ASSET PRICING AND CONSUMPTION-PORTFOLIO CHOICE WITH RECURSIVE UTILITY AND UNSPANNED RISK

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We study consumption-portfolio and asset pricing frameworks with recursive preferences and unspanned risk. We show that in both cases, portfolio choice and asset pricing, the value function of the investor/representative agent can be characterized by a specific semilinear partial differential equation. To date, the solution to this equation has mostly been approximated by Campbell-Shiller techniques, without addressing general issues of existence and uniqueness. We develop a novel approach that rigorously constructs the solution by a fixed point argument. We prove that under regularity conditions a solution exists and establish a fast and accurate numerical method to solve consumption-portfolio and asset pricing problems with recursive preferences and unspanned risk. Our setting is not restricted to affine asset price dynamics. Numerical examples illustrate our approach.

Key Words: consumption-portfolio choice · asset pricing · stochastic differential utility · incomplete markets · fixed point approach · FBSDE

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1. Introduction

In economic models, decision making of agents is described by utility functionals. For instance, in the representative agent model that has dominated macroeconomics for the last three decades there is one individual, the representative agent, whose preferences have to be described. In the classical version the agent is assumed to have a time-separable von-Neumann Morgenstern utility function (e.g. CRRA utility) and to have access to a financial market that is complete. Both these specifications are potential sources of why the classical framework is not able to explain several empirical facts about asset prices (e.g. large equity premium, low riskfree rate, excess volatility) that are referred to as asset pricing puzzles.[1] Economists have responded to these challenges by assuming that agents have more general preferences (e.g. recursive preferences) and/or by postulating more involved asset or endowment processes (e.g. jumps, disasters, unspanned diffusion). Papers proposing such models include Rietz (1988), Bansal and Yaron (2004), Barro (2006), Benzoni, Collin-Dufresne, and Goldstein (2011), Gabaix (2012), Wachter (2013), among others.

To calculate the values of cash flows in such a representative agent economy, the investor's intertemporal marginal rate of substitution, also known as the stochastic discount factor (pricing kernel, deflator), is the key ingredient. A stochastic discount factor induces a pricing rule that determines all asset prices in an economy. It is thus of crucial importance that the agent's utility can be described in a tractable way. In every continuous-time model, this utility satisfies a certain partial differential equation that can be derived by applying a stochastic representation theorem for expectations. For recursive preferences, this can be a reduced to an equation belonging to a particular class of semilinear partial differential equations. Such equations are inherently difficult to solve and, in general, it is not even clear whether they admit (unique smooth) solutions. So far researchers usually resort to approximation techniques of unclear precision (e.g. Campbell-Shiller approximation for non-unit elasticity of intertemporal substitution) and consider affine frameworks. In this context, our paper makes a significant contribution: For possibly non-affine models, we prove the existence of a solution and develop a fast and accurate numerical method to compute this solution. Our scheme solves the nonlinear partial differential equation by iteratively solving certain linear partial differential equations. We also derive worst-case bounds for the accuracy of our methodology. Therefore, our results provide a solid basis for future research in asset pricing with recursive preferences.

[1] See Campbell (2003).

Furthermore, we also contribute to the extensive literature on dynamic incomplete-market portfolio theory, a research area that according to Cochrane (2014) is at the same time “important” and “hard”. This area is concerned with an agent’s consumption-portfolio choice problem where returns are not necessarily independent and identically distributed. We study an incomplete-market consumption-portfolio problem that nests classical frameworks such as Kim and Omberg (1996), Campbell and Viceira (1999), Barberis (2000), Wachter (2002), Chacko and Viceira (2005) and Liu (2007), among others. With the exception of Chacko and Viceira (2005), these authors assume time-additive CRRA preferences. Following Chacko and Viceira (2005), we also allow for recursive preferences. Moreover, in contrast to the above-mentioned papers, we do not restrict our analysis to affine models. It is well-known that the associated Hamilton-Jacobi-Bellman equation (short: Bellman or HJB equation) characterizes the agent’s indirect utility, which in turn determines the agent’s optimal consumption-portfolio choice. We reduce the Bellman equation to a partial differential equation that belongs to the same class as the above-mentioned equation in asset pricing. Researchers have so far relied on approximative methods in this context as well, although in general not even the issue of existence has been resolved. Therefore, as an additional contribution of this article, we provide a verification theorem demonstrating that a suitable smooth solution of the reduced Bellman equation is the solution to the consumption-portfolio problem. Our proof is based on a combination of dynamic programming methods and utility gradient inequalities for recursive utility. Following the same agenda as outlined above, we then establish existence of a solution and construct this solution by fixed point arguments. Again our numerical method provides a fast and accurate way of calculating the investor’s indirect utility and optimal strategies. Our results thus also establish a tractable approach to incomplete-market consumption-portfolio choice problems with recursive preferences.

From a formal point of view, our contributions can be summarized as follows: First, we establish a verification theorem which demonstrates that a suitable $C^{1,2}$ solution of the reduced HJB equation provides the solution to the consumption-portfolio problem. This equation is also of crucial importance in asset pricing applications since it characterizes the consumption-wealth ratio of the representative agent. Second, we provide an explicit construction of such a $C^{1,2}$ solution based on fixed point arguments for the associated forward-backward stochastic differential equation. More precisely, we study the Feynman-Kac representation mapping Φ that is associated to a power transform of the HJB equation and obtain a fixed point in the space of continuous functions as a limit of iterations of Φ. Using the probabilistic representation of this solution we are able to improve this to convergence in $C^{0,1}$. This not only yields a theoretical convergence result, but also leads directly to an efficient numerical method to determine optimal strategies via iteratively solving linear partial differential equations.

The remainder of the paper is structured as follows: Section 2 summarizes the related literature. Section 3 studies the performance of the Campbell-Shiller approximation and shows that it might fail by a large margin. Section 4 defines the agent's utility functional that is modeled via recursive preferences. Section 5 analyzes the consumption-portfolio problem with dynamic programming methods and derives a candidate solution. Section 6 provides a verification result, i.e. establishes conditions under which the candidate solution is optimal. Section 7 relates our findings to the asset pricing literature and demonstrates that in common asset pricing frameworks the representative agent's value function satisfies the same semilinear partial differential equation as the value function in our consumption-portfolio optimization framework. Section 8 shows that this function can be characterized as the fixed point of a nonlinear Feynman-Kac operator. Section 9 establishes our numerical method to determine this fixed point. Section 10 considers examples of consumption-portfolio problems and asset pricing models and applies our solution method to these frameworks. This section also offers an informal user's guide on how to apply our theoretical results to concrete problems. Section 11 concludes. The Appendix contain proofs left out of the main text as well as auxiliary results.

2. Links to the Literature

Our paper is related to several strands of literature. First, our paper adds to the asset pricing literature by establishing a novel solution method for the agent's value function and the consumption-wealth ratio. In particular, this includes research on long-run risk and disasters such as Rietz (1988), Bansal and Yaron (2004), Barro (2006), Benzoni, Collin-Dufresne, and Goldstein (2011), Gabaix (2012), Wachter (2013).

Second, our paper contributes to the literature on dynamic incomplete-market portfolio theory by establishing a new general solution method, and by providing new solutions. Chacko and Viceira (2005) study a consumption-portfolio problem with affine stochastic volatility and recursive preferences. They find an explicit solution for unit elasticity of intertemporal substitution (EIS) and approximate the solution for non-unit EIS using the Campbell-Shiller technique. By contrast, our approach enables us to solve the problem without relying on an approximation, even for non-affine specifications of stochastic volatility. Liu (2007) considers portfolio problems with unspanned risk and time-additive utility. Our approach can be used to generalize several of his solutions to settings where the agent has recursive utility and where asset dynamics are non-affine or non-quadratic. Notice that Liu (2007)'s framework nests the models by Kim and Omberg (1996), Campbell and Viceira (1999), Barberis (2000), and Wachter (2002), among others, as special cases. Our results are also related to Schroder and Skiadas (1999), who focus on complete markets, and to Schroder and Skiadas (2003), who obtain explicit solutions for unit EIS by applying duality methods. Both papers consider agents with recursive utility. Recursive utility is developed in Kreps and Porteus (1978), Kreps and Porteus (1979), Epstein and Zin (1989), and Duffie and Epstein (1992b).

Third, our solution approach has ties to several strands of the literature. The verification argument used to solve the consumption-portfolio problem builds on the so-called utility gradient approach that has been developed in a series of papers by Duffie, Schroder, and Skiadas including Duffie and Skiadas (1994), Schroder and Skiadas (1999), Schroder and Skiadas (2003), Schroder and Skiadas (2005) and Schroder and Skiadas (2008). We generalize the verification results in Duffie and Epstein (1992b) who derive a verification result for aggregators satisfying a Lipschitz condition and in Kraft, Seifried, and Steffensen (2013) who allow for Epstein-Zin preferences, but restrict their analysis to certain relations between risk aversion and EIS. Our results are also related to the findings of Duffie and Lions (1992) who study the existence of stochastic differential utility with PDE methods. In discrete time, the existence and uniqueness of recursive utility is studied by Marinacci and Montrucchio (2010). Our fixed point arguments are related to the work by Berdjane and Pergamenshchikov (2013). These authors, however, focus on the special case where the agent has time-additive utility with low risk aversion (less risk averse than logarithmic) and where the state process has constant volatility.

Finally, our analysis of existence and uniqueness also contributes to the literature on semilinear partial differential equations (PDEs) and backward and forward-backward stochastic differential equations (BSDEs and FBSDEs, respectively). We demonstrate that the FBSDE associated to the semilinear PDE which is relevant for our applications in consumption-portfolio choice and asset pricing admits a unique bounded solution. Importantly, the driver of this FBSDE is not Lipschitz, so standard results (see, e.g., Pardoux and Peng (1990), Ma, Protter, and Yong (1994) or El Karoui, Peng, and Quenez (1997)) do not apply. We thus contribute to the growing literature on non-Lipschitz BSDEs and FBSDEs, including, among others, Kobylanski (2000), Briand and Carmona (2000), Briand and Hu (2008), and Delbaen, Hu, and Richou (2011). In addition, by deriving an associated Feynman-Kac representation, this paper adds to the literature that connects FBSDEs to semilinear Cauchy problems; see, e.g., Pardoux and Peng (1992), Delarue (2002) and Ma, Yin, and Zhang (2012) and the references therein.

3. Exact Solution vs. Approximation

To emphasize the need for a rigorous approach, we study the accuracy of the well-known Campbell-Shiller (CS) approximation technique[2] by comparing it with the exact solution that is calculated by applying the method developed in this paper.

To illustrate our point, we consider the disaster model in Wachter (2013) and use the calibration provided in her paper.[3] Notice that Wachter (2013) assumes that the representative agent has recursive preferences with unit elasticity of intertemporal substitution. For this special case, an exact solution is available and no approximation is needed. The CS approximation can be thought of as a first-order Taylor expansion around this exact solution. This approximation is frequently used in the literature if the EIS is not one. To evaluate its reliability, we perform comparative statics where we vary the elasticity of intertemporal substitution or the risk aversion (for fixed EIS of 1.5). First, we calculate the consumption-wealth ratio and the value function of the representative agent using the CS approximation. Then we compare the results with the exact solutions obtained from our method. Figure 1 depicts these two exercises. From Figure 1 (right panel), it can be seen that a small variation in the EIS can lead to a significant increase in the deviation of the exact from approximate solution (e.g., EIS of 1.3 vs. EIS of 1.4). Notice that the approximation is accurate for unit EIS. Besides, from Figure 1 (left panel) it can be seen that the approximation works fine for small risk aversions even if we fix the EIS at 1.5. However, it fails by a large margin if the risk aversion is 5 or bigger. Figure 2 quantifies the implied losses in consumption units. More precisely, we equate the agent's value functions (exact, approximate) and calculate the relative change $\ell \in (-\infty, 1]$ in the initial value of the endowment processes that leads to the same utility, i.e.

$$w^{\mathrm{CS}}(0, c, y) = w(0, (1-\ell)c, y). \tag{1}$$

[2]The discrete-time version of this approximation technique goes back to Campbell and Shiller (1988). The continuous-time analog, which we analyze in this section, is developed in Campbell and Viceira (2002) and Chacko and Viceira (2005).

[3]Section 10 contains a detailed description of this model.

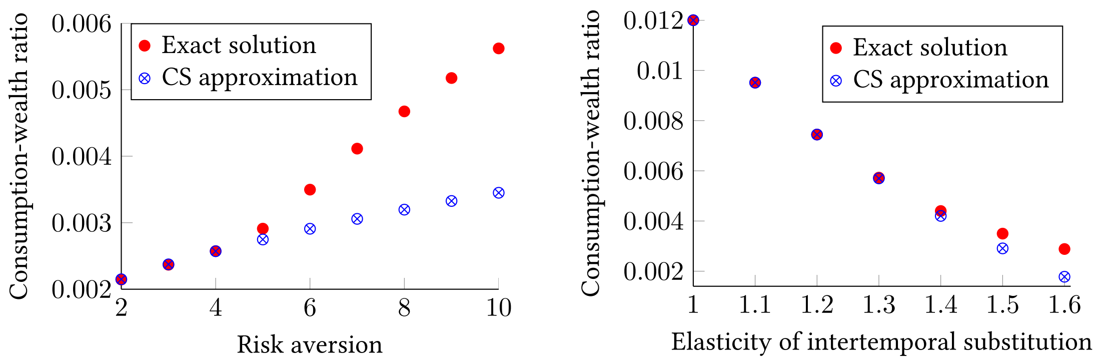

FIGURE 1. **Consumption-Wealth Ratio and CS Approximation.** This figure depicts the exact consumption-wealth ratio and the associated CS approximation as functions of the relative risk aversion γ for EIS $\psi = 1.5$ (left panel) and as functions of the EIS ψ for relative risk aversion $\gamma = 6$ (right panel). We use Wachter's model with $\rho = 0$ and $p = 0.5$. The other parameters can be found below equation (32).

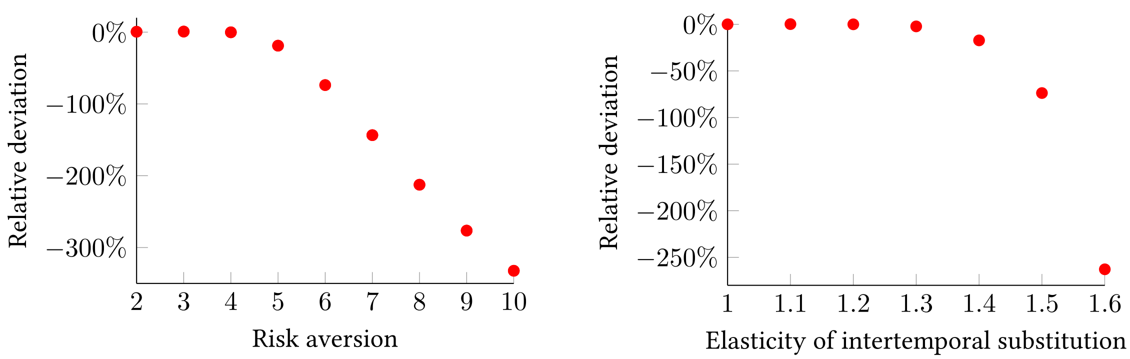

FIGURE 2. **Relative Error of the CS Approximation.** This figure shows the relative error of the CS approximation, as defined in (1), as a function of the relative risk aversion γ and EIS $\psi = 1.5$ (left panel) and as function of the EIS ψ and risk aversion $\gamma = 6$ (right panel). We use Wachter's model with $\rho = 0$ and $p = 0.5$. The other parameters can be found below equation (32).

Here w^{CS} and w are the agent's indirect utility if the CS-approximation or our method is applied, respectively. The variables c and y denote the current values of the endowment and state process. Apparently, the deviations can be tremendous in relative terms. The main issue is now the following: In a specific application and for a specific calibration, it is impossible to know a priori whether the approximation is reasonably accurate (e.g., EIS=1.2 vs. EIS=1.4 in Figure 1 (right panel)). This highlights the importance of the method developed in this paper, which provides the correct solution without resorting to approximations.

4. Consumption Plans and Epstein-Zin Preferences

We fix a probability space $(\Omega, \mathcal{F}, P)$ with a complete right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ that is generated by a Wiener process $(W, \bar{W})$. We denote the *consumption space* by $\mathsf{C} \triangleq (0, \infty)$. In the following, we are interested in an agent's preferences on the space of dynamic consumption plans.

Definition 4.1 (Consumption Plans)**.** A progressively measurable C-valued process c is a *consumption plan* if

$$c \in \mathcal{C} \triangleq \left\{ c \in \mathcal{D}^+ \,:\, \mathrm{E}\left[\int_0^T c_t^p \,\mathrm{d}\, t + c_T^p \right] < \infty \text{ for all } p \in \mathbb{R} \right\}.$$

Here we denote the set of square-integrable progressively measurable processes by

$$\mathcal{D} \triangleq \left\{ X = (X_t)_{t\in[0,T]} \text{ progressively measurable } : \mathrm{E}\left[\int_0^T X_t^2 \,\mathrm{d}\, t + X_T^2 \right] < \infty \right\}$$

and write $\mathcal{D}^+ \triangleq \{X \in \mathcal{D} \,:\, X_t > 0 \text{ for } t \in [0,T]\}$ for its strictly positive cone.

The agent's preferences on $\mathcal{C}$ are described by a continuous-time recursive utility index $\nu : \mathcal{C} \to \mathbb{R}$ so that

$$c \in \mathcal{C} \text{ is weakly preferred to } \bar{c} \in \mathcal{C} \quad \text{if and only if} \quad \nu(c) \geq \nu(\bar{c}),$$

see Epstein and Zin (1989) and Duffie and Epstein (1992b). To construct the Epstein-Zin utility index, let $\delta > 0$, $1 \neq \gamma > 0$ and $1 \neq \psi > 0$ be given and put $\phi \triangleq \frac{1}{\psi}$. If $\gamma < 1$ set $\mathsf{V} \triangleq (0, \infty)$ and for $\gamma > 1$ set $\mathsf{V} \triangleq (-\infty, 0)$. Then the continuous-time Epstein-Zin aggregator is given by $f : \mathsf{C} \times \mathsf{V} \to \mathbb{R}$,

$$f(c, v) \triangleq \delta\theta v \left[\left(\frac{c}{((1-\gamma)v)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right]$$

where $\theta = \frac{1-\gamma}{1-\phi}$. Here γ represents the agent's relative risk aversion, ψ is his elasticity of intertemporal substitution (EIS) and δ is his rate of time preference. One can show (see e.g., Schroder and Skiadas (1999)) that for every consumption plan $c \in \mathcal{C}$ there exists a unique process V^c satisfying

$$V_t^c = \mathrm{E}_t \left[\int_t^T f(c_s, V_s^c) \,\mathrm{d}\, s + U(c_T) \right] \text{ for all } t \in [0,T] \tag{2}$$

where $U : \mathsf{C} \to \mathbb{R}$, $U(x) \triangleq \varepsilon^{1-\gamma} \frac{1}{1-\gamma} x^{1-\gamma}$ is a CRRA utility function for bequest and $\varepsilon \in (0, \infty)$ is a weight factor. This leads to the following definition:

Definition 4.2 (Utility Index)**.** The Epstein-Zin utility index $\nu : \mathcal{C} \to \mathsf{V}$ is given by $\nu(c) \triangleq V_0^c$ where V^c is the unique process satisfying (2).

The classical additive utility specification

$$\nu(c) = \mathrm{E}\left[\int_0^T e^{-\delta s} u(c_s)\,\mathrm{d}\,s + e^{-\delta T} U(c_T)\right]$$

where $u:\ \mathsf{C} \to \mathbb{R}$, $u(x) \triangleq \frac{1}{1-\gamma} x^{1-\gamma}$ is subsumed as the special case of the Epstein-Zin parametrization where $\gamma = \phi$. Hence the analysis of this article applies in particular to consumption-portfolio optimization with additive CRRA preferences and arbitrary risk aversion parameter $\gamma \neq 1$.

Remark. The specifications $\gamma = 1$ or $\phi = 1$ correspond to unit relative risk aversion or unit EIS, respectively; $\gamma = \phi = 1$ represents additive logarithmic utility. The case of unit EIS, $\phi = 1$, is well-understood and has been studied extensively in the literature; see, e.g., Schroder and Skiadas (2003) and Chacko and Viceira (2005). The analysis of this article applies *mutatis mutandis* to these special cases. $\diamond$

5. Consumption-Portfolio Selection with Epstein-Zin Preferences

5.1 Financial Market Model

Two securities are traded. The first is a locally risk-free asset (e.g., a money market account) M with dynamics

$$\mathrm{d}\,M_t = r(Y_t) M_t\,\mathrm{d}\,t,$$

while the second asset (e.g., a stock or stock index) S is risky and follows the dynamics

$$\mathrm{d}\,S_t = S_t\left[(r + \lambda(Y_t))\,\mathrm{d}\,t + \sigma(Y_t)\,\mathrm{d}\,W_t\right].$$

The interest rate $r:\ \mathbb{R} \to \mathbb{R}$ and the stock's excess return and volatility $\lambda, \sigma : \mathbb{R} \to \mathbb{R}$ are assumed to be measurable functions of a state process Y satisfying

$$\mathrm{d}\,Y_t = \alpha(Y_t)\,\mathrm{d}\,t + \beta(Y_t)\left(\rho\,\mathrm{d}\,W_t + \sqrt{1-\rho^2}\,\mathrm{d}\,\bar{W}_t\right), \quad Y_0 = y.$$

Here $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are measurable functions and $\rho \in [-1, 1]$ denotes the correlation between stock returns and the state process. Throughout this article, we assume:

- (A1) The coefficients $r, \lambda, \sigma, \alpha$ are bounded and Lipschitz continuous; the coefficient β is bounded and has a bounded Lipschitz continuous derivative.
- (A2) Ellipticity condition: $\inf_{y\in\mathbb{R}} \sigma(y) > 0$ and $\inf_{y\in\mathbb{R}} \beta(y) > 0$.

The investor's wealth dynamics are given by

$$\mathrm{d}\,X_t^{\pi,c} = X_t^{\pi,c}\left[(r(Y_t) + \pi_t\lambda(Y_t))\,\mathrm{d}\,t + \pi_t\sigma(Y_t)\,\mathrm{d}\,W_t\right] - c_t\,\mathrm{d}\,t, \quad X_0 = x \tag{3}$$

where π_t denotes the fraction of wealth invested in the risky asset at time t, the constant $x > 0$ is the investor's initial wealth and c his consumption plan.

Definition 5.1 (Admissible Strategies)**.** The pair of strategies (π, c) is *admissible* for initial wealth $x > 0$ if it belongs to the set

$$\mathcal{A}(x) \triangleq \{(\pi, c) \in \mathcal{D} \times \mathcal{C} \,:\, X_t^{\pi,c} > 0 \text{ for all } t \in [0,T] \text{ and } c_T = X_T^{\pi,c}\}\,.$$

Since the investor's dynamic risk preferences are described by a recursive utility functional of Epstein-Zin type, an admissible pair $(\pi, c) \in \mathcal{A}(x)$ yields utility

$$\nu(c) \triangleq V_0^c, \quad \text{where } V_t^c \triangleq \mathrm{E}_t\left[\int_t^T f(c_s, V_s^c)\,\mathrm{d}\,s + U(X_T^{\pi,c})\right] \text{ for } t \in [0,T].$$

Definition 5.2 (Consumption-Portfolio Problem)**.** Given the initial wealth $x > 0$, the investor's *consumption-portfolio problem* is to maximize utility over the class of admissible strategies $\mathcal{A}(x)$, i.e. to

$$\text{find} \quad (\pi^\star, c^\star) \in \mathcal{A}(x) \quad \text{such that} \quad \nu(c^\star) = \sup_{(\pi,c)\in\mathcal{A}(x)} \nu(c). \tag{P}$$

Remark. Problem (P) has been widely studied in the literature: Schroder and Skiadas (1999) investigate the case of complete markets. Schroder and Skiadas (2003) derive abstract first-order conditions and solve the consumption-portfolio problem for an investor with unit EIS. Schroder and Skiadas (2003), Schroder and Skiadas (2005), and Schroder and Skiadas (2008) characterize optimal strategies via the associated abstract first-order conditions, but do not address existence. Chacko and Viceira (2005) obtain closed-form solutions for an investor with unit EIS in an inverse Heston stochastic volatility model, and Kraft, Seifried, and Steffensen (2013) derive explicit solutions for a non-unit EIS investor whose preference parameters satisfy the condition

$$\psi = 2 - \gamma + \tfrac{(1-\gamma)^2}{\gamma}\rho^2. \tag{H}$$

Berdjane and Pergamenshchikov (2013) study the above-described consumption-portfolio problem in the special case where the investor has additive preferences with relative risk aversion $\gamma \in (0,1)$. Figure 3 depicts the parametrizations for which solutions to the problem are known in the literature. ◇

5.2 The HJB Equation

We consider the dynamic programming equation that corresponds to the optimization problem (P)

$$\begin{aligned} 0 = \sup_{\pi\in\mathbb{R},\, c\in(0,\infty)} \Big\{ & w_t + x(r + \pi\lambda)w_x - cw_x + \tfrac{1}{2}x^2\pi^2\sigma^2 w_{xx} + \alpha w_y \\ & + \tfrac{1}{2}\beta^2 w_{yy} + x\pi\sigma\beta\rho w_{xy} + f(c, w)\Big\} \end{aligned} \tag{4}$$

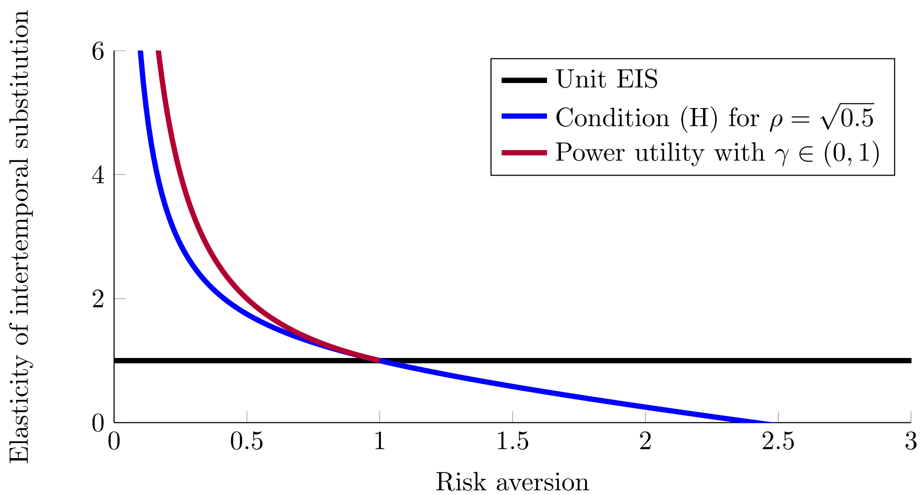

FIGURE 3. **Known Solutions.** This figure depicts combinations of risk aversion γ and elasticity of intertemporal substitution ψ for which solutions to consumption-portfolio problems with unspanned risk are known.

subject to the boundary condition $w(T,x,y) = \varepsilon^{1-\gamma}\frac{1}{1-\gamma}x^{1-\gamma}$. We conjecture

$$w(t,x,y) = \tfrac{1}{1-\gamma}x^{1-\gamma}h(t,y)^k, \quad (t,x,y) \in [0,T] \times (0,\infty) \times \mathbb{R} \tag{5}$$

where k is a constant, $h \in C^{1,2}([0,T] \times \mathbb{R})$ and $h(T,\cdot) = \hat{\varepsilon} \triangleq \varepsilon^{\frac{1-\gamma}{k}}$. Choosing $k \triangleq \frac{\gamma}{\gamma+(1-\gamma)\rho^2}$ and solving the first-order conditions leads to the following definition:

Definition 5.3. The *candidate optimal strategies* are given by

$$\hat{\pi} \triangleq \frac{\lambda}{\gamma\sigma^2} + \frac{k}{\gamma}\frac{\beta\rho}{\sigma}\frac{h_y}{h} \quad \text{and} \quad \hat{c} \triangleq \delta^\psi h^{q-1}x \tag{6}$$

where $q \triangleq 1 - \frac{\psi k}{\theta}$ and where h is a solution of the semilinear partial differential equation (PDE)

$$0 = h_t - \tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta^\psi}{1-q}h^q, \quad h(T,\cdot) = \hat{\varepsilon} \tag{7}$$

with $\tilde{r} \triangleq -\frac{1}{k}\left[r(1-\gamma) + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{\lambda^2}{\sigma^2} - \delta\theta\right]$ and $\tilde{\alpha} \triangleq \alpha + \frac{1-\gamma}{\gamma}\frac{\lambda\beta\rho}{\sigma}$. In the following we refer to (7) as the *reduced HJB equation.*

Remark. The function h in the separation (5) is closely related to the candidate for the agent's optimal consumption-wealth ratio (see, e.g., Campbell, Chacko, Rodriguez, and Viceira (2004), Campbell and Viceira (2002), and Chacko and Viceira (2005)). More precisely, by (6) we have $\frac{\hat{c}}{x} = \delta^\psi h^{-\frac{\psi k}{\theta}}$ so we can represent the candidate for the value function as $w(t,x,y) = \frac{1}{1-\gamma}x^{1-\gamma}\delta^\theta\left(\frac{\hat{c}}{x}\right)^{-\frac{\theta}{\psi}}$.

Lemma 5.4. *If $h \in C^{1,2}([0,T] \times \mathbb{R})$ is a solution of* (7)*, then the function given by $w(t,x,y) = \frac{1}{1-\gamma} x^{1-\gamma} h(t,y)^k$ solves the HJB equation* (4)*.*

Lemma 5.5. *The functions $\tilde{r}$ and $\tilde{\alpha}$ are bounded and Lipschitz continuous.*

Remark. Note that for all $\rho \in [-1,1]$ we have

$$q = 1 - \frac{1-\phi}{1-\gamma} C \quad \text{where} \quad C \triangleq \frac{\psi\gamma}{\gamma(1-\rho^2)+\rho^2} > 0.$$

Thus $q < 1$ if and only if $\frac{1-\phi}{1-\gamma} > 0$ and $q > 1$ if and only if $\frac{1-\phi}{1-\gamma} < 0$; see also Table 1 and Figure 4. ◇

$q < 1$	$q = 1$	$q > 1$
$\frac{1-\phi}{1-\gamma} > 0$	$\phi = 1$	$\frac{1-\phi}{1-\gamma} < 0$

Table 1. **Ranges of q.** This table reports the range of the exponent q in (7) depending on the risk aversion γ and the reciprocal of the elasticity of intertemporal substitution ϕ.

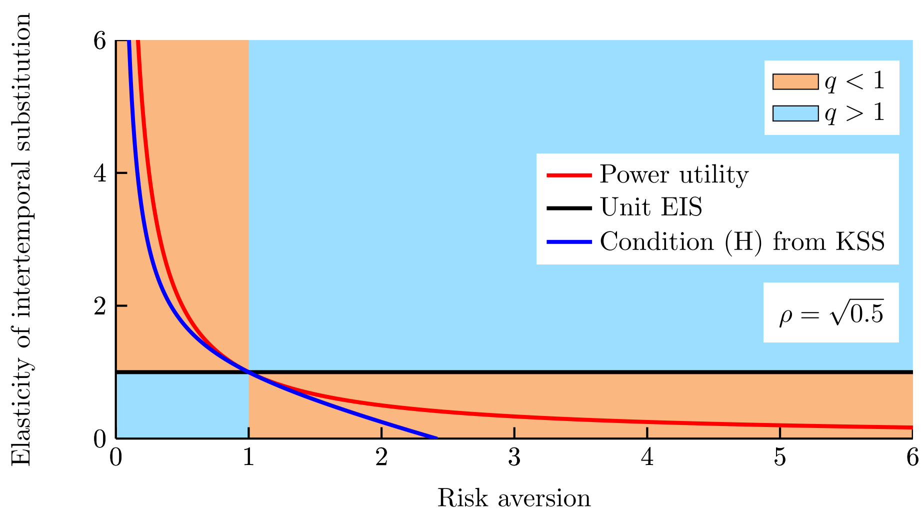

Figure 4. **Range of q.** This figure depicts the range of the exponent q in (7) depending on the risk aversion γ and the elasticity of intertemporal substitution ψ. Condition (H) is calculated for $\rho = \sqrt{0.5}$.

We now state following general existence result for the semilinear PDE (7):

Theorem 5.6. *There exists a solution $h \in C^{1,2}([0,T] \times \mathbb{R})$ to the reduced HJB equation* (7) *and positive constants $0 < \underline{h} < \overline{h}$ such that*

$$\underline{h} \le h \le \overline{h} \quad \textit{and} \quad \|h_y\|_\infty < \infty. \tag{8}$$

Theorem 5.6 is one of the main results of this article. A large part of Section 8 below is dedicated to its proof. Before that, we demonstrate in Sections 6 and 7 how Theorem 5.6 is fundamental for the solutions of both consumption-portfolio choice and asset pricing problems.

6. Verification

In this section, we establish the following verification theorem:

Theorem 6.1 (Solution of the Consumption-Portfolio Problem (P)). *Let h be a solution to the reduced HJB equation* (7) *as in Theorem 5.6. Then the corresponding candidate strategies* $(\hat\pi, \hat c)$*,*

$$(9)\qquad \hat\pi_t = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)^2} + \frac{k}{\gamma}\frac{\beta(Y_t)\rho}{\sigma(Y_t)}\frac{h_y(t,Y_t)}{h(t,Y_t)}, \quad \hat c_t = \delta^\psi h(t,Y_t)^{q-1}X_t^{\hat\pi,\hat c} \quad \text{for } t\in[0,T)$$

and $\hat c_T \triangleq X_T^{\hat\pi,\hat c}$ *are optimal for the consumption-portfolio selection problem* (P).

Here we slightly abuse notation by setting $\hat\pi_t = \hat\pi(t, X_t^{\hat\pi,\hat c}, Y_t)$ and $\hat c_t \triangleq \hat c(t, X_t^{\hat\pi,\hat c}, Y_t)$ for $t\in[0,T)$. This will not give rise to confusion in the following.

6.1 Abstract Utility Gradient Approach

Let $(\bar\pi,\bar c)\in\mathcal{A}(x)$ be a given fixed consumption-portfolio strategy (below we take the candidate solution in (9), but the abstract argument here does not rely on that specific choice). We put

$$\nabla_t(\bar c) \triangleq \begin{cases} f_c(\bar c_t, V_t^{\bar c}) & \text{if } t<T,\\ U'(\bar c_T) & \text{if } t=T,\end{cases}$$

and define the corresponding *utility gradient* by

$$(10)\qquad m_t(\bar c) \triangleq \exp\left(\textstyle\int_0^t f_v(\bar c_s, V_s^{\bar c})\,\mathrm{d}\,s\right)\nabla_t(\bar c).$$

If $\bar c$ satisfies the integrability condition $\mathrm{E}\left[\int_0^T f_c(\bar c_s, V_s^{\bar c})^p\,\mathrm{d}\,s + \exp\left(p\int_0^T f_v(\bar c_s,V_s^{\bar c})\,\mathrm{d}\,s\right)\right]<\infty$ for all $p>0$, then we have the utility gradient inequality

$$V_0^c \le V_0^{\bar c} + \langle m(\bar c), c-\bar c\rangle \quad \text{for all } c\in\mathcal{C}$$

see Duffie and Skiadas (1994) and Schroder and Skiadas (1999). Here the inner product on $\mathcal{D}$ is given by $\langle X,Y\rangle = \mathrm{E}[\int_0^T X_tY_t\,\mathrm{d}\,t + X_TY_T]$.

For every strategy $(\pi,c)\in\mathcal{A}(x)$ we now introduce the *deflated wealth processes*

$$Z_t^{\pi,c} \triangleq \bar m_t X_t^{\pi,c} + \textstyle\int_0^t \bar m_s c_s\,\mathrm{d}\,s, \quad \text{where } \bar m \triangleq m(\bar c).$$

With this we can state the following general verification theorem:

Theorem 6.2 (Abstract Verification)**.** *Suppose that for every admissible strategy $(\pi, c) \in \mathcal{A}(x)$ the deflated wealth process $Z^{\pi,c}$ is a local martingale and that $Z^{\bar{\pi},\bar{c}}$ is a true martingale. Moreover assume that*

$$\mathrm{E}\left[\int_0^T f_c(\bar{c}_s, V_s^{\bar{c}})^p \,\mathrm{d}\, s + \exp\left(p\int_0^T f_v(\bar{c}_s, V_s^{\bar{c}}) \,\mathrm{d}\, s\right)\right] < \infty \quad \textit{for all } p > 0.$$

Then $(\bar{\pi}, \bar{c})$ is optimal.

Proof. The utility gradient inequality at $\bar{c}$ implies

$$V_0^c \leq V_0^{\bar{c}} + \langle \bar{m}, c - \bar{c} \rangle = V_0^{\bar{c}} + \mathrm{E}\left[\int_0^T \bar{m}_s(c_s - \bar{c}_s) \,\mathrm{d}\, s + \bar{m}_T(X_T^{\pi,c} - X_T^{\bar{\pi},\bar{c}})\right].$$

The process $Z^{\pi,c}$ is a positive local martingale and hence a supermartingale. Moreover

$$\int_0^T \bar{m}_s(c_s - \bar{c}_s) \,\mathrm{d}\, s + \bar{m}_T(X_T^{\pi,c} - X_T^{\bar{\pi},\bar{c}}) = Z_T^{\pi,c} - Z_T^{\bar{\pi},\bar{c}},$$

and thus using $X_0^{\pi,c} = X_0^{\bar{\pi},\bar{c}} = x$, we obtain

$$\mathrm{E}[Z_T^{\pi,c} - Z_T^{\bar{\pi},\bar{c}}] \leq \mathrm{E}[Z_0^{\pi,c} - Z_0^{\bar{\pi},\bar{c}}] = f_c(\bar{c}_0, V_0^{\bar{c}})(X_0^{\pi,c} - X_0^{\bar{\pi},\bar{c}}) = 0. \qquad \square$$

6.2 Admissibility of the Candidate Solution $(\hat{\pi}, \hat{c})$

In the proof of Theorem 6.1 below, we will apply the abstract verification result in Theorem 6.2 to the candidate optimal strategy (9). Therefore in the following we verify that the conditions of Theorem 6.2 are satisfied for that strategy.

We first establish admissibility of $(\hat{\pi}, \hat{c})$. Thus suppose that h is a solution of the reduced HJB equation (7), whose existence is guaranteed by Theorem 5.6, and let $(\hat{\pi}, \hat{c})$ be given by (9). For simplicity of notation we write

$$\hat{V} \triangleq V^{\hat{c}}, \qquad \hat{X} \triangleq X^{\hat{\pi},\hat{c}}, \qquad \hat{m} \triangleq m(\hat{c})$$

for the utility process, the wealth process and the utility gradient associated to $(\hat{\pi}, \hat{c})$. The proofs of the following two results are deferred to Appendix A.

Lemma 6.3. *The candidate optimal wealth process has all moments, i.e.*

$$\mathrm{E}\left[\sup\nolimits_{t\in[0,T]} \hat{X}_t^p\right] < \infty \quad \textit{for all } p \in \mathbb{R}.$$

In particular $\hat{X}_t > 0$ for all $t \in [0, T]$ a.s.

As a consequence we can show that $\hat{c} \in \mathcal{C}$ and $\hat{V}_t = w(t, \hat{X}_t, Y_t)$, where by Lemma 5.4 the function $w(t, x, y) \triangleq \frac{1}{1-\gamma} x^{1-\gamma} h(t, y)^k$ solves the HJB equation (4):

Lemma 6.4 (Admissibility of $\hat{c}$)**.** *Let* $V_t \triangleq w(t, \hat{X}_t, Y_t)$, $t \in [0, T]$. *Then* $V = \hat{V}$ *and* $w_x(t, \hat{X}_t, Y_t) = f_c(\hat{c}_t, \hat{V}_t)$. *Moreover we have*

$$\mathrm{E}\left[\sup_{t\in[0,T]}|\hat{c}_t|^p + \sup_{t\in[0,T]}|\hat{V}_t|^p\right] < \infty \quad \textit{for all } p \in \mathbb{R}$$

and in particular $\hat{c} \in \mathcal{C}$.

Combining Lemmas 6.3 and 6.4 it follows in particular that $(\hat{\pi}, \hat{c}) \in \mathcal{A}(x)$.

6.2 Optimality of the Candidate Solution

Next we show that the deflated wealth process $Z^{\pi,c}$ is a local martingale for every admissible consumption-portfolio strategy $(\pi, c) \in \mathcal{A}(x)$. As above, proofs are delegated to Appendix A.

Lemma 6.5 (Dynamics of $Z^{\pi,c}$)**.** *For all* $(\pi, c) \in \mathcal{A}(x)$ *the deflated wealth process* $Z^{\pi,c}$ *is a local martingale with dynamics*

$$\mathrm{d}\, Z_t^{\pi,c} = \hat{m}_t X_t^{\pi,c}\left[\left(\pi_t\sigma(Y_t) - \frac{\lambda(Y_t)}{\sigma(Y_t)}\right)\mathrm{d}\, W_t + k\sqrt{1-\rho^2}\beta(Y_t)\frac{h_y(t,Y_t)}{h(t,Y_t)}\,\mathrm{d}\,\bar{W}_t\right].$$

For the candidate optimal process $(\hat{\pi}, \hat{c})$ *this implies*

$$\mathrm{d}\, Z_t^{\hat{\pi},\hat{c}} = \hat{m}_t \hat{X}_t\left[\left(\frac{1-\gamma}{\gamma}\frac{\lambda(Y_t)}{\sigma(Y_t)} + \frac{k}{\gamma}\beta(Y_t)\rho\frac{h_y(t,Y_t)}{h(t,Y_t)}\right)\mathrm{d}\, W_t + k\sqrt{1-\rho^2}\beta(Y_t)\frac{h_y(t,Y_t)}{h(t,Y_t)}\,\mathrm{d}\,\bar{W}_t\right].$$

To prove Theorem 6.1 it remains to verify that $Z^{\hat{\pi},\hat{c}}$ is in fact a true martingale, and that the utility gradient inequality holds at $\hat{c}$.

Lemma 6.6. *For any* $p > 0$ *we have*

$$\mathrm{E}\left[\int_0^T f_c(\hat{c}_s, \hat{V}_s)^p\,\mathrm{d}\, s + \exp\left(p\int_0^T f_v(\hat{c}_s, \hat{V}_s)\,\mathrm{d}\, s\right)\right] < \infty, \quad \mathrm{E}\left[\sup_{t\in[0,T]}|\hat{m}_t|^p\right] < \infty.$$

Moreover the process $Z^{\hat{\pi},\hat{c}}$ *is a martingale.*

Putting together the above results, we can complete the

Proof of Theorem 6.1. By Lemmas 6.5 and 6.6 the assumptions of Theorem 6.2 are fulfilled. Thus $(\hat{\pi}, \hat{c})$ is optimal for the consumption-portfolio problem (P). □

7. Asset Pricing with Epstein-Zin Preferences

The purpose of this section is to demonstrate the significance of our results for asset pricing applications. Therefore, we introduce a model that nests a continuous-time version of Barro (2006)'s disaster model as well as the model by Wachter (2013) as special cases.

Endowment Process. We assume an endowment economy populated by a representative agent. His endowment (aggregate consumption) has the dynamics

$$\mathrm{d}\,C_t = C_{t-}[\mu(Y_t)\,\mathrm{d}\,t + \sigma(Y_t)\,\mathrm{d}\,W_t + (e^{Z_t}-1)\,\mathrm{d}\,N_t]$$

where $\mathrm{d}\,Y_t = \alpha(Y_t)\,\mathrm{d}\,t + \beta(Y_t)\big(\rho\,\mathrm{d}\,W_t + \sqrt{1-\rho^2}\,\mathrm{d}\,\bar{W}_t\big)$, $Y_0 = y$, are the dynamics of a state process. The process N is a counting process with intensity $\Lambda_t = \Lambda(Y_t)$. We assume that the function $\Lambda(\,\cdot\,)$ satisfies the conditions in (A1) from Section 5. Besides, the random variables Z_t are independent of W, $\bar{W}$ and N with time-invariant distribution ν. We also assume that, with $\gamma > 0$ denoting the agent's relative risk aversion, we have $\mathrm{E}^\nu[e^{(1-\gamma)Z_t}] < \infty$ where $\mathrm{E}^\nu[\,\cdot\,]$ denotes the expectation with respect to the ν-distribution (i.e., $\int e^{(1-\gamma)z}\nu(\mathrm{d}\,z) < \infty$).

Value Function and State-Price Deflator. The representative agent's utility functional is given by[4]

$$V_t^C = \mathrm{E}_t\left[\int_t^T f(C_s, V_s^C)\,\mathrm{d}\,s + U(C_T)\right] \text{ for all } t \in [0,T]$$

where f is the continuous-time Epstein-Zin aggregator and $U(c) \triangleq \varepsilon^{1-\gamma}\frac{1}{1-\gamma}c^{1-\gamma}$. Similar as in Section 5 the agent's value function satisfies a PDE of the form

$$0 = w_t + \mu c w_c + \tfrac{1}{2}c^2\sigma^2 w_{cc} + \alpha w_y + \tfrac{1}{2}\beta^2 w_{yy} + c\sigma\beta\rho w_{cy} + f(c,w) + \Lambda\,\mathrm{E}^\nu[\Delta w]$$

where $w(T,c,y) = \varepsilon^{1-\gamma}\frac{1}{1-\gamma}c^{1-\gamma}$ and $V_t^C = w(t, C_t, Y_t)$. Here

$$\mathrm{E}^\nu[\Delta w](t,c,y) = \mathrm{E}^\nu[w(t, ce^{Z_t}, y)] - w(t,c,y)$$

is the expected change of the value function upon a jump of the endowment process. As in Section 5 the solution takes the form

$$w(t,c,y) = \tfrac{1}{1-\gamma}c^{1-\gamma}h(t,y)^k, \quad (t,c,y) \in [0,T]\times(0,\infty)\times\mathbb{R}$$

where we now set $k = 1$. This leads to the following semilinear PDE for h:

$$0 = h_t - \tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta}{1-q}h^q, \quad h(T,\,\cdot\,) = \hat{\varepsilon} = \varepsilon^{1-\gamma} \tag{11}$$

with $q \triangleq 1 - 1/\theta$, $\tilde{r} \triangleq -[(1-\gamma)\mu - \frac{1}{2}\gamma(1-\gamma)\sigma^2 - \delta\theta + \Lambda(\mathrm{E}^\nu[e^{(1-\gamma)Z_t}] - 1)]$, and $\tilde{\alpha} \triangleq \alpha + (1-\gamma)\sigma\beta\rho$. Since $\mathrm{E}^\nu[e^{(1-\gamma)Z_t}]$ is a time-independent constant and Λ satisfies the conditions in (A1), the PDE (11) takes exactly the same form as (7). Hence, it can be solved with the methods developed in this article. Given the solution h of (11) the state-price deflator m in this economy (i.e., the representative agent's utility

[4]Note that we use a finite time horizon here. By choosing a large T and a suitable weight on bequest, this can be used to approximate the infinite horizon case; see Algorithm 10.2 and the results in Section 10.

gradient) can be expressed in closed form via[5]

$$m_t = \exp\Big(\delta \textstyle\int_0^t \frac{\phi-\gamma}{1-\gamma} h(s, Y_s)^{-\frac{1}{\theta}} - \theta \,\mathrm{d}\, s\Big) C_t^{-\gamma} h(t, Y_t). \tag{12}$$

Using the state-price deflator equilibrium asset prices can now be calculated in a straightforward manner. For instance, the value of the claim to aggregate consumption, i.e. the present value of all future consumption, is given by

$$P_t^C = \textstyle\int_t^T \mathrm{E}_t[\frac{m_s}{m_t} C_s] \,\mathrm{d}\, s + \mathrm{E}_t[\frac{m_T}{m_t} C_T].$$

In particular, we obtain the consumption-wealth ratio[6] as $\frac{C_t}{P_t^C} = \delta h(t, Y_t)^{-\frac{1}{\theta}}$.

8. Feynman-Kac Fixed Point Approach to the HJB Equation

The goal of this section is to prove Theorem 5.6. More precisely, we present a constructive method to obtain a classical solution of the reduced HJB equation (7),

$$0 = h_t - \tilde{r} h + \tilde{\alpha} h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta\psi}{1-q} h^q, \quad h(T, \cdot\,) = \varepsilon^{\frac{1-\gamma}{k}} = \hat{\varepsilon}.$$

To this end, we study the following forward-backward stochastic differential equation (FBSDE) that is associated to the reduced HJB equation:

$$\mathrm{d}\,\eta_t^{t_0,y_0} = \tilde{\alpha}(\eta_t^{t_0,y_0}) \,\mathrm{d}\, t + \beta(\eta_t^{t_0,y_0}) \,\mathrm{d}\, W_t, \quad \eta_{t_0}^{t_0,y_0} = y_0 \tag{13}$$

$$\mathrm{d}\, X_t^{t_0,y_0} = -\left[\tfrac{\delta\psi}{1-q}(X_t^{t_0,y_0})^q - \tilde{r}(\eta_t^{t_0,y_0}) X_t^{t_0,y_0}\right] \mathrm{d}\, t + Z_t^{t_0,y_0} \,\mathrm{d}\, W_t, \quad X_T^{t_0,y_0} = \hat{\varepsilon} \tag{14}$$

where $t_0 \in [0, T]$ and $y_0 \in \mathbb{R}$. We will demonstrate that there exists a unique family $(X^{t,y})_{t\in[0,T]}^{y\in\mathbb{R}}$ of bounded positive solutions that yield a solution to the reduced HJB equation via the generalized Feynman-Kac formula

$$h(t, y) = X_t^{t,y} = \mathrm{E}_t\left[\textstyle\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau^{t,y}) \,\mathrm{d}\,\tau} \tfrac{\delta\psi}{1-q} (X_s^{t,y})^q \,\mathrm{d}\, s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau^{t,y}) \,\mathrm{d}\,\tau}\right].$$

Remark. In this context, a natural way to think of the function h is as the fixed point of the *Feynman-Kac operator* $\Phi:\ C_b([0, T] \times \mathbb{R}) \to C_b([0, T] \times \mathbb{R})$,

$$(\Phi h)(t, y) \triangleq \mathrm{E}_t\left[\textstyle\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau^{t,y}) \,\mathrm{d}\,\tau} \tfrac{\delta\psi}{1-q} h(s, \eta_s^{t,y})^q \,\mathrm{d}\, s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau^{t,y}) \,\mathrm{d}\,\tau}\right].$$

In Section 9 we will elaborate this perspective in detail. ◇

The connection between semilinear PDEs and (F)BSDEs is well-established in the mathematical literature. While classical results, including Pardoux and Peng (1992), Ma, Protter, and Yong (1994) and Ma, Yin, and Zhang (2012), impose a Lipschitz condition on the generator, recent research focuses on relaxing that assumption. Starting

[5] See, e.g., Duffie and Epstein (1992a) and the utility gradient in (10).
[6] See, e.g., Benzoni, Collin-Dufresne, and Goldstein (2011).

from Kobylanski (2000) existence and uniqueness results for BSDEs with quadratic and convex drivers have been obtained. Thus Briand and Carmona (2000), Delarue (2002), Briand and Hu (2008) and Delbaen, Hu, and Richou (2011) replace the Lipschitz assumption by a so-called monotonicity condition, while retaining a polynomial growth condition. In general, however, the driver in the FBSDE system (13), (14) is neither Lipschitz, nor does it satisfy monotonicity or polynomial growth conditions. Hence, results from that literature cannot be applied to this equation. By establishing suitable a priori estimates for (13), (14) and (7), we prove the relevant existence, uniqueness and representation results in the following.

8.1 Solving the FBSDE System: A Fixed Point Approach

Until further notice, we fix $t_0 \in [0,T]$ and $y_0 \in \mathbb{R}$ and let $\eta \triangleq \eta^{t_0,y_0}$ be given by (13). For a progressively measurable process $(X)_{t\in[t_0,T]}$ we write

$$\|X\|_\infty = \operatorname{ess\,sup}_{\mathrm{d}\,t\otimes P} |X_t|$$

and denote by $\mathcal{D}_\infty$ the collection of all progressively measurable processes $(X_t)_{t\in[t_0,T]}$ with $\|X\|_\infty < \infty$. Clearly $(\mathcal{D}_\infty, \|\cdot\|_\infty)$ forms a Banach space. In the following we construct a fixed point of the operator $\Psi : \mathcal{D}_\infty \to \mathcal{D}_\infty$, $X \mapsto \Psi X$ defined via

$$(\Psi X)_t \triangleq \mathrm{E}_t\left[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta^\psi}{1-q}(0\vee X_s)^q\,\mathrm{d}\,s + \hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right]. \tag{15}$$

Remark. The process ΨX is continuous and thus has a progressive modification. Indeed, setting

$$M_t \triangleq \mathrm{E}_t\left[\int_0^T e^{-\int_0^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta^\psi}{1-q}(0\vee X_s)^q\,\mathrm{d}\,s + \hat{\varepsilon}e^{-\int_0^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right]$$

we have that M is a bounded continuous martingale and

$$(\Psi X)_t = e^{\int_0^t \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}M_t - \int_0^t e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta^\psi}{1-q}(0\vee X_s)^q\,\mathrm{d}\,s.$$

In the following we always work with that version of ΨX. ◇

Lemma 8.1. *Let $X \in \mathcal{D}_\infty$ with $\Psi X = X$. Then X solves the BSDE*

$$\mathrm{d}\,X_t = -\left[\tfrac{\delta^\psi}{1-q}(0\vee X_t)^q - \tilde{r}(\eta_t)X_t\right]\mathrm{d}\,t + Z_t\,\mathrm{d}\,W_t, \quad X_T = \hat{\varepsilon}. \tag{16}$$

In particular, if X is positive then it is a solution of (14).

Proof. Let $X \in \mathcal{D}_\infty$ with $\Psi X = X$ and set

$$Y_t \triangleq e^{-\int_{t_0}^t \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}X_t = \mathrm{E}_t\left[\int_t^T e^{-\int_{t_0}^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta^\psi}{1-q}(0\vee X_s)^q\,\mathrm{d}\,s + \hat{\varepsilon}e^{-\int_{t_0}^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right]$$

and
$$M_t = \mathrm{E}_t\left[\int_{t_0}^T e^{-\int_{t_0}^s \tilde r(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta\psi}{1-q}(0\vee X_s)^q\,\mathrm{d}\,s + \hat\varepsilon e^{-\int_{t_0}^T \tilde r(\eta_\tau)\,\mathrm{d}\,\tau}\right].$$
Then M is a bounded martingale and we have
$$Y_t = M_t - \int_{t_0}^t e^{-\int_{t_0}^s \tilde r(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta\psi}{1-q}(0\vee X_s)^q\,\mathrm{d}\,s.$$
With integration by parts it follows that X solves (16). If X is positive then $X = 0\vee X$ and X also solves (14). □

Our construction of a fixed point of Ψ is based on the following ramification of the classical Banach fixed point argument for the space $\mathcal{D}_\infty$:

Proposition 8.2 (Fixed Point Iteration in $\mathcal{D}_\infty$)**.** *Let $S : A \to A$ be an operator on a closed, non-empty subset A of $\mathcal{D}_\infty$ and assume that there are constants $c > 0$, $\varrho \geq 0$ such that for all $X, Y \in A$ we have a Lipschitz condition of the form*
$$|(SX)_t - (SY)_t| \leq c\int_t^T \mathrm{E}_t\left[e^{(s-t)\varrho}|X_s - Y_s|\right]\mathrm{d}\,s \quad \textit{a.s. for all } t\in[t_0,T].$$
Then S has a unique fixed point. Moreover, the iterative sequence $X_{(n)} \triangleq SX_{(n-1)}$ $(n = 1, 2, \dots)$ with an arbitrarily chosen $X_{(0)} \in A$ satisfies
$$\|X_{(n)} - X\|_\infty \leq e^{T\varrho}(\|X_{(0)}\| + \|X\|_\infty)\left(\tfrac{ecT}{n}\right)^n \quad \textit{for all } n > cT.$$

Proof. The proof is provided in Appendix A. □

The following general convergence theorem is the main result of Section 8.

Theorem 8.3 (Fixed Point and Convergence)**.** *Let $t_0 \in [0,T]$ and $y_0 \in \mathbb{R}$. Then there is a unique progressively measurable process $X^{t_0,y_0} \in \mathcal{D}_\infty$ that solves (14). Moreover there are constants $0 < \underline{h} < \overline{h}$ such that $\underline{h} \leq X^{t_0,y_0} \leq \overline{h}$ for all $(t_0,y_0) \in [0,T]\times\mathbb{R}$. The sequence defined by $X^{t_0,y_0}_{(0)} \triangleq \hat\varepsilon$ and $X^{t_0,y_0}_{(n)} \triangleq \Psi X^{t_0,y_0}_{(n-1)}$ $(n = 1, 2, \dots)$ satisfies*
$$\|X^{t_0,y_0}_{(n)} - X^{t_0,y_0}\|_\infty \leq C\left(\tfrac{c}{n}\right)^n \;\textit{ for all } n > \tfrac{c}{e}$$
where the constants $C, c > 0$ are explicitly given by $C \triangleq e^{T\|r\|_\infty}(\hat\varepsilon + \overline{h})$ and
$$c \triangleq eTq|\tfrac{\delta\psi}{1-q}|\underline{h}^{q-1} \textit{ for } q < 1, \quad c \triangleq eTq|\tfrac{\delta\psi}{1-q}|\overline{h}^{q-1} \textit{ for } q > 1. \tag{17}$$

In order to prove Theorem 8.3 we distinguish the cases $q < 1$ and $q > 1$.

Proof of Theorem 8.3 for $\mathbf{q < 1}$. Throughout this paragraph we assume that $q < 1$. We consider the operator Ψ defined in (15) on the Banach space
$$A^{<1} \triangleq \{X \in \mathcal{D}_\infty : X_t \geq \underline{h},\ \mathrm{d}\,t\otimes P\text{-a.e.}\}, \quad \text{where } \underline{h} \triangleq \hat\varepsilon e^{-T\|\tilde r\|_\infty} > 0. \tag{18}$$

Lemma 8.4. *The operator $\Psi : A^{<1} \to A^{<1}$ is well-defined and satisfies*

$$|(\Psi X)_t - (\Psi \tilde{X})_t| \leq c \int_t^T e^{(s-t)\|\tilde{r}\|_\infty} \mathrm{E}_t[|X_s - \tilde{X}_s|] \,\mathrm{d}\, s \quad \text{for all } X, \tilde{X} \in A^{<1},$$

where $c \triangleq |\frac{\delta^\psi}{1-q}| q \underline{h}^{q-1}$.

Proof. For $X \in A^{<1}$ we obviously have $0 \vee X = X$ and thus

$$(\Psi X)_t = \mathrm{E}_t \left[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau} \frac{\delta^\psi}{1-q} X_s^q \,\mathrm{d}\, s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau} \right] \geq \mathrm{E}_t \left[\hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau} \right] \geq \underline{h}.$$

Moreover, $(\Psi X)_t \leq T e^{T\|\tilde{r}\|_\infty} \frac{\delta^\psi}{1-q} (\underline{h}^q + \|X\|_\infty^q) + \hat{\varepsilon} e^{T\|\tilde{r}\|_\infty}$ and it follows that $\Psi : A^{<1} \to A^{<1}$ is well-defined. For the second part of the claim, note that the mapping $[\underline{h}, \infty) \to \mathbb{R}, x \mapsto x^q$ is Lipschitz continuous with Lipschitz constant $L \triangleq q \underline{h}^{q-1}$. Thus we have

$$|(\Psi X)_t - (\Psi \tilde{X})_t| \leq \frac{\delta^\psi}{1-q} L \int_t^T e^{(s-t)\|\tilde{r}\|_\infty} \mathrm{E}_t[|X_s - \tilde{X}_s|] \,\mathrm{d}\, s. \qquad \square$$

Theorem 8.5 (Fixed Point and Convergence: $q < 1$)**.** *Suppose that $q < 1$. There exists a progressively measurable process $X \in \mathcal{D}_\infty$ with*

$$X_t = (\Psi X)_t = \mathrm{E}_t \left[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau} \frac{\delta^\psi}{1-q} X_s^q \,\mathrm{d}\, s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau} \right].$$

Starting from $X^{(0)} = \hat{\varepsilon}$ the iterative sequence $\underline{h} \leq X^{(n)} \triangleq \Psi X^{(n-1)}$ $(n = 1, 2, \dots)$ converges to X in $\mathcal{D}_\infty$. In addition we have $\underline{h} \leq X \leq \overline{h}$ where both $\underline{h}$, given by (18), and $\overline{h} > 0$ are independent of (t_0, y_0). Finally, the process X is the unique fixed point of Ψ that is bounded below by $\underline{h}$.

Proof. It is clear that $\underline{h} \leq \hat{\varepsilon} = X^{(0)}$ and thus $X^{(0)} \in A^{<1}$. Lemma 8.4 implies that $X^{(n)} \in A^{<1}$ for each member of the iterative sequence $X^{(n)} = \Psi X^{(n-1)}$. Applying Proposition 8.2 to the mapping $\Psi : A^{<1} \to A^{<1}$ it follows that the iterative sequence converges in norm to the unique fixed point $X = \Psi X$. In particular $0 < \underline{h} \leq X$ and

$$X_t = (\Psi X)_t = \mathrm{E}_t[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau} \frac{\delta^\psi}{1-q} X_s^q \,\mathrm{d}\, s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau}].$$

To establish the upper bound observe that by Lemma 8.1, X satisfies

$$\mathrm{d}\, X_t = - \left[\frac{\delta^\psi}{1-q} X_t^q - \tilde{r}(t, \eta_t) X_t \right] \mathrm{d}\, t + \mathrm{d}\, M_t, \quad X_T = \hat{\varepsilon}$$

where M is an L^2-martingale. Hence for every stopping time τ we have

$$\begin{aligned} 1_{\{\tau > t\}} X_t &= \mathrm{E}_t \left[1_{\{\tau > t\}} \int_t^\tau \left(\frac{\delta^\psi}{1-q} X_s^q - \tilde{r}(s, \eta_s) X_s \right) \mathrm{d}\, s + 1_{\{\tau > t\}} X_\tau \right] \\ &\leq \mathrm{E}_t \left[1_{\{\tau > t\}} \int_t^\tau (a X_s + b) \,\mathrm{d}\, s + 1_{\{\tau > t\}} X_\tau \right] \end{aligned}$$

where $a \triangleq \frac{\delta\psi}{1-q} + \|\tilde{r}\|_\infty > 0$ and $b \triangleq \frac{\delta\psi}{1-q}(1 + \underline{h}^q)$. Thus we can apply a variant of the stochastic Gronwall-Bellman inequality, see Proposition A.1, and we obtain

$$X_t \leq \mathrm{E}_t\left[\int_t^T e^{a(s-t)} b \,\mathrm{d}\, s + e^{a(T-t)}\hat{\varepsilon}\right] \leq Te^{aT}b + e^{aT}\hat{\varepsilon} \triangleq \overline{h}$$

where $\overline{h}$ is a constant depending only on $\delta, \psi, q, \tilde{r}, \hat{\varepsilon}$ and T. □

Proof of Theorem 8.3 for $q < 1$. Theorem 8.5 yields a unique process X^{t_0,y_0} that satisfies $X^{t_0,y_0} = \Psi X^{t_0,y_0}$ and $0 < \underline{h} \leq X^{t_0,y_0} \leq \overline{h} < \infty$, where the constants $\underline{h}, \overline{h}$ are independent of (t_0, y_0). By Lemma 8.1 the process X^{t_0,y_0} is a solution of (14). Proposition 8.2 shows that the convergence rate of the iterative sequence $X^{t_0,y_0}_{(0)} = \hat{\varepsilon}$, $X^{t_0,y_0}_{(n)} \triangleq \Psi X^{t_0,y_0}_{(n-1)}$ $(n = 1, 2, \dots)$ is given by

$$\|X^{t_0,y_0}_{(n)} - X^{t_0,y_0}\|_\infty \leq e^{T\varrho}(\|X^{t_0,y_0}_{(0)}\| + \|X^{t_0,y_0}\|_\infty)\left(\tfrac{ecT}{n}\right)^n$$

where $\varrho \triangleq \|\tilde{r}\|_\infty$ and $c \triangleq |\frac{\delta\psi}{1-q}|q\underline{h}^{q-1}$ by Lemma 8.4. In view of the fact that $\|X^{t_0,y_0}_{(0)}\|_\infty = \hat{\varepsilon}$ and $\|X^{t_0,y_0}\|_\infty \leq \overline{h}$, this completes the proof. □

Proof of Theorem 8.3 for $\mathbf{q} > 1$. In the following we suppose that $q > 1$. In contrast to the previous paragraph, we now consider the operator Ψ on

$$(19) \qquad A^{>1} \triangleq \{X \in \mathcal{D}_\infty \,:\, X_t \leq \overline{h},\ \mathrm{d}\, t \otimes P\text{-a.e.}\}, \quad \text{where } \overline{h} \triangleq \hat{\varepsilon}e^{T\|\tilde{r}\|_\infty}.$$

Lemma 8.6. *The operator $\Psi : A^{>1} \to A^{>1}$ is well-defined and satisfies*

$$|(\Psi X)_t - (\Psi\tilde{X})_t| \leq c\int_t^T e^{(s-t)\|\tilde{r}\|_\infty}\, \mathrm{E}_t[|X_s - \tilde{X}_s|]\,\mathrm{d}\, s \quad \textit{for all } X, \tilde{X} \in A^{>1},$$

where $c \triangleq |\frac{\delta\psi}{1-q}|q\overline{h}^{q-1}$.

Proof. For any $X \in \mathcal{D}_\infty$ we have

$$(\Psi X)_t = \mathrm{E}_t\left[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\frac{\delta\psi}{1-q}(0 \vee X_s)^q \,\mathrm{d}\, s + \hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right] \leq \mathrm{E}_t\left[\hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right] \leq \overline{h}.$$

In addition we have $(\Psi X)_t \geq -Te^{T\|\tilde{r}\|_\infty}|\frac{\delta\psi}{1-q}|\|X\|^q_\infty$ so that $\Psi : A^{>1} \to A^{>1}$ is well-defined. Since the function $[0, \overline{h}] \to \mathbb{R}, x \mapsto (0 \vee x)^q$ is Lipschitz with Lipschitz constant $L \triangleq q\overline{h}^{q-1}$ we obtain

$$|(\Psi X - \Psi\tilde{X})(t, y)| \leq |\tfrac{\delta\psi}{1-q}|L\int_t^T e^{(s-t)\|\tilde{r}\|_\infty}\, \mathrm{E}_t[|X_s - \tilde{X}_s|]\,\mathrm{d}\, s. \qquad □$$

Theorem 8.7 (Fixed Point and Convergence: $q > 1$)**.** *Let $q > 1$. There is a progressively measurable process $X \in \mathcal{D}_\infty$ with*

$$X_t = (\Psi X)_t = \mathrm{E}_t[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta\psi}{1-q}X_s^q \,\mathrm{d}\, s + \hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}]$$

and the iterative sequence $\overline{h} \geq X^{(n)} \triangleq \Psi X^{(n-1)}$ $(n = 1, 2, \dots)$ *with* $X^{(0)} \triangleq \hat{\varepsilon}$ *converges to* X *in* $\mathcal{D}_\infty$*. Besides we have* $\underline{h} \leq X \leq \overline{h}$ *where* $\underline{h} > 0$ *and* $\overline{h}$ *in* (19) *are independent of* (t_0, y_0)*, and* X *is the unique positive fixed point of* Ψ *that is bounded above by* $\overline{h}$*.*

Proof. We have $X^{(0)} = \hat{\varepsilon} \leq \overline{h}$ and thus $X^{(0)} \in A^{>1}$. By Lemma 8.6 each member of the iterative sequence satisfies $X^{(n)} = \Psi X^{(n-1)} \in A^{>1}$ and in particular $X^{(n)} \leq \overline{h}$. Proposition 8.2 applies to $\Psi : A^{>1} \to A^{>1}$ to show that there is a unique $X \in A^{>1}$ with $\Psi X = X$ and $\|X^{(n)} - X\|_\infty \to 0$. In particular X satisfies $X \leq \overline{h}$.

To demonstrate that $X \geq 0$, recall from Lemma 8.1 that

$$\mathrm{d}\, X_t = -\left[\tfrac{\delta\psi}{1-q}(0 \vee X_t)^q - \tilde{r}(t, \eta_t) X_t\right] \mathrm{d}\, t + \mathrm{d}\, M_t, \quad X_T = \hat{\varepsilon}$$

with an L^2-martingale M. Thus for all stopping times τ we have

$$1_{\{\tau > t\}} X_t = \mathrm{E}_t \left[1_{\{\tau > t\}} \textstyle\int_t^\tau \left(\tfrac{\delta\psi}{1-q}(0 \vee X_s)^q - \tilde{r}(s, \eta_s) X_s\right) \mathrm{d}\, s + 1_{\{\tau > t\}} X_\tau\right].$$

With $L \triangleq q\overline{h}^{q-1}$, the Lipschitz constant of $(-\infty, \overline{h}) \to \mathbb{R}$, $x \mapsto (0 \vee x)^q$, we obtain

$$\begin{aligned} 1_{\{\tau > t\}} X_t &\geq \mathrm{E}_t \left[1_{\{\tau > t\}} \textstyle\int_t^\tau \left(\tfrac{\delta\psi}{1-q} L 1_{\{X_s > 0\}} X_s - \tilde{r}(s, \eta_s) X_s\right) \mathrm{d}\, s + 1_{\{\tau > t\}} X_\tau\right] \\ &= \mathrm{E}_t \left[1_{\{\tau > t\}} \textstyle\int_t^\tau a_s X_s \,\mathrm{d}\, s + 1_{\{\tau > t\}} X_\tau\right] \end{aligned}$$

where the process $a_s \triangleq \frac{\delta\psi}{1-q} L 1_{\{X_s > 0\}} - \tilde{r}(s, \eta_s)$ is bounded and progressively measurable. Now Proposition A.1 yields

$$X_t \geq \mathrm{E}_t \left[e^{\int_t^T a_s \,\mathrm{d}\, s} \hat{\varepsilon}\right] \geq e^{T(\frac{\delta\psi}{1-q} L - \|r\|_\infty)} \hat{\varepsilon} \triangleq \underline{h} > 0$$

where $\underline{h}$ is a constant that depends only on $\delta, \psi, q, \tilde{r}, \hat{\varepsilon}$ and T. In particular X is positive and we have

$$X_t = (\Psi X)_t = \mathrm{E}_t\left[\textstyle\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau} \tfrac{\delta\psi}{1-q} X_s^q \,\mathrm{d}\, s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau) \,\mathrm{d}\, \tau}\right]. \qquad \square$$

Proof of Theorem 8.3 for $q > 1$*.* The proof is the same as in the case $q < 1$, with Theorem 8.5 replaced by Theorem 8.7 and Lemma 8.4 replaced by Lemma 8.6. $\square$

8.2 Differentiability of the Fixed Point

In this section we demonstrate that the solutions X^{t_0, y_0} of (14) provided by Theorem 8.3 yield a solution h to the reduced HJB equation (7)

$$h_t - \tilde{r} h + \tilde{\alpha} h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta\psi}{1-q} h^q = 0, \quad h(T, \cdot\,) = \hat{\varepsilon}.$$

For that purpose we cut off the nonlinearity using the *a priori* estimates provided by Theorem 8.3, which leads us to a PDE that is known to have a classical solution

$g \in C_b^{1,2}([0,T]\times\mathbb{R})$. We then conclude by proving that $g = h$, where $h(t_0, y_0) = X_{t_0}^{t_0,y_0}$. Here and in the following $C_b^{1,2}([0,T]\times\mathbb{R})$ denotes the Banach space of all functions $u : [0,T]\times\mathbb{R}, (t,y) \to u(t,y)$ that are once continuously differentiable with respect to t and twice continuously differentiable with respect to y and have norm $\|u\|_{C^{1,2}} < \infty$. The norm is given by

$$\|u\|_{C^{1,2}} \triangleq \|u\|_\infty + \|u_t\|_\infty + \|u_y\|_\infty + \|u_{yy}\|_\infty \quad \text{for } u \in C_b^{1,2}([0,T]\times\mathbb{R}).$$

Theorem 8.8 (Differentiability, Probabilistic Representation)**.** *Let X^{t_0,y_0} denote the solutions to the FBSDEs* (14) *given by Theorem 8.3 and define*

$$h(t_0, y_0) \triangleq X_{t_0}^{t_0,y_0} \quad \textit{for } (t_0,y_0) \in [0,T]\times\mathbb{R}.$$

Then $h \in C_b^{1,2}([0,T]\times\mathbb{R})$, h satisfies the reduced HJB equation (7)*, and h admits the probabilistic representation*

$$h(t,y) = \mathrm{E}_t\left[\int_t^T \left(-\tilde{r}(\eta_s^{t,y})h(s,\eta_s^{t,y}) + \tfrac{\delta\psi}{1-q}h(s,\eta_s^{t,y})^q\right)\mathrm{d}\,s + \hat{\varepsilon}\right]. \tag{20}$$

Proof. We take $\underline{h}$ and $\overline{h}$ as in Theorem 8.3 and choose a smooth cut-off function $\varphi \in C_b^1(\mathbb{R})$ such that

$$\varphi(v) = \tfrac{1}{2}\underline{h} \text{ for } v \le \tfrac{1}{2}\underline{h}, \quad \varphi(v) = v \text{ for } v \in [\underline{h}, \overline{h}], \quad \varphi(v) = \overline{h} + 1 \text{ for } v \ge \overline{h}.$$

We set $f(v) \triangleq \frac{\delta\psi}{1-q}\varphi(v)^q$ and consider the semilinear Cauchy problem

$$g_t - \tilde{r}g + \tilde{\alpha}g_y + \tfrac{1}{2}\beta^2 g_{yy} + f(g) = 0, \quad g(T,\,\cdot\,) = \hat{\varepsilon}. \tag{21}$$

The function f is clearly continuously differentiable and bounded with a bounded derivative. Hence by a classical result on semilinear PDEs, see e.g., Theorem 8.1 in Ladyzenskaja, Solonnikov, and Ural'ceva (1968), p. 495, there exists a classical solution $g \in C_b^{1,2}([0,T]\times\mathbb{R})$ to (21).

To demonstrate that $g = h$ we fix $(t_0,y_0) \in [0,T]\times\mathbb{R}$ and set $\bar{X}_t^{t_0,y_0} \triangleq \bar{X}_t \triangleq g(t,\eta_t)$, $t \in [t_0,T]$, where $\eta \triangleq \eta^{t_0,y_0}$ is given by (13). By Itō's formula and (21) we have

$$\mathrm{d}\,\bar{X}_t = -[f(\bar{X}_t) - \tilde{r}(\eta_t)\bar{X}_t]\,\mathrm{d}\,t + \bar{Z}_t\,\mathrm{d}\,W_t, \quad \bar{X}_T = \hat{\varepsilon}, \tag{22}$$

where $\bar{Z}_t \triangleq g_y(t,\eta_t)\beta(\eta_t^{u,y})$ is bounded. On the other hand, Theorem 8.3 yields a unique solution $X \triangleq X^{t_0,y_0}$ of (14), i.e.

$$\mathrm{d}\,X_t = -\left[\tfrac{\delta\psi}{1-q}X_t^q - \tilde{r}(\eta_t)X_t\right]\mathrm{d}\,t + Z_t\,\mathrm{d}\,W_t, \quad X_T = \hat{\varepsilon}.$$

Since $\underline{h} \le X \le \overline{h}$ we have $f(X_t) = \frac{\delta\psi}{1-q}X_t^q$ and therefore X also satisfies

$$\mathrm{d}\,X_t = -[f(X_t) - \tilde{r}(\eta_t)X_t]\,\mathrm{d}\,t + Z_t\,\mathrm{d}\,W_t, \qquad X_T = \hat{\varepsilon}.$$

Thus we conclude that X solves (22), too. Since (22) is a BSDE with a Lipschitz driver, it follows from Theorem 2.1 in El Karoui, Peng, and Quenez (1997) that $X = \bar{X}$. In particular we have $h(t_0, y_0) = X_{t_0}^{t_0,y_0} = \bar{X}_{t_0}^{t_0,y_0} = g(t_0, y_0)$. □

Using Theorem 8.8 we are finally in a position to complete the

Proof of Theorem 5.6. This follows from Theorem 8.3 and Theorem 8.8. □

9. PDE Iteration Approach

In this section we establish an explicit constructive method to obtain the solution of the reduced HJB equation. Existence and uniqueness of the solution are guaranteed by Theorem 5.6 above. More precisely, we will demonstrate that $h_n \triangleq \Phi^n \hat{\varepsilon} \overset{n\to\infty}{\longrightarrow} h$ in $C^{0,1}$, where the operator Φ is given by

$$\Phi : D(\Phi) \subset C_b^{1,2}([0,T] \times \mathbb{R}) \to C_b^{1,2}([0,T] \times \mathbb{R}), \quad f \mapsto \Phi f$$

and $g \triangleq \Phi f$ is the unique classical solution of the linear Cauchy problem

$$0 = g_t - \tilde{r} g + \tilde{\alpha} g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta^\psi}{1-q}(0 \vee f)^q \quad \text{with} \quad g(T, \cdot\,) = \hat{\varepsilon}.$$

Thus h can be determined by iteratively solving linear PDEs.

9.1 PDE Iteration

Our first step is to show that the iteration of PDEs as above is feasible. Thus we verify that the operator Φ is well-defined on its domain $D(\Phi)$ where

$$\begin{aligned} D(\Phi) &\triangleq \{f \in C_b^{1,2}([0,T] \times \mathbb{R}) \,:\, f \geq \underline{h}\} \quad \text{for } q < 1, \quad \text{and} \\ D(\Phi) &\triangleq \{f \in C_b^{1,2}([0,T] \times \mathbb{R}) \,:\, f \leq \overline{h}\} \quad \text{for } q > 1. \end{aligned}$$

Here $\underline{h}, \overline{h}$ are the constants specified in Theorem 8.3.

Lemma 9.1. *Assume $u \in D(\Phi)$. Then there exists a unique $g \in C^{1,2}([0,T] \times \mathbb{R})$ with*

$$0 = g_t - \tilde{r} g + \tilde{\alpha} g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta^\psi}{1-q}(0 \vee u)^q, \quad g(T, \cdot\,) = \hat{\varepsilon}. \tag{23}$$

Proof. If $q < 1$ and $u \geq \underline{h} > 0$, then $f \triangleq \frac{\delta^\psi}{1-q}(0 \vee u)^q \in C^{1,2}([0,T] \times \mathbb{R})$. If $q > 1$ with $u \leq \overline{h} < \infty$, then f is Lipschitz continuous since

$$|f(t,y) - f(t',y')| \leq |\tfrac{\delta^\psi}{1-q}| q \overline{h}^{q-1} |u(t,y) - u(t',y')|.$$

In either case, by classical results, see e.g., Theorem 5.1 in Ladyzenskaja, Solonnikov, and Ural'ceva (1968), p. 320, there is a unique $g \in C^{1,2}([0,T] \times \mathbb{R})$ satisfying (23). □

To establish the link between the iterated solutions h_n of the Cauchy problem and the stochastic processes $X^{t_0,y_0}_{(n)}$ of Section 8, we first record a uniqueness result:

Lemma 9.2. *For every $n \in \mathbb{N}$ the process $X^{(n)} \triangleq X^{t_0,y_0}_{(n)}$ defined in Theorem 8.3 is the unique solution of the linear BSDE*

$$\mathrm{d}\, X^{(n)}_t = -\left[\tfrac{\delta\psi}{1-q}\big(0 \vee X^{(n-1)}_t\big)^q - \tilde{r}(\eta^{t_0,y_0}_t)X^{(n)}_t\right]\mathrm{d}\, t + Z^{(n)}_t \,\mathrm{d}\, W_t, \quad X^{(n)}_t = \hat{\varepsilon}. \tag{24}$$

Proof. With $\varphi \triangleq \frac{\delta\psi}{1-q}(0 \vee X^{(n-1)})^q$, by definition of $X^{(n)}$, we have

$$X^{(n)}_t = \mathrm{E}_t\left[\int_t^T e^{-\int_t^s \tilde{r}(\eta^{t_0,y_0}_\tau)\,\mathrm{d}\,\tau}\varphi_s \,\mathrm{d}\, s + \hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta^{t_0,y_0}_\tau)\,\mathrm{d}\,\tau}\right].$$

By Proposition 2.2 in El Karoui, Peng, and Quenez (1997), $X^{(n)}$ is the unique solution of the linear backward equation $\mathrm{d}\, X^{(n)}_t = -[\varphi_t - \tilde{r}(\eta^{t_0,y_0}_t)X^{(n)}_t]\,\mathrm{d}\, t + Z^{(n)}_t\,\mathrm{d}\, W_t$. □

The connection between h_n and $X^{t_0,y_0}_{(n)}$ is now given as follows:

Theorem 9.3. *For each $n \in \mathbb{N}$ we have $h_n = \Phi^n\hat{\varepsilon} \in D(\Phi)$ and*

$$h_n(t, \eta^{t_0,y_0}_t) = \big(X^{t_0,y_0}_{(n)}\big)_t \quad \textit{for all } t \in [t_0, T],\ (t_0, y_0) \in [0, T] \times \mathbb{R}.$$

Proof. The assertion is clearly true for $n = 0$ since $h_0 = \Phi^0\hat{\varepsilon} = \hat{\varepsilon}$ and $X^{t_0,y_0}_{(0)} = \hat{\varepsilon}$. Assume by induction that $h_{n-1} = \Phi^{n-1}\hat{\varepsilon} \in D(\Phi)$ with

$$h_{n-1}(t, \eta^{t_0,y_0}_t) = \big(X^{t_0,y_0}_{(n-1)}\big)_t \quad \text{for all } t \in [t_0, T],\ (t_0, y_0) \in [0, T] \times \mathbb{R}. \tag{25}$$

By Lemma 9.1 $g \triangleq h_n = \Phi h_{n-1} \in C^{1,2}_b([0, T] \times \mathbb{R})$ is well-defined and satisfies

$$0 = g_t - \tilde{r}g + \tilde{\alpha}g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta\psi}{1-q}(0 \vee h_{n-1})^q, \quad g(T, \cdot\,) = \hat{\varepsilon}. \tag{26}$$

Let $(t_0, y_0) \in [0, T] \times \mathbb{R}$ and $\eta \triangleq \eta^{t_0,y_0}$ be given by (13) and set $X_t \triangleq g(t, \eta_t)$. By (25), (26) and Itō's formula we have

$$\mathrm{d}\, X_t = -\left[\tfrac{\delta\psi}{1-q}\left(0 \vee \big(X^{t_0,y_0}_{(n-1)}\big)_t\right)^q - \tilde{r}(\eta_t)X_t\right]\mathrm{d}\, t + Z_t\,\mathrm{d}\, W_t,$$

where $Z_t \triangleq g_y(t, \eta_t)\beta(\eta_t)$ is bounded. Consequently X is a solution of (24), so by Lemma 9.2 we must have $X = X^{t_0,y_0}_{(n)}$. Hence it follows that

$$h_n(t, \eta^{t_0,y_0}_t) = \big(X^{t_0,y_0}_{(n)}\big)_t \quad \text{for all } t \in [t_0, T],\ (t_0, y_0) \in [0, T] \times \mathbb{R}.$$

For $q < 1$ Theorem 8.5 implies $\underline{h} \leq X^{t_0,y_0}_{(n)}$, whereas for $q > 1$ we have $X^{t_0,y_0}_{(n)} \leq \overline{h}$ by Theorem 8.7. Thus $h_n \in D(\Phi)$, and the induction is complete. □

The convergence $h_n \to h$ can now be established as a corollary of the analysis in Section 8.

Corollary 9.4. *Let* $h \in C_b^{1,2}([0,T] \times \mathbb{R})$ *be the unique solution to the reduced HJB equation* (7)*. Moreover let* $h_n \triangleq \Phi^n \hat{\varepsilon} \in C_b^{1,2}([0,T] \times \mathbb{R})$ *be defined recursively as the unique bounded solution of the Cauchy problem*

$$0 = (h_n)_t - \tilde{r} h_n + \tilde{\alpha}(h_n)_y + \tfrac{1}{2}\beta^2 (h_n)_{yy} + \tfrac{\delta\psi}{1-q}(0 \vee h_{n-1})^q, \quad h_n(T, \cdot\,) = \hat{\varepsilon}.$$

Then, with the constants $C, c > 0$ *given by* (17)*, we have*

$$\|h_n - h\|_\infty \le C \left(\tfrac{c}{n}\right)^n \quad \text{for all } n > \tfrac{c}{e}.$$

Proof. By Theorem 9.3 we have $h_n(t, \eta_t^{t_0,y_0}) = \left(X_{(n)}^{t_0,y_0}\right)_t$ for all $t \in [t_0, T]$, $(t_0, y_0) \in [0,T] \times \mathbb{R}$. Thus Theorem 8.3 yields

$$|h_n(t_0,y_0) - h(t_0,y_0)| = |\left(X_{(n)}^{t_0,y_0}\right)_{t_0} - X_{t_0}^{t_0,y_0}| \le \|X_{(n)}^{t_0,y_0} - X^{t_0,y_0}\|_\infty \le C\left(\tfrac{c}{n}\right)^n$$

for all $n > \frac{c}{e}$ uniformly in $(t_0, y_0) \in [0,T] \times \mathbb{R}$. □

9.2 Convergence Rate of the PDE Iteration in $C^{0,1}$

In this section we use the probabilistic representation (20) of h established in Theorem 8.8 to demonstrate that both h_n and $(h_n)_y$ converge uniformly to h and h_y. We also identify the relevant convergence rate. We replace (A1) by the slightly stronger regularity condition

(A1') The coefficients $r, \lambda, \sigma, \alpha, \beta$ are bounded with bounded, Lipschitz continuous derivatives.

Similarly as in Lemma 5.5 this assumption guarantees that $\tilde{\alpha}$ and β have a bounded Lipschitz continuous derivative. This implies the following estimate for the derivative of the semigroup $(P_s)_{s\in[0,T]}$ generated by $\eta^{0,\cdot}$:

Proposition 9.5 (Derivative of the Semigroup)**.** *Assume that (A1') and (A2) are satisfied and let* $(P_s)_{s\in[0,T]}$ *be the semigroup associated to the process* $\eta^{0,\cdot}$ *given by* (13)*. Then there exists a constant* $M > 0$ *such that for all* $f \in C_b(\mathbb{R})$ *we have*

$$\|D(P_t f)\|_\infty \le M t^{-\frac{1}{2}} \|f\|_\infty \text{ for all } t \in [0,T].$$

Proof. See Theorem 1.5.2 in Cerrai (2001) or Theorem 3.3 in Bertoldi and Lorenzi (2005). □

Remark. See also Elworthy and Li (1994) and Cerrai (1996) for related results. For Hölder-continuous $f \in C_b(\mathbb{R})$, results like Proposition 9.5 are well-known in the literature on parabolic PDEs; see Ladyzenskaja, Solonnikov, and Ural'ceva (1968). ⋄

We are now in a position to establish the convergence of our fixed point iteration in $C^{0,1}([0,T]\times\mathbb{R})$ endowed with the norm $\|h\|_{C^{0,1}} \triangleq \|h\|_\infty + \|\frac{\partial}{\partial y}h\|_\infty$. This provides the rigorous basis for the numerical method we develop in Section 10.

Theorem 9.6 (Convergence in $C^{0,1}$)**.** *The functions h_n $(n=1,2,\dots)$ are uniformly bounded in $C^{0,1}([0,T]\times\mathbb{R})$ and we have*

$$\|h_n - h\|_{C^{0,1}} \leq 2cM\sqrt{T}\left(\|r\|_\infty \tfrac{C}{n} + \tfrac{1}{eT}\right)\left(\tfrac{c}{n-1}\right)^{n-1} \quad \textit{for all } n > \tfrac{c}{e}+1,$$

where $C, c > 0$ are given by (17) *and $M > 0$ is the constant from Proposition 9.5.*

Proof. First note that for each $n \in \mathbb{N}$ Theorem 9.3 implies that $h_n(t, \eta_t^{t_0,y_0}) = \big(X_{(n)}^{t_0,y_0}\big)_t$ so by Lemma 9.2

$$h_n(t,\eta_t^{t_0,y_0}) = \mathrm{E}_t\Big[\int_t^T \tfrac{\delta\psi}{1-q}(0\vee h_{n-1}(s,\eta_s^{t_0,y_0}))^q - \tilde{r}(\eta_s)h_n(s,\eta_s^{t_0,y_0})\,\mathrm{d}\,s + \hat{\varepsilon}\Big]$$

for all $t\in[t_0,T]$. Hence with $f_n \triangleq \frac{\delta\psi}{1-q}(0\vee h_n)^q$ we can represent h_n via

$$h_n(t_0,y_0) = \int_0^{T-t_0}(P_s\tilde{h}_n(t_0,s,\,\cdot\,))(y_0)\,\mathrm{d}\,s + \hat{\varepsilon}$$

where $(P_s)_{s\in[0,T]}$ denotes the semigroup corresponding to $\eta^{0,\cdot}$ and $\tilde{h}_n(t,s,y) \triangleq f_{n-1}(s+t,y) - \tilde{r}(y)h_n(s+t,y)$. Analogously, by Theorem 8.8 we obtain

$$h(t_0,y_0) = \int_0^{T-t}(P_s\tilde{h}(t_0,s,\,\cdot\,))(y)\,\mathrm{d}\,s + \hat{\varepsilon}, \text{ with } \tilde{h}(t,s,\cdot) \triangleq \tfrac{\delta\psi}{1-q}h(s+t,\cdot)^q - \tilde{r}h(s+t,\cdot).$$

Setting $v_n \triangleq \tilde{h}_n - \tilde{h}$, we thus we have $h_n(t_0,\,\cdot\,) - h(t_0,\,\cdot\,) = \int_0^{T-t_0}P_s v_n(t_0,s,\,\cdot\,)\,\mathrm{d}\,s$. Thus with $C, c > 0$ given by (17) and Corollary 9.4 it follows that

$$\|v_n\|_\infty \leq \|r\|_\infty\|h_n - h\|_\infty + L_q\left|\tfrac{\delta\psi}{1-q}\right|\|h_{n-1}-h\|_\infty \leq C\|r\|_\infty(\tfrac{c}{n})^n + L_q\left|\tfrac{\delta\psi}{1-q}\right|(\tfrac{c}{n-1})^{n-1},$$

where $L_q \triangleq 1_{\{q<1\}}q\left|\frac{\delta\psi}{1-q}\right|\underline{h}^{q-1} + 1_{\{q>1\}}q\left|\frac{\delta\psi}{1-q}\right|\overline{h}^{q-1}$. Proposition 9.5 implies

$$\|\tfrac{\partial}{\partial y}h_n(t_0,\,\cdot\,) - \tfrac{\partial}{\partial y}h(t_0,\,\cdot\,)\|_\infty \leq M\|v_n\|_\infty\int_0^{T-t_0}\tfrac{1}{\sqrt{s}}\,\mathrm{d}\,s \leq 2\sqrt{T}M\|v_n\|_\infty.$$

The observations $L_q\left|\frac{\delta\psi}{1-q}\right| = \frac{c}{eT}$ and $(\frac{c}{n})^{n-1} \leq (\frac{c}{n-1})^{n-1}$ complete the proof. □

10. Numerical Results

10.1 User's Guide

Before we study specific applications, we provide a general outline that explains how to apply our theoretical results to concrete consumption-portfolio problems and asset pricing models. By Theorem 6.1, the solution to the consumption-portfolio problem (P) is given by the optimal policies $(\hat{\pi},\hat{c})$ in (9). These depend on the solution of the reduced HJB equation

$$0 = h_t - \tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta^\psi}{1-q}h^q, \quad h(T,\,\cdot\,) = \hat{\varepsilon}, \tag{7}$$

see also Definition 5.3. Analogously, in the asset pricing framework studied in Section 7 the state-price deflator is given by (12),

$$m_t = \exp\left(\delta\int_0^t \tfrac{\phi-\gamma}{1-\gamma}h(s,Y_s)^{-\frac{1}{\theta}} - \theta \,\mathrm{d}\,s\right) C_t^{-\gamma} h(t,Y_t)$$

where h satisfies the semilinear partial differential equation (11), which is also of the form (7). Theorem 5.6 implies that both PDEs have unique bounded classical solutions. Algorithm 10.1 below provides a step-by-step method for the construction of solutions to PDEs of the form (7). This algorithm is easy to implement and relies solely on an efficient method for solving linear PDEs as a prerequisite. Consistency of this approach is guaranteed by Theorem 9.6, which demonstrates that the sequence of solutions provided by Algorithm 10.1 converges to the solution of (7). Theorem 9.6 also implies that the same is true for the associated derivatives. Additionally, Theorem 9.6 ensures a superexponential speed of convergence.

Algorithm 10.1.

(1) Set $h_0 \triangleq \hat{\varepsilon}$ and $n \triangleq 1$.

(2) Compute h_n as the solution g of the linear inhomogeneous PDE

$$0 = g_t - \tilde{r}g + \tilde{\alpha}g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta^\psi}{1-q}(0 \vee h_{n-1})^q, \quad g(T,\,\cdot\,) = \hat{\varepsilon}. \tag{$*$}$$

(3) If h_n is not yet sufficiently close to h_{n-1}, increase n by 1 and return to (2).

To solve the linear PDE ($*$) in Step (2), we use a semi-implicit Crank-Nicolson scheme. Notice that the relevant finite-difference matrices depend on the linear part of the PDE ($*$) only. Therefore, the construction and LU decomposition of these matrices must be carried out only once in a precomputation step. This is one important feature that contributes to the excellent numerical performance of our method.

Remark. We emphasize one important point. Formally, our analysis requires the coefficients of the state process to satisfy assumptions (A1') and (A2) of Section 5. These are standard regularity conditions, but may not be satisfied in specific models such as the Heston (1993) model below. Notice, however, that once such a model is discretized, it is indistinguishable from a model that satisfies (A1') and (A2). $\diamond$

In some applications (e.g., asset pricing), the solution to an infinite-horizon problem is needed. In this case the following extension of Algorithm 10.1 can be used:

Algorithm 10.2.

(1) Fix $\varepsilon > 0$ and a *moderate* time horizon T and set $h^0 \triangleq \hat{\varepsilon}$ and $n \triangleq 1$.

(2) Use Algorithm 10.1 to compute h^n as the solution h of the finite-horizon semi-linear PDE

$$0 = h_t - \tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta\psi}{1-q}h^q, \quad h(T,\,\cdot\,) = h^{n-1}(0,\,\cdot\,).$$

(3) If $h^n(0,\,\cdot\,)$ is not yet sufficiently close to $h^{n-1}(0,\,\cdot\,)$, increase n by 1 and return to (2); otherwise return $h \triangleq h^n(0,\,\cdot\,)$.

In Step (1) one may take, e.g., $\varepsilon = 1$ and $T = 1$. By construction, it is clear that $g^n : [0, nT] \times \mathbb{R} \to \mathbb{R}$ with $g^n(t, y) \triangleq h^{n-k}(t - kT, y)$ for $t \in [kT, (k+1)T]$ solves

$$0 = g_t - \tilde{r}g + \tilde{\alpha}g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta}{1-q}g^q, \quad g(nT,\,\cdot\,) = \hat{\varepsilon}.$$

Under a suitable transversality condition,[7] the limit $h \triangleq \lim_{n\to\infty} g^n(0,\,\cdot\,)$ is a solution of the infinite-horizon equation

$$0 = -\tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta}{1-q}h^q.$$

The specific choice of ε and T becomes irrelevant in the limit $n \to \infty$.

10.2 Consumption-Portfolio Optimization with Stochastic Volatility

Generalized Square-Root and GARCH Diffusions. We first illustrate our approach for the model specification

$$\mathrm{d}\,S_t = S_t[(r + \bar{\lambda}Y_t)\,\mathrm{d}\,t + \sqrt{Y_t}\,\mathrm{d}\,W_t] \tag{27}$$

with constant interest rate r and constant $\bar{\lambda}$, i.e. we consider a stochastic volatility model with stochastic excess return. The state process satisfies

$$\mathrm{d}\,Y_t = (\vartheta - \kappa Y_t)\,\mathrm{d}\,t + \bar{\beta}Y_t^p(\rho\,\mathrm{d}\,W_t + \sqrt{1-\rho^2}\,\mathrm{d}\,\bar{W}_t) \tag{28}$$

with mean reversion level ϑ/κ, mean reversion speed κ, and $p \in [0.5, 1]$. For $p = 0.5$ we obtain the Heston (1993) model and for $p = 1$ a GARCH diffusion model. Christoffersen, Jacobs, and Mimouni (2010) test the empirical performance of stochastic volatility models and find that models with $p = 1$ outperform the Heston model. Note that closed-form solutions for consumption-portfolio problems with such dynamics are only available in the special case $p = 0.5$, but solely with specific parameter choices. Further note that for $p > 0.5$ the model is not affine, i.e. explicit solutions cannot be expected. We choose the model coefficients as follows:

$$r = 0.02, \quad \kappa = 5, \quad \tfrac{\vartheta}{\kappa} = 0.15^2, \quad \bar{\lambda} = 3.11, \quad \rho = -0.5, \quad \text{and} \quad \bar{\beta} = 0.25 \tag{29}$$

[7] See Duffie and Lions (1992) or Appendix C of Duffie and Epstein (1992b) with C. Skiadas.

so that for $p = 0.5$ the calibration is similar to that of Liu and Pan (2003). Furthermore, we assume that the agent's rate of time preference is $\delta = 0.05$ and that the bequest motive is $\varepsilon = 1$. The time horizon is set to $T = 10$ years. We begin with numerical examples for the Heston model (i.e., $p = 0.5$ in (28)).

Computational Efficiency. The theoretical convergence rate identified in Theorem 9.6 materializes quickly in practice. Typical running times for the solutions reported below are well under 5 seconds.[8] To quantify the convergence speed, Figure 5 depicts

[8]Machine: Intel® Core™ i3-540 Processor (4M Cache, 3.06 GHz), 4 GB RAM.

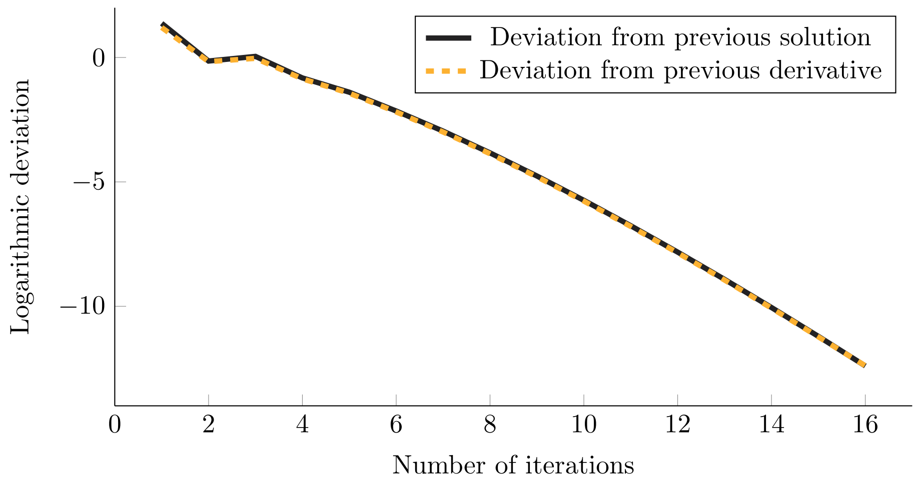

FIGURE 5. **Logarithmic Deviation from Previous Solution.** This figure depicts the convergence speed (30) of the value function. This figure is based on a Heston model with parameters (29).

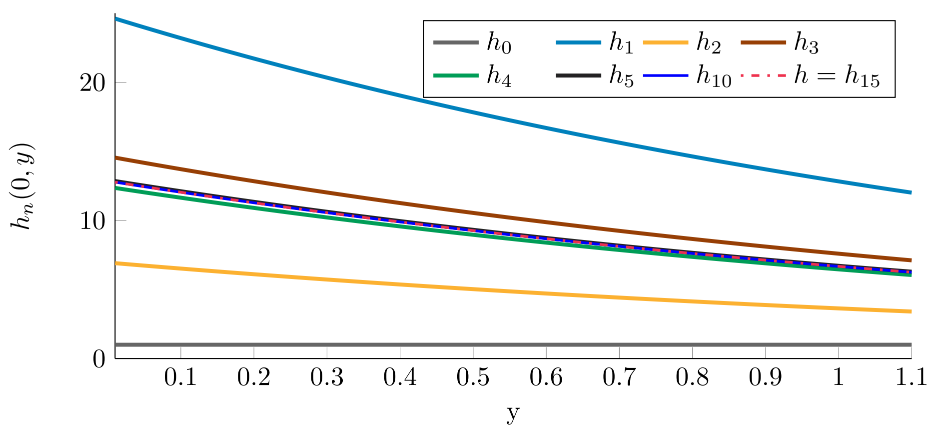

FIGURE 6. **Approximation after n Iteration Steps.** The functions h_n described in Algorithm 10.1 converge to the solution h of the reduced HJB equation. This figure is based on a Heston model with parameters (29).

the logarithmic relative deviations

$$\log_{10}\left(\frac{\|h_n - h_{n-1}\|_\infty}{\|h_{n-1}\|_\infty}\right) \quad \text{and} \quad \log_{10}\left(\frac{\|\frac{\partial}{\partial y}h_n - \frac{\partial}{\partial y}h_{n-1}\|_\infty}{1+\|\frac{\partial}{\partial y}h_{n-1}\|_\infty}\right) \tag{30}$$

as a function of the number of iterations n. Figure 5 clearly illustrates the superlinear convergence of our method. Figure 6 shows the convergence of Algorithm 10.1. We plot the intermediate solutions after $n = 1, 2, \dots, 5, 10, 15$ steps of the iteration. It is apparent that the algorithm converges quickly: After $n = 5$ steps the solution is visually indiscernible from subsequent iterations; the solutions for $n \geq 15$ are even numerically indistinguishable.

Optimal Strategies. Figure 7 illustrates the optimal consumption-wealth ratio $(c/x)^\star$ at time $t = 0$ as a function of initial volatility σ_0 for a risk aversion of $\gamma = 5$ and an EIS of $\psi \in \{0.5, 1, 1.5\}$. For reasonable risk aversions, the optimal stock allocations as a function of σ_0 are almost flat. For instance, for $\gamma \in \{3, 4, 5, 6, 10\}$ and $\psi = 0.5$ the demands vary between about 110% and 30%.

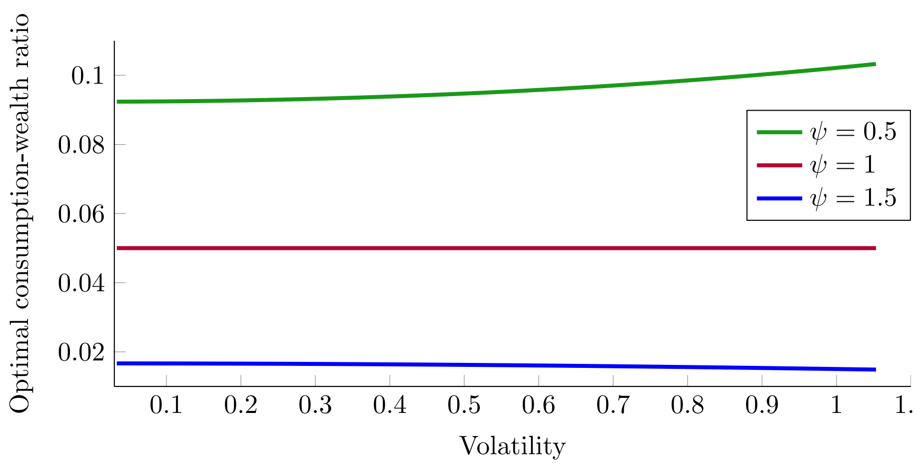

FIGURE 7. **Optimal Consumption-Wealth Ratio.** This figure depicts the optimal consumption-wealth ratio $(c/x)^\star$ at time $t = 0$ as a function of initial volatility σ_0 for a risk aversion of $\gamma = 5$. This figure is based on a Heston model with parameters (29).

Comparison with Known Solutions. Figure 8 shows a range of solutions of (7) as the EIS ψ varies. Here we have chosen $\gamma = 2$ so that for $\psi = 0.125$ (the lowest graph in Figure 8) an explicit solution is available (see Kraft, Seifried, and Steffensen (2013)). For $\psi = 1$ we use the finite-horizon analog of the explicit solution in Chacko and Viceira (2005). The solutions for the other values of the EIS are computed by applying Algorithm 10.1. Note that Figure 8 depicts $g \triangleq h^{\frac{k}{1-\gamma}}$ so that the value function can be represented as $w(t, x, y) = \frac{1}{1-\gamma}x^{1-\gamma}h(t, y)^k = \frac{1}{1-\gamma}\left(g(t, y)x\right)^{1-\gamma}$ where in this context

g can be interpreted as a cash multiplier. Finally, we present comparative statics for the model (27) where we vary the power p. Figure 9 shows the value of the stock demand $\pi^\star$ at time $t = 0$ as a function of the initial volatility σ_0 and the power p. Here $\gamma = 5$ and the EIS is $\psi = 1.5$.

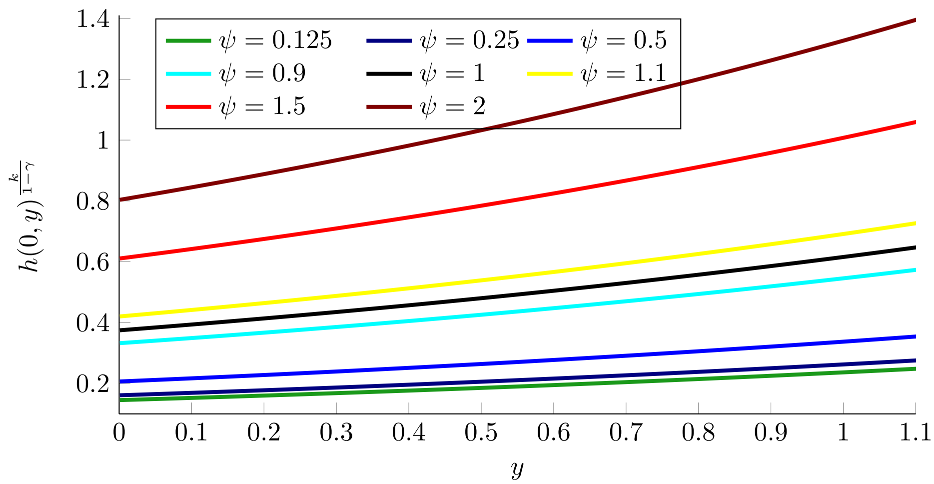

FIGURE 8. **Value Function for Different EIS.** This figure compares the function $h^{\frac{k}{1-\gamma}}$ at time $t = 0$ for a risk aversion of $\gamma = 2$ and an EIS of $\psi \in \{0.125, 0.25, 0.5, 0.9, 1, 1.1, 1.5, 2\}$. It is based on a Heston model with parameters (29).

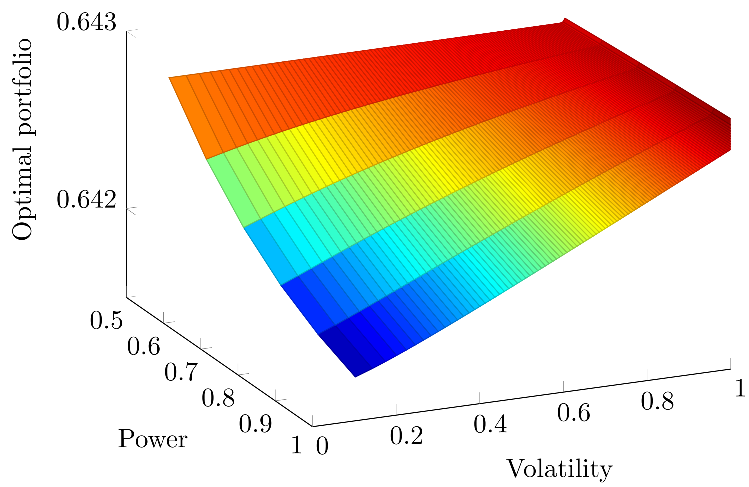

FIGURE 9. **Optimal Stock Demand and Power.** This figure depicts the optimal stock demand $\pi^\star$ at time $t = 0$ as a function of initial volatility σ_0 and the power p. The model is (27) so $p = 0.5$ corresponds to the Heston model. The calibration is given by parameters (29), the agent's risk aversion is $\gamma = 5$ and his EIS is $\psi = 1.5$.

Exponential Vasicek. As another application, we consider a stochastic volatility model where the volatility is lognormally distributed. The asset price dynamics are

$$\mathrm{d}\, S_t = S_t[(r + \bar{\lambda} e^{2Y_t})\,\mathrm{d}\, t + e^{Y_t}\,\mathrm{d}\, W_t]$$

with interest rate $r = 0.05$ and $\bar{\lambda} = 3.11$. The state process has Vasicek dynamics $\mathrm{d}\, Y_t = (\vartheta - \kappa Y_t)\,\mathrm{d}\, t + \bar{\beta}(\rho\,\mathrm{d}\, W_t + \sqrt{1-\rho^2}\,\mathrm{d}\,\bar{W}_t)$ with mean reversion speed $\kappa = 5$ and mean reversion level $\vartheta/\kappa = -1.933$. The correlation is set to $\rho = -0.5$ and we put $\bar{\beta} = 0.57$. These parameters are chosen in such a way that the long-term mean and variance of the squared-volatility process $\sigma_t = e^{2Y_t}$ coincide with those of the squared volatility process in the Heston model (28) calibrated according to (29). We continue to use the rate of time preference $\delta = 0.05$ and the bequest motive $\varepsilon = 1$.

Optimal Strategies. The left panel of Figure 10 depicts the optimal consumption-wealth ratio at time $t = 0$ as a function of initial volatility for a risk aversion of $\gamma = 5$ and an EIS of $\psi \in \{0.5, 1, 1.5\}$. The right panel shows optimal stock allocations as a function of initial volatility for $\gamma \in \{3, 4, 5, 6, 10\}$ and $\psi = 0.5$.

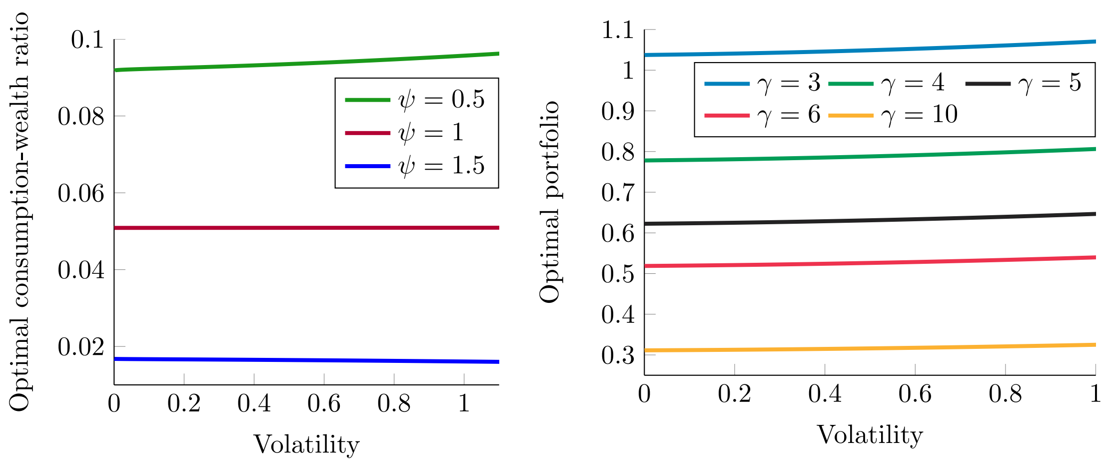

FIGURE 10. **Optimal Consumption and Portfolio Strategies.** The left panel depicts the optimal consumption-wealth ratio $(c/x)^\star$ at time $t = 0$ as a function of initial volatility σ_0 for $\gamma = 5$. The right panel shows the optimal stock allocation $\pi^\star$ at time $t = 0$ as a function of initial volatility σ_0 for $\psi = 0.5$. Both are based on an exponential Vasicek model with $\kappa = 5$, $\vartheta/\kappa = -1.933$, $\rho = -0.5$, and $\bar{\beta} = 0.57$.

10.3 Asset Pricing in Disaster Models

Generalized Square-Root and GARCH Diffusions. In this subsection, we illustrate our general approach for disaster models, which play an important role in asset pricing (see, e.g., Barro (2006)). The endowment process is given by

$$\mathrm{d}\, C_t = C_{t-}[\mu \,\mathrm{d}\, t + \sigma \,\mathrm{d}\, W_t + (e^{Z_t} - 1) \,\mathrm{d}\, N_t] \tag{31}$$

where N is a counting process with intensity $\lambda_t = Y_t$. For $p \in [0.5, 1]$ the state process Y is assumed to satisfy

$$\mathrm{d}\, Y = \kappa(\bar{\lambda} - Y_t) \,\mathrm{d}\, t + \bar{\beta} Y_t^p (\rho \,\mathrm{d}\, W_t + \sqrt{1-\rho^2} \,\mathrm{d}\, \bar{W}_t) \tag{32}$$

with mean reversion speed $\kappa = 0.080$ and mean reversion level $\bar{\lambda} = 0.0355$. Moreover, we set $\mu = 0.0252$, $\sigma = 0.02$ and $\bar{\beta} = 0.067$. The time preference rate is $\delta = 0.012$. The random variables Z_t that model the sizes of disaster events are independent of W, $\bar{W}$ and N and satisfy $\mathrm{E}^\nu[e^{(1-\gamma)Z_t}] = e^{(1-\gamma)0.15}$. Notice that the parameters are calibrated such that for $p = 0.5$ the model by Wachter (2013) obtains. Unless stated otherwise we fix $p = 0.5$.

In the following, we present results for an infinite-horizon economy by applying Algorithm 10.2. Depending on the choice of the model parameters, typical computation times until a steady state is reached vary between 30 and 90 seconds.[9] To demonstrate the efficiency of the algorithm, we first study the convergence to the steady state for bequest motives $\varepsilon \in \{0.1, 1, 10\}$. Figure 11 shows the maximal distance of the corresponding finite time-horizon PDE solution to the infinite-horizon stationary solution if $\gamma = 3$, $\psi = 1.5$, and $\rho = 0$. As expected, the steady-state solutions are independent of the weight on the bequest motive.

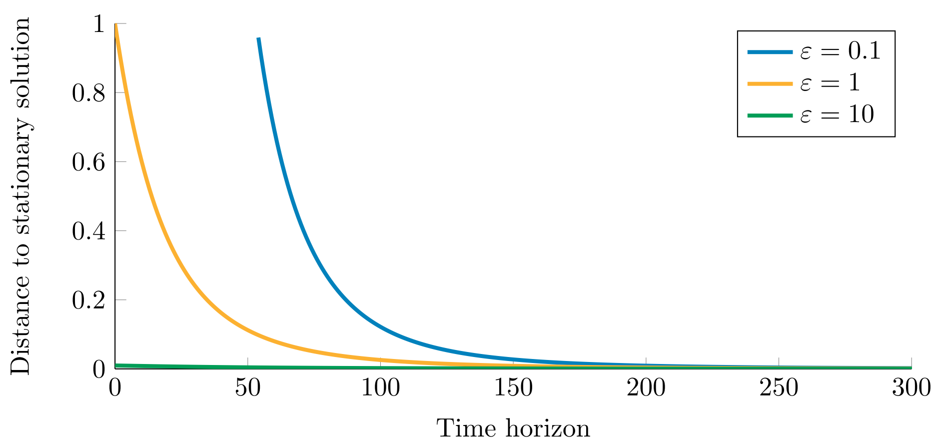

FIGURE 11. **Maximal distance to the stationary solution.** This figure illustrates the speed of convergence for alternative values of the bequest motive ε. We show the maximal distance to the stationary solution as a function of the time horizon.

[9]Machine: Intel® Core™ i3-540 Processor (4M Cache, 3.06 GHz), 4 GB RAM

Figure 12 depicts the consumption-wealth ratio as a function of risk aversion for an initial intensity of $\lambda_0 = \bar{\lambda}$, a correlation of $\rho = 0$, and an EIS of $\psi \in \{0.5, 1, 1.5\}$. Figure 13 shows the consumption-wealth ratio as a function of ρ and λ_0. Here the representative agent's EIS is set to $\psi = 0.5$ and his risk aversion is $\gamma = 3$. Note that Wachter (2013) focuses on $\psi = 1$.

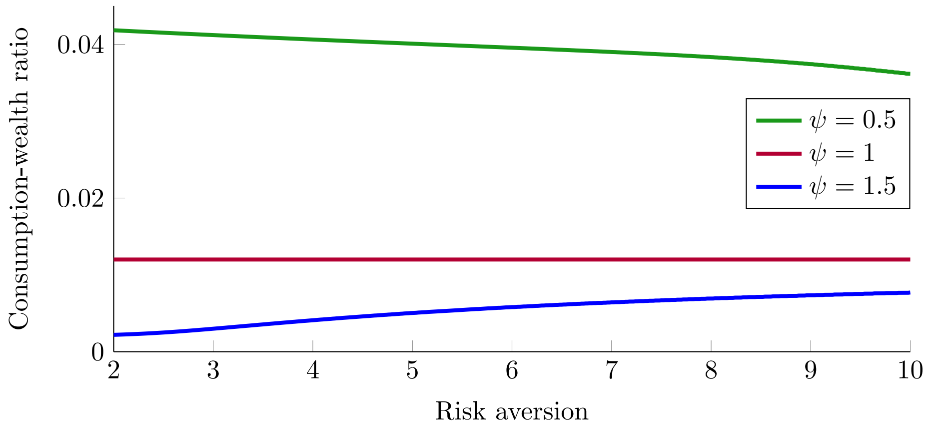

FIGURE 12. **Consumption-Wealth Ratio in Wachter's Model.** This figure depicts the consumption-wealth ratio as a function of the agent's risk aversion for alternative levels of EIS ψ. We set $\rho = 0$ and $p = 0.5$. The other parameters can be found below equation (32). Note that $\psi = 1$ is the case analyzed by Wachter (2013).

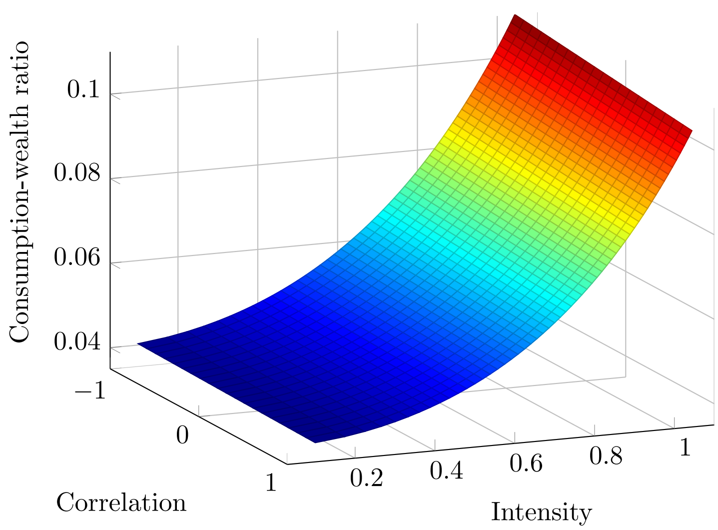

FIGURE 13. **Consumption-Wealth Ratio in Wachter's Model.** This figure shows the consumption-wealth ratio as a function of correlation ρ and initial intensity λ_0. The other parameters can be found below equation (32). The representative agent's risk aversion is $\gamma = 3$ and his EIS is $\psi = 0.5$. Wachter (2013) focuses on $\psi = 1$ and $\rho = 0$.

Finally, we analyze the influence of the power p in (32). Figure 14 shows the consumption-wealth ratio as a function of the power p and the initial intensity λ_0. Here we set $\gamma = 3$, $\psi = 1.5$ and $\rho = 0$. Notice that for all values $p > 0.5$ the model fails to be affine and closed-form solutions are not available.

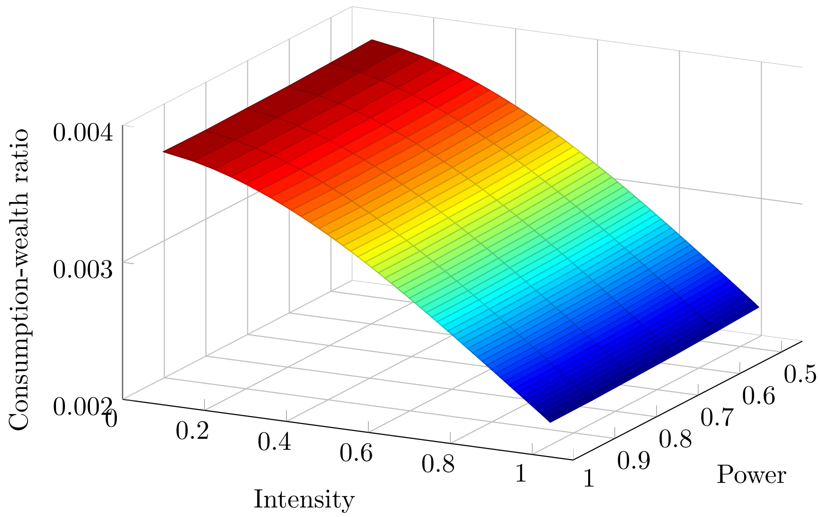

Figure 14. **Consumption-Wealth Ratio in the Generalized Square-Root and GARCH Models**. This figure depicts the consumption-wealth ratio as a function of the power p and the initial intensity λ_0. The other parameters can be found below equation (32). The correlation is $\rho = 0$. The representative agent's risk aversion is $\gamma = 3$ and his EIS is $\psi = 1.5$. Wachter (2013) focuses on $\psi = 1$ and $p = 0.5$.

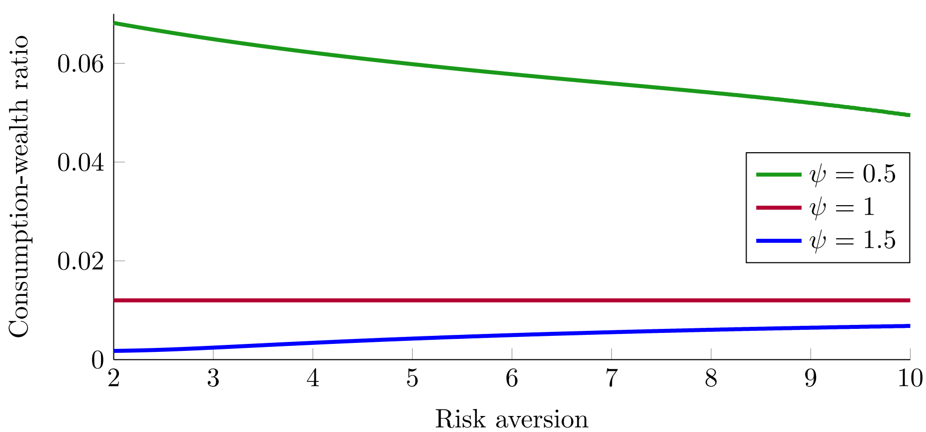

Figure 15. **Consumption-Wealth Ratio in Exponential Vasicek Model.** For different values of EIS, this figure shows the consumption-wealth ratio as a function of the agent's risk aversion γ for $\rho = 0$. The remaining coefficients are chosen to match the calibration of Wachter (2013).

Exponential Vasicek. Finally, we consider a variant of Wachter's model where the intensity process follows an exponential Vasicek process. Aggregate consumption follows the dynamics (31) where the counting process N has intensity $\lambda_t = e^{Y_t}$ and the state process Y satisfies $\mathrm{d}\, Y = (\vartheta - \kappa Y_t)\,\mathrm{d}\, t + \bar{\beta}(\rho\,\mathrm{d}\, W_t + \sqrt{1-\rho^2}\,\mathrm{d}\,\bar{W}_t)$. The mean reversion speed is $\kappa = 0.080$ and the mean reversion level $\bar{y} \triangleq \vartheta/\kappa = -0.058$. Moreover, we set $\mu = 0.0252$, $\sigma = 0.02$ and $\bar{\beta} = 0.305$. These parameters are chosen such that the long-term mean and variance of the intensity process λ match those of the previous disaster model (32) for $p = 0.5$. The time preference rate is set to $\delta = 0.012$ and we assume $\mathrm{E}^{\nu}[e^{(1-\gamma)Z_t}] = e^{(1-\gamma)0.15}$. Figure 15 depicts the consumption-wealth ratio as a function of γ for $\psi \in \{0.5, 1, 1.5\}$, $\lambda_0 = e^{\bar{y}}$, and $\rho = 0$.

11. Conclusion

This paper develops a new method to solve incomplete-market consumption-portfolio problems and asset pricing models with unspanned risk and recursive preferences. We demonstrate that in both settings the agent's value function is characterized by a semilinear partial differential equation. In the literature, solutions of this equation have only been obtained in special cases, and general existence and uniqueness results have not been available. Researchers have thus resorted to approximative methods. This article establishes both theoretical existence and uniqueness results and an efficient numerical method for this equation. Our results are neither restricted to affine asset dynamics, nor do we have to impose any constraints on the agent's risk aversion or elasticity of intertemporal substitution. Based on our theoretical results, we offer an easily-implemented computational method to find the agent's value function by a fixed point iteration. This numerical approach is substantiated by a rigorous convergence analysis. We illustrate our approach using a range of examples from both consumption-portfolio choice and asset pricing. Our findings open up a new avenue of future research: No approximations are needed and models outside the affine class can be studied.

Appendix A. Proofs Omitted from the Main Text

Proof of Lemma 5.4. Since h solves the reduced HJB equation (7) it follows that

$$\begin{aligned} H(z,\hat{\pi},\hat{c}) \triangleq w_t &+ x(r+\hat{\pi}\lambda)w_x - \hat{c}w_x + \tfrac{1}{2}x^2\hat{\pi}^2\sigma^2 w_{xx} + \alpha w_y \\ &+ \tfrac{1}{2}\beta^2 w_{yy} + x\hat{\pi}\sigma\beta\rho w_{xy} + f(\hat{c},w) = 0 \end{aligned}$$

where $z \triangleq (t, x, y, w_x, w_y, w_{x_y} w_{xx}, w_{yy})$. Separating $H(z,\pi,c) \triangleq q(z) + u(z,\pi) + s(z,c)$, it is easy to see that the candidate solutions $\hat{\pi}$ and $\hat{c}$ defined in (6) are the unique solutions of the associated first-order conditions

$$(33)\ 0 = s_c(z,c) = -w_x + f_c(c,w), \quad 0 = u_\pi(z,\pi) = x\lambda w_x + \pi x^2\sigma^2 w_{xx} + x\sigma\beta\rho w_{xy}.$$

By concavity of u and s, it follows that $H(z,\hat\pi,\hat c) = \sup_{\pi\in\mathbb{R},\, c\in(0,\infty)} H(z,\pi,c)$. □

Proof of Lemma 5.5. By (A1) and (A2), $\tilde\alpha$ and $\tilde r$ are bounded. Moreover

$$\begin{aligned}|\tilde\alpha(y) - \tilde\alpha(\bar y)| \le |\alpha(y) - \alpha(\bar y)| + |\tfrac{1-\gamma}{\gamma}|\rho\Big(|\tfrac{\lambda(y)}{\sigma(y)}||\beta(y) - \beta(\bar y)| + |\tfrac{\beta(\bar y)}{\sigma(y)}||\lambda(y) - \lambda(\bar y)|\Big)\\ + |\beta(\bar y)\lambda(\bar y)||\tfrac{\sigma(\bar y)-\sigma(y)}{\sigma(y)\sigma(\bar y)}|\end{aligned}$$

so $\tilde\alpha$ is Lipschitz continuous. Finally,

$$\begin{aligned}k|\tilde r(y) - \tilde r(\bar y)| \le |1-\gamma||r(y) - r(\bar y)| + |\tfrac{1-\gamma}{\gamma}|\|\lambda\|_\infty(\inf_{x\in\mathbb{R}}\sigma(x))^{-2}|\lambda(y) - \lambda(\bar y)|\\ + |\tfrac{1-\gamma}{\gamma}|\|\lambda\|_\infty^2\|\sigma\|_\infty(\inf_{x\in\mathbb{R}}\sigma(x))^{-4}|\sigma(\bar y) - \sigma(y)|.\end{aligned}$$ □

Proof of Lemma 6.3. The candidate optimal wealth process $\hat X$ follows the dynamics

$$\mathrm{d}\,\hat X_t = \hat X_t\Big[(r_t + \tfrac{1}{\gamma}\tfrac{\lambda_t^2}{\sigma_t^2} + \tfrac{k}{\gamma}\tfrac{\lambda_t\beta_t\rho}{\sigma_t}\tfrac{h_y}{h} - \delta^\psi h^{q-1})\,\mathrm{d}\,t + (\tfrac{1}{\gamma}\tfrac{\lambda_t}{\sigma_t} + \tfrac{k}{\gamma}\beta_t\rho\tfrac{h_y}{h})\,\mathrm{d}\,W_t\Big].$$

Put $a_t \triangleq r_t + \frac{1}{\gamma}\frac{\lambda_t^2}{\sigma_t^2} + \frac{k}{\gamma}\frac{\lambda_t\beta_t\rho}{\sigma_t}\frac{h_y}{h} - \delta^\psi h^{q-1}$ and $b_t \triangleq \frac{1}{\gamma}\frac{\lambda_t}{\sigma_t} + \frac{k}{\gamma}\beta_t\rho\frac{h_y}{h}$. Our assumptions on the coefficients and on h_y and h imply that both a and b are bounded. By Itō's formula

$$\hat X_t^p = x^p\,\exp\Big(p\textstyle\int_0^t\big(a_s + \tfrac12(p-1)b_s^2\big)\,\mathrm{d}\,s\Big)\,\mathcal{E}_t\big(p\textstyle\int_0^\cdot b_s\,\mathrm{d}\,W_s\big)$$

where $\mathcal{E}_t(\,\cdot\,)$ denotes the stochastic exponential. Choose $M > 0$ such that $|pa_t| + |p(p-1)b_t^2|, |pb_t| < M$ for all $t \in [0,T]$. By Novikov's condition $\mathcal{E}_t\big(p\int_0^\cdot b_s\,\mathrm{d}\,W_s\big)$ is an L^2-martingale, so using Doob's L^2-inequality we obtain $\mathrm{E}[\sup_{t\in[0,T]}\hat X_t^p] \le 2x^pe^{MT}\,\mathrm{E}[\mathcal{E}_T\big(p\int_0^\cdot b_s\,\mathrm{d}\,W_s\big)^2]^{\frac12} < \infty$. □

Proof of Lemma 6.4. By Itō's formula we have

$$\begin{aligned}\mathrm{d}\,V_t = \big[w_t + \hat X_t(r_t + \hat\pi_t\lambda_t)w_x - \hat c_t w_x + \tfrac12\hat X_t^2\hat\pi_t^2\sigma_t^2 w_{xx} + \alpha_t w_y\\ + \tfrac12\beta_t^2 w_{yy} + \hat X_t\hat\pi_t\sigma_t\beta_t\rho w_{xy}\big]\,\mathrm{d}\,t + \mathrm{d}\,M_t\end{aligned}$$

where M is a local martingale. Hence $\mathrm{d}\,V_t = -f(\hat c_t, V_t)\,\mathrm{d}\,t + \mathrm{d}\,M_t$, by Lemma 5.4. Moreover, exploiting the special form of w we get

$$\mathrm{d}\,M_t = V_t\Big[\tfrac{1-\gamma}{\gamma}\tfrac{\lambda_t}{\sigma_t} + \tfrac{\rho k}{\gamma}\beta_t\tfrac{h_y}{h}\Big]\,\mathrm{d}\,W_t + V_t k\sqrt{1-\rho^2}\beta_t\tfrac{h_y}{h}\,\mathrm{d}\,\bar W_t.$$

Here V_t can be rewritten as $V_t = w(t,\hat X_t,Y_t) = \frac{1}{1-\gamma}\hat X_t^{1-\gamma}h(t,Y_t)^k$. By (8) the function h is bounded and bounded away from zero. Thus $\mathrm{E}[\sup_{t\in[0,T]}|V_t|^p] < \infty$ for all $p \in \mathbb{R}$, by Lemma 6.3. As h_y, λ, β and σ^{-1} are bounded and h is bounded away from zero, the local martingale part in the Itō decomposition of V is an L^2-martingale. By the uniqueness of the stochastic differential utility process, V is the unique utility process $V^{\hat{c}}$ associated to $\hat{c}$. The first-order condition (33) for optimal consumption implies $w_x(t, \hat{X}_t, Y_t) = f_c(t, w(t, \hat{X}_t, Y_t)) = f_c(\hat{c}_t, \hat{V}_t)$. From Lemma 6.3 and the boundedness of $\delta^\psi h(t, Y_t)^{q-1}$, we obtain $\mathrm{E}[\sup_{t\in[0,T]}|\hat{c}_t|^p] < \infty$ for all $p \in \mathbb{R}$. In particular $\hat{c} \in \mathcal{C}$. $\square$

Proof of Lemma 6.5. For simplicity of notation we set $r_t \triangleq r(Y_t)$, $\lambda_t \triangleq \lambda(Y_t)$ and $\sigma_t \triangleq \sigma(Y_t)$. We have $\mathrm{d}\, Z_t^{\pi,c} = \hat{m}_t c_t \,\mathrm{d}\, t + \hat{m}_t \,\mathrm{d}\, X_t^{\pi,c} + X_t^{\pi,c} \,\mathrm{d}\, \hat{m}_t + \mathrm{d}[\hat{m}_t, X_t^{\pi,c}]$, by the product rule. Inserting the dynamics of $X^{\pi,c}$ from (3) we get

$$\mathrm{d}\, Z_t^{\pi,c} = \hat{m}_t X_t^{\pi,c}[(r_t + \pi_t\lambda_t)\,\mathrm{d}\, t + \pi_t\sigma_t \,\mathrm{d}\, W_t] + X_t^{\pi,c} \,\mathrm{d}\, \hat{m}_t + \mathrm{d}[\hat{m}_t, X_t^{\pi,c}].$$

Lemma 6.4 implies that $\hat{V}_t = w(t, \hat{X}_t, Y_t)$ and $\hat{m}_t = e^{\int_0^t f_v(\hat{c}_s, \hat{V}_s)\,\mathrm{d}\, s} w_x(t, \hat{X}_t, Y_t)$. From here on we abbreviate $f_v = f_v(\hat{c}_t, \hat{V}_t)$, $w_x = w_x(t, \hat{X}_t, Y_t)$ etc. Clearly $\mathrm{d}\, \hat{m}_t = \hat{m}_t[f_v \,\mathrm{d}\, t + \frac{\mathrm{d}\, w_x}{w_x}]$. Since $f_v(c, v) = \delta\frac{\phi-\gamma}{1-\phi}c^{1-\phi}[(1-\gamma)v]^{\frac{\phi-1}{1-\gamma}} - \delta\theta$, we get $f_v(\hat{c}_t, w(t, \hat{X}_t, Y_t)) = \frac{\phi-\gamma}{1-\phi}\delta^\psi h^{q-1} - \delta\theta$. By Itō's formula

$$\mathrm{d}\, w_x = w_x\left[\tfrac{w_{xt}}{w_x}\,\mathrm{d}\, t + \tfrac{w_{xx}}{w_x}\,\mathrm{d}\, \hat{X}_t + \tfrac{1}{2}\tfrac{w_{xxx}}{w_x}\,\mathrm{d}[\hat{X}_t] + \tfrac{1}{2}\tfrac{w_{xyy}}{w_x}\,\mathrm{d}[Y_t] + \tfrac{w_{xxy}}{w_x}\,\mathrm{d}[\hat{X}_t, Y_t]\right].$$

Substituting for w yields

$$\tfrac{\mathrm{d}\, w_x}{kw_x} = \tfrac{h_t}{h}\,\mathrm{d}\, t - \tfrac{\gamma}{k}\tfrac{\mathrm{d}\, \hat{X}_t}{\hat{X}_t} + \tfrac{h_y}{h}\,\mathrm{d}\, Y_t + \tfrac{1}{2}\tfrac{\gamma(1+\gamma)}{k}\tfrac{\mathrm{d}[\hat{X}_t]}{\hat{X}_t^2} + \tfrac{1}{2}\left((k-1)\tfrac{h_y^2}{h^2} + \tfrac{h_{yy}}{h}\right)\mathrm{d}[Y_t] - \tfrac{\gamma}{\hat{X}_t}\tfrac{h_y}{h}\,\mathrm{d}[\hat{X}_t, Y_t].$$

Plugging in the candidate $\hat{\pi}$ from (9) and the dynamics of $\hat{X}$ and Y we obtain

$$\tfrac{\mathrm{d}\, w_x}{kw_x} = A_t^1\,\mathrm{d}\, t + A_t^2\,\mathrm{d}\, t - \tfrac{1}{k}\tfrac{\lambda_t}{\sigma_t}\,\mathrm{d}\, W_t + \sqrt{1-\rho^2}\beta_t\tfrac{h_y}{h}\,\mathrm{d}\, \bar{W}_t, \quad \text{where}$$

$$A_t^1 \triangleq \tfrac{h_t}{h} - \tfrac{\gamma}{k}r_t + \tfrac{1}{2}\tfrac{1}{k}\tfrac{1-\gamma}{\gamma}\tfrac{\lambda_t^2}{\sigma_t^2} + \tfrac{1}{\gamma}\tfrac{\lambda_t\beta_t\rho}{\sigma}\tfrac{h_y}{h} + \tfrac{\gamma}{k}\delta^\psi h^{q-1} + \tfrac{k}{2}\tfrac{1+\gamma}{\gamma}\beta_t^2\rho^2\tfrac{h_y^2}{h^2}$$

$$A_t^2 \triangleq \tfrac{h_y}{h}\left(\alpha_t - \tfrac{\rho\beta_t\lambda_t}{\sigma_t}\right) + \tfrac{h_y^2}{h^2}\left(\tfrac{k-1}{2}\beta_t^2 - k\beta_t^2\rho^2\right) + \tfrac{\beta_t^2}{2}\frac{h_{yy}}{h}.$$

For the sum of the $\frac{h_y^2}{h^2}$-terms we have

$$\tfrac{k}{2}\tfrac{1+\gamma}{\gamma}\beta_t^2\rho^2\tfrac{h_y^2}{h^2} + \tfrac{h_y^2}{h^2}\left(\tfrac{k-1}{2}\beta_t^2 - k\beta_t^2\rho^2\right) = \beta_t^2\tfrac{h_y^2}{h^2}\left(\tfrac{k}{2}\rho^2\tfrac{1+\gamma}{\gamma} + \tfrac{k-1}{2} - \rho^2 k\right) = 0$$

by our choice of k. Combining the above we obtain

$$\begin{aligned}\mathrm{d}\, \hat{m}_t = k\hat{m}_t&\left[\tfrac{h_t}{h} + \tfrac{1}{k}\left(-\gamma r_t + \tfrac{1}{2}\tfrac{1-\gamma}{\gamma}\tfrac{\lambda_t^2}{\sigma_t^2} - \delta\theta\right) + \tilde{\alpha}_t\tfrac{h_y}{h} + \tfrac{\beta_t^2}{2}\tfrac{h_{yy}}{h} + \tfrac{\phi\theta}{k}\delta^\psi h^{q-1}\right]\\ &+k\hat{m}_t\left[-\tfrac{1}{k}\tfrac{\lambda_t}{\sigma_t}\,\mathrm{d}\, W_t + \sqrt{1-\rho^2}\beta_t\tfrac{h_y}{h}\,\mathrm{d}\, \bar{W}_t\right]\end{aligned}$$

and it follows that $\mathrm{d}[\hat{m}_t, X_t^{\pi,c}] = -\lambda_t \pi_t \hat{m}_t X_t^{\pi,c}\,\mathrm{d}\,t$. Since h solves (7) we get

$$\begin{aligned}\mathrm{d}\,Z_t^{\pi,c} &= \hat{m}_t X_t^{\pi,c}[(r_t+\pi_t\lambda_t)\,\mathrm{d}\,t + \pi_t\sigma_t\,\mathrm{d}\,W_t] + X_t^{\pi,c}\,\mathrm{d}\,\hat{m}_t + \mathrm{d}[\hat{m}_t, X_t^{\pi,c}]\\ &= \hat{m}_t X_t^{\pi,c}\tfrac{1}{h}\left[h_t - \tilde{r}_t h + \tilde{\alpha}_t h_y + \tfrac{1}{2}\beta_t^2 h_{yy} + \tfrac{\delta^\psi}{1-q}h^q\right]\mathrm{d}\,t + \mathrm{d}\,M_t = \mathrm{d}\,M_t\end{aligned}$$

where $\mathrm{d}\,M_t \triangleq \hat{m}_t X_t^{\pi,c}[(\pi_t\sigma_t - \frac{\lambda_t}{\sigma_t})\,\mathrm{d}\,W_t + k\sqrt{1-\rho^2}\beta_t\frac{h_y}{h}\,\mathrm{d}\,\bar{W}_t]$ defines a local martingale M. A direct calculation using the definition of $\hat{\pi}$ yields the statement for $Z^{\hat{\pi},\hat{c}}$. □

Proof of Lemma 6.6. Recall that $\underline{h} \le h \le \overline{h}$ so

$$f_v(\hat{c}_s, \hat{V}_s) = \tfrac{\phi-\gamma}{1-\phi}\delta^\psi h(s,Y_s)^{q-1} - \delta\theta \le |\tfrac{\phi-\gamma}{1-\phi}|\delta^\psi\left(\underline{h}^{q-1} + \overline{h}^{q-1}\right) + |\delta\theta| \triangleq m_1$$

and we get $0 \le \exp(p\int_0^T f_v(\hat{c}_s,\hat{V}_s)\,\mathrm{d}\,s) \le e^{Tpm_1}$. On the other hand, it follows from Lemma 6.4 that $\mathrm{E}[\sup_{t\in[0,T]} f_c(\hat{c}_t,\hat{V}_t)^p] < \infty$ for all $p\in\mathbb{R}$. This proves the first part of the claim and implies the asserted estimate for $\hat{m}_t = \exp(\int_0^t f_v(\hat{c}_s,\hat{V}_s)\,\mathrm{d}\,s) f_c(\hat{c}_s,\hat{V}_s)$. To show that $Z^{\hat{\pi},\hat{c}}$ is a martingale, note that $\frac{1-\gamma}{\gamma}\frac{\lambda_t}{\sigma_t} + \frac{k}{\gamma}\beta_t\rho\frac{h_y}{h}$ is uniformly bounded by some $c>0$. By Lemma 6.6 and Lemma 6.3 we have

$$\int_0^T \mathrm{E}\left[\hat{m}_t^2\hat{X}_t^2(\tfrac{1-\gamma}{\gamma}\tfrac{\lambda_t}{\sigma_t} + \tfrac{k}{\gamma}\beta_t\rho\tfrac{h_y}{h})^2\right]\mathrm{d}\,t \le c^2\int_0^T\sqrt{\mathrm{E}[\hat{m}_t^4]\,\mathrm{E}[\hat{X}_t^4]}\,\mathrm{d}\,t < \infty.$$

Analogously we obtain $\int_0^T \mathrm{E}[\hat{m}_t^2\hat{X}_t^2(k\sqrt{1-\rho^2}\beta_t\frac{h_y}{h})^2]\,\mathrm{d}\,t < \infty$. From this and Lemma 6.5, we conclude that $Z^{\hat{\pi},\hat{c}}$ is an L^2-martingale. □

Proof of Proposition 8.2. For $\kappa > c+\varrho$, define a metric d which is equivalent to $\|\cdot\|_\infty$, by $d(X,Y) \triangleq \operatorname{ess\,sup}_{\mathrm{d}\,t\otimes P} e^{-\kappa(T-t)}|X_t - Y_t|$. Then (A,d) is a complete metric space. By definition of d we get $|X_s - Y_s| \le e^{\kappa(T-s)}d(X,Y)$ $\mathrm{d}\,t\otimes P$-a.e., so

$$e^{-\kappa(T-t)}|(SX)_t - (SY)_t| \le e^{-\kappa(T-t)}c\int_t^T e^{(s-t)\varrho}e^{\kappa(T-s)}d(X,Y)\,\mathrm{d}\,s \le \tfrac{c}{\kappa-\varrho}d(X,Y)$$

and we conclude that $d(SX,SY) \le \frac{c}{\kappa-\varrho}d(X,Y)$, where $\frac{c}{\kappa-\varrho} < 1$. Hence S is a contraction on (A,d). Thus by Banach's Fixed Point Theorem there is a unique $X\in A$ with $SX = X$, and for all $n\in\mathbb{N}$ we have $d(X_{(n)},X) \le (\frac{c}{\kappa-\varrho})^n d(X_{(0)},X)$. The preceding argument is valid for every choice of $\kappa > c+\varrho$. To find the optimal choice of κ note that by the above

$$|(X_{(n)})_t - X_t| \le e^{\kappa T}d(X_{(n)},X) \le (\tfrac{c}{\kappa-\varrho})^n e^{\kappa T}d(X_{(0)},X) \le e^{\kappa T}(\|X_{(0)}\|_\infty + \|X\|_\infty)(\tfrac{c}{\kappa-\varrho})^n$$

and thus the error in the n^{th} step is bounded by $e_n(\kappa) \triangleq e^{\kappa T}(\|X_{(0)}\|_\infty + \|X\|_\infty)(\frac{c}{\kappa-\varrho})^n$. Differentiating e_n with respect to κ we see that with $\hat{\kappa}(n) \triangleq \frac{n+T\varrho}{T}$ we get $e_n(\hat{\kappa}(n)) = \min_{\kappa>0} e_n(\kappa)$ for all $n > cT$ and we obtain the asserted error bound. □

Proposition A.1. *Let $A = (A_t)_{t\in[0,T]}$ be bounded and progressive, let $Z \in L^p(P)$ and let B be a progressive process in $L^p(\mathrm{d}\,t \otimes P)$ for some $p > 1$. Moreover let $X = (X_t)_{t\in[0,T]}$ be right-continuous and adapted with $\mathrm{E}[\sup_{t\in[0,T]} |X_t|] < \infty$. If*

$$1_{\{\tau>t\}}X_t \geq \mathrm{E}_t\left[1_{\{\tau>t\}}\int_t^\tau (A_sX_s + B_s)\,\mathrm{d}\,s + 1_{\{\tau>t\}}X_\tau\right] \quad \textit{a.s. for } t \in [0,T]$$

for every stopping time τ and $X_T \geq Z$ a.s., then

$$X_t \geq \mathrm{E}_t\left[\int_t^T e^{\int_t^s A_u\,\mathrm{d}\,u}B_s\,\mathrm{d}\,s + e^{\int_t^T A_s\,\mathrm{d}\,s}Z\right] \quad \textit{for all } t \in [0,T] \textit{ a.s.}$$

Proof. See Lemma C2 in Schroder and Skiadas (1999).[10] □

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[10] An alternative proof is also available from the authors upon request.

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