On the Critical Exponent of Infinitely Generated Veech Groups

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Introduction

> Der Weg zum besseren Billard ist der Weg zur Vereinfachung – einfaches Billard ist gutes Billard. (Andreas Huber, Billiard Coach)

One of the most simple ways to look at billiards is to consider just one ball as a point without extension that moves without any friction in unit speed on straight lines satisfying "angle of incidence equals angle of reflection" at the boundary. While the dynamics of these idealized billiards are quite easy on a rectangular billiard table, they become more complicated and interesting if one permits tables of more general shapes, e.g. the shape of an arbitrary polygon whose angles are rational multiples of π. Such a polygon is called a *rational polygon*.

Applying an unfolding construction first described by Katok and Zemlyakov ([ZK75]) we obtain a surface called translation surface and the trajectory of the ball becomes a geodesic with a constant direction on this surface. The unfolding construction and an accurate definition of translation surfaces are given in Chapter 2. For now, we can think of a translation surface as the Riemann surface X obtained from a rational polygon with pairs of parallel sides by identifying these sides by translations.

The next step in describing the dynamics of the billiard ball is to define the Veech group of a translation surface X: we consider first the *translation group* $\mathrm{Trans}(X)$ consisting of all orientation-preserving homeomorphisms $f : X \to X$ which are translations in each chart. Then the *affine group* $\mathrm{Aff}(X)$ consists of all orientation-preserving homeomorphisms $f : X \to X$ which are affine in each chart. The derivative of such an affine self-homeomorphism is globally constant a 2×2-matrix with determinant 1. The *Veech group* is the image of $\mathrm{Aff}(X)$ in $\mathrm{SL}_2(\mathbb{R})$ under the derivation map:

$$\mathrm{SL}(X) := \{\gamma \in \mathrm{SL}_2(\mathbb{R}) \mid \exists f \in \mathrm{Aff}(X) \text{ with derivative } \gamma\}$$

These groups give a short exact sequence $\mathrm{Trans}(X) \hookrightarrow \mathrm{Aff}(X) \twoheadrightarrow \mathrm{SL}(X)$. Since $\mathrm{SL}(X)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, Veech groups are Fuchsian groups, i.e. they act properly discontinuously on the hyperbolic plane.

For the hyperbolic plane we use the upper half plane model $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{im}(z) > 0\}$ with the metric $\rho_{\mathbb{H}}$. The orbit of a base point under the action of a Fuchsian group Γ accumulates only at points in the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. The set of all these limit points is called the limit set $\Lambda(\Gamma)$. There is a rough sizing of Fuchsian groups regarding the limit set: If $|\Lambda|$ is finite, it is 0, 1 or 2 and Γ is called *elementary*. If $|\Lambda|$ is infinite, Γ is *non-elementary*. Additionally, if the limit set is the whole boundary, Γ is called Fuchsian group *of the first kind*, otherwise *of the second kind*. A Fuchsian group of finite co-volume is called a *lattice*. Lattices are exactly the finitely generated Fuchsian groups of the first kind.

A translation surface whose Veech group is a lattice is called a *Veech surface*. They are of special interest because of the famous Veech dichotomy: on a Veech surface for any direction θ either every trajectory in direction θ is periodic or every trajectory in this direction is equidistributed. A big set of Veech surfaces are the so-called *square-tiled surfaces*, which are coverings of the square torus branched over one point. Their Veech group is arithmetic and in the space of translation surfaces of genus g the square-tiled surfaces form a dense subset. The first examples of non-arithmetic Veech surfaces were the regular double n-gons for odd $n \geq 5$ and the regular n-gons for even $n \geq 8$ given by Veech ([Vee89]).

The next "smaller" Fuchsian groups are groups that are still of the first kind, but not a lattice and thus infinitely generated. Until the beginning of this millennium not even the existence of translation surfaces with infinitely generated Veech groups was known. In [McM03] McMullen and in [HS04] Hubert and Schmidt give two different constructions of such surfaces with an infinitely generated Veech group. In Chapter 2 we will briefly present their constructions. We will focus on the approach of Hubert and Schmidt who use coverings of Veech surfaces branched at the singularities and one special marked point – a non-periodic connection point. Their Veech group is commensurable to the subgroup of the unmarked surface's Veech group consisting of all elements stabilizing the marked point. The surfaces we will consider are the translation surfaces L_D (with $D > 0$ and $D \equiv 0 \mod 4$ not a square) obtained from an L-shaped polygon with vertical side lengths (from left to right) $w := \sqrt{\frac{D}{4}}$, $w - 1$, and 1 and horizontal side lengths (from top to bottom) 1, w and $1 + w$. In [McM05] McMullen shows that L_D is a Veech surface of genus 2 and that in some sense many of the Veech surfaces of genus 2 are more or less of this form (for more details see Chapter 2).

Another way to associate some kind of "size" to a Fuchsian group Γ is to look at its critical exponent $\delta(\Gamma)$, which is defined as

$$\delta(\Gamma) := \inf\{a \in \mathbb{R} : \sum_{\gamma \in \Gamma} e^{-a\rho_{\mathbb{H}}(i,\gamma(i))} < \infty\}.$$

There exist some general bounds on the critical exponent of Fuchsian groups:

- For all Fuchsian groups the critical exponent is at most 1.
- If Γ is a lattice, $\delta(\Gamma) = 1$.
- If Γ is finitely generated and of the second kind, $\delta(\Gamma) < 1$
- If Γ is non-elementary and contains a parabolic element, $\delta(\Gamma) > \frac{1}{2}$.

In this context, we can state our main result:

Main Theorem A. *For every non-periodic connection point P on the Veech surface L_D (with $D \equiv 0 \mod 4$ not a square) the (infinitely generated) stabilizer subgroup $\mathrm{SL}(L_D; P) := \mathrm{Stab}_{\mathrm{SL}(L_D)}(P)$ has critical exponent strictly between $\frac{1}{2}$ and 1.*

In [GJ00] Gutkin and Judge show that the Veech groups of coverings like the one analyzed by Hubert and Schmidt are commensurable to $\mathrm{SL}(L_D; P)$. Since commensurable groups have the same critical exponent, we obtain the following theorem:

Main Theorem B. *There exist translation surfaces whose Veech group is infinitely generated with critical exponent strictly between* $\frac{1}{2}$ *and* 1. *More precisely this is the case for every affine covering of* L_D *(with* $D \equiv 0 \mod 4$ *not a square) branched at any non-periodic connection point* P.

Note that the restriction to $D \equiv 0 \mod 4$ is made for technical reasons, and the result propably is true also for $D \equiv 1(4)$ and the surfaces $L_{D,\pm 1}$ defined in Definition 2.17. For details see Section 6.2.3.

The proof of Main Theorem A can be found in Chapter 6. We use a result of Roblin and Tapie ([RT13]), which implies that $\delta(\mathrm{SL}(L_D; P)) < 1$ if the Schreier graph of $\mathrm{SL}(L_D)$ with respect to $\mathrm{SL}(L_D; P)$ and any finite generating set S of $\mathrm{SL}(L_D)$ is non-amenable. Their proof is described in Chapter 5, after we give some background on the critical exponent of Fuchsian groups.

Studying the group $\mathrm{SL}(L_D; P)$ and its critical exponent one easily checks that for all points Q in the $\mathrm{SL}(L_D)$-orbit of P the groups $\mathrm{SL}(L_D; P)$ and $\mathrm{SL}(L_D; Q)$ are conjugate and thus have the same critical exponent. Hence the question which points belong to the orbit of P arises very naturally.

We will examine this question in Chapter 4:

Theorem C. *Given a connection point** $P = \big(x_r + x_i w; y_r + y_i w\big) \in L_D$ *with reduced fractions* $x_r, x_i, y_r, y_i \in \mathbb{Q}$, *set* $N(P)$ *to be the least common denominator of* x_r, x_i, y_r, *and* y_i. *All points* Q *in the* $\mathrm{SL}(L_D)$*-orbit of* P *also have both coordinates in* $\mathbb{Q}(w)$. *Let* $N(Q)$ *be the least common denominator of the four reduced fractions describing* Q. *Then*

$$N(Q) = N(P).$$

In particular, there are infinitely many distinct orbits of connection points.

In the special case $D = 8$ we can be more precise:

Theorem D. *Let* $D = 8$ *and* $w = \sqrt{2}$ *and fix* $N \in \mathbb{N}$. *The set of all non-periodic†* *points* P *of the form* $P = (x_r + x_i w; y_r + y_i w)$ *with* x_r, x_i, y_r, y_i *reduced fractions with least common denominator* N *decomposes into a* ***finite*** *number of orbits under the action of* $\langle A, B\rangle = \mathrm{SL}(L_8)$.

The remaining chapters are dedicated to the description of the action of $\mathrm{SL}(L_D)$ on L_D (Chapter 3), background on translation surfaces and Veech groups (Chapter 2) and Schreier graphs (Chapter 1). In the last-mentioned chapter we explore a method to show non-amenability of (Schreier) graphs:

Theorem E. *Let* G *be a graph. If it is possible to omit edges of* G *obtaining a forest* G' *in which every connected component is an infinite simple tree without leaves, then*

$$\mathfrak{c}(G) \geq \frac{1}{2n(G') + 2},$$

where $n(G')$ *is the supremum of the lengths of connected pieces of* G', *where every vertex has valency* 2. *In particular, if* $n(G') < \infty$ *then* G *is non-amenable.*

*All points of this form are connection points, see Proposition 2.22.

†this excludes exactly the 6 Weierstraß points

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1 Graphs and Amenability

In this chapter we will define some vocabulary about graphs – especially Schreier graphs – and amenability of graphs. Amenability originally was defined as a property of groups. A countable group Γ is called *amenable* if there exists a left-invariant mean $\lambda : \ell^\infty(\Gamma) \to \mathbb{R}$. There is a nice criterion sufficient for a group to be non-amenable, namely the existence of a free non-abelian subgroup. Given a normal subgroup $\Pi \triangleleft \Gamma$ the existence of a free non-abelian subgroup $\mathcal{F}$ that has trivial intersection with Π is sufficient for the non-amenability of the factor group.

Amenability of graphs is defined via the Cheeger constant of the graph and it is well-known (see e.g. [Woe00], Proposition 12.4) that Cayley graphs with respect to finite generating sets are amenable if and only if the corresponding finitely generated group is amenable. Schreier graphs are a generalization of Cayley graphs of factor groups – the subgroup does not need to be normal. Hence one could hope that also for the Schreier graph of Γ with respect to $\Pi < \Gamma$ in order to be non-amenable it is sufficient to find a non-abelian free subgroup of Γ with trivial intersection with Π. But unfortunately this is not true and we will give a counterexample. In this counterexample the subgroup Π is not finitely generated. The question, whether the condition for factor graphs can be extended to Schreier graphs with finitely generated subgroups remains open.

Thus we have to find another method to prove non-amenability of (Schreier) graphs. Such a method is provided in the second section of this chapter and summarized by Theorem E.

1.1 Basics About Graphs

We will use two different notions of graphs and start by defining them.

Definition 1.1. A *directed graph* G is a tuple (V, E, α, β) consisting of a nonempty set $V = V(G)$, a set $E = E(G)$ and two maps α and $\beta : E \to V$. We call V the *vertex set*, its elements v *vertices*, E the *edge set* and the elements $e \in E$ *edges*. The vertex $\alpha(e)$ is called the *origin* and $\beta(e)$ the *destination* of the edge e. Both together are the *endpoints* of e.

Definition 1.2. A *simple graph* is a pair (V, E) consisting of a nonempty vertex set V and an edge set $E \subseteq \{\{v, w\} \mid v, w \in V,\ v \neq w\}$. For an edge $\{v, w\}$ the vertices v and w are called the *endpoints* of this edge.

In particular, in a simple graph the edges do not have a direction or orientation. And there are no edges with only one endpoint and no multi-edges, both allowed in a directed graph. In most cases it will be clear out of context, which kind of graph we mean, hence we denote also simple graphs by G.

Convention 1.3. Throughout the thesis we only consider *locally finite graphs* – i.e. graphs, in which every vertex is endpoint of only finitely many edges.

For both, directed and simple graphs, we introduce the terms paths and connectivity and then equip connected graphs with a metric. First we will do this for simple graphs:

Definition 1.4. A *path* π of *length* $l(\pi) = n$ in a simple graph $G = (V, E)$ is a sequence of vertices $[v_0, v_1, \dots, v_n]$ with the property that $\{v_i, v_i + 1\} \in E$ for all $i = 0 \dots n-1$. The vertex v_0 is called *starting point* and v_n *terminal point* of the path. Furthermore for all vertices $v \in V$ there is a path $[v]$ of length 0, called the empty path from v to v (because it contains no edges). If all the vertices $v_0, \dots, v_n$ are different, we call π a *simple path.* If $v_0, \dots, v_{n-1}$ are different vertices and $v_n = v_0$, the path π is called a *circle.*

Definition 1.5. Let $G = (V, E)$ be a simple graph.

- For a path $\pi = [v_0, \dots, v_n]$ let $\overline{\pi}$ be the *inverse path* $[v_n, \dots, v_0]$.
- If for two vertices $v, w \in V$ there is a path with starting point v and terminal point w, we say v and w are *connected.* If all pairs $(v, w) \in V$ are connected, the graph G is called *connected.*
- For connected graphs G we define a map

  $$\begin{aligned} d_G : V \times V &\to \mathbb{N} \\ (v, w) &\mapsto \min\{l(\pi) \mid \pi \text{ a path from } v \text{ to } w\}. \end{aligned}$$

  The map d_G is a metric, called the *edge metric*, so (V, d_G) becomes a metric space and a path from v to w of length $d_G(v, w)$ is called a *geodesic.*
- A graph that does not contain a circle is called a *forest.* A connected forest is called a *tree.*

Remark 1.6. If G is not connected, it decomposes into *connected components* and we define d as above for vertices in the same connected components and extend it to $V \times V$ by defining $d(v, w) := \infty$ if v and w are not connected.

Before we define the same terms analogically for directed graphs we have to introduce inverse edges, because we want to allow a path to use a directed edge in the inverse direction, too. We can then see a pair of edge and inverse edge as a “geometrical edge”

Definition 1.7. Given a directed graph $G = (V, E, \alpha, \beta)$ for every edge $e \in E$ we formally define an *inverse edge* e^-. This leads to the sets $E^- = \{e^- | e \in E\}$ and $E^\pm = E \sqcup E^-$. We extend inversion, α and β from E to $E^\pm$ by setting $(e^-)^- = e$, $\alpha(e^-) = \beta(e)$ and $\beta(e^-) = \alpha(e)$.

Since we may have multi-edges in directed graphs, the definition of a path is slightly different because the sequence of edges cannot be recovered from the sequence of vertices. However, knowing the edges we also know the sequence of vertices.

Definition 1.8. An *edge path* π of *length* $l(\pi) = n$ in a directed graph $G = (V, E, \alpha, \beta)$ is a finite sequence $[e_1, e_2, \dots, e_n]$ with $e_i \in E^{\pm}$ and the property that $\alpha(e_{i+1}) = \beta(e_i)$ for $i = 1, \dots, n-1$. We can extend α and β to the set of all edge paths by setting $\alpha([e_1, e_2, \dots, e_n]) := \alpha(e_1)$ and $\beta([e_1, e_2, \dots, e_n]) := \beta(e_n)$. Also in this case for every vertex v we define an *empty edge path* $[]_v$ with $\alpha([]_v) = \beta([]_v) = v$ and for π there is an *inverse edge path* $\pi^- = [e_n^-, \dots, e_1^-]$. The terms circle, connectivity, tree, the distance function d_G and the term geodesic are defined as above. In directed graphs there may be circles of length 1, which are called *loops*.

Sometimes we will not only look at the graph G as a whole, but also at a "smaller part" of it, a subgraph:

Definition 1.9. An *induced subgraph* $G' = (V', E', \alpha', \beta')$ of a graph $G = (V, E, \alpha, \beta)$ is a graph with $V' \subset V$, $E' = \{e \in E \,|\, \alpha(e), \beta(e) \in V'\}$, $\alpha' = \alpha|_{E'}$ and $\beta' = \beta|_{E'}$. An induced subgraph of a simple graph $G = (V, E)$ is defined analogically.

The last basic definitions about graphs deal with the local structure of a graph at a vertex v:

Definition 1.10. Let G be a simple graph and $v \in V(G)$ a vertex.

- A vertex $w \neq v$ is called a *neighbor* of v if $\{v, w\} \in E$.
- The number of edges that have v as one endpoint (= the number of neigbors of v) is called the *valency* of v, denoted by $\mathrm{val}_G(v)$. The valency of v in a subgraph G' is called the *G'-valency* of v, denoted by $val_{G'}(v)$.

Let G be a directed graph and $v \in V(G)$ a vertex.

- A vertex w (not necessary $\neq v$) is called a *neighbor* of v if there is a edge $e \in E^{\pm}$ with $\alpha(e) = v$ and $\beta(e) = w$, i.e. if there is an (geometrical) edge with endpoints v and w.
- The number of edges $e \in E^{\pm}$ with $\alpha(e) = v$ is called the (total) *valency* of v, denoted by $\mathrm{val}_G(v)$. Note that this is the number of geometrical edges with endpoints v and some other vertex plus twice the number of loops at the vertex v. The valency of v in a subgraph G' is called the *G'-valency* of v, denoted by $val_{G'}(v)$.

For both – directed and simple graphs G – we define:

- A vertex with valency 1 is called a *leaf*.
- If the valency of vertices in $V(G)$ is bounded, we call
$$\mathrm{val}_{\max}(G) := \max\{\mathrm{val}_G(v) \,|\, v \in V(G)\}$$
the *maximal valency* of G.
- If all vertices have the same valency ρ, the graph is said to be *ρ-regular* or *ρ-valent*. A finite tree T always contains leaves, so we call T already *ρ-valent* if every vertex that is not a leaf has valency ρ.

Now we introduce special types of directed graphs that are attached to groups, first Cayley graphs and then Schreier coset graphs:

Definition 1.11. For a group Γ and a nonempty finite subset $S \subset \Gamma$ consider the following directed graph $G_{\Gamma,S}$:

- The vertex set V is the set of elements of Γ.
- The edge set E^+ is $\Gamma \times S$ with $\alpha((\gamma, s)) = \gamma$ and $\beta((\gamma, s)) = \gamma s$.
- Moreover we add to every edge (γ, s) the label s. So $G_{\Gamma,S}$ is $2|S|$-regular, at every vertex for each $s \in S$ there is one outgoing edge labeled with s and one incoming s-edge. Walking an edge incoming at the vertex g labeled with s backwards means to multiply g by s^{-1}.

Obviously this graph is connected if and only if $S^{\pm} := S \cup S^{-1}$ is a generating set for Γ. In this case $G_{\Gamma,S}$ is called the *Cayley graph* of Γ with respect to S.

Remark 1.12. We already mentioned that a connected graph together with the edge metric becomes a metric space. Also for finitely generated groups there is a metric – the word metric with respect to a generating set S:

$$d_{\Gamma,S}(\gamma_1, \gamma_2) := |\gamma_1^{-1}\gamma_2| := \min\{\text{length of } w \,|\, w \text{ a word in } S \text{ with } w =_\Gamma \gamma_1^{-1}\gamma_2\}$$

By definition the word metric for a group Γ with respect to S is exactly the edge metric of $G_{\Gamma,S}$.

As an example for a Cayley graph – especially viewed as metric space – we look at the free abelian group of rank k with respect to a basis $\{x_1, \ldots, x_k\}$ and investigate the ball $B_n(0)$ of radius n with center 0:

Example 1.13. For a free abelian group of rank k the following is a presentation: $\langle x_1, \ldots x_k \,|\, [x_i, x_j] = 1\rangle$. We can identify the generators with the standard basis vectors $\mathbf{e_i}$ of $\mathbb{R}^k$ and the elements of the free abelian group – i.e. the vertices of its Cayley graph – with the points of $\mathbb{Z}^k \subset \mathbb{R}^k$. In the Cayley graph there is an edge from $\mathbf{a} \in \mathbb{Z}^k$ to $\mathbf{b} \in \mathbb{Z}^k$ labeled with x_i if and only if $\mathbf{b} - \mathbf{a} = \mathbf{e_i}$.

The ball $B_n(0)$ of radius n with center 0 is the set of all vertices v with $d_G(0, v) \leq n$. The vertices with distance exactly n form the sphere $S_n(0)$. We want to estimate $|S_n(0)|$ and $|B_n(0)|$, the number of different vertices with distance to 0 exactly n, respectively at most n. Clearly $|S_n(0)|$ is bounded below by the number of words that can be build by choosing n of the k generators with repetitions allowed but without taking order into account, which is

$$|S_n(0)| \geq \binom{n+k-1}{k-1} = \frac{(n+k-1)\cdot(n+k-2)\cdot \ldots \cdot (n+1)}{(k-1)!},$$

a polynomial in n of degree $k-1$. This would count the words with only "positive" generators x_i, but there are also the inverse letters x_i^{-1}. There are 2^k ways to choose the sign of the exponents of the k letters $x_i^{\pm 1}$. This does not give the exact value of $|S_n(0)|$, because not in all words all letters occur. But we obtain the upper bound

$$|S_n(0)| \leq 2^k \cdot \binom{n+k-1}{k-1},$$

which is a polynomial in n of degree $k-1$, too. By Faulhaber's formula ([Knu93]) the number $|B_n(0)| = \sum_{k=0}^{n} |S_n(0)|$ can be bounded below and above by two polynomials in n of degree $k-1+1=k$.

These estimates show, what one might have guessed already: that the "volume" of the n-ball in $\mathbb{Z}^k$ is asymptotically $c_B n^k$ and the "area" of the "boundary" – the n-sphere – is asymptotically $c_S n^{k-1}$ for some constants c_B and c_S.

In the example we have chosen a special vertex 0 – the vertex representing the empty word or the identity element – as center for the balls. But we could choose any vertex and would obtain the same result, which is a consequence of the following

Remark 1.14. Any group Γ acts on its Cayley graph – with respect to any generating set – by left-multiplication. This means there is a homomorphism

$$\begin{aligned} \phi : \Gamma &\to \operatorname{Aut} G_{\Gamma,S} \\ \gamma &\mapsto (f_\gamma : G_{\Gamma,S} \to G_{\Gamma,S},\ G \ni v \mapsto \gamma v), \end{aligned}$$

where γv means the product in the group Γ, which is defined because every vertex of G is a group element of Γ. In particular Cayley graphs are vertex-transitive – for all vertices v and $w \in V$ there is a automorphism f with $f(v) = w$. Roughly spoken this means that the graph "looks the same" at every vertex.

For many applications this is a very usefull fact, but we will see, that this in general is not true anymore, if we replace Cayley graph by the objects we are interested in and that seem to be very similar to Cayley graphs – the Schreier coset graphs. But let us first define the latter:

Definition 1.15. To a group Γ, a subgroup $\Pi < \Gamma$ and a nonempty finite subset $S \subset \Gamma$ we assign the following directed graph $G_{\Gamma,\Pi,S}$:

- The vertex set V is the set of right cosets of Π in Γ:

$$V(G_{\Gamma,\Pi,S}) := \{\Pi\gamma | \gamma \in \Gamma\}.$$

- The edge set is

$$E^+ := V \times S \text{ with } \alpha((\Pi\gamma, s)) = \Pi\gamma \text{ and } \beta((\Pi\gamma, s)) = \Pi\gamma s.$$

- Moreover we add to every edge $(\Pi\gamma, s)$ the label s.

If $S^{\pm}$ is a generating set for Γ, the graph $G_{\Gamma,\Pi,S}$ is called the *Schreier coset graph** of Γ with respect to Π and S.

Comparing Definition 1.11 and Definition 1.15 one easily sees that Cayley graphs and Schreier graphs are quite similar. Like Cayley graphs Schreier graphs are $2|S|$-regular, the Schreier graph $G_{\Gamma,\{id\},S}$ is exactly the Cayley graph $G_{\Gamma,S}$ and if Π is a normal subgroup of Γ, then $G_{\Gamma,\Pi,S}$ equals the Cayley graph of the factor group Γ/Π with respect to $\{s\Pi | s \in S\}$.

But as mentioned above a crucial difference concerns the automorphisms of Schreier graphs:

*in the following just: Schreier graph

Example 1.16. In general $G_{\Gamma,\Pi,S}$ is not vertex-transitive. In particular though left-multiplication by Γ is an action on the cosets, in general it is not an action on the graph $G_{\Gamma,\Pi,S}$:

Let $\Gamma = \langle a, b | - \rangle$ be the free group of rank 2. We set $\Pi = \langle a \rangle$, the subgroup generated by a, and $S = \{a, b\}$. Then $G_{\Gamma,\Pi,S}$ looks as presented in Figure 1.1.

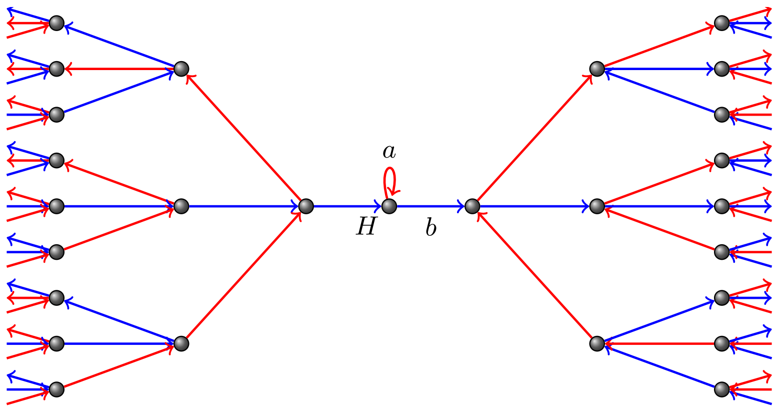


Figure 1.1: The *root-looped* 4*-valent tree*: a Schreier graph that is not vertex-transitive.

In particular the edge (Π, a) is a loop and it is the only loop. Hence every automorphism of $G_{\Gamma,\Pi,S}$ has to fix the vertex Π, which implies that $G_{\Gamma,\Pi,S}$ is not vertex transitive and left-multiplication by Γ is not an action.

Remark 1.17. We name the graph shown in Figure 1.1 *root-looped* 4*-valent tree*, even though as a directed graph it is not a tree because of the loop at one vertex. This vertex is called the root.

Remark 1.18. We are particularly interested in the following setting: The group Γ is the Veech group of an L-shaped Veech surface L and acts on the points of the surface L_D (from the right). The Schreier graph we investigate is $G = G_{\Gamma,\Gamma_P,S}$ where Γ_P is the stabilizer of a non periodic connection point P and S is a finite generating set of Γ. In this setting the vertices of the Schreier graph, the right cosets, can be identified with the points of the Γ-orbit of P:

- the subgroup Γ_P itself is identified with P.
- the vertex $\Gamma_P\gamma^{-1}$ is identified with $\gamma(P) =: \gamma \circ P$

This is well-defined because $\Gamma_P\gamma_1^{-1} = \Gamma_P\gamma_2^{-1}$ is equivalent to $\gamma_1^{-1}\gamma_2 \in \Gamma_P$ and thus to $\gamma_1 \circ P = \gamma_2 \circ P$. Because of the left-right twist the label s of an edge from $\Gamma_P\gamma^{-1}$ to $\Gamma_P\gamma^{-1}s$ becomes an s^{-1} on the edge from $\gamma \circ P$ to $s^{-1}\gamma \circ P$. But for the question of amenability the direction of the edges does not matter, so we do not have to be too worried about.

1.2 Amenability

1.2.1 Amenability of Groups

In this section we give a short overview about amenability of countable groups and also recommend [Pat00] as well as the nice blogpost [Tao11].

Given a countable group Γ and a map $f : \Gamma \to \mathbb{R}$ for any $x \in \Gamma$ the *left-translation* by x is defined as $\tau_x f(\gamma) \coloneqq f(x^{-1}\gamma)$. A *mean* on Γ is a linear functional $\lambda : \ell^\infty(\Gamma) \to \mathbb{R}$ such that $\lambda(\mathbf{1}) = 1$ and $\lambda(f) \geq 0$ for all f with $f(\gamma) \geq 0$ for all γ.

Definition 1.19. A countable group Γ is called *amenable*, if there exists a left-invariant mean on Γ.

This definition was given by von Neumann when he analyzed the Banach-Tarski paradox. Obviously, the following subgroup-criterion holds:

Proposition 1.20. *Closed subgroups of amenable groups are amenable. Hence a discrete group containing a non-amenable subgroup is non-amenable.*

Følner found an equivalent definition of amenability:

Theorem 1.21 ([Fol55])**.** *A countable group Γ is amenable, if and only if there exists a so called Følner sequence: a sequence (A_i) of non-empty finite subsets Γ such that*

$$\frac{|(\gamma \cdot A_i)\Delta A_i|}{|A_i|} \to 0 \text{ as } i \to \infty \text{ for all } \gamma \in \Gamma.$$

Here, Δ denotes the symmetric difference.

Examples of amenable groups are finite groups, abelian groups and virtually solvable groups. Proofs can be found in [Tao11] and in [Pat00].

From the criterion of Følner one easily sees that groups of polynomial growth are amenable, since the i-balls around 1 (with respect to the word metric) build a Følner sequence. For details on the growth of a group and a proof that groups of intermediate growth are supramenable and thus amenable see for instance [Wag93].

On the other hand not every group of exponential growth is non-amenable:

Example 1.22. for example the Baumslag-Solitar group $\mathrm{BS}(1,2) = \langle a, b \mid ab = b^2 a\rangle$ is a semi-direct product of $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}$ and thus metabelian and solvable. But it contains the free monoid $\langle a, ba\rangle_{\text{mon}}$, and hence is of exponential growth. That this monoid is indeed free can be easily seen, when one views the Cayley graph of $\mathrm{BS}(1,2)$ in the form shown in Figure 1.2.1[†]. The horizontal edges are labeled with b, while the upgoing edges are labeled with a. Two different words in a^+ and $(ba)^+$ lead to different vertices of the Cayley graph, since a and ba lead to different limbs of the underlying infinite binary tree.

Let us finish this section with an example of a non-amenable group:

[†]The picture is taken from `en.wikipedia.org/wiki/Baumslag-Solitar_group`.

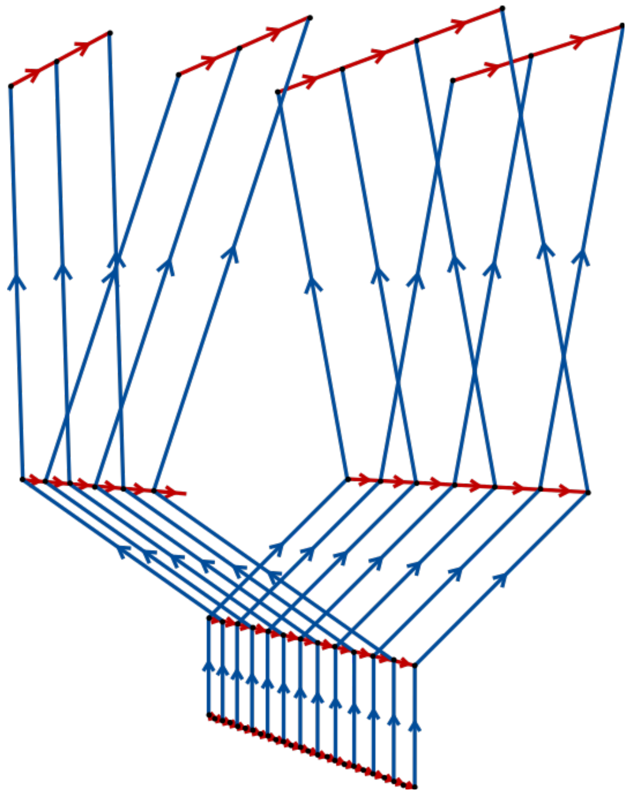


Figure 1.2: Cayley graph of BS(1, 2).

Example 1.23. The free group $F = \langle a, b \mid - \rangle$ of rank 2 is non-amenable. If it was amenable, there would be an invariant mean λ. Denote by A_1, A_{-1}, B_1 and $B_{-1} \subset F$ the set of all words beginning with a, a^{-1}, b or b^{-1}, respectively. Since $B_1 \subset (a^{-1}A_1) - A_1$, we conclude $\lambda(1_{B_1}) \leq \lambda(\tau_{a^{-1}} 1_{A_1}) - \lambda(1_{A_1}) = 0$. Similarly also A_1,A_{-1} and B_{-1} and obviously also the empty word have zero mean. Thus λ would be identically zero, what is a contradiction to $\lambda(1) = 1$.

Hence a way to show non-amenability of a group is to find a free subgroup of rank ≥ 2. Until 1980 there was the von Neumann conjecture, that possibly a group is amenable if and only if it contains a free non-abelian subgroup. This conjecture was disproved by Alexander Ol'shanskii in [Ol'82]: he showed that the Tarski monster group, which was known to not contain a free non-abelian subgroup, is non-amenable. For linear groups however, there is the Tits alternative [Tit72], which shows that the von Neumann conjecture is true in this case.

1.2.2 Amenability of Graphs

Let us now define amenability for graphs. In addition to the basic definitions from the beginning of this chapter we need the notion of the (vertex-) *boundary* of a set $M \subset V$:

$$\partial M = \{\text{vertices of } M \text{ that have a neighbor in } M^c\}$$

Accordingly the *interior* is $\mathring{M} := M - \partial M$.

Definition 1.24. Let $G = (V, E, \alpha, \beta)$ respectively $G = (V, E)$ be a graph.

- For a finite nonempty set of vertices $M \subset V$ we define
$$c(M) := \frac{|\partial M|}{|M|}.$$
- The *Cheeger constant* of G is
$$\mathfrak{c}(G) := \inf_{\text{finite } M \subset V} c(M).$$
- If $\mathfrak{c}(G) = 0$, the graph G is called *amenable*; otherwise it is *non-amenable.*
- If we focus on the interior of M instead of counting the boundary vertices we define
$$\iota(M) := \frac{|\mathring{M}|}{|M|} = 1 - c(M) \quad \text{and} \quad \mathfrak{i}(G) := \sup_{\text{finite } M \subset V} \iota(M) = 1 - \mathfrak{c}(G).$$

Remark 1.25. The Cheeger constant depends not only on the vertex set but also on the edges. This dependence can be found in the numerator $|\partial M|$. But it does not depend on the direction or orientation of edges. Hence in the case of directed graphs (as Cayley or Schreier graphs) we can replace the oriented geometrical edges by undirected and unoriented edges. Also loops and multiedges do not change the number of boundary vertices or the Cheeger constant, so we can omit them, too, to get a simple graph with the same Cheeger constant.

As a first example of a graph for which we can calculate the Cheeger constant we take the free abelian group of rank k:

Example 1.26. As in Example 1.13 let Γ be the free abelian group of rank k, presented by $\langle x_1, \dots x_k \,|\, [x_i, x_j] = 1 \rangle$ and let $G_{\Gamma, \{x_1, \dots, x_k\}}$ be its Cayley graph (identified with $\mathbb{Z}^k$). Since the group Γ is abelian and therefore also amenable, we would hope that also its Cayley graph is amenable and so want to find a sequence (M_n) of finite vertex sets such that $c(M_n)$ tends to 0 as n goes to infinity: We already estimated the size of the n-ball and the n-sphere with center 0, so let us try the sequence $M_n := B_n(0)$. Obviously ∂M_n is exactly the sphere $S_n(0)$. Hence comparing with the results from Example 1.13 we get
$$c(M_n) = \frac{|\partial M_n|}{|M_n|} = \frac{|S_n(0)|}{|B_n(0)|} \sim \frac{c_S n^{k-1}}{c_B n^k} = \frac{c_S}{c_B} \cdot \frac{1}{n}.$$
So $c(M_n)$ tends to zero for $n \to \infty$ and $G_{\Gamma, \{x_1, \dots, x_k\}}$ is indeed amenable as we guessed.

Unfortunately there are not many results on (non-) amenability of graphs. As described and used in [Kap02] the main technique is the following due to Bartholdi. He generalizes a result of Grigorchuk about Cayley graphs ([Gri80]) to all regular graphs:

Theorem 1.27 ([Bar99])**.** *Let $G = (V, E)$ be a connected d-regular graph, choose a point $v_0 \in V$ and let a_n be the number of reduced edge-paths of length n from v_0 to v_0. Then G is amenable if and only if $\limsup_{n \to \infty} \sqrt[n]{a_n} = d - 1$.*

The graph we want to analyze is a Schreier graph and thus regular, but the groups that come into play are too complicated to count the reduced edge-paths. Hence we want to find another (more elementary) method to prove non-amenability of a Schreier graph. Let us begin by collecting some facts about Cheeger constants that will help us:

First it is obvious that a graph containing a finite connected component C is amenable since $|\partial C| = 0$. But on the other hand we have

Proposition 1.28. *If $G = (V, E)$ is an amenable graph without any finite connected component, then for any finite subset $F \subset V$ the subgraph induced by $V' = V - F$ is amenable too.*

Proof. Since G is amenable there exists a sequence of finite sets $M_i \subset V$ with $c(M_i) \to 0$ as $i \to \infty$. There is no finite connected component, hence each M_i has at least one boundary vertex, $|\partial M_i| \geq 1$. Thus $|M_i|$ has to tend to infinity to make $c(M_i) = \frac{|\partial M_i|}{|M_i|} \to 0$ possible.

Now let us set $M_i' := M_i - P = M_i \cap V'$ and look at the sequence $(c(M_i'))$:

$$c(M_i') = \frac{|\partial M_i'|}{|M_i'|} \leq \frac{|\partial M_i|}{|M_i| - |F|}.$$

The inverse of this term is

$$\frac{|M_i| - |F|}{|\partial M_i|} = \frac{|M_i|}{|\partial M_i|} - \frac{|F|}{|\partial M_i|}.$$

The minuend tends to infinity, the subtrahend is bounded above by $|F|$ and hence

$$c(M_i') \to 0 \text{ as } i \to \infty.$$

Thus the subgraph induced by V' is amenable. □

Proposition 1.29. *Let $G = (V, E)$ be graph with connected components K_i and let M be a finite subset of the vertex set V. Setting $M_i := M \cap K_i$ the following holds:*

$$\min_{i|M_i \neq \emptyset} \frac{|\partial M_i|}{|M_i|} \leq \frac{|\partial M|}{|M|} \leq \max_{i|M_i \neq \emptyset} \frac{|\partial M_i|}{|M_i|}$$

Proof. Since the M_i are in different connected components the boundary ∂M_i is the intersection of M_i with ∂M. First consider the case of two components, so M is the disjoint union $M_1 \sqcup M_2$, and $\partial M = (\partial M \cap M_1) \sqcup (\partial M \cap M_2) = \partial M_1 \sqcup \partial M_2$. Without loss of generality we assume $c(M_1) = \frac{|\partial M_1|}{|M_1|} \leq \frac{|\partial M_2|}{|M_2|} = c(M_2)$. This is equivalent to $|\partial M_1| \leq \frac{|M_1| \cdot |\partial M_2|}{|M_2|}$. Thus we obtain

$$c(M) = \frac{|\partial M|}{|M|} = \frac{|\partial M_1| + |\partial M_2|}{|M_1| + |M_2|} \leq \frac{\frac{|M_1| \cdot |\partial M_2| + |M_2| \cdot |\partial M_2|}{|M_2|}}{|M_1| + |M_2|} = \frac{|\partial M_2|}{|M_2|} = c(M_2).$$

Analogically one gets $c(M) \geq c(M_1)$. Note that M is finite and thus only finitely many M_i are non-empty. Since $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$, inductively we obtain the general case of arbitrary many connected components. □

Corollary 1.30. *Let G be a graph with connected components K_i. Then*

$$\mathfrak{c}(G) = \inf_i \mathfrak{c}(K_i).$$

Proof. That $\mathfrak{c}(G) \leq \inf_i \mathfrak{c}(K_i)$ is clear by definition of the Cheeger constant.

To show $\mathfrak{c}(G) \geq \inf_i \mathfrak{c}(K_i)$ let M be an arbitrary (nonempty) finite set of vertices and $M_i := M \cap K_i$ as in Proposition 1.29. Thus $c(M_i) \leq c(M)$ for at least one i.

Now let $(M_j)_{j\in\mathbb{N}}$ be a sequence of finite vertex sets with $\inf_j c(M_j) = \mathfrak{c}(G)$ and let $M_{j,i}$ be $M_j \cap K_i$. Since for all j there exists an i such that $c(M_j) \geq c(M_{j,i})$, it follows

$$\mathfrak{c}(G) = \inf_j c(M_j) \geq \inf_j \inf_i c(M_{j,i}) = \inf_{i,j} c(M_{j,i}) = \inf_i \inf_j c(M_{j,i}) \geq \inf_i \mathfrak{c}(K_i).$$

□

Proposition 1.31. *If a graph G' arises from a graph G by ommiting some edges, then the Cheeger constants satisfy:*

$$\mathfrak{c}(G') \leq \mathfrak{c}(G).$$

Proof. For every finite set M clearly the number $|\partial' M|$ is at most $|\partial M|$, where $\partial' M$ denotes the (vertex) boundary of M in G'. Hence

$$\mathfrak{c}(G') = \inf_{\text{finite } M\subset V} \frac{|\partial' M|}{|M|} \leq \inf_{\text{finite } M\subset V} \frac{|\partial M|}{|M|} = \mathfrak{c}(G).$$

□

Later on we want to show non-amenability of a Schreier graph and for this want to choose the generating set such that it contains special elements. This will be allowed by a result of Woess about metrically equivalent graphs.

Definition 1.32. Let $G = (V, E)$ and $G' = (V', E')$ be graphs with edge metrics d and d', respectively. The graphs G and G' are called *metrically equivalent*, if there exists a surjective map $\varphi : V \twoheadrightarrow V'$ and a constant $A \geq 1$ such that

$$\frac{d(v,w)}{A} \leq d'(\varphi(v), \varphi(w)) \leq Ad(v,w)$$

for all $v, w \in V$.

Proposition 1.33. *Let $G = G_{\Gamma,\Pi,S}$ and $G' = G'_{\Gamma,\Pi,S'}$ be Cayley or Schreier graphs with respect to finite generating sets S and S' of Γ and let d and d' be the edge metrics of G and G', respectively. Then G and G' are metrically equivalent.*

Proof. Since $S = \{s_1, \ldots, s_n\}$ and $S' = \{s'_1, \ldots s'_m\}$ are finite and G and G' are connected also $\max_{i=1,\ldots n; j=1,\ldots m}(d'(\Pi, \Pi s_i), d(\Pi, \Pi s'_j))$ is finite. Set A to be this number and φ as the identity map on $V(G) = V(G')$. □

Theorem 1.34 ([Woe00], Theorem 4.7)**.** *Let G and G' be connected graphs with bounded vertex degrees. If G and G' are metrically equivalent, then G is amenable if and only if G' is amenable. In particular, for Cayley graphs and Schreier graphs amenability is independent of the choice of a finite generating set.*

Let us now consider an example of a non-amenable Cayley graph, the Cayley graph of a free group of finite rank ≥ 2:

Example 1.35. Let $G_{\mathcal{F}_k}$ be the Cayley graph of a free group of rank $k \geq 2$ with respect to a free generating set: $\mathcal{F}_k = \langle a_1, \ldots, a_k | - \rangle$:

As described in Remark 1.25 we can ignore the direction of the edges and get an infinite $2k$-regular tree. We want to find a lower bound for $c(M)$ for finite vertex sets M. By Proposition 1.29 we can assume that M is connected. If the interior $\mathring{M}$ is empty, $c(M) = 1$. So let us assume $|\mathring{M}| = n \geq 1$ and let v be an inner vertex. We mark v as the root of M and divide the other vertices of M in levels corresponding to their distance to v. Now we can view the tree M from the root and estimate the total number of vertices: There are v and its $2k$ neighbors in M and every further inner vertex has $2k-1$ neigbors on a higher level that are not already counted. Hence $|M| \geq 1 + 2k + (n-1)(2k-1) = n(2k - 1 + \frac{2}{n})$ and

$$c(M) = 1 - \frac{|\mathring{M}|}{|M|} \geq 1 - \frac{n}{n(2k-1+\frac{2}{n})} = \frac{2k-2+\frac{2}{n}}{2k-1+\frac{2}{n}}. \tag{1.1}$$

For increasing n this expression is decreasing, whence we get the lower bound for $n \to \infty$:

$$c(M) \geq \frac{2k-2}{2k-1}.$$

Since for balls with center 0 the inequality (1.1) becomes an equality, the sequence B_n converges to the last expression and we get the exact value of the Cheeger constant

$$\mathfrak{c}(G_{\mathcal{F}_k}) = \frac{2k-2}{2k-1}$$

and thus the non-amenability of the infinite $2k$-regular tree.

Remark 1.36. In particular, for the Cayley graph of the free group of rank 1 – which is $\mathbb{Z}$ – this value is 0 and we again see amenability of this graph. Of course the above calculation can be made analog for k-regular infinite trees $T_{k\text{-reg}}$ also for k odd and we get

$$\mathfrak{c}(T_{k\text{-reg}}) = \frac{k-2}{k-1}.$$

Knowing this result we could be tempted to ask the following question about an analog statement to the fact that a group containing a free non-abelian subgroup is non-amenable:

Question 1.37. *Let $\Pi < \Gamma$ be groups. Is it true that if there exists a free non-abelian subgroup $\mathcal{F} < \Gamma$ with trivial intersection with Π, the Schreier graph $G_{\{\Gamma,\Pi,S\}}$ is non-amenable?*

The existence of such a free group implies that the Schreier graph contains a subgraph isomorphic to the Cayley graph of the free group, which is non-amenable as we just saw. For arbitrary graphs one can easily construct a counterexample by combining a graph isomorphic to the Cayley graph of $\mathbb{Z}$ and a graph isomorphic to the Cayley graph of a free group of rank two. For example take both as seperated connected components or connect them at a single common vertex. Surely this graph is amenable.

But Schreier graphs are more special and in some aspects quite similar to Cayley graphs, where the question can be answered with "yes" (cf. Theorem 1.34). If we additionally assume that Π is finitely generated, we do not know the answer. But we now give a counterexample with an infinitely generated subgroup Π.

Example 1.38. Let $\Gamma = \langle a, b \mid - \rangle$ be the free group in two generators and Π the subgroup $\left\langle \{a^k b a^{-k}\}_{k \in \mathbb{Z}-\{0\}} \right\rangle$. Then the Schreier graph $G_{\Gamma,\Pi,\{a,b\}}$ is amenable, but there is a free non-abelian subgroup $\digamma$ with trivial intersection with Π.

Proof. All cosets Πa^k are different, because

$$\Pi a^m = \Pi a^n \Leftrightarrow \Pi a^{m-n} = \Pi \Leftrightarrow a^{m-n} \in \Pi \Leftrightarrow m = n.$$

From Πa^k there is an outgoing edge labeled with a to Πa^{k+1} and an incoming from Πa^{k-1}. Since $\Pi a^k b = \Pi a^k \Leftrightarrow a^k b a^{-k} \in \Pi$ the b-edges from Πa^k are loops at Πa^k for all $k \in \mathbb{Z} - \{0\}$. Summed up this part of the coset graph looks like the Cayley graph of $\mathbb{Z}$ with an additional loop at almost every vertex. Only at the origin, the vertex $\Pi = \Pi 1$, there are an incoming and an outgoing b-edge connecting this part to the following part:

All cosets Πw and $\Pi w'$ with $w \neq w'$ and w and w' reduced words starting with $b^{\pm 1}$ are different vertices since $\Pi w = \Pi w' \Leftrightarrow w w'^{-1} \in \Pi$, which is not true since every (nontrivial) element in the subgroup Π begins and ends with a or a^{-1}. Furthermore no Πw with w starting with $b^{\pm 1}$ is adjacent to one of the Πa^k ($k \neq 0$), because of the 4-regularity of the Schreier graph. So the Schreier graph looks as illustrated in Figure 1.3.

Now we can easily give a sequence of finite vertex sets with the property that the quotient of boundary vertices to all vertices of one set tends to zero:

$$(M_i)_{i \in \mathbb{N}} \text{ with } M_i = \{\Pi a^k \colon |k| \leq i\}\}$$

In M_i there are three boundary vertices Π, Πa^i and Πa^{-i}, whereas $|M_i| = 2i + 1$, so $c(M) = \frac{3}{2i+1} \to 0$ for $i \to \infty$.

It remains to find a (free non-abelian) subgroup $\digamma < \Gamma$ that intersects with Π only trivially: One can easily check that the subgroup $\digamma = \left\langle bab, b^2 \right\rangle$ is such a group: As a subgroup of a free group it is free, obviously of rank > 1, and every nontrivial word in $\digamma$ begins with $b^{\pm 1}$ and thus is not in Π. $\square$

So we have seen that the answer to Question 1.37 in general is "no", if infinitely generated subgroups are allowed.

Furthermore, nearly directly from the definition of amenability the following subgraph criterion – inspired by the last example – is clear:

Proposition 1.39. *If a graph G contains an infinite connected amenable subgraph G' that is connected to $G - G'$ only at finitely many vertices of G', then also G is amenable.*

Proof. Since the subgraph G' is amenable there is a sequence of finite sets $M_i \subset V(G')$ with $c_{G'}(M_i) = \frac{|\partial_{G'} M_i|}{|M_i|} \to 0$. Since G' is connected and infinite, every M_i has at least one boundary vertex. Hence $|M_i|$ has to tend to infinity. Thus the same sequence

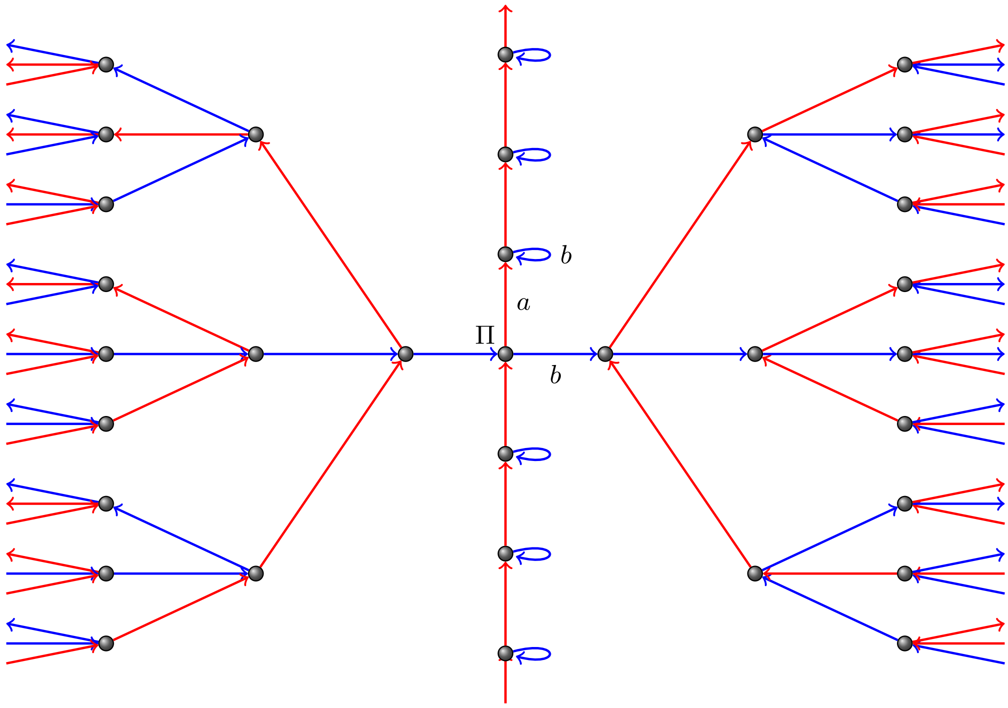


Figure 1.3: Schreier graph $G_{\Gamma,\Pi,S}$: amenable although Γ contains a free subgroup intersecting Π only trivially.

regarded as subsets of $V(G)$ still has the property that $c_G(M_i) = \frac{|\partial_G M_i|}{|M_i|} \to 0$, because only finitely many of the vertices of $\bigcup M_i$ are boundary vertices with respect to G but not with respect to G'. $\square$

But now having in mind Corollary 1.30, Proposition 1.31 and Example 1.35 we want to get back to trees and obtain a criterion for the (non-)amenability of an arbitrary infinite tree that does not contain a leaf. In order to do this we first have to introduce some notation and investigate simple finite trees.

Definition 1.40. We call a subgraph Z of a (simple) graph G a $\mathbb{Z}$-*piece*, if it is connected and consitsts only of vertices v with $\mathrm{val}_G(v) = 2$. The number $|V(Z)|$ is called the *length* of the $\mathbb{Z}$-piece. Let $n(G)$ denote the supremum of $|V(Z)|$ over all $\mathbb{Z}$-pieces Z.

Lemma 1.41. *A finite tree T with k leaves not containing vertices of valency 2 has at most $2k - 2$ vertices.*[‡]

Proof. Let us first prove the statement for 3-valent finite trees with k leaves, i.e. all vertices but the leaves have valency 3. This is done by induction on k. For $k = 3$ there is only one trivalent finite tree, that has $4 = 3 \cdot 2 - 2$ vertices. Since the tree is finite, for $k > 3$ there exist two leaves adjacent to the same vertex v. If we remove these two leaves and the corresponding edges, v becomes a leave and we get a trivalent finite tree

[‡]exactly $2k - 2$ if and only if T is 3-valent

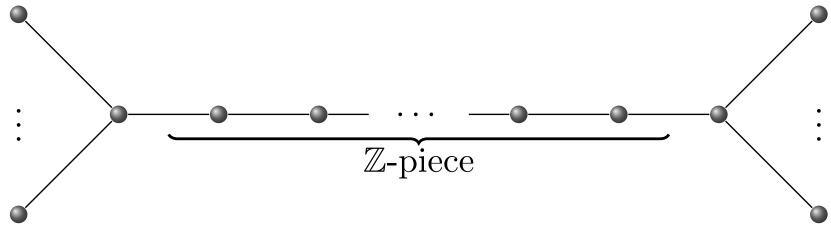


Figure 1.4: A $\mathbb{Z}$-piece.

with $k-1$ leaves. By induction this has $2(k-1)-2$ vertices. Thus the tree with k leaves has 2 more vertices, which is $2k-2$.

An arbitrary finite tree T with k leaves that does not contain vertices of valency 2 can be transformed to a trivalent one with the same number of leaves by repeating the following procedure: For every vertex w of valency greater than 3 replace this vertex by a segment of one edge and two vertices w' and w'', such that exactly two of the former neighbors of w are adjacent to w' and the others to w''.

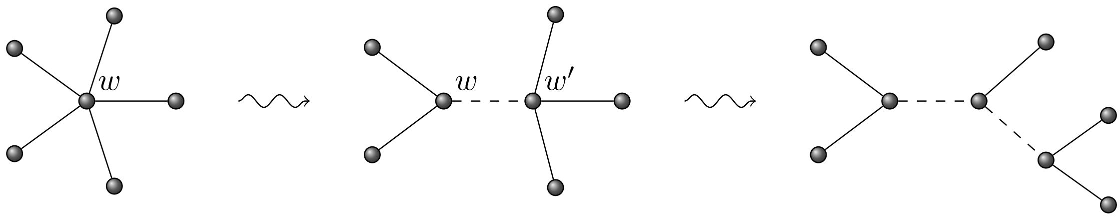


Figure 1.5: From an arbitrary tree to a 3-valent tree.

In each step the number of vertices increases while the number of leaves stays constant. Hence T has at most $2k-2$ vertices. $\square$

Remark 1.42. With an analog induction we can prove that a k-valent finite tree with n leaves[§] has exactly $k+1+\frac{k-1}{k-2}(n-k)$ vertices.

Remark 1.43. In the following proposition we will use the well-known fact (easily provable by induction), that in a finite simple tree T the number of vertices $|V(T)|$ and the number of edges $|E(T)|$ differ by 1, i.e. $|E(T)| = |V(T)| - 1$.

Proposition 1.44. *A (infinite[¶]) simple tree T that does not contain leaves is non-amenable if and only if $n(T) < \infty$.*

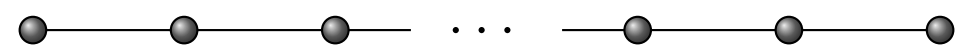
Proof. If $n(T) = \infty$ then there exists a sequence of $\mathbb{Z}$-pieces Z_i with $|Z_i| \to \infty$. Since $|\partial Z_i| = 2$ for all i, the sequence $c(Z_i)$ tends to 0 and T is amenable.

For the other implication let $n(T)$ be $n < \infty$. For an arbitrary finite vertex set $M \subset V(T)$ we want to give a lower bound for $c(M)$. We already know that it is sufficient to find such a bound for sets such that the induced finite subgraph T' is

[§] In such trees n has to be at least k and $n-k$ has to be divisible by $(k-2)$

[¶] obsolete because finite trees always have leaves connected (Proposition 1.29). Obviously T' is a finite tree. Let k denote the number of leaves of T'.

First consider the case that T' consists only of vertices of valency 1 or 2. Then $k = 2$ and T' looks as follows:



We notice that although apart from the 2 leaves T' looks like a $\mathbb{Z}$-piece the number of vertices does not need to be bounded, because there might be vertices that have valency greater than 2 in T, but only 2 of the neighbors are also in T'. Let $l \geq 0$ denote the number of such vertices of valency 2 with neighbors in $T - T'$. So these vertices and the two leaves are the boundary vertices of M.

If we replace every part of T' that is a $\mathbb{Z}$-piece in T together with its two boundary edges by an edge, the resulting tree T'' has the same number of boundary vertices $|\partial T''| = |\partial T'| = l + 2$. This is also the number $|T''|$ of all vertices of T''. Hence $c(T'') = \frac{l+2}{l+2}$. The fraction $c(T')$ differs only in the denominator $|T'|$. To get back from T'' to T' we have to add at most n vertices per edge in T''. Remember that a finite tree with $l+2$ vertices has $l+1$ edges. Hence we get the upper bound $|T'| \leq l + 2 + (l+1)n$ and the lower bound

$$c(T') \geq \frac{l+2}{l+2+(l+1)n} = \frac{1}{1 + \frac{l+1}{l+2}n}.$$

Regarded as a function in l this expression is increasing, which implies

$$c(T') \geq \frac{1}{1+\frac{n}{2}} = \frac{2}{n+2} > 0.$$

Now we want to get a lower bound for $c(T')$ in the case that T' has at least one vertex of T'-valency at least 3. To do this we partition the tree in disjoint parts and apply Proposition 1.29.

1. The $\mathbb{Z}$-pieces $Z_{l,i}$ of T' that contain $l \geq 1$ vertices with valency ≥ 3 in T. These are basically trees of the form analyzed in the first step of this proof[‖], but the leaves are missing. Thus
$$\frac{|\partial T' \cap Z_i|}{|Z_i|} \geq \frac{l}{l + (l+1)n} \geq \frac{1}{1+2n}.$$
2. The rest, i.e. the other $\mathbb{Z}$-pieces – that are $\mathbb{Z}$-pieces also in T – and the vertices of valency 1 (the k leaves) or at least 3 in T'. We want to work with a tree, so we connect the rest by adding an edge joining the two neighbors (in T') of each $Z_{l,i}$ we have removed. Let T'' denote the resulting tree and T''' the tree arising from T'' by replacing all remaining $\mathbb{Z}$-pieces by an edge between the respective neighbors. This tree has k leaves and no vertices of valency 2. So we can apply Lemma 1.41 and get the upper bound $2k - 2$ for the total number of vertices of T'''. Since the number of vertices of a $\mathbb{Z}$-piece in T'' is bounded by n and there are $2k - 3$ edges in T''', we get
$$|\text{Vertex set of } T''| \leq 2k - 2 + (2k-3) \cdot n = k\left(2 + 2n - \frac{2+3n}{k}\right).$$

[‖]with $l \geq 1$ instead of $l \geq 0$

At least every leaf of T'' has a neighbor in $T - T'$ and thus belongs to $\partial T'$ and $\partial T''$. Hence we conclude

$$\frac{|\partial T' \cap T''|}{|T''|} \geq \frac{k}{k\left(2 + 2n - \frac{2+3n}{k}\right)}.$$

This expression is decreasing for increasing k, hence it is bounded below by

$$\frac{|\partial T' \cap T''|}{|T''|} \geq \frac{1}{2+2n}.$$

With Proposition 1.29 we conclude

$$c(T') \geq \min\left\{\frac{1}{2+2n}, \frac{1}{1+2n}\right\} = \frac{1}{2+2n} > 0,$$

which means that also in this case the tree T is indeed non-amenable.

□

This proposition confirms what one might have guessed already: Among the infinite trees with bounded $n(T) < \infty$ and without leaves, the infinite 3-valent tree with all edges replaced by Z-pieces of the maximal allowed length has the smallest Cheeger constant. In this tree the balls of radius i with any center $B_i(*)$ have $c(B_i(*)) \to \frac{1}{2+2n(T)}$. Therefore, considering also the results of the proof of the theorem this tree has exactly this term as Cheeger constant, which is the smallest possible.

Combining Corollary 1.30, Proposition 1.31 and Proposition 1.44 we get the following

Theorem E. *Let G be a graph. If it is possible to omit edges of G and get a forest G' with every connected component an infinite simple tree without leaves, then*

$$\mathfrak{c}(G) \geq \frac{1}{2n(G')+2},$$

where $n(G')$ is the supremum of the lengths of the Z-pieces in G'. In particular if $n(G') < \infty$ then G is non-amenable.

A special graph that will play an important role in Chapter 6 is the root-looped 4-valent tree T_{rl4} seen in Figure 1.1. Proposition 1.44 implies that this graph is non-amenable and the lower bound on the Cheeger constant given in Theorem E yields $\mathfrak{c}(T_{rl4}) \geq \frac{1}{4}$. But for this graph we can get a better result:

Remark 1.45. The Cheeger constant of the root-looped 4-valent tree T_{rl4} is $\mathfrak{c}(T_{rl4}) = \frac{2}{3}$.

Proof. First we observe that the $\mathbb{Z}$-piece at the root has length 1, so if we take M to be the root and it's two neighbors, then $c(M) = \frac{2}{3}$, whence $\mathfrak{c}(T_{rl4}) \leq \frac{2}{3}$ follows.

For the other direction let M be any finite vertex set. If the root is not in $\mathring{M}$, we have seen in Example 1.35 that $c(M) \geq \frac{2}{3}$. If otherwise the root is one of n inner vertices of M, then (using the same arguments as in Example 1.35) M has at least $1+2+3\cdot(n-1)$ vertices. Hence $c(M) \geq 1 - \frac{n}{1+2+3(n-1)} = \frac{2}{3}$ and we get $\mathfrak{c}(T_{rl4}) \geq \frac{2}{3}$, what finishes the proof. □

1.3 The Combinatorial Spectrum of a Graph

In Chapter 5 we will need some information on the spectrum of a graph G with infinite vertex set, in particular on its bottom $\mu_0(G)$. This is why we provide some basic facts about the combinatorial Laplacian in this section.

Let $G = (V, E)$ be a connected graph with maximal valency $k > 0$. We define the *gradient* ∇ as an operator sending a map $b : V \to \mathbb{R}$ to the map

$$\nabla b : E \to \mathbb{R}, \quad (i, j) \mapsto b(i) - b(j)$$

and the *combinatorial Laplacian* Δ_G as

$$\Delta_G b(i) = \sum_{j:\{i,j\}\in E} b(i) - b(j)$$

for all maps b, which are in $\mathfrak{l}^2(V)$, i.e. the maps with $\langle b, b\rangle := \sum_i b(i)^2 < \infty$. The combinatorial Laplacian is a bounded operator self-adjoint with respect to the quadratic form $q(b) = \sum_{\{i,j\}\in E}(b(i)-b(j))^2 =: \langle\Delta_G b, b\rangle$. If G has a finite vertex set V, the constant maps are eigenfunctions with eigenvalue $\lambda_0(G) = 0$. For connected graphs the eigenvalue 0 has multiplicity 1 and the difference $\mu_0(G) = \lambda_1(G) - \lambda_0(G)$ is called *spectral gap*. If V is infinite, we set the *spectral gap* $\mu_0(G)$ to be the smallest eigenvalue $\mu_0(G)$ of Δ_G.

Let us now assume that G has infinitely many vertices. By the Min-Max principle the spectral gap satisfies

$$\mu_0(G) = \inf_b \frac{\sum_{\{i,j\}\in E}(b(i) - b(j))^2}{\sum_{i\in V} b(i)^2} = \inf_b \frac{||\nabla b||^2}{||b||^2}. \tag{1.2}$$

There are upper and lower bounds on $\mu_0(G)$ in terms of the Cheeger constant $\mathfrak{c}(G)$:

Theorem 1.46 (Cheeger). *Let $G = (V, E)$ be a connected graph with $|V| = \infty$ and maximal vertex valency k. Then the following inequalities for the spectral gap $\mu_0(G)$ hold:*

$$\frac{\mathfrak{c}(G)^2}{2k} \leq \mu_0(G) \leq k\mathfrak{c}(G).$$

A proof of a similar theorem can be found in [Ver93], Section 2. But note that he defines the Cheeger constant via the edge boundary $\partial_E A$, which consists of all edges that connect vertices of A and A^c. We will just use the first inequality and thus also prove just this one:

Proof. Let $b : V \to \mathbb{R}$ have finite support. We define

$$\mathcal{S} := \sum_{\{i,j\}\in E} |b^2(i) - b^2(j)|$$

and by the Cauchy-Schwartz inequality for $\langle\nabla b, b(i) + b(j)\rangle$ combined with the estimate $\sum_{\{i,j\}\in E} |b(i)| + |b(j)| \leq 2k \sum_{i\in V} |b(i)|$ we obtain

$$\mathcal{S} \leq \sqrt{2k}||\nabla b|| \cdot ||b||.$$

On the other hand $\mathcal{S} = \sum b^2(i) - b^2(j)$, where the sum is taken over all (oriented) edges (i,j) with $b^2(i) \geq b^2(j)$. The image of b is finite and we sort the values of b^2 obtaining $a_0 = 0 < a_1 < \ldots < a_r$ and define

$$A_l \coloneqq \{i \in V \mid b^2(i) \geq a_l\}.$$

Then we can write the sum $\mathcal{S}$ as $\sum(a_l - a_{l-1})$, where each summand $a_l - a_{l-1}$ appears with multiplicity equal to the number of edges (i,j) with $b^2(i) \geq a_l$ and $b^2(j) < a_l$. This number is bounded below by $|\partial A_l|$. With the definition of the Cheeger constant $\mathfrak{c}(G) < \frac{|\partial A_l|}{|A_l|}$ and thus

$$\mathcal{S} \geq \mathfrak{c}(G) \sum_{l=1}^{r} (a_l - a_{l-1})|A_l| = \mathfrak{c}(G)||b||^2$$

follows. All together we finish the proof by obtaining

$$\frac{||\nabla b||^2}{||b||^2} \geq \frac{\mathfrak{c}(G)^2}{2k}.$$

□

Thus for G having a strictly positive spectral gap $\mu_0(G)$ is equivalent to having a strictly positive Cheeger constant $\mathfrak{c}(G)$ and hence also to being non-amenable.

2 Background on Translation Surfaces

After we dealt with graphs and the question when they are non-amenable in the previous chapter we now introduce the objects we are mainly interested in: translation surfaces. There are three equivalent ways to define them: via translation structures, i.e. surfaces with an atlas such that all transition maps are translations, as a Riemann surface equipped with a holomorphic 1-form or – as we will do – as polygons with parallel sides glued together. For details of the first two definitions and the equivalence of those see for instance the survey [Mas06]. The second part of this chapter is a short summary of known results concerning the size of Veech groups with focus on surfaces with infinitely generated Veech group and on Veech surfaces of genus 2.

2.1 Translation Surfaces and Veech Groups

2.1.1 Translation Surfaces

Definition 2.1. Let $\{P_1, \dots P_n\}$ be a finite set of polygons embedded in the Euclidean plane such that

- the boundary of every polygon is oriented counterclockwise, i.e. such that the polygon lies on the left of each side.
- the set of sides decomposes into pairs of sides, each pair consisting of parallel, reverse oriented sides of the same length.

The surface X obtained by gluing together these pairs of sides by translations is called a *translation surface*.

In the following we will additionally assume that translation surfaces are connected. Furthermore we will sometimes permit also *translation surfaces with boundary*, i.e. a set of polygons, where not every side is glued to another one.

Remark 2.2. Immediately from the definition we see:

- Moving along a glued pair of sides $\{s, t\}$ one of the corresponding polygons appears to the left and the other one to the right.
- The total angle around each vertex v is $2\pi m_v$ with m_v a positive integer. The vertex v is called a *singularity* and the number $m_v - 1$ its *order*. The set of all singularities is denoted by $S(X)$.
- The group $\mathrm{GL}_2(\mathbb{R})$ acts as linear maps on the plane and thus also on the polygons. This action maps parallel reverse oriented sides with the same length to parallel reverse oriented sides with the same length. Hence the action on the polygons induces an action on translation surfaces.

Remark 2.3. By identifying $\mathbb{R}^2$ with $\mathbb{C}$ this definition of translation surfaces yields a Riemann surface X together with

- an atlas such that away from the singularities all transition maps are translations

or

- a holomorphic one-form η coming from the differential $\mathrm{d}z$ on the Euclidean plane.

Definition 2.4.
- The collection of polygons glued at some of the identified sides and embedded in the plane together with markings for the identified sides, which are not glued yet, is called a *pattern* of the translation surface.
- A geodesic emanating from a singularity is called a *separatrix*. A geodesic, without singularities in its interior, which connects two singularities is called a *saddle connection*.
- A nonsingular point P is a *connection point* of a translation surface X if every separatrix without singularities in its interior that passes through P can be extended to a saddle connection.
- A *translation/affine covering* of X by Y is a continuous map $p : Y \to X$, such that $p^{-1}(S(X)) = S(Y)$ and $p|_{Y-S(Y)} : Y - S(Y) \to X - S(X)$ is locally a translation/affine map.

By the Gauß-Bonnet theorem the genus of the translation surface X can be calculated from the equation $\sum_{v \in S(X)}(m_v - 1) = 2g - 2$. The space ΩM_g of all pairs (X, η) with X a Riemann surface of genus g and η a holomorphic one-form can be stratified by the orders of the singularities, e.g. for $g = 2$ there are two strata: $\Omega M_2(1, 1)$ – the translation surfaces with two singularities of order 1 – and $\Omega M_2(2)$ – the translation surfaces with a single singularity of order 2. We will deal with these two spaces – especially the latter – later in Section 2.2.2. Note that the strata are preserved under the action of $\mathrm{GL}_2(\mathbb{R})$ described above.

2.1.2 From Billiards to Translation Surfaces

A comprehensive survey about rational billiards and translation surfaces is the article [MT02] of Masur and Tabachnikov. As mentioned in the introduction there is an unfolding process that assigns a translation surface X to the billiard dynamics on a given rational polygon P. We shortly describe this construction from [ZK75]:

Let the boundary of $P_1 := P$ be oriented counterclockwise, set $X' := \{P\}$ and choose a side $e \in \partial P_1$. Reflect P_1 in this side, call the resulting polygon P_2, reverse the orientation of its boundary (such that it is oriented counterclockwise), add P_2 to X' and mark the side $e \in \partial P_1$ and its image $\overline{e} \in \partial P_2$ as one of the pairs of sides which are parallel, reverse oriented and of the same length. Continue with all sides that are not yet marked, but if the reflected polygon differs from one that is already in the set X' just by a translation, do not add it. Since all angles of P are rational multiples of π, this procedure yields a finite set X' of polygons and thus finitely many sides, partitioned in pairs of parallel sides of the same length with reversed orientation, resulting in a translation surface X.

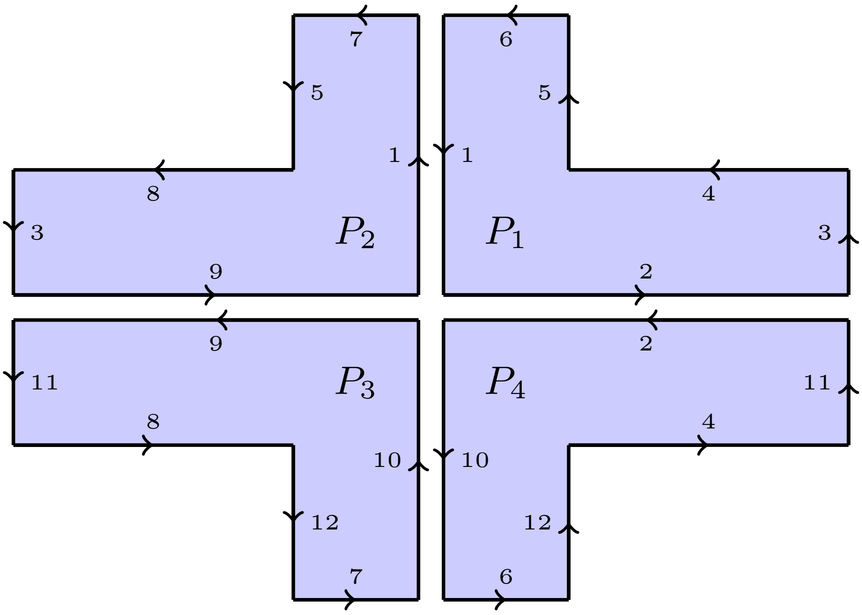


Figure 2.1: Translation surface L from unfolding an L-shaped polygon.

The big advantage of this construction is that the path of an billiard ball on the billiard table becomes a straight line on the translation surface. In Figure 2.1 the resulting translation surface L of the unfolding process of an L-shaped table is shown. Since the sides are glued, we can cut and glue the pattern of the surface – without changing the surface – and get an L-shaped pattern with side identifications representing the same translation surface (see Figure 2.2). We see that this pattern has the form of the original L-shaped billiard table and is just scaled by a factor 2 in the horizontal and in the vertical direction each.

2.1.3 Veech Groups

Definition 2.5. Let X and Y be translation surfaces (or translation surfaces with boundary) with singularities $S(X)$ and $S(Y)$, respectively. A map $f : X \to Y$ is called a *affine map*, if it is locally affine on $X - S(X) \to Y - S(Y)$. Accordingly f is called a *translation map*, if it is locally a translation on $X - S(X) \to Y - S(Y)$.

Now we can define three groups assigned to a translation surface X:

Definition 2.6. Let X be a connected translation surface (with finite volume).

1. The *translation group* $\mathrm{Trans}(X)$ consists of all translation self-homeomorphisms $f : X \to X$, which map $S(X)$ to itself.

2. The *affine group* $\mathrm{Aff}(X)$ consists of all orientation-preserving affine self - homeomorphisms $f : X \to X$, which map $S(X)$ to itself. Note that, since X is connected

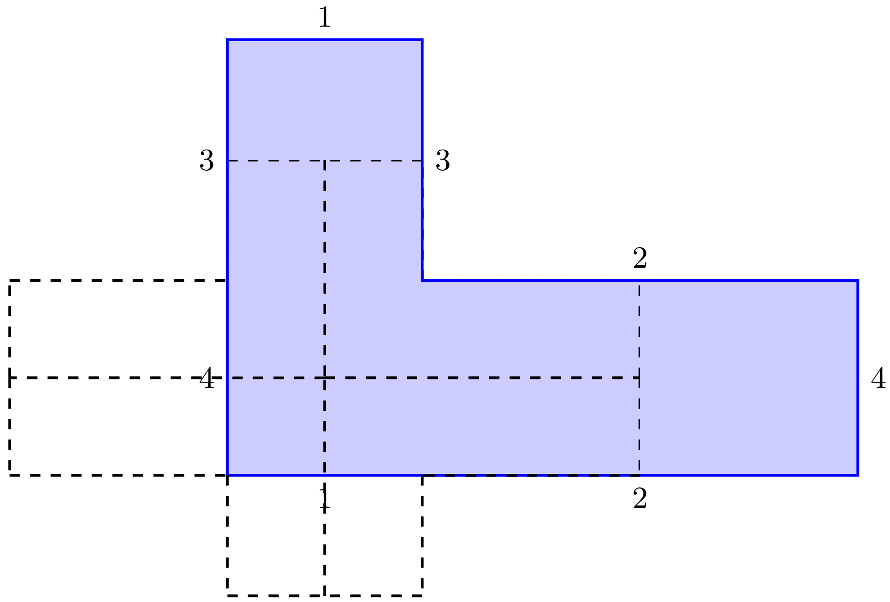


Figure 2.2: Another pattern for the translation surface from Figure 2.1.

and of finite volume, the linear part – i.e. the derivative – of f is a globally constant 2×2-matrix $Df = A$ of determinant 1.

3. The *Veech group* $\mathrm{SL}(X)$ is the image of $\mathrm{Aff}(X)$ in $\mathrm{SL}_2(\mathbb{R})$ under the derivation map.

Remark 2.7.
- Note that the Veech group $\mathrm{SL}(X)$ also can be characterized by the action of $\mathrm{SL}_2(\mathbb{R})$ on the space of translation surfaces. The Veech group $\mathrm{SL}(X)$ is the stabilizer of X under this action, because $A \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(X)$ acts by an affine self-homeomorphism and an affine self-homeomorphism stabilizes X. This means, that A is in $\mathrm{SL}(X)$, if and only if it is possible to apply A to a pattern of X and get this pattern (with correct side identifications) back by a "cut & glue" process.
- Sometimes the Veech group is not defined as above, but as the image of the above group in $\mathrm{PSL}_2(\mathbb{R})$, because the latter is a group acting faithfully on the hyperbolic plane.

Proposition 2.8. *Let X be a translation surface. Then the following holds:*

1. *The groups* $\mathrm{Trans}(X)$, $\mathrm{Aff}(X)$ *and* $\mathrm{SL}(X)$ *form a short exact sequence:*
$$\mathrm{Trans}(X) \hookrightarrow \mathrm{Aff}(X) \twoheadrightarrow \mathrm{SL}(X).$$
2. *The Veech group* $\mathrm{SL}(X)$ *is a discrete subgroup of* $\mathrm{SL}_2(\mathbb{R})$.
3. *The translation group* $\mathrm{Trans}(X)$ *is finite; it is trivial, if X is in $\Omega M_2(2)$.*
4. *Given any $M \in \mathrm{GL}_2(\mathbb{R})$ and any translation surface X the Veech group* $\mathrm{SL}(M \circ X)$ *satisfies* $\mathrm{SL}(M \circ X) = M\,\mathrm{SL}(X) M^{-1}$.

The statement of item 3 is the reason why for $X \in \Omega M_2(2)$ we can and will identify elements γ of the Veech group with the corresponding affine map.

Proof. We give sketches of the proofs:

1. Obviously every translation self-homeomorphism is also affine and an affine self-homeomorphism ϕ is in the kernel of the derivation map if and only if the linear part is the identity matrix and thus $\phi \in \mathrm{Trans}(X)$.

2. A proof of this statement can be found for example in [HS06a], Section 1.3. It is first shown that the set of saddle connection vectors is discrete in $\mathbb{R}^2$. Thus for a sequence $\{\gamma_n\}$ of Veech group elements tending to the identity there exist two linearly independent saddle connection vectors which are fixed by γ_n for all n large enough. But this means that γ_n already is the identity.

3. The second statement is Proposition 4.4 of [HL06]: Let $f \in \mathrm{Trans}(X)$ be a non-trivial translation. Then f fixes the singularity – there is only one for $X \in \Omega M_2(2)$. For ε smaller than the length of the shortest saddle connection consider the points at distance ε from the singularity in a given direction. Since the total angle around the singularity is $3 \cdot 2\pi$, there are exactly 3 such points. The translation f has to permute them. This permutation has no fixed points, otherwise f would be the identity. This implies that the orbit of every point beside the singularity has size 3. But on the other hand by [Möl06], Theorem 5.1, there are exactly 6 points on X with finite $\mathrm{SL}(X)$-orbit and f also has to permute them with the singularity as the only fixed point. This is a contradiction.

   The general statement is proven similar, using that there are only finitely many singularities of X.

4. Any $\gamma \in \mathrm{SL}(X)$ stabilizes X, and thus $M\gamma M^{-1} \circ (M \circ X) = M\gamma \circ X = M \circ X$. Hence $M\,\mathrm{SL}(X)M^{-1} \subseteq \mathrm{SL}(M \circ X)$. By the same argument with M and M^{-1} switched also $M^{-1}\,\mathrm{SL}(X)M \subseteq \mathrm{SL}(M \circ X)$ and thus $M\,\mathrm{SL}(X)M^{-1} = \mathrm{SL}(M \circ X)$ hold.

□

We call an element $A \in \mathrm{SL}_2(\mathbb{R})$ *elliptic*, if $|\operatorname{tr}(A)| < 2$, *parabolic*, if $A \neq \pm I$ and $|\operatorname{tr}(A)| = 2$, and *hyperbolic*, if $|\operatorname{tr}(A)| > 2$. To find parabolic elements of the Veech group, it is useful to decompose the translation surface into cylinders. We will define the notion of a *cylinder* and discuss how to use them in order to find parabolic elements of the Veech group:

Definition 2.9. A *cylinder* C of *heigth* h and *circumference* c in a translation surface X is the image of a $[0,c] \times (0,h)$ rectangle with the identification $(0;y) \sim (c;y)$ (see Figure 2.3) under a map $\phi = f \circ R$ with R a rotation and f a translation homeomorphism. Its *modulus* μ is the quotient $\mu := \frac{c}{h}$.

For a cylinder we call the image of $\pm\binom{1}{0}$ under the rotation R the *direction* of the cylinder. If the direction is $\pm\binom{1}{0}$ or $\pm\binom{0}{1}$, we call the cylinder *horizontal* or *vertical*, respectively. This might be a bit confusing because it seems as a cylinder with $R = \mathrm{id}$

should be vertical and not horizontal (see Figure 2.3), but it is the horizontal direction, that is special here, since all horizontal straight lines in the interior of the cylinder build closed geodesics.

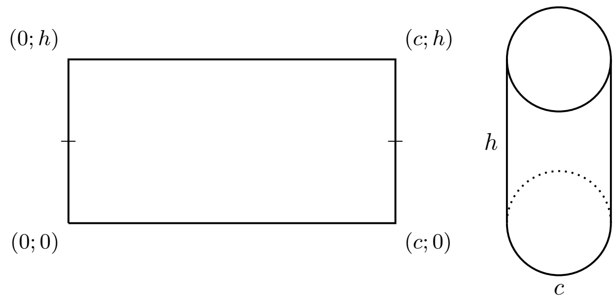


Figure 2.3: Preimage of a cylinder.

Definition 2.10. A *cylinder decomposition* of a translation surface X in direction θ is a finite collection of disjoint cylinders of direction θ such that their closures cover X.

Next, we will discuss when and how we get a parabolic element B with eigenvector $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ of the Veech group from a horizontal cylinder decomposition. For an arbitrary direction $\theta = R(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ we can rotate the translation surface, such that θ becomes the horizontal direction – i.e. we act by R^{-1}. If we find a parabolic element B of the Veech group of the rotated surface, the element RBR^{-1} is the corresponding element of the original surface's Veech group fixing the direction θ.

Proposition 2.11. *Let $\{C_i\}_{i\in I}$ be a horizontal cylinder decomposition of a translation surface X and μ_i the modulus of C_i. If for a fixed $j \in I$ and all $i \in I$ the quotient $\frac{\mu_i}{\mu_j} = q_i$ is rational*, we set k to be the least common multiple of the numerators of the q_i (written as reduced fractions). Then the matrix $B = \left(\begin{smallmatrix}1 & k\mu_j\\0 & 1\end{smallmatrix}\right)$ is in the Veech group* $\mathrm{SL}(X)$.

Proof. We first consider just a single horizontal cylinder C_i of height h_i, circumference c_i. The matrix $\left(\begin{smallmatrix}1 & x\\0 & 1\end{smallmatrix}\right)$ shears the $[0, c] \times (0, h)$ rectangle and thus twists the cylinder. A full twist is reached for $x = \mu_i = \frac{c_i}{h_i}$ (see Figure 2.4), accordingly $x = k_i\mu_i$ performs k_i full twists. Note that for full twists the horizontal sides of the cylinder's closure are fixed pointwise. This is why we can perform full twists on each cylinder and they will "fit together" at the intersection of their closures, the horizontal sides.

If for all $i \in I$ the quotient $\frac{\mu_i}{\mu_j} = q_i$ is rational and we set k as the lcm of the numerators of the reduced fractions q_i, the parabolic matrix $B = \left(\begin{smallmatrix}1 & k\mu_j\\0 & 1\end{smallmatrix}\right)$ is in the Veech group. □

2.1.4 Fuchsian Groups

As seen in Proposition 2.8 the Veech group of a translation surface is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. Hence it is a Fuchsian group, i.e. it acts properly discontinuously on the

*in this case, we say the moduli of the cylinder are *commensurable*.

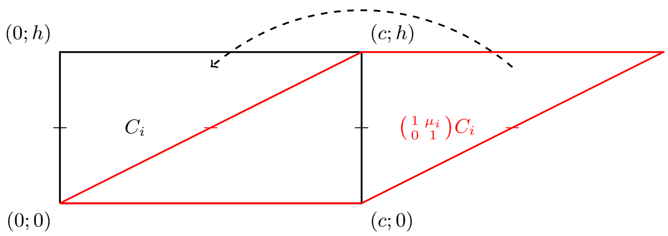


Figure 2.4: Sheared cylinder and cutting.

(hyperbolic) upper half plane $\mathbb{H}$ by Moebius transformations. Fuchsian groups are a well-studied subject – see for instance [Bea83] and [Kat92] – and we will now recall some important notions.

Fix a point of $\mathbb{H}$ – we choose the point i – as a base point and let Γ be a Fuchsian group. Note that all accumulation points of the orbit Γi have to be at the boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$, because otherwise the action would not be properly discontinuous.

Definition 2.12. Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group.

- For a Fuchsian group Γ the *limit set* Λ is the set of all accumulation points $r \in \partial\mathbb{H}$ of the orbit Γi. If the limit set is finite, Γ is called *elementary*, otherwise *non-elementary*. Non-elementary groups whose limit set is the whole boundary are called Fuchsian groups *of the first kind*, other non-elementary groups *of the second kind*.

- If Γ has a convex fundamental domain with finitely many sides, it is called *geometrically finite*, if it has a fundamental domain, which has finite hyperbolic area, it is called *of finite covolume* or *lattice*.

- Two Fuchsian groups Γ and Γ' are said to be *commensurate*, if they have a commom subgroup of finite index in both, Γ and Γ'. They are called *commensurable*, if they have subgroups of finite index, which are conjugate by an element of $\mathrm{SL}_2(\mathbb{R})$.

The following lemma plays a crucial role in the construction of infinitely generated Veech groups by McMullen as well as in the construction by Hubert and Schmidt:

Lemma 2.13 ([HS04], Lemma 3)**.** *A Fuchsian group of the first kind either is a lattice or it is infinitely generated.*

Proof. Let Γ be of the first kind and finitely generated. By Theorem 4.6.1 of [Kat92], Γ is geometrically finite. And by Theorem 4.5.1 also of [Kat92], a geometrically finite Fuchsian group of the first kind is a lattice. □

Another important result comparing different Veech groups is the following by Gutkin and Judge:

Theorem 2.14 ([GJ00], Theorem 4.9)**.** *Let $p : Y \to X$ be an affine covering of translation surfaces.Then the groups $\mathrm{SL}(Y)$ and $\mathrm{SL}(X)$ are commensurable. If p is a translation covering, then they are commensurate.*

2.2 The “Size” of a Veech Group

2.2.1 Small Veech Groups

When talking about Veech groups and their size we should first mention a result of Möller describing the Veech group of a translation surface with genus at least 2, which is “generic” in its stratum. Here *generic in its stratum* means lying outside a countable union of real co-dimension one submanifolds in its stratum. Furthermore, surfaces admitting a hyperelliptic involution (having $-I$ as linear part), are called hyperelliptic.

Theorem 2.15 ([Möl09], Theorem 1.1)**.** *For $g(X) \geq 2$, the Veech group of a generic translation surface X is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or trivial, depending on whether X belongs to a hyperelliptic component of its stratum or not.*

The next larger groups are cyclic groups. Whereas in every stratum there exists a translation surface, whose Veech group is cyclically generated by a parabolic element ([Möl09], Proposition 1.4), it is still an open question, if there exists a translation surface with Veech group cyclically generated by a hyperbolic element.

2.2.2 Lattices

There exist translation surfaces with a very large Veech group: Fuchsian groups of the first kind and in particular the finitely generated ones – lattices in $\mathrm{SL}_2(\mathbb{R})$. A translation surface whose Veech group is a lattice is called *Veech surface* and these are the objects of the famous Veech dichotomy:

Theorem 2.16 ([Vee89])**.** *If X is a Veech surface, then for each direction either*

1. *all geodesics are uniformly distributed, in particular dense, or*
2. *all geodesics are closed or saddle connections.*

The directions in 2. are called *periodic*.

For genus 2 there in ([Cal04]) Calta constructs Vecch surfaces and independently in ([McM05] and [McM06]) McMullen gives a classification of all Veech surfaces of genus 2.

In [McM05] he defines a prototype for each $\mathrm{GL}_2(\mathbb{R})$-orbit of Veech surfaces in $\Omega M_2(2)$. We choose another normalization in each orbit – we want the lower left part to be a 1×1-square - and define (cf. Figure 2.5):

Definition 2.17. Let $D \geq 5$ be a positive integer $\equiv 0$ or $1 \mod 4$.

- If $D \equiv 0 \mod 4$, set $w := \sqrt{\frac{D}{4}}$ and define $\boldsymbol{L_D}$ to be the translation surface obtained from the L-shaped polygon with horizontal side lengths (from top to bottom) 1, w, and $1+w$, and vertical side lengths (from left to right) w, $w-1$, and 1 by gluing together the opposite sides[†].
- If $D \equiv 1 \mod 4$, set $w := \frac{1+\sqrt{D}}{2}$ and define $\boldsymbol{L_{D,-1}}$ to be the translation surface obtained from the L-shaped polygon with horizontal and vertical side lengths (from

[†]Treat the left/bottom side as two sides each

bottom to top and from left to right) w, $w-1$, and 1 by gluing together the opposite sides[‡].

- Additionaly, if $D \equiv 1 \mod 8$ and $D \neq 9$, define $\boldsymbol{L_{D,+1}}$ to be the translation surface obtained from the L-shaped polygon with horizontal side lengths (from top to bottom) 1, w, and $1+w$, and vertical side lengths (from left to right) $w-1$, $w-2$, and 1 by gluing together the opposite sides[§].

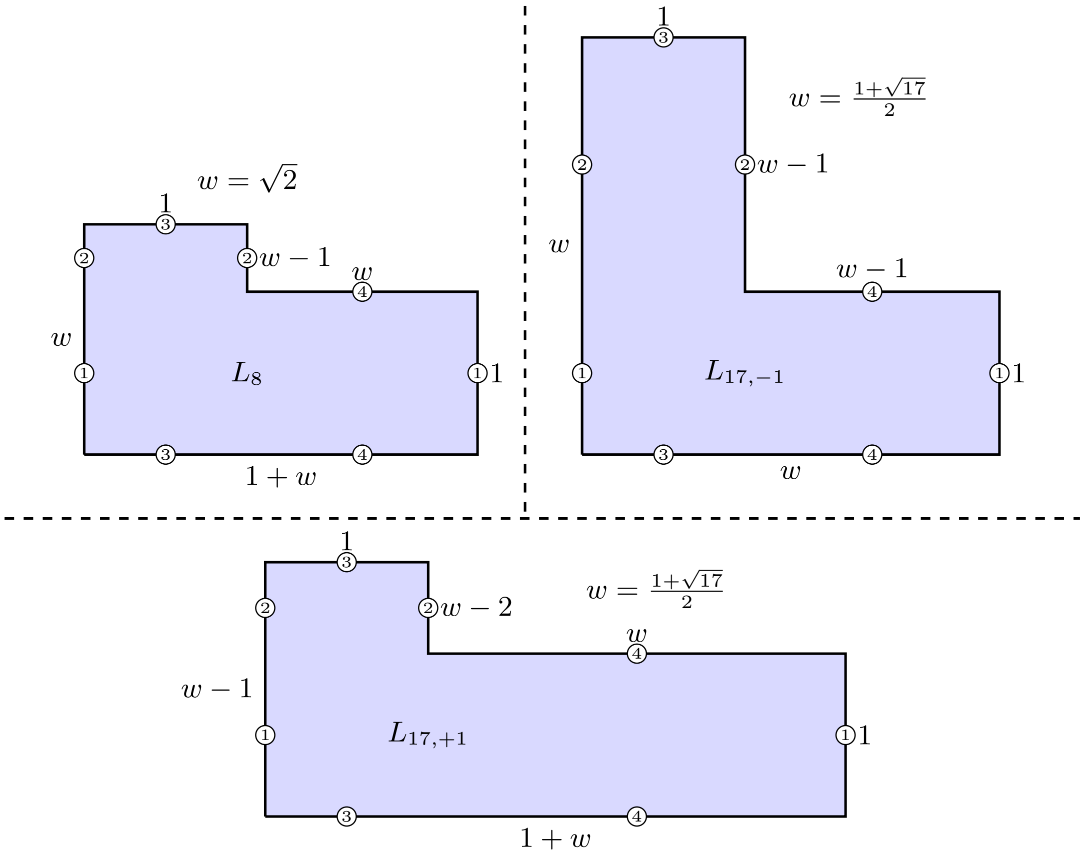


Figure 2.5: The surfaces L_8, $L_{17,-1}$, and $L_{17,+1}$.

Theorem 2.18 ([McM05], Corollary 1.3)**.** *The translation surfaces L_D, $L_{D,-1}$, and $L_{D,+1}$ are Veech surfaces. Every Veech surface X of $\Omega M_2(2)$ is in the* $\mathrm{GL}_2(\mathbb{R})$*-orbit of one of these for an suitable D.*

The number D in the above theorem is the discriminant of the *trace field* of X, the number field $\mathbb{Q}$ adjoint all traces of elements of the Veech group $\mathrm{SL}(X)$.

One year later he succeeded in classifying the *primitive* Veech surfaces in $\Omega M_2(1,1)$, i.e. the Veech surfaces that are not coverings of surfaces of lower genus.

[‡] Treat the left/bottom side as two sides each

[§] Treat the left/bottom side as two sides each

Theorem 2.19 ([McM06], Theorem 1.1)**.** *All primitive Veech surfaces in* $\Omega M_2(1,1)$ *are in the* $\mathrm{GL}_2(\mathbb{R})$*-orbit of the surface obtained from a regular decagon by gluing opposite sides.*

Remark 2.20. If D is a square, then all side lengths of L_D (or $L_{D,\pm 1}$) are rational numbers. Thus in this case the translation surface is square-tiled. Since in the following we will need primitive Veech surfaces, we will assume from now on that D is not a square. Furthermore, we will concentrate on the discriminants $D \equiv 0 \mod 4$ and the surfaces L_D.

The surfaces L_D for $D \equiv 0 \mod 4$ not a square are an important ingredient of our Main Theorem A. Thus we need to understand which points are *periodic points* – i.e. have a finite orbit under the action of the Veech group – and which points are connection points.

Theorem 2.21 ([Möl06], Theorem 5.1)**.** *The only periodic points on a primitive Veech surface in* $\Omega M_2(2)$ *are the fixed points of the hyperelliptic involution.*

We choose the coordinates of L_D, such that the origin $(0;0)$ is in the lower left corner. For the points of the glued sides we take the coordinates of the left or lower side. The action of the hyperelliptic involution's derivative $-I$ on L_D is illustrated in Figure 2.6. It has 6 fix points shown in Figure 2.7.

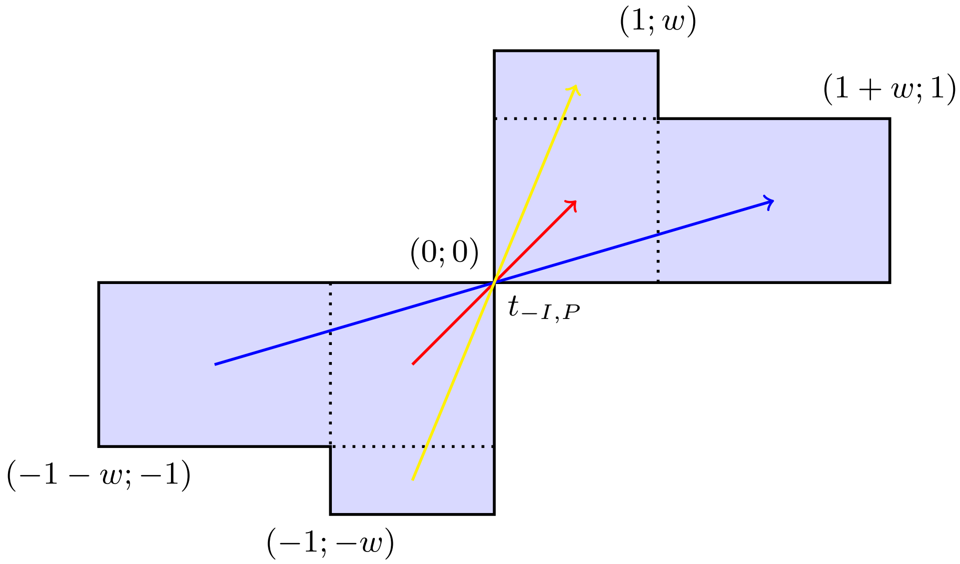


Figure 2.6: The action of $-I$ on L_8: $P \mapsto -P + t_{-I,P}$.

In Section 4.1 we will define the *group of periods* as a subgroup of $(\mathbb{R}^2,+)$ and see that all periods have both components in the field $\mathbb{Q}(w)$. The same holds for all saddle connection vectors since L_D has only one singularity and thus saddle connection vectors are periods. Moreover, by Theorem A.1 from [McM03] the set of periodic directions of L_D is precisely $\mathbb{P}^1(\mathbb{Q}(w))$. In Section 3.2 of [HS04] translation surfaces with these

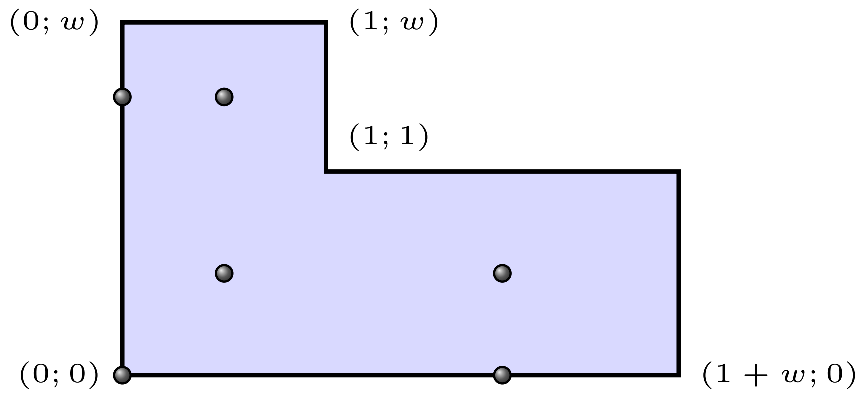


Figure 2.7: The 6 periodic points of L_D.

properties are called *of strong holonomy type* and it is shown that exactly the points with both coordinates in the field $\mathbb{Q}(w)$ are connection points. Thus we obtain:

Proposition 2.22. *The connection points of L_D are exactly the points of the form $P = \big(x_r + x_i w; y_r + y_i w\big)$ with $w = \sqrt{\frac{D}{4}}$ and $x_r, x_i, y_r, y_i \in \mathbb{Q}$.*

2.2.3 Infinitely Generated Veech Groups

At the beginning of this millennium McMullen and independently Hubert and Schmidt could prove the existence of translation surfaces with infinitely generated Veech groups. We will shortly describe the two different approaches:

In [McM03], Theorem 10.1 McMullen shows that for translation surfaces of genus 2 the limit set is either empty, a single point or the whole boundary $\partial\mathbb{H}$. In particular, if the Veech group contains a hyperbolic element, the group is of the first kind. By Lemma 2.13 a Fuchsian group of the first kind is either a lattice or infinitely generated. Using this he gives concrete examples (Figure 2.8) of translation surfaces $X \in \Omega M_2(1,1)$ with infinitely generated Veech group by showing that they contain a hyperbolic element (with irrational trace), but are not Veech surfaces.

Later he completed the classification of primitive Veech surfaces of genus 2 (Theorem 2.19), that states that all primitive Veech surfaces with two singularities of order 1 are in the same $\mathrm{GL}_2(\mathbb{R})$-orbit – the orbit of the regular decagon surface. Every non-primitive Veech surface of genus 2 is in the orbit of a square-tiled surface and for these surfaces all elements of the Veech group have rational trace. This implies:

Theorem 2.23 ([McM06], Theorem 1.3)**.** *Every translation surface $X \in \Omega M_2(1,1)$ which contains a hyperbolic element with irrational trace either is in the $\mathrm{GL}_2(\mathbb{R})$-orbit of the regular decagon surface or has an infinitely generated Veech group.*

Also Hubert and Schmidt constructed translation surfaces, such that the Veech group is of the first kind, but not a lattice. We sketch this construction from [HS04] which we will use during this thesis.

$$a = b - 1 + \sqrt{b^2 - b + 1} \notin \mathbb{Q} \text{ with } b \in \mathbb{Q}$$

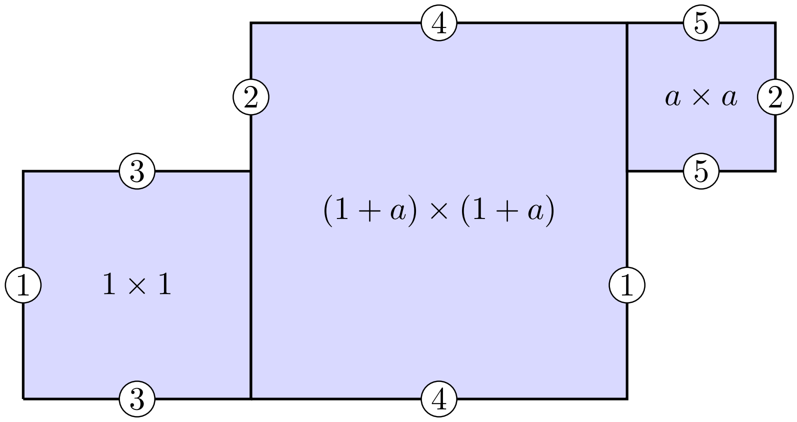


Figure 2.8: McMullen's examples of translation surfaces with infinitely generated Veech group.

Definition 2.24. Given a translation surface X with singularities $S(X)$ and a non-singular point P define the *marking of* X *at* P as a new translation surface $(X; P)$ by adding P to the set of singularities. Let the group of affine diffeomorphisms of $(X; P)$ be the subgroup of $\mathrm{Aff}(X)$ consisting of the maps that fix P. Accordingly define the Veech group $\mathrm{SL}(X; P)$ as the stabilizer subgroup of P in $\mathrm{SL}(X)$.

As seen in Remark 1.18 there is a bijection between the right cosets of $\mathrm{SL}(X)$ modulo $\mathrm{SL}(X; P)$ and the orbit points of P under the action of $\mathrm{SL}(X)$. Thus the non-periodicity of P guarantees, that $\mathrm{SL}(X; P)$ is of infinite index in $\mathrm{SL}(X)$ and hence not a lattice.

To show that $\mathrm{SL}(X; P)$ is of the first kind, Hubert and Schmidt use Proposition 3.1 of [Vor96], which states that the set of directions of geodesic segments emanating from P and encountering a singularity is dense in $S^1 = \partial\mathbb{H}$. If P is a connection point, this means the set of directions of saddle connections through P is dense in S^1. To each of these saddle connections belongs a parabolic element of $\mathrm{SL}(X)$ fixing the saddle connection pointwise, particularly fixing P. A group containing a parabolic element with eigenvector $\binom{x}{y}$ has the fixed direction $\frac{y}{x}$ in its limit set. Hence the limit set of $\mathrm{SL}(X; P)$ is dense in S^1, which is not possible for groups of the second kind (see for instance Theorem 3.4.6 of [Kat92]). This yields

Theorem 2.25 ([HS04], Proposition 1)**.** *Let P be a non-periodic connection point on a Veech surface X, then* $\mathrm{SL}(X; P)$ *is infinitely generated.*

By Theorem 2.14 the Veech groups of affine coverings of $(X; P)$ are commensurable to $\mathrm{SL}(X; P)$. Thus they are also infinitely generated and it remains to find Veech surfaces with non-periodic connection points. But we have seen in Proposition 2.22 that the Veech surfaces in $\Omega M_2(2)$ have infinitely many connection points; thus the following proposition holds and gives us candidates for Veech surfaces of the first kind with critical exponent strictly smaller than 1 (for the definition of the critical exponent see Chapter 5):

Proposition 2.26. *An affine covering of the Veech surface L_D ramified over the singularity and a point P of the form $P = \big(x_r + x_i w; y_r + y_i w\big)$ with $w = \sqrt{\frac{D}{4}}$ and $x_r, x_i, y_r, y_i \in \mathbb{Q}$ is of the first kind and has infinitely generated Veech group.*

In this context note that the question, whether there exist translation surfaces whose Veech group is of the second kind, is still open. But as mentioned above, by [McM03], Theorem 10.1, such surfaces cannot have genus 2. For an short overview, which types of Fuchsian groups appear as Veech groups, see Table 2.1.

Veech group	$\|\mathbf{\Lambda}\|$	**Existence**
finite	0	✓ [Möl09]
(virtually) cyclic parabolic	1	✓ [Möl09]
(virtually) cyclic hyperbolic	2	open
finitely generated of 2nd kind	∞, $\Lambda \neq \partial\mathbb{H}$	open, “no” in genus 2 [McM03]
infinitely generated of 2nd kind	∞, $\Lambda \neq \partial\mathbb{H}$	open, “no” in genus 2 [McM03]
lattices	∞, $\Lambda = \partial\mathbb{H}$	✓ [Vee89] & square-tiled surfaces
infinitely generated of 1st kind	∞, $\Lambda = \partial\mathbb{H}$	✓ [HS04], [McM03]

Table 2.1: Overview which types of Fuchsian groups appear as Veech groups.

3 The Prototypes L_D

In this chapter we take a closer look at the Veech surfaces L_D for $D \equiv 0 \mod 4$ not a square which we defined in the previous chapter. In Proposition 2.11 we have seen that we can obtain a parabolic element of the Veech group $\mathrm{SL}(L_D)$ from a cylinder decomposition of L_D. There are two directions for which it is very easy to see a cylinder decomposition of L_D, namely the horizontal and the vertical direction (cf. the first pictures of Figure 3.3 and Figure 3.4).

We need to know not only the corresponding parabolic Veech group elements B and A, but also how they act on the points of L_D. Therefore we introduce coordinates for the points P of L_D and obtain formulas for the coordinates of $B^l \circ P$ and $A^k \circ P$. Afterwards, we will concentrate on points of the form $P = \big(x_r + x_i w; y_r + y_i w\big)$ with $x_r, x_i, y_r, y_i \in \mathbb{Q}$ and prove some quite technical lemmata concerning the change of $s(P) := |x_i| + |y_i|$ by the action of A^k and B^l and concerning the points which are periodic under A or B. These lemmata will be used in Chapter 6.

3.1 Horizontal and Vertical Direction of the Prototype L_D

As mentioned in Remark 2.20 we are interested in the surface L_D obtained from the L-shaped polygon with horizontal side lengths (from top to bottom) 1, w, and $1 + w$, and vertical side lengths (from left to right) w, $w - 1$, and 1 by gluing together the opposite sides* (see Figure 3.1). Here D is a positive integer congruent 0 modulo 4 that is not a square and $w = \sqrt{\frac{D}{4}}$. We choose the lower left corner as the origin of our coordinate system. Then the coordinates of the other points are uniquely defined by the differential $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ on the Euclidean plane, since the L-shaped polygon is simply connected.

We now want to find the parabolic elements A and B corresponding to the vertical, respectively the horizontal, cylinder decomposition. Moreover, we need a description of the action of A and B on $P \in L_D$, i.e. we want to know the coordinates of $A \circ P$ and $B \circ P$ given the coordinates of P. Therefore, we analyze the cylinder decomposition in more detail than in Chapter 2 and start with one cylinder.

3.1.1 A Single Cylinder

Since we are not just interested in the Veech group element but also in its action on the points, let us again analyze Figure 2.4 and follow a point $\big(x; y\big)$ of the cylinder under the action of the matrix $\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$. In the first step, $\big(x; y\big)$ is mapped to $\big(x + \mu y; y\big)$. Then, if $x + \mu y \geq c$, we have to push it back into the original rectangle to express it in the original coordinates, i.e. translate it by $\left(\begin{smallmatrix} -c \\ 0 \end{smallmatrix}\right)$. If $x + \mu y < c$, the point already lies in

*For the gluing treat the left/bottom side as two sides each.

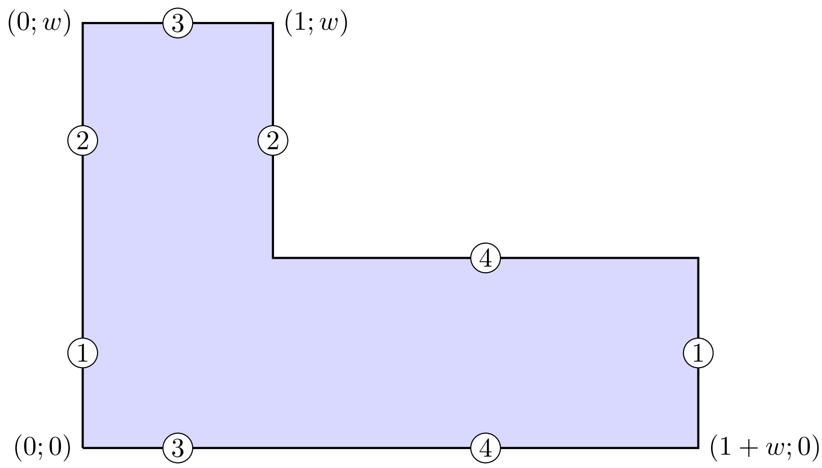


Figure 3.1: The L-shaped polygon L_D with side identifications.

the original rectangle and we do not translate it. Doing this translation is the same as picking the representative of $x + \mu y \mod c$, that is between 0 and c.

For a multiple twist $\left(\begin{smallmatrix} 1 & k\mu \\ 0 & 1 \end{smallmatrix}\right)$ the point $(x; y)$ is moved to $(x + k\mu y \mod c; y)$. The translation to push $(x + k\mu y; y)$ back in the rectangle is $(l \cdot c; 0)$ for some $0 \leq l < k$.

To divide the rectangle into parts with the same translation, consider the points $(x; y)$, such that $\left(\begin{smallmatrix} 1 & k\mu \\ 0 & 1 \end{smallmatrix}\right)(x; y)$ has x-component exactly $(l + 1) \cdot c$. These are the lines $y = -\frac{1}{k\mu}x + \frac{(l+1)h}{k}$. The resulting division is illustrated in Figure 3.1.1.

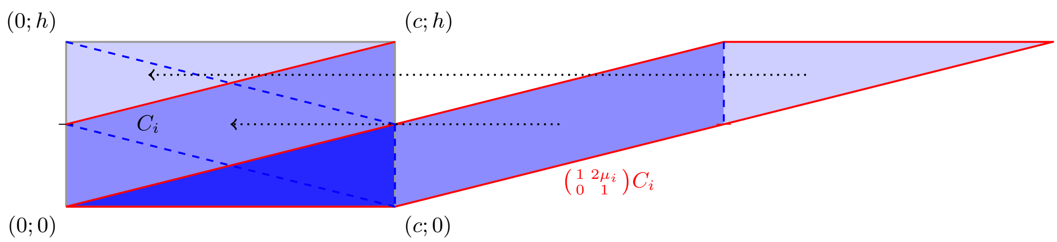


Figure 3.2: Division of the rectangle into parts with the same translation part for the action of $\left(\begin{smallmatrix} 1 & 2\mu \\ 0 & 1 \end{smallmatrix}\right)$.

Note that these translations are not the same as the elements of the translation group $\operatorname{Trans}(X)$ of a translation surface X or the translations of the affine maps $z \mapsto Az + b$, which depend only on the chosen chart.

3.1.2 The Horizontal Direction

Using the results of the previous subsection we can now find a parabolic element of the Veech group $\operatorname{SL}(L_D)$ fixing the horizontal direction $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and can describe its action on the points of L_D.

The surface L_D decomposes into two cylinders whose circumferences are in the horizontal direction: the upper cylinder $C_u = [0,1] \times (1,w)/_\sim$ and the lower cylinder $C_d = [0,1+w] \times (0,1)/_\sim$. The moduli are $\mu_u = \frac{1}{w-1}$ and $\mu_d = \frac{1+w}{1}$. The quotient of the moduli is $\frac{\mu_d}{\mu_u} = (w+1)(w-1) = w^2-1 = \frac{d-1}{1}$. Hence the desired parabolic element in $\mathrm{SL}(L_D)$ is

$$\underline{\underline{B := \begin{pmatrix} 1 & 1+w \\ 0 & 1 \end{pmatrix}}} = \begin{pmatrix} 1 & 1 \cdot \mu_d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (d-1) \cdot \mu_u \\ 0 & 1 \end{pmatrix}.$$

This means B twists the lower cylinder once and the upper cylinder $d-1$ times.

The computations of the previous subsection imply, for $\big(x;y\big) \in C_d$:

$$\begin{pmatrix} 1 & 1+w \\ 0 & 1 \end{pmatrix} \circ \big(x;y\big) = \big(x + (1+w)y \mod 1+w; y\big)$$

For the action on points of C_u we have to bear in mind that the lower left corner of the upper cylinder's closure has coordinates $\big(0;1\big)$, not $\big(0;0\big)$. This is why we have to replace y by $y-1$, apply the action described in the previous section and then shift the image point 1 upwards again:

$$\begin{pmatrix} 1 & 1+w \\ 0 & 1 \end{pmatrix} \circ \big(x;y-1\big) = \big(x + (1+w)(y-1) \mod 1; y-1+1\big)$$

Note that, as the y-coordinate is not changed by B, the image points always lie in the same cylinder as $\big(x;y\big)$. Knowing this and the description of the action of B, we also know the action of B^l for all $l \in \mathbb{Z}$:

$$B^l \circ \big(x;y\big) = \begin{cases} \big(x + l(1+w)y \mod 1+w; y\big) & \text{if } y \leq 1 \\ \big(x + l(1+w)(y-1) \mod 1; y\big) & \text{if } y > 1. \end{cases} \tag{3.1}$$

Figure 3.3 illustrates the action of B considering the cylinders divided into parts having equal translation part. Viewing the action in this way will be useful in Chapter 4.

The points of L_D we are interested in are the connection points which by Proposition 2.22 are the points with coordinates in $\mathbb{Q}(w)$. We will write them in the form $(x,y) = (x_r + x_i w, y_r + y_i w)$ and call $x_r, y_r \in \mathbb{Q}$ the *rational parts* and $x_i, y_i \in \mathbb{Q}$ the *irrational parts*.

For the horizontal parabolic element $B = \left(\begin{smallmatrix} 1 & 1+w \\ 0 & 1 \end{smallmatrix}\right)$, a point $Q = (x_r + x_i w, y_r + y_i w)$ and an integer l the x-coordinate of the point $B^l \circ Q$ is denoted by x_{B^l}. For its rational part and irrational part we write $x_{B^l,r}$ and $x_{B^l,i}$, respectively. The difference $x_{B^l,i} - x_i$ is denoted by $\Delta_{B^l}(Q)$.

For points $Q = (x,y) = (x_r + x_i w, y_r + y_i w) \in L_D$ with $y \leq 1$, i.e. $Q \in C_d$, Equation 3.1 states $x_{B^l} = x + l(1+w)y \mod 1+w$. Hence the difference of the irrational parts amounts to

$$\Delta_{B^l}(Q) = l(y_r + y_i) - q_{y,l} \text{ with } q_{y,l} = \lfloor ly \rfloor \text{ or } \lceil ly \rceil,$$

which implies that for all $Q \in C_d$ and all $l \in \mathbb{Z}$ there exists an $r \in (-1,1)$ such that $\Delta_{B^l}(Q) = ly_r + ly_i - ly - r = ly_i(1-w) - r$.

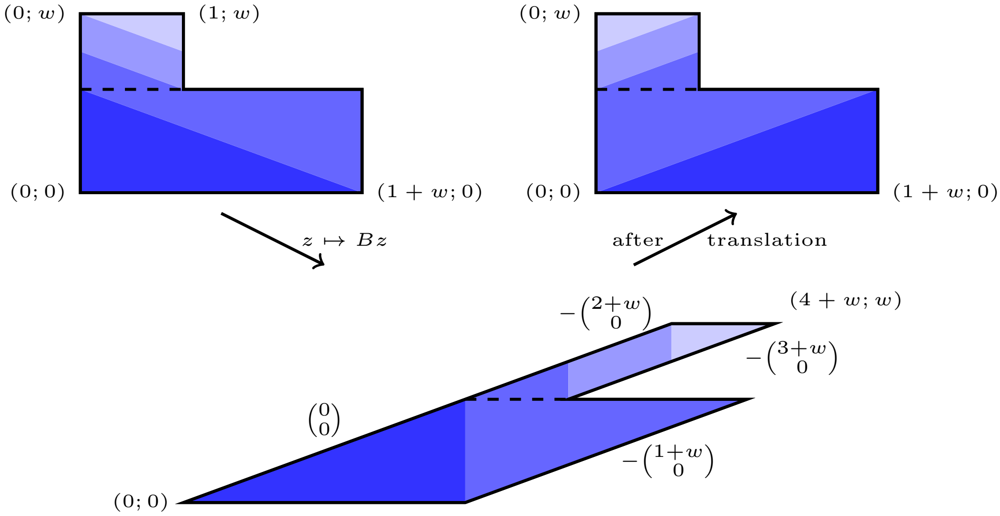


Figure 3.3: The action of B considering the cylinders divided into parts having equal translation part.

For points Q of the upper cylinder C_u – the points with $y > 1$ – Equation 3.1 states $x_{B^l} = x + l(1+w)y - l(1+w) \mod 1$. Therefore, in this case the difference of the irrational parts is

$$\Delta_{B^l}(Q) = l(y_r + y_i - 1).$$

In summary we have the following results for $\Delta_{B^l}(Q)$:

$$\Delta_{B^l}(Q) = \begin{cases} ly_i(1-w) - r \text{ for an } r \in (-1,1) & \text{if } y \leq 1 \\ l(y_r + y_i - 1) & \text{if } y > 1. \end{cases} \tag{3.2}$$

3.1.3 The Vertical Direction

We now want to analyze the parabolic elements of $\mathrm{SL}(L_D)$ with eigenvector $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ using the vertical cylinder decomposition of L_D.

Also in the vertical direction, the surface L_D decomposes into two cylinders: the left cylinder $C_l = (0,1) \times [0,w]/_\sim$, and the right one $C_r = (1, 1+w) \times [0,1]/_\sim$. Here $\sim$ is the identification of top and bottom of the rectangles, so compared to Section 3.1.1 the cylinders are rotated by 90°. We could apply the rotation $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, act as described in Section 3.1.1 and rotate back. But in this case it seems easier to do the analogous calculations as above just with horizontal and vertical direction as well as circumference and height swapped appropriately. The desired matrix has the form $\left(\begin{smallmatrix} 1 & 0 \\ \mu & 1 \end{smallmatrix}\right)$.

For the moduli we get $\mu_l = \frac{w}{1}$ and $\mu_r = \frac{1}{w}$. The quotient of the moduli is $\frac{\mu_l}{\mu_r} = w^2 = d$, yielding the Veech group element

$$\underline{\underline{A := \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}}} = \begin{pmatrix} 1 & 0 \\ 1 \cdot \mu_l & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d \cdot \mu_r & 1 \end{pmatrix}.$$

For the action of A^k on $(x;y)$ we have to look at the two cylinders separately. Remember that the right cylinder has its "origin" in $(1;0)$ instead of $(0;0)$ and note that under the action of A the x-coordinate stays unchanged resulting in the following description:

$$A^k \circ (x;y) = \begin{cases} (x; y + kxw \mod w) & \text{if } x \leq 1 \\ (x; y + k(xw - w) \mod 1) & \text{if } x > 1. \end{cases} \tag{3.3}$$

Figure 3.4 illustrates the action of A considering the cylinders divided into parts having equal translation part.

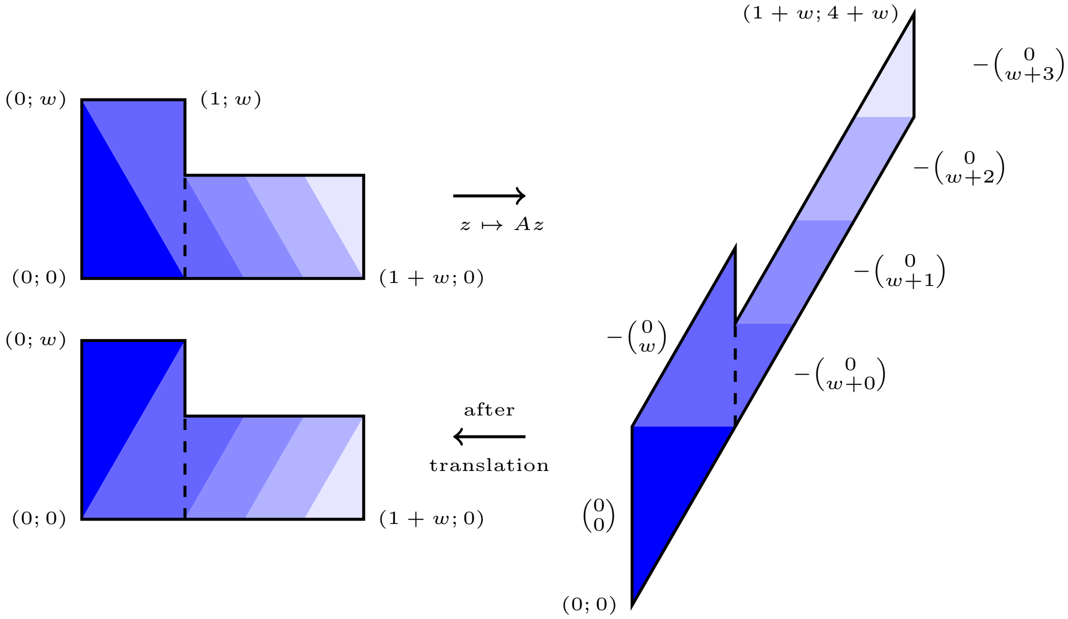


Figure 3.4: The action of A considering the cylinders divided into parts having equal translation part.

Similarly to the notation x_{B^l} we introduce the notations y_{A^k} for the y-coordinate of $A^k \circ Q$ as well as $y_{A^k,r}$ and $y_{A^k,i}$ for the rational part and the irrational part of y_{A^k}, respectively. Furthermore we define $\Delta_{A^k}(Q) \coloneqq y_{A^k,i} - y_i$ and compute this value for points Q of the left and right cylinder separately:

For points $Q = (x, y) = (x_r + x_i w; y_r + y_i w) \in L_D$ with $x \leq 1$, i.e. $Q \in C_l$, Equation 3.3 states $y_{A^k,i} = y_i + kx_r \mod w$. Hence we see that the difference of the irrational parts is

$$\Delta_{A^k}(Q) = kx_r - q_{x,k} \text{ with } q_{x,k} = \lfloor kx \rfloor \text{ or } \lceil kx \rceil,$$

which implies that for all $Q \in C_l$ and all $k \in \mathbb{Z}$ there is an $r \in (-1, 1)$ such that $\Delta_{A^k}(Q) = kx_r - kx - r = -kx_i w - r$.

For points Q of the right cylinder C_r – the points with $x > 1$ – Equation 3.3 states $y_{A^k,i} = y_i + k(x_r - 1)$. Hence in this case the difference of the irrational parts is

$$\Delta_{A^k}(Q) = k(x_r - 1).$$

In summery we have the following results for $\Delta_{A^k}(Q)$:

$$\Delta_{A^k}(Q) = \begin{cases} -kx_i w - r \text{ for an } r \in (-1,1) & \text{if } x \leq 1 \\ k(x_r - 1) & \text{if } x > 1. \end{cases} \tag{3.4}$$

3.2 Points Periodic Under A or B

Given a parabolic element γ of the Veech group of a Veech surface, it is easy to check whether a point $(x; y)$ is periodic under the action of this element: by the Veech dichotomy, parabolic elements correspond to cylinder decompositions of the surface in the direction of the corresponding eigenvector. The well-known fact of Proposition 3.3 states that a point is periodic under γ, if and only if the splitting ratio of the point in its cylinder is rational. In this section, the term *splitting ratio* is defined and conditions on the points periodic under A and the points periodic under B are deduced.

Definition 3.1. Consider a point Q in the closure of a cylinder C and its preimage $(x; y)$ in $[0, c] \times [0, h]/_{\sim}$ as in Section 3.1.1. The *splitting ratio* of Q in C is the point's height compared to the height of the cylinder: $\mathrm{sr}_C(Q) := \frac{y}{h}$.

Remark 3.2 ([HS06b] Lemma 4 and Corollary 1)**.** The splitting ratio is $\mathrm{GL}_2(\mathbb{R})$-invariant and preserved under affine diffeomorphisms.

Proposition 3.3. *Let $\bigcup C_i$ be a cylinder decomposition of a translation surface with corresponding parabolic Veech group element γ. Then a point Q in a cylinder C_j is periodic under γ if and only if the splitting ratio $\mathrm{sr}_{C_j}(Q)$ is rational.*

Proof. Without loss of generality we assume the cylinders to have their circumference in the horizontal direction and the cylinder C_j to be the cylinder $C = [0, c_j] \times [0, h_j]/_{\sim}$. Then γ is of the form $\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$, where μ is a multiple of the modulus of each cylinder, in particular there is an $n \in \mathbb{Z}$ with $\mu = n \cdot \frac{c_j}{h_j}$.

Let $\mathrm{sr}_{C_j}(Q) = \alpha$. This means, the y-coordinate of Q is $\alpha \cdot h_j$. The image under the action of $\left(\begin{smallmatrix} 1 & k\mu \\ 0 & 1 \end{smallmatrix}\right)$ is $\left(x; y + kn\frac{c_j}{h_j}\alpha h_j \mod c_j\right) = \left(x; y + kn\alpha c_j \mod c_j\right)$.

If $\alpha = \frac{p}{q}$ is rational, the y-coordinate is shifted by $\frac{knp}{q} c_j$. This shift is $0 \mod c_j$ if $\frac{knp}{q}$ is an integer, which is the case if k is a multiple of $\frac{q}{\gcd(q,n)}$. The behavior of the sequence of shifts for $k \in \mathbb{Z}$ is obviously the same as the behavior of the sequence $k(n\alpha) \mod 1$ and $n\alpha$ is irrational if and only if α is irrational. It is well known that for irrational α this sequence is equidistributed modulo 1 and in particular not periodic. $\square$

Lemma 3.4. *A point $Q = (x; y) = (x_r + x_i w; y_r + y_i w) \in L_D$ is periodic under the action of B if and only if one of the following two conditions holds:*

1. $y \leq 1$ *and* $y_i = 0$.
2. $y > 1$ *and* $y_r = 1 - y_i$

In particular if Q is periodic under B then $0 \leq y_i < 1$.

Proof. By Proposition 3.3 the point Q is periodic under B if and only if its splitting ratio in the corresponding cylinder is rational.

If $y \leq 1$, the point is in the lower cylinder C_d which has height 1. The height of Q in C_d is y (see Figure 3.2), therefore the splitting ratio is $\mathrm{sr}_{C_d}(Q) = \frac{y}{1} = y_r + y_i w$. This is rational if and only if $y_i = 0$.

If $y > 1$, the point is in the upper cylinder C_u which has height $w - 1$. The height of Q in C_u is $y - 1$, therefore the splitting ratio is

$$\mathrm{sr}_{C_u}(Q) = \frac{y-1}{w-1} = \frac{(y-1)(w+1)}{d-1} = \frac{1}{d-1}(y_r + dy_i - 1 + (y_r + y_i - 1)w).$$

This is rational if and only if $y_r = 1 - y_i$.

For the additional conclusion we observe that solving the inequality $1 \leq y_r + y_i w < w$ for y_r and setting $y_r = 1 - y_i$ yields $1 - y_i w \leq 1 - y_i < w - y_i w$, which is equivalent to $0 \leq y_i < 1$. ☐

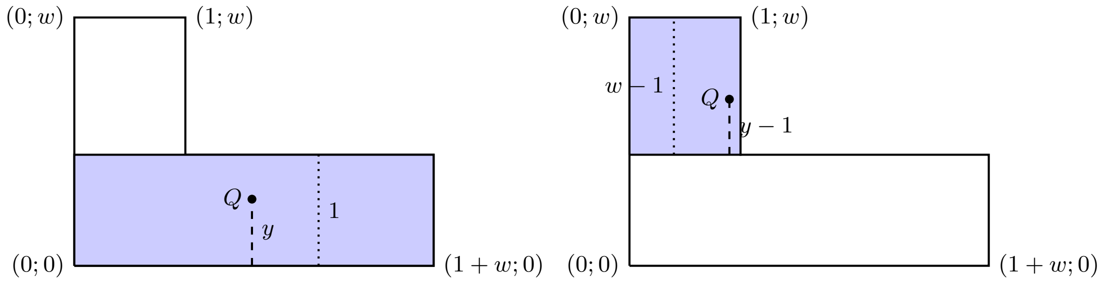


Figure 3.5: The splitting ratio of Q in the horizontal cylinders.

Lemma 3.5. *A point $Q = \big(x; y\big) = \big(x_r + x_i w; y_r + y_i w\big) \in L_D$ is periodic under the action of A if and only if one of the following two conditions holds:*

1. *$x \leq 1$ and $x_i = 0$.*

2. *$x > 1$ and $x_r = 1$*

In particular if Q is periodic under A then $0 \leq x_i < 1$.

Proof. By Proposition 3.3 the point Q is periodic under A if and only if its splitting ratio in the corresponding cylinder is rational. Note that the corresponding cylinders are C_l and C_r, whose circumferences are in the vertical direction and whose heights are in x-direction.

If $x \leq 1$, the point is in the left cylinder C_l which has height 1. The height of Q in C_l is x (see Figure 3.6). Therefore, the splitting ratio is $\mathrm{sr}_{C_l}(Q) = \frac{x}{1} = x_r + x_i w$. This is rational if and only if $x_i = 0$.

If $x > 1$, the point is in the right cylinder C_r which has height $1 + w - 1 = w$. The height of Q in C_r is $x - 1$. Therefore, the splitting ratio is

$$\mathrm{sr}_{C_r}(Q) = \frac{x-1}{w} = \frac{(x_r - 1)}{w} + x_i.$$

This is rational if and only if $x_r = 1$.

For the additional conclusion we solve the inequality $1 \le x_r + x_i w < 1 + w$ for x_i and obtain $\frac{1-x_r}{w} \le x_i < \frac{1+w-x_r}{w}$. For $x_r = 1$ this becomes $0 \le x_i < 1$. $\square$

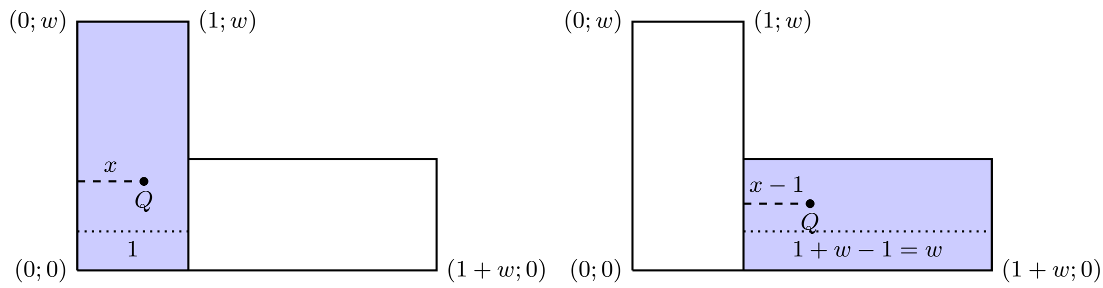


Figure 3.6: The splitting ratio of Q in the vertical cylinders.

3.3 The Action of A and B in More Detail

To obtain some information about the structure of the Schreier graph from the action of A^k and B^l on a point Q, we will analyze how the absolute values of the coordinates change. Since the x-coordinate and the y-coordinate are bounded below by 0 and above by w and $1 + w$, respectively, if the absolute value of an irrational part grows, usually also the corresponding rational part's absolute value grows. This is why we will look at the sum of the irrational part's absolute values, which will be denoted by

$$s(Q) := |x_i| + |y_i|.$$

Another notation needed is the following: Suppose $Q = \big(x_r + x_i w; y_r + y_i w\big)$ and x_r, x_i, y_r, y_i are reduced fractions with denominators q_{xr}, q_{xi}, q_{yr} and q_{yi} respectively. Then we define $N(Q)$ to be the least common denominators of x_r, x_i, y_r and y_i,

$$N(Q) := \operatorname{lcm}(q_{xr}, q_{xi}, q_{yr}, q_{yi}).$$

Now we can formulate some quite technical lemmata, which will help to prove the non-amenability of the Schreier graph of $\mathrm{SL}(L_D)$ modulo $\mathrm{SL}(L_D; P)$ with respect to any finite generating set in Chapter 6.

Lemma 3.6. *Let $Q = \big(x_r + x_i w; y_r + y_i w\big)$ and set $N = N(Q)$. If Q is periodic under A but not under B, there exists an l_0, such that for all l with $|l| \ge l_0$ the following inequality holds:*

$$s(Q) < s(B^l \circ Q).$$

Lemma 3.7. *Let $Q = \big(x_r + x_i w; y_r + y_i w\big)$ and set $N = N(Q)$. If Q is periodic under B but not under A, there exists a k_0, such that for all $k \in \mathbb{Z}$ with $|k| \ge k_0$ the following inequality holds:*

$$s(Q) < s(A^k \circ Q)$$

Proof of Lemma 3.6. Since B^l does not change y_i, it increases $s(Q)$ if and only if it increases $|x_i|$. But we know the bounds $0 \leq x_i < 1$ for a point periodic under A by Lemma 3.5. Hence $s(Q) < s(B^l \circ Q)$ is guaranteed if the absolute value of the change of x_i by B^l – which is $|\Delta_{B^l}(Q)|$ – is greater than 2. We will show the existence of l_0 for Q in the lower and upper cylinder separately:

If Q is in the lower cylinder C_d, Equation 3.2 states that B^l changes the irrational part of x by

$$\Delta_{B^l}(Q) = ly_i(1-w) - r \text{ for a } r \in (-1, 1).$$

By the reverse triangle inequality we obtain $|\Delta_{B^l}(Q)| = |ly_i(1-w)-r| \geq |ly_i(1-w)|-|r|$ and $|ly_i(1-w)| \geq 3$ would imply $|\Delta_{B^l}(Q)| > 2$. Since Q is not periodic under B, we know $y_i \neq 0$ (Lemma 3.4) and thus $|y_i| \geq \frac{1}{N}$. Thus we can choose

$$l_{0,d} = \frac{3N}{w-1}$$

and we obtain $|l| \geq \frac{3N}{w-1} \geq \frac{3}{(w-1)|y_i|}$ for all l with $|l| \geq l_{0,d}$. This finally implies $|ly_i(1-w)| = |l|(w-1)|y_i| \geq 3$ as required.

If Q is in the upper cylinder C_u, by Equation 3.2 $\Delta_{B^l}(Q)$ equals $l(y_r+y_i-1)$. Since Q is not periodic under B, we know $y_r+y_i-1 \neq 0$ (Lemma 3.4) and thus $|y_r+y_i-1| \geq \frac{1}{N}$. Then

$$l_{0,u} = 2N+1$$

implies $|\Delta_{B^l}(Q)| \geq |l|\frac{1}{N} > 2$ for all l with $|l| \geq l_{0,u}$.

Hence we get the statement of the lemma with

$$l_0 = \max\{l_{0,d}, l_{0,u}\} = \max\left\{\frac{3N}{w-1}, 2N+1\right\}.$$

□

Proof of Lemma 3.7. By analogy to the proof of Lemma 3.6 this lemma is certainly true if $|\Delta_{A^k}(Q)| > 2$. We will show the existence of k_0 for Q in the left and right cylinder separately:

If Q is in the left cylinder C_l, Equation 3.4 states that A^k changes the irrational part of y by

$$\Delta_{A^k}(Q) = -kx_iw - r \text{ for a } r \in (-1, 1).$$

From the reverse triangle inequality $|\Delta_{A^k}(Q)| = |-kx_iw-r| \geq |kx_iw|-|r|$ follows and $|kx_iw| \geq 3$ would imply $|\Delta_{A^k}(Q)| > 2$. Since Q is not periodic under A, we know $x_i \neq 0$ (Lemma 3.5) and thus $|x_i| \geq \frac{1}{N}$. Thus we can choose

$$k_{0,l} = \frac{3N}{w}$$

and obtain $|k| \geq \frac{3N}{w} \geq \frac{3}{|x_i|w}$ for all k with $|k| \geq k_{0,l}$. This implies $|kx_iw| = |k|w|x_i| \geq 3$ as desired.

If Q is in the right cylinder, $\Delta_{A^k}(Q) = k(x_r-1)$ by Equation 3.4. Since Q is not periodic under A, we know $x_r - 1 \neq 0$ (Lemma 3.5) and thus $|x_r-1| \geq \frac{1}{N}$. Then

$$k_{0,r} = 2N+1$$

implies for all k with $|k| \geq k_{0,r}$, that $|\Delta_{A^k}(Q)| \geq |k|\frac{1}{N} > 2$ holds.

Hence we get the statement of the lemma with

$$k_0 = \max\{k_{0,l}, k_{0,r}\} = \max\left\{\frac{3N}{w}, 2N+1\right\}.$$

$\square$

For the proof of the next two lemmata, which will help prove Lemma 3.10, we will use the fact that $|a \pm b| > |b|$ implies $\operatorname{sgn}(a \pm b) = \operatorname{sgn}(a)$.

Lemma 3.8. *If $Q \in L_D$ is not periodic under A, then the signs of $\Delta_{A^k}(Q)$ and $\Delta_{A^{-k}}(Q)$ are different for all $k > k_0$.*

Proof. If Q is in the right cylinder, $\Delta_{A^{\pm k}}(Q) = \pm k(x_r - 1)$. Since Q is not periodic under A, we have $x_r \neq 1$ and $k(x_r - 1)$ and $-k(x_r - 1)$ have different signs.

If the point Q is in the left cylinder, $\Delta_{A^{+k}}(Q) = -kx_iw - r$ for some $r \in (-1, 1)$ and $\Delta_{A^{-k}}(Q) = kx_iw + r'$ for an $r' \in (-1, 1)$. As seen in the proof of Lemma 3.6, the absolute value of these changes is greater than 2 for $k > k_0$, hence $|-kx_iw - r|$ and $|kx_iw + r'|$ are bigger than $|r|$ and $|r'|$. Thus the first term – $\Delta_{A^{+k}}(Q)$ – has sign $\operatorname{sgn}(-kx_i)$, which is not 0 because Q is not periodic under A, whereas the second one – $\Delta_{A^{-k}}(Q)$ – has sign $\operatorname{sgn}(kx_i) = -\operatorname{sgn}(-kx_i)$. $\square$

Lemma 3.9. *If Q is a point not periodic under B, then the signs of $\Delta_{B^l}(Q)$ and $\Delta_{B^{-l}}(Q)$ are different for all $l > l_0$.*

Proof. If Q is in the upper cylinder, $\Delta_{B^{\pm l}}(Q) = \pm l(y_r + y_i - 1)$. Since Q is not periodic under B, we have $y_r + y_i \neq 1$ and $l(y_r + y_i - 1)$ and $-l(y_r + y_i - 1)$ have different signs.

If the point Q is in the lower cylinder $\Delta_{B^{+l}}(Q) = ly_i(1 - w) - r$ for a $r \in (-1, 1)$ and $\Delta_{B^{-l}}(Q) = ly_i(1 - w) + r'$ for some $r' \in (-1, 1)$. For $l > l_0$ the absolute value of these changes is at least 2 – as shown in the proof of Lemma 3.7 – hence $|ly_i(1 - w) - r|$ and $|ly_i(1 - w) + r'|$ are bigger than $|r|$ and $|r'|$. Since $1 - w$ is negative, the first term – $\Delta_{B^{+l}}(Q)$ – has sign $\operatorname{sgn}(-ly_i)$, whereas $\Delta_{B^{-l}}(Q)$ has sign $\operatorname{sgn}(ly_i) = -\operatorname{sgn}(-ly_i)$. $\square$

Lemma 3.10. *Let $Q = \big(x_r + x_iw; y_r + y_iw\big)$ be a point periodic neither under A nor under B and set $N = N(Q)$. Then there are numbers k_1 and l_1 such that for all pairs (k, l) with $k > k_1$ and $l > l_1$ at least three of the following four inequalities hold:*

$$\begin{aligned} s(Q) &< s(A^k \circ Q), \\ s(Q) &< s(A^{-k} \circ Q), \\ s(Q) &< s(B^l \circ Q), \\ s(Q) &< s(B^{-l} \circ Q). \end{aligned}$$

Proof. As a first observation, we see that since A does not change x_i and B does not change y_i, we only have to look at the effect of $A^{\pm k}$ on $|y_i|$ and of $B^{\pm l}$ on $|x_i|$ to see the effect on $s(Q) = |x_i| + |y_i|$.

The second step is to apply Lemma 3.8 and Lemma 3.9: Since the changes of y_i by A^k and by A^{-k} have different signs, one of them has the same sign as y_i itself and thus the corresponding $A^{\pm k}$ increases $|y_i|$. By analogy one of $B^{\pm l}$ has $\operatorname{sgn} \Delta_{B^{\pm l}}(Q) = \operatorname{sgn}(x_i)$ and thus increases $|x_i|$. Hence at least two of the four inequalities hold. For the third one we have to show that one of the two Δ's with the opposite sign from y_i respectively x_i is big enough to nevertheless increase $|y_i|$ respectively $|x_i|$, i.e. is bigger than $2|y_i|$ respectively $2|x_i|$:

If $|x_i| \geq |y_i|$ this will be $\Delta_{A^k}(Q)$ for k with $|k|$ big enough, otherwise it will be $\Delta_{B^l}(Q)$ for l with $|l|$ big enough. To finish the proof, we distinguish these two cases:

First Case $|x_i| \geq |y_i|$: Let us first find a k_1 with the property that for all $k \in \mathbb{Z}$ with $|k| > k_1$ the inequality $|\Delta_{A^k}(Q)| > 2|x_i|$ holds. In particular for points with $|x_i| \geq |y_i|$ this inequality implies $|y_{A^k,i}| = |y_i + \Delta_{A^k}(Q)| > |x_i| \geq |y_i|$.

We distinguish by the cylinder containing Q:

If Q is in the left cylinder, $\Delta_{A^k}(Q) = -kx_iw - r$ for an $r \in (-1, 1)$. With the reversed triangle inequality we get $|-kx_iw - r| > 2|x_i|$, if $|k||x_i|w > 2|x_i| + |r|$. This again would be implied by $|k||x_i|w \geq 2|x_i| + 1$, which is equivalent to $|k| \geq \frac{2}{w} + \frac{1}{|x_i|w}$. Since Q is not periodic under A, we know (from Lemma 3.5) that $x_i \neq 0$ and thus $|x_i| \geq \frac{1}{N}$. Hence the above inequality is fulfilled if

$$|k| \geq \frac{2+N}{w}.$$

If Q is in the right cylinder, the x-coordinate satisfies $1 < x_r + x_iw < 1 + w$ and therefore there is a $q \in (0, w)$, such that $x_r = -x_iw + 1 + q$. The change of y_i is $\Delta_{A^k}(Q) = k(x_r - 1)$. Using the reversed triangle inequality we obtain for the absolute value $|\Delta_{A^k}| = |k| \cdot |-x_iw+q| \geq |k|(|x_iw|-|q|) > |k|(|x_i|w-w)$. Hence $|\Delta_{A^k}(Q)| > 2|x_i|$, if $|k| \geq \frac{2}{w} \cdot \frac{|x_i|}{|x_i|-1}$. This term decreases for $|x_i| > 1$.

The smallest possible value for $|x_i| > 1$ is $|x_i| = 1 + \frac{1}{N}$. Thus for points Q with $|x_i| > 1$ the inequality $|\Delta_{A^k}(Q)| > 2|x_i|$ is guaranteed, if

$$|k| \geq \frac{2(N+1)}{w}.$$

For points Q with $|x_i| \leq 1$ already $|k| > k_0$ suffices, because the same computations as in the proof of Lemma 3.7 show $|\Delta_{A^k}(Q)| > 2 \geq 2|x_i|$ for all points Q not periodic under A and all k with $|k| > k_0$.

Summarizing, $|\Delta_{A^k}(Q)| > 2|x_i|$ holds for all k with $|k| > k_1$, where

$$k_1 = \max\left\{\frac{2+N}{w}, k_0, \frac{2(N+1)}{w}\right\},$$

where $\frac{2+N}{w}$ can be omited because it is always smaller than $\frac{2(N+1)}{w}$.

Second Case $|x_i| < |y_i|$: we have to find an l_1 satisfying $|\Delta_{B^l}(Q)| > 2|y_i|$ for all $l \in \mathbb{Z}$ with $|l| > l_1$. In particular, for points with $|x_i| < |y_i|$ this inequality implies $|x_{B^l,i}| = |x_i + \Delta_{B^l}(Q)| > |y_i| > |x_i|$.

We distinguish by the cylinder that contains Q:

If Q is in the lower cylinder, $|\Delta_{B^l}|$ is greater than $|ly_i(1-w)|-1$ by the same arguments as in the proof of Lemma 3.6. Therefore, it is greater than $2|y_i|$, if $|l|(w-1)|y_i| \geq 2|y_i|+1$, and this is equivalent to $|l| \geq \frac{2}{w-1} + \frac{1}{(w-1)|y_i|}$. Since Q is in the lower cylinder but not periodic under B, we know that $y_i \neq 0$ and thus $|y_i| \geq \frac{1}{N}$. Hence the inequality holds if

$$|l| \geq \frac{2+N}{w-1}.$$

If Q is in the upper cylinder, the y-coordinate satisfies $1 < y_r + y_i w < w$ and therefore there is a $q \in (0, w-1)$, such that $y_r = -y_i w + 1 + q$. The change of x_i is $\Delta_{B^l}(Q) = l(y_r + y_i - 1)$. Again, the reversed triangle inequality implies for the absolute value $|\Delta_{B^l}| = |l| \cdot |(y_i(1-w) + q)| \geq |l|(|y_i(1-w)| - |q|) > |l|((w-1)|y_i| - (w-1))$. Hence $|\Delta_{B^l}(Q)| > 2|y_i|$, if $|l| \geq \frac{2}{w-1} \cdot \frac{|y_i|}{|y_i|-1}$. This term is decreasing for $|y_i| > 1$.

The smallest possible value for $|y_i| > 1$ is $|y_i| = 1 + \frac{1}{N}$. Thus for points Q with $|y_i| > 1$ the inequality $|\Delta_{B^l}(Q)| > 2|y_i|$ is guaranteed, if

$$|l| \geq \frac{2(N+1)}{w-1}.$$

For points Q with $|y_i| \leq 1$, already $|l| > l_0$ suffices, because the same computations as in the proof of Lemma 3.6 show $|\Delta_{B^l}(Q)| > 2 \geq 2|y_i|$ for all points Q not periodic under B and all l with $|l| > l_0$.

Summarizing $|\Delta_{B^l}(Q)| > 2|y_i|$ holds for all l with $|l| > l_1$, where

$$l_1 = \max\{\frac{2+N}{w-1}, l_0, \frac{2(N+1)}{w-1}\},$$

where $\frac{2+N}{w-1}$ can be omited because it is always smaller than $\frac{2(N+1)}{w-1}$.

All in all we proved that for all points Q with $|x_i| \geq |y_i|$ the following inequalities hold for all pairs (k,l) with $k > k_1$ and $l > l_1$:

$$\begin{gathered} s(Q) < s(A^k \circ Q), \\ s(Q) < s(A^{-k} \circ Q), \\ \text{at least one of } s(Q) < s(B^l \circ Q) \text{ and } s(Q) < s(B^{-l} \circ Q). \end{gathered}$$

For points Q with $|x_i| < |y_i|$ the following inequalities hold for all pairs (k,l) with $k > k_1$ and $l > l_1$:

$$\begin{gathered} s(Q) < s(B^l \circ Q), \\ s(Q) < s(B^{-l} \circ Q), \\ \text{at least one of } s(Q) < s(A^k \circ Q) \text{ and } s(Q) < s(A^{-k} \circ Q). \end{gathered}$$

$\square$

4 $\mathrm{SL}(L_D)$-Orbits of Connection Points

Hubert and Schmidt's construction of translation surfaces with infinitely generated Veech groups (described in Chapter 2) uses a marking of a non-periodic connection point P on a Veech surface X. Let us instead consider a point $\gamma \circ P$ for an element $\gamma \in \mathrm{SL}(X)$. Clearly, the corresponding stabilizer subgroups are conjugate: $\mathrm{SL}(X; \gamma \circ P) = \gamma\, \mathrm{SL}(X; P)\gamma^{-1}$. Since conjugate Fuchsian groups have the same dynamics at the boundary of the hyperbolic plane and the same critical exponent (cf. Chapter 5) the following question arises naturally:

Question 4.1. *What are the orbits of the non-periodic connection points under the action of the Veech group?*

In our context the answer to this question is important for another reason already seen in Remark 1.18: The vertices of the Schreier graph $G_{\mathrm{SL}(L_D),\mathrm{SL}(L_D;P),S}$ can be identified with the points in the orbit of P. This is why we dedicate this chapter to – at least partially – answering the question for the surfaces L_D.

In the first section we get a necessary condition for two connection points to be in the same orbit: For a point $Q = \big(x_r + x_i w; y_r + y_i w\big)$ with $x_r, x_i, y_r\ y_i$ reduced fractions let $N(Q)$ be the least common denominator of these fractions. Then two points P and P' can be in the same orbit only if $N(P) = N(P')$.

In the second section we analyze the case $D = 8$ and derive finite bounds on the number of orbits with fixed number $N(P)$.

4.1 General Restrictions on the $\mathrm{SL}(L_D)$-Orbit of $P \in L_D$

Following the idea of the proof of Lemma 5 of [HS06b] we get a first observation concerning the orbit of points of L_D.

Proposition 4.2. *The $\mathrm{SL}(L_D)$-orbit of any point $P \in L_D$ that is not periodic under both, A and B, is dense in L_D.*

Proof. Without loss of generality assume $P = \big(x; y\big)$ is not periodic under A. Let Q be an arbitrary point of L_D. The y-coordinate of $A^k \circ P$ is $y + kxw \mod 1$ or $\mod w$, depending on which vertical cylinder contains P. The associated sequence can only be either periodic or equidistributed in the interval $[0, 1]$ or $[0, w]$, respectively. Since P is not periodic under A, it is equidistributed and in particular it is dense in $[0, 1]$ or $[0, w]$. In the proof of Lemma 5 in [HS06b] it is shown that only finitely many members of this sequence have rational splitting ratio in their horizontal cylinder. This also follows from Theorem C because there exist only finitely many points with fixed x-coordinate, fixed least common denominator, and rational splitting ratio in the horizontal cylinder (see also Lemma 3.4).

Hence with a suitable power k_1 the point $A^{k_1} \circ P$ is in the same horizontal cylinder as Q and has an irrational splitting ratio in it. Thus it is not periodic under B and with a suitable power l the point $B^l A^{k_1} \circ P$ has x-coordinate arbitrary close to the x-coordinate of Q. Additionally we can choose l such that the resulting point has irrational splitting ratio in its vertical cylinder and thus is not periodic under A. Now in the last step we find a number k_2, such that $A^{k_2} B^l A^{k_1} \circ P$ is arbitrary close to Q. $\square$

Since this does not tell us anything about the number of orbits, we have to get further information: We need to consider the homology $H_1(L_D, \mathbb{Z})$. A basis for this homology consists of the $2g = 4$ paths h_1, h_2, v_1 and v_2 shown in Figure 4.1. Remember that we identify $\mathbb{R}^2$ and $\mathbb{C}$ and that the differential form $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$ on the plane yields a holomorphic one-form η on L_D. The *group of periods* of L_D (more precisely (L_D, η)) is the additive subgroup $\{\int_\pi \eta \mid \pi \in H_1(L_D, \mathbb{Z})\} < \mathbb{C}$. For every closed path π on L_D this gives a complex number or – after identifying $\mathbb{C} = \mathbb{R}^2$ – a vector in $\mathbb{R}^2$, that is a $\mathbb{Z}$-linear combination of the vectors $\binom{1}{0}, \binom{1+w}{0}, \binom{0}{1}$ and $\binom{0}{w}$ corresponding to the paths h_1, h_2, v_1 and v_2, respectively.

Knowing this we can prove the following two propositions:

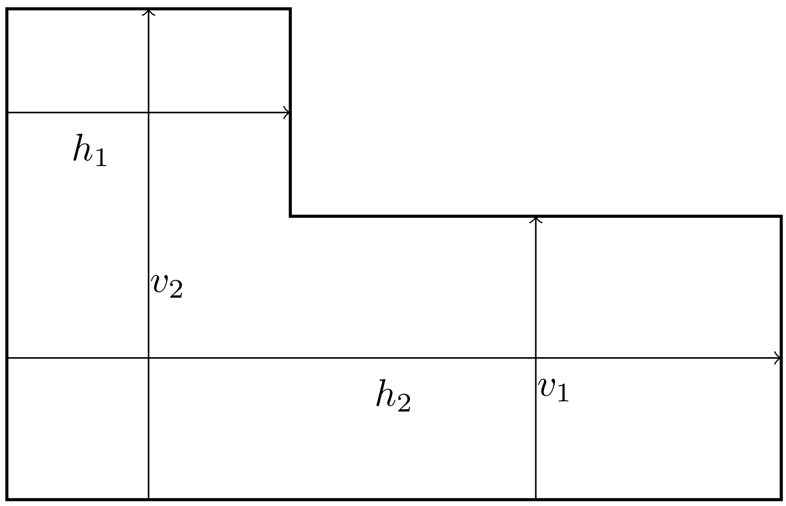


Figure 4.1: A basis of $H_1(L_D, \mathbb{Z})$.

Proposition 4.3. *The Veech group* $\mathrm{SL}(L_D)$ *is contained in* $\mathrm{SL}_2(\mathbb{Z}[w])$.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of the Veech group $\mathrm{SL}(L_D)$. Thus $\gamma \circ L_D = L_D$ and in particular the group of periods of $\gamma \circ L_D$ equals the set of periods of L_D. The image of a period is again a period; hence $\gamma \cdot \binom{1}{0} = \binom{a}{c}$ as well as $\gamma \cdot \binom{0}{1} = \binom{b}{d}$ can be written as a $\mathbb{Z}$-linear combination of $\binom{1}{0}, \binom{1+w}{0}, \binom{0}{1}$ and $\binom{0}{w}$. Since all these vectors have their components in $\mathbb{Z}[w]$, also a, b, c and d have to be in $\mathbb{Z}[w]$. $\square$

Proposition 4.4. *For all* $\gamma \in \mathrm{SL}(L_D)$ *and all* $Q = (x; y) \in L_D$ *the translation part* $t_{\gamma,Q}$ *added to* $\gamma \cdot (x; y)$ *to describe the point* $\gamma \circ Q$ *in the L-coordinates (see Chapter 3) has both components in* $\mathbb{Z}[w]$.

Proof. To understand the action of $\gamma \in \mathrm{SL}(L_D)$, we want to describe the coordinates of $\gamma \circ Q$ in the coordinates from the L-shaped polygon (as in Chapter 3). We get the coordinates $\bigl(x; y\bigr)$ of Q by integrating a path π starting at the singularity and ending in Q, which does not leave the L-polygon, against η. For the image $\gamma \circ Q$ we do the same: choose a path starting in the singularity ending in $\gamma \circ Q$, which does not leave the L. This path differs from $\gamma \circ \pi$ by some element $h \in H_1(L_D, \mathbb{Z})$. Hence the coordinates of $\gamma \circ Q$ are $\gamma \cdot \bigl(x; y\bigr) + t_{\gamma,Q}$ with $t_{\gamma,Q} = -\int_h \eta$ a period.

But as seen in the proof of the last proposition all periods are $\mathbb{Z}$-linear combinations of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1+w\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\w\end{smallmatrix}\right)$ and thus in $\mathbb{Z}[w]^2$.

This completes the proof, but we want to give another point of view to understand which role the periods play here: As mentioned in Chapter 2, for γ to be in the Veech group means that there is a way to "cut & glue" the polygon $\gamma \cdot L$ and get back the polygon L with the correct side identifications. The periods corresponding to the basis of the homology (Figure 4.1) are exactly the translation vectors of the side identifications of the original L-polygon. The idea of the proof is that all translations needed to glue the polygon after cutting are γ-images of these four vectors. Since γ and these vectors have all of their components in $\mathbb{Z}[w]$, so does the translation part $t_{\gamma,Q}$ for every point. $\square$

These two propositions together prove:

Theorem C. *Given a connection point* $P = \bigl(x_r + x_i w; y_r + y_i w\bigr) \in L_D$ *with reduced fractions* $x_r, x_i, y_r, y_i \in \mathbb{Q}$*, set* $N(P)$ *to be the least common denominator of* x_r, x_i, y_r*, and* y_i*. All points* Q *in the* $\mathrm{SL}(L_D)$*-orbit of* P *also have both coordinates in* $\mathbb{Q}(w)$*. Let* $N(Q)$ *be the least common denominator of the four reduced fractions describing* Q*. Then*

$$N(Q) = N(P).$$

In particular, there are infinitely many distinct orbits of connection points.

Proof. By the two last propositions, $\gamma \in \mathrm{SL}(L_D)$ maps P to $\gamma \circ P = \gamma \cdot P + t_{\gamma,P}$ with all entries of γ and of $t_{\gamma,P}$ in $\mathbb{Z}[w]$. Hence also $\gamma \circ P$ has both components in $\mathbb{Q}[w]$ and the least common denominator will not increase. But it will also not decrease; otherwise $\gamma^{-1} \in \mathrm{SL}(L_D)$ would increase it. $\square$

4.2 The Special Case $D = 8$

In the special case $D = 8$ (and thus $w = \sqrt{2}$) the group generated by A and B – the parabolic elements fixing the vertical or horizontal direction, respectively – is conjugate to a subgroup of the Hecke triangle group H_8 of index 2. Let H be the group conjugate to H_8 containing $\langle A, B\rangle$. An element of $H - \langle A, B\rangle$ is $R = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, the rotation by $-\frac{\pi}{2}$. But this element is not in the Veech group $\mathrm{SL}(L_8)$. If it was, the rotated L-shaped polygon would have the same Veech group as the original, but $\mathrm{SL}(L_8)$ contains the element $B = \left(\begin{smallmatrix}1&1+w\\0&1\end{smallmatrix}\right)$, whereas $\mathrm{SL}(R \circ L_8)$ contains $RAR^{-1} = \left(\begin{smallmatrix}1&w\\0&1\end{smallmatrix}\right)$. These two elements generate the group $\left(\begin{smallmatrix}1&\mathbb{Z}[\sqrt{2}]\\0&1\end{smallmatrix}\right)$, which is not discrete in $\mathrm{SL}_2(\mathbb{R})$. Moreover, the Hecke triangle groups are maximal Fuchsian groups, i.e. after adding any element of $\mathrm{SL}_2(\mathbb{R})$ the resulting group would no longer be discrete. This shows that $\mathrm{SL}(L_8)$ is generated by A and B.

Recall that the action of A and B can be described as follows:
$A^{\pm 1} \circ \left(x_r + x_i\sqrt{2}; y_r + y_i\sqrt{2}\right) = \left(x_r + x_i\sqrt{2}; y'_r + y'_i\sqrt{2}\right)$ with

$$y'_r = y_r \pm 2x_i \mp t_{A,r} \text{ with } t_{A,r} \in \{0, 1, 2\}$$
$$y'_i = y_i \pm x_r \mp t_{A,i} \text{ with } t_{A,i} \in \{0, 1\}$$

$B^{\pm 1} \circ \left(x_r + x_i\sqrt{2}; y_r + y_i\sqrt{2}\right) = \left(x'_r + x'_i\sqrt{2}; y_r + y_i\sqrt{2}\right)$ with

$$x'_r = x_r \pm y_r \pm 2y_i \mp t_{B,r} \text{ with } t_{B,r} \in \{0, 1, 2\}$$
$$x'_i = x_i \pm y_r \pm y_i \mp t_{B,i} \text{ with } t_{B,i} \in \{0, 1\}$$

As seen in the previous section, there are infinitely many orbits of connection points under the action of SL(L_8), because all points $\left(x_r + x_i\sqrt{2}; y_r + y_i\sqrt{2}\right)$ in the same orbit share the least common denominator of the four components $x_r, x_i, y_r, y_i \in \mathbb{Q}$ of any point – denoted by N. But even for a fixed N there is not always just one orbit. To find a lower bound on the number of orbits with fixed N, consider the following equivalence relation on the points $P = \left(x_r + x_i\sqrt{2}; y_r + y_i\sqrt{2}\right)$ with the four components x_r, x_i, y_r and y_i reduced fractions with least common denominator N: for all such points expand the four components to fractions with N as denominator. Then two points are equivalent, if and only if the two vectors consisting of the numerators modulo N are the same. This yields the set $V = \{[a, b, c, d] \in \mathbb{Z}/_{N\mathbb{Z}} \mid \gcd(a, b, c, d, N) = 1\}$ as quotient. Since all the $t_{A/B,r/i}$ in the description of the action of A and B are integers, adding them corresponds to adding multiples of N to the numerators of the expanded fractions and the action descends to an action on V. Let G_N be the graph describing this action, i.e. the following graph:

- The vertex set is $V = \{[a, b, c, d] \in \mathbb{Z}/_{N\mathbb{Z}} \mid \gcd(a, b, c, d, N) = 1\}$.
- The edges E are labeled with A or B and go from $v \in V$ to $v' \in V$, if and only if for any point P represented by v the point $A \circ P$ or $B \circ P$, respectively, is represented by v'.

Obviously, the number of connected components $C(N)$ of this graph G_N is a lower bound on the number of orbits of the action of SL(L_8) for fixed N, since SL(L_8) $= \langle A, B\rangle$. The graph G_2 is illustrated in Figure 4.2, the number of connected components for small values N is listed in Table 4.1.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$C(N)$	1	5	1	8	1	5	3	8	1	5	1	8	1	15
N	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$C(N)$	1	8	3	5	1	8	3	5	3	8	1	5	1	24

Table 4.1: The number of connected components $C(N)$ of the graph G_N.

Conjecture 4.5. *The values of $C(N)$ for small N shown in Table 4.1 indicate that the function $C : N \mapsto C(N)$ is weakly multiplicative, i.e. $C(NM) = C(N)C(M)$ for all co-prime natural numbers N and M. Furthermore, we conjecture that, if $N = p^k$ is a power of an odd prime, $C(N)$ equals $C(p)$.*

$$
\begin{tikzcd}
{[1,0,0,0]} \arrow[loop above, "B"] \arrow[r, leftrightarrow, "\mathrm{A}"] & {[1,0,0,1]} \arrow[r, leftrightarrow, "\mathrm{B}"] & {[1,1,0,1]} \arrow[r, leftrightarrow, "\mathrm{A}"] & {[1,1,0,0]} \arrow[loop above, "B"] \\
{[0,0,1,0]} \arrow[loop above, "A"] \arrow[r, leftrightarrow, "\mathrm{B}"] & {[1,1,1,0]} \arrow[r, leftrightarrow, "\mathrm{A}"] & {[1,1,1,1]} \arrow[r, leftrightarrow, "\mathrm{B}"] & {[0,1,1,1]} \arrow[loop above, "A"] \\
{[0,1,1,0]} \arrow[loop above, "A"] \arrow[r, leftrightarrow, "\mathrm{B}"] & {[1,0,1,0]} \arrow[r, leftrightarrow, "\mathrm{A}"] & {[1,0,1,1]} \arrow[r, leftrightarrow, "\mathrm{B}"] & {[0,0,1,1]} \arrow[loop above, "A"] \\
{[0,1,0,1]} \arrow[loop above, "A"] \arrow[r, leftrightarrow, "\mathrm{B}"] & {[0,0,0,1]} \arrow[loop above, "A"] & {[0,1,0,0]} \arrow[loop above, "A"] \arrow[loop below, "B"] &
\end{tikzcd}
$$

Figure 4.2: The graph G_2 with 5 connected components.

Remark 4.6. The action on V can be described as follows:

$$
\begin{aligned}
A \circ [a,b,c,d] &= [a, b, c + 2 \cdot b, d + a] \mod N \\
B \circ [a,b,c,d] &= [a + c + 2 \cdot d, b + c + d, c, d] \mod N
\end{aligned}
$$

So the edges can be computed very easy, but notice that the vertex set V grows quite fast for growing N: it has approximately N^4 elements.

On the other hand the number of orbits for every fixed N is finite:

Theorem D. *Let $D = 8$ and $w = \sqrt{2}$ and fix $N \in \mathbb{N}$. The set of all non-periodic* points P of the form $P = \big(x_r + x_i w; y_r + y_i w\big)$ with x_r, x_i, y_r, y_i reduced fractions with least common denominator N decomposes into a **finite** number of orbits under the action of $\langle A, B\rangle = \mathrm{SL}(L_8)$.*

Proof. Consider the following set of points of L_8:

$$S = \{\big(x_r + x_i w; y_r + y_i w\big) \in L_8 \mid |x_i|, |y_i| \leq 35 + 24w\}.$$

Since N is fixed, there are only finitely many possible values of x_i and y_i. Moreover, with $0 \leq x = x_r + x_i w < 1 + w$ and $0 \leq y = y_r + y_i w < w$, also the number of possible values of x_r and y_r is bounded. Thus the set S is finite. The idea of the proof is to give an algorithm which finds a word in $A^{\pm}$ and $B^{\pm}$, that connects an arbitrary point P with least common denominator N to a point in S. Then $|S|$ is an upper bound on the number of orbits for fixed N.

In Algorithm 1 we describe how we find a word W in $A^{\pm}$ and $B^{\pm}$ with $W \circ P$ is an element of S, when a non-periodic point $P\big(x_r + x_i w; y_r + y_i w\big) \in L_8$ is given as input.

*this excludes exactly the 6 Weierstraß points

Algorithm 1: Finding a word W mapping P to an element of S.

Input: A non-periodic point $P = \big(x_r + x_i w; y_r + y_i w\big)$ with x_r, x_i, y_r, y_i reduced fractions with least common denominator N
Output: A word W in $A^{\pm}$ and $B^{\pm}$ with $W \circ P \in S$
$W \leftarrow$ empty word;
while $P \notin S$ **do**
  if P *is periodic under* B*;* // Case 1
    then $P \leftarrow A^{-1}B^{-1}A \circ P$, $W \leftarrow \text{concat}(A^{-1}B^{-1}A, W)$;
  else if P *is periodic under* A*;* // Case 2
    then $P \leftarrow B^{-1}A^{-1}B \circ P$, $W \leftarrow \text{concat}(B^{-1}A^{-1}B, W)$;
  else if $|x_i| < |y_i|$ *;* // Case 3
    then **if** $x < 1$ **then** $k \leftarrow \lceil \frac{1}{|x_i|w} \rceil$ **else** $k \leftarrow 1$;
    if $|y_{A^k,i}| \leq |y_{A^{-k},i}|$ **then** $P \leftarrow A^k \circ P$, $W \leftarrow \text{concat}(A^k, W)$;
    else $P \leftarrow A^{-k} \circ P$, $W \leftarrow \text{concat}(A^{-k}, W)$;
  else if $|x_i| \geq |y_i|$ *;* // Case 4
    then **if** $y < 1$ **then** $l \leftarrow \lceil \frac{1}{|y_i|(w-1)} \rceil$ **else** $l \leftarrow 1$;
    if $|x_{B^l,i}| \leq |x_{B^{-l},i}|$ **then** $P \leftarrow B^l \circ P$, $W \leftarrow \text{concat}(B^l, W)$;
    else $P \leftarrow B^{-l} \circ P$, $W \leftarrow \text{concat}(B^{-l}, W)$;
end
return W

Obviously, the conditions $|x_i| < |y_i|$ and $|x_i| \geq |y_i|$ of case 3 and 4, respectively, guarantee that all points are covered by the algorithm. To see that the algorithm terminates and thus yields a word W with $W \circ P \in S$ we consider the values $|x_i|$, $|y_i|$ and $m := \max(|x_i|, |y_i|)$ and will prove that in all cases – with one exception – m is reduced. The exception is in case 4, if $|x_i| = |y_i|$: since B does not change y, after applying $B^{\pm l}$ the value of m will still be $|y_i|$, but then, in the next step P will be in case 3 or 2 and thus m will be reduced. Hence after finitely many steps we get a point in the orbit of P, which is in S.

In the following we will often use these two inequalities that hold for every point in L_8:

$$0 \leq x_r + x_i w < 1 + w \tag{4.1}$$
$$0 \leq y_r + y_i w < w \tag{4.2}$$

Case 1: Let us begin with the case that $P \notin S$ *is periodic under* B. As seen in Section 3.2, in this case for the irrational part y_i the inequalities $0 \leq y_i < 1$ hold. Since P is not in S, the absolute value of x_i is at least $35 + 24w$ and $m = |x_i|$.

The algorithm tells us to compute $A^{-1}B^{-1}A \circ P$. We will do this step by step, beginning with $A \circ P$:

$$A \circ P = \big(x; y_r + 2x_i - t_1 + (y_i + x_r - t_2)w\big) \text{ with } t_1 \in \{0,1,2\} \text{ and } t_2 \in \{0,1\}$$

The y-coordinate of $B^{-1}A \circ P$ is the same as for $A \circ P$ and the x-coordinate has

$$\text{rational part } x'_r = -x_r - 2x_i - y_r - 2y_i + t_1 + 2t_2 + t_3 \text{ with } t_3 \in \{0,1,2\} \text{ and}$$
$$\text{irrational part } x'_i = -x_r - x_i - y_r - y_i + t_1 + t_2 + t_4 \text{ with } t_4 \in \{0,1\}$$

The last step – applying B^{-1} – does not change the x-coordinate, but the y-coordinate: If we write the point $A^{-1}B^{-1}A \circ P$ in the form $\left(x'_r + x'_i w; y'_r + y'_i w\right)$, we get x'_r and x'_i as above and for y:

$$y'_r = 2x_r + 4x_i + 3y_r + 2y_i - 3t_1 - 2t_2 - 2t_4 + t_5 \text{ with } t_5 \in \{0,1,2\} \text{ and}$$
$$y'_i = 2x_r + 2x_i + y_r + 3y_i - t_1 - 3t_2 - t_3 + t_6 \text{ with } t_6 \in \{0,1\}$$

We have to show that $m' := \max(|x'_i|, |y'_i|) < m = \max(|x_i|, |y_i|) = |x_i|$ and we do this by showing:

a) if $x_i > 35 + 24w$, then $|x'_i| < |x_i|$.

b) if $x_i < -(35 + 24w)$, then $|x'_i| < |x_i|$.

c) if $x_i > 35 + 24w$, then $|y'_i| < |x_i|$.

d) if $x_i < -(35 + 24w)$, then $|y'_i| < |x_i|$.

Note that since $|x_i| > 35 + 24w$ the sign of $x_r + x_i$ is $-\operatorname{sgn}(x_i)$ and that this implies $|x_r + x_i| = \operatorname{sgn}(x_i)(x_r + x_i)$.

a) $x_i > 35 + 24w > \frac{4}{2-w}$: This implies $x_i > wx_i - x_i + 4$ and with Equation 4.1

$$x_i > -x_r - x_i + 4.$$

Furthermore we know that the possible values of $-y_r - y_i + t_1 + t_2 + t_4$ lie between 0 and 4. Hence we can estimate

$$|x_i| = x_i > -x_r - x_i + 4 \geq |-x_r - x_i| + |-y_r - y_i + t_1 + t_2 + t_4| \geq |x'_i|.$$

b) $x_i < -(35 + 24w) < -\frac{5+w}{2-w}$: This inequality together with Equation 4.1 implies

$$-x_i > -x_i w + 1 + w + x_i + 4 > x_r + x_i + 4$$

and

$$|x_i| = -x_i > |-x_r - x_i| + |-y_r - y_i + t_1 + t_2 + t_4| \geq |x'_i|.$$

c) $x_i > 35 + 24w > \frac{7}{3-2w}$: Together with Equation 4.1 this inequality implies

$$x_i > 2x_i w - 2x_i + 7 \geq -2x_r - 2x_i + 7.$$

Since the value of $y_r + 3y_i - t_1 - 3t_2 - t_3 + t_6$ is between -7 and 4 we can estimate

$$|x_i| = x_i > |2x_r + 2x_i| + |y_r + 3y_i - t_1 - 3t_2 - t_3 + t_6| \geq |y'_i|.$$

d) $x_i < -(35 + 24w) = -\frac{9+2w}{3-2w}$: Together with Equation 4.1 this implies

$$-x_i > 2 + 2w - 2x_i w + 2x_i + 7 > 2x_r + 2x_i + 7$$

and

$$|x_i| = -x_i > |2x_r + 2x_i| + |y_r + 3y_i - t_1 - 3t_2 - t_3 + t_6| \geq |y'_i|.$$

This completes case 1 and we continue with

Case 2: Now $P \notin S$ *is periodic under* A and we have to compare P with $B^{-1}A^{-1}B \circ P$. The latter is of the form

$$\begin{aligned}
x_r' &= 3x_r + 2x_i + 4y_r + 6y_i - 3t_1 - 2t_2 - t_3 - 2t_4 + t_5 \\
x_i' &= x_r + 3x_i + 3y_r + 4y_i - t_1 - 3t_2 - t_3 - t_4 + t_6 \\
y_r' &= -2x_i - y_r - 2y_i + 2t_2 + t_3 \\
y_i' &= -x_r - y_r - y_i + t_1 + t_4 \\
&\text{with } t_1, t_3, t_5 \in \{0,1,2\} \text{ and } t_2, t_4, t_6 \in \{0,1\}
\end{aligned}$$

Since in this case all computations are very similar to the previous case – just use Equation 4.2 instead of Equation 4.1 – we will just give the bounds, that guarantee $m' < m$. Note that points periodic under A have $|x_i| < 1$ and thus $m = |y_i|$. Furthermore we use $\operatorname{sgn}(3y_r + 4y_i) = -\operatorname{sgn}(y_i) = \operatorname{sgn}(y_r + y_i)$.

a) if $y_i > 35 + 24w > \frac{8}{5-3w}$, then $|x_i'| < |y_i|$.

b) if $y_i < -(35 + 24w) < -\frac{8+3w}{5-3w}$, then $|x_i'| < |y_i|$.

c) if $y_i > 35 + 24w > \frac{3}{2-w}$, then $|y_i'| < |y_i|$.

d) if $y_i < -(35 + 24w) < -\frac{3+w}{2-w}$, then $|y_i'| < |y_i|$.

Case 3: Now $|x_i| < |y_i|$ and we want to show that the power of A given by the algorithm reduces $|y_i|$ and thus reduces m.

Remember that the difference between the irrational parts of the y-coordinate of P and of $A^k \circ P$ is denoted by $\Delta_{A^k}(P)$ and by Equation 3.4 this difference is

$$\Delta_{A^k}(P) = \begin{cases} -kx_i w - r \text{ for an } r \in (-1,1) & \text{if } x \leq 1 \\ k(x_r - 1) & \text{if } x > 1. \end{cases}$$

- If $x \leq 1$ and $k = \lceil \frac{1}{|x_i|w} \rceil = \frac{1}{|x_i|w} + s$ for an $s \in (0,1)$ we get

  $$\Delta_{A^k}(P) = -\operatorname{sgn}(x_i) - sx_i w - r_+ \quad \text{and} \quad \Delta_{A^{-k}}(P) = \operatorname{sgn}(x_i) + sx_i w - r_-.$$

  Since $|r| < 1$, the signs of $\Delta_{A^k}(P)$ and $\Delta_{A^{-k}}(P)$ are different: one is positive, the other is negative. Moreover, $|\Delta_{A^{\pm k}}(P)| \leq 1 + s|x_i|w + 1 < 2|y_i|$. Thus either A^k or A^{-k} reduces the absolute value $|y_i|$ and m.

- If $x > 1$ and $k = 1$, we have $\Delta_{A^{\pm k}}(P) = \pm(x_r - 1)$. Since P is not periodic under the action of A, the rational part $x_r \neq 1$. Moreover,

  $$|\Delta_{A^{\pm k}}(P)| \leq |x_r| + 1 < 1 + w + w|x_i| + 1 < 2|y_i|.$$

  Thus either A or A^{-1} reduces the absolute value $|y_i|$ and m.

Case 4: Now $|y_i| \leq |x_i|$ and we have to show that the power of B given by the algorithm reduces $|x_i|$. Remember Equation 3.2:

$$\Delta_{B^l}(Q) = \begin{cases} ly_i(1-w) - r \text{ for an } r \in (-1,1) & \text{if } y \leq 1 \\ l(y_r + y_i - 1) & \text{if } y > 1. \end{cases}$$

- If $y \leq 1$ and $l = \lceil \frac{1}{|y_i|(w-1)} \rceil = \frac{1}{|y_i|(w-1)} + s$ for an $s \in (0,1)$ we get

  $\Delta_{B^l}(P) = -\operatorname{sgn}(y_i) - sy_i(w-1) - r_+$ and $\Delta_{B^{-l}}(P) = \operatorname{sgn}(y_i) + sy_i(w-1) - r_-$.

  Since $|r| < 1$, the signs of $\Delta_{B^l}(P)$ and $\Delta_{B^{-l}}(P)$ are different. Moreover,

  $$|\Delta_{B^{\pm l}}(P)| \leq 1 + s|y_i|(w-1) + 1 < 2|y_i| \leq 2|x_i|.$$

  Thus either B^l or B^{-l} reduces $|x_i|$.

- If $y > 1$ and $l = 1$, we have $\Delta_{B^{\pm l}}(P) = \pm(y_r + y_i - 1)$. Since P is not periodic under B, the change Δ is not 0. Moreover, y_r and y_i have different signs and $|y_i| < |y_r|$ unless both $|y_r|$ and $|y_i|$ are much smaller than $35+24w$. Hence $|\Delta_{B^{\pm l}}(P)| < 2|x_i|$. Thus either B or B^{-1} reduces $|x_i|$.

This completes the proof, since in all cases $m = \max(|x_i|, |y_i|)$ is reduced: in the case $|x_i| = |y_i|$ after two steps, otherwise in each step. This finally leads to a point with $|x_i|, |y_i| < 35 + 24w$, which is in the finite set S. □

Remark 4.7. Points in S can have approximately $((35+24w)\cdot 2N)^2$ different values for x_i and y_i and for each such pair there might be up to $((1+w)\cdot N)\cdot(w\cdot N)$ possible values for x_r and y_r. Thus the upper bound on the number of orbits with fixed N given by the theorem is about $(35+24w)^2(w+2)N^4 = (8114+5737w)N^4 \approx 16227N^4$. This is a very bad upper bound, supposably the correct number of orbits is very close (or equal) to the lower bound given by the number of connected components of the graph described in the beginning of this section. In particular for $N = 1$ with a little bit more work one can show that, indeed, all non-periodic points are in the same orbit. This means that for all non-periodic points $P = \big(x_r + x_i w; y_r + y_i w\big)$ with $x_r, x_i, y_r, y_i \in \mathbb{Z}$ the groups SL($L_8; P$) are conjugate and have the same critical exponent.

Remark 4.8 (Other discriminants D)**.** As mentioned in Theorem 2.18 there exist L-shaped Veech surfaces also for $D \equiv 1 \mod 4$. For $D = 5$ set $w_5 := \frac{1+\sqrt{5}}{2}$. One easily checks that $A_5 = \left(\begin{smallmatrix} 1 & 0 \\ w_5 & 1 \end{smallmatrix}\right)$ and $B_5 = \left(\begin{smallmatrix} 1 & w_5 \\ 0 & 1 \end{smallmatrix}\right)$ are the vertical respectively horizontal primitive parabolic elements of the Veech group SL($L_{5,-1}$). It is known that also in this case the Veech group is generated by these two elements. All methods of this section should apply also to the orbits of SL($L_{5,-1}$) on $L_{5,-1}$.

For bigger D however, one should not expect too much: The bigger D gets, the smaller is $\langle A_D, B_D \rangle$ compared to SL(L_D). Thus the method to find a lower bound on the number of orbits for fixed N will not work, because even though we can define the graph G_N as described at the beginning of this chapter one would need to know the action of **all** generators of SL(L_D). Also Algorithm 1 will not work as stated: For big D there exist (many) points, such that no non-trivial element of $\langle A, B \rangle$ – which by the ping-pong lemma is a free subgroup for D big enough – decreases the absolute values of the irrational parts of the coordinates.

5 Critical Exponent and Graph-Periodic Manifolds

The *critical exponent* of a Fuchsian group serves as a way to describe the dynamics of the group's action at the boundary of the upper half plane or more precisely at the limit set. Roughly spoken it measures how fast the orbit of a base point tends to the boundary. For instance in [BJ97] Bishop and Jones show that the critical exponent of non-elementary Fuchsian groups equals the Hausdorff dimension of the conical limit set – the set of points $x \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ for which there exists a sequence of group elements transporting the base point to x within a bounded distance to a geodesic ending in x.

In the first section of this chapter we define the critical exponent of Fuchsian groups and collect some known results bounding it. Further background on the critical exponent viewed from different perspectives can be found in [Nic89].

The aim of the second section is to build a bridge from specific Fuchsian groups – subgroups of lattices – (see Chapter 2) and their critical exponent to amenability of Schreier graphs (described in Chapter 1). We use a concept introduced by Tapie ([Tap10]): *graph-periodic manifolds* M over a cell C, i.e. manifolds consisting of isometric copies of another manifold glued together according to the structure given by a graph. Following [RT13] we compare the bottom of the spectrum of C and of $M = \mathbb{H}/\mathrm{SL}(L_D; P)$ and in doing so we prove that the critical exponent of $\mathrm{SL}(L_D; P) < \mathrm{SL}(L_D)$ is strictly smaller than 1, if the Schreier graph $G_{\mathrm{SL}(L_D),\mathrm{SL}(L_D;P),S}$ is non-amenable for a finite set S generating $\mathrm{SL}(L_D)$.

5.1 The Critical Exponent

Let us first define the term critical exponent and collect some basic properties. Therefore let $\rho_{\mathbb{H}}$ be the hyperbolic metric on the upper half plane $\mathbb{H}$.

Definition 5.1. Let Γ be a Fuchsian group and $* \in \mathbb{H}$. The *Poincaré series* to the exponent $a \in \mathbb{R}$ and the base point $*$ is the series $\sum_{\gamma\in\Gamma} e^{-a\rho_{\mathbb{H}}(*,\gamma(*))}$. The infimum of exponents a, for which the Poincaré series converges is called the *critical exponent* $\delta(\Gamma)$:

$$\delta(\Gamma) := \inf\big\{a \in \mathbb{R} \mid \sum_{\gamma\in\Gamma} e^{-a\rho_{\mathbb{H}}(*,\gamma(*))} < \infty\big\}$$

Remark 5.2. Because of the triangle inequality the convergence of the Poincaré series and thus also the critical exponent are independent of the choice of base point. Usually we will set $* = i$.

Our first observations concern the critical exponent of commensurable groups. Recall that two subgroups Γ and Γ' of $\mathrm{SL}_2(\mathbb{R})$ are called commensurable if there exist subgroups $\Pi < \Gamma$ and $\Pi' < \Gamma'$ of finite index each, that are conjugate in $\mathrm{SL}_2(\mathbb{R})$.

Proposition 5.3. *If Γ and Γ' are conjugate by $g \in \mathrm{GL}_2(\mathbb{R})$, then $\delta(\Gamma) = \delta(\Gamma')$.*

Proof. Let $\Gamma' = g\Gamma g^{-1}$ and $\alpha \geq \delta(\Gamma')$. Then the series $\sum_{\gamma' \in \Gamma'} e^{-\alpha\rho_{\mathbb{H}}(i,\gamma'(i))}$ converges absolute and equals $\sum_{\gamma\in\Gamma} e^{-\alpha\rho_{\mathbb{H}}(i,g\gamma g^{-1}(i))}$. Since g and g^{-1} act as isometries, this series equals $\sum_{\gamma\in\Gamma} e^{-\alpha\rho_{\mathbb{H}}(g^{-1}(i),\gamma g^{-1}(i))}$, which – because of the independence of the choice of the base point – converges if and only if $\sum_{\gamma\in\Gamma} e^{-\alpha\rho_{\mathbb{H}}(i,\gamma(i))}$ converges. Hence $\delta(\Gamma) \leq \delta(\Gamma')$. A symmetric argument yields $\delta(\Gamma) \geq \delta(\Gamma')$ and thus $\delta(\Gamma) = \delta(\Gamma')$ □

Proposition 5.4. *If $\Pi < \Gamma$ is a subgroup of finite index $[\Gamma : \Pi] = n < \infty$, then $\delta(\Pi) = \delta(\Gamma)$.*

Proof. The inequality $\delta(\Pi) \leq \delta(\Gamma)$ is clear by definition.

To show $\delta(\Pi) \geq \delta(\Gamma)$, assume that $\delta(\Pi) < \delta(\Gamma)$ and fix a $\delta \in (\delta(\Pi), \delta(\Gamma))$ as well as a transversal $\{\gamma_1, \ldots, \gamma_n\}$ for the left cosets of Γ with respect to Π. Since $\delta > \delta(\Pi)$, the series $\sum_{\pi\in\Pi} e^{-\delta\rho_{\mathbb{H}}(i,\pi(i))}$ converges. But then also the series $\sum_{\pi\in\gamma_i^{-1}\Pi} e^{-\delta\rho_{\mathbb{H}}(i,\gamma_i\pi(i))}$ converges, because $\gamma_i\pi$ runs through Π as π runs through $\gamma_i^{-1}\Pi$. The index of Π in Γ is finite, thus also $\sum_{i=1}^{n}\sum_{\pi\in\gamma_i^{-1}\Pi} e^{-\delta\rho_{\mathbb{H}}(i,\gamma_i\pi(i))}$ converges and as all summands are positive, it converges absolutely. Hence we can rearrange the summands and (remember that $\Gamma = \bigsqcup_{i=1}^{n} \gamma_i\Pi$) get the convergence of $\sum_{\gamma\in\Gamma} e^{-\delta\rho_{\mathbb{H}}(i,\gamma(i))}$, in contradiction to $\delta < \delta(\Gamma)$. □

Corollary 5.5. *Commensurable Fuchsian groups have the same critical exponent.*

In general it is quite hard to find the exact critical exponent of a group. But there are some general bounds that can be found in an elementary way:

Proposition 5.6. *Let Γ be a Fuchsian group. Then the following holds:*

1. *If $|\Gamma| = \infty$, then $\delta(\Gamma) \geq 0$.*
2. *A cyclic group generated by a hyperbolic element has critical exponent 0.*
3. *Let $\Pi < \Gamma$ be a subgroup, then $\delta(\Pi) \leq \delta(\Gamma)$.*
4. *If Γ contains a parabolic element, then $\delta(\Gamma) \geq \frac{1}{2}$.*

Proof. 1. If Γ is an infinite group, then the Poincaré series with exponent $a = 0$ obviously diverges.

2. In Proposition 5.3 we showed that conjugate groups have the same critical exponent. Thus we can assume that $\Gamma = \langle\gamma\rangle$ for some $\gamma = \left(\begin{smallmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{smallmatrix}\right)$ with $\lambda > 1$. The image of the base point i under γ^n is $\lambda^n i$. The hyperbolic distance from i to $\gamma^n \circ i$ is $\ln \lambda^{|n|}$ and thus the Poincaré series to the base point i becomes

$$\sum_{\gamma\in\Gamma} e^{-a\rho_{\mathbb{H}}(i,\gamma(i))} = \sum_{n\in\mathbb{Z}} \lambda^{-|n|a} = 1 + 2 \cdot \sum_{n\in\mathbb{N}} \left(\frac{1}{\lambda^a}\right)^n$$

This is a geometric series and converges if and only if $\frac{1}{\lambda^a} < 1$, which is the case if and only if $a > 0$.

3. Clear by definition of δ.

4. Consider the cyclic group Π generated by the parabolic element. After conjugation we can assume that $\Pi = \langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \rangle$ and compute $\delta(\Pi) = \frac{1}{2}$. Now the statement follows from 3.

□

Further bounds on the critical exponent of Fuchsian groups are given in the following four theorems. They are summarized in Table 5.1.

Theorem 5.7 ([Nic89], Theorem 1.6.1)**.** *For all Fuchsian groups* Γ *the critical exponent is at most* 1*.*

Proof. We define the *orbital counting function* as

$$N(r, x, y) := |\{\gamma \in \Gamma \mid \rho_{\mathbb{H}}(x, \gamma(y)) < r\}|.$$

It was already shown in [Hop36] that for all Fuchsian groups there is a constant k_1 such that $N(r, x, y) < k_1 e^r$. Let us consider the partial sums

$$\sum_{\gamma \in \Gamma : \rho_{\mathbb{H}}(i,\gamma(i)) < R} e^{-a\rho_{\mathbb{H}}(i,\gamma(i))} = \int_0^R e^{-at} \mathrm{d}N(t, i, i) = N(R, i, i)e^{-aR} + a \int_0^R N(t, i, i)e^{-at} \mathrm{d}t.$$

Using the above estimate $N(t, i, i) < k_1 e^t$ we obtain the convergence of the Poincaré series for all $a > 1$. □

Theorem 5.8 ([Nic89], Theorem 1.6.3)**.** *If* Γ *is a lattice, then* $\delta(\Gamma) = 1$*.*

Proof. In Theorem 1.5.2 of [Nic89] Nicholls proves that if Γ is a lattice there exists a constant k_2 and a r_0 such that for all $r > r_0$ the orbital counting function is bounded below by

$$N(r, x, y) > k_2 e^r.$$

If we again consider the partial sums as in the last proof, we see that for lattices the Poincaré series diverges for $a = 1$. Together with Theorem 5.8 the proof is completed. □

Theorem 5.9 ([Bea68]/[Pat76a])**.** *If* Γ *is non-elementary and contains a parabolic element, then* $\delta(\Gamma) > \frac{1}{2}$*.*

Proof. For $\gamma \in \Gamma$ we define $f(\gamma) := e^{\rho_{\mathbb{H}}(i,\gamma(i))}$. Moreover, we set $\phi(\Gamma, a) := \sum_{\gamma \in \Gamma - I} f(\gamma)^{-a}$. Thus the Poincaré series of Γ to the exponent a equals $1+\phi(\Gamma, a)$. Finally, for all $a > \delta(\Gamma)$ we define

$$\psi(\Gamma, a) := \frac{\phi(\Gamma, a)}{1 + \phi(\Gamma, a)}.$$

Since Γ is non-elementary, there exists a subgroup $\Gamma_0 < \Gamma$ that is a free product of two cyclic groups Γ_1 and Γ_2. We choose Γ_1 to be generated by a parabolic element and conclude that $\delta(\Gamma_1) = \frac{1}{2}$ (Proposition 5.6).

Now fix a number $a_0 > \delta(\Gamma_2)$ and choose $\varepsilon < \psi(\Gamma_2, a_0)$. For elementary groups and thus in particular for the cyclic group Γ_1 the Poincaré series to its critical exponent diverges: $\phi(\Gamma_1, a) \to \infty$ for $a \to \frac{1}{2} = \delta(\Gamma_1)$ from above. For decreasing a the function ϕ

and because of the monotony of $\psi = \frac{\phi}{1+\phi}$ in ϕ also ψ increase. This implies that there is an a_1 with $a_0 \geq a_1 > \frac{1}{2}$ such that $\psi(\Gamma_1, a_1) > 1 - \varepsilon$. Hence

$$\psi(\Gamma_1, a_1) + \psi(\Gamma_2, a_1) > 1 - \varepsilon + \varepsilon = 1.$$

In the following lemma we show that $\delta(\Gamma_0) \geq a$ for all a with $\psi(\Gamma_1, a) + \psi(\Gamma_2, a) \geq 1$. This finishes the proof by setting $a = a_1$. □

Lemma 5.10. *Let Γ be a Fuchsian group and the free product of Γ_1 and Γ_2. Then*

$$\psi(\Gamma, a) \geq \psi(\Gamma_1, a) + \psi(\Gamma_2, a).$$

If $\psi(\Gamma_1, a) + \psi(\Gamma_2, a) \geq 1$, then the Poincaré series to the exponent a diverges and thus the critical exponent satisfies $\delta(\Gamma) \geq a$.

Proof. We observe that by the triangle inequality the function $f(\gamma) = e^{\rho_{\mathbb{H}}(i, \gamma(i))}$ satisfies

$$f(\gamma\gamma') \leq f(\gamma) f(\gamma').$$

Using this together with the normal form of elements of the free product we obtain

$$\phi(\Gamma_1 * \Gamma_2, a) \geq (1 + \phi(\Gamma_1, a))(1 + \phi(\Gamma_2, a)) \left(\sum_{n \geq 0} (\phi(\Gamma_1, a)\phi(\Gamma_2, a))^n \right) - 1.$$

This expression diverges for $\phi(\Gamma_1, a)\phi(\Gamma_2, a) \geq 1$.

For $\phi(\Gamma_1, a)\phi(\Gamma_2, a) < 1$ we have a geometric series and obtain

$$\phi(\Gamma_1 * \Gamma_2, a) \geq \frac{(1 + \phi(\Gamma_1, a))(1 + \phi(\Gamma_2, a))}{1 - \phi(\Gamma_1, a)\phi(\Gamma_2, a)} - 1.$$

By the definition and the monotony of $\psi(\Gamma, a)$ this yields

$$\psi(\Gamma, a) \geq \psi(\Gamma_1, a) + \psi(\Gamma_2, a)$$

and the condition $\phi(\Gamma_1, a)\phi(\Gamma_2, a) \geq 1$ is equivalent to $\psi(\Gamma_1, a) + \psi(\Gamma_2, a) \geq 1$. □

Since we will not deal with Fuchsian groups of the second kind, we will not prove the following theorem but refer to [Bea68] or [Pat75].

Theorem 5.11 ([Bea68]/[Pat75])**.** *If Γ is finitely generated and of the second kind, then $\delta(\Gamma) < 1$.*

Another very useful result concerns a connection between the critical exponent $\delta(\Pi)$ of a Fuchisan group Π and the smallest eigenvalue $\lambda_0(M)$ of the Laplacian Δ_M on the hyperbolic manifold $M = \mathbb{H}/\Pi$. It was first proven by Patterson in [Pat76b] for geometrically finite Fuchsian groups and generalized by Sullivan in [Sul79] to all discrete groups acting on the hyperbolic space $\mathbb{H}^{d+1}$ by isometries*:

*Note that $\mathbb{H} = \mathbb{H}^2 = \mathbb{H}^{1+1}$.

$\lvert\Gamma\rvert = \infty$	$\delta(\Gamma) \geq 0$
Γ contains a parabolic element	$\delta(\Gamma) \geq \frac{1}{2}$
Γ contains parabolic & is non-elementary	$\delta(\Gamma) > \frac{1}{2}$
Γ is finitely generated & of second kind	$\delta(\Gamma) < 1$
all Fuchsian groups Γ	$\delta(\Gamma) \leq 1$
Γ is a lattice	$\delta(\Gamma) = 1$

Table 5.1: General bounds on the critical exponent of Fuchsian groups.

Theorem 5.12 ([Pat76b]/[Sul79])**.** *Let Π be a Fuchsian group and $M = {}^{\mathbb{H}}/_{\Pi}$. Then the smallest eigenvalue λ_0 of the Laplacian on M is*

$$\lambda_0(M) = \begin{cases} \delta(\Pi)(1-\delta(\Pi)), & \text{if } \delta(\Pi) \geq \frac{1}{2} \\ \frac{1}{4}, & \text{if } \delta(\Pi) \leq \frac{1}{2}. \end{cases}$$

In particular, $\lambda_0(M) > 0$ implies $\delta(\Pi) < 1$. This is why we will take a closer look at the Laplacian on hyperbolic manifolds and the bottom of its spectrum in the next section.

5.2 From Hyperbolic Surfaces to Schreier Graphs

In this section we follow the ideas of Tapie and Roblin ([Tap10], [RT13]). They discuss Riemannian coverings and a generalization – graph-periodic manifolds – in order to obtain a lower bound on the smallest eigenvalue of the manifold's Laplacian. Roughly spoken, a graph-periodic manifold consists of isometric copies of a smaller manifold with boundary, which are glued together according to the structure given by a regular graph. The lower bound is given in Théorème 0.2 of [RT13] for Riemannian coverings and in Théorème 1 of [Tap10] for graph-periodic manifolds. We will introduce the notion of graph-periodic manifolds, explain how our situation fits into this concept and prove a variation of these theorems giving a lower bound for our special case.

Definition 5.13. Let $k \in \mathbb{N}$ be fixed. A *marked cell of valency k* is a set $\{C, H_1, \dots, H_k\}$ where C is a smooth Riemannian manifold with piecewise $\mathcal{C}^1$ boundary and $H_i \subset \partial C$ are pairwise disjoint compact codimension 1 submanifolds which are $\mathcal{C}^1$ with piecewise $\mathcal{C}^1$ boundary (of codimension 2). The H_i are called *transition zones*, C is called the *cell.*

Definition 5.14. Let $G = (V, E)$ be a graph of constant valency k and $\{C, H_1, \dots, H_k\}$ be a marked cell of valency k. A manifold M is called *marked G-periodic over C* if it satisfies the following conditions:

1. For all $v \in G$ there exist submanifolds $C_v \subset M$ with pairwise disjoint interiors such that $M = \bigcup_{v \in V(G)} C_v$. Furthermore there exists an isometry $J_v : C_v \to C$.
2. For all $v, w \in V$ there is an edge $\{v, w\} \in E$ if and only if $J_v(C_v \cap C_w) \subset \partial C$ contains a unique transition zone, which we will denote by H_{vw}.

3. For all edges $\{v, w\} \in E$ the map $J_w \circ J_v^{-1} : J_v(C_v \cap C_w) \to J_w(C_v \cap C_w)$ induces an isometry from H_{vw} to H_{wv}.

Let us now describe how the situation of this thesis fits into the concept of graph-periodic manifolds. Afterwards we will look at a small example using well-known groups:

Given a lattice Fuchsian group Γ and a subgroup Π there are the corresponding quotients $N = {}^{\mathbb{H}}\!/_{\Gamma}$ and $M = {}^{\mathbb{H}}\!/_{\Pi}$. Thus there are three covering maps: $p : M \to N$ and the universal coverings $\pi_\Gamma : \mathbb{H} \to N$ and $\pi_\Pi : \mathbb{H} \to M$. Since Γ is a lattice, there is a fundamental domain $\mathcal{F}_\Gamma \subset \mathbb{H}$ of finite volume and with finitely many sides for the action of Γ on $\mathbb{H}$, such that pairs of sides correspond to a set $S = \{\gamma_1, \ldots, \gamma_n\}$ of elements generating Γ (usually a Dirichlet fundamental domain). We set $C = \mathcal{F}_\Gamma$ and mark C by choosing for any of these pairs a transition zone H_i on one of the sides and the γ_i-image of H_i on the other side. Then C is a marked cell of valency $2 \cdot n$ and M consists of Γ-images of $\pi_\Pi(C)$, one for each coset $\Pi\gamma$. Setting G to be the Schreier graph $G_{\Gamma,\Pi,S}$, we see that M is a G-periodic manifold over C.

Example 5.15. Let Γ be the modular group $\mathrm{SL}_2(\mathbb{Z})$ and Π be the principal congruence subgroup $\Gamma[2]$. The fundamental domains $\mathcal{F}_\Gamma$, $\mathcal{F}_{\Gamma[2]}$ and the Schreier coset graph $G_{\Gamma,\Gamma[2],\{S,T\}}$ for $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are shown in Figure 5.1. The main difference between this example and the situation we are interested in is that in this example the subgroup has finite index; hence the Schreier graph G is finite and thus in particular amenable. Moreover, $N = {}^{\mathbb{H}}\!/_{\Gamma}$ and $M = {}^{\mathbb{H}}\!/_{\Gamma[2]}$ have finite volume and thus the smallest eigenvalues $\lambda_0(N)$ and $\lambda_0(M)$ are 0, since any constant function $\phi_0 : N$ or $M \to \mathbb{R}$ is a valid eigenfunction with eigenvalue 0.

Before we state the main result of this section we collect some facts about the Laplacian Δ_N on a hyperbolic manifold $N = {}^{\mathbb{H}}\!/_{\Gamma}$. For the definitions of divergence $\mathrm{div} f$, gradient ∇f and the Laplacian $\Delta = \mathrm{div}(\nabla f)$ see Chapter 1 of [Cha84].

For a hyperbolic manifold M let $L^2(M)$ be the space of measurable maps $f : M \to \mathbb{R}$ with $\int_M |f|^2 < \infty$. On $L^2(M)$ there is the *inner product* $\langle f, g \rangle_M := \int_M fg$ and the *norm* $||f||_M^2 := \langle f, f \rangle_M$. Furthermore, we define the *Rayleigh quotient* as

$$\mathcal{R}_M(f) = \frac{||\nabla f||_M^2}{||f||_M^2}.$$

The Min-Max-principle implies that the smallest eigenvalue $\lambda_0(M)$ satisfies

$$\lambda_0(M) = \inf_{f \in \mathcal{H}^1(M)} \mathcal{R}_M(f),$$

where $\mathcal{H}^1(M) \subset L^2(M)$ is the Sobolev space of maps $f \in L^2(M)$, whose gradient is a L^2 vector field. Moreover, if $\phi_0 : M \to \mathbb{R}$ is a eigenfunction to the eigenvalue $\lambda_0(M)$, then

$$\lambda_1(M) = \inf\{\mathcal{R}_M(f) \mid f \in \mathcal{H}^1(M), \langle f, \phi_0 \rangle_M = 0\}. \tag{5.1}$$

The lower bound on $\lambda_0({}^{\mathbb{H}}\!/_{\Pi})$ obtained in the main result will depend on the spectral gap $\nu := \lambda_1(C) - \lambda_0(C)$ of the Laplacian on C (with Neumann boundary conditions). Therefore it is important that in our case, where C is a (Dirichlet) fundamental domain of a lattice Γ, the smallest eigenvalue $\lambda_0(C)$ is isolated and thus the spectral gap is strictly positive. This is shown in three steps:

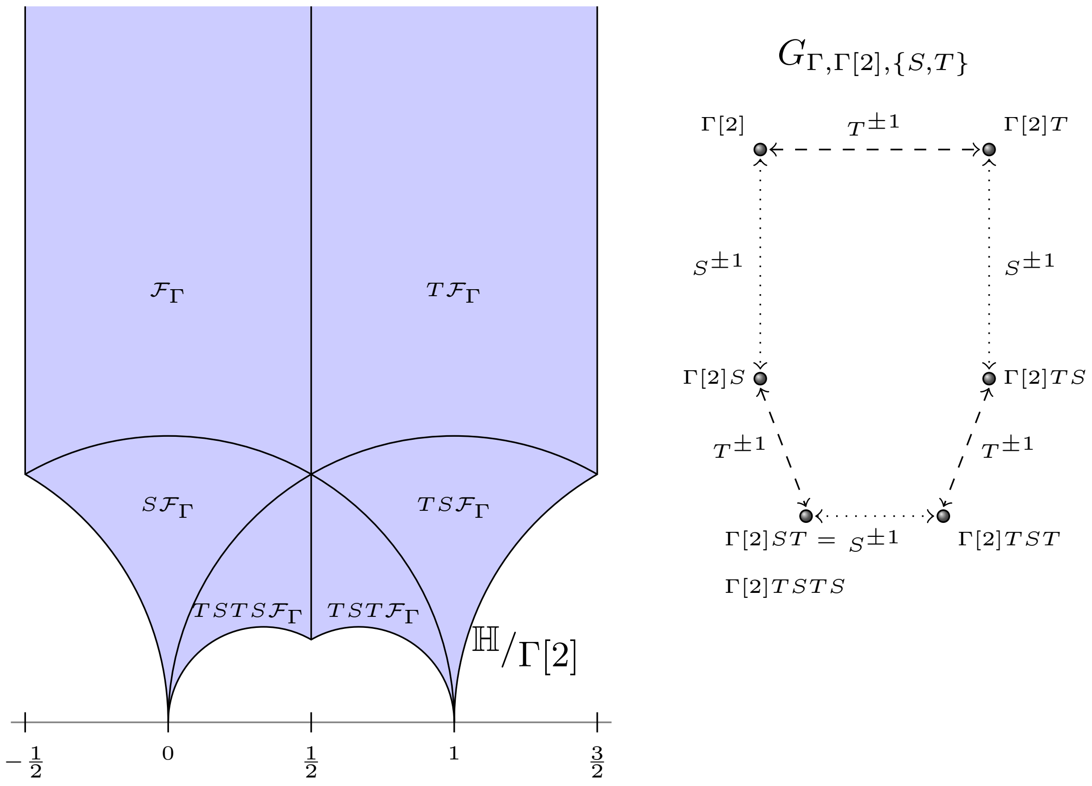


Figure 5.1: $\mathbb{H}/\Gamma[2]$ as G-periodic manifold over a fundamental domain $\mathcal{F}_\Gamma$ of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

- By Lemme 5.2 of [RT13] the eigenvalue $\lambda_0(C)$ is an isolated eigenvalue of multiplicity 1, if $\lambda_0(C) < \lambda_0^{ess}(C)$, where $\lambda_0^{ess}(C)$ is the bottom of the essential spectrum of C, i.e. the infimum of the real numbers for which a so-called Weyl's sequence exist.
- Since C has finite volume, the constant functions are in $\mathcal{H}^1(C)$ and thus $\lambda_0(C) = 0$.
- It holds $\lambda_0^{ess}(C) > 0$:

Lemma 5.16 ([RT13], Lemme 5.4)**.** *Let C be a Dirichlet fundamental domain of a lattice. The bottom of the essential spectrum $\lambda_0^{ess}(C)$ is at least $\frac{1}{4}$.*

Proof. By Lemme 5.3 of [RT13], if there exists a compact subset $K \subset C$ and a function $\phi : C - K \to (0, \infty)$ with gradient $\nabla\phi$ tangent on $\partial C - K$ that satisfies $\Delta\phi \geq \lambda\phi$, then $\lambda_0^{ess}(C) \geq \lambda \in \mathbb{R}$.

Since C is the Dirichlet fundamental domain of a lattice, there exists a compact set K such that $C - K$ is a disjoint union of finitely many cusps. We define such a function ϕ only on a cusp at ∞, since all cusps are conjugate to such a cusp. It has the form $\{(x, y) \in \mathbb{H} \mid a \leq x \leq b \ \wedge y \geq c\}$ for some $a, b, c \in \mathbb{R}$ with $c > 0$. Setting $\phi((x, y)) = y^{\frac{1}{2}}$, one easily sees that ϕ is strictly positive, $\nabla\phi$ is tangent on $\partial C - K$ and that $\Delta\phi = \frac{1}{4}\phi$. Hence $\lambda_0^{ess}(C) \geq \frac{1}{4}$. $\square$

Summarized this yields:

Proposition 5.17. *Let C be a Dirichlet fundamental domain of a lattice. The spectral gap $\nu := \lambda_1(C) - \lambda_0(C) = \lambda_1(C)$ of the Laplacian on C (with Neumann boundary conditions) is strictly positive: $\nu > 0$.*

Let us now state the main result of this section. The proof is basically a special case of the proof of Théorème 4.3 of [RT13].

Proposition 5.18. *Let Γ be a lattice, $\Pi < \Gamma$ a subgroup, C a Dirichlet fundamental domain of Γ and $G = G_{\Gamma,\Pi,S}$ the Schreier graph of Γ with respect to Π and the finite generating set S obtained from C. Then $M = \mathbb{H}/\Pi$ is a marked G-periodic manifold over C as described above. The bottom of the spectrum $\lambda_0(M)$ can be bounded above by*

$$\lambda_0(M) \geq \min\left\{ \frac{\frac{A}{V}\nu}{\nu + \frac{A}{V}\mu_0(G)}\mu_0(G), \nu \right\},$$

where $A > 0$ is a constant depending on the geometry of C in a neighborhood of the transition zones, $\nu > 0$ is the spectral gap of C, V is the (finite) volume of C and $\mu_0(G)$ is the spectral gap of the combinatorial Laplacian on the graph G.

Combining this proposition with Theorem 5.12, Theorem 1.46, and Theorem 1.34 this implies:

Corollary 5.19. *Let Γ be a lattice and $\Pi < \Gamma$ a subgroup. The critical exponent $\delta(\Pi)$ is strictly smaller than 1, if the Schreier graph $G_{\Gamma,\Pi,S}$ is non-amenable for any finite set S generating Γ.*

Proof. [Proposition 5.18] First observe that since Γ is a lattice, C has finite volume V and thus the constant map $\mathbf{1}_C : C \to \{1\} \subset \mathbb{R}$ is an eigenfunction to the eigenvalue $\lambda_0(C) = 0$. We can lift $\mathbf{1}_C$ to the map $\mathbf{1}_M : M \to \{1\}$, but in the following we will just write $\mathbf{1}$ for both maps.

In Equation 5.1 we saw that $\lambda_0(M) = \inf_{f \in \mathcal{H}^1(M)} \mathcal{R}(f)$. Let f_ϵ be a smooth map with compact support satisfying $\mathcal{R}(f_\epsilon) = \frac{||\nabla f_\epsilon||_M}{||f_\epsilon||_M} \leq \lambda_0(M) + \epsilon$.

For every vertex $i \in V(G)$ we write C_i for the image of the cell C corresponding to this vertex and define

- $f_{\epsilon,i} := f_\epsilon|_{C_i}$,
- $b_i := \frac{1}{V}\langle f_{\epsilon,i}, \mathbf{1}\rangle_{C_i} = \frac{1}{V}\int_{C_i} f_{\epsilon,i} \cdot \mathbf{1}$, and
- $g_i := f_{\epsilon,i} - b_i\mathbf{1}$.

One easily checks that g_i and $\mathbf{1}$ are orthogonal (with respect to the inner product on C_i) and hence $\mathcal{R}(g_i) \geq \lambda_1(C) = \nu$. Moreover by bilinearity of the inner product we have $||g_i||_C^2 = ||f_{\epsilon,i}||_C^2 - b_i^2 V$.

We want to estimate $\lambda_0(M) + \epsilon$ and know that it is at least $\frac{||\nabla f_\epsilon||_M^2}{||f_\epsilon||_M^2} = \frac{\sum_{i \in V(G)} ||\nabla f_\epsilon||_{C_i}^2}{\sum_{i \in V(G)} ||f_\epsilon||_{C_i}^2}$. We continue by giving a lower bound on the numerator and an upper bound on the denominator.

The first is done in Lemme 4.8 of [RT13], which states, that there is a constant $A > 0$ depending on neighborhoods of the transition zones such that

$$\sum_{i \in V(G)} ||\nabla f_{\epsilon,i}||^2_{C_i} \geq A \sum_{\{i,j\} \in E(G)} (b_i - b_j)^2.$$

The proof is done by estimating $||\nabla f_{\epsilon,i}||^2_{H_{ij}\times[0,R]}$ in terms of the Newtonian Capacity of the "rectangles" $H_{ij} \times [-R, R]$. These are tubular neigborhoods of the transition zones, which have one part in the cell C_i and one in the cell C_j for each edge $\{i, j\} \in E(G)$. This is technical and we will not give the details here.

For the upper bound on $\sum_{i\in V(G)} ||f_\epsilon||^2_{C_i}$ let us estimate $||\nabla f_{\epsilon,i}||^2_{C_i} = ||\nabla(b_i\mathbf{1}) + \nabla g_i||^2_{C_i}$. The first summand obviously vanishes and we obtain

$$\begin{aligned}
(\lambda_0(M) + \epsilon) \sum_{i \in V(G)} ||f_{\epsilon,i}||^2_{C_i} &\geq \sum_{i \in V(G)} ||\nabla f_{\epsilon,i}||^2_{C_i} \\
&\geq \sum_{i \in V(G)} ||\nabla g_i||^2_{C_i} \\
&\geq \sum_{i \in V(G)} \nu ||g_i||^2_{C_i} \\
&\geq \nu \sum_{i \in V(G)} ||\nabla f_{\epsilon,i}||^2_{C_i} - \nu \sum_{i \in V(G)} b_i^2 V.
\end{aligned}$$

Assume that $\lambda_0(M) + \epsilon < \nu$. Otherwise the proposition is clearly true. Then the above inequality is equivalent to

$$\sum_{i \in V(G)} ||f_{\epsilon,i}||^2_{C_i} \leq V \left(1 - \frac{\lambda_0(M) + \epsilon}{\nu}\right)^{-1} \sum_{i \in V(G)} b_i^2.$$

With these bounds on the numerator and denominator we summarize:

$$\lambda_0(M) + \epsilon \geq \frac{\sum_{i\in V(G)} ||\nabla f_\epsilon||^2_{C_i}}{\sum_{i\in V(G)} ||f_\epsilon||^2_{C_i}} \geq \frac{A}{V} \frac{\sum_{\{i,j\}\in E(G)} (b_i - b_j)^2}{\sum_{i\in V(G)} b_i^2} \left(1 - \frac{\lambda_0(M) + \epsilon}{\nu}\right).$$

By Equation 1.2 the quotient $\frac{\sum_{\{i,j\}\in E(G)}(b_i-b_j)^2}{\sum_{i\in V(G)} b_i^2}$ is greater or equal to the spectral gap μ_0 of the graph G. Inserting this observation and solving for $\lambda_0(M) + \epsilon$ yields

$$\lambda_0(M) + \epsilon \geq \frac{\frac{A}{V}\nu}{\nu + \frac{A}{V}\mu_0(G)} \mu_0(G).$$

Since all constants on the right-hand side are independent of ϵ this finishes the proof. □

6 Conclusion

6.1 Proofs of the Main Theorems

After we provided the auxiliary material in the previous chapters in this final chapter we can reap the fruit of our labor and prove our main theorems:

Main Theorem A. *For every non-periodic connection point P on the Veech surface L_D (with $D \equiv 0 \mod 4$ not a square) the (infinitely generated) stabilizer subgroup $\mathrm{SL}(L_D; P) := \mathrm{Stab}_{\mathrm{SL}(L_D)}(P)$ has critical exponent strictly between $\frac{1}{2}$ and 1.*

As already mentioned in the introduction by Theorem 2.14 the Veech groups of affine coverings of L_D branched at the singularity and P are commensurable to $\mathrm{SL}(L_D; P)$. By Corollary 5.5 commensurable Fuchsian groups have the same critical exponent and the other main theorem follows:

Main Theorem B. *There exist translation surfaces whose Veech group is infinitely generated with critical exponent strictly between $\frac{1}{2}$ and 1. More precisely this is the case for every affine covering of L_D (with $D \equiv 0 \mod 4$ not a square) branched at the singularity and one non-periodic connection point P.*

In Chapter 5 we have seen, that $\delta(\mathrm{SL}(L_D; P)) > \frac{1}{2}$, since $\mathrm{SL}(L_D; P)$ is non-elementary and contains a parabolic element (Proposition 5.6). Furthermore, by Corollary 5.19 the critical exponent $\delta(\mathrm{SL}(L_D; P))$ is strictly smaller than 1 if the Schreier graph of $\mathrm{SL}(L_D)$ with respect to $\mathrm{SL}(L_D; P)$ and any finite generating set S of $\mathrm{SL}(L_D)$ is non-amenable.

So let us now show that for every non-periodic connection point P and any finite set S generating $\Gamma = \mathrm{SL}(L_D)$ the graph $G_{\Gamma,\Pi,S}$ is non-amenable, where in order to improve readability Π denotes the subgroup $\mathrm{SL}(L_D; P)$. Theorem 1.34 tells us that we can change the finite generating set. Thus $G_{\Gamma,\Pi,S}$ is non-amenable if and only if $G = G_{\Gamma,\Pi,S\cup\{A^k,B^l\}}$ is non-amenable.

Let G' be the graph induced by all vertices of G that correspond* to points that are not periodic under both A and B. Remember that by Proposition 2.22 the point P is of the form $P = \big(x_r + x_i w; y_r + y_i w\big)$ with $x_r, x_i, y_r, y_i \in \mathbb{Q}$. If all these rational numbers are reduced fractions, we denoted the least common denominator by $N(P)$. In Theorem C we have seen that all points Q in the orbit of P are of the same form and have the same least common denominator. Hence all vertices of the Schreier graph correspond to points $Q \in L_D$ with $N(Q) = N(P)$. The points periodic under A and under B are described in Section 3.2, and since the denominators of x_r, x_i, y_r and y_i are bounded by $N(P)$, there are only finitely many such points. Thus we can apply Proposition 1.28 and non-amenability of G' implies non-amenability of G.

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*$\Pi\gamma \leftrightarrow \gamma \circ P$

The last step to simplify the graph is to omit all edges that are not labeled with A^k or B^l. We call this graph G''. Omitting edges does not increase the Cheeger constant (Proposition 1.31). Hence the graphs G', G and $G_{\Gamma,\Pi,S}$ are non-amenable if G'' is non-amenable.

That G'' is indeed non-amenable follows from:

Proposition 6.1. *There are numbers k and l depending only on $N := N(P)$ and w, such that every connected component of G'' is either isomorphic to the infinite 4-valent tree or to the root-looped 4-valent tree (cf. Figure 6.1).*

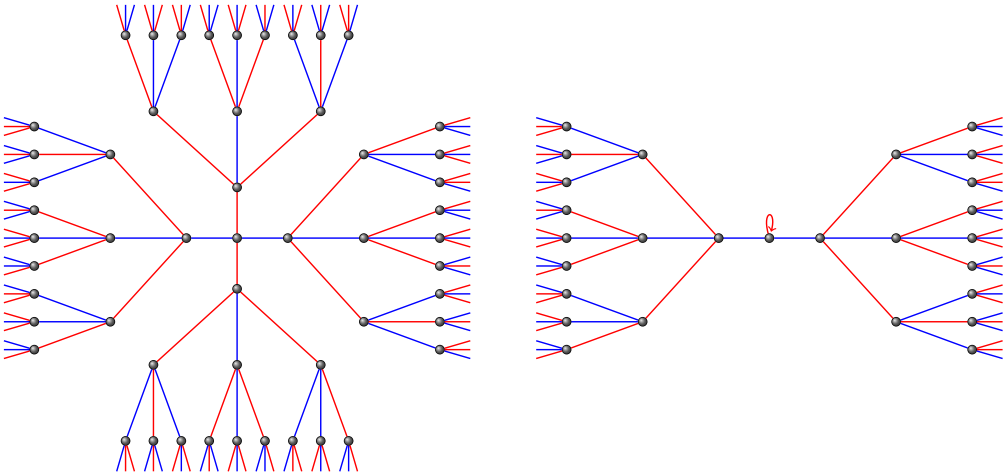

Figure 6.1: The infinite 4-valent and the root-looped 4-valent tree.

As seen in Example 1.35 and Remark 1.45 both graphs have Cheeger constant $\frac{2}{3}$, hence this Proposition together with Corollary 1.30 implies the non-amenability of G'' and thus gives also:

Corollary 6.2. *The graph $G_{\{\Gamma,\Pi,S\}}$ is non-amenable for any nonperiodic connection point P and any finite generating set S of Γ.*

Proof of Proposition 6.1. We will analyze the points corresponding to the vertices of G'' and the sum of the absolute values of their irrational parts. Remember that for a point $Q = \big(x_r + x_i w; y_r + y_i w\big)$ this sum $|x_i| + |y_i|$ is denoted by $s(Q)$.

1. For every point Q in the orbit of P periodic under A (resp. B) the period divides N since $N(Q) = N$ (Section 3.2). Hence in G'' the circle at the periodic point Q is in fact a loop, i.e. of length 1, if we choose k (resp. l) to be a multiple of N.

   Thus Q has exactly two neighbors $\neq Q$, namely $B^l \circ Q$ and $B^{-l} \circ Q$ (resp. $(A^k \circ Q$ and $A^{-k} \circ Q)$). Lemma 3.6 and Lemma 3.7 state that for $l > l_0 = \max\left\{\frac{3N}{w-1}, 2N+1\right\}$

   $$s(Q) < s(B^l \circ Q) \text{ and } s(Q) < s(B^{-l} \circ Q)$$

   (resp. for $k > k_0 = \max\left\{\frac{3N}{w}, 2N+1\right\}$ the analogous statement for $s(A^{\pm k} \circ Q))$ holds.

2. In Lemma 3.10 we have seen that if we choose $k > k_1 = \max\left\{k_0, \frac{2(N+1)}{w}\right\}$ and $l > l_1 = \max\left\{l_0, \frac{2(N+1)}{w-1}\right\}$, for all points Q neither periodic under A nor under B at least three of the following inequalities hold:

$$\begin{aligned} s(Q) &< s(A^k \circ Q) \\ s(Q) &< s(A^{-k} \circ Q) \\ s(Q) &< s(B^l \circ Q) \\ s(Q) &< s(B^{-l} \circ Q). \end{aligned}$$

All these conditions on the choice of k and l only depend on N and w. Hence we can and do choose k as the smallest multiple of N bigger than k_1 and l as the smallest multiple of N bigger than l_1 and analyze which graphs can occur as connected components of G'' in this case:

First Case: in the connected component there is a point Q periodic under A or B:

Let us remove the loop at Q and look at an path of length n starting in $Q = Q_0$ visiting the points $Q_0, Q_1, \dots Q_n$ with $Q_1 \neq Q_0$ and $Q_{j+2} \neq Q_j$. From item 1 we know that $s(Q_1) > s(Q)$. Thus Q_1 has a neighbor with smaller s-value and can not be periodic under A or B. Furthermore we also know the one and only neighbor of Q_1 that has a lower s-value than Q_1 is Q_0 (because of item 2) and we get $s(Q_2) > s(Q_1)$. The same argument for the following points show that none of the points Q_j, $j \geq 1$ is periodic under A or B and $(s(Q_j))_{j=0,\dots,n}$ is strictly increasing. In particular there is no circle in this connected component, all vertices beside Q in the same component have valency 4. This component is isomorphic to the root-looped 4-valent tree.

Second Case: in the connected component there is no point periodic under A or B, so all vertices have valency 4:

Suppose in the connected component there is a circle $[v_0, v_1, \dots, v_{n-1}, v_0]$ with corresponding points $[Q_0, Q_1, \dots, Q_{n-1}, Q_0]$. Then the set $\{s(Q_0), \dots, s(Q_{n-1})\}$ is finite and thus contains its maximum. Let Q_j be a point with $s(Q_j)$ this maximum. This implies $s(Q_{j-1}) \leq s(Q_j)$ and $s(Q_{j+1}) \leq s(Q_j)$, which is a contradiction to item 2. Thus in this case the connected component is the infinite 4-valent tree. □

This completed the proof of our main theorems and we will finish with some remarks in the next section.

6.2 Final Remarks

6.2.1 Effective Bounds on the Critical Exponent

Since the constant $A > 0$ of Proposition 5.18 is given explicitly in [RT13], we would obtain an effective estimate on the upper bound of the critical exponent $\delta(\mathrm{SL}(L_D; P))$, if we have lower bounds on the spectral gap $\eta = \lambda_1(C)$ of the cell C and on the spectral gap $\mu_0(G)$ of the combinatorial Laplacian on the graph G.

By Theorem 1.46 $\mu_0(G) \geq \frac{\mathfrak{c}(G)^2}{2\,\mathrm{val}_{\max}}$, so one would like to have a lower bound for the Cheeger constant $\mathfrak{c}(G)$.

Using the notation of Section 6.1 we saw in the proof of Proposition 6.1 that the graph G'' has Cheeger constant $\mathfrak{c}(G'') = \frac{2}{3}$ and $\mathfrak{c}(G') \geq \mathfrak{c}(G'')$. The next step to prove non-amenability of G and the first problem for getting an quantitative bound on $\mathfrak{c}(G')$ is applying Proposition 1.28, which is just qualitative. The graphs G and G' differed by a finite number of vertices – the vertices corresponding to non-periodic connection points in the orbit of P periodic under both A and B. But for $N(P) = 1$ there are no such points (see Lemma 3.5 and Lemma 3.4).

But the really crucial step is the change of the (finite) generating set of $\mathrm{SL}(L_D)$, where we use Theorem 1.34 to see that this does not concern amenability of the Schreier graph. Also this step, which has to be done no matter what $N(P)$ is, is just qualitative and we do not know any estimates on the change of the Cheeger constant.

This is why we unfortunately do not obtain a effective bound on the critical exponent $\delta(\mathrm{SL}(L_D; P))$.

6.2.2 McMullen's Infinitely Generated Veech Groups

In Section 2.2.3 we described not only Hubert and Schmidt's approach but also the one of McMullen ([McM03] and [McM06]) to find translation surfaces with infinitely generated Veech group. Obviously, it would be interesting to analyze also the critical exponent of the groups found by McMullen. But unfortunately our method using the concept of graph-periodic manifolds of Roblin and Tapie is not applicable here.

6.2.3 The Veech Surfaces $L_{D,\pm 1}$ for $D \equiv 1 \mod 4$

In McMullen's classification of primitive Veech surfaces in $\Omega M_2(2)$ (Theorem 2.18) the surfaces L_D for $D \equiv 0 \mod 4$ that we analyzed are just one of three types of L-shaped Veech surfaces. The whole theory of this thesis works analogously for the two other types, if we set w to be $\frac{1+\sqrt{D}}{2}$. The only thing one has to check is, that the technical lemmata of Section 3.3 also hold for the corresponding vertical and horizontal primitive parabolic elements $A_{D,\pm 1}$ and $B_{D,\pm 1}$ of $\mathrm{SL}(L_{D,\pm 1})$, which differ a little bit from the matrices A and B we used.

For the surfaces $L_{D,-1}$ these matrices are $A_{D,-1} = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}$ and $B_{D,-1} = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$. Because of the symmetry of the horizontal and vertical direction of $L_{D,-1}$ (see Figure 2.5) some of the computations become even a little bit easier. The remaining surfaces are the surfaces $L_{D,-1}$, where the vertical and horizontal primitive parabolic elements are $A_{D,+1} = \begin{pmatrix} 1 & 0 \\ w-1 & 1 \end{pmatrix}$ and $B_{D,+1} = \begin{pmatrix} 1 & w+1 \\ 0 & 1 \end{pmatrix}$.

After proving the corresponding variants of Lemma 3.6 – Lemma 3.10, the proof that $\delta(\mathrm{SL}(L_{D,\pm 1}; P)$ is strictly between $\frac{1}{2}$ and 1 works exactly as the proof for $D \equiv 0 \mod 4$ described in Section 6.1 and the main theorems can be extended to all affine coverings of Veech surfaces of $\Omega M_2(2)$ branched at the singularity and a non-periodic connection point.

List of Figures

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Zusammenfassung

Die zentralen Objekte dieser Dissertation sind *Translationsflächen* X. Dabei handelt es sich um Riemann'sche Flächen, die aus einer endlichen Menge von in die euklidische Ebene eingebetteten Polygonen durch Verkleben von parallelen gleichlangen Seiten entstehen. Zwei Translationsflächen sind gleich, wenn es möglich ist, die Polygone durch „Zerschneiden und mittels Translationen neu Zusammenkleben" ineinander zu überführen. Die Gruppe $\mathrm{GL}_2(\mathbb{R})$ operiert auf der Menge der Translationsflächen via der linearen Abbildungen auf den Polygonen. Der Stabilisator einer Translationsfläche X unter dieser Operation wird die *Veech-Gruppe* von X genannt und mit $\mathrm{SL}(X)$ bezeichnet. Die Veech-Gruppe ist eine diskrete Untergruppe von $\mathrm{SL}_2(\mathbb{R})$ und damit eine Fuchs'sche Gruppe.

Fuchs'sche Gruppen operieren durch Möbius-Transformationen auf der oberen Halbebene $\mathbb{H}$ und werden je nach ihrer Limesmenge in *elementare* und *nicht-elementare* Gruppen eingeteilt. Letztere wiederum unterteilt man in Gruppen *erster* oder *zweiter Art*. Fuchs'sche Gruppen mit endlichem co-Volumen heißen *Gitter* und sind genau die endlich erzeugten Gruppen erster Art. Translationsflächen X, deren Veech-Gruppe ein Gitter ist, heißen *Veech-Flächen* und sind von besonderem Interesse, da für sie die *Veech Alternative* gilt, die besagt, dass für eine Richtung θ entweder alle geodätischen Strahlen auf X in Richtung θ periodisch oder alle solche Strahlen gleichverteilt sind.

Ein feineres Maß für die „Größe" einer Fuchs'schen Gruppe Γ ist der *kritische Exponent* $\delta(\Gamma)$. Er ist definiert als das Infimum aller reellen Zahlen a, für die die *Poincaré Reihe* $\sum_{\gamma\in\Gamma}\exp(-a\rho_{\mathbb{H}}(i,\gamma\circ i))$ konvergiert. Es gilt $0 \leq \delta(\Gamma) \leq 1$ für alle unendlichen Fuchs'schen Gruppen Γ.

Hauptziel der Dissertation ist der Beweis von

Theorem 1. *Es gibt Translationsflächen, für die der kritische Exponent ihrer Veech-Gruppe echt zwischen* $\frac{1}{2}$ *und* 1 *liegt.*

Der kritische Exponent von elementaren Gruppen ist höchstens $\frac{1}{2}$. Translationsflächen mit elementaren Veech-Gruppen sind also als Kandidaten für das Theorem ausgeschlossen. Ist Γ ein Gitter, so gilt $\delta(\Gamma) = 1$. Also scheiden auch Veech-Flächen für das Theorem aus.

Bis zum Jahr 2003 waren Gitter die einzigen bekannten nicht-elementaren Veech-Gruppen. McMullen klassifizierte die Veech-Flächen vom Geschlecht 2 und zeigte, dass jede solche Fläche, die nur eine Singularität besitzt, in der $\mathrm{GL}_2(\mathbb{R})$-Bahn der Fläche L_D liegt, die aus einem L-förmigen Polygon mit geeigneten von D abhängigen Seitenlängen entsteht.

Während auch heute noch keine Translationsfläche mit Veech-Gruppe von zweiter Art bekannt ist, fanden McMullen und unabhängig davon Hubert und Schmidt Konstruktionen unendlich erzeugter Veech-Gruppen von erster Art. Eine Abschätzung des kritischen

Exponenten dieser Gruppen war 10 Jahre lang eine wichtige offene Frage, die nun durch Theorem 1 beantwortet wird.

Punkte $P \in X$ mit der Eigenschaft, dass alle geodätischen Strahlen, die in einer Singularität von X starten und durch P verlaufen, so erweitert werden können, dass sie in einer Singularität enden, heißen *Verbindungspunkte*. Diese sind zentral in der Konstruktion von Hubert und Schmidt. Sie konstruieren Translationsflächen, deren Veech-Gruppen kommensurabel zu der Stabilisatoruntergruppe $\mathrm{SL}(X; P)$ von P sind und damit den gleichen kritischen Exponenten haben. Für Verbindungspunkte mit unendlicher $\mathrm{SL}(X)$-Bahn (diese Punkte heißen *nicht-periodisch*) ist $\mathrm{SL}(X; P)$ unendlich erzeugt und von erster Art.

Wir zeigen Theorem 1, indem wir zeigen, dass für jedes $D \equiv 0 \mod 4$, welches keine Quadratzahl ist, und jeden nicht-periodischen Verbindungspunkt P der kritische Exponent der Gruppe $\mathrm{SL}(L_D; P)$ echt zwischen $\frac{1}{2}$ und 1 liegt.

Eine natürliche Frage in diesem Zusammenhang ist die Abhängigkeit von P: Liegt ein Punkt P' in der $\mathrm{SL}(L_D)$-Bahn von P, so ist auch er ein nicht-periodischer Verbindungspunkt und die zugehörigen Gruppen $\mathrm{SL}(L_D; P)$ und $\mathrm{SL}(L_D; P')$ sind konjugiert zueinander, haben also die gleiche Dynamik am Rand der hyperbolischen Ebene. Daher widmen wir uns in Kapitel 4 der Bestimmung der $\mathrm{SL}(L_D)$-Bahn von nicht-periodischen Verbindungspunkten P.

Diese haben die Form $P = (x_r + x_i w; y_r + y_i w)$ mit $x_r, x_i, y_r, y_i \in \mathbb{Q}$ und $w := \frac{\sqrt{D}}{2}$. Wir zeigen, dass jede Bahn dicht in L_D liegt und dass der Hauptnenner $N(P)$ der (gekürzten) Brüche x_r, x_i, y_r, y_i dieser Darstellung eine Invariante der Bahn ist. Daraus folgt:

Theorem 2. *Es gibt unendlich viele verschiedene Bahnen von nicht-periodischen Verbindungspunkten von L_D. Diese liegen alle dicht.*

Wir kennen die Operation der zwei Elemente $A := \left(\begin{smallmatrix} 1 & 0 \\ w & 1 \end{smallmatrix}\right)$ und $B := \left(\begin{smallmatrix} 1 & 1+w \\ 0 & 1 \end{smallmatrix}\right)$ der Veech-Gruppe $\mathrm{SL}(L_D)$. Im Spezialfall $D = 8$ erzeugen diese beiden Elemente die ganze Gruppe und wir geben je ein Verfahren an, um eine untere und eine obere Schranke an die Anzahl der Bahnen von nicht-periodischen Verbindungspunkten P mit fixiertem Hauptnenner $N(P)$ zu finden. Damit zeigen wir:

Theorem 3. *Die Menge der Verbindungspunkte $P = (x_r + x_i w; y_r + y_i w)$ mit festem Wert $N(P)$ zerfällt in eine* **endliche** *Anzahl von Bahnen der Operation von $\mathrm{SL}(L_8)$.*

Im Verlauf des Beweises von Theorem 1 ist es nötig, die Nichtmittelbarkeit eines Graphen G zeigen. Da wir nur sehr wenige Informationen über die Struktur von G in unserer konkreten Situation haben, entwickeln wir in Kapitel 1 die folgende Methode:

Theorem 4. *Sei G ein Graph, den man durch Weglassen von Kanten in einen Wald G' ohne Blätter überführen kann, bei dem das Supremum der Längen von zusammenhängenden Valenz-2-Teilgraphen von G' beschränkt ist. Dann ist G nichtmittelbar.*

Um diese Methode anzuwenden, ordnen wir jeder Ecke P von G ein Komplexitätsmaß $s(P)$ zu und weisen nach, dass dieser Wert für Worte in A- und B-Potenzen mit wachsender Wortlänge „tendenziell wächst“.

Im Folgenden werden wir die wichtigsten Resultate der einzelnen Kapitel zusammenfassen, um schließlich den Beweis aus Kapitel 6 skizzieren zu können.

Graphen und Mittelbarkeit

Es tauchen zwei verschiedene Arten von Graphen auf: Schreier-Graphen sind gerichtete reguläre Graphen mit Kantenbeschriftungen und gegebenenfalls Multikanten und Schleifen, während die Cheeger-Konstante für einfache Graphen definiert wird, da sie unabhängig von der Richtung und Beschriftung der Kanten sowie dem Auftauchen von Multikanten und Schleifen ist. Daher werden wir auch ohne weitere Erwähnung zwischen diesen beiden Arten von Graphen wechseln.

Sei nun also $G = (V, E)$ ein Graph mit Eckenmenge V und Kantenmenge E. Für eine Teilmenge $M \subset V$ seien

$$\partial M := \{v \in M \mid v \text{ hat einen Nachbarn in } M^c\} \quad \text{und} \quad c(M) := \frac{|\partial M|}{|M|}.$$

Die *Cheeger-Konstante* von G ist definiert als

$$\mathfrak{c}(G) := \inf_{M \subset V \text{ endlich}} c(M)$$

und G heißt genau dann *mittelbar*, wenn $\mathfrak{c}(G) = 0$ ist.

Mit Hilfe der folgenden Propositionen wird eine Methode entwickelt, um für bestimmte Graphen nachzuweisen, dass sie nichtmittelbar sind:

Proposition 5. *Sei $G = (V, E)$ ein mittelbarer Graph ohne endliche Zusammenhangskomponenten. Dann is für jede endliche Menge $F \subset V$ auch der von $V - F$ induzierte Untergraph mittelbar.*

Proposition 6. *Sei G ein Graph mit Zusammenhangskomponenten K_i. Dann gilt*

$$\mathfrak{c}(G) = \inf_i \mathfrak{c}(K_i)$$

.

Proposition 7. *Entsteht ein Graph G' aus einem Graphen G durch Weglassen von Kanten, so gilt $\mathfrak{c}(G') \leq \mathfrak{c}(G)$.*

Daraufhin bestimmen wir eine untere Schranke für die Cheeger-Konstante eines Baums ohne Blätter und erhalten damit Theorem 4.

Außerdem stellen wir fest, dass die Mittelbarkeit von Schreier-Graphen unabhängig von der Wahl eines endlichen Erzeugendensystems der Obergruppe ist.

Abschließend sei noch erwähnt, dass für Cayley-Graphen von Faktorgruppen Γ/Π ein Kriterium für Nichtmittelbarkeit existiert: Enthält Γ eine freie Gruppe $\mathcal{F}$ vom Rang mindestens 2, die den Normalteiler Π nur trivial schneidet, so ist der Cayley-Graph von Γ/Π nichtmittelbar. Da Schreier-Graphen in mancherlei Hinsicht Cayley-Graphen sehr ähneln, könnte man hoffen, dass das gleiche Kriterium auch für Schreier-Graphen gilt. In Kapitel 1 geben wir allerdings ein Gegenbeispiel an: Dabei ist $\Gamma = \langle a, b \rangle$ und die Untergruppe $\Pi = \left\langle \{a^k b a^{-k}\}_{k \in \mathbb{Z} - \{0\}} \right\rangle$. Die freie Untergruppe mit trivialem Schnitt mit Π ist $\mathcal{F} = \left\langle bab, b^2 \right\rangle$.

Grundlagen zu Translationsflächen

Dieses Kapitel widmet sich dem Sammeln grundlegender Definitionen und Sätze über Translationsflächen und Veech-Gruppen. Insbesondere wird McMullens Klassifikation aller primitiven Veech-Flächen vom Geschlecht 2 vorgestellt. Die Veech-Flächen L_D, die in den weiteren Kapiteln behandelt werden sollen, entstehen aus einem L-förmigen Polygon mit den folgenden Seitenlängen durch Verkleben der gegenüberliegenden Seiten. Die vertikalen Seitenlängen sind von links nach rechts w, $w-1$ und 1, die horizontalen sind von unten nach oben $1+w$, w und 1. Dabei sind D eine positive Zahl kongruent 0 modulo 4 (die kein Quadrat ist) und $w = \sqrt{\frac{D}{4}}$.

Die Stabilisatoruntergruppe $\mathrm{SL}(L_D; P) < \mathrm{SL}(L_D)$ ist von erster Art, aber kein Gitter (also unendlich erzeugt), wenn P ein nicht-periodischer Verbindungspunkt ist. Diese sind von der Form $(x_r + x_i w; y_r + y_i w)$ mit $x_r, x_i, y_r, y_i \in \mathbb{Q}$.

Die Prototypen L_D

Zwei parabolische Elemente A und B der Veech-Gruppe $\mathrm{SL}(L_D)$ sind besonders leicht zu bestimmen: diejenigen primitiven parabolischen Elemente, die die vertikale bzw. horizontale Richtung fixieren. Auch die Operation von A und B auf den Punkten von L_D wird in diesem Kapitel beschrieben. Dies geschieht mittels einer Zerlegung der Fläche L_D in vertikale bzw. horizontale Zylinder. Für diese Elemente $A = \left(\begin{smallmatrix} 1 & 0 \\ w & 1 \end{smallmatrix}\right)$ und $B = \left(\begin{smallmatrix} 1 & 1+w \\ 0 & 1 \end{smallmatrix}\right)$ und ihre Operation auf einem Punkt $Q = (x_r + x_i w; y_r + y_i w)$ mit $x_r, x_i, y_r, y_i \in \mathbb{Q}$ werden dann einige technische Lemmata bewiesen, die den Wert $s(Q) := |x_i| + |y_i|$ betreffen. Dabei wird ein weiterer Wert verwendet, der jedem Punkt obiger Form zugeordnet ist, nämlich der Hauptnenner $N(Q)$ der gekürzten Brüche x_r, x_i, y_r und y_i.

Die folgenden Lemmata präzisieren, was wir in der Einleitung mit dem „tendenziellen Wachstum" von $s(Q)$ bezeichnet haben.

Lemma 8. *Falls Q periodisch unter A, nicht aber unter B ist, dann existiert ein l_0 abhängig von $N(Q)$ und w, sodass für alle l mit $|l| \geq l_0$ die Ungleichung $s(Q) < s(B^l \circ Q)$ erfüllt ist.*

Lemma 9. *Falls Q periodisch unter B, nicht aber unter A ist, existiert ein k_0 abhängig von $N(Q)$ und w, sodass für alle k mit $|k| \geq k_0$ die Ungleichung $s(Q) < s(A^k \circ Q)$ erfüllt ist.*

Lemma 10. *Falls Q weder periodisch unter A noch unter B ist, dann existieren k_1 und l_1 abhängig von $N(Q)$ und w, sodass für alle Paare (k, l) mit $k > k_1$ und $l > l_1$ mindestens drei der folgenden vier Ungleichungen erfüllt sind:*

$$\begin{aligned} s(Q) &< s(A^k \circ Q), \\ s(Q) &< s(A^{-k} \circ Q), \\ s(Q) &< s(B^l \circ Q), \\ s(Q) &< s(B^{-l} \circ Q). \end{aligned}$$

$\mathrm{SL}(L_D)$-Bahnen

Hauptresultat dieses Kapitels ist, dass die oben definierte Zahl $N(P)$ eine Invariante der $\mathrm{SL}(L_D)$-Bahn von P ist. Für jeden Punkt P hat die Operation von $\gamma \in \mathrm{SL}(L_D)$ die Form $\gamma \circ P = \gamma \cdot P - t$ für ein $t \in \mathbb{R}^2$, wobei $\gamma \cdot P$ die gewöhnliche lineare Abbildung der 2×2-Matrix γ bezeichnet. Wir beweisen dieses Hauptresultat, indem wir zeigen, dass sowohl alle Einträge von γ als auch von t Elemente von $\mathbb{Z}[w] = \mathbb{Z} + \mathbb{Z}w$ sind.

Daraus folgt insbesondere, dass es unendlich viele verschiedene $\mathrm{SL}(L_D)$-Bahnen gibt (Theorem 2).

Der Rest des Kapitels widmet sich dem Spezialfall $D = 8$ und dem Nachweis, dass die Anzahl der $\mathrm{SL}(L_8)$-Bahnen von Punkten P mit fest gewähltem $N(P)$ endlich ist:

Dies geschieht durch die Angabe eines Algorithmus', der zu einem beliebigen Punkt $Q = (x_r + x_i w; y_r + y_i w)$ mit $x_r, x_i, y_r, y_i \in \mathbb{Q}$ mit Hauptnenner $N(Q) = N$ ein Wort in den Gruppenerzeugern A und B findet, das Q auf einen Punkt einer Menge S abbildet. Die Menge S besteht dabei aus den Punkten, für die $|x_i|$ und $|y_i|$ höchstens gleich $35+24w$ sind. Da N fest gewählt ist, liegen in S nur endlich viele Punkte und Theorem 3 folgt.

Kritischer Exponent und Graph-periodische Mannigfaltigkeiten

Um die Suche nach dem kritischen Exponenten von $\mathrm{SL}(L_D; P) < \mathrm{SL}(L_D)$ mit der Nichtmittelbarkeit von Graphen zu verbinden, betrachten wir in diesem Kapitel – den Ideen von Roblin und Tapie folgend – Graph-periodische Mannigfaltigkeiten: Dies sind Mannigfaltigkeiten M, die aus isometrischen Kopien einer anderen Mannigfaltigkeit C mit Rand bestehen, die entsprechend der von einem regulären Graphen G vorgegebenen Struktur verklebt sind. Konkret handelt es sich in unserer Situation um die hyperbolische Fläche $M = \mathbb{H}/\mathrm{SL}(L_D; P)$, einen Dirichlet-Fundamentalbereich C des Gitters $\mathrm{SL}(L_D)$ und den Schreier-Graph G von $\mathrm{SL}(L_D)$ bezüglich $\mathrm{SL}(L_D; P)$ und dem durch den Dirichlet-Fundamentalbereich gegebenen Erzeugendensystem von $\mathrm{SL}(L_D)$.

Mit den folgenden Ergebnissen lässt sich so ein Zusammenhang zwischen $\delta(\mathrm{SL}(L_D; P))$ und $\mathfrak{c}(G)$ finden.

Theorem 11 (Patterson/Sullivan)**.** *Sei Γ eine Fuchs'sche Gruppe und M die hyperbolische Fläche $\mathbb{H}/\Gamma$. Für den kleinsten Eigenwert $\lambda_0(M)$ des Laplace-Operators Δ auf M gilt:*

$$\lambda_0(M) = \begin{cases} \delta(\Pi)(1 - \delta(\Pi)), & \text{wenn } \delta(\Pi) \geq \frac{1}{2} \\ \frac{1}{4}, & \text{wenn } \delta(\Pi) \leq \frac{1}{2}. \end{cases}$$

Der kritische Exponent $\delta(\Gamma)$ ist also genau dann kleiner als 1, wenn der kleinste Eigenwert $\lambda_0(M)$ echt positiv ist.

Im Folgenden verwenden wir, dass der kleinste Eigenwert $\lambda_0(C)$ des Laplace-Operators auf C (mit Neumann-Randbedingungen) 0 ist und mit Vielfachheit 1 auftritt. Die *spektrale Lücke* $\eta \coloneqq \lambda_1(C) - \lambda_0(C)$ ist also strikt positiv.

Theorem 12 (Roblin/Tapie)**.** *Seien M, C und G wie oben. Dann gilt*

$$\lambda_0(M) \geq \lambda_0(C) + A\eta\mu_0(G),$$

wobei $A > 0$ eine Konstante abhängig von einer Markierung von Übergangszonen des Randes von C ist, und $\mu_0(G)$ der kleinste Eigenwert des kombinatorischen Laplace-Operators auf G ist.

Der Wert $\mu_0(G)$ kann durch Terme in $\mathfrak{c}(G)$ abgeschätzt werden und ist genau dann positiv, wenn G nichtmittelbar ist, womit wir die Brücke zwischen dem kritischen Exponenten von $\mathrm{SL}(L_D; P)$ und der Cheeger-Konstante von G geschlagen haben.

Endergebnis

In diesem Kapitel schließlich werden alle Resultate zusammengeführt, um die Nichtmittelbarkeit des Schreier-Graphen G und damit $\delta(\mathrm{SL}(L_D; P)) < 1$ zu beweisen. Wir skizzieren die Beweisstrategie:

Da es sich bei der Untergruppe um den Stabilisator eines Punktes P handelt, können wir die Ecken von G mit den Punkten in der $\mathrm{SL}(L_D)$-Bahn von P identifizieren: die Rechtsnebenklasse $\mathrm{SL}(L_D; P)\gamma$ entspricht dem Punkt $\gamma \circ P$. Um die technischen Lemmata aus Kapitel 3 benutzen zu können, fügen wir dem Erzeugendensystem der Obergruppe die Elemente A^k und B^l für feste, von $N(P)$ abhängige, Werte k und l hinzu und erhalten einen Graphen G', der genau dann mittelbar ist, wenn G mittelbar ist. Im nächsten Schritt erhalten wir G'', indem wir zunächst aus technischen Gründen eine bestimmte – wiederum von $N(P)$ abhängige – endliche Menge von Ecken und anschließend alle Kanten, die nicht mit A^k oder B^l beschriftet sind, entfernen. Aus den Propositionen des ersten Kapitels folgt, dass G nichtmittelbar ist, falls G'' nichtmittelbar ist.

Mit den technischen Lemmata aus Kapitel 3 kann man schließlich zeigen, dass jede Zusammenhangskomponente von G'' isomorph ist entweder zum 4-valenten unendlichen Baum oder zu einem unendlichen Baum, der eine Wurzel von Valenz 2 hat, ansonsten aber 4-valent ist. In beiden Fällen ergibt sich eine Cheeger-Konstante von $\frac{2}{3}$. Also sind G'' und damit auch G nichtmittelbar und $\delta(\mathrm{SL}(L_D; P)) < 1$.

Abschließend bleibt noch zu bemerken, dass die Einschränkung auf Veech-Flächen mit $D \equiv 0 \mod 4$ nicht nötig ist und das gleiche Verfahren auch für die anderen Prototypen L_D aus McMullens Klassifikation, für die $D \equiv 1 \mod 4$ ist, funktioniert. Was dazu zu überprüfen ist, ist, dass die technischen Lemmata bezüglich des Komplexitätsmaßes $s(Q) = |x_i| + |y_i|$ auch für diese Fälle ihre Gültigkeit behalten.