \overline{Gr}_{n-k}(V)$, $E \mapsto tE$, $t > 0$.
Then the map $\mu : \mathcal{K}(V) \longrightarrow \mathbb{R}$ defined by

$$\mu(K) := \int_{\overline{Gr}_{n-k}(V)} \chi(K \cap E) d\eta(E)$$

is an even translation invariant, continuous valuation of degree k. η is called **Crofton measure** for μ. A valuation admitting such a representation is called **smooth**. This definition coincide with the Definition 1.1.6 given above ([11]).

1.4 Another description of smooth valuations

Smooth valuations can be described in terms of integration of differential forms over some submanifold of the sphere bundle SV. More precisely :
Let $K \in \mathcal{K}(V)$, $x \in \partial K$ and E an affine hyperplane.

Definition 1.4.1. *E is a* **support plane** *of K at x if $x \in E$ and $K \subset E^+$ or $K \subset E^-$, where E^+ and E^- are the two half spaces bounded by E.*
E is an **oriented support plane** *of K at x if $x \in E$ and $K \subset E^-$.*

If ∂K is smooth, there exists exactly one oriented support plane through each boundary point x. For $x \in \partial K$ and E an associated oriented support plane, the couple (x, E) is an element of the oriented $n-1$-dimensional Grassmann bundle $Gr_{n-1}^+(TV)$. We can see each pair (x, E) as an element $(x, [\xi])$ of S^*V, where $x \in V$, $[\xi] \in T_x^*V$ and $[\xi] = [\tilde{\xi}]$ if and only if $\xi = \lambda\tilde{\xi}$ for some $\lambda > 0$. V being an Euclidean vector space, we can identify S^*V with the sphere bundle $SV = V \times S(V)$, where $S(V)$ is the unit sphere in V. The manifold

$$Gr_{n-1}^+(TV) = S^*V \cong SV$$

is in fact a contact manifold. Contact manifolds are defined as follows ([29]) :

Definition 1.4.2. *A* **k-dimensional distribution** *in a smooth manifold M is a smooth section Q of the Grassmannian Gr_kTM, i.e. Q_p is a k-dimensional subspace of T_pM for $p \in M$.*

A **contact manifold** *is a smooth manifold M of dimension $2n-1$ with a codimension 1 distribution Q which is completely non-integrable, i.e. if locally $Q = \ker\alpha$ for some 1-form α, then $\alpha \wedge d\alpha^{n-1} \neq 0$. If α' is another 1-form with $\ker\alpha' = Q$, then $\alpha' = f\alpha$ for some non-vanishing function f and $\alpha' \wedge d\alpha'^{n-1} = f\alpha \wedge (df \wedge \alpha + f d\alpha)^{n-1} = f^n \alpha \wedge d\alpha^{n-1} \neq 0$.*
α is called **contact form**. *In general, α exists only locally.*

Since $\alpha \wedge d\alpha^{n-1} \neq 0$ and $Q = \ker\alpha$, we have $d\alpha^{n-1}|_{Q_p} \neq 0$ for $p \in M$. Hence $d\alpha|_{Q_p}$ is a symplectic form on Q_p, and Q is a symplectic bundle over M.

The canonical contact form α on the sphere bundle SV is defined by

$$\alpha_{(x,v)}(w) = \langle v, d\pi(w)\rangle,$$

where $\pi : SV \to V$ is the canonical projection.
The kernel of α defines the contact distribution $Q = \ker\alpha$.

Definition 1.4.3. *The* **conormal cycle** *of K is defined by*

$$N(K) := \{(x, E) \in Gr^+_{n-1}(TV) \mid E \text{ is an oriented support plane of } K \text{ through } x\}.$$

The image of the conormal cycle of K under the identification $S^*V \cong SV$ is called the **normal cycle** *of K :*

$$N(K) := \{(x, v) \in SV \mid v \text{ is an outer unit normal vector of } K \text{ through } x\}.$$

Theorem 1.4.4 (Properties of $N(K)$, [26])**.**

i) $N(K)$ is a Lipschitz submanifold of SV of dimension $n-1$,

ii) $N(K)$ is a **cycle** *: $\partial N(K) = 0$, i.e. $\int_{N(K)} d\rho = 0$ for $\rho \in \Omega^{n-2}(SV)$,*

iii) $N(K)$ is **horizontal** *: $\int_{N(K)} \alpha \wedge \rho = 0$ for all ρ, where α is the contact form on SV,*

iv) $N(K)$ is **legendrian**, *i.e. $\int_{N(K)} d\alpha \wedge \rho = 0$ for all ρ,*

v) $\pi_* N(K) = \partial K$, where π is the projection $\pi : SV \to V$, $(x, v) \mapsto x$,

vi) If $K, L, K \cup L \in \mathcal{K}(\mathcal{V})$, then $N(K \cup L) + N(K \cap L) = N(K) + N(L)$.

The last property implies that if $\omega \in \Omega^{n-1}(SV)$ is translation invariant, then the map

$$K \mapsto \int_{N(K)} \omega$$

is a translation invariant continuous valuation.

Definition 1.4.5. *A valuation $\mu \in$ Val is called* **smooth** *if there exists $\omega \in \Omega^{n-1}(SV)^{tr}$ and $\varphi \in \Lambda^n V^* = \Omega^n(V)^{tr}$ such that*

$$\mu(K) = \int_{N(K)} \omega + \int_K \varphi,$$

where the superscript ^{tr} means translation invariant.

This definition of smoothness is equivalent to definition 1.1.6 ([7]).
We denote by Val^{sm} the space of smooth valuations in Val. By Alesker's irreducibility theorem, Val^{sm} is a dense subspace of Val. The map

$$\begin{aligned} \Psi : \Omega^{n-1}(SV)^{tr} \times \Omega^n(V)^{tr} &\longrightarrow \mathrm{Val}^{sm} \\ (\omega, \varphi) &\mapsto \Psi_{(\omega,\varphi)}, \end{aligned}$$

defined by

$$\Psi_{(\omega,\varphi)}(K) := \int_{N(K)} \omega + \int_K \varphi,$$

is surjective. Its kernel is given by the following theorem :

Theorem 1.4.6 ([17])**.** $\Psi_{(\omega,\varphi)} = 0$ *if and only if*

i) $\mathrm{D}\omega + \pi^*\varphi = 0$,

ii) $\pi_*\omega = 0$,

where $\pi : SV \longrightarrow V$ *is the canonical projection and* D *is the Rumin operator, which will be introduced in the next section.*

Remark 1.4.7. *The construction of the normal cycle and of* Ψ *can be extended to the more general setting of valuations on manifolds, as we will see in Chapter 6.*

Chapter 2

The Rumin operator and valuations

2.1 The Rumin operator

Let V be as before an Euclidean vector space. The sphere bundle $SV = V \times S(V)$ is a contact manifold with contact form given by

$$\alpha_{(x,v)}(w) = \langle v, d\pi(w) \rangle,$$

where $\pi : SV \to V$ is the canonical projection.
The space of differential forms on $SV = V \times S(V)$ has a bigrading

$$\Omega^*(SV) = \bigoplus_{k,l} \Omega^{k,l}(SV),$$

where $\Omega^{k,l}(SV)$ denotes the space of differential forms of bidegree (k, l) on SV.

Definition 2.1.1. *A differential form $\omega \in \Omega^{k,l}(SV)$ is called* **vertical** *if $\omega(v_1, ..., v_{k+l}) = 0$ whenever v_i are horizontal, i.e. $v_i \in Q$.*

Equivalently : $\omega \wedge \alpha = 0$ or $\omega = \alpha \wedge \varphi$ for some $\varphi \in \Omega^{k-1,l}(SV)$.

Proposition 2.1.2 ([38])**.** *Let $\omega \in \Omega^{k,n-k-1}(SV)$. Then there is a unique vertical form ω_v such that $d(\omega + \omega_v)$ is vertical.*

Definition 2.1.3 ([38])**.** *The* **Rumin operator** *is defined by*

$$D\omega = d(\omega + \omega_v).$$

Remark 2.1.4 ([38])**.** *If ω is vertical, then $D\omega = d(\omega - \omega) = 0$.*
If $\omega = d\alpha \wedge \varphi$, then $D\omega = d(\omega - \alpha \wedge d\varphi) = 0$.
If ω is closed, then $D\omega = 0$ ($\omega_v = 0$ is vertical).

Let G be a compact subgroup of $\mathrm{SO}(V)$ acting transitively on $S(V)$.
We define the following subspaces of $\Omega^{k,l}(SV)$. As before the superscript G denotes the space of G-invariant forms. We define

$$\mathcal{I}^{k,l}(SV) := \{\omega \in \Omega^{k,l}(SV) \,|\, \omega = \alpha \wedge \xi + d\alpha \wedge \psi,\ \xi \in \Omega^{k-1,l}(SV),\ \psi \in \Omega^{k-1,l-1}(SV)\},$$

$\Omega^{k,l}_v(SV)^G := \{\omega \in \Omega^{k,l}(SV)^G \,|\, \alpha \wedge \omega = 0\}$ space of **vertical forms**,

$\Omega^{k,l}_h(SV)^G := \Omega^{k,l}(SV)^G \big/ \Omega^{k,l}_v(SV)^G$ space of **horizontal forms**,

$\Omega^{k,l}_p(SV)^G := \Omega^{k,l}(SV)^G \big/ \mathcal{I}^{k,l}(SV)^G$ space of **primitive forms**.

Multiplication by the symplectic form $d\alpha$ induces an operator $L : \Omega_h^{k,l}(SV)^G \to \Omega_h^{k+1,l+1}(SV)^G$ which is injective for $k+l \leq n-2$ and surjective for $k+l \geq n-2$ ([42]).
Moreover :

$$\Omega_p^{k,l}(SV)^G = \Omega_h^{k,l}(SV)^G \Big/ L(\Omega_h^{k-1,l-1}(SV)^G) \, .$$

The exterior derivative $d : \Omega^{k,l}(SV)^G \to \Omega^{k,l+1}(SV)^G$ induces an operator

$$d_Q : \Omega_p^{k,l}(SV)^G \to \Omega_p^{k,l+1}(SV)^G,$$

since $d(\mathcal{I}^{k,l}(SV)^G) \subset \mathcal{I}^{k,l+1}(SV)^G$:

$$d(\alpha \wedge \xi + d\alpha \wedge \psi) = d\alpha \wedge \xi - \alpha \wedge d\xi + d\alpha \wedge d\psi \in \mathcal{I}^{k,l+1}(SV). \qquad \checkmark$$

Moreover, since multiples of α and of $d\alpha$ lie in the kernel of D (cf. Remark 2.1.4),

$$D : \Omega_p^{k,n-k-1}(SV)^G \to \Omega_v^{k,n-k}(SV)^G.$$

Lemma 2.1.5 ([16]). *Let $\omega \in \Omega^{k,l}(SV)^G$ with $d\omega = 0$. Then*

i) If $0 < l < n-1$, there exists $\varphi \in \Omega^{k,l-1}(SV)^G$ with $d\varphi = \omega$,

ii) If $l = 0$, then $\omega \in (\Lambda^k V)^G \subset \Omega^{k,0}(SV)^G$.

Proof. We write

$$\omega := \sum c_i \phi_i \wedge \tau_i,$$

where $\phi_i \in \Lambda^k V$ and $\tau_i \in \Omega^l(S(V))$.
Then

$$d\omega = \sum c_i \phi_i \wedge d\tau_i = 0,$$

implies $d\tau_i = 0$, i.e. the τ_i's are closed.
For $0 < l < n-1$: Since the l-th de Rham cohomology of the sphere $S(V)$ is 0, we find $\rho_i \in \Omega^{l-1}(S(V))$ with $d\rho_i = \tau_i$.
Then $\tilde{\varphi} := \sum c_i \phi_i \wedge \rho_i$ satisfies $d\tilde{\varphi} = \omega$, but is not G-invariant in general.
Define $\varphi := \int_G g^* \tilde{\varphi} dg$. Then $\varphi \in \Omega^{k,l-1}(SV)^G$ and $d\varphi = \omega$.

If $l = 0$, then all the τ_i's are constant, hence $\omega \in (\Lambda^k V)^G$. □

2.2 Link to valuations

The method presented in this section was developed by Bernig in [16] to compute the dimensions of the spaces Val^G with $G = \mathrm{Sp}(n), \mathrm{Sp}(n) \cdot \mathrm{U}(1), \mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$.

For $K \in \mathcal{K}(V)$ let as above $N(K)$ be its normal cycle.
We define the operator

$$\begin{aligned} nc : \Omega_p^{k,n-k-1}(SV)^G &\to Val_k^G \\ \omega &\mapsto \int_{N(K)} \omega. \end{aligned}$$

This map is well defined, since $N(K)$ is horizontal and legendrian.

Theorem 2.2.1 ([16]). *For $0 < k < n$, the sequence*

$$0 \longrightarrow (\Lambda^k V)^G \longrightarrow \Omega_p^{k,0}(SV)^G \xrightarrow{d_Q} \ldots \xrightarrow{d_Q} \Omega_p^{k,n-k-1}(SV)^G \xrightarrow{nc} Val_k^G \longrightarrow 0$$

is exact.

Proof. By Lemma 2.1.5, ii) the injectivity of the map $(\Lambda^k V)^G \longrightarrow \Omega_p^{k,0}(SV)^G$ is clear. ✓

For $0 < l < n-1$, let $[\omega] \in \Omega_p^{k,l}(SV)^G$ with $d_Q[\omega] = 0$, i.e. $\omega \in \Omega^{k,l}(SV)^G$ with $d\omega = \alpha\wedge\xi + d\alpha\wedge\psi \in \mathcal{I}^{k,l+1}(SV)^G$.
Define $\omega' := \omega - \alpha\wedge\psi$ and $\xi' := \xi + d\psi$.
We obtain

$$d\omega' = \alpha\wedge\xi + d\alpha\wedge\psi - d\alpha\wedge\psi + \alpha\wedge d\psi = \alpha\wedge\xi'.$$

Differentiation yields

$$0 = d\alpha\wedge\xi' - \alpha\wedge d\xi',$$

hence $L(\xi'\big|_Q) = 0$ where L is the multiplication by $d\alpha$, which is injective for $k + l \leq n - 2$. Therefore $\xi'\big|_Q = 0$, so we can write $\xi' = \alpha\wedge\xi''$, and so $d\omega' = \alpha\wedge\xi' = 0$. By Lemma 2.1.5 there exists $\varphi' \in \Omega^{k,l-1}(SV)^G$ such that $d\varphi' = \omega'$.
Hence $[\omega] = [\omega'] = [d\varphi'] = d_Q[\varphi']$, i.e. $[\omega]$ is d_Q-exact. ✓
The surjectivity of the map nc follows from Alesker's irreducibility theorem 1.1.5. Let μ be a G-invariant valuation of degree k. We may approximate μ by a sequence of valuations μ_ω of the form

$$\mu_\omega : K \mapsto \int_{N(K)} \omega.$$

Averaging these valuations over G, we may in fact approximate μ by a sequence of G-invariant such valuations μ_ω^G. But the space of G-invariant valuations of this type is finite dimensional, hence closed. So μ itself is of the form $K \mapsto \int_{N(K)} \omega$ for some $\omega \in \Omega^{k,n-k}(SV)^G$. ✓

Finally, if $[\omega] \in \Omega_p^{k,n-k-1}(SV)^G$ with $nc[\omega] = 0$, then by Theorem

$$D[\omega] = d(\omega + \omega_v) = 0$$

for some vertical form $\omega_v \in \Omega_v^{k,n-k-1}(SV)^G$. Then $\omega' := \omega + \xi$ is a closed translation invariant G-invariant form of bidegree $(k, n-k-1)$, hence by Lemma 2.1.5 there exists $\varphi \in \Omega^{k,n-k-2}(SV)^G$ with $d\varphi = \omega$, i.e. $[\omega] = [\omega'] = d_Q[\varphi]$ is d_Q-exact. ✓

□

Corollary 2.2.2 ([16]). *For $0 \leq k, l \leq n$, set*

$$b_k := \dim(\Lambda^k V)^G,$$
$$b_{k,l} := \dim \Omega_h^{k,l}(SV)^G,$$

and $b_k = 0$, $b_{k,l} = 0$ for other values of k and l.
Then for $0 \leq k \leq n$:

$$\dim Val_k^G = \sum_{l=0}^{n-k-1} (-1)^{n-k-l-1}(b_{k,l} - b_{k-1,l-1}) + (-1)^{n-k} b_k.$$

Proof. For $k = n$, the equation becomes

$$\dim Val_n^G = 1 = b_n$$

which is correct ([30]).

For $k = 0$, we have $b_{0,l} = \dim \Omega^l(S(V))^G$ and the cohomology of the complex
$0 \longrightarrow \Omega^0(S(V))^G \xrightarrow{d} \Omega^1(S(V))^G \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{n-1}(S(V))^G \longrightarrow 0$
is the same as the cohomology of the usual de Rham complex, hence

$$\sum_{l=0}^{n-1}(-1)^{n-l-1} b_{0,l} + (-1)^n b_0 = (-1)^{n-1} \underbrace{\chi(S(V))}_{=0} + (-1)^n = 1,$$

which equals $\dim Val_0^G$.

For $0 < k < n$, by Theorem 2.2.1 we have

$$\begin{aligned}\dim Val_k^G &= \sum_{l=0}^{n-k-1} (-1)^{n-k-l-1} \dim \Omega_p^{k,l}(SV)^G + (-1)^{n-k} \dim(\Lambda^k V)^G \\ &= \sum_{l=0}^{n-k-1} (-1)^{n-k-l-1} (b_{k,l} - b_{k-1,l-1}) + (-1)^{n-k} b_k,\end{aligned}$$

using the fact that for $l \leq n-k-1$, L is injective and

$$\Omega_p^{k,l}(SV)^G = \Omega_h^{k,l}(SV)^G \Big/ L(\Omega_h^{k-1,l-1}(SV)^G)\ . \qquad \square$$

Example : $G = \mathrm{SO}(V)$.

As an example, we use corollary 2.2.2 to verify the classical Hadwiger's theorem. Let therefore $V = \mathbb{R}^n$ and $G = \mathrm{SO}(n)$.
The coefficients $b_k = \dim(\Lambda^k \mathbb{R}^n)^{\mathrm{SO}(n)}$ are 1 for $k = 0, n$ and 0 for other values of k.
We can hence already compute the dimension of $\mathrm{Val}_n^{\mathrm{SO}(n)}$:

$$\dim \mathrm{Val}_n^{\mathrm{SO}(n)} = (-1)^0 b_n = 1,$$

which is correct.
To compute the coefficients $b_{k,l}$ for $0 \leq k \leq n-1$, we use the description of $\Omega_h^{k,l}(SV)^G$ which will be explained in Section 5.2

$$\Omega_h^{k,l}(SV)^G \cong \Lambda^{k,l}(T \oplus T)^{Stab},$$

where $T = T_v(S(V))$ and $Stab$ is the stabilizer of $(0, v) \in SV$. In our case, $Stab = \mathrm{SO}(n-1)$ and $T = \mathbb{R}^{n-1}$. So we have

$$b_{k,l} = \dim \Lambda^{k,l}(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})^{\mathrm{SO}(n-1)}.$$

Fu gives in [25] a set of generators of $\Lambda^*(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})^{\mathrm{SO}(n-1)}$ in the following way :
Choose generators $dx_1, ..., dx_{n-1}, dy_1, ..., dy_{n-1}$ of $\Lambda^*(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})$ such that

$$dx_i(e_j, 0) = dy_i(0, e_j) = \delta_{ij},$$

where $e_1, ..., e_{n-1}$ is the standard orthonormal basis of $\mathbb{R}^{n-1}$, and

$$dx_i(0, v) = dy_i(v, 0) = 0 \qquad \text{for all } v \in \mathbb{R}^{n-1}.$$

Define

$$\omega := \sum_{i=1}^{n-1} dx_i \wedge dy_i,$$

and

$$\kappa_* := \bigwedge_{i=1}^{n-1} (dx_i + dy_i).$$

ω and κ_* are invariant under the action of $\mathrm{SO}(n-1)$. Furthermore, since the subspaces

$$\Lambda(i, n-i-1) := \Lambda^i(\mathbb{R}^{n-1} \oplus 0) \wedge \Lambda^{n-i-1}(0 \oplus \mathbb{R}^{n-1})$$

are invariant, with $\Lambda^{n-1}(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}) = \bigoplus_{i=0}^{n-1} \Lambda(i, n-i-1)$, it follows that the components $\kappa_i \in \Lambda(i, n-i-1)$, $i = 0, ..., n-1$, of κ_* are all invariant.

Lemma 2.2.3. *The exterior algebra of invariant elements of* $\Lambda^*(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})$ *is generated by* $\omega, \kappa_1, ..., \kappa_{n-1}$.

Remark that $\omega \in \Lambda(1,1)$ with the same notation as before, and that $\kappa_i^2 = 0$ for $i \neq \frac{n-1}{2}$, since $\kappa_i^2 \in \Lambda(2i, 2n-2i-2) = 0$ and either $2i$ or $2n-2i-2$ is strictly bigger than $n-1$ ($n \geq 2$). The element $\kappa^2_{\frac{n-1}{2}}$ is of bidegree $(n-1, n-1)$, and plays therefore no role in our computation, since $b_{n-1,n-1} = \dim \Lambda^{n-1,n-1}(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})^{\mathrm{SO}(n-1)}$ does not appear in the formula of corollary 2.2.2. An element of $\Lambda^{k,l}(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})^{\mathrm{SO}(n-1)}$ is therefore of the form

$$\eta := r \cdot \omega^\alpha + \sum_{i=1}^{n-1} \beta_i \kappa_i,$$

where $\alpha \in \{0, 1, ..., n-1\}$ and $r, \beta_i \in \mathbb{R}$. η is of bidegree (k,l) if and only if

$$\begin{cases} \alpha + \sum_{i=1}^{n-1} \beta_i' \cdot i = k \\ \alpha + \sum_{i=1}^{n-1} \beta_i' \cdot (n-i-1) = l \end{cases}$$

where β_i' is 1 if $\beta_i \neq 0$ and 0 if $\beta_i = 0$.

There are two possible cases.
First case : $\sum \beta_i' \neq 0$. The second equation becomes

$$\alpha + (n-1) \sum \beta_i' - l = \sum i \beta_i',$$

which inserted in the first equation implies

$$2\alpha + (n-1) \sum \beta_i' = k + l. \tag{2.1}$$

Since $0 \leq k \leq n-1$ and $0 \leq l \leq n-k-1$

$$0 \leq k + l \leq n - 1.$$

It follows that $\alpha = 0$, and only one β_j' can be 1 and so $\eta = \beta_j \kappa_j$. Equation (2.1) becomes $k + l = n - 1$, hence the bidegree of η is $(k,l) = (k, n-k-1)$, so $\eta = \beta_k \kappa_k$.

Second case : $\sum \beta_i' = 0$. Since $\beta_i' \in \{0, 1\}$, this implies that all β_i' are 0, and the system of equations becomes

$$k = \alpha = l,$$

and $\eta = r\omega^k$ is of bidegree $(k,l) = (k,k)$.

Remark that for $n = 2m+1$ odd, the space of invariant (m,m)-forms is 2-dimensional and spanned by the forms

$$\omega^m \qquad \text{and} \qquad \kappa_m.$$

The dimension of $\Lambda^{k,l}(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})^{\mathrm{SO}(n-1)}$ is therefore

$$b_{k,l} = \begin{cases} 1, & \text{if } k \neq \frac{n-1}{2} \text{ and either } k = l \text{ or } k + l = n-1, \\ 2, & \text{if } k = l = \frac{n-1}{2}, \\ 0, & \text{else.} \end{cases}$$

The formula from corollary 2.2.2 becomes for $1 \leq k \leq n-1$

$$\begin{aligned} \mathrm{Val}_k^{\mathrm{SO}(n)} &= (-1)^{n-k} \underbrace{b_k}_{=0} + \sum_{l=0}^{n-k-1} (-1)^{n-k-l-1} (b_{k,l} - b_{k-1,l-1}) \\ &= \begin{cases} (-1)^0 b_{k,n-k-1} + (-1)^{n-2k-1}(b_{k,k} - b_{k-1,k-1}), & \text{if } 1 \leq k < \frac{n-1}{2} \\ (-1)^0 b_{k,n-k-1}, & \text{if } \frac{n-1}{2} < k \leq n-1 \\ (-1)^0 (b_{k,k} - b_{k-1,k-1}), & \text{if } k = \frac{n-1}{2} \end{cases} \\ &= 1, \end{aligned}$$

which is the expected result.

Finally for $k = 0$ we get

$$\begin{aligned} \dim \mathrm{Val}_0^{\mathrm{SO}(n)} &= (-1)^n b_0 + \sum_{l=0}^{n-1} (-1)^{n-l-1} b_{0,l} \\ &= (-1)^n + (-1)^{n-1} b_{0,0} + (-1)^0 b_{0,n-1} \\ &= 1, \end{aligned}$$

which is also true.

Chapter 3

Representation theory of Lie groups

3.1 Lie groups and Lie algebras

The representation theory of Lie groups is a wide subject. We give here only the necessary elements for this work. We refer to [22] for proofs.

Definition 3.1.1. *A group G is a* **Lie group** *if G is a manifold and the multiplication $G \times G \to G,\ (g,h) \mapsto gh$ and the inversion $G \to G,\ g \mapsto g^{-1}$ are differentiable.*
A closed subgroup $H \subseteq G$ which is also a submanifold is called **Lie subgroup** *of G.*

Examples of Lie groups are all finite groups or the so called classical Lie groups :

$$Gl(n), Sl(n), SO(n), O(n), U(n), SU(n).$$

Definition 3.1.2. *A* **Lie algebra** *is a vector space $\mathfrak{g}$ with a bilinear product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying*

i) $[X,Y] = -[Y,X]$ *for all* $X, Y \in \mathfrak{g}$

ii) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ *for all* $X, Y, Z \in \mathfrak{g}$ *(Jacobi identity).*

A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called **Lie subalgebra** *of $\mathfrak{g}$ if $\mathfrak{h}$ is closed under the Lie bracket $[.,.]$.*

Theorem 3.1.3. *(i) If G is a Lie group, then $\mathfrak{g} = T_eG$ is a Lie algebra. $\mathfrak{g}$ is also isomorphic to the space of left-invariant vector fields in G.*
(ii) If $\mathfrak{g}$ is a Lie algebra, there exists a unique simply connected Lie group G with Lie algebra $\mathfrak{g}$.
(iii) If G and H are Lie groups with G connected and simply connected, the maps from G to H are in one-to-one correspondence with maps of the associated Lie algebras, by associating to $\rho : G \to H$ its differential $(d\rho)_e : \mathfrak{g} \to \mathfrak{H}$.

There is an exponential map $exp : \mathfrak{g} \to G, X \mapsto exp(X) = \gamma^X(1)$, where γ^X is the integral curve of the left-invariant vector field associated with X.
If G is compact and connected, then exp is surjective. In general, $exp(\mathfrak{g})$ generate the connected component of e in G.

3.2 Representation theory

Definition 3.2.1. *A* **representation of a Lie group** *G is a vector space V together with a homomorphism $\rho : G \to \mathrm{GL}(V)$. We will write gv for $\rho(g)(v)$.*
A **representation of a Lie algebra** *$\mathfrak{g}$ is a vector space V together with a linear map $\phi : \mathfrak{g} \to End(V)$ such that*

$$[\phi X, \phi Y] = \phi([X, Y]).$$

Remark that statement (iii) in the above theorem implies in particular that representations of a connected and simply connected Lie group are in one-to-one correspondence with representations of its associated Lie algebra.

Definition 3.2.2. *A G-***linear map** *φ between two representations V and W is a vector space map $\varphi : V \longrightarrow W$ such that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow g & & \downarrow g \\ V & \xrightarrow{\varphi} & W \end{array}$$

commutes for every $g \in G$. The set of G-linear maps between V and W is a vector space denoted by $Hom_G(V, W)$.
A **subrepresentation** *of a representation V of G is a G-invariant vector subspace.*
V is called **irreducible** *if there is no non-zero G-invariant subspace.*

Proposition 3.2.3. *If V and W are two representations of G, then the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also representation of G, via respectively*

$$g(v \oplus w) = gv \oplus gw \qquad \text{and} \qquad g(v \otimes w) = gv \otimes gw.$$

In particular, the n-th tensor power $V^{\otimes n}$ is also a representation of G, in which the exterior power $\Lambda^n(V)$ and the symmetric power $Sym^n(V)$ are subrepresentations.
$Hom(V, W)$ is also a representation of G through

$$(g\varphi)(v) = g(\varphi(g^{-1}v)),$$

or equivalently

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \downarrow g & & \downarrow g \\ V & \xrightarrow{g\varphi} & W \end{array}$$

commutes. The space $Hom(V, W)^G$ of elements of $Hom(V, W)$ fixed by the action of G is therefore the space of G-linear maps between V and W, i.e.

$$Hom(V, W)^G = Hom_G(V, W).$$

The special case where W is $\mathbb{C}$ implies that the dual space V^* is also a representation with the G-action given by

$$(g\xi)(v) = \xi(g^{-1}v).$$

Proposition 3.2.4. *The following usual identities for vector spaces are also true for representations of Lie groups :*

$$U \otimes (V \oplus W) = (U \otimes V) \oplus (U \otimes W)$$

$$\Lambda^k(V \oplus W) = \bigoplus_{a+b=k} \Lambda^a(V) \otimes \Lambda^b(W)$$

$$\Lambda^k(V^*) = \Lambda^k(V)^*.$$

If V is a representation of G, then it induces a representation of its Lie algebra $\mathfrak{g}$ in the following way : suppose γ_t is a curve in G with $\gamma_0 = e$ and $\gamma_0' = X \in \mathfrak{g}$. Then the action of X on V is given by

$$X(v) = \frac{d}{dt}\Big|_{t=0} \gamma_t(v).$$

Theorem 3.2.5 (Complete reducibility theorem)**.** *Any representation of a compact Lie group G is a direct sum of irreducible representations.*

Corollary 3.2.6. *For a representation V of a Lie group G, the dimension of the subspace V^G of G-invariant elements of V is the coefficient of the trivial representation in the decomposition of V in irreducible representations.*

Proof. Let V be an irreducible G-representation. Then the space

$$V^G = \{v \in V \mid gv = v \ \forall g \in G\}$$

is a closed G-invariant subspace of V. Since V is irreducible, either $V^G = V$ or $V^G = 0$. The first case is only possible if $V = V_{triv} = \mathbb{C}$ is the trivial representation.
Let now V be an arbitrary representation. Then by the complete reducibility theorem

$$V = \bigoplus_\alpha n_\alpha V_\alpha,$$

where V_α are irreducible representations. Then

$$V^G = \bigoplus_\alpha n_\alpha V_\alpha^G = n_{triv} V_{triv},$$

hence

$$\dim V^G = n_{triv}.$$

□

Theorem 3.2.7 (Schur's lemma)**.** *If V and W are irreducible representations of a Lie group G, then*

$$\dim Hom_G(V, W) = \begin{cases} 1 & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases}$$

3.3 Irreducible representations of so$(2m+1)$

We will here no longer present the general definitions but only give the necessary tools in order to compute the dimensions of the G-invariant spaces of interest in Chapter 5. Further details and general definitions can be found e.g. in [27].
Let us therefore consider only representations of the Lie algebra so$(2m+1)$.
We can associate to any irreducible representation V an element of a lattice in $\mathbb{R}^m$, called **weight lattice**. For the irreducible representations of so$(2m+1)$, this lattice Λ is generated by vectors $L_1, L_2, ..., L_m$ and $(L_1 + ... + L_m)/2$. The elements of this lattice which are associated to irreducible representations of so$(2m+1)$ are of the form

$$\sum_{i=1}^m \lambda_i L_i$$

where $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0$ and the λ_i are either all integers or all half-integers.
This vector is called the **highest weight** of the irreducible representation.
The irreducible representation with highest weight $\sum \lambda_i L_i$ will also be denoted by $\Gamma_{[\lambda_1,...,\lambda_m]}$.

Moreover, the exterior powers of the standard representation generate all the irreducible representations whose highest weights are in the sublattice $\mathbb{Z}\{L_1, ..., L_m\}$. In fact we have :

Theorem 3.3.1 ([27])**.** *For $k = 1, ..., m-1$, the exterior power $\Lambda^k(V_{st})$ of the standard representation V_{st} of* so$(2m+1)$ *is the irreducible representation with highest weight*

$$L_1 + ... + L_k.$$

The spin representation S is the irreducible representation with highest weight

$$\frac{1}{2}(L_1 + ... + L_m).$$

These highest weights are called **fundamental weights** *and the associated representations* **fundamental representations**.

A useful tool to study representations is the notion of character of a given representation. The general definition is somewhat technical, therefore we choose here to use another approach which is sufficient for our aim : we use Weyl character formula to define the character of irreducible representations and deduce from those the character of any representation.

We form the **representation ring** R of so$(2m+1)$ by taking the free abelian group on the isomorphism classes $[V]$ of finite-dimensional representations V and dividing by the equivalence relation $[V] = [U] + [W]$ whenever $V \cong U \oplus W$. By the complete reducibility theorem 3.2.5, it follows that R is a free abelian group on the classes $[V]$ of irreducible representations. The tensor product of representations makes R into a ring : $[V] \cdot [W] = [V \otimes W]$.
In fact, R is a polynomial ring on the classes $[V_{st}], [\Lambda^2 V_{st}], ..., [\Lambda^{m-1} V_{st}], [S]$ of the fundamental representations

$$R = \mathbb{Z}[[V_{st}], [\Lambda^2 V_{st}], ..., [\Lambda^{m-1} V_{st}], [S]].$$

Let Λ be as before the lattice generated by the vectors $L_1, ..., L_m$ and $(L_1 + ... + L_m)/2$, and let $\mathbb{Z}[\Lambda]$ be the integral group ring on the abelian group Λ.

Definition 3.3.2 (Weyl character formula)**.** *The* **character of an irreducible representation** $\Gamma_{[\lambda_1,...,\lambda_m]} = \Gamma_\lambda$ *is defined by*

$$Char(\Gamma_\lambda) := \frac{\left| x_j^{\lambda_i+m-i+1/2} - x_j^{-(\lambda_i+m-i+1/2)} \right|}{\left| x_j^{m-i+1/2} - x_j^{-(m-i+1/2)} \right|} \in \mathbb{Z}[\Lambda],$$

where $x_j^{\pm 1}$ and $x_j^{\pm 1/2}$ are the elements of $\mathbb{Z}[\Lambda]$ corresponding to the weights $\pm L_j$ and $\pm\frac{1}{2}L_j$ respectively.

For the fundamental representations of so$(2m+1)$, this formula yields that the character of $\Lambda^k V_{st}$ is the k-th elementary symmetric polynomial of the $2m+1$ elements $x_1, x_1^{-1}, ..., x_m, x_m^{-1}$ and 1; denote it by B_k. The character of the spin representation S is $B := \sum x_1^{\pm 1/2} \cdot ... \cdot x_m^{\pm 1/2}$ which we can see as the m-th elementary symmetric polynomial in the variables $x_i^{1/2} + x_i^{-1/2}$, $i = 1, ..., m$.

Then the above considerations together with the following theorem allow us to define the character of any so$(2m+1)-$representation through :

Theorem 3.3.3. *The homomorphism*

$$Char : R = \mathbb{Z}[[V_{st}], [\Lambda^2 V_{st}], ..., [\Lambda^{m-1} V_{st}], [S]] \longrightarrow \mathbb{Z}[B_1, ..., B_{m-1}, B] \subset \mathbb{Z}[\Lambda]$$

is an isomorphism.

In practice, there is no easy way to find the decomposition of an arbitrary representation as an element of $R = \mathbb{Z}[[V_{st}], [\Lambda^2 V_{st}], ..., [\Lambda^m V_{st}], [S]]$. But in Section 5.4 we will see that the only representations of interest in our case are exterior powers of irreducible representations or sums of irreducible representations. To compute the character of a representation given in this form, we can use the following recurrence formula :

Theorem 3.3.4 (Adams formula, [21]). *Define the* **Adams operator** $\psi^k : \mathbb{Z}[\Lambda] \longrightarrow \mathbb{Z}[\Lambda]$ *by* $\psi^k(x_j) = x_j^k$.
Then we have for any so$(2m+1)-$*representation* V

$$Char(\Lambda^d V) = \frac{1}{d} \sum_{k=1}^{d} (-1)^{k-1} \psi^k(CharV) Char(\Lambda^{d-k} V).$$

This formula allows us to compute inductively the character of $\Lambda^k V$. The next step is to write the obtained polynomial as linear combination of characters of irreducible representations. Since a character determines completely the associated representation, if

$$Char(V) = \bigoplus n_\lambda Char(\Gamma_\lambda),$$

then

$$V = \bigoplus n_\lambda \Gamma_\lambda.$$

To decompose the character of a representation in characters of irreducible representations, we need two observations :

- The leading monomial of the character of the irreducible representation $\Gamma_{[\lambda_1,...,\lambda_m]}$ is $x_1^{\lambda_1} \cdot ... \cdot x_m^{\lambda_m}$, where the leading monomial of a polynomial is the monomial of highest degree (with respect to the lexicographic order).
- If the leading monomial of the character of a representation V is $n_\lambda x_1^{\lambda_1} \cdot ... \cdot x_m^{\lambda_m}$, then the leading monomial of the character of $V - n_\lambda \Gamma_{[\lambda_1,...,\lambda_m]}$ is of strictly lower degree.

Therefore we apply the following algorithm to decompose the character of a representation V in irreducible characters :

a) Find the leading monomial of $Char(V)$: $n_\lambda x_1^{\lambda_1} \cdot ... \cdot x_m^{\lambda_m}$,

b) Compute $Char(\Gamma_{[\lambda_1,...,\lambda_m]})$ with Weyl character formula,

c) Compute $Char(V - n_\lambda \Gamma_{[\lambda_1,...,\lambda_m]})$,

d) Find the leading monomial of the new polynomial. If it is not a constant, start over with b), else we have the decomposition of V.

After at most $deg\, Char(V)$ steps, we obtain the decomposition of $Char(V)$.

Chapter 4

The spin groups

4.1 The spin groups in terms of Clifford algebras

We present here a description from [27] of the spin groups in terms of Clifford algebras. This representation will not be the one we use to describe the Spin(9) representation, but it gives some more information about the structure of the spin groups.

Let V be a finite dimensional vector space, with dimension n, and let Q be a symmetric bilinear form on V.
Consider the special orthogonal group $\mathrm{SO}(Q)$ of V.
It is known that $\pi_1(\mathrm{SO}(Q)) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. This result induces that, for $n \geq 3$, $\mathrm{SO}(Q)$ has a connected double covering, and this double covering is called the **spin group** $\mathrm{Spin}(Q)$, for which holds the following short exact sequence :

$$\{1\} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{Spin}(Q) \longrightarrow \mathrm{SO}(Q) \longrightarrow \{e\}.$$

Definition 4.1.1. *The* **Clifford algebra** $C = C(Q)$ *of* V *is an associative algebra with unit* 1*, which contains and is generated by* V*, with* $v \cdot v = -Q(v,v) \cdot 1$ *for all* $v \in V$*. By polarization, we get equivalently*

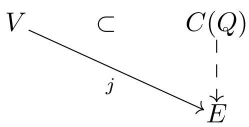
$$v \cdot w + w \cdot v = -2Q(v,w) \cdot 1, \qquad \forall v, w \in V.$$

Remark 4.1.2. *We could also use the condition* $v \cdot v = Q(v,v) \cdot 1$ *(as in [27]). We preferred to use the one with which* Q *is positive definite.*

The Clifford algebra can also be defined as the universal algebra with the following property :
if E is any associative algebra with unit, and there is a linear mapping $j : V \to E$ such that $j(v)^2 = -Q(v,v) \cdot 1$ or equivalently

$$j(v) \cdot j(w) + j(w) \cdot j(v) = -2Q(v,w) \cdot 1, \qquad \forall v, w \in V,$$

then there should be a unique homomorphism of algebras from $C(Q)$ to E extending j :



The Clifford algebra can be constructed by taking the tensor algebra $T^\bullet(V) = \bigoplus_{\nu \geq 0} V^{\otimes \nu}$ and setting

$$C(Q) = T^\bullet(V) \big/ I(Q) \ ,$$

where $I(Q)$ is the ideal generated by all elements of the form $v \otimes v + Q(v,v) \cdot 1$.
Then holds

Lemma 4.1.3. *If $e_1, ..., e_n$ form a basis of V, then the products $e_{i_1} \cdot ... \cdot e_{i_k}$ with $1 \leq i_1 < ... < i_k \leq n$ and 1 form a basis of $C(Q)$. In particular, $\dim C(Q) = 2^n$. Moreover we have the following relations if we take the canonical basis $e_1, ..., e_n$ of V :*

$$e_i \cdot e_j = -e_j \cdot e_i, \qquad e_i \cdot e_i = -1.$$

On $C(Q)$ we have different operations :

1. An anti-involution called the **conjugation** $^-: C(Q) \to C(Q)$ defined by

   $$\overline{v_1 \cdot ... \cdot v_r} = (-1)^r v_r \cdot ... \cdot v_1, \qquad \forall v_1, ..., v_r \in V,$$

   which is the composition of

2. the **canonical anti-automorphism** $t : C(Q) \to C(Q)$ defined by

   $$t(v_1 \cdot ... \cdot v_r) = v_r \cdot ... \cdot v_1, \qquad \forall v_1, ..., v_r \in V,$$

   and

3. the **canonical automorphism** $\alpha : C(Q) \to C(Q)$ defined by

   $$\alpha(v_1 \cdot ... \cdot v_r) = (-1)^r v_1 \cdot ... \cdot v_r, \qquad \forall v_1, ..., v_r \in V.$$

For $\nu = 0, 1$, we consider the eigenspaces $C(Q)^\nu$ of α for the eigenvalues $(-1)^\nu$,
i.e. $C(Q)^\nu = \{x \in C(Q) \mid \alpha(x) = (-1)^\nu x\}$, and we see that
$C(Q)^0 =: C(Q)^{\text{even}}$ is spanned by the products of an even number of elements in V, and
$C(Q)^1 =: C(Q)^{\text{odd}}$ is spanned by the products of an odd number of elements in V.
It follows

$$C(Q) = C(Q)^{\text{even}} \oplus C(Q)^{\text{odd}},$$

and $C(Q)^{\text{even}}$ is a subalgebra of $C(Q)$.

We start now with an abstract definition of the spin group and verify that it is the double covering of $\mathrm{SO}(V)$.

Definition 4.1.4.

$$\mathrm{Spin}(Q) := \{x \in C(Q)^{\text{even}} \mid x \cdot \bar{x} = 1 \text{ and } x \cdot V \cdot \bar{x} \subset V\}.$$

Any $x \in \mathrm{Spin}(Q)$ determines an endomorphism $\rho(x)$ on V by

$$\rho(x)v = x \cdot v \cdot \bar{x},$$

and we have

Proposition 4.1.5. *For $x \in \mathrm{Spin}(Q)$, $\rho(x)$ is in $\mathrm{SO}(Q)$. The mapping*

$$\rho : \mathrm{Spin}(Q) \to \mathrm{SO}(Q)$$

*is an homomorphism, making $\mathrm{Spin}(Q)$ a connected double covering of $\mathrm{SO}(Q)$.
The kernel of ρ is $\{\pm 1\}$.*

Proof. Consider the larger subgroup, called **pin group**,

$$\mathrm{Pin}(Q) := \{x \in C(Q) \mid x \cdot \bar{x} = 1 \text{ and } x \cdot V \cdot \bar{x} \subset V\},$$

and the homomorphism

$$\tilde{\rho} : \mathrm{Pin}(Q) \to \mathrm{O}(Q), \qquad \tilde{\rho}(x)v = \alpha(x) \cdot v \cdot \bar{x}.$$

Then we have $\mathrm{Spin}(Q) = \mathrm{Pin}(Q) \cap C(Q)^{\mathrm{even}}$, and $\tilde{\rho}|_{\mathrm{Spin}(Q)} = \rho$.

1st claim : for $x \in \mathrm{Pin}(Q)$: $\tilde{\rho}(x) \in \mathrm{O}(Q)$, i.e. $\tilde{\rho}$ preserves the quadratic form Q.
First note that for $w \in V$, $\bar{w} = -w$, and therefore $Q(w,w) = -w \cdot w = w \cdot \bar{w}$. So we can compute

$$\begin{aligned} Q(\tilde{\rho}(x)v, \tilde{\rho}(x)v) &= \alpha(x) \cdot v \cdot \bar{x} \cdot \overline{(\alpha(x) \cdot v \cdot \bar{x})} \\ &= \alpha(x) \cdot v \cdot \bar{x} \cdot x \cdot \bar{v} \cdot \overline{\alpha(x)} \\ &= \alpha(x) \cdot v \cdot \bar{v} \cdot \alpha(\bar{x}) \\ &= Q(v,v)\alpha(x) \cdot \alpha(\bar{x}) \\ &= Q(v,v)\alpha(x \cdot \bar{x}) \\ &= Q(v,v). \qquad \checkmark \end{aligned}$$

2nd claim : $\tilde{\rho}$ is surjective.
It is a well known fact that the orthogonal group $\mathrm{O}(Q)$ is generated by reflections. Taking R_w as the reflection in the hyperplane perpendicular to w, normalizing w so that $Q(w,w) = 1$, we see

$$w \cdot \bar{w} = w \cdot (-w) = Q(w,w) \cdot 1 = 1,$$

so $w \in \mathrm{Pin}(Q)$,

$$\tilde{\rho}(w)w = \alpha(w) \cdot w \cdot \bar{w} = -w \cdot 1 = -w,$$

and for $v \in V$ with $Q(v,w) = 0$ (v perpendicular to w)

$$\tilde{\rho}(w)v = \alpha(w) \cdot v \cdot \bar{w} = -w \cdot v \cdot \bar{w} = (v \cdot w - 2Q(v,w) \cdot 1) \cdot \bar{w} = v \cdot w \cdot \bar{w} = v. \qquad \checkmark$$

3rd claim : the kernel of $\tilde{\rho}$ is $\{\pm 1\}$.
Suppose x is in the kernel, and decompose $x = x_0 + x_1$ with $x_0 \in C(Q)^{\mathrm{even}}$ and $x_1 \in C(Q)^{\mathrm{odd}}$. Then

$$\alpha(x) \cdot v = v \cdot \alpha(x) \ \forall v \in V,$$

and so

$$x_0 \cdot v = v \cdot x_0 \text{ and } -x_1 \cdot v = v \cdot x_1 \ \forall v \in V. \tag{4.1}$$

Next we can write x_0 as a linear combination of monomials in the canonical basis of V (cf. Lemma 4.1.3), so

$$x_0 = a_0 + e_1 b_1, \qquad \text{with } a_0 \in C(Q)^{\mathrm{even}},\ b_1 \in C(Q)^{\mathrm{odd}},$$

where neither a_0 nor b_1 contain a summand with a factor e_1.
Applying the first relation of (4.1) to $v = e_1$, we get

$$a_0 + e_1 b_1 = e_1^{-1}(a_0 + e_1 b_1)e_1.$$

Since each monomial in a_0 is of even degree and contains no factor e_1, we have $e_1 a_0 = a_0 e_1$. Similarly $e_1 b_1 = -b_1 e_1$, and so it becomes

$$a_0 + e_1 b_1 = a_0 - e_1 b_1.$$

We conclude that $e_1 b_1 = 0$ and therefore that x_0 contains no monomial with a factor e_1.
The same argument applied successively to the other basis elements proves that x_0 can be written as linear combination of monomials with no $e_i, i = 1, ..., n$ as factor, i.e. $x_0 \in \mathbb{R} \cdot 1$.
Proceeding similarly with the second relation in (4.1), with $x_1 = a_1 + e_1 b_0$ and $v = e_1$, we get

$$a_1 + e_1 b_0 = -e_1 a_1 e_1 - b_0 e_1 = a_1 - e_1 b_0.$$

Thus $b_0 = 0$ and $x_1 \in \mathbb{R} \cdot 1$. But $\mathbb{R} \cdot 1 \subset C(Q)^{\mathrm{even}}$, so $x_1 = 0$.
In conclusion $x = x_0 \in \mathbb{R} \cdot 1$, and with the condition $x \cdot \bar{x} = x^2 = 1$, we get $x = \pm 1$. $\checkmark$

It follows that if $R \in \mathrm{O}(Q)$ is written as product of reflections $R_{w_1} \cdot ... \cdot R_{w_r}$, then the two elements in $\tilde{\rho}^{-1}(R)$ are $\pm w_1 \cdot ... \cdot w_r$.
In particular, this gives us another description of the spin group

$$\begin{aligned} \mathrm{Spin}(Q) &= \mathrm{Pin}(Q) \cap C(Q)^{\mathrm{even}} = \tilde{\rho}^{-1}(\mathrm{SO}(Q)) \\ &= \{\pm w_1 \cdot ... \cdot w_{2k} \mid w_i \in V, Q(w_i, w_i) = -1\}. \end{aligned}$$

To complete the proof, we must check that $\mathrm{Spin}(Q)$ is connected, or equivalently that the two elements in the kernel of ρ can be connected by a path in $\mathrm{Spin}(Q)$.
Consider the path between $+1$ and -1

$$\gamma : t \mapsto \cos t + \sin t\ e_1 \cdot e_2, \qquad 0 \leq t \leq \pi.$$

We have

$$\overline{\gamma(t)} = \cos t - \sin t\ e_1 \cdot e_2,$$

so $\gamma(t) \cdot \overline{\gamma(t)} = 1$, and therefore $\gamma(t) \in \mathrm{Pin}(Q)$.
Since $\tilde{\rho}(\gamma(t))$ must stay in a connected component of $\mathrm{O}(Q)$ and $\tilde{\rho}(\gamma(0)) = E \in \mathrm{SO}(Q)$:

$$\gamma(t) \in \mathrm{Spin}(Q) \qquad \forall t.$$

□

4.2 Octonions

To be able to give the description of the spin groups in the next section, we present here the octonions as well as some of their useful properties.
We will only give a few basic facts about octonions. We refer to Baez's paper ([12]) for more information.

The first way of constructing the division algebra $\mathbb{O}$ of the octonions can be seen as an extension of the construction of the complex numbers as pairs of real numbers.
We start with the algebra $\mathbb{R}$ of the real numbers. $\mathbb{R}$ is a real commutative associative division algebra.
Any complex number can be identified with a pair of real numbers through

$$z = a + ib,$$

where $a, b \in \mathbb{R}$ and i satisfy the relation $i^2 = -1$. The division algebra $\mathbb{C}$ of the complex numbers is not ordered anymore, but still commutative and associative.
Analogously we can see any quaternion as a pair of complex numbers

$$h = y + jz,$$

where $y, z \in \mathbb{C}$ and j satisfy the same relation $j^2 = -1$. So the quaternions form a division algebra $\mathbb{H}$ of real dimension 4 with basis $1, i, j, i \cdot j =: k$ satisfying

$$i^2 = j^2 = k^2 = ijk = -1.$$

$\mathbb{H}$ is associative but not commutative anymore.
If we repeat this procedure, we get the algebra $\mathbb{O}$ of the octonions : it is a division algebra of real dimension 8 and basis denoted by $1 =: e_0, e_1, ..., e_7$ satisfying

$$e_i^2 = -1, \qquad e_i e_j = -e_j e_i.$$

$\mathbb{O}$ is not associative anymore, only alternative, i.e. any subalgebra generated by any two elements is associative.
However we cannot construct a 16-dimensional division algebra with this procedure. In fact, we have

Proposition 4.2.1 ([20]). *All division algebras have dimension 1,2,4 or 8.*

And also

Proposition 4.2.2 ([31]). *$\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are the only normed division algebras.*

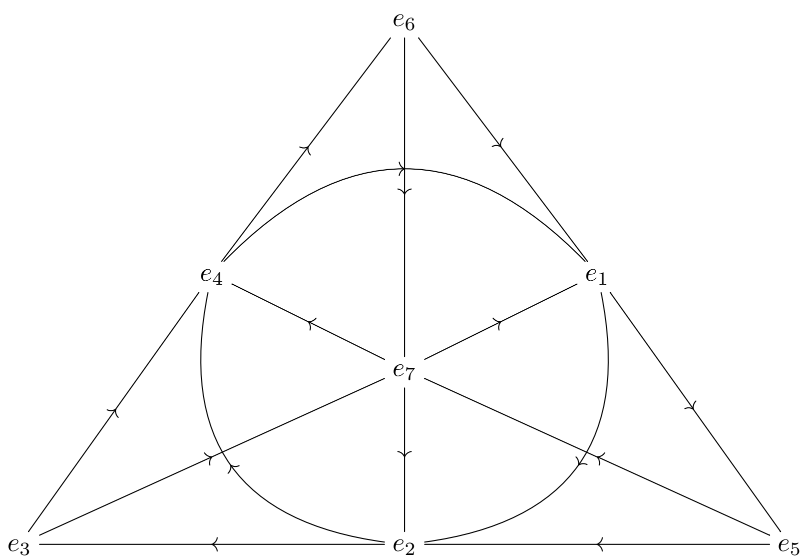
Since $\mathbb{O}$ is no longer commutative nor associative, the computation with octonions has to be performed carefully, using their multiplication table :
Let $e_0 = 1, e_1, ..., e_7$ be the basis elements of $\mathbb{O}$. Then their multiplication is given in this table, whose element on the i-th row and the j-th column is the result of the multiplication of e_i with e_j.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1

We can see a range of properties from the table:

1. $e_i^2 = -1$
2. $e_i e_j = -e_j e_i$ for $i \neq j$
3. $e_i e_j = e_k \Rightarrow e_{i+1} e_{j+1} = e_{k+1}$
4. $e_i e_j = e_k \Rightarrow e_{2i} e_{2j} = e_{2k}$

where $i, j, k > 0$ and the indices in 3. and 4. are considered as elements of $\mathbb{Z}_7$.
With these properties and one non trivial product like $e_1 e_2 = e_4$, we can recover the whole table. This table can be nicely represented in a picture called the **Fano plane** :



Each pair of distinct points lies on a unique line. Each line contains three points, and each of these triples has a cyclic ordering shown by the arrows. If e_i, e_j and e_k are cyclically ordered in this way then

$$e_i e_j = e_k, \qquad e_j e_i = -e_k.$$

These rules, together with

1. $e_0 = 1$ is the identity element,

2. $e_i^2 = -1$ for $i > 0$,

define the algebra structure of $\mathbb{O}$.
The center of $\mathbb{O}$ is equal to $\mathbb{R}$ and we denote by $\mathbb{O}'$ its orthogonal complement in $\mathbb{O}$, i.e. the subspace of pure octonionic elements of $\mathbb{O}$. Every octonion can be written in the form

$$q = \sum_{i=0}^{7} x_i e_i,$$

with $x_i \in \mathbb{R}$.
There is a conjugation map $\bar{}$ in $\mathbb{O}$ defined by

$$q \mapsto \bar{q} = x_0 e_0 - \sum_{i=1}^{7} x_i e_i,$$

and a scalar product $\langle .,. \rangle : \mathbb{O}^2 \longrightarrow \mathbb{R}$

$$\langle x, y \rangle = \left\langle \sum_{i=0}^{7} x_i e_i, \sum_{i=0}^{7} y_i e_i \right\rangle = \sum_{i=0}^{7} x_i y_i = \frac{1}{2}(x\bar{y} + y\bar{x}).$$

As already mentioned, $\mathbb{O}$ is not associative, but alternative. By a theorem of Artin ([39]), an algebra A is alternative if and only if for all $a, b \in A$

$$a(ab) = (aa)b, \qquad (ab)a = a(ba), \qquad (ba)a = b(aa).$$

In fact, any two of these equations imply the third, so we can take only two of them as definition of alternative.
An equivalent definition can be given in the following way. Define on A the trilinear map

$$[\cdot,\cdot,\cdot] : A^3 \longrightarrow A$$

by

$$[a, b, c] = (ab)c - a(bc).$$

This map is called the **associator**. It measures the failure of associativity, analogously to the commutator which measures the failure of commutativity.
Then A is alternative precisely when the associator is an alternating map, i.e. for all $a, b, c \in A$

$$[a, b, c] = -[b, a, c], \text{ or equivalently } [a, a, b] = 0.$$

4.3 An explicit description of the Spin(9) representation

All the results in this section are due to Sudbery ([41],[40]). They lead to an explicit description of the action of Spin(9) on $\mathbb{R}^{16} = \mathbb{O}^2$.

Let $H_2(\mathbb{O})$ be the set of hermitian 2×2 matrices with entries in $\mathbb{O}$, defined by the condition $X^* := \overline{X}^T = X$, where the bar denotes the componentwise conjugation in $\mathbb{O}$. Let $A_2(\mathbb{O})$ be the set of antihermitian 2×2 matrices with entries in $\mathbb{O}$, defined by the condition $X^* = -X$.

$H_2(\mathbb{O})$ forms a 10-dimensional Jordan algebra with the product given by the anticommutator, i.e. if we define

$$X \cdot Y := \frac{1}{2}(XY + YX),$$

then $X \cdot Y$ is a commutative but not associative product satisfying

$$X \cdot (X^2 \cdot Y) = X^2 \cdot (X \cdot Y).$$

We can decompose $H_2(\mathbb{O})$ as

$$\begin{aligned} H_2(\mathbb{O}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{O}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \in M(2,\mathbb{O}) \,\middle|\, a, d \in \mathbb{R}, b \in \mathbb{O} \right\} \\ &= \left\{ \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a-d}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} \,\middle|\, a, d \in \mathbb{R},\, b \in \mathbb{O} \right\} \\ &= \left\{ \alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & b \\ \bar{b} & 0 \end{pmatrix} \,\middle|\, \alpha, \beta \in \mathbb{R},\, b \in \mathbb{O} \right\}. \end{aligned}$$

Let W be the 9-dimensional subspace of $H_2(\mathbb{O})$ generated by

$$P := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S(x) := \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix} \qquad x \in \mathbb{O},$$

with inner product given by

$$g(\alpha P + S(x), \beta P + S(y)) = \alpha\beta + \langle x, y \rangle \qquad \alpha, \beta \in \mathbb{R}, x, y \in \mathbb{O}.$$

Then $H_2(\mathbb{O}) = \mathbb{R} \cdot I \oplus W$. I acts as an identity in the Jordan algebra, and the Jordan product of two elements of W is

$$\begin{aligned} (\alpha P + S(x)) \cdot (\beta P + S(y)) &= \frac{1}{2} \left(\begin{pmatrix} \alpha & x \\ \bar{x} & -\alpha \end{pmatrix} \begin{pmatrix} \beta & y \\ \bar{y} & -\beta \end{pmatrix} + \begin{pmatrix} \beta & y \\ \bar{y} & -\beta \end{pmatrix} \begin{pmatrix} \alpha & x \\ \bar{x} & -\alpha \end{pmatrix} \right) \\ &= \frac{1}{2} \left(\begin{pmatrix} 2\alpha\beta + x\bar{y} + y\bar{x} & 0 \\ 0 & 2\alpha\beta + \bar{x}y + \bar{y}x \end{pmatrix} \right) \\ &= (\alpha\beta + \langle x, y \rangle) I \\ &= g(\alpha P + S(x), \beta P + S(y)) I. \end{aligned}$$

The set of derivations of the Jordan algebra $H_2(\mathbb{O})$, i.e. the set of linear maps $D : H_2(\mathbb{O}) \to H_2(\mathbb{O})$ satisfying

$$D(xy) = (Dx)y + x(Dy), \tag{4.2}$$

is the set of antisymmetric linear maps on W, since derivations on $H_2(\mathbb{O})$ acts only on W (I is the identity of the algebra) and the condition (4.2) is equivalent to

$$g(Dv, w) + g(v, Dw) = 0, \qquad \forall v, w \in W,$$

thus

$$\mathrm{Der} H_2(\mathbb{O}) \cong \mathrm{so}(W) = \mathrm{so}(9). \tag{4.3}$$

On the other hand :
For any associative normed division algebra $\mathbb{K}$, $H_2(\mathbb{K})$ also forms a Jordan algebra with the product given by the anticommutator, and the derivations are all of the form

$$X \mapsto \mathrm{ad}A(X) := [A, X]$$

for some antihermitian matrix A. This is the zero derivation if and only if $A = \lambda I$ with a λ in the center of $\mathbb{K}$.
Although $\mathbb{O}$ is not associative, the identity

$$[A, X \cdot Y] = [A, X] \cdot Y + X \cdot [A, Y]$$

which makes ad$A$ a derivation holds for $A \in A_2(\mathbb{O})$ and $X, Y \in H_2(\mathbb{O})$.
Derivations of $\mathbb{O}$ also act as derivations of $H_2(\mathbb{O})$ by acting on each entries of the matrices. These are all the derivations of $H_2(\mathbb{O})$ and so we have

$$\mathrm{Der}H_2(\mathbb{O}) = \mathrm{ad}A_2(\mathbb{O}) + \mathrm{Der}(\mathbb{O}).$$

The space of antihermitian matrices can be decomposed as

$$\begin{aligned}
A_2(\mathbb{O}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2,\mathbb{O}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\bar{a} & -\bar{c} \\ -\bar{b} & -\bar{d} \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & d \end{pmatrix} \in M(2,\mathbb{O}) \;\middle|\; a, d \in \mathbb{O}',\, b \in \mathbb{O} \right\} \\
&= \left\{ \begin{pmatrix} \frac{a-d}{2} & b \\ -\bar{b} & \frac{-a+d}{2} \end{pmatrix} + \begin{pmatrix} \frac{a+d}{2} & 0 \\ 0 & \frac{a+d}{2} \end{pmatrix} \;\middle|\; a, d \in \mathbb{O}',\, b \in \mathbb{O} \right\} \\
&= A_2'(\mathbb{O}) \oplus \mathbb{O}' \cdot I,
\end{aligned}$$

where $A_2'(\mathbb{O})$ is the subspace of traceless matrices in $A_2(\mathbb{O})$, and $\mathbb{O}' \cdot I$ is the subspace of pure octonionic multiples of the identity matrix I.
For $a \in \mathbb{O}'$, $\mathrm{ad}(aI)$ acts on an element X of $H_2(\mathbb{O})$ by acting as the commutator map $C_a : \mathbb{O} \longrightarrow \mathbb{O}, C_a(x) := [a, x]$ on each entries of X:

$$\begin{aligned}
\mathrm{ad}(aI)(X) &= [aI, X] \\
&= \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ \bar{x}_{12} & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} \\ \bar{x}_{12} & x_{22} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \\
&= \begin{pmatrix} ax_{11} - x_{11}a & ax_{12} - x_{12}a \\ a\bar{x}_{12} - \bar{x}_{12}a & ax_{22} - x_{22}a \end{pmatrix} \\
&= \begin{pmatrix} C_a(x_{11}) & C_a(x_{12}) \\ C_a(\bar{x}_{12}) & C_a(x_{22}) \end{pmatrix}.
\end{aligned}$$

We therefore have $ad(\mathbb{O}'I) = C(\mathbb{O}')$.

Furthermore, since $H_2(\mathbb{O})$ is an irreducible set, $\mathrm{ad}A_2'(\mathbb{O}) = A_2'(\mathbb{O})$. Hence

$$\mathrm{Der}H_2(\mathbb{O}) = A_2'(\mathbb{O}) + C(\mathbb{O}') + \mathrm{Der}(\mathbb{O}). \tag{4.4}$$

In order to obtain a decomposition of $\mathrm{so}(\mathbb{O}')$, we need to give an explicit description of

$$\mathrm{Der}(\mathbb{O}) =: G_2.$$

In fact, we have:

Lemma 4.3.1. *For any $a, b \in \mathbb{O}$, the linear map*

$$D(a,b)x = [a,b,x] + \frac{1}{3}[[a,b],x]$$

is a derivation of $\mathbb{O}$.

Proof. $D(a,b)$ is a derivation if and only if

$$\begin{aligned}
& D(a,b)(xy) = (D(a,b)x)y + x(D(a,b)y) \\
\Leftrightarrow\quad & D(a,b) \circ R_y = R_y \circ D(a,b) + R_{D(a,b)y} \\
\Leftrightarrow\quad & [D(a,b), R_y] = R_{D(a,b)y},
\end{aligned}$$

where R_y is the right multiplication by y.
We prove this last equality.
To do that, we need some formulas: Recall that $\mathbb{O}$ is alternative, i.e. the associator

$$[x,y,z] := x(yz) - (xy)z$$

is an alternating function. This alternative law is equivalent to the following equalities :

$$\begin{aligned} L_xL_y - L_{xy} &= L_{yx} - L_yL_x \\ &= R_{xy} - R_yR_x \\ &= R_xR_y - R_{yx} \\ &= [L_y, R_x] \\ &= -[L_x, R_y], \end{aligned} \tag{4.5}$$

where L_x is the left multiplication by x.
We therefore get the following formulas for the commutators of two right multiplications, respectively left multiplications :

$$\begin{aligned} [R_x, R_y] &= R_xR_y - R_yR_x \\ &= R_xR_y + (R_xR_y - R_{yx} - R_{xy}) \\ &= -2[L_x, R_y] + 2R_{yx} - R_{yx} - R_{xy} \\ &= R_{[x,y]} - 2[L_x, R_y], \end{aligned}$$

and

$$\begin{aligned} [L_x, L_y] &= L_xL_y - L_yL_x \\ &= L_xL_y + L_xL_y - L_{xy} - L_{yx} \\ &= 2(-[L_x, R_y] + L_{xy}) - L_{xy} - L_{yx} \\ &= -2[L_x, R_y] + L_{[x,y]}. \end{aligned}$$

So we can finally compute

$$\begin{aligned} 2[R_y, D(a,b)] &= \frac{2}{3}[R_y, L_{[a,b]} - R_{[a,b]} - 3[L_y, R_b]] \\ &= [R_y, -R_{[a,b]} - 2[L_a, R_b]] + \frac{1}{3}[R_y, R_{[a,b]}] + \frac{2}{3}[R_y, L_{[a,b]}] \\ &= [R_y, [R_a, R_b]] + \frac{1}{3}([R_y, R_{[a,b]}] + [R_{[a,b]}, R_y] + R_{[[a,b],y]}) \\ &= [R_y, [R_a, R_b]] + \frac{1}{3}R_{[[a,b],y]}. \end{aligned}$$

Furthermore for an associative algebra (such as $R(\mathbb{O})$ the set of all R_a with $a \in \mathbb{O}$), we have

$$\begin{aligned} [A, [B,C]] &= A(BC - CB) - (BC - CB)A \\ &= C(AB + BA) + (AB + BA)C - B(AC + CA) - (AC + CA)B. \end{aligned}$$

We have also, by the alternative law :

$$R_xR_y + R_yR_x = R_{xy+yx}.$$

So we get

$$\begin{aligned} [R_y, [R_a, R_b]] &= R_bR_{ay+ya} + R_{ay+ya}R_b - R_aR_{yb+by} - R_{yb+by}R_a \\ &= R_{b(ya)+b(ay)+(ya)b+(ay)b-a(yb)-a(by)-(yb)a-(by)a} \\ &= R_{-2[a,b,y]+[[b,a],y]}. \end{aligned}$$

So it becomes

$$\begin{aligned} 2[R_y, D(a,b)] &= R_{-2[a,b,y]+[[b,a],y]} + \frac{1}{3} R_{[[a,b],y]} \\ &= -2R_{[a,b,y]+\frac{1}{3}[[a,b],y]} \\ &= -2R_{D(a,b)y}. \end{aligned}$$

□

The proof shows that $D(a,b)$ is generated by left and right multiplication maps. In fact it is generated by the commutator maps $C_d := L_d - R_d$, namely

$$D(a,b) = \frac{1}{6}([C_a, C_b] + C_{[a,b]}). \tag{4.6}$$

This equality follows also from the above formulas :

$$\begin{aligned} \frac{1}{6}([C_a, C_b] + C_{[a,b]}) &= \frac{1}{6}([L_a - R_a, L_b - R_b] + L_{[a,b]} - R_{[a,b]}) \\ &= \frac{1}{6}([L_a, L_b] + [R_a, R_b] - [L_a, R_b] - [R_a, L_b] + [L_a, R_b] + 2[L_a, R_b] \\ &\quad +[R_a, R_b] + 2[L_a, R_b]) \\ &= \frac{1}{3}([L_a, L_b] + [R_a, R_b] + [L_a, R_b]) \\ &= \frac{1}{3}(-3[L_a, R_b] + L_{[a,b]} - R_{[a,b]}) \\ &= D(a,b). \end{aligned}$$

Such a derivation, and sums of such derivations, are called **inner derivations**. It can be shown ([39]) that all the derivations of $\mathbb{O}$ are of this type.

In general the antisymmetric maps of $\mathbb{O}$ are given by the derivations and the left and right multiplication maps :

$$\mathrm{so}(\mathbb{O}) = \mathrm{Der}(\mathbb{O}) + L(\mathbb{O}') + R(\mathbb{O}').$$

Let $C_a = L_a - R_a$, be as before the commutator map, and let $C(\mathbb{O}')$ be the set of all C_a with $a \in \mathbb{O}'$. Then each C_a maps $\mathbb{O}'$ to itself. The derivations in $Der(\mathbb{O})$ have the same property, so the Lie algebra $\mathrm{so}(7) = \mathrm{so}(\mathbb{O}')$ of antisymmetric maps of $\mathbb{O}'$ is

$$\mathrm{so}(\mathbb{O}') = \mathrm{Der}(\mathbb{O}) + C(\mathbb{O}').$$

Since $\mathbb{O}$ is neither commutative nor associative, these sums are direct sums (as vector spaces, not as Lie algebras), and the Lie brackets are given by

$$\begin{aligned} [D, L_a] &= L_{Da} \\ [D, R_a] &= R_{Da} \\ [L_a, L_b] &= 2D(a,b) + \frac{1}{3}L_{[a,b]} + \frac{2}{3}R_{[a,b]} \\ [R_a, R_b] &= 2D(a,b) - \frac{2}{3}L_{[a,b]} - \frac{1}{3}R_{[a,b]} \\ [L_a, R_b] &= -D(a,b) + \frac{1}{3}L_{[a,b]} - \frac{1}{3}R_{[a,b]}, \end{aligned} \tag{4.7}$$

with $D \in \mathrm{Der}(\mathbb{O})$, $a, b \in \mathbb{O}'$. The proof of these formulas is already contained in the computations above.

With this decomposition of $\mathrm{so}(\mathbb{O}')$, (4.4) becomes

$$\mathrm{Der}H_2(\mathbb{O}) = A'_2(\mathbb{O}) \oplus \mathrm{so}(\mathbb{O}'), \tag{4.8}$$

and so, with (4.3),

$$\mathrm{so}(9) = A_2'(\mathbb{O}) \oplus \mathrm{so}(\mathbb{O}'). \tag{4.9}$$

$\mathrm{so}(\mathbb{O}')$ is a subalgebra in this algebra, the Lie bracket of $T \in \mathrm{so}(\mathbb{O}')$ and a matrix $A \in A_2'(\mathbb{O})$ is given by the action of T on the entries of A and the Lie bracket between two matrices A and B in $A_2'(\mathbb{O})$ is

$$[A, B] = (AB - BA - aI) \oplus (C_a + E(A, B)), \tag{4.10}$$

where $a = \frac{1}{2}\mathrm{tr}(AB - BA)$ and

$$E(A, B)x = \sum_{i,j} [a_{ij}, b_{ij}, x] \qquad (x \in \mathbb{O}).$$

This decomposition of so(9) allows us to give explicitly the spin action of so(9) on $\mathbb{O}^2 = \mathbb{R}^{16}$.
We start with one of the most remarkable properties of the octonions, which comes from considering a generalization of the derivation equation (4.2) in the form

$$T(xy) = (T^\sharp x)y + x(T^\flat y), \tag{4.11}$$

where $T, T^\sharp, T^\flat$ are antisymmetric maps. The **principle of triality** asserts that if $T \in \mathrm{so}(8) = \mathrm{so}(\mathbb{O})$ is given, then there exist unique maps $T^\sharp, T^\flat$ satisfying (4.11).
These mappings $^\sharp$ and $^\flat$ verify for all $T_1, T_2 \in \mathrm{so}(8)$:

$$\begin{aligned} [T_1, T_2]^\sharp &= [T_1^\sharp, T_2^\sharp], \\ \text{and} \quad [T_1, T_2]^\flat &= [T_1^\flat, T_2^\flat], \end{aligned}$$

since : $[T_1, T_2]$ is also an element of so(8), so the principle of triality implies that it exists unique maps $[T_1, T_2]^\sharp$ and $[T_1, T_2]^\flat$ s.t.

$$\begin{aligned} ([T_1, T_2]^\sharp x)y + x([T_1, T_2]^\flat y) &= [T_1, T_2](x, y) \\ &= T_1(T_2(xy)) - T_2(T_1(xy)) \\ &= T_1((T_2^\sharp x)y + x(T_2^\flat y)) - T_2((T_1^\sharp x)y + x(T_1^\flat y)) \\ &= (T_1^\sharp(T_2^\sharp x))y + (T_2^\sharp x)(T_1^\flat y) + (T_1^\sharp x)(T_2^\flat y) + x(T_1^\flat(T_2^\flat y)) \\ &\quad -(T_2^\sharp(T_1^\sharp x))y - (T_1^\sharp x)(T_2^\flat y) - (T_2^\sharp x)(T_1^\flat y) - x(T_2^\flat(T_1^\flat y)) \\ &= ((T_1^\sharp T_2^\sharp - T_2^\sharp T_1^\sharp)x)y + x((T_1^\flat T_2^\flat - T_2^\flat T_1^\flat)y) \\ &= ([T_1^\sharp, T_2^\sharp]x)y + x([T_1^\flat, T_2^\flat]y). \qquad \checkmark \end{aligned}$$

The correspondences $T \mapsto T^\sharp$ and $T \mapsto T^\flat$ define therefore 8-dimensional representations of so(8). These are not equivalent to the defining representation . A further representation, which is equivalent to the defining representation, is given by $T \mapsto \bar{T}$ where

$$\overline{T}x = \overline{(T\bar{x})}.$$

Explicitly, $T^\sharp, T^\flat$ and $\overline{T}$ are given as follows, since $\mathrm{so}(8) = Der(\mathbb{O}) \oplus L(\mathbb{O}') \oplus R(\mathbb{O}')$:

$$\begin{array}{lll} D^\sharp = D^\flat = D & & (D \in \mathrm{Der}(\mathbb{O})) \\ L_a^\sharp = L_a + R_a & L_a^\flat = -L_a & \\ R_a^\sharp = -R_a & R_a^\flat = L_a + R_a & (a \in \mathbb{O}') \end{array}$$

Remark 4.3.2. *Note that for $T = D + C_a \in \mathrm{so}(\mathbb{O}')$ we have*

$$\overline{T} = T \qquad \text{and} \qquad \overline{T^\sharp} = T^\flat,$$

so $T \mapsto T^\sharp$ and $T \mapsto T^\flat$ are equivalent as representations of $\mathrm{so}(\mathbb{O}')$.

We can compute for $T \in \mathrm{so}(\mathbb{O})$: $(T^\sharp)^\sharp = T$, $(T^\flat)^\flat = T$, $(T^\sharp)^\flat = \overline{T^\flat}$, $(T^\flat)^\sharp = \overline{T^\sharp}$, so we get the following equations

$$\begin{aligned} T(xy) &= (T^\sharp x)y + x(T^\flat y) && (4.12)\\ T^\sharp(xy) &= (Tx)y + x(\overline{T^\flat} y) && (4.13)\\ T^\flat(xy) &= (\overline{T^\sharp} x)y + x(Ty). && (4.14) \end{aligned}$$

Hence we have the final statement :

Theorem 4.3.3 ([41])**.** *The Lie algebra so*(9) *of Spin*(9) *can be represented as*

$$\mathrm{so}(9) = A_2'(\mathbb{O}) \oplus \mathrm{so}(\mathbb{O}'),$$

and the action ρ *of* so(9) *on* $S := \{2 \times 1 \text{ column vectors with entries in } \mathbb{O}\} = \mathbb{O}^2 \cong \mathbb{R}^{16}$ *(spin representation) is given by*

$$\begin{aligned} A \in A_2'(\mathbb{O}) \quad &\Rightarrow \quad \rho(A)(x) := A \cdot x \qquad \text{(matrix multiplication)}\\ T \in \mathrm{so}(\mathbb{O}') \quad &\Rightarrow \quad \rho(T)(x) := T^\sharp x \qquad \text{(componentwise action)} \end{aligned}$$

Proof. The first part was proved above.
For the second part, it remains to show that

$$[\rho(T), \rho(A)] = \rho([T, A]).$$

Let $T \in \mathrm{so}(\mathbb{O}')$, $A = \begin{pmatrix} p & q \\ -\bar{q} & -p \end{pmatrix} \in A_2'(\mathbb{O})$ and $(s_1, s_2)^T \in S = \mathbb{O}^2$. Then we have:

$$\begin{aligned} \rho(TA)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} &= \begin{pmatrix} Tp & Tq \\ -T(\bar{q}) & -T(p) \end{pmatrix}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}\\ &= \begin{pmatrix} T(p)s_1 + T(q)s_2 \\ -T(\bar{q})s_1 - T(p)s_2 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} [\rho(T), \rho(A)]\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} &= \rho(T)\rho(A)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} - \rho(A)\rho(T)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}\\ &= \rho(T)\begin{pmatrix} ps_1 + qs_2 \\ -\bar{q}s_1 - ps_2 \end{pmatrix} - \rho(A)\begin{pmatrix} T^\sharp(s_1) \\ T^\sharp(s_2) \end{pmatrix}\\ &= \begin{pmatrix} T^\sharp(ps_1 + qs_2) \\ T^\sharp(-\bar{q}s_1 - ps_2) \end{pmatrix} - \begin{pmatrix} p(T^\sharp s_1) + q(T^\sharp s_2) \\ -\bar{q}(T^\sharp s_1) - p(T^\sharp s_2) \end{pmatrix}\\ &= \begin{pmatrix} (Tp)s_1 + p(\overline{T^\flat}s_1) + (Tq)s_2 + q(\overline{T^\flat}s_2) - p(T^\sharp s_1) - q(T^\sharp s_2) \\ -(T\bar{q})s_1 - \bar{q}(\overline{T^\flat}s_1) - (Tp)s_2 - p(\overline{T^\flat}s_2) + \bar{q}(T^\sharp s_1) + p(T^\sharp s_2) \end{pmatrix} \quad \text{cf. (4.13)}\\ &= \begin{pmatrix} (Tp)s_1 + (Tq)s_2 \\ -(T\bar{q})s_1 - (Tp)s_2 \end{pmatrix} \qquad \text{since Remark 4.3.2 implies } \overline{T^\flat} = T^\sharp \end{aligned}$$

□

Chapter 5

Computation of the coefficients b_k and $b_{k,l}$

5.1 Stabilizer

First we can verify that the stabilizer of the Spin(9) representation is Spin(7) in the explicit description given in Section 4.3.

Since the action of Spin(9) is transitive on the sphere $\mathbb{S}^{15}$, it is sufficient to compute the stabilizer of the point $(1,0) \in \mathbb{O}^2 = \mathbb{R}^{16}$.

Given the description of so(9) in Section 4.3, an element of so(9) is given by a pair (A,T) with

$$A = \begin{pmatrix} p & q \\ -\bar{q} & -p \end{pmatrix} \text{ with } p \in \mathbb{O}',\ q \in \mathbb{O}, \qquad T \in \mathrm{so}(\mathbb{O}') = \mathrm{so}(7).$$

Such an element is in the stabilizer of $(1,0)$ if and only if the equation

$$(A,T)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds, where the action on the left is given in theorem 4.3.3. It becomes therefore

$$\begin{pmatrix} p \\ -\bar{q} \end{pmatrix} + \begin{pmatrix} T^\sharp(1) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

or equivalently

$$q = 0 \text{ and } p = -T^\sharp(1).$$

An element of the stabilizer Stab is therefore of the form

$$\begin{pmatrix} -T^\sharp(1) & 0 \\ 0 & T^\sharp(1) \end{pmatrix} \oplus T$$

for some $T \in \mathrm{so}(\mathbb{O}')$, and we get the following bijective map

$$\begin{aligned} \psi : \mathrm{so}(\mathbb{O}') &\longrightarrow \mathrm{Stab} \\ T &\mapsto \begin{pmatrix} -T^\sharp(1) & 0 \\ 0 & T^\sharp(1) \end{pmatrix} \oplus T. \end{aligned}$$

However this map is not an isomorphism of Lie algebra, since it does not conserve the Lie brackets. We introduce therefore a new map

$$\begin{array}{rrcl} \varphi : & \mathrm{so}(\mathbb{O}') & \longrightarrow & \mathrm{so}(\mathbb{O}') \\ & D \in \mathrm{Der}(\mathbb{O}) & \mapsto & D \\ & C_a \in C(\mathbb{O}') & \mapsto & -\dfrac{1}{2} C_a . \end{array}$$

where we use the isomorphism (as vector spaces) $\mathrm{so}(\mathbb{O}') = \mathrm{Der}(\mathbb{O}) \oplus C(\mathbb{O}')$.
The new map $\Phi := \psi \circ \varphi$ is now an isomorphism of Lie algebras between $\mathrm{so}(\mathbb{O}') = \mathrm{so}(7)$ and the stabilizer of the spin action on so(16). We can indeed verify :
For $D_1, D_2 \in \mathrm{Der}(\mathbb{O})$:

$$\begin{aligned} [\Phi(D_1), \Phi(D_2)] &= [\psi(D_1), \psi(D_2)] \\ &= \left[\begin{pmatrix} D_1^\sharp(1) & 0 \\ 0 & -D_1^\sharp(1) \end{pmatrix} \oplus D_1 , \begin{pmatrix} D_2^\sharp(1) & 0 \\ 0 & -D_2^\sharp(1) \end{pmatrix} \oplus D_2 \right] \\ &= [0 \oplus D_1, 0 \oplus D_2] \\ &= 0 \oplus [D_1, D_2] \\ &= \Phi([D_1, D_2]), \end{aligned}$$

since $[D_1, D_2]$ is again a derivation. ✓

For $D \in \mathrm{Der}(\mathbb{O}), C_a \in C(\mathbb{O}')$:

$$\begin{aligned} [\Phi(D), \Phi(C_a)] &= \left[\psi(D), -\frac{1}{2}\psi(C_a) \right] \\ &= \left[0 \oplus D, \begin{pmatrix} \frac{1}{2}C_a^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_a^\sharp(1) \end{pmatrix} \oplus -\frac{1}{2}C_a \right] \\ &= \left[0 \oplus D, \begin{pmatrix} \frac{1}{2}C_a^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_a^\sharp(1) \end{pmatrix} \oplus 0 \right] + 0 \oplus -\frac{1}{2}[D, C_a] \\ &= \begin{pmatrix} D(\frac{3}{2}a) & 0 \\ 0 & D(-\frac{3}{2}a) \end{pmatrix} \oplus 0 + 0 \oplus [D, L_a - R_a] \\ &= \begin{pmatrix} \frac{3}{2}Da & 0 \\ 0 & -\frac{3}{2}Da \end{pmatrix} \oplus -\frac{1}{2}(L_{Da} - R_{Da}) \qquad \text{(cf. (4.7))} \\ &= \begin{pmatrix} \frac{1}{2}C_{Da}^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_{Da}^\sharp(1) \end{pmatrix} \oplus -\frac{1}{2}C_{Da} \\ &= \psi(-\frac{1}{2}C_{Da}) \\ &= \Phi(C_{Da}) \\ &= \Phi([D, C_a]) \qquad ✓ \end{aligned}$$

For $C_a, C_b \in C(\mathbb{O}')$:

$$
\begin{aligned}
[\Phi(C_a), \Phi(C_b)] &= [-\frac{1}{2}\psi(C_a), -\frac{1}{2}\psi(C_b)] \\
&= \left[\begin{pmatrix} \frac{1}{2}C_a^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_a^\sharp(1) \end{pmatrix} \oplus -\frac{1}{2}C_a, \begin{pmatrix} \frac{1}{2}C_b^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_b^\sharp(1) \end{pmatrix} \oplus -\frac{1}{2}C_b\right] \\
&= \underbrace{\left[\begin{pmatrix} \frac{1}{2}C_a^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_a^\sharp(1) \end{pmatrix} \oplus 0, \begin{pmatrix} \frac{1}{2}C_b^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_b^\sharp(1) \end{pmatrix} \oplus 0\right]}_{=(1)} \\
&\quad + \underbrace{\left[\begin{pmatrix} \frac{1}{2}C_a^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_a^\sharp(1) \end{pmatrix} \oplus 0, 0 \oplus -\frac{1}{2}C_b\right]}_{=(2)} \\
&\quad + \underbrace{\left[0 \oplus -\frac{1}{2}C_a, \begin{pmatrix} \frac{1}{2}C_b^\sharp(1) & 0 \\ 0 & -\frac{1}{2}C_b^\sharp(1) \end{pmatrix} \oplus 0\right]}_{=(3)} \\
&\quad + \underbrace{\left[0 \oplus -\frac{1}{2}C_a, 0 \oplus -\frac{1}{2}C_b\right]}_{=(4)}.
\end{aligned}
$$

With the rule (4.10) for the computation of commutators between two elements of $A_2'(\mathbb{O})$, we compute :

$$
\begin{aligned}
(1) &= \Bigg[\underbrace{\begin{pmatrix} \frac{3}{2}a & 0 \\ 0 & -\frac{3}{2}a \end{pmatrix}}_{=:A} \oplus 0, \underbrace{\begin{pmatrix} \frac{3}{2}b & 0 \\ 0 & -\frac{3}{2}b \end{pmatrix}}_{=:B} \oplus 0\Bigg] \\
&= (AB - BA - \alpha I) \oplus (C_\alpha + E(A,B)),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha &= \frac{1}{2}\mathrm{tr}(AB - BA) \\
&= \frac{1}{2}\mathrm{tr}\begin{pmatrix} \frac{9}{4}[a,b] & 0 \\ 0 & \frac{9}{4}[a,b] \end{pmatrix} \\
&= \frac{9}{4}[a,b],
\end{aligned}
$$

and

$$
\begin{aligned}
E(A,B)x &= \sum_{ij}[a_{ij}, b_{ji}, x] \\
&= \left[\frac{3}{2}a, \frac{3}{2}b, x\right] + \left[-\frac{3}{2}a, -\frac{3}{2}b, x\right] \\
&= \frac{9}{2}[a,b,x] \\
&= -\frac{9}{2}[L_a, R_b]x \qquad \text{(cf. (4.5))}.
\end{aligned}
$$

(1) becomes :

$$
\begin{aligned}
(1) &= 0 \oplus \frac{9}{4}C_{[a,b]} - \frac{9}{2}[L_a, R_b] \\
&= 0 \oplus \frac{9}{4}C_{[a,b]} - \frac{9}{2}(-D(a,b) + \frac{1}{3}C_{[a,b]}) \qquad \text{(cf. (4.7))} \\
&= 0 \oplus \frac{9}{2}D(a,b) + \frac{3}{4}C_{[a,b]}.
\end{aligned}
$$

Since the commutator between an element T of so(7) and A of $A'_2(\mathbb{O})$ is given by the action of T on the entries of A, (2) and (3) becomes :

$$\begin{aligned}
(2) &= \left[\begin{pmatrix} \frac{3}{2}a & 0 \\ 0 & -\frac{3}{2}a \end{pmatrix} \oplus 0, 0 \oplus -\frac{1}{2}C_b\right] \\
&= -\left[0 \oplus -\frac{1}{2}C_b, \begin{pmatrix} \frac{3}{2}a & 0 \\ 0 & -\frac{3}{2}a \end{pmatrix} \oplus 0\right] \\
&= -\begin{pmatrix} -\frac{1}{2}C_b(\frac{3}{2}a) & 0 \\ 0 & -\frac{1}{2}C_b(-\frac{3}{2}a) \end{pmatrix} \oplus 0 \\
&= -\begin{pmatrix} -\frac{3}{4}[b,a] & 0 \\ 0 & \frac{3}{4}[b,a] \end{pmatrix} \oplus 0 \\
&= \begin{pmatrix} -\frac{3}{4}[a,b] & 0 \\ 0 & \frac{3}{4}[a,b] \end{pmatrix} \oplus 0,
\end{aligned}$$

$$\begin{aligned}
(3) &= \left[0 \oplus -\frac{1}{2}C_a, \begin{pmatrix} \frac{3}{2}b & 0 \\ 0 & -\frac{3}{2}b \end{pmatrix} \oplus 0\right] \\
&= \begin{pmatrix} -\frac{1}{2}C_a(\frac{3}{2}b) & 0 \\ 0 & -\frac{1}{2}C_a(-\frac{3}{2}b) \end{pmatrix} \oplus 0 \\
&= \begin{pmatrix} -\frac{3}{4}[a,b] & 0 \\ 0 & \frac{3}{4}[a,b] \end{pmatrix} \oplus 0.
\end{aligned}$$

Finally, since $\mathrm{so}(\mathbb{O}')$ is a Lie subalgebra of so(9), (4) is

$$\begin{aligned}
(4) &= 0 \oplus \frac{1}{4}[C_a, C_b] \\
&= 0 \oplus \frac{1}{4}([L_a, R_a] - [L_a, R_b] - \underbrace{[R_a, L_b]}_{\substack{=[L_a,R_b] \\ \text{cf. (4.5)}}} + [R_a, R_b]) \\
&= 0 \oplus \frac{1}{4}\Big(2D(a,b) + \frac{1}{3}L_{[a,b]} + \frac{2}{3}R_{[a,b]} \\
&\qquad -2(-D(a,b) + \frac{1}{3}L_{[a,b]} - \frac{1}{3}R_{[a,b]}) \\
&\qquad +(2D(a,b) - \frac{2}{3}L_{[a,b]} - \frac{1}{3}R_{[a,b]})\Big) \\
&= 0 \oplus \frac{1}{4}(6D(a,b) - L_{[a,b]} + R_{[a,b]}) \\
&= 0 \oplus \frac{3}{2}D(a,b) - \frac{1}{4}C_{[a,b]}.
\end{aligned}$$

We therefore get :

$$
\begin{aligned}
[\Phi(C_a),\Phi(C_b)] &= (1)+(2)+(3)+(4)\\
&= 0\oplus\frac{9}{2}D(a,b)+\frac{3}{4}C_{[a,b]}\\
&\quad+\begin{pmatrix} -\frac{3}{4}[a,b] & 0\\ 0 & \frac{3}{4}[a,b]\end{pmatrix}\oplus 0\\
&\quad+\begin{pmatrix} -\frac{3}{4}[a,b] & 0\\ 0 & \frac{3}{4}[a,b]\end{pmatrix}\oplus 0\\
&\quad+0\oplus\frac{3}{2}D(a,b)-\frac{1}{4}C_{[a,b]}\\
&= \begin{pmatrix} -\frac{3}{2}[a,b] & 0\\ 0 & \frac{3}{2}[a,b]\end{pmatrix}\oplus 6D(a,b)+\frac{1}{2}C_{[a,b]}\\
&= 0\oplus 6D(a,b)-\begin{pmatrix} -\frac{1}{2}C^\sharp_{[a,b]}(1) & 0\\ 0 & \frac{1}{2}C^\sharp_{[a,b]}(1)\end{pmatrix}\oplus -\frac{1}{2}C_{[a,b]}\\
&= \psi(6D(a,b))-\psi(-\frac{1}{2}C_{[a,b]})\\
&= \Phi(6D(a,b))-\Phi(C_{[a,b]})\\
&= \Phi([C_a,C_b]). \qquad\qquad (\text{cf. } (4.6)) \qquad \checkmark
\end{aligned}
$$

Moreover the action of Stab $\cong$ so(7) on the tangent space $T_{(1,0)}\mathbb{S}^{15}=\mathbb{O}'\oplus\mathbb{O}$ is the sum of the standard representation of so(7) on $\mathbb{O}'=\mathbb{R}^7$ and the spin representation of so(7) on $\mathbb{O}=\mathbb{R}^8$.
To show this, let $D\oplus C_a\in$ so(7) and $p\in\mathbb{O}'$, $q\in\mathbb{O}$. Then we have :

$$
\begin{aligned}
\Phi(D\oplus C_a)\begin{pmatrix} p\\ 0\end{pmatrix} &= \left(\begin{pmatrix} \frac{3}{2}a & 0\\ 0 & -\frac{3}{2}a\end{pmatrix}\oplus D-\frac{1}{2}C_a\right)\begin{pmatrix} p\\ 0\end{pmatrix}\\
&= \begin{pmatrix} \frac{3}{2}ap+D^\sharp(p)-\frac{1}{2}C^\sharp_a(p)\\ 0\end{pmatrix}\\
&= \begin{pmatrix} \frac{3}{2}ap+D(p)-\frac{1}{2}(L_a+2R_a)(p)\\ 0\end{pmatrix}\\
&= \begin{pmatrix} \frac{3}{2}ap+D(p)-\frac{1}{2}(ap+2pa)\\ 0\end{pmatrix}\\
&= \begin{pmatrix} D(p)+ap-pa\\ 0\end{pmatrix}\\
&= \begin{pmatrix} (D+C_a)(p)\\ 0\end{pmatrix}, \qquad \checkmark
\end{aligned}
$$

and

$$\begin{aligned}
\Phi(D \oplus C_a)\begin{pmatrix} 0 \\ q \end{pmatrix} &= \left(\begin{pmatrix} \frac{3}{2}a & 0 \\ 0 & -\frac{3}{2}a \end{pmatrix} \oplus D - \frac{1}{2}C_a\right)\begin{pmatrix} 0 \\ q \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -\frac{3}{2}aq + D^\sharp(q) - \frac{1}{2}C_a^\sharp(q) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -\frac{3}{2}aq + D(q) - \frac{1}{2}(L_a + 2R_a)(q) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -\frac{3}{2}aq + D(q) - \frac{1}{2}(aq + 2qa) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ D(q) - 2aq - qa \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (D + C_a)^\flat(q) \end{pmatrix},
\end{aligned}$$

and since, by Remark 4.3.2, for $T \in \mathrm{so}(\mathbb{O}')$, the representations $T \mapsto T^\sharp$ and $T \mapsto T^\flat$ are equivalent, this is really the spin representation of so(7) on $\mathbb{O} = \mathbb{R}^8$. ✓

5.2 Preliminary observations

In Section 2.1, we proved the following formula for the dimensions of Val_k^G :

Corollary 2.2.2. *For $0 \le k, l \le n$, set*

$$b_k := \dim(\Lambda^k V)^G,$$
$$b_{k,l} := \dim \Omega_h^{k,l}(SV)^G,$$

and $b_k = 0$, $b_{k,l} = 0$ for other values of k and l.
Then for $0 \le k \le n$:

$$\dim Val_k^G = \sum_{l=0}^{n-k-1} (-1)^{n-k-l-1}(b_{k,l} - b_{k-1,l-1}) + (-1)^{n-k} b_k.$$

So we want to compute the coefficients b_k and $b_{k,l}$.
The coefficients b_k's can be computed with the Adams operator (cf. Theorem 3.3.4).
For the computation of the coefficients $b_{k,l}$, we first need to describe the spaces $\Omega_h^{k,l}(SV)^{\mathrm{Spin}(9)}$.
Let $v = (1,0)^T \in \mathbb{S}^{15} \subset \mathbb{O}^2$. We have the isomorphism :

$$\begin{aligned}
\Psi \ : \ \Omega^{k,l}(SV)^{\mathrm{Spin}(9)} &\longrightarrow \Lambda^{k,l}(T_{(0,v)}SV)^{\mathrm{Stab}((0,v))} = \Lambda^{k,l}(T_{(0,v)}SV)^{\mathrm{Spin}(7)} \\
\omega &\mapsto \omega(0,v)
\end{aligned}$$

Then with the decomposition :

$$\begin{aligned}
T_{(0,v)}(SV) &= T_{(0,v)}(V \times \mathbb{S}^{15}) \\
&= T_0 V \times T_v \mathbb{S}^{15} \\
&= V \times T, \qquad \text{with } T := T_v\mathbb{S}^{15} = \mathbb{O}' \oplus \mathbb{O} \\
&= \mathbb{R}v \oplus T \oplus T,
\end{aligned}$$

the isomorphism becomes

$$\begin{aligned}
\Psi \ : \ \Omega^{k,l}(SV)^{\mathrm{Spin}(9)} &\longrightarrow \Lambda^{k,l}(\mathbb{R} \oplus \mathbb{O}' \oplus \mathbb{O} \oplus \mathbb{O}' \oplus \mathbb{O})^{\mathrm{Spin}(7)} \\
\omega &\mapsto \omega(0,v).
\end{aligned}$$

Recalling that

$$\Omega_h^{k,l}(SV)^G := \Omega^{k,l}(SV)^G \Big/ \Omega_v^{k,l}(SV)^G$$

where

$$\Omega_v^{k,l}(SV)^G := \{\omega \in \Omega^{k,l}(SV)^G \,|\, \alpha \wedge \omega = 0\},$$

and since

$$\begin{aligned} \Psi(\alpha)(\lambda v + w) &= \alpha_{(0,v)}(\lambda v + w) \\ &= \langle v, \lambda v\rangle \\ &= \lambda, \end{aligned}$$

Ψ induces an isomorphism on the space of horizontal forms by

$$\begin{aligned} \Omega_h^{k,l}(SV)^{\mathrm{Spin}(9)} &\longrightarrow \Lambda^{k,l}(\mathbb{O}' \oplus \mathbb{O} \oplus \mathbb{O}' \oplus \mathbb{O})^{\mathrm{Spin}(7)} \\ \omega &\mapsto \omega(0, v). \end{aligned}$$

So the coefficients $b_{k,l}$ can be computed as

$$b_{k,l} = \dim \Lambda^{k,l}(\mathbb{O}' \oplus \mathbb{O} \oplus \mathbb{O}' \oplus \mathbb{O})^{\mathrm{Spin}(7)}.$$

This expression can be decomposed as :

$$\Lambda^{k,l}(\mathbb{O}' \oplus \mathbb{O} \oplus \mathbb{O}' \oplus \mathbb{O})^{\mathrm{Spin}(7)} = (\Lambda^k(\mathbb{O}' \oplus \mathbb{O}) \otimes \Lambda^l(\mathbb{O}' \oplus \mathbb{O}))^{\mathrm{Spin}(7)}. \tag{5.1}$$

For two G-representations V, W,

$$V \otimes W = Hom(V, W^*)$$

and

$$(V \otimes W)^G = Hom_G(V, W^*).$$

By the complete reducibility theorem 3.2.5, V and W^* can be decomposed as

$$V = \bigoplus_\alpha n_\alpha \Gamma_\alpha, \qquad W^* = \bigoplus_\beta m_\beta \Gamma_\beta,$$

where Γ_λ are irreducible representations. Then

$$\begin{aligned} (V \otimes W)^G &= Hom_G(V, W^*) \\ &= \bigoplus_{\alpha,\beta} n_\alpha m_\beta \underbrace{Hom_G(\Gamma_\alpha, \Gamma_\beta)}_{\substack{=0 \text{ if } \alpha \neq \beta \\ (\text{Schur's lemma 3.2.7})}} \\ &= \bigoplus_\alpha n_\alpha m_\alpha Hom_G(\Gamma_\alpha, \Gamma_\alpha), \end{aligned}$$

and hence

$$\begin{aligned} \dim(V \otimes W)^G &= \bigoplus_\alpha n_\alpha m_\alpha \underbrace{\dim(Hom_G(\Gamma_\alpha, \Gamma_\alpha))}_{=1 \text{ (Schur's lemma 3.2.7)}} \\ &= \bigoplus_\alpha n_\alpha m_\alpha. \end{aligned}$$

Therefore, if we can decompose

$$\Lambda^i(\mathbb{O}' \oplus \mathbb{O})$$

for $i = 0, ..., 7$, with (5.1) we find the coefficients $b_{k,l}$.

5.3 Computation of the b_k

Proposition 5.3.1.

$$b_k = 1 \qquad \textit{for } k = 0, 8, 16,$$
$$b_k = 0 \qquad \textit{for other values of } k.$$

Proof. *Case* $k = 0$ *:* $\Lambda^0(V) = \mathbb{C}$ is the trivial representation, so $b_0 = 1$.
Case $k = 1$ *:* $\Lambda^1(V) = V = \Gamma_{[1/2,1/2,1/2,1/2]}$ is irreducible, so $b_1 = 0$.
Case $2 \leq k \leq 8$ *:* With the Adams formula, we have :

$$Char(\Lambda^k(V)) = \frac{1}{k}\sum_{j=1}^{k}(-1)^{j-1}\left(\psi^j(Char(V))Char(\Lambda^{k-j}V)\right),$$

and with the algorithm given in Section 3.3, we find

$$\begin{aligned}
\Lambda^2 V &= \Gamma_{[1,1,1,0]} + \Gamma_{[1,1,0,0]}\\
\Lambda^3 V &= \Gamma_{[3/2,3/2,1/2,1/2]} + \Gamma_{[3/2,1/2,1/2,1/2]}\\
\Lambda^4 V &= \Gamma_{[2,2,0,0]} + \Gamma_{[2,1,1,1]} + \Gamma_{[2,1,0,0]} + \Gamma_{[2,0,0,0]} + \Gamma_{[1,1,1,1]}\\
\Lambda^5 V &= \Gamma_{[5/2,3/2,1/2,1/2]} + \Gamma_{[5/2,1/2,1/2,1/2]} + \Gamma_{[3/2,3/2,3/2,3/2]} + \Gamma_{[3/2,3/2,1/2,1/2]} + \Gamma_{[3/2,1/2,1/2,1/2]}\\
\Lambda^6 V &= \Gamma_{[3,1,1,0]} + \Gamma_{[3,1,0,0]} + \Gamma_{[2,2,1,1]} + \Gamma_{[2,1,1,1]} + \Gamma_{[2,1,1,0]} + \Gamma_{[2,1,0,0]} + \Gamma_{[1,1,1,0]} + \Gamma_{[1,1,0,0]}\\
\Lambda^7 V &= \Gamma_{[7/2,1/2,1/2,1/2]} + \Gamma_{[5/2,3/2,3/2,1/2]} + \Gamma_{[5/2,3/2,1/2,1/2]} + \Gamma_{[5/2,1/2,1/2,1/2]} + \Gamma_{[3/2,3/2,3/2,1/2]}\\
&\quad + \Gamma_{[3/2,3/2,1/2,1/2]} + \Gamma_{[3/2,1/2,1/2,1/2]} + \Gamma_{[1/2,1/2,1/2,1/2]}\\
\Lambda^8 V &= \Gamma_{[4,0,0,0]} + \Gamma_{[3,1,1,1]} + \Gamma_{[3,1,1,0]} + \Gamma_{[3,0,0,0]} + \Gamma_{[2,2,2,0]} + \Gamma_{[2,2,1,0]} + \Gamma_{[2,2,0,0]} + \Gamma_{[2,1,1,1]}\\
&\quad + \Gamma_{[2,1,1,0]} + \Gamma_{[2,0,0,0]} + \Gamma_{[1,1,1,1]} + \Gamma_{[1,1,1,0]} + \Gamma_{[1,0,0,0]} + \underbrace{\Gamma_{[0,0,0,0]}}_{\text{trivial representation}}
\end{aligned}$$

hence

$$b_k = 1 \;\Leftrightarrow k = 8.$$

Case $9 \leq k \leq 16$ *:* Since

$$\Lambda^k V \cong \Lambda^{16-k} V,$$

the only non-zero b_k is b_{16}. □

5.4 Computation of the $b_{k,l}$

As established in Section 5.2,

$$b_{k,l} = \dim \Omega_h^{k,l}(SV)^{\mathrm{Spin}(9)} = \dim(\Lambda^k(\mathbb{O}' \oplus \mathbb{O}) \otimes \Lambda^l(\mathbb{O}' \oplus \mathbb{O}))^{\mathrm{Spin}(7)},$$

and since

$$\dim(V \otimes W)^G = \bigoplus_\alpha n_\alpha m_\alpha$$

where

$$V = \bigoplus_\alpha n_\alpha \Gamma_\alpha \qquad W^* \cong W = \bigoplus_\beta m_\beta \Gamma_\beta$$

are the decompositions in irreducible representations, it only remains to find the decomposition in irreducible representations of

$$\Lambda^k(\mathbb{O}' \oplus \mathbb{O})$$

for $0 \leq k \leq 7$.
We first compute with Weyl character formula 3.3.2 the character

$$Char(\Gamma_{[1,0,0]}) + Char(\Gamma_{[1/2,1/2,1/2]})$$

of $\mathbb{O} \oplus \mathbb{O}'$, then apply Adam's formula 3.3.4 and the same algorithm as before to find

$$\begin{aligned}
\Lambda^0(\mathbb{O} \oplus \mathbb{O}') &= \Gamma_{[0,0,0]} \\
\Lambda^1(\mathbb{O} \oplus \mathbb{O}') &= \Gamma_{[1,0,0]} + \Gamma_{[1/2,1/2,1/2]} \\
\Lambda^2(\mathbb{O} \oplus \mathbb{O}') &= \Gamma_{[3/2,1/2,1/2]} + \Gamma_{[1,0,0]} + 2 \times \Gamma_{[1,1,0]} + \Gamma_{[1/2,1/2,1/2]} \\
\Lambda^3(\mathbb{O} \oplus \mathbb{O}') &= \Gamma_{[2,1,0]} + \Gamma_{[2,0,0]} + \Gamma_{[3/2,3/2,1/2]} + 2 \times \Gamma_{[3/2,1/2,1/2]} + 2 \times \Gamma_{[1,1,1]} + \Gamma_{[1,1,0]} + \Gamma_{[1,0,0]} \\
&\quad +2 \times \Gamma_{[1/2,1/2,1/2]} + \Gamma_{[0,0,0]} \\
\Lambda^4(\mathbb{O} \oplus \mathbb{O}') &= \Gamma_{[5/2,1/2,1/2]} + \Gamma_{[2,2,0]} + \Gamma_{[2,1,1]} + \Gamma_{[2,1,0]} + 2 \times \Gamma_{[2,0,0]} + \Gamma_{[3/2,3/2,3/2]} + 2 \times \Gamma_{[3/2,3/2,1/2]} \\
&\quad +3 \times \Gamma_{[3/2,1/2,1/2]} + 4 \times \Gamma_{[1,1,1]} + \Gamma_{[1,1,0]} + 2 \times \Gamma_{[1,0,0]} + 3 \times \Gamma_{[1/2,1/2,1/2]} + 2 \times \Gamma_{[0,0,0]} \\
\Lambda^5(\mathbb{O} \oplus \mathbb{O}') &= \Gamma_{[3,0,0]} + \Gamma_{[5/2,3/2,1/2]} + \Gamma_{[5/2,1/2,1/2]} + \Gamma_{[2,2,1]} + 3 \times \Gamma_{[2,1,1]} + 2 \times \Gamma_{[2,1,0]} \\
&\quad +2 \times \Gamma_{[3/2,3/2,3/2]} + 3 \times \Gamma_{[3/2,3/2,1/2]} + 5 \times \Gamma_{[3/2,1/2,1/2]} + 3 \times \Gamma_{[1,1,1]} + 5 \times \Gamma_{[1,1,0]} \\
&\quad +3 \times \Gamma_{[1,0,0]} + 4 \times \Gamma_{[1/2,1/2,1/2]} + \Gamma_{[0,0,0]} \\
\Lambda^6(\mathbb{O} \oplus \mathbb{O}') &= \Gamma_{[3,1,0]} + \Gamma_{[5/2,3/2,3/2]} + \Gamma_{[5/2,3/2,1/2]} + 2 \times \Gamma_{[5/2,1/2,1/2]} + 2 \times \Gamma_{[2,2,1]} + 4 \times \Gamma_{[2,1,1]} \\
&\quad +3 \times \Gamma_{[2,1,0]} + \Gamma_{[2,0,0]} + 2 \times \Gamma_{[3/2,3/2,3/2]} + 5 \times \Gamma_{[3/2,3/2,1/2]} + 6 \times \Gamma_{[3/2,1/2,1/2]} \\
&\quad +4 \times \Gamma_{[1,1,1]} + 6 \times \Gamma_{[1,1,0]} + 5 \times \Gamma_{[1,0,0]} + 5 \times \Gamma_{[1/2,1/2,1/2]} \\
\Lambda^7(\mathbb{O} \oplus \mathbb{O}') &= \Gamma_{[3,1,1]} + \Gamma_{[5/2,3/2,3/2]} + 2 \times \Gamma_{[5/2,3/2,1/2]} + 2 \times \Gamma_{[5/2,1/2,1/2]} + \Gamma_{[2,2,2]} + \Gamma_{[2,2,1]} \\
&\quad +2 \times \Gamma_{[2,2,0]} + 4 \times \Gamma_{[2,1,1]} + 4 \times \Gamma_{[2,1,0]} + 3 \times \Gamma_{[2,0,0]} + 3 \times \Gamma_{[3/2,3/2,3/2]} \\
&\quad +5 \times \Gamma_{[3/2,3/2,1/2]} + 7 \times \Gamma_{[3/2,1/2,1/2]} + 7 \times \Gamma_{[1,1,1]} + 4 \times \Gamma_{[1,1,0]} + 3 \times \Gamma_{[1,0,0]} \\
&\quad +6 \times \Gamma_{[1/2,1/2,1/2]} + 4 \times \Gamma_{[0,0,0]}
\end{aligned}$$

We can summarize this results in the following table whose kth column contains in the line indexed by $[\lambda_1, \lambda_2, \lambda_3]$ the coefficient of $\Gamma_{[\lambda_1,\lambda_2,\lambda_3]}$ in the decomposition of $\Lambda^k(\mathbb{O} \oplus \mathbb{O}')$:

	$\Lambda^k(\mathbb{O} \oplus \mathbb{O}')$							
	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$\Gamma_{[\lambda_1,\lambda_2,\lambda_3]}$								
$[0,0,0]$	1			1	2	1		4
$[\frac{1}{2},\frac{1}{2},\frac{1}{2}]$		1	1	2	3	4	5	6
$[1,0,0]$		1	1	1	2	3	5	3
$[1,1,0]$			2	1	1	5	6	4
$[1,1,1]$				2	4	3	4	7
$[\frac{3}{2},\frac{1}{2},\frac{1}{2}]$			1	2	3	5	6	7
$[\frac{3}{2},\frac{3}{2},\frac{1}{2}]$					2	3	5	5
$[\frac{3}{2},\frac{3}{2},\frac{3}{2}]$					1	2	2	3
$[2,0,0]$				1	2	1	1	3
$[2,1,0]$				1	1	2	3	4
$[2,1,1]$					1	3	4	4
$[2,2,0]$					1			2
$[2,2,1]$						1	2	1
$[2,2,2]$								1
$[\frac{5}{2},\frac{1}{2},\frac{1}{2}]$					1	1	2	2
$[\frac{5}{2},\frac{3}{2},\frac{1}{2}]$							1	2
$[\frac{5}{2},\frac{3}{2},\frac{3}{2}]$							1	1
$[3,0,0]$						1		
$[3,1,0]$							1	
$[3,1,1]$								1

Then the coefficient $b_{k,l}$ is the component-wise multiplication of the kth column with the lth column, e.g.

$$b_{1,2} = 0 \cdot 0 + 1 \cdot 1 + 1 \cdot 1 + 0 \cdot 2 + 0 \cdot 0 + 0 \cdot 1 = 2.$$

So finally we obtain for the coefficients $b_{k,l}$ the following symmetric table

	k	0	1	2	3	4	5	6	7
l									
0		1	0	0	1	2	1	0	4
1		0	2	2	3	5	7	10	9
2		0	2	7	7	10	22	28	24
3		1	3	7	18	30	39	50	63
4		2	5	10	30	56	68	88	116
5		1	7	22	39	68	116	150	162
6		0	10	28	50	88	150	204	210
7		4	9	24	63	116	162	210	266

and for the coefficients where $8 \leq k \leq 15$ or $8 \leq l \leq 15$ we have the symmetry relations

$$b_{k,l} = b_{15-k,15-l} = b_{k,15-l} = b_{15-k,l}.$$

Hence, with Corollary 2.2.2, we get for the dimensions of the spaces of Spin(9)−invariant k-homogeneous valuations :

$$\begin{aligned}
\dim \mathrm{Val}_{16}^{\mathrm{Spin}(9)} &= b_{16} = 1 \\
\dim \mathrm{Val}_{15}^{\mathrm{Spin}(9)} &= -b_{15} + b_{15,0} \\
&= b_{0,0} = 1 \\
\dim \mathrm{Val}_{14}^{\mathrm{Spin}(9)} &= b_{14} - (b_{14,0} - b_{13,-1}) + (b_{14,1} - b_{13,0}) \\
&= -b_{1,0} + b_{1,1} - b_{2,0} \\
&= 2 \\
\dim \mathrm{Val}_{13}^{\mathrm{Spin}(9)} &= -b_{13} + (b_{13,0} - b_{12,-1}) - (b_{13,1} - b_{12,0}) + (b_{13,2} - b_{12,1}) \\
&= b_{2,0} - b_{2,1} + b_{3,0} + b_{2,2} - b_{3,1} \\
&= 3 \\
\dim \mathrm{Val}_{12}^{\mathrm{Spin}(9)} &= b_{12} - (b_{12,0} - b_{11,-1}) + (b_{12-1} - b_{11,0}) - (b_{12,2} - b_{11,1}) + (b_{12,3} - b_{11,2}) \\
&= -b_{3,0} + b_{3,1} - b_{4,0} - b_{3,2} + b_{4,1} + b_{3,3} - b_{4,2} \\
&= 6 \\
\dim \mathrm{Val}_{11}^{\mathrm{Spin}(9)} &= -b_{11} + (b_{11,0} - b_{10,-1}) - (b_{11,1} - b_{10,0}) + (b_{11,2} - b_{10,1}) - (b_{11,3} - b_{10,2}) \\
&\quad +(b_{11,4} - b_{10,3}) \\
&= b_{4,0} - b_{4,1} + b_{5,0} + b_{4,2} - b_{5,1} - b_{4,3} + b_{5,2} + b_{4,4} - b_{5,3} \\
&= 10 \\
\dim \mathrm{Val}_{10}^{\mathrm{Spin}(9)} &= b_{10} - (b_{10,0} - b_{9,-1}) + (b_{10,1} - b_{9,0}) - (b_{10,2} - b_{9,1}) + (b_{10,3} - b_{9,2}) \\
&\quad -(b_{10,4} - b_{9,3}) + (b_{10,5} - b_{9,4}) \\
&= -b_{5,0} + b_{5,1} - b_{6,0} - b_{5,2} + b_{6,1} + b_{5,3} - b_{6,2} - b_{5,4} + b_{6,3} + b_{5,5} - b_{6,4} \\
&= 15 \\
\dim \mathrm{Val}_{9}^{\mathrm{Spin}(9)} &= -b_9 + (b_{9,0} - b_{8,-1}) - (b_{9,1} - b_{8,0}) + (b_{9,2} - b_{8,1}) - (b_{9,3} - b_{8,2}) + (b_{9,4} - b_{8,3}) \\
&\quad -(b_{9,5} - b_{8,4}) + (b_{9,6} - b_{8,5}) \\
&= b_{6,0} - b_{6,1} + b_{7,0} + b_{6,2} - b_{7,1} - b_{6,3} + b_{7,2} + b_{6,4} - b_{7,3} - b_{6,5} + b_{7,4} + b_{6,6} - b_{7,5} \\
&= 20 \\
\dim \mathrm{Val}_{8}^{\mathrm{Spin}(9)} &= b_8 - (b_{8,0} - b_{7,-1}) + (b_{8,1} - b_{7,0}) - (b_{8,2} - b_{7,1}) + (b_{8,3} - b_{7,2}) - (b_{8,4} - b_{7,3}) \\
&\quad +(b_{8,5} - b_{7,4}) - (b_{8,6} - b_{7,5}) + (b_{8,7} - b_{7,6}) \\
&= 1 - 2b_{7,0} + 2b_{7,1} - 2b_{7,2} + 2b_{7,3} - 2b_{7,4} + 2b_{7,5} - 2b_{7,6} + b_{7,7} \\
&= 1 + 26 = 27
\end{aligned}$$

and for $0 \leq k \leq 7$, we have by the Hard Lefschetz theorem 1.1.7,

$$\dim \mathrm{Val}_k^{\mathrm{Spin}(9)} = \dim \mathrm{Val}_{16-k}^{\mathrm{Spin}(9)}.$$

So the dimensions of $\mathrm{Val}_k^{\mathrm{Spin}(9)}$ are :

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim \mathrm{Val}_k^{\mathrm{Spin}(9)}$	1	1	2	3	6	10	15	20	27	20	15	10	6	3	2	1	1

Chapter 6

Smooth valuations on manifolds

6.1 Valuations on smooth manifolds

The following section is based on the two papers [6], [7] and the lectures notes [10], due to Alesker.

Let M be a smooth oriented n-dimensional manifold. We would like to define on M a similar concept as convex valuations on a vector space.
In order to be able to do that, we define on M another class of bodies (as convexity does not make sense on an arbitrary manifold), namely the class of submanifolds with corners.

Definition 6.1.1. *A closed subset $N \subset M$ is a* **submanifold with corners** *if it is locally diffeomorphic to $\mathbb{R}^i_{\geq 0} \times \mathbb{R}^j$, with integers i, j.*

Denote by $\mathcal{P}(M)$ the space of compact submanifolds with corners in M.
For $K \in \mathcal{P}(M)$, we consider the following subsets of T^*M :

Definition 6.1.2. *The* **characteristic cycle** *of K is given by*

$$CC(K) = \bigcup_{x \in K} (T_x K)^\circ,$$

where $T_x K$ denotes the tangent cone to K at x

$$T_x K := \{\xi \mid \exists \text{ curve } c : \mathbb{R} \longrightarrow K \text{ with } c(0) = x, c'(0) = \xi\},$$

$(T_x K)^\circ$ is the dual tangent cone

$$(T_x K)^\circ := \{\eta \in T^*_x M \mid \langle \eta, \xi \rangle \leq 0 \ \forall \xi \in T_x K\}.$$

The **normal cycle** *of K is given by*

$$N(K) := \bigcup_{x \in K} ((T_x K)^\circ \backslash 0)/\mathbb{R}_{\geq 0}$$

It is known that $N(K)$ is a $n-1$-dimensional Lipschitz submanifold of the oriented projectivization $\mathbb{P}_M$ of T^*M

$$\mathbb{P}_M := (T^*M \backslash 0)/\mathbb{R}_{\geq 0} \cong SM,$$

where 0 is the zero section of T^*M.
An element of $N(K)$ can be thought of as a pair (p, H) with $p \in M$ and $H \subset T_p M$ an oriented hyperplane.
If $M = V$ is a vector space, this definition coincides with the definition of Section 1.4 and $N(\cdot)$ satisfies the properties of Theorem 1.4.4.

Definition 6.1.3. *A* **valuation on a manifold** *M is a finitely additive functional $\mu : \mathcal{P}(M) \longrightarrow \mathbb{R}$, i.e. for any $A, B \in \mathcal{P}(M)$ such that $A \cup B,\ A \cap B \in \mathcal{P}(M)$,*

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

A valuation is said to be **smooth** *if there exists $\omega \in \Omega^{n-1}(SM),\ \varphi \in \Omega^n(M)$ such that*

$$\mu(K) = \int_{N(K)} \omega + \int_K \varphi.$$

We denote by $\mathcal{V}^\infty(M)$ the space of smooth valuations on M. Note that by the properties of the normal cycle, any pair $(\varphi, \omega) \in \Omega^n(M) \times \Omega^{n-1}(SM)$ defines a valuation.

Let K be a compact submanifold with corners in M. We associate to K a submanifold of TM by:

Definition 6.1.4. *For a compact submanifold with corners $K \subset M$, its* **disc bundle** *$N_1(K) \subset TM$ is obtained by summing $K \times \{0\}$ and the image of $[0,1] \times N(K)$ under the homothety in the second factor :*

$$N_1(K) = \iota_*(K) + F_*([0,1] \times N(K)),$$

where $\iota : M \hookrightarrow TM,\ p \mapsto (p, 0)$ is the natural inclusion and $F : \mathbb{R} \times SM \to TM,\ (t, (p, v)) \mapsto (p, tv)$ is the homothety map.

$N_1(K)$ is a n-dimensional submanifold of TM, and we have

$$\partial N_1(K) = N(K).$$

If a smooth form $\varphi \in \Omega^{n-1}(SM)$ extends to an $n-1$-form on TM, then Stoke's theorem implies

$$\int_{N(\cdot)} \varphi = \int_{N_1(\cdot)} d\varphi.$$

Conversely, we have the following :

Lemma 6.1.5. *Any n-form ω on TM defines a smooth valuation by*

$$\mu(K) = \int_{N_1(K)} \omega,$$

for $K \in \mathcal{P}(M)$.

Proof. Such a valuation can be written as

$$\begin{aligned}
\mu(K) &= \int_{N_1(K)} \omega \\
&= \int_K \iota^*\omega + \int_{[0,1]\times N(K)} F^*\omega \\
&= \int_K \underbrace{\iota^*\omega}_{=:\varphi} + \int_{N(K)} \underbrace{\int_0^1 F^*\big|_{(t,\cdot)} \left(\frac{\partial}{\partial t}, \cdot\right) dt}_{=:\tilde{\omega}} \\
&= \int_K \varphi + \int_{N(K)} \tilde{\omega},
\end{aligned}$$

with $\varphi \in \Omega^n(M)$ and $\omega \in \Omega^{n-1}(SM)$. □

There is a canonical filtration by closed subspaces on the space of smooth valuations.
For every point $p \in M$, the space $\mathrm{Val}^\infty(T_pM)$ of translation invariant smooth convex valuations on T_pM has, by McMullen's theorem, a grading by the degree of homogeneity:

$$\mathrm{Val}^\infty(T_pM) = \bigoplus_{i=0}^{n} \mathrm{Val}_i^\infty(T_pM).$$

Let us denote by $\mathrm{Val}^\infty(TM)$ the bundle whose fiber over a point p is the space $\mathrm{Val}^\infty(T_pM)$. Then we have the grading

$$\mathrm{Val}^\infty(TM) = \bigoplus_{i=0}^{n} \mathrm{Val}_i^\infty(TM).$$

Theorem 6.1.6 ([10])**.** *There exists a canonical filtration of $\mathcal{V}^\infty(M)$ by closed subspaces*

$$\mathcal{V}^\infty(M) = \mathcal{W}_0 \supset \mathcal{W}_1 \supset \ldots \supset \mathcal{W}_n,$$

such that the associated graded space $gr_W\mathcal{V}^\infty(M) := \bigoplus_{i=0}^{n} \mathcal{W}_i/\mathcal{W}_{i+1}$ is canonically isomorphic to the space $C^\infty(M, \mathrm{Val}^\infty(TM))$ of smooth sections of the infinite-dimensional vector bundle $\mathrm{Val}^\infty(TM) \longrightarrow M$.

Let G be a Lie group acting isotropically on M, i.e. $\bar{G}$ acts transitively on the sphere bundle SM. Then the isomorphism above restricts to

$$gr_W\mathcal{V}^\infty(M)^G \cong C^\infty(M, \mathrm{Val}^\infty(TM))^G \cong \mathrm{Val}^H(T_pM)$$

where $H \subset G$ is the stabilizer of the point p; in particular, we have the following isomorphism

$$(\mathcal{W}_i/\mathcal{W}_{i+1})^G \cong \mathrm{Val}_i^H(T_pM). \tag{6.1}$$

We can also introduce a filtration on the space of n-forms on T^*M, following [6].
For every $y \in T^*M$ we define

$$\begin{aligned}(W_i(\Omega^n(T^*M)))_y \quad := \quad & \big\{\omega \in \Lambda^n T_y^*(T^*M) \mid \omega|_F \equiv 0 \text{ for all } F \subset T_y(T^*M) \\ & \qquad \text{with } \dim(F \cap T_y(\pi^{-1}\pi(y))) > n-i\big\},\end{aligned}$$

where $\pi : T^*M \to M$ is the projection $(p,\xi) \mapsto p$. Then we have the filtration

$$\Omega^n(T^*M) = W_0(\Omega^n(T^*M)) \supset W_1(\Omega^n(T^*M)) \supset \ldots \supset W_n(\Omega^n(T^*M)) \supset W_{n+1}(\Omega^n(T^*M)) = 0,$$

and

Theorem 6.1.7 ([6])**.** *The map $\Xi : \Omega^n(T^*M) \to \mathcal{V}^\infty(M)$ given by*

$$(\Xi(\omega))(K) := \int_{CC(K)} \omega$$

is surjective. Moreover, for $i = 0, 1, \ldots, n$, the map Ξ maps $W_i(\Omega^n(T^*M))$ onto $\mathcal{W}_i$ surjectively.

Let $\mu \in \mathcal{W}_k$ and $p \in M$. Let $\tau : U \to V \subset \mathbb{R}^n$ be a coordinate chart around p. The differential $d\tau_p : T_pM \to T_{\tau(p)}V \cong \mathbb{R}^n$ is a linear isomorphism. For $K \in \mathcal{P}(T_pM)$, we define

$$T_p^k\mu(K) := \frac{1}{k!}\frac{d^k}{dt^k}\bigg|_{t=0} (\tau^{-1})^*\mu\left(\tau(p) + t(d\tau_p(K) - \tau(p))\right).$$

Proposition 6.1.8 ([7])**.** *$T_p^k\mu$ is independent of the choice of τ and belongs to $\mathrm{Val}_k^\infty(T_pM)$.*

6.2 A canonical valuation on Riemannian manifolds

All the notions of Riemannian geometry presented and used in this section can be found with more details in [28].

Let M be an n-dimensional differentiable manifold, and $\Gamma(M)$ the space of vector fields on M.

Definition 6.2.1. *An* **affine connection** *on M is a map*

$$\nabla : \Gamma(M) \times \Gamma(M) \longrightarrow \Gamma(M),\ (X, Y) \mapsto \nabla_X Y$$

with the following properties :

(i) $\nabla_{f_1 X + f_2 Y} Z = f_1 \nabla_X Z + f_2 \nabla_Y Z$,

(ii) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X (f \cdot Y) = f \nabla_X Y + X(f) \cdot Y$.

If (M, g) is a Riemannian manifold, we define a particular affine connection :

Definition 6.2.2. *An affine connection ∇ on (M, g) is a* **Levi-Civita connection** *if and only if*

(i) ∇ *is* **torsion-free**, *i.e.* $\nabla_X Y - \nabla_Y X = [X, Y]$,

(ii) ∇ *is* **metric**, *i.e.* $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ *for all* $X; Y, Z \in \Gamma(M)$.

There exists on every Riemannian manifold a unique Levi-Civita connection.

Definition 6.2.3. *For $X, Y \in \Gamma(M)$, ∇ the Levi-Civita connection on M, we define*

$$\begin{array}{rcl} R(X,Y) : \Gamma(M) & \longrightarrow & \Gamma(M) \\ Z & \mapsto & R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z. \end{array}$$

This map R induces a (1,3)-tensor on T_pM through

Definition 6.2.4. *For $x, y, z \in T_pM$, define $R_p(x, y)z \in T_pM$ by $(R(X, Y)Z)(p)$ where $X, Y, Z \in \Gamma(M)$ are vector fields which extend x, y, z. The (1,3)-tensor*

$$\begin{array}{rcl} R_p : T_pM \times T_pM \times T_pM & \longrightarrow & T_pM \\ (x, y, z) & \mapsto & R_p(x, y)z \end{array}$$

is called **curvature tensor**.

This (1,3)-tensor R_p is equivalent to the (0,4)-tensor (again denoted by R_p) defined by

$$R_p(x, y, z, w) := g_p(R_p(x, y)z, w).$$

Definition 6.2.5. *For $p \in M$, $v, w \in T_pM$, we define*

$$K(v, w) = \frac{R_p(v, w, v, w)}{\langle v, v\rangle\langle w, w\rangle - \langle v, w\rangle^2}.$$

In particular, if v, w are orthonormal, then

$$K(v, w) = R_p(v, w, v, w).$$

K depends in fact only on the 2-plane $E_{v,w}$ in T_pM generated by v and w. K is called the **sectional curvature** *of $E_{v,w}$.*

Lemma 6.2.6 (Properties of the curvature tensor)**.** *For $x, y, z, w \in T_pM$, we have*

1. $R_p(x, y, z, w) = -R_p(y, x, z, w)$
2. $R_p(x, y, z, w) = -R_p(x, y, w, z)$
3. $R_p(x, y, z, w) = R_p(z, w, x, y)$

R_p is thus an element of $Sym^2\Lambda^2 T_pM \subset \Lambda^2 T_pM \otimes \Lambda^2 T_pM$.

Definition 6.2.7. *On the set of k-forms on an oriented vector space W, the* **Hodge-$*$ operator** *is defined by*

$$\begin{array}{ccc} \Omega^k W & \longrightarrow & \Omega^{n-k} W \\ \omega & \mapsto & *\omega, \end{array}$$

where

$$*\omega(v_1, ..., v_{n-k}) = \omega(v_{n-k+1}, ..., v_n),$$

for an oriented orthonormal basis $v_1, ..., v_n$ of W.

The Hodge-$*$ operator is an isomorphism. Hence, we can see R_p as an element R_p^* of $\Lambda^2(T_pM) \otimes \Lambda^{n-2}(T_pM) \subset \Lambda^n(T_pM \oplus T_pM)$.

Let $(M, g), (N, h)$ be two Riemannian manifolds, $y \in N$, and $\varphi : (N, h) \longrightarrow (M, g)$.
We define two subspaces of T_yN :
the **vertical subspace** of T_yN

$$V_y := \ker d\varphi_y,$$

and the **horizontal subspace** of T_yN

$$H_y := V_y^\perp = (\ker d\varphi_y)^\perp.$$

Definition 6.2.8. *A map $\varphi : (N, h) \longrightarrow (M, g)$ is a* **Riemannian submersion** *if φ is a smooth submersion, and for any $y \in N$, $d\varphi_y$ is an isometry between H_y and $T_{\varphi(y)}M$.*

We would like to prove that the projection $\pi : TM \longrightarrow M$ is a Riemannian submersion. First we equip TM with an appropriate Riemannian metric.

Definition 6.2.9. *A* **vector field along a curve** *$c : I \subset \mathbb{R} \longrightarrow M$ is a curve $X : I \longrightarrow TM$ such that $X(t) \in T_{c(t)}M$ for all $t \in I$.*

Theorem 6.2.10 ([28])**.** *Let ∇ be as before the Levi-Civita connection of M, and c be a curve on M. There exists a unique operator $\frac{\nabla}{dt}$ defined on the vector space of vector fields along c, which satisfies the following conditions :*

i) for any real function f on I

$$\frac{\nabla}{dt}(fY)(t) = f'(t)Y(t) + f(t)\frac{\nabla}{dt}Y(t),$$

ii) if there exists a neighborhood of t_0 in I such that Y is the restriction to c of a vector fields X defined on a neighborhood of $c(t_0)$ in M, then

$$\frac{\nabla}{dt}Y(t_0) = (D_{c'(t_0)}X)\big|_{c(t_0)}.$$

Then TM is a Riemannian manifold with the metric given by the following : for $c_i(t) := (p_i(t), v_i(t)),\ i = 1, 2$, two curves in TM with $c_1(0) = c_2(0)$, we define

$$\tilde{g}(c_1'(0), c_2'(0)) := g(p_1'(0), p_2'(0)) + g\left(\frac{\nabla}{dt}v_1(0), \frac{\nabla}{dt}v_2(0)\right)$$

With this metric, the projection $\pi : TM \longrightarrow M$ is a Riemannian submersion :

$$V_{(p,v)} = \{c'(0) \mid c = (x, w) : I \longrightarrow TM \text{ with } c(0) = (p, v),\ d\pi_{(p,v)}c'(0) = x'(0) = 0\}$$

and

$$H_{(p,v)} = (V_{(p,v)})^{\perp} = \{c'(0) \mid c = (x, w) : I \longrightarrow TM \text{ with } c(0) = (p, v),\ \frac{\nabla}{dt}w(0) = 0\},$$

hence for two curves $c_i = (x_i, w_i) : I \longrightarrow TM, i = 1, 2$, s.t. $c_i'(0) \in H_{(p,v)}$

$$\begin{aligned} g\big(d\pi_{(p,v)}(c_1'(0)), d\pi_{(p,v)}(c_2'(0))\big) &= g(x_1'(0), x_2'(0)) \\ &= \tilde{g}(c_1'(0), c_2'(0)), \end{aligned}$$

and so $d\pi_{(p,v)}$ is an isometry between $H_{(p,v)}$ and T_pM.

Moreover we have

$$V_{(p,v)} = T_{(p,v)}\pi^{-1}(p) = T_{(p,v)}T_pM$$

and since T_pM is a vector space, we can identify

$$V_{(p,v)} \cong T_pM.$$

It follows therefore that, for $(p, v) \in TM$, the vertical subspace $V_{(p,v)}$ and the horizontal subspace $H_{(p,v)}$ are n-dimensional subspaces of $T_{(p,v)}TM$ with

$$T_{(p,v)}TM = V_{(p,v)} \oplus H_{(p,v)} \cong T_pM \oplus T_pM,$$

hence

$$T^*_{(p,v)}TM \cong T^*_pM \oplus T^*_pM.$$

Therefore we can see the curvature tensor as an element of

$$R^*_p \in \Lambda^n(T_pM \oplus T_pM) \cong \Lambda^n(T_{(p,v)}TM),$$

and we can define an n-form $\omega \in \Omega(TM)$ by

$$\omega_{(p,v)} := \frac{1}{C_{n-2}} R^*_p \in \Lambda^n(T_{(p,v)}TM),$$

where the normalization constant C_{n-2} is the volume of the $n-2$-dimensional unit ball :

$$C_{n-2} = \frac{\pi^{(n-2)/2}}{\Gamma\left(\frac{n-2}{2} + 1\right)}.$$

By Lemma 6.1.5, such an n-form on TM yields a smooth valuation $\mu \in \mathcal{V}^\infty(M)$ through integration over the disc bundle of K :

$$K \in \mathcal{P}(M) \mapsto \mu(K) := \int_{N_1(K)} \omega.$$

Theorem 6.2.11. *The valuation μ defined above has filtration index 2, i.e. $\mu \in \mathcal{W}_2$.*

Proof. By defining the n-form

$$\tilde{\omega} := \begin{cases} \omega & \text{on } N_1(K) \\ 0 & \text{on } T^*M \backslash N_1(K) \end{cases}$$

we can rewrite μ as

$$\mu(K) = \int_{CC(K)} \tilde{\omega}.$$

Although $\tilde{\omega}$ is not smooth, we can approach $\tilde{\omega}$ by smooth forms whose restriction to $N_1(K)$ is ω. Hence we can apply Theorem 6.1.7 which gives the condition : $\mu \in \mathcal{W}_2$ if and only if $\tilde{\omega} \in W_2(\Omega^n(TM))$.
Therefore, for $y = (p, v) \in TM$, we have to show that

$$R_p^*\big|_F \equiv 0$$

for all $F \subset T_y(TM)$ with $\dim(F \cap T_y(\pi^{-1}\pi(y))) = \dim(F \cap V_{(p,v)}) > n - 2$, where $V_{(p,v)} \cong T_pM$ is the vertical subspace of $T_{(p,v)}(TM)$.
If $v_1, ..., v_{n-1} \in V_{(p,v)}$, $w \in T_{(p,v)}(TM) \cong T_pM$, then :

$$R_p^*(w, v_1, ..., v_{n-1}) = 0,$$

since by definition $R_p^* \in \Lambda^2(T_pM) \otimes \Lambda^{n-2}(T_pM)$. Therefore $\tilde{\omega} \in W_2(\Omega^n(TM))$, and $\mu \in \mathcal{W}_2$. □

Hence Proposition 6.1.8 implies that we can associate to μ a valuation $T_p^2\mu$ on T_pM of homogeneity degree 2. The isomorphism $\mathcal{W}_2/\mathcal{W}_3 \cong \mathrm{Val}_2(T_pM)$ is given by

$$T_p^2\mu(K) := \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} (\tau^{-1})^*\mu\left(\tau(p) + t(d\tau_p(K) - \tau(p))\right),$$

for $K \in \mathcal{P}(T_pM)$, where $\tau : U \subset M \longrightarrow V \subset \mathbb{R}^n$ is a coordinate chart around p.
Since ω is invariant under the action of the isometry group G of M, the valuation μ is also G-invariant. Hence, equation (6.1) implies that $T_p^2\mu$ is invariant under $H = Stab_pG$.

Proposition 6.2.12. *The valuation $T_p^2\mu$ is even.*

Proof. Recall the definition of the Euler-Verdier involution, introduced by Alesker in [7].
Let $s : S^*M \to S^*M$ be the natural involution $(q, [\xi]) \mapsto (q, [-\xi])$.

Definition 6.2.13. *Let $\psi = \psi_\eta$ be a valuation given by*

$$\psi_\eta(K) = \int_{N(K)} \eta$$

with $\eta \in \Omega^{n-1}(S^*M)$. The **Euler-Verdier involution** *is then defined as $(-1)^n\psi_{s^*\eta}$.*

The valuation $T_p^k\mu$ on the vector space T_pM is even, if and only if it belongs to the $(-1)^k$-eigenspace of the Euler-Verdier involution ([13], [7]).
Analogously, consider the involution on TM

$$\tilde{s} : TM \to TM, \qquad (q, v) \mapsto (q, -v).$$

The valuation $T_p^2\mu$ is even if and only if it belongs to the $(-1)^2$-eigenspace of the involution induced by $\tilde{s}$.
Since $\tilde{s}^*\omega = (-1)^{n-2}\omega$, μ belongs to the $(-1)^2$-eigenspace of the involution induced by $\tilde{s}$; hence $T_p^2\mu$ is even.

□

Theorem 6.2.14. *The Klain function of the valuation $T_p^2\mu \in \mathrm{Val}_2(T_pM)$ is the sectional curvature of M.*

Proof. Let $E \in Gr_2(T_pM)$ be a 2-dimensional vector subspace of T_pM, and D^2 be the 2-dimensional unit ball in E. Consider the exponential map $\exp : T_pM \to M$, and set $\tau := \exp^{-1}$. Then we have $\tau(p) = 0$ and $d\tau_p = \mathrm{id}|_{T_pM}$. Hence :

$$\begin{aligned} T_p^2\mu(D^2) &= \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} (\tau^{-1})^*\mu\left(\tau(p) + t(d\tau_p(D^2) - \tau(p))\right) \\ &= \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \mu\left(\tau^{-1}(tD^2)\right) \\ &= \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \int_{N_1(\tau^{-1}(tD^2))} \omega. \end{aligned}$$

Define $S_t := \tau^{-1}(tD^2) \in \mathcal{P}(M)$. Then

$$\begin{aligned} \mu(S_t) &= \int_{N_1(S_t)} \omega \\ &= \int_{N_1(S_t)} \omega_{(q,v)}(v_1^h, v_2^h, v_3^v, ..., v_n^v)dqdv \end{aligned}$$

where v_1^h, v_2^h are horizontal lifts (i.e. lifts in the horizontal subspace $H_{(q,v)}$) of an orthonormal ba– sis $\{v_1, v_2\}$ of T_qS_t and $v_3^v, ..., v_n^v$ are vertical lifts (i.e. lifts in the vertical subspace $V_{(q,v)}$) of an orthonormal basis of the orthogonal complement of T_qS_t in T_qM,

$$= \frac{1}{C_{n-2}} \mathrm{vol}_{n-2}(D^{n-2}) \int_{S_t} R_q^*(v_1, v_2, v_3, ..., v_n)dq,$$

where D^{n-2} denotes the $n-2$-dimensional unit ball, since, by definition, R_q^* is constant on each fiber. It becomes through the isomorphism given by the Hodge-$*$ operator

$$\begin{aligned} \frac{1}{C_2t^2}\mu(S_t) &= \frac{1}{C_2t^2}\int_{S_t} R_q(v_1, v_2, v_1, v_2)dq \\ &= \frac{1}{C_2t^2}\int_{S_t} K(T_qS_t)dq \\ &\longrightarrow K(E), \qquad \text{as } t \to 0, \end{aligned}$$

where we recall that E is the 2-dimensional vector subspace of T_pM and K denote the sectional curvature.
Hence

$$\begin{aligned} T_p^2\mu(D^2) &= \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \mu\left(S_t\right) \\ &= C_2K(E) = \mathrm{vol}_2(D^2)K(E), \end{aligned}$$

therefore the Klain function of $T_p^2\mu$ is the sectional curvature K of M. □

Examples

Example 1. The n-dimensional sphere $M = \mathbb{S}^n$. Then we have $T_pM = \mathbb{R}^n$, $G = Isom(M, g) = \mathrm{O}(n+1)$, $H = Stab_p = \mathrm{O}(n)$, and the construction above allows us to construct a valuation $T_p^2\mu$ on T_pM whose Klain function

$$\mathrm{Kl}_{T_p^2\mu} = K_{\mathbb{S}^n} \equiv 1$$

is constant 1, hence, by the injectivity of the Klain embedding, $T_p^2\mu$ is the second intrinsic volume μ_2.

Example 2. The complex projective space $M = \mathbb{C}P^2$. Then we have $T_pM = \mathbb{C}^n$, $G = Isom(M,g) = \mathrm{U}(n+1)/\mathrm{U}(1)$, $Stab_p = \mathrm{U}(n)$, and

$$\mathrm{Kl}_{T_p^2\mu} = 1 + 3\cos^2\varphi(x,y),$$

where φ is the Kähler angle : for a plane generated by two vectors v, w, the angle φ satisfies

$$\cos^2\varphi(x,y) = \langle v, iw\rangle^2.$$

φ measures the angle between the complex planes spanned by $\{v, iv\}$ and $\{w, iw\}$ (cf. [33]).
Several bases of $\mathrm{Val}^{\mathrm{U}(n)}$ have been given by Bernig-Fu in [18]. A basis for $\mathrm{Val}_2^{\mathrm{U}(n)}$ is given by the Tasaki valuations $\tau_{2,i}$, $i = 0, 1$. Their Klain functions are

$$\mathrm{Kl}_{\tau_{2,0}} = 1, \qquad \mathrm{Kl}_{\tau_{1,0}} = \cos^2\varphi,$$

where φ is the Kähler angle.
Hence we can express our new valuation $T_p^2\mu$ in the basis of the Tasaki valuations by

$$T_p^2\mu = \tau_{2,0} + 3\tau_{2,1}.$$

Example 3. The quaternionic projective space $M = \mathbb{H}P^n$ Then we have $T_pM = \mathbb{H}^n$, $G = Isom(M,g) = \mathrm{Sp}(n+1)\cdot\mathrm{Sp}(1)$ and $H = Stab_p = \mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$.
The sectional curvature of the quaternionic projective space is given by

$$K(E_{x,y}) = 1 + 3\cos^2\alpha(x,y),$$

where the angle α is defined by

$$\cos^2\alpha(x,y) = \langle x, yi\rangle + \langle x, yj\rangle + \langle x, yk\rangle,$$

with $1, i, j, k$ the usual basis of $\mathbb{H}$ over $\mathbb{R}$ (cf. [34]).

Example 4. The octonionic projective plane $M = \mathbb{O}P^2$. Due to the non-associativity of $\mathbb{O}$, the concept of octonionic projective space $\mathbb{O}P^n$ only makes sense for $n = 2$ ([12]). In this case, we have $T_pM = \mathbb{O}^2$, $G = Isom(M,g) = F_4$ and $H = Stab_p = \mathrm{Spin}(9)$. For $(a,b),(c,d) \in \mathbb{O}^2$ with $\|(a,b)\| = \|(c,d)\| = 1$ and $\langle (a,b),(c,d)\rangle = 0$, the sectional curvature of the plane generated by $(a,b),(c,d)$ is given by (cf. [23])

$$K(E_{(a,b),(c,d)}) = 4\left[\|a\wedge c\|^2 + \|b\wedge d\|^2 + \frac{1}{4}\|a\|^2\|d\|^2 + \frac{1}{4}\|b\|^2\|c\|^2 + \frac{1}{2}\langle ab, cd\rangle - \langle ad, bc\rangle\right];$$

hence :

i) $K(E_{(a,0),(b,0)}) = 4$,

ii) $K(E_{(a,0),(0,b)}) = 1$.

Since Spin(9) acts transitively on $\mathbb{S}^{15}$, we may assume $(a,b) = (1,0)$. Then it becomes

$$K(E_{(1,0),(c,d)}) = 4\|c\|^2 + \|d\|^2.$$

The Klain function of $T_p^2\mu$ is therefore between 1 and 4. Alesker constructed in [9] a valuation τ which is Spin(9)-invariant and of homogeneity degree 2, called the octonionic pseudo-volume (in analogy to the quaternionic pseudo-volume defined in [5]). For the octonionic pseudo-volume holds

i) $\mathrm{Kl}_\tau(E_{(a,0),(b,0)}) = 0$,

ii) $\mathrm{Kl}_\tau(E_{(1,0),(0,1)}) = 1$.

Since we have shown that $\mathrm{Val}_2^{\mathrm{Spin}(9)}$ is of dimension 2, the valuation $T_p^2\mu$ can be expressed as a linear combination of the second intrinsic volume and the octonionic pseudo-volume. Comparing the values given above for the Klain functions of $T_p^2\mu$ and τ, we get for $T_p^2\mu$:

$$T_p^2\mu = 4\mu_2 - 3\tau.$$

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