Capital gains taxes: modeling in continuous time and impacts on investment decisions

Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften

vorgelegt beim Fachbereich 12, Informatik und Mathematik
der Johann Wolfgang Goethe-Universität
in Frankfurt am Main

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

von
Björn Ulbricht
geboren in Ruthweiler/Kusel

Frankfurt 2014
(D 30)

vom Fachbereich 12 (Informatik und Mathematik)
der Johann Wolfgang Goethe–Universität als Dissertation angenommen.

DEKAN
Prof. Dr. Uwe Brinkschulte

GUTACHTER
Prof. Dr. Christoph Kühn
Johann Wolfgang Goethe-Universität Frankfurt am Main

Prof. Dr. Mathias Beiglböck
Universität Wien

DATUM DER DISPUTATION
26.06.2015

Abstract

In the first part of the thesis, we show that the payment flow of a linear tax on trading gains from a security with a semimartingale price process can be constructed for all càglàd and adapted trading strategies. It is characterized as the unique continuous extension of the tax payments for elementary strategies w.r.t. the convergence "uniformly in probability". In this framework, we prove that under quite mild assumptions dividend payoffs have almost surely a negative effect on investor's after-tax wealth if the riskless interest rate is always positive. In addition, we give an example for tax-efficient strategies for which the tax payment flow can be computed explicitly.

In the second part of the thesis, we investigate the impact of capital gains taxes on optimal investment decisions in a quite simple model. Namely, we consider a risk neutral investor who owns one risky stock from which she assumes that it has a lower expected return than the riskless bank account and determine the optimal stopping time at which she sells the stock to invest the proceeds in the bank account up to the maturity date. In the case of linear taxes and a positive riskless interest rate, the problem is nontrivial because at the selling time the investor has to realize book profits which triggers tax payments. We derive a boundary that is continuous and increasing in time, and decreasing in the volatility of the stock such that the investor sells the stock at the first time its price is smaller or equal to this boundary.

Acknowledgement

Foremost, I would like to express my sincere gratitude to my advisor Prof. Dr. Christoph Kühn for leading me to my research area and for the continuous support of my Ph.D. study, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I am grateful to him for helping me to work out many details of our joint works.

Besides my advisor, I would especially like to thank Prof Dr. Budhi Arta Surya for his support during his stay in Frankfurt and his ideas for parts of this thesis.

Thanks go to Prof. Dr. Mathias Beiglböck for agreeing to act as second referee.

Furthermore, I am grateful for having attended great lectures and seminars of Prof. Dr. Götz Kersting, Prof Dr. Peter Kloeden, Prof Dr. Anton Wakolbinger who drew me into probability theory and related fields.

I am deeply greatful the whole stochastics and discrete mathematics group at Goethe University who made the last few years such a great time for me. Especially, I want to thank Sadik Iliman and Ute Lenz for reading parts of this work.

Last but not least, I especially want to express my gratitude towards my family for always supporting me during my studies and throughout my life.

Contents

Chapter 1

Introduction

The main object of this thesis is to model capital gains taxes for continuous time trading strategies and to examine the impact of taxes on investment decisions.

In real markets, capital gains taxes have a major impact on the investor's wealth after taxes. In Germany, e.g., interest earnings, dividends, and realized stock gains are taxed at 25%. In the USA, capital gains taxes are sometimes even higher and reach from 0% up to 39.6%, depending on several factors like the differentiation of short-term (up to 39.6 %) and long-term capital (up to 20%)[1].
In most countries, an important feature of the tax code is the fact that trading gains are not taxed before the asset is liquidated, i.e., the gain is realized. Thus, the investor can influence the timing of the tax payments, namely she holds a *deferral option*.

Most of the articles dealing with capital gains taxes face the question how to model the so-called tax basis. The tax basis is the relevant reference value for calculating capital gains. Capital gains taxes of a sold stock are calculated by

"tax rate" × ("sale price" – "tax basis").

One can see the difficulty of modeling the tax basis even in a simple example. If the investor buys, e.g., 100 General Motors stocks at time t_1, another 100 at time t_2, and sells 100 at time t_3, it matters *which* of the stocks she sells, as in general $\alpha \cdot 100(S_{t_3} - S_{t_2}) \neq \alpha \cdot 100(S_{t_3} - S_{t_1})$. When the portfolio is liquidated at some time t_4, the difference of the accumulated tax payments disappears because $\alpha \cdot 100(S_{t_3} - S_{t_2}) + \alpha \cdot 100(S_{t_4} - S_{t_1}) = \alpha \cdot 100(S_{t_3} - S_{t_1}) + \alpha \cdot 100(S_{t_4} - S_{t_2})$. But, the order of sales still matters for discounted payments if the riskless interest rate does not vanish.

The regulation described above that leaves it up to the taxpayer to choose which trading gain to realize first when a stock position is reduced is called the *exact tax basis*. An example is the U.S. tax law that allows investors to

[1]the numbers are based on the datas of the Tax Foundation for 2013

use a separate tax basis for each security.

Capital gains taxes also have a major impact on investment decisions. In consideration of the feature that capital profits/losses are not taxed before the asset liquidated, the investor has some incentive to realize profits as late as possible to avoid tax-payments, but this can be at odds with portfolio regroupings in order to earn higher returns before taxes.

Although, in practice, capital gains taxes may be the most relevant market friction, there is only little literature on capital gains taxes in advanced continuous time models. Ben Tahar et al. [BST10; BST07] solve the Merton problem with proportional transaction costs and a tax based on the average of past purchasing prices. This approach has the advantage that the optimization problem is Markov with the one-dimensional tax basis as additional state variable.
Cadenillas and Pliska [CP99] and Buescu et al. [BCP07] maximize the long-run growth rate of investor's wealth in a model with taxes and transaction costs. Here, after each portfolio regrouping, the investor has to pay capital gains taxes for her total portfolio.
Jouini et al. [JKT99; JKT00] consider the first-in-first-out priority rule with one nondecreasing asset price, but with a quite general tax code, and derive first-order conditions for the optimal consumption problem. The problem consists of injecting cash from the income stream into the single asset and withdrawing it for consumption. Consequently, all admissible strategies are of finite variation. Dybvig and Koo [DK96] and DeMiguel and Uppal [DU05] use the exact tax basis in a discrete time model and relate the portfolio optimization problem to nonlinear programming.

Modeling Capital Gains Taxes for Trading Strategies of Infinite Variation

In *Chapter 2*, we want to answer the following question: Can tax payments on capital gains be modeled for continuous time trading strategies of the kind they generally appear in mathematical finance? Most of these strategies possess infinite variation, as e.g., the optimal stock position in the Merton problem or the replicating portfolio of an option in the Black Scholes model. A straight forward construction of the tax payment flow, analogous to time-discrete models, would be based both on accumulated purchases and accumulated sales of assets, but of course, these quantities explode if strategies are of infinite variation.

For simplicity, we consider a *linear* taxing rule with tax rate $\alpha \in (0, 1)$, i.e., if an asset with stochastic price process S is purchased at time t_1 and sold at time t_2, the trading gains $S_{t_2} - S_{t_1}$ are taxed at $\alpha(S_{t_2} - S_{t_1})$. Negative tax payments for losses, so-called tax credits, can be interpreted as a refund of former tax payments or a deduction against future tax payments.

In some countries, the tax basis is the average purchase price of all stocks of the same firm (e.g., in Canada) or the price of the stock which was bought first ("first-in-first-out", a procedure followed, e.g., in Germany). For a comparison of modeling these different tax bases, see Section 2.11.
Of course, the exact tax basis offers the investor the maximal possible flexibility to make use of her tax-timing option. Economically, the exact tax basis seems to be the most reasonable one because highly correlated stocks of different firms are anyhow considered separately.

In the case of a positive riskless interest rate, it is more favorable to realize smaller trading gains first. Moreover, if the stock falls below its purchasing price, it is worthwhile to sell it in order to realize the trading loss and rebuy it immediately, which is called a *wash sale*. These facts were already observed in Dybvig and Koo [DK96], see Properties 1 and 2 on page 6. For a rigorous proof of these seemingly obvious statements considering arbitrary dynamic trading strategies and a motivation for the continuous time modeling, see Section 2.10.

For investors, wash sales are a method to claim a capital loss without actually changing their position. But, e.g., the U.S. tax law disallows loss deductions if the same stock is repurchased within thirty days. However, this regulation can easily be bypassed by purchasing a similar stock.

The *exact tax basis* is hard to manage when not considering a discrete time model (as in Dybvig and Koo [DK96] and DeMiguel and Uppal [DU05]) where the purchase price of each single stock in the portfolio can be identified. In the transition to continuous-time models, one faces the problem that the exact tax-basis becomes an infinite dimensional state variable. But, there are also other tax codes, specifying the basis to which the price of a security has to be compared in order to evaluate the capital gains (or losses).
A continuous-time model circumventing the problems of an exact tax basis is proposed by Ben Tahar et al. [BST07; BST10]. To calculate the book profits, only one additional state variable is necessary. This state variable, which is used as tax basis, models the average purchase price of the stocks in the portfolio. Their modeling yields the restriction of only considering right-continuous, finite variation strategies. Despite all this, the optimal investment problem is not analytically solvable.
An alternative to the LIFO taxation priority rule is proposed by Jouni et al. [JKT99; JKT00]. In their approach, the so-called first-in-first-out (FIFO) taxation priority rule is used, i.e., the stocks which spent the longest time in the portfolio are sold first. In both articles, they consider only one riskless asset and finite variation strategies with continuous paths.
Another (but not very common) way to handle taxes is proposed by Cadenillas and Pliska [CP99] and Buescu et al. [BCP07] where all capital gains have to be taxed when rearranging the portfolio. This kind of taxation equals a wash sale for all stocks in the portfolio at the rearranging dates.

Whereas in models with proportional transaction costs, it is quite obvious

that strategies of exploding variation lead to exploding costs and thus to an immediate ruin for sure, capital gains taxes do not explode. Namely, taxes are not triggered by portfolio regroupings alone if there are no price changes. In addition, even if the investment strategy forces that gains from upward movements of the stock are realized, there is to some extent an offset by losses due to tax credits.
On the other hand, a straightforward generalization of the model by [DK96; DU05] to continuous time is only available for finite variation strategies – as not only the number of shares held in the portfolio enters in the self-financing condition, but it is based on both purchases and sells. In *Chapter 2*, we show how tax payments can nevertheless be constructed under the condition that stocks are semimartingales.

One application is to compare different dividend policies. As dividend payoffs, in contrast to (unrealized) book profits, have to be taxed immediately, capital gains taxes are also relevant for dividend policies. Among economists, there have been extensive discussions about optimal dividend policies. In the famous article by Modigliani and Miller [MM61], their effect on the current stock price is considered, and their irrelevance for the firm valuation is shown in perfect markets (i.e., without taxes). A question arising from [MM61] is: "Why do firms pay dividends?". The so-called *dividend puzzle*, at first appearing in Black [Bla76], states that there are no rational reasons for a firm to pay dividends. Bernheim [Ber91] solves this puzzle considering a model (with taxes) in which firms attempt to signal profitability by distributing cash to shareholders. For a survey on these general, but mainly less formal, discussions on dividend policies, we refer to the book of Lease et al. [Lea+00].

Anyway, it seems to be quite obvious that dividends have, in principle, a negative impact on investors' after-tax wealths. Indeed, let $r_t > 0$ be the floating rate. By strict convexity of the exponential function, one has

$$1 + (1-\alpha)\left(\exp\left(\int_0^t r_s\,ds\right) - 1\right) > \exp\left((1-\alpha)\int_0^t r_s\,ds\right). \tag{1.1}$$

The LHS of (1.1) can be interpreted as the value of a bank account with initial capital 1 when capital gains are taxed at time t with factor α. The RHS corresponds to the same situation, but capital gains are already taxed at the time they occur. This tax regulation takes effect if interest is paid out as a continuous, positive dividend (the after-tax dividend is then reinvested in the bank account).

However, considering dynamic trading strategies and asset price processes that are not increasing with probability 1, a proof of the conjecture that the effect of dividends is always negative, is, even in discrete time, much trickier than (1.1). In Section 2.6, we give a proof of this assertion in the continuous time framework.
Finally, to demonstrate the tractability of the model, we give an example for

tax-efficient dynamic trading strategies for which the tax payment flow can be computed explicitly and is easy to interpret.

Optimal Selling Time of a Stock under Capital Gains Taxes

If profits are only taxed when the asset is liquidated, even in the simplest case of a *linear* taxing rule (which we consider in *Chapter 3*), there is a nontrivial interrelation between creating trading gains and tax liabilities by dynamic investment strategies. The investor can influence the timing of the tax payments, i.e., she holds a deferral option. In the case of a positive riskless interest rate, there is some incentive to realize profits as late as possible, but this can be at odds with portfolio regroupings in order to earn higher returns before taxes.

Solving a portfolio optimization problem with taxes allowing for arbitrary continuous time trading strategies is a rather daunting task, especially for the so-called exact tax basis and the first-in-first-out priority rule. Namely, shares having the same price but being purchased at different times possess, in general, different book profits, and hence, their liquidation triggers different tax payments. Thus, the book profits of the shares in the portfolio become an infinite-dimensional state variable (cf. Jouini, Koehl, and Touzi [JKT00; JKT99] for the first-in-first-out priority rule and *Chapter 2* of this thesis for the exact tax basis).

In practise, an investor is usually interested in much simpler optimization problems. Therefore, we want to analyze a typical and analytically quite tractable investment decision problem to determine exemplarily the impact of capital gains taxes and to see how model parameters, as the volatility of the stock, enter into the solution. Often, the investor wants to maximize her trading profits within a certain finite period of time by exchanging one asset for another one only once. This means that she has to solve an optimal stopping problem. In this simple setting, different tax bases, as, e.g., the exact tax basis, the average tax basis, or the first-in-first-out priority rule, coincide. To investigate the impact of taxes on investment decisions, we look at an investor owning an asset which she would sell immediately to buy another one if she *was not* subject to taxation. Under risk neutrality, this just means that the asset the investor holds at the beginning has a lower expected return than the alternative asset. Then, we investigate to what extent she is prevented from this transaction by the obligation to pay taxes at the time she liquidates the first asset. The price of the first asset is modeled as *stochastic* process in the Black-Scholes market to see the impact of the volatility on the deferral option the tax payer holds. We prove the plausible supposition that the possibility to time the tax payments is more worthwhile for holders of more volatile assets and, consequently, the risky asset is sold later (see Proposition 2.4). We assume that the second asset is then kept in the portfolio up to maturity. Thus, it is no essential restriction to model it as a riskless bank account.

In contrast to the model of Constantinides [Con83], we do *not* assume that the investor can both defer the tax payments and divest the stock from her portfolio at the same time by trading in a market for short sell contracts (see Subsection 4.1 of [Con83]). To our mind, it is mainly an interesting gedanken-experiment to shorten the stock, instead of selling it, in order to defer tax payments, but under real-world tax legislation, it is no option for private investors. By assuming the existence of such a market for short sell contracts, Constantinides can price the timing option in the Black-Scholes model by no-arbitrage arguments and without solving a *free*-boundary problem. Another essential difference to [Con83] is that we have a deterministic finite time horizon, whereas in [Con83], liquidation is forced at independent Poisson times. Thus, as in problems with infinite time horizon, as in the latter article, one gets rid of "time" as a state variable.

In *Chapter 3*, standard techniques from the theory of optimal stopping are used, especially an approach that turns the problem with a terminal payoff to one with a *running* payoff, see Peskir and Shiryaev [PS06]. The objective function is much simpler than in other recent papers on the optimal selling time of a stock without taxes, as Shiryaev et al. [SXZ08], where the stock is sold at the stopping time which maximizes the expected ratio between the stock price and its maximum over the entire horizon. Du Toit and Peskir [DTP09] complement this by determining a stopping time that minimizes the expected ratio of the ultimate maximum and the current stock price. Dai and Zhong [DZ12] consider a similar problem in which the average stock price is used as reference. In addition to the above selling problems, in their recent work, Baurdoux et al. [Bau+14] discuss a "buy low and sell high" problem as a sequential optimal stopping of a Brownian bridge modeling stock pinning. This is a phenomenon where a stock price tends to end up in the vicinity of the strike of its option near its expiry, see [AL03] for a detailed explanation.

1.1 Structure of this Thesis

Chapter 2 is based on the article [KU14]. In Section 2.2, we present the model and the first main result, Theorem 2.2.12, which shows how to construct tax payment processes for adapted, left-continuous trading strategies. The construction is based on automatic wash sales and the rule to sell shares with shorter residence time first. The optimality of this procedure is proven in Chapter 2.10 for the discrete time model of Dybvig and Koo [DK96]. In Section 2.3, basic properties of the book profits of a portfolio are discussed. They are used in the proof of Theorem 2.2.12 in Section 2.4. In Section 2.5, the self-financing condition of the model is introduced. In Section 2.6, the second main result, Theorem 2.6.3, showing that the investor is always better off in a model with a stock which does not pay dividends is stated and proven. Section 2.8 is about tax-efficient strategies, and Section 2.7 gives examples that show the necessity of some assumptions. In Section 2.11, we compare the modeling of different tax bases in financial mathematics.

Chapter 3 is based on the article [KSU14]. In Section 3.1, we formulate the optimal stopping problem and present its solution (Theorem 3.1.2), which is, accompanied by Proposition 3.1.4, the main result of the chapter. Afterwards, the results are related to other contributions in the literature. Section 3.2 introduces the applied method to solve the problem and prepares the proofs which are given in Section 3.3. Section 3.4 examines the same stopping problem as in Section 3.1 but with a bond where earnings are taxed at maturity and compares the resulting stopping boundaries in both problems.

Published Contents

The articles “Modeling Capital Gains Taxes for Trading Strategies of Infinite Variation” [KU14] and ”Optimal Selling Time of a Stock under Capital Gains Taxes“ [KSU14] are submitted for publication.

Chapter 2

Modeling Capital Gains Taxes for Trading Strategies of Infinite Variation

In this chapter, we model capital gains taxes for trading strategies of infinite variation. To be able to define an accumulated tax-payment process, we have to consider the rule that stocks with the longest residence time in the portfolio are sold first and stocks with negative book profits are sold and immediately repurchased (wash sale). These rules are motivated by proving their optimality in the time-discrete model of Dybvig and Koo [DK96]. We show that in our model, dividend payments negatively influence the investor's after-tax wealth if the dividend policy has no effect on the stochastic return process. As a further application, we find out tax-efficient strategies which try to defer tax-payments as long as possible.

2.1 Notation

Throughout the chapter, we fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions. Denote by $\mathcal{O}$ (resp. by $\mathcal{P}$) the optional σ-algebra (resp. the predictable σ-algebra) on $\Omega \times [0,T]$. For optional processes $X, X^n, n \in \mathbb{N}$, we write $X^n \stackrel{\text{up}}{\to} X$ iff X^n converges uniformly in probability to X, i.e., $\sup_{t\in[0,T]} |X^n_t - X_t|$ converges to 0 in probability. Equality of processes is understood up to evanescence.
A process X is called làglàd iff all paths possess finite left and right limits (but they may have double jumps). We set $\Delta^+ X := X_+ - X$ and $\Delta X := \Delta^- X := X - X_-$, where $X_{t+} := \lim_{s\downarrow t} X_s$ and $X_{t-} := \lim_{s\uparrow t} X_s$. For a random variable Y, we define $Y^+ := \max(Y, 0)$ and $Y^- := \max(-Y, 0)$.

2.2 Construction of the Tax Payment Process

For an investor trading in finitely many different stocks, the total tax payment is just the sum of the tax payments considering only gains from one type of stock. Thus, it is sufficient to consider only one risky asset (sometimes called stock). Its price process is given by the semimartingale $(S_t)_{t\in[0,T]}$ (thus the paths are càdlàg). The stock pays out nonnegative dividends. Accumulated dividends per share are modeled by the nondecreasing adapted càdlàg process $(D_t)_{t\in[0,T]}$. All capital gains (positive or negative) are taxed with the rate $\alpha \in (0,1)$. But, whereas dividends are taxed immediately, trading gains arising from stock price movements are not taxed before they are realized. Denote by $\mathbb{L}$ the set of all left-continuous adapted processes with finite right limits. The investor's strategy is the number of identical stocks she holds, and it is modeled by some $\varphi \in \mathbb{L}$ with $\varphi_0 = 0$ and $\varphi \geq 0$. Short-selling is forbidden as otherwise the investor can hold *one* long and *one* short position of the same stock at the same time, and this can lead to an arbitrage opportunity under a linear tax rule and a positive riskless interest rate (losses are immediately realized, and the corresponding gains are deferred, cf. Constantinides [Con83]).

Example 2.2.1 (Arbitrage with a long and a short position)**.** *Let* $\alpha = 10\%$. *Consider a linear taxation rule and a riskless investment opportunity, called bond, with interest rate* $r = 5\%$. *Capital gains in the stock position are taxed when realized, whereas (w.l.o.g.) earnings in the bond are continuously taxed. The initial wealth is* $v_0 = \$0$. *An investor buys* 100 *Amazon stocks financed by short selling* 100 *Amazon stocks at the same time* $t = 0$ *at a price of* \$100. *Let us now distinguish two cases. At time* t_1, *the stock price either (a) increases by* \$10, *or (b) decreases by* \$10. *In both cases, the portfolio is liquidated at time* t_2.
In case (a), the investor makes losses with the short position. Realizing these losses by wash sales means a tax credit of $100 \cdot \alpha(\$110 - \$100) = \$100$. *These tax credits are then invested in the riskless bond. Liquidating the portfolio at time* t_2 *implies a terminal wealth of*

$$\begin{aligned} V_{t_2} =& 100\,(S_{t_2} - \alpha(S_{t_2} - 100) - S_{t_3} + \alpha(S_{t_2} - 110)) \\ &+ 100\exp(0.05(1-\alpha)\cdot(t_2 - t_1)) \\ =& -100 + 100\exp(0.05(1-\alpha)\cdot(t_2-t_1)) \\ >& 0. \end{aligned} \tag{2.1}$$

In case (b), the investor makes losses with the long position and the calculations yield the same results as in (2.1), namely

$$V_{t_2} = -100 + 100\exp(0.05\cdot(t_2 - t_1)) > 0.$$

As positivity of the wealth does not depend on the size of increase or decrease of the stock, respectively, allowing simultaneous short and long positions in *a market with linear capital gains taxes implies* ***arbitrage*** *when the investor realizes losses by wash sales.*

The assumption $\varphi_0 = 0$ is solely for notational convenience (cf. (2.5)). It does not rule out that the investor starts with a bulk trade $\varphi_{0+} > 0$.

Remark 2.2.2. *In general, the tax payment flow cannot be derived from the process φ alone as payments depend on* which *shares the investor sells when φ is reduced and on the occurrence of wash sales that do not enter in φ. Given some φ, we work with a special procedure that dictates which of the shares to sell. In Section 2.10, for a nonnegative interest rate, the optimality of this procedure is proven in the discrete time model of Dybvig and Koo [DK96] where arbitrary shares can be sold. We use that a payment obligation in the future is preferred to a payment obligation today. With this intuition in mind, the constructions in the current section are well-founded, but there are also good reasons to read Section 2.10 first.*

To construct the tax payment process, several mathematical objects have to be introduced. For every t, we sort the φ_t stocks by the time spending in the portfolio and label them by x: the larger x the longer the residence time in the portfolio. We follow the above-mentioned procedure:

$$\text{"latest purchased stocks are sold first".} \tag{2.2}$$

With this procedure, the purchasing time of the x^{th} stock is defined by

$$\tau_{t,x} := \begin{cases} \sup M_{t,x} & \text{if } M_{t,x} \neq \emptyset \\ t & \text{otherwise} \end{cases}, \quad t \in [0,T], x \in \mathbb{R}_+, \tag{2.3}$$

where $M_{t,x} := \{u \in \mathbb{R}_+ \mid (u \leq t \text{ and } x - \varphi_t + \varphi_u \leq 0) \text{ or } (u < t \text{ and } x - \varphi_t + \varphi_{u+} \leq 0)\}$. By $\varphi_0 = 0$ and $\varphi \geq 0$, one has

$$M_{t,x} = \emptyset \Leftrightarrow x > \varphi_t \tag{2.4}$$

and thus

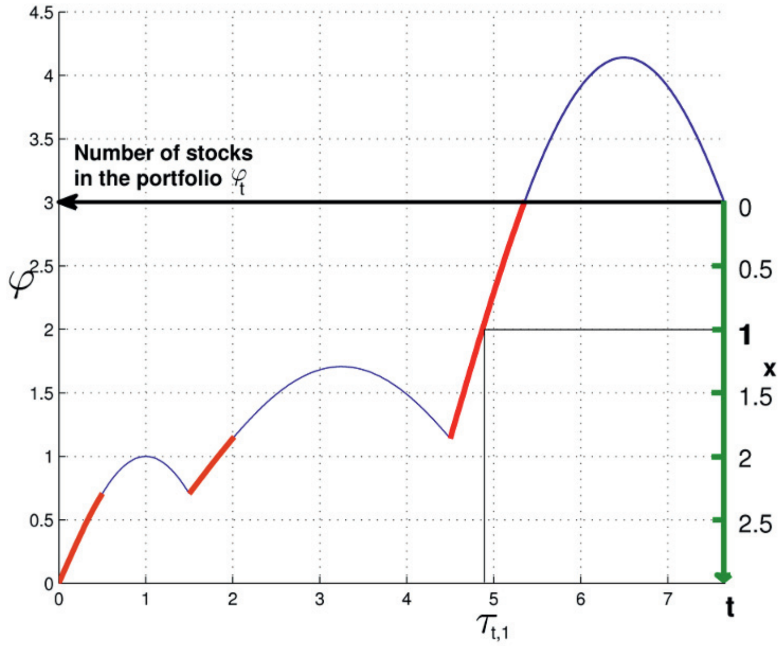
$$\tau_{t,x} = 1_{(x \leq \varphi_t)} \sup M_{t,x} + 1_{(x > \varphi_t)} t. \tag{2.5}$$

The construction is illustrated in Figure 2.1.

In the next step, an automatic loss realization is modeled. The trading gain of piece x is decomposed into

$$S_t - S_{\tau_{t,x}} = \underbrace{\inf_{\tau_{t,x} \leq u \leq t} S_u - S_{\tau_{t,x}}}_{\text{realized losses by wash sales}} + \underbrace{S_t - \inf_{\tau_{t,x} \leq u \leq t} S_u}_{\text{unrealized book profits}}. \tag{2.6}$$

This is motivated as follows: if a stock falls below its purchasing price, it is sold and repurchased in order to declare a loss. Then, in the continuous time limit, the realized loss is the first summand on the RHS of (2.6). The residual second summand equals the unrealized book profits.



On the ordinate, the stocks that are in the portfolio at time t are sorted by descending label x (see the green axis). $\tau_{t,x}$, the purchasing time of stock x, is the last time u before t with $\varphi_u = \varphi_t - x$ (see the case $x = 1$). The pieces that are marked in red symbolize the stocks (and their purchasing times) which are still in the portfolio at time t. If the position is reduced, stocks with lower residence time in the portfolio are sold first.

Figure 2.1: Determination of $\tau_{t,x}$

Definition 2.2.3 (Book profits)**.** *Let $\varphi \in \mathbb{L}$ with $\varphi_0 = 0$ and $\varphi \geq 0$. The mapping $F : \Omega \times [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ with*

$$F_\omega(t, x) := S_t(\omega) - \inf_{\tau_{t,x}(\omega) \leq u \leq t} S_u(\omega), \tag{2.7}$$

where $\tau_{t,x}$ is defined in (2.3), is called the book profit function.

A book profit is a gain that is demonstrated on paper, but not actually real yet. By the wash sales and the fact that a newly bought share starts with book profit zero, a share with a longer stay in the portfolio possesses a higher (or equal) book profit, i.e., $x \mapsto F_\omega(t, x)$ is nondecreasing.

Note that wash sales neither enter into the strategy φ (implying that these transactions have no impact on the trading gains) nor in the purchasing times $\tau_{t,x}$. The latter means that $\tau_{t,x}$ is the time at which the share possessing at time t with label x is bought and kept in the portfolio afterwards at least up to time t, *apart from later rebuys caused by wash sales.*

Example 2.2.4 (Automatic loss realization)**.** *Consider a 3−period model with stock price $S = (100, 102, 98, 101)$ and a constant strategy $\varphi = (1, 1, 1, 1)$. So,*

the book profit function at t_0, t_1, t_2, t_3 is given by $F(t_0, x) = 0$, $F(t_1, x) = 2$, $F(t_2, 1) = 0$ and $F(t_3, 1) = 3$ for $x \in (0, 1]$.
The constant strategy yields that $\tau_{t_i,x} = t_0$ for $i = 0, 1, 2, 3$, $x \in (0, 1]$. Although, the purchasing time $\tau_{t,x}$ does not change, the stocks in the portfolio are sold and immediately repurchased (wash sale) at t_2 because they have negative book profits. So, for calculating the unrealized book profits at t_3, the new purchase price at t_2, which is given by $\inf_{\tau_{t_0,x} \leq u \leq t_3} S_u$, is relevant.

Remark 2.2.5. *The book profit function (2.7) that depends on the paths of the stock price and the total number of shares turns out to be the key object to construct tax payments for strategies of infinite variation and to find out tax-efficient strategies.*

Proposition 2.2.6. *$F(t, x)$ and $\tau_{t,x}$ fulfill the following properties:*

(i) The mapping $x \mapsto \tau_{t,x}$ is nonincreasing on $[0, \varphi_t]$.

(ii) $F(t, x) = 0$ for $x > \varphi_t$.

(iii) $x \mapsto F(t, x)$ is nondecreasing on $[0, \varphi_t]$.

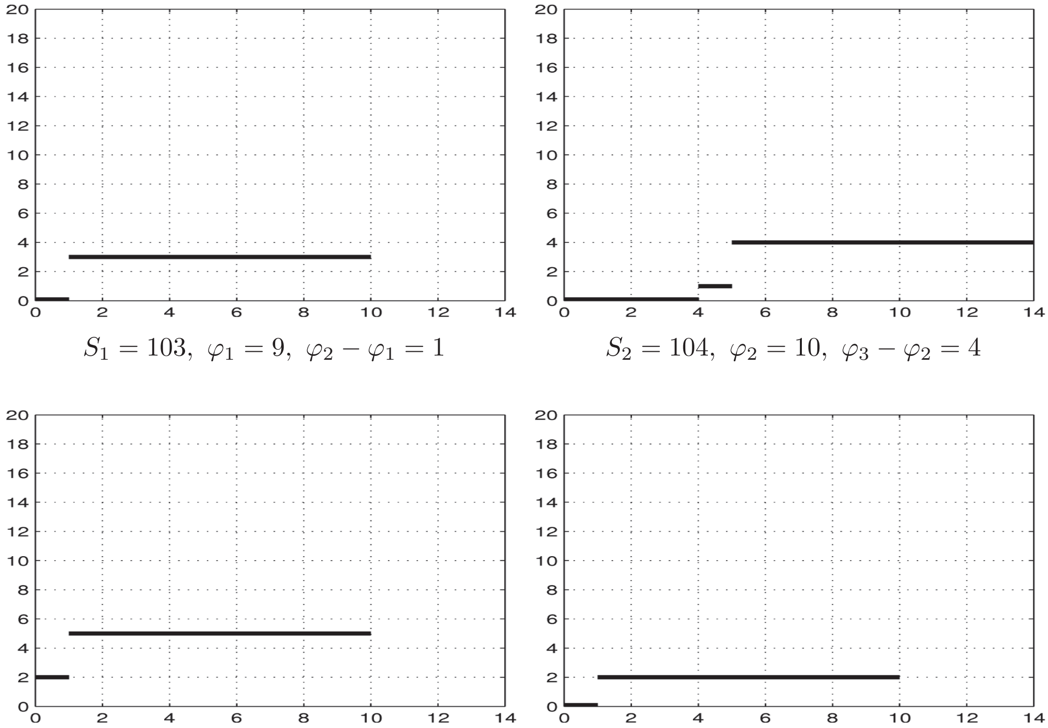
(iv) $x \mapsto F(t, x)$ is left-continuous.

(v) If φ is an elementary strategy, then $\lim_{s \downarrow t} F(s, x)$ exists for all t, x.

The proof can be found at the beginning of Section 2.3. Of course, $F(t, x)$ is only used for $x \leq \varphi_t$. Possible states and developments of F over time can be seen in Figure 2.2.

Remark 2.2.7. *To ensure that the function $x \mapsto F(t, x)$ is left-continuous, besides φ_u, also φ_{u+} has to be considered in the definition of $M_{t,x}$. It is convenient that $x \mapsto F(t, x)$ does not possess double jumps, but for the following construction of the tax payment process the values of F at the (countably many) points of discontinuity do not matter. $F_\omega(t, \cdot) \mid_{(0,\varphi_t(\omega)]}$ can also be seen as the left-continuous inverse of the distribution function of the book profits over all shares that are in the portfolio at time t (here, "distribution function" means the number of shares with book profits lower than or equal to a given bound).*

Whereas the book profit function in (2.7) is directly defined for all $\varphi \in \mathbb{L}$, it turns out that a straight forward construction of the tax payment process, analogous to time-discrete models, would be based on both the accumulated purchases and the accumulated sales (this is as both effects are quite different). Thus, in a first step, the tax payments are only defined for elementary strategies. Then, in Theorem 2.2.12 we show that it can be extended to all left-continuous adapted processes. However, this extension is not obvious and relies, among other things, on the assumption that S is a semimartingale (see Remark 2.7.2). With the help of (2.7), a process Π can be defined which reflects the accumulated tax payments up to time t.



$S_1 = 103,\ \varphi_1 = 9,\ \varphi_2 - \varphi_1 = 1$ $S_2 = 104,\ \varphi_2 = 10,\ \varphi_3 - \varphi_2 = 4$

$S_3 = 105,\ \varphi_3 = 14,\ \varphi_4 - \varphi_3 = -4$ $S_4 = 102,\ \varphi_4 = 10,\ \varphi_5 - \varphi_4 = 0$

An example how $x \mapsto F(t, x)$ can evolve in a 4−period model, i.e., $t \in \{0, 1, 2, 3, 4\}$. In this example, the stock price is given by $S = (S_0, \dots, S_4) = (100, 103, 104, 105, 102)$, and the investor chooses the strategy $\varphi = (\varphi_1, \dots, \varphi_5) = (9, 10, 14, 10, 10)$, following the standard notation in discrete time, i.e., φ_1 shares are purchased at price S_0 etc. On the abscissa there are the shares ordered by their book profits and on the ordinate the book profits $F(t+, x)$, i.e., after the portfolio regrouping at time t. Observe that at time $t = 4$, i.e., in the fourth picture, one share (at the very left) is sold and immediately bought back to realize a loss of one monetary unit (wash sale).

Figure 2.2: Evolvement of $x \mapsto F(t, x)$

Definition 2.2.8 (Accumulated tax payments for elementary strategies)**.** *Let φ be a nonnegative elementary strategy s.t. $\varphi = \sum_{i=1}^{k} H_{i-1} 1_{]\!]\kappa_{i-1},\kappa_i]\!]}$, where $0 = \kappa_0 \le \kappa_1 \le \ldots \le \kappa_k = T$ are stopping times and H_{i-1} is $\mathcal{F}_{\kappa_{i-1}}$−measurable. Let τ and F be as in Definition 2.2.3. Then,*

$$
\begin{aligned}
\Pi_t(\varphi) :=& \alpha \sum_{i=1}^{k} 1_{(\kappa_{i-1}<t)} \int_0^{(H_{i-1}-H_{i-2})^-} F(\kappa_{i-1}, x)\, dx \\
&+ \alpha \sum_{i=1}^{k} 1_{(\kappa_{i-1}<t)} \int_0^{\varphi_t} \left(F(\kappa_{i-1}+, x) + \inf_{\kappa_{i-1} \le u \le t \wedge \kappa_i} (S_u - S_{\kappa_{i-1}}) \right) \wedge 0\, dx \\
&+ \alpha \int_0^t \varphi_u dD_u, \qquad (2.8)
\end{aligned}
$$

where $H_{-1} := 0$, is the tax payment process *of the elementary strategy φ (The* *limit* $F(\kappa_{i-1}+, x) := \lim_{s \downarrow \kappa_{i-1}} F(s, x)$ *exists by Proposition 2.2.6(v)).*

Π is obviously well-defined, i.e., it does not depend on the representation of φ.

Remark 2.2.9. *Let us explain the three components of* $\Pi_t(\varphi)$.

$\alpha \sum_{i=1}^{k} 1_{(\kappa_{i-1}<t)} \int_0^{(H_{i-1}-H_{i-2})^-} F(\kappa_{i-1}, x)\, dx$ *are the tax payments that are triggered by selling stocks in order to follow the strategy* φ. *A downward jump of* φ *forces the investor to realize book profits. She takes the shares with the smallest label* x, *which is in line with (2.2) and (2.3). As* $x \mapsto F(s, x)$ *is nondecreasing, the sold shares possess the lowest book profits of all shares in the portfolio. By* $F \geq 0$, *this term is nonnegative.*

$\alpha \sum_{i=1}^{k} 1_{(\kappa_{i-1}<t)} \int_0^{\varphi_t} \big(F(\kappa_{i-1}+, x) + \inf_{\kappa_{i-1} \leq u \leq t \wedge \kappa_i}(S_u - S_{\kappa_{i-1}})\big) \wedge 0\, dx$ *is always less or equal to zero. The* i^{th} *summand models the tax credits due to realized losses by wash sales between the trading times* κ_{i-1} *and* κ_i. *This equals minus the local time of* S *at different levels (in the sense of Asmussen [Asm03], page 251). Namely, the book profit of piece* x *is the solution of a Skorokhod problem started at* $F(\kappa_{i-1}+, x)$ *in which the stock price movements are reflected at* 0 *(however, this interpretation is only valid in between portfolio regroupings). The local time we consider has jumps iff downward price jumps dominate previous book profits. It is different from the semimartingale local time, see (5.47) in Jacod [Jac79] for a definition. But, for* S *being a continuous local martingale, the semimartingale local time of the reflected stock price is twice the local time in [Asm03], see the appendix of Yor [Yor79].*

$\alpha \int_0^t \varphi_u dD_u$ *are taxes on dividends, which have to be paid immediately.*

Remark 2.2.10. *Given an elementary process* φ *modeling the total number of shares in the portfolio,* $\Pi_t(\varphi)$ *are the minimal accumulated tax payments up to time* t. *This statement results from Theorem 2.10.1 together with Subsection 2.10.1.*

(2.8) can generally not be formulated for strategies of infinite variation.

Remark 2.2.11. *It is quite natural that the tax payment process has double jumps. Namely, the stock price is right-continuous whereas the strategy is left-continuous, and the tax payments are triggered both by downward jumps of the stock (through wash sales) and by sales of stocks following the strategy* φ.

Theorem 2.2.12. *Let* $\varphi \in \mathbb{L}$ *and* $(\varphi^n)_{n\in\mathbb{N}}$ *be a sequence of elementary strategies with* $\varphi^n_0 = 0$, $\varphi^n \geq 0$, *and* $\varphi^n \stackrel{up}{\to} \varphi$. *Then, the accumulated tax payments* Π^n *for* φ^n *(as defined in Definition 2.2.8) are optional processes with làglàd paths. In addition, there exists an optional process* Π *possessing almost surely làglàd paths such that* $\Pi^n \stackrel{up}{\to} \Pi$. *Different choices of up-approximating sequences of* φ *lead to the same* Π *up to evanescence.*

Consequently, the mapping $\varphi \mapsto \Pi(\varphi)$ *from Definition 2.2.8 possesses an up to evanescence unique extension*

$$\{\varphi \in \mathbb{L} \mid \varphi_0 = 0,\ \varphi \geq 0\} \to \{X : \Omega \times [0,T] \to \mathbb{R} \mid X \text{ is optional and làglàd}\},$$

which is continuous w.r.t. the convergence uniformly in probability. The extension, also denoted by Π*, possesses the jumps*

$$\Delta\Pi_t = \alpha \int_0^{\varphi_t} \left(\limsup_{s<t, s\to t} F(s,x) + \Delta S_t \right) \wedge 0\, dx + \alpha\varphi_t \Delta D_t \quad \text{and} \tag{2.9}$$

$$\Delta^+\Pi_t = \alpha \int_0^{(\Delta^+\varphi_t)^-} F(t,x)dx. \tag{2.10}$$

Note that any $\varphi \in \mathbb{L}$ with $\varphi \geq 0$ can be approximated uniformly in probability by a sequence of nonnegative elementary strategies (see, e.g., Theorem II.10 in [Pro04]).

Definition 2.2.13. *For* $\varphi \in \mathbb{L}$ *with* $\varphi \geq 0$*, the tax payment process* $\Pi(\varphi)$ *is defined as the limit process in Theorem 2.2.12.*

2.3 Properties of the Book Profit Function

In this section, we state some properties of $F(t,x)$. These are needed in the next section for showing convergence of Π^n.

Proof of Proposition 2.2.6. (i): Let $y \leq x \leq \varphi_t$. By (2.4), we have $M_{t,x} \neq \emptyset$. By the left-continuity of φ, $\sup M_{t,x}$ is attained, i.e., $x-\varphi_t+\varphi_{\tau_{t,x}} \leq 0$ or $x-\varphi_t+\varphi_{\tau_{t,x}+} \leq 0$. We conclude that $y-\varphi_t+\varphi_{\tau_{t,x}} \leq 0$ or $y-\varphi_t+\varphi_{\tau_{t,x}+} \leq 0$. Thus $\tau_{t,x} \leq \tau_{t,y}$.

(ii): Follows immediately from (2.5).

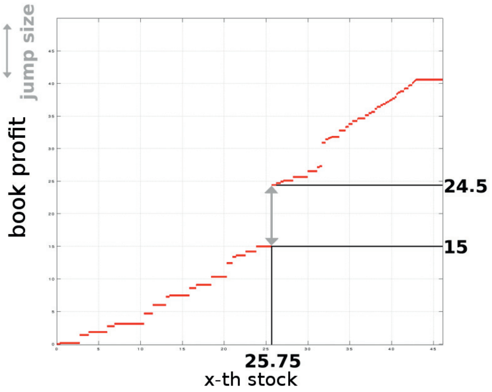
(iii): Due to $\tau_{t,y} \geq \tau_{t,x}$ for $y \leq x \leq \varphi_t$, one has $F(t,x)-F(t,y) = \inf_{\tau_{t,y}\leq u\leq t} S_u - \inf_{\tau_{t,x}\leq u\leq t} S_u \geq 0$.

(iv): By (ii), it is enough to show left-continuity at $x \in (0,\varphi_t]$. One has $x-\varphi_t+\varphi_u > 0$ for all $u \in (\tau_{t,x}, t]$ and $x-\varphi_t+\varphi_{u+} > 0$ for all $u \in (\tau_{t,x}, t)$. Because the infimum of a càglàd process on a compact interval is attained in a right or a left limit, one has that

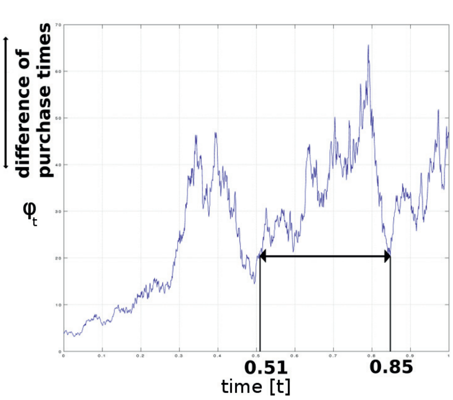
$$\inf\{x-\varphi_t+\varphi_u \mid u \in [\tau_{t,x}+\varepsilon, t]\} > 0, \quad \forall\varepsilon > 0.$$

Therefore, there exists $\delta_0 > 0$ s.t. for all $\delta \in (0,\delta_0]$

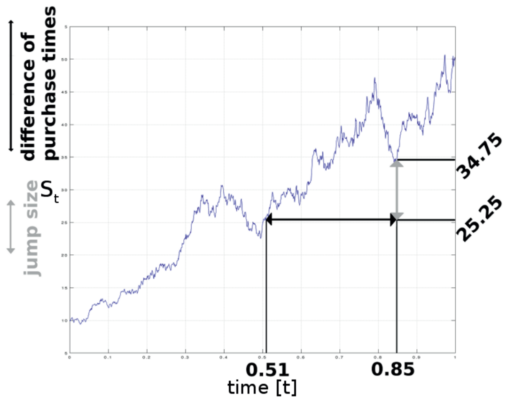
$$x-\delta-\varphi_t+\varphi_u > 0\ \forall u \in [\tau_{t,x}+\varepsilon, t] \text{ and } x-\delta-\varphi_t+\varphi_{u+} > 0\ \forall u \in [\tau_{t,x}+\varepsilon, t).$$



(a) Book profits $x \mapsto F(1,x)$. The grey arrow shows a jump at $x_0 = 25.75$ of size \$ 19.5



(b) Path of the Merton strategy $(\varphi_t)_{t\in[0,1]}$



(c) Path of the stock price $(S_t)_{t\in[0,1]}$

An example of $x \mapsto F(1,x)$ for a continuous time strategy. The paths are simulated for the Merton problem [Mer71], i.e., the stock price and the exact optimal strategy are continuous and of infinite variation. Observe that, nevertheless, $x \mapsto F(1,x)$ has jumps. Let us explain the jump of $x \mapsto F(1,x)$ in figure (a) at $x_0 = 25.75$ (looking at figures (b) and (c)). As $\varphi_1 = 46.25$, the piece $x_0 = 25.75$ is purchased at $\tau_{1,25.75} = \sup\{u \mid \varphi_u = 46.25 - 25.75 = 20.5\} = 0.85$. φ possesses a local minimum at $\tau_{1,0.85}$. In addition, there are no wash sales of piece $x_0 = 25.75$ after $\tau_{1,0.85}$. Define $\tau^* := \sup\{u < \tau_{1,0.85} \mid \varphi_u = 46.25 - 25.75\} = 0.51$. $\tau^* = \lim_{x\downarrow 25.75} \tau_{1,x}$ is the time at which x marginally bigger than $x_0 = 25.75$ is purchased. Then, the jump size $\lim_{x\downarrow 25.75} F(1,x) - F(1, 25.75)$ results from $S_{0.85} - \inf_{0.51\leq u\leq 0.85} S_u = 9.5$, the book profits of piece 25.75+ at time 0.85 (cf. the black arrows in figure (b) and (c)).

Figure 2.3: $x \mapsto F(1,x)$ for a continuous time strategy

Thus, either $M_{t,x-\delta} = \emptyset$ or $0 \leq \sup M_{t,x-\delta} \leq \tau_{t,x} + \varepsilon$. If the first holds for some $\delta \in (0, \delta_0]$, it also holds for all smaller positive numbers and zero. In this case, left-continuity is obvious because $\tau_{t,y} = \tau_{t,x} = t$ for all y in a left neighborhood of x. In the second case, one has $\tau_{t,x-\delta} - \tau_{t,x} \leq \varepsilon$ and, by (i), $\tau_{t,x-\delta} \in [\tau_{t,x}, \tau_{t,x} + \varepsilon]$ for all $\delta \in (0, \delta_0]$. By right-continuity of S we are done.

(v): Let φ be an elementary strategy with representation as in Definition 2.2.8. Let $t \in [\kappa_{i-1}, \kappa_i)$ and $s_1, s_2 \in (t, \kappa_i]$, i.e., $\varphi_{s_1} = \varphi_{s_2}$. For $x = 0$, one has $F(s_1, 0) = F(s_2, 0) = 0$. For $x \in (0, \varphi_{s_1}]$, one has $M_{s_1,x}, M_{s_2,x} \subset [0, \kappa_{i-1}]$ which leads, again by $\varphi_{s_1} = \varphi_{s_2}$, to $M_{s_1,x} = M_{s_2,x}$. By $x \leq \varphi_{s_1}$ and (2.4), one has $M_{s_1,x} \neq \emptyset$ and arrives at $\tau_{s_1,x} = \tau_{s_2,x} \leq \kappa_{i-1}$ and thus $F(s_1, x) = F(s_2, x)$. For $x > \varphi_{s_1} = \varphi_{s_2}$ one has that $M_{s_1,x} = M_{s_2,x} = \emptyset$ and thus $F(s_1, x) = F(s_2, x) = 0$. Consequently, the limit $\lim_{s \downarrow t} \tau_{s,x} =: \tau_{t+,x}$ exists for all $x \in \mathbb{R}_+$. ■

In the next lemma, we examine the behavior of the book profit function for two strategies whose paths are close together.

Lemma 2.3.1. *Let $\varphi, \widetilde{\varphi} \in \mathbb{L}$ with $\varphi_0 = \widetilde{\varphi}_0 = 0$ and $\varphi, \widetilde{\varphi} \geq 0$. $\widetilde{\tau}_{t,x}$, $\widetilde{F}$, and $\widetilde{M}_{t,x}$ denote the quantities from Definition 2.2.3 for $\widetilde{\varphi}$ instead of φ. Fix some $\omega \in \Omega$ and $t \in [0, T]$. If*

$$\sup_{0 \leq u \leq t} |\varphi_u(\omega) - \widetilde{\varphi}_u(\omega)| \leq \varepsilon, \tag{2.11}$$

then

$$F_\omega(t, x) \leq \widetilde{F}_\omega(t, x + 2\varepsilon) \quad \text{for all } x \leq \widetilde{\varphi}_t(\omega) - 2\varepsilon \tag{2.12}$$

and

$$\left| \int_0^{\varphi_t(\omega)} F_\omega(t, x)\, dx - \int_0^{\widetilde{\varphi}_t(\omega)} \widetilde{F}_\omega(t, x)\, dx \right| \leq 3\varepsilon \left(\sup_{0 \leq u \leq t} S_u(\omega) - \inf_{0 \leq u \leq t} S_u(\omega) \right). \tag{2.13}$$

Proof. We fix some $\omega \in \Omega$ satisfying (2.11) and omit it in the rest of the proof. Let $x \leq \widetilde{\varphi}_t - 2\varepsilon$. By (2.11), one has $\widetilde{M}_{t,x+2\varepsilon} \subset M_{t,x}$. This gives $\sup \widetilde{M}_{t,x+2\varepsilon} \leq \sup M_{t,x}$. Furthermore, by (2.4), one has $\widetilde{M}_{t,x+2\varepsilon} \neq \emptyset$ and thus $\widetilde{\tau}_{t,x+2\varepsilon} = \sup \widetilde{M}_{t,x+2\varepsilon} \leq \sup M_{t,x} \leq \tau_{t,x}$, which implies

$$F(t, x) - \widetilde{F}(t, x + 2\varepsilon) = \inf_{\widetilde{\tau}_{t,x+2\varepsilon} \leq u \leq t} S_u - \inf_{\tau_{t,x} \leq u \leq t} S_u \leq 0.$$

As obviously $F(t, x) = S_t - \inf_{\tau_{t,x} \leq u \leq t} S_u \leq \sup_{0 \leq u \leq t} S_u - \inf_{0 \leq u \leq t} S_u$ for all $x \in \mathbb{R}_+$, (2.12) implies

$$\begin{aligned}
&\int_0^{\varphi_t} F(t, x)\, dx \\
&\leq \int_0^{(\widetilde{\varphi}_t - 2\varepsilon) \vee 0} \widetilde{F}(t, x + 2\varepsilon)\, dx + (\varphi_t - \widetilde{\varphi}_t + 2\varepsilon) \left(\sup_{0 \leq u \leq t} S_u - \inf_{0 \leq u \leq t} S_u \right) \\
&\leq \int_0^{\widetilde{\varphi}_t} \widetilde{F}(t, x)\, dx + 3\varepsilon \left(\sup_{0 \leq u \leq t} S_u - \inf_{0 \leq u \leq t} S_u \right).
\end{aligned}$$

By symmetry, we obtain (2.13). ■

In the next section, we prove that Π is an optional process. For this purpose, some measurability of F has to be checked.

Proposition 2.3.2. *F is $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+) - \mathcal{B}(\mathbb{R}_+)-$measurable.*

Proof. Because $x \mapsto F_\omega(t,x)$ is left-continuous and on $[0, \varphi_t]$ also nondecreasing, one gets

$$F_\omega(t,x) = 1_{(x \leq \varphi_t(\omega))} \sup_{q \in \mathbb{Q}_+} \left\{ F_\omega(t,q) - 1_{(x<q)} \infty \right\}.$$

As $\{(\omega,t,x) \mid x \leq \varphi_t(\omega)\} \in \mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$, it remains to show that $(\omega,t) \mapsto F_\omega(t,q)$ is $\mathcal{O} - \mathcal{B}(\mathbb{R}_+)-$measurable for every fixed q.

Step 1: Let us show that $(\omega,t) \mapsto \tau_{t,q}(\omega)$ is $\mathcal{P} - \mathcal{B}(\mathbb{R}_+)-$measurable. Define the (random) sets

$$M^n_{t,q} := \{u \in [0,t] \mid q - \varphi_t + \varphi_u \leq 1/n, \ u \in \mathbb{Q}\}, \quad n \in \mathbb{N}.$$

By

$$\sup M^n_{\cdot,q} \quad = \quad \sup_{u \in \mathbb{Q}_+} u 1_{\{(\omega,t) \ \mid \ q - \varphi_t(\omega) + \varphi_u(\omega) \leq 1/n \text{ and } u<t\}}$$

and the predictability of φ, the mapping $\sup M^n_{\cdot,q} : \Omega \times [0,T] \to \mathbb{R}_+$, $(\omega,t) \mapsto \sup M^n_{t,q}(\omega)$ is written as a pointwise supremum over countably many predictable functions, and thus it is also predictable. Now, we show that

$$\sup M^n_{t,q} 1_{(q \leq \varphi_t)} \to \tau_{t,q} 1_{(q \leq \varphi_t)} \quad \text{pointwise for } n \to \infty. \tag{2.14}$$

Let $n \in \mathbb{N}$, $u \in M_{t,q}$. There exists $v \in \mathbb{Q}$ arbitrary close to u with $v \in M^n_{t,q}$ and thus

$$\sup M_{t,q} \leq \sup M^n_{t,q}, \quad \forall n \in \mathbb{N}. \tag{2.15}$$

Assume that $q \leq \varphi_t$, i.e., $\tau_{t,q} = \sup M_{t,q}$ by (2.5). First note that $q - \varphi_t + \varphi_u > 0$ for all $u \in (\tau_{t,q}, t]$ and $q - \varphi_t + \varphi_{u+}$ for all $u \in (\tau_{t,q}, t)$. As the infimum of a càdlàg process is attained in the right or the left limit on a compact interval, one has

$$\inf\{q - \varphi_t + \varphi_u \mid u \in [\tau_{t,q} + \varepsilon, t]\} > 0, \quad \forall \varepsilon > 0.$$

Therefore, there exists $N \in \mathbb{N}$ s.t.

$$q - \frac{1}{n} - \varphi_t + \varphi_u > 0 \quad \forall u \in [\tau_{t,q} + \varepsilon, t], \ n \geq N.$$

This implies $\sup M^n_{t,q} \leq \tau_{t,q} + \varepsilon = \sup M_{t,q} + \varepsilon$ for all $n \geq N$. Together with (2.15) one obtains (2.14). (2.14), the predictability of $\sup M^n_{\cdot,q}$, and (2.5) imply the predictability of $(\omega,t) \mapsto \tau_{t,q}(\omega)$.

Step 2: One has

$$F(t,q) = S_t - \inf_{\tau_{t,q} \leq u \leq t} S_u = 0 \vee \sup_{y \in \mathbb{Q}} (S_t - S_y) 1_{(\tau_{t,q} < y < t)}$$

and by Step 1, $\{(\omega,t) \mid \tau_{t,q}(\omega) < y\} \in \mathcal{P}$. Because S is optional, $F(\cdot,q)$ is also optional, which completes the proof. ■

2.4 Proof of Theorem 2.2.12

Proposition 2.4.1. *For any elementary strategy φ, it holds that*

$$\alpha \int_0^t \varphi_u dS_u + \alpha \int_0^t \varphi_u dD_u = \alpha \int_0^\infty F(t,x)dx + \Pi_t, \quad \forall t \in [0,T]. \tag{2.16}$$

This proposition is the key step to prove Theorem 2.2.12. Namely, by the semimartingale property of S and D the integrals converge if $\varphi^n \to \varphi$, and with Lemma 2.3.1 it can be shown that also the corresponding book profits $\int_0^\infty F(t,x)dx$ converge. For the latter one needs that φ^n converges uniformly in probability and not only pointwise. To prove the proposition one needs the following lemma.

Lemma 2.4.2. *Let φ be an elementary strategy, s.t. $\varphi = \sum_{i=1}^k H_{i-1} 1_{]\!]\kappa_{i-1},\kappa_i]\!]}$, where $0 = \kappa_0 \leq \kappa_1 \leq \ldots \leq \kappa_k = T$ are stopping times and H_{i-1} is $\mathcal{F}_{\kappa_{i-1}}$–measurable. For all $t \in (\kappa_{i-1}, \kappa_i]$, $x \in (0, \varphi_t]$, we have*

$$S_t - S_{\kappa_{i-1}} = \left(F(\kappa_{i-1}+, x) + \inf_{\kappa_{i-1} \leq u \leq t} (S_u - S_{\kappa_{i-1}}) \right) \wedge 0 + F(t,x) - F(\kappa_{i-1}+, x).$$

Proof. Let $t_1, t_2 \in (\kappa_{i-1}, \kappa_i]$, i.e., $\varphi_{t_1} = \varphi_{t_2}$. As $x > 0$, one has $M_{t_1,x}, M_{t_2,x} \subset [0, \kappa_{i-1}]$, which leads, again by $\varphi_{t_1} = \varphi_{t_2}$, to $M_{t_1,x} = M_{t_2,x}$. By $x \leq \varphi_{t_1}$, we have $0 \in M_{t_1,x} \neq \emptyset$ and arrive at

$$\tau_{t_1,x} = \tau_{t_2,x} \leq \kappa_{i-1}. \tag{2.17}$$

By (2.17), the limit $\lim_{s \downarrow \kappa_{i-1}} \tau_{s,x} =: \tau_{\kappa_{i-1}+,x}$ exists and coincides with $\tau_{t,x}$, $t \in (\kappa_{i-1}, \kappa_i]$. This yields

$$\begin{aligned}
&\left(- \inf_{\tau_{\kappa_{i-1}+,x} \leq u \leq \kappa_{i-1}} S_u + \inf_{\kappa_{i-1} \leq u \leq t} S_u \right) \wedge 0 \\
&= \left(- \inf_{\tau_{t,x} \leq u \leq \kappa_{i-1}} S_u + \inf_{\kappa_{i-1} \leq u \leq t} S_u \right) \wedge 0 \\
&= - \inf_{\tau_{t,x} \leq u \leq \kappa_{i-1}} S_u + \inf_{\tau_{t,x} \leq u \leq t} S_u \\
&= - \inf_{\tau_{\kappa_{i-1}+,x} \leq u \leq \kappa_{i-1}} S_u + \inf_{\tau_{t,x} \leq u \leq t} S_u,
\end{aligned} \tag{2.18}$$

where for the second equality we use that, by (2.17), $[\tau_{t,x}, t] = [\tau_{t,x}, \kappa_{i-1}] \cup [\kappa_{i-1}, t]$, and we distinguish the cases $\inf_{\tau_{t,x} \leq u \leq \kappa_{i-1}} S_u \geq \inf_{\kappa_{i-1} \leq u \leq t} S_u$ and $\inf_{\tau_{t,x} \leq u \leq \kappa_{i-1}} S_u < \inf_{\kappa_{i-1} \leq u \leq t} S_u$. Using (2.18), the right-continuity of S, and the definition of F, it can immediately be seen that the LHS of (2.18) equals

$$\left(F(\kappa_{i-1}+, x) + \inf_{\kappa_{i-1} \leq u \leq t} (S_u - S_{\kappa_{i-1}}) \right) \wedge 0,$$

and the RHS of (2.18) equals

$$F(\kappa_{i-1}, x) - F(t,x) + S_t - S_{\kappa_{i-1}}.$$

So, we are done. ∎

Proof of Proposition 2.4.1. Let φ be as in Lemma 2.4.2. First, consider increments of (2.16) on $(\kappa_{i-1}, \kappa_i]$, $i \in \{1, \dots, k\}$. Let $t_1, t_2 \in (\kappa_{i-1}, \kappa_i]$. Because $\varphi_{t_1} = \varphi_{t_2}$ on $(\kappa_{i-1}, \kappa_i]$, one has, by definition of Π,

$$\begin{aligned}\Pi_{t_2} - \Pi_{t_1} =& \alpha \int_0^{\varphi_{t_1}} \left(F(\kappa_{i-1}+, x) + \inf_{\kappa_{i-1} \le u \le t_2 \wedge \kappa_i} (S_u - S_{\kappa_{i-1}}) \right) \wedge 0 \; dx \\ & - \alpha \int_0^{\varphi_{t_1}} \left(F(\kappa_{i-1}+, x) + \inf_{\kappa_{i-1} \le u \le t_1 \wedge \kappa_i} (S_u - S_{\kappa_{i-1}}) \right) \wedge 0 \; dx \\ & + \alpha \varphi_{t_1} (D_{t_2} - D_{t_1}).\end{aligned}$$

By Lemma 2.4.2, one arrives at

$$\begin{aligned}\Pi_{t_2} - \Pi_{t_1} =& \alpha \int_0^{\varphi_{t_1}} \left(S_{t_2} - S_{\kappa_{i-1}} - F(t_2, x) + F(\kappa_{i-1}+, x) \right) dx \\ & - \alpha \int_0^{\varphi_{t_1}} \left(S_{t_1} - S_{\kappa_{i-1}} - F(t_1, x) + F(\kappa_{i-1}+, x) \right) dx \\ & + \alpha \varphi_{t_1} (D_{t_2} - D_{t_1}) \\ =& \alpha \varphi_{t_1} (S_{t_2} + D_{t_2} - S_{t_1} - D_{t_1}) - \alpha \int_0^\infty (F(t_2, x) - F(t_1, x)) dx \\ =& \alpha \int_0^{t_2} \varphi_s d(S+D)_s - \alpha \int_0^{t_1} \varphi_s d(S+D)_s - \alpha \int_0^\infty F(t_2, x) dx \\ & + \alpha \int_0^\infty F(t_1, x) dx,\end{aligned}$$

where in the last equality we use that $\varphi_s = \varphi_{t_1}$ for all $s \in (t_1, t_2]$. This means that (2.16) holds true for all increments on $(\kappa_{i-i}, \kappa_i]$. As it obviously holds for $t = 0$, it remains to show that the right jumps of the processes $t \mapsto \int_0^\infty F(t, x) dx$ and Π at κ_{i-1} sum up to 0 as the LHS of (2.16) is right-continuous. By similar arguments as in the proof of Proposition 2.2.6(v), one obtains

$$\tau_{\kappa_{i-1}+, x} = \tau_{\kappa_{i-1}, x - (\varphi_{\kappa_{i-1}+} - \varphi_{\kappa_{i-1}})} \; \forall x \in \mathbb{R}_+ \text{ with the convention } \tau_{\kappa_{i-1}, y} := t \; \forall y < 0. \tag{2.19}$$

With the convention $F(\kappa_{i-1}, y) = 0$ for $y < 0$, one obtains

$$\begin{aligned}\lim_{t \downarrow \kappa_{i-1}} \int_0^{\varphi_t} F(t, x) dx &= \int_0^{\varphi_{\kappa_{i-1}+}} \left(S_{\kappa_{i-1}} - \inf_{\tau_{\kappa_{i-1}+, x} \le u \le \kappa_{i-1}} S_u \right) dx \\ &\overset{(2.19)}{=} \int_0^{\varphi_{\kappa_{i-1}+}} F(\kappa_{i-1}, x - \Delta^+ \varphi_{\kappa_{i-1}}) dx \\ &= \int_{-\Delta^+ \varphi_{\kappa_{i-1}}}^{\varphi_{\kappa_{i-1}+} - \Delta^+ \varphi_{\kappa_{i-1}}} F(\kappa_{i-1}, x) dx \\ &= \int_0^{\varphi_{\kappa_{i-1}}} F(\kappa_{i-1}, x) dx - \int_0^{-\Delta^+ \varphi_{\kappa_{i-1}}} F(\kappa_{i-1}, x) dx\end{aligned}$$

$$= \int_0^{\varphi_{\kappa_{i-1}}} F(\kappa_{i-1}, x)dx - \int_0^{\left(\Delta^+\varphi_{\kappa_{i-1}}\right)^-} F(\kappa_{i-1}, x)dx, \tag{2.20}$$

where the first equality follows from the definition of F using that S is right-continuous and $\tau_{t,x} = \tau_{\kappa_{i-1}+,x}$ for all $t \in (\kappa_{i-1}, \kappa_i]$ and $x > 0$. (2.20) means that

$$-\Delta^+\Pi_{\kappa_{i-1}} = \Delta^+ \left(\int_0^{\varphi_{\kappa_{i-1}}} F(\kappa_{i-1}, x)dx \right) = - \int_0^{\left(\Delta^+\varphi_{\kappa_{i-1}}\right)^-} F(\kappa_{i-1}, x)dx, \tag{2.21}$$

and we are done. ■

Proof of Theorem 2.2.12. The proof is divided into 3 steps.

Step 1: Let $(\varphi^n)_{n\in\mathbb{N}}$ be a sequence of nonnegative elementary strategies with $\varphi_0^n = 0$ and $\varphi^n \overset{\text{up}}{\to} \varphi$. From Proposition 2.3.2, one knows that $(\omega, t, x) \mapsto F_\omega^n(t, x)$ is $\mathcal{O} \otimes \mathcal{B}(\mathbb{R}_+) - \mathcal{B}(\mathbb{R}_+)$−measurable. So, $(\omega, t) \mapsto \int_0^\infty F_\omega^n(t, x)dx$ is $\mathcal{O} - \mathcal{B}(\mathbb{R}_+)$−measurable. Together with Proposition 2.4.1 and the fact that $\varphi^n \bullet S$ and $\varphi^n \bullet D$ are optional, this implies that Π^n is also optional.

In the next step, it is shown that $(\Pi^n)_{n\in\mathbb{N}}$ is an up-Cauchy sequence. Again by Proposition 2.4.1, it is enough to show that $(\varphi^n \bullet S)_{n\in\mathbb{N}}$, $(\varphi^n \bullet D)_{n\in\mathbb{N}}$, and $\left(\int_0^\infty F^n(\cdot, x)dx\right)_{n\in\mathbb{N}}$ are up-Cauchy sequences. Because $\varphi^n \overset{\text{up}}{\to} \varphi$ and S, D are semimartingales, it is known, e.g., from Theorem II.11 in [Pro04], that $(\varphi^n \bullet S)_{n\in\mathbb{N}}$, $(\varphi^n \bullet D)_{n\in\mathbb{N}}$ are up-Cauchy sequences. So, it remains to consider $\int_0^\infty F^n(t, x)dx$.

Let $\varepsilon > 0$. As S possesses càdlàg paths, there exists $K \in \mathbb{R}_+$ s.t.

$$P\left(\sup_{0\le t\le T} S_t - \inf_{0\le t\le T} S_t \ge K \right) \le \frac{\varepsilon}{2}.$$

As $\varphi^n \overset{\text{up}}{\to} \varphi$, there exists $N_\varepsilon \in \mathbb{N}$ s.t.

$$P\left(\sup_{0<t\le T} |\varphi_t^n - \varphi_t^m| > \frac{\varepsilon}{3K} \right) \le \frac{\varepsilon}{2}, \quad \forall n, m \ge N_\varepsilon.$$

By Lemma 2.3.1, we have

$$\left\{ \sup_{0\le t\le T} \left| \int_0^\infty F^n(t, x) - F^m(t, x)dx \right| > \frac{\varepsilon}{K} (\sup_{0\le t\le T} S_t - \inf_{0\le t\le T} S_t) \right\}$$
$$\subset \left\{ \sup_{0<t\le T} |\varphi_t^n - \varphi_t^m| > \frac{\varepsilon}{3K} \right\},$$

and one gets

$$
\begin{aligned}
&P\left(\sup_{0\le t\le T}\left|\int_0^\infty F^m(t,x)-F^n(t,x)\right|>\varepsilon\right)\\
\le& P\left((\sup_{0\le t\le T} S_t-\inf_{0\le t\le T} S_t)\frac{\varepsilon}{K}>\varepsilon\right)+P\left(\sup_{0<t\le T}|\varphi^n_t-\varphi^m_t|\ge\frac{\varepsilon}{3K}\right)\\
\le&\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\quad\forall n,m\ge N_\varepsilon.
\end{aligned}
$$

So, $(\Pi^n)_{n\in\mathbb{N}}$ is an up-Cauchy sequence. Because the space of làglàd functions (also called "regulated functions") mapping from $[0,T]$ to $\mathbb{R}$ is complete w.r.t. the supremum norm, there exists an optional làglàd process Π s.t. $\Pi^n\overset{\text{up}}{\to}\Pi$ (optionality follows from pointwise convergence up to evanescence of a suitable subsequence and the usual conditions).

Step 2: Let us now show (2.9). Let $t\in(0,T]$, $x_0\in(0,\varphi_t)$ and assume that

$$x\mapsto\widetilde{F}(t,x):=S_{t-}-\inf_{\tau_{t,x}\le u<t}S_u$$

is continuous at x_0. $\widetilde{F}(t,\cdot)$ is the time-t book profit function under the modified stock price process $\widetilde{S}_u:=1_{(u<t)}S_u+1_{(u\ge t)}S_{t-}$ (this modification removes the impact of ΔS_t on the book profits).

Let $\varepsilon\in(0,\varphi_t-x_0)$. By the left-continuity of φ and by $\tau_{t,x_0+\varepsilon}\le\tau_{t,x_0}<t$, one has for s smaller but close enough to t that

$$|\varphi_s-\varphi_t|\le\varepsilon\quad\text{and}\quad s>\tau_{t,x_0+\varepsilon}. \tag{2.22}$$

For s satisfying (2.22), one has $M_{s,x_0}\neq\emptyset$, $M_{t,x_0+\varepsilon}\cap[0,s]\neq\emptyset$, and the two implications

$$u\in M_{s,x_0}\Rightarrow u\in M_{t,x_0-\varepsilon},\qquad u\in M_{t,x_0+\varepsilon}\cap[0,s]\Rightarrow u\in M_{s,x_0}$$

hold; see (2.3) for the definition of M. This implies

$$\tau_{t,x_0-\varepsilon}\ge\tau_{s,x_0}\ge\tau_{t,x_0+\varepsilon}.$$

It follows that

$$\inf_{\tau_{t,x_0+\varepsilon}\le u<t}S_u\le\inf_{\tau_{s,x_0}\le u<t}S_u\le\inf_{\tau_{t,x_0-\varepsilon}\le u<t}S_u.$$

By the continuity of $\widetilde{F}(t,\cdot)$ in x_0, the left and the right bound are close together for ε small. We conclude that $\lim_{s<t,s\to t}F(s,x_0)=:F(t-,x_0)$ exists and

$$F(t-,x_0)=S_{t-}-\inf_{\tau_{t,x_0}\le u<t}S_u \tag{2.23}$$

(For elementary strategies, one has $\tau_{s,x} = \tau_{t,x}$ for s smaller but close to t, and therefore the limit $F(t-,x)$ exists for all $x \in \mathbb{R}_+$). By (2.23) and a distinction of the cases $S_t < \inf_{\tau_{t,x_0} \leq u < t} S_u$ and $S_t \geq \inf_{\tau_{t,x_0} \leq u < t} S_u$, one obtains $F(t,x_0) = 0 \vee (F(t-,x_0) + \Delta S_t)$ and thus

$$\Delta F(t,x_0) = (-F(t-,x_0)) \vee \Delta S_t = \Delta S_t + (-F(t-,x_0) - \Delta S_t) \vee 0.$$

By monotonicity, the mapping $x \mapsto \widetilde{F}(t,x)$ has at most countably many discontinuities, so that $\lim_{s<t,s\to t} \int_0^\infty F(s,x)\,dx$ exists and

$$\begin{aligned} &\Delta \int_0^\infty F(t,x)\,dx \\ =&\Delta \int_0^{\varphi_t} F(t,x)\,dx = \varphi_t \Delta S_t + \int_0^{\varphi_t} \left(- \limsup_{s<t,s\to t} F(s,x) - \Delta S_t \right) \vee 0\,dx \end{aligned} \quad (2.24)$$

(interchanging integral and limit is possible as F and S are bounded for ω fixed). By construction of Π, Proposition 2.4.1 holds for all $\varphi \in \mathbb{L}$. Together with (2.24) and $\Delta(\varphi \bullet (S+D)) = \varphi\Delta(S+D)$, this implies (2.9).

Step 3: It remains to prove (2.10). For the approximating elementary trading strategies φ^n, (2.10) follows immediately from Definition 2.2.8. As $(\Delta^+\varphi^n)^-$ converges to $(\Delta^+\varphi)^-$ uniformly in probability,

$$\int_0^{(\Delta^+\varphi_t^n)^-} F^n(t,x)dx \overset{\text{up}}{\longrightarrow} \int_0^{(\Delta^+\varphi_t)^-} F(t,x)dx \quad (2.25)$$

follows by the same arguments as in the proof of Lemma 2.3.1. Putting everything together, the assertion follows. ■

Example 2.4.3 (Counterexample for the existence of $F(t-,x)$)**.** *Consider a book profit function F at time $t=1$ such that $x \mapsto F(1,x)$ has a jump at $x_0 = 2$ (i.e., $F(1,2+) > F(1,2)$) and $F(1,x) > 0$ for all $x \in [1,3]$ with $\varphi_1 = 3$*[1]*. Suppose that for $t \geq 1$, the stock price S is constant, and the strategy is given by $\varphi_u = 3 + (2-u)\sin\left(\frac{\pi}{2-u}\right)$ for $u \in [1,2)$ and $\varphi_u = 3$ for $u \geq 2$. As the strategy φ_s is oscillating infinitely often around φ_t for s smaller but close to $t=2$, this yields*

$$\limsup_{s<2,s\to 2} F(s,2) = F(2,2+) > F(2,2) = \liminf_{s<2,s\to 2} F(s,2).$$

(the values of $F(s,2)$ are toggling infinitely often between the sets $A := \{y : y \in [F(2,2+),\infty)\}$ and $B := \{y : y \in (0,F(2,2)]\}$, where, according to the assumption, $d(A,B) = F(2,2+) - F(2,2) > 0$.)

[1]Note that by a proper choice of the stock price S and corresponding strategy φ in the time interval $[0,1]$, one can easily construct an example which fulfills these conditions on F

2.5 Self-financing Condition

To prepare Section 2.6, we introduce the self-financing condition of the model which is a natural generalization of the standard continuous time self-financing condition without taxes.

Besides the risky stock with price process S and dividend process D, the market consists of a so-called bank account. Formally, the bank account can be seen as a security with price process 1 and *dividend process*

$$B_t = \int_0^t r_s \, ds, \quad t \in [0, T], \tag{2.26}$$

where the locally riskless interest rate r is a predictable, *nonnegative*, and integrable process. This simplifies the analysis as increments of B are taxed immediately, and one needs not consider unrealized book profits of the bank account (as for the risky stock).

Definition 2.5.1 (Wealth process and self-financing condition)**.** *Let X be an optional process modeling the number of monetary units in the bank account, and $\varphi \in \mathbb{L}$ models the number of stocks the investor holds in her portfolio. The* wealth process V *of the strategy (X, φ) is defined as*

$$V = V(X, \varphi) := X + \varphi S. \tag{2.27}$$

A strategy (X, φ) is called self-financing with initial wealth v_0 iff

$$V = v_0 + (1 - \alpha)X \bullet B + \varphi \bullet D + \varphi \bullet S - \Pi \tag{2.28}$$

with Π from Definition 2.2.13.

Remark 2.5.2. *As B is continuous, it is sufficient to assume that X is optional instead of predictable. Thus, the after-tax dividend $(1 - \alpha)\varphi_t \Delta D_t$ of the stock can be included in the number of monetary units X_t. Note that an immediate reinvestment of the payoff in the stock would only affect φ_{t+}, but not φ_t.*

Remark 2.5.3. *For any $\varphi \in \mathbb{L}$, $v_0 \in \mathbb{R}$, there exists a unique optional process X s.t. (X, φ) is self-financing. Indeed, plugging (2.27) into (2.28) yields*

$$X = v_0 + (1 - \alpha)X \bullet B + \varphi \bullet D + \varphi \bullet S - \Pi - \varphi S. \tag{2.29}$$

Now, an optional process X solves (2.29) iff X is làglàd, the càdlàg process X_+ solves the SDE

$$Z = v_0 + (1 - \alpha)Z_- \bullet B + \varphi \bullet D + \varphi \bullet S - \Pi_+ - \varphi_+ S$$

(which has a unique solution Z, cf., e.g., Theorem V.7 in [Pro04]), and

$$X = Z - \Delta^+ X = Z + \Delta^+ \Pi + S \Delta^+ \varphi.$$

(2.28) means that increments of the wealth process solely result from trading gains and tax payments. An alternative condition is to assume that portfolio regroupings do not involve costs. The latter condition may be more intuitive, but it has the drawback that it can only be stated for strategies that can be used as integrators (thus, trading strategies that are no semimartingales would be excluded even though they could economically make sense). Let φ and Π be as in Definition 2.2.8. The alternative self-financing condition reads

$$X_t = v_0 - \sum_{i=1}^{k} 1_{(\kappa_{i-1}<t)} S_{\kappa_{i-1}}(\varphi_{\kappa_{i-1}+} - \varphi_{\kappa_{i-1}}) + \int_0^t (1-\alpha) X_s r_s ds - \Pi_t + \varphi \bullet D_t. \tag{2.30}$$

It is an easy exercise to prove equivalence of (2.30) and (2.28) for elementary strategies.

Set $n := \sup\{i : \kappa_{i-1} < t\}$ and note that $\varphi_{\kappa_i} = \varphi_{\kappa_{i-1}+}$ and $\varphi_{\kappa_n} = \varphi_t$. Plugging in the definition of X_t into V_t and rearranging the terms gives, together with $\varphi_0 = 0$,

$$\begin{aligned}
V_t =& v_0 - \sum_{i=1}^{n} S_{\kappa_{i-1}}(\varphi_{\kappa_i} - \varphi_{\kappa_{i-1}}) + \int_0^t (1-\alpha) X_s r_s ds - \Pi_t + \varphi_t S_t + \varphi \bullet D_t \\
=& v_0 - \sum_{i=1}^{n} S_{\kappa_{i-1}}(\varphi_{\kappa_i} - \varphi_{\kappa_{i-1}}) + (1-\alpha) X \bullet B_t - \Pi_t + \varphi_t S_t - \varphi_{\kappa_n} S_{\kappa_n} \\
& + \sum_{i=1}^{n} \left(\varphi_{\kappa_i} S_{\kappa_i} - \varphi_{\kappa_{i-1}} S_{\kappa_{i-1}}\right) + \varphi \bullet D_t \\
=& v_0 - \varphi_{\kappa_n}\left(S_{\kappa_n} - S_t\right) + \sum_{i=1}^{n} \varphi_{\kappa_i}\left(S_{\kappa_i} - S_{\kappa_{i-1}}\right) + (1-\alpha) X_s \bullet B_t - \Pi_t + \varphi \bullet D_t \\
=& v_0 + \varphi \bullet S_t + (1-\alpha) X_s \bullet B_t - \Pi_t + \varphi \bullet D_t.
\end{aligned}$$

2.6 Comparison of Different Dividend Policies

In this section, we investigate the effect of different dividend policies on the investor's after-tax wealth. In particular, we show that under the mild condition that the dividend policy has no effect on the *stochastic return process*, the effect of dividends is always negative. This assumption is formalized by the following definition.

Definition 2.6.1. *Let R be a semimartingale with $\Delta R \geq -1$ and $s_0 \in \mathbb{R}_+$. Then, for any nondecreasing càdlàg process D, define S^D as the unique solution of*

$$S^D = s_0 + S^D_- \bullet R - D. \tag{2.31}$$

We call D admissible iff $S^D \geq 0$, i.e., we only consider dividend payoffs that do not exceed the stock price. R is the return process *modeling the stochastic profit per invested capital.*

Observe that for any admissible D, the stock price S^D stays at zero once the process or its left limit hits it. Note that by $\Delta R \geq -1$, $D = 0$, which corresponds to the model without dividends, is admissible. Alternatively, one can start with an arbitrary nondecreasing process $\widetilde{D}$ with

$$\Delta \widetilde{D} \leq 1 + \Delta R \tag{2.32}$$

modeling accumulated dividends as multiples of the current stock price and consider the SDE

$$S = s_0 + S_- \bullet (R - \widetilde{D}). \tag{2.33}$$

Then, $S^D = S$ for $D := S_- \bullet \widetilde{D}$, and, by (2.32), the stock price is nonnegative. But, as for an arbitrary admissible dividend process D the integral $\frac{1}{S^D_-} 1_{\{S^D_- > 0\}} \bullet D$ may explode, Definition 2.6.1 is slightly more general.

Remark 2.6.2. *(2.31) says that one has the same R for all processes D, i.e., there holds a scaling invariance of the stochastic investment opportunities. The negative effect of dividends on the after-tax wealth is essentially based on this property. It is, e.g., not satisfied in the Bachelier model with dividends.*

Note that we do not assume that dividend payoffs are accompanied by downward jumps of the same size of the stock price. Such a behavior can be explained by no-arbitrage arguments if dividends are predictable. However, the framework also allows for a spontaneous dividend payment ΔD_t, e.g., if ΔR_t is large.

Recall that we consider a market model with two investment opportunities: a risky stock with price process S^D and dividend process D (interrelated by Condition (2.31)) and a locally riskless bank account. The latter is an asset with price process 1 and the nondecreasing dividend process B defined in (2.26). We denote the model by $((S^D, D), (1, B))$. Now, we compare the situation of an arbitrary admissible dividend process D with the situation of no dividends. In the latter model, we use the subscript 0, i.e., S^0, Π^0, V^0, etc. The following theorem is the main result of this section.

Theorem 2.6.3. *Let (X^D, φ^D) be a self-financing strategy with initial wealth v_0 in the model with dividends $((S^D, D), (1, B))$, and let V^D be the corresponding wealth process. Then, there exists a self-financing strategy (X^0, φ^0) with initial wealth v_0 in the model without dividends $((S^0, 0), (1, B))$, where V^0 is the corresponding wealth process, s.t. $V^D \leq V^0$.*

To prove the theorem, we use the following two lemmata.

Lemma 2.6.4. *The process*

$$\frac{S^D}{S^0} 1_{\{S^0>0\}}$$

is nonincreasing.

Proof. The case $s_0 = 0$ is obvious. Let $s_0 > 0$ and define

$$\tau := \inf\{t \geq 0 \mid \Delta R_t = -1\}.$$

By the formula of Yoeurp-Yor [YY77] (see also [Jas03]), one has

$$S^D = S^0 \left(1 - \frac{1}{S^0_-} \bullet D + \sum_{0<s\leq \cdot} \frac{1}{S^0_{s-}} \frac{\Delta D_s \Delta R_s}{1 + \Delta R_s}\right) \tag{2.34}$$

on the stochastic interval $[\![0, \tau[\![$. The second factor of the RHS of (2.34) is obviously a nonincreasing process. As $S^D = S^0 = 0$ on $[\![\tau, \infty[\![$, we are done. ∎

The key step to prove Theorem 2.6.3 is the following lemma.

Lemma 2.6.5. *Let* $\varphi^D \in \mathbb{L}$ *and* $\varphi^0 := \varphi^D \frac{S^D_-}{S^0_-} 1_{\{S^0_->0\}}$. *Then,* $\varphi^0 \in \mathbb{L}$,

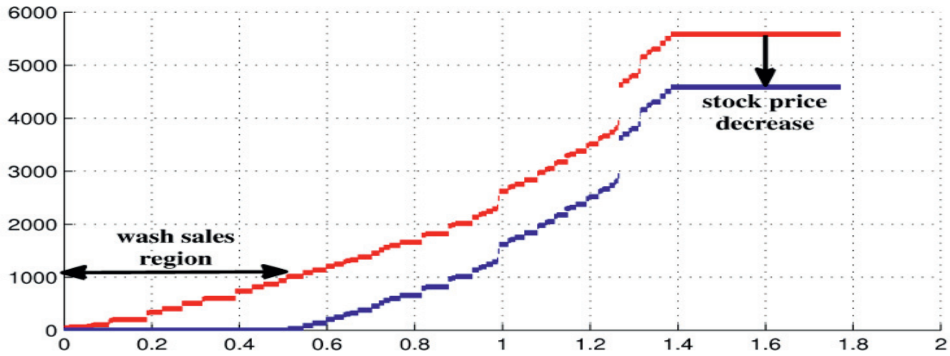
$$\varphi^0 \bullet S^0 = \varphi^D \bullet (S^D + D), \quad \textit{and} \quad \Pi^0 \leq \Pi^D. \tag{2.35}$$

The lemma indicates that for an arbitrary strategy in the model with dividends, there exists a strategy in the model without dividends leading to the same trading gains in the risky stock but not exceeding accumulated tax payments. The money invested in the stock is the same for both strategies. If price processes do not vanish, one can recover φ^D from φ^0 by investing the dividend payoffs in new stocks. This is illustrated in Figure 2.4.

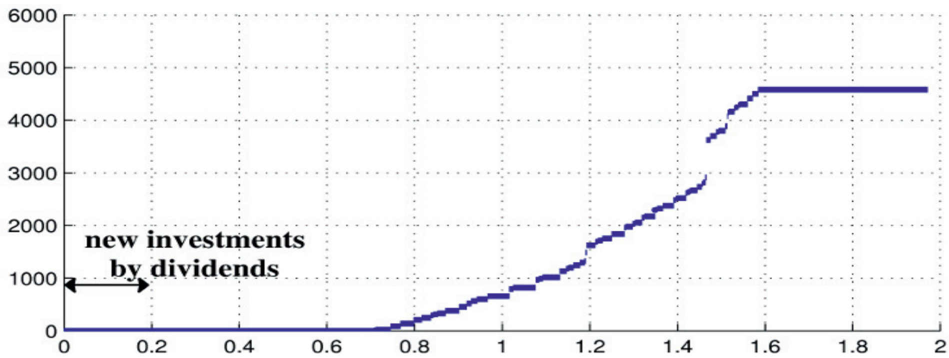
Proof of Lemma 2.6.5. The proof is divided into 3 steps. We first show that the difference of the accumulated tax payments Π^D and Π^0 can be expressed in terms of the difference of integrals over the corresponding book profit functions. Then, we show how to compare the purchase times in both models. In the last step, we conclude, together with the first two steps, that Π^D dominates Π^0.

Step 1: As $\varphi^0 \leq \varphi^D$, one obviously has $\varphi^0 \in \mathbb{L}$. Because $S^D = S^D_- \bullet R - D$ and $S^0 = S^0_- \bullet R$, one obtains

$$\begin{aligned}
\varphi^0 \bullet S^0 =& \varphi^D \frac{S^D_-}{S^0_-} 1_{\{S^0_->0\}} \bullet (S^0_- \bullet R) \\
=& \varphi^D S^D_- 1_{\{S^0_->0\}} \bullet R \\
=& \varphi^D 1_{\{S^0_->0\}} \bullet (S^D_- \bullet R) \\
=& \varphi^D \bullet (S^D + D),
\end{aligned} \tag{2.36}$$



Book profit functions $x \mapsto F^D(t_1-, x)$ and $x \mapsto F^D(t_1, x)$ modeling book profits immediately before and after the predictable dividend payoff $\Delta D_{t_1} = 1000$, respectively, associated with $\Delta S_{t_1} = -1000$. One has $F^D(t_1-, x) + \Delta S_{t_1} < 0$ iff $x < 55$. This means that 55 stocks are sold and immediately repurchased (wash sale).



Book profit function $x \mapsto F^D(t_1+, x)$ after portfolio regrouping. According to φ^D, the dividend payoff is invested in 20 new stocks which start with zero book profits, and the function is shifted about 20 units to the right.

Figure 2.4: Reinvestment of dividends

where for the last equality we use that $\{S^0_- = 0\} \subset \{S^D_- = 0\}$ and the process $S^D_- 1_{\{S^D_- = 0\}} \bullet R$ vanishes.

By construction of Π, Proposition 2.4.1 holds for all strategies in $\mathbb{L}$, i.e.,

$$\alpha \int_0^\infty F^D(t,x)dx + \Pi^D_t = \alpha\varphi^D \bullet (S^D + D)$$

(and the same without dividends). Together with (2.36), one obtains

$$\Pi^D_t - \Pi^0_t = \alpha \int_0^\infty F^0(t,x)dx - \alpha \int_0^\infty F^D(t,x)dx. \tag{2.37}$$

Step 2: Let us show that for $\varphi^0_t > 0$ (implying that $\varphi^D_t > 0$ and $S^D_{t-} > 0$)

$$\tau^D_{t,x} \geq \tau^0_{t, x\frac{\varphi^0_t}{\varphi^D_t}}, \quad \forall x \in \mathbb{R}_+. \tag{2.38}$$

First note that

$$M^0_{t,x\frac{\varphi^0_t}{\varphi^D_t}} = \emptyset \Leftrightarrow M^D_{t,x} = \emptyset \tag{2.39}$$

(cf. Definition 2.2.3). It is sufficient to consider x s.t. both sets are not empty. One has

$$x - \varphi^D_t + \varphi^D_u > 0 \quad \forall u \in (\tau^D_{t,x}, t] \quad \text{and} \quad x - \varphi^D_t + \varphi^D_{u+} > 0 \quad \forall u \in (\tau^D_{t,x}, t).$$

We conclude

$$0 < \frac{\varphi^0_t}{\varphi^D_t}\left(x - \varphi^D_t + \varphi^D_u\right) = \frac{\varphi^0_t}{\varphi^D_t}\left(x - \varphi^0_t \frac{S^0_{t-}}{S^D_{t-}} + \varphi^0_u \frac{S^0_{u-}}{S^D_{u-}}\right) \le \frac{\varphi^0_t}{\varphi^D_t} x - \varphi^0_t + \varphi^0_u$$

for all $u \in (\tau^D_{t,x}, t]$, where for the last inequality we use that $\frac{\varphi^D}{\varphi^0} = \frac{S^0_-}{S^D_-}$ is nondecreasing by Lemma 2.6.4.
By $\frac{\varphi^D_+}{\varphi^0_+} = \frac{S^0}{S^D}$, one obtains analogously for φ^0_{u+} that

$$0 < \frac{\varphi^0_t}{\varphi^D_t}\left(x - \varphi^D_t + \varphi^D_{u+}\right) = \frac{\varphi^0_t}{\varphi^D_t}\left(x - \varphi^0_t \frac{S^0_{t-}}{S^D_{t-}} + \varphi^0_{u+} \frac{S^0_u}{S^D_u}\right) \le \frac{\varphi^0_t}{\varphi^D_t} x - \varphi^0_t + \varphi^0_{u+}$$

for all $u \in (\tau^D_{t,x}, t)$.
As $M^0_{t,x\frac{\varphi^0_t}{\varphi^D_t}} \neq \emptyset$, it can be concluded that $\tau^0_{t,x\frac{\varphi^0_t}{\varphi^D_t}} = \sup M^0_{t,x\frac{\varphi^0_t}{\varphi^D_t}} \le \tau^D_{t,x}$.

Step 3: For $\varphi^0_t > 0$ (implying $S^0_{t-} > 0$ and $\varphi^D_t > 0$), we have

$$\begin{aligned}
F^D(t,x) &= S^D_t - \inf_{\tau^D_{t,x} \le u \le t} S^D_u \\
&\overset{\text{Lemma 2.6.4}}{\le} \left(\frac{S^D_{t-}}{S^0_{t-}} S^0_t - \inf_{\tau^D_{t,x} \le u < t} \frac{S^D_u}{S^0_u} S^0_u\right) \vee 0 \\
&\overset{\text{Lemma 2.6.4}}{\le} \frac{S^D_{t-}}{S^0_{t-}}\left(S^0_t - \inf_{\tau^D_{t,x} \le u < t} S^0_u\right) \vee 0 \\
&\overset{(2.38)}{\le} \frac{S^D_{t-}}{S^0_{t-}}\left(S^0_t - \inf_{\tau_{t,\varphi^0_t/\varphi^D_t x} \le u < t} S^0_u\right) \vee 0 \\
&= \frac{\varphi^0_t}{\varphi^D_t} F^0\left(t, \frac{\varphi^0_t}{\varphi^D_t} x\right).
\end{aligned} \tag{2.40}$$

Observe that for the second inequality, we use $S^D_{t-}/S^0_{t-} \le S^D_u/S^0_u$ for u *strictly* smaller than t (all considered prices are nonzero). For $\varphi^0_t > 0$, it follows from (2.40) that

$$\begin{aligned}
&\alpha \int_0^\infty F^0(t,x)dx - \alpha \int_0^\infty F^D(t,x)dx \\
\ge &\alpha \int_0^\infty F^0(t,x)dx - \alpha \int_0^\infty \frac{\varphi^0_t}{\varphi^D_t} F^0\left(t, x\frac{\varphi^0_t}{\varphi^D_t}\right) dx = 0.
\end{aligned} \tag{2.41}$$

If $\varphi_t^0 = 0$, then either $\varphi_t^D = 0$ or $S_{t-}^D = 0$. Both equalities imply $F^D(t,\cdot) = 0$, and, consequently, the first difference in (2.41) is nonnegative. Putting (2.37) and (2.41) together yields the assertion. ■

Proof of Theorem 2.6.3. Let $\varphi^D \in \mathbb{L}$. φ^0 is defined as in Lemma 2.6.5 and X^D, X^0 are the unique positions in the bank account to meet the self-financing condition (cf. Remark 2.5.3). Let us first examine the right limits V_+^0 and V_+^D. By the self-financing condition, one has

$$\begin{aligned} V_+^0 =& v_0 + (1-\alpha)X^0 \bullet B + \varphi^0 \bullet S^0 - \Pi_+^0 \\ V_+^D =& v_0 + (1-\alpha)X^D \bullet B + \varphi^D \bullet S^D + \varphi^D \bullet D - \Pi_+^D. \end{aligned}$$

On the other hand,

$$V_+^D = X_+^D + \varphi_+^D S^D = X_+^D + \varphi_+^0 S^0 = V_+^0 + X_+^D - X_+^0$$

Together with $\varphi^0 \bullet S = \varphi^D \bullet (S_t^D + D)$, one arrives at

$$\begin{aligned} X_+^D - X_+^0 =& V_+^D - V_+^0 \\ =& (1-\alpha)(X_+^D - X_+^0) \bullet B - \Pi_+^D + \Pi_+^0 \\ \leq& (1-\alpha)(X^D - X^0) \bullet B \end{aligned} \tag{2.42}$$

By Gronwall's lemma in the form of Lemma 2.1 in [Kat91] applied to the nonnegative càdlàg process $(X_+^D - X_+^0) \vee 0$ and the nondecreasing process B (here, one needs $r \geq 0$), one obtains $X_+^D \leq X_+^0$, and thus

$$V_+^D \leq V_+^0. \tag{2.43}$$

Note that the lemma cannot be applied directly to X^D and X^0 as these processes are not càdlàg. Thus, the right jumps of $V^D - V^0$ have to be analyzed. For $\varphi_t^D = 0$, one also has $\varphi_t^0 = 0$, and the jump at time t vanishes. Otherwise, one argues

$$\begin{aligned} & \Delta^+(V^D - V^0)_t \\ =& \Delta^+(\Pi^0 - \Pi^D)_t \\ \overset{(2.10)}{=}& \alpha \int_0^{(\Delta^+\varphi_t^0)^-} F^0(t,x)dx - \alpha \int_0^{(\Delta^+\varphi_t^D)^-} F^D(t,x)dx \\ \overset{(2.40)}{\geq}& \alpha \int_0^{(\Delta^+\varphi_t^0)^-} F^0(t,x)dx - \alpha \int_0^{(\Delta^+\varphi_t^D)^-} \frac{\varphi_t^0}{\varphi_t^D} F^0\left(t, \frac{\varphi_t^0}{\varphi_t^D}x\right) dx \\ =& \alpha \int_0^{(\Delta^+\varphi_t^0)^-} F^0(t,x)dx - \alpha \int_0^{\left(\frac{\varphi_t^0}{\varphi_t^D}\Delta^+\varphi_t^D\right)^-} F^0(t,x)\,dx \\ =& \alpha \int_0^{(\Delta^+\varphi_t^0)^-} F^0(t,x)dx - \alpha \int_0^{\left(\frac{S_{t-}^D}{S_{t-}^0}\left(\varphi_{t+}^0 \frac{S_t^0}{S_t^D} - \varphi_t^0 \frac{S_{t-}^0}{S_{t-}^D}\right)\right)^-} F^0(t,x)\,dx \end{aligned}$$

$$\begin{aligned} &\geq \alpha \int_0^{\left(\Delta^+\varphi_t^0\right)^-} F^0(t,x)dx - \alpha \int_0^{\left(\frac{S_{t-}^D}{S_{t-}^0}\left(\varphi_{t+}^0 \frac{S_{t-}^0}{S_{t-}^D} - \varphi_t^0 \frac{S_{t-}^0}{S_{t-}^D}\right)\right)^-} F^0\left(t,x\right)dx \\ &= 0. \end{aligned} \tag{2.44}$$

The last inequality uses that $S_t^0/S_t^D \geq S_{t-}^0/S_{t-}^D$ by Lemma 2.6.4. Putting (2.43) and (2.44) together, one obtains

$$V_t^D = V_{t+}^D - \Delta^+ V_t^D \leq V_{t+}^0 - \Delta^+ V_t^D \leq V_{t+}^0 - \Delta^+ V_t^0 = V_t^0.$$

■

2.7 (Counter-)Examples

In this section, we give some examples that illustrate problems with the construction of the tax payment process and show the necessity of some assumptions.

We start with an example showing that for an adapted, càglàd strategy the tax payment process Π defined in (2.2.8) can be of infinite variation. This means that the tax credits from wash sales, which are the downward movements of Π, are infinite and it indicates that Π can in general not be constructed in a direct way as in (2.2.8) for elementary strategies. The example is based on frequent updates of the position invested in the stock s.t. upward movements have to be taxed before they are to some extent compensated by downward movements.

Example 2.7.1 (A tax payment process of infinite variation)**.** *Let $T = 1$ and S be a standard Brownian motion. In the interval $[0, 1/2]$ we buy and resell k_1 times a_1 stocks whereas each holding period takes $1/(4k_1)$ time units. We proceed analogously on $[1/2, 3/4], [3/4, 7/8], \ldots$. The sequences $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ and $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ specifying the number of purchasing times and the number of stocks, respectively, in the interval $[1 - 2^{-(n-1)}, 1 - 2^{-n}]$ should satisfy*

$$a_n \to 0, n \to \infty \text{ and } \sqrt{\frac{1}{2^{n+1}k_n}} k_n a_n = \frac{\sqrt{k_n}}{2^{(n+1)/2}} a_n \geq 1, \ \forall n \in \mathbb{N}. \tag{2.45}$$

Of course, such a choice is possible. This strategy is plotted in Figure 2.5. Denote by $d_n := 1/(2^{n+1}k_n)$ the holding period of purchases in the interval $[1 - 2^{-(n-1)}, 1 - 2^{-n}]$. Then, the strategy is formally given by

$$\varphi_t := \sum_{n=1}^{\infty} a_n \sum_{k=1}^{k_n} 1_{(1-2^{-(n-1)}+2(k-1)d_n, 1-2^{-(n-1)}+(2(k-1)+1)d_n]}(t).$$

As $a_n \to 0$ *we have that* $\lim_{t\to 1} \varphi_t = 0$ *and thus* $\varphi \in \mathbb{L}$*. The accumulated tax credits are given by the nondecreasing process*

$$Y_t = -\,\alpha \sum_{n=1}^{\infty} a_n \sum_{k=1}^{k_n} 1_{(1-2^{-(n-1)}+2(k-1)d_n<t)} \times \inf_{1-2^{-(n-1)}+2(k-1)d_n \le u \le t\wedge(1-2^{-(n-1)}+(2(k-1)+1)d_n)} \left(S_u - S_{1-2^{-(n-1)}+2(k-1)d_n}\right)$$

(cf. the second item in Remark 2.2.9). For different n, k *the summands are stochastically independent and by the scaling property of the Brownian motion they coincide in law with*

$$\frac{a_n}{\sqrt{2^{n+1}k_n}}(-\alpha)\inf_{0\le s\le 1} B_s$$

where B *is again a Brownian motion. By the law of large numbers, we have*

$$P\left(\sum_{k=1}^{k_n}(-\alpha)\inf_{k\le s\le k+1}(B_s - B_k) > k_n E\left(\frac{-\alpha}{2}\inf_{0\le s\le 1} B_s\right)\right) \to 1,\ k_n \to \infty.$$

Together with (2.45), this implies the existence of a $K \in \mathbb{N}$ *s.t.*

$$\begin{aligned} & P\left(Y_{1-2^{-n}} - Y_{1-2^{-(n-1)}} > E\left(\frac{-\alpha}{2}\inf_{0\le s\le 1} B_s\right)\right) \\ = \ & P\left(\sum_{k=1}^{k_n}\frac{a_n}{\sqrt{2^{n+1}k_n}}(-\alpha)\inf_{k\le s\le k+1}(B_s - B_k) > E\left(\frac{-\alpha}{2}\inf_{0\le s\le 1} B_s\right)\right) \\ \overset{(2.45)}{\ge} \ & P\left(\sum_{k=1}^{k_n}(-\alpha)\inf_{k\le s\le k+1}(B_s - B_k) > k_n E\left(\frac{-\alpha}{2}\inf_{0\le s\le 1} B_s\right)\right) \ge 1/2, \end{aligned}$$

for all $n \in \mathbb{N}$ *s.t.* $k_n \ge K$*. By* $k_n \to \infty$ *for* $n \to \infty$ *and the second Borel-Cantelli lemma this shows that* $P(Y_1 = \infty) = 1$.

Remark 2.7.2. *If the stock price process is not a semimartingale, different sequences of up-approximating elementary strategies of a left-continuous strategy* φ *can lead to different limits of the actual tax payments* Π^n*. Namely, if* S *is not a semimartingale, there exists a sequence of nonnegative elementary strategies* $(\varphi^n)_{n\in\mathbb{N}}$ *s.t.*

$$||\varphi^n||_\infty \to 0, \quad E(1 \wedge \sup_{t\in[0,T]}(\varphi^n \bullet S_t)^-) \to 0, \quad n \to \infty,$$

but

$$E(1 \wedge \sup_{t\in[0,T]}(\varphi^n \bullet S_t)^+) \not\to 0, \quad n \to \infty,$$

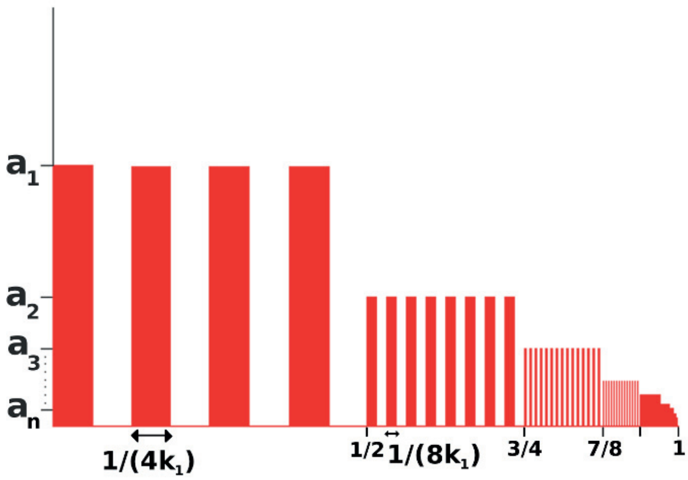


Figure 2.5: Trading strategy from Example 2.7.1

see Theorem 1.7 of [BSV11] (shifting the strategies by the constants $||\varphi^n||_\infty$ *shows that they can be chosen nonnegative). By* $||\varphi^n||_\infty \to 0$*, the book profits vanish, i.e.,* $\int_0^\infty F^n(\cdot, x)\, dx \to 0$ *uniformly in probability, but the trading gains do not tend to zero. Thus, by Proposition 2.4.1,* $(\Pi^n)_{n\in\mathbb{N}}$ *does not tend to zero. On the other hand, the elementary strategy* $\varphi = 0$*, the uniform limit of* $(\varphi^n)_{n\in\mathbb{N}}$*, leads to zero tax payments.*

Remark 2.7.3. *Tax payments are not continuous w.r.t. pointwise convergence of elementary strategies. Indeed, let* $\varphi^n = 1_{(0,1/2]\cup(1/2+1/n,1]}$*. Then,* φ^n *converges pointwise to* $\varphi = 1_{(0,1]}$ *and* $\varphi^n \bullet S \to \varphi \bullet S$ *uniformly in probability. But, in contrast to* φ*, the strategy* φ^n *realizes current book profits at time* $1/2$*. Thus, it is not possible to define the tax payment process as unique continuous extension w.r.t. pointwise convergence to the space of all predictable locally bounded strategies as it is done for the stochastic integral, cf. Theorem I.4.31 in [JS03]. It seems that the convergence "uniformly in probability" for trading strategies is tailor-made for modeling capital gains taxes. The strategy set* $\mathbb{L}$ *is still rich enough to cover almost all relevant strategies in applications.*

2.8 Tax-efficient Strategies

Let $S \geq 0$ be a continuous semimartingale and $\varphi_t = g(S_t)$ for all $t > 0$, where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a *nondecreasing* and twice continuously differentiable function. This means that the "initial" position is $\varphi_{0+} = g(S_0)$, and the investor increases (reduces) her position after an increase (decrease) of the stock price. Denote by g^{-1} the right-continuous inverse of g, i.e.,

$$g^{-1}(y) := \sup\{s \mid g(s) \leq y\}.$$

Let us show that the book profit function reads

$$F(t,x) := S_t - \inf_{\tau_{t,x}\le u\le t} S_u = \begin{cases} S_t - g^{-1}(\varphi_t - x), & x \le \varphi_t - \inf_{0<u\le t}\varphi_u \\ S_t - \inf_{0\le u\le t} S_u, & x > \varphi_t - \inf_{0<u\le t}\varphi_u \end{cases} \quad (2.46)$$

for all $t > 0$.
This means that the infinite-dimensional stochastic process F is a direct function of the two-dimensional stochastic process $(S_t, \inf_{0\le u\le t} S_u)_{t\ge 0}$. Note that $\inf_{0<u\le t}\varphi_u = g(\inf_{0\le u\le t} S_u)$.

To prove (2.46), first consider the case that $x \le \varphi_t - \inf_{0<u\le t}\varphi_u$. By definition of $\tau_{t,x}$, one has that $g(S_u) = \varphi_u > \varphi_t - x$ for all $u \in (\tau_{t,x}, t]$. Together with the monotonicity and the continuity of g, this implies $S_u > g^{-1}(\varphi_t - x)$. On the other hand, we have $\varphi_{\tau_{t,x}+} = \varphi_t - x$ and thus, by $g(S_{\tau_{t,x}}) = \varphi_{\tau_{t,x}+}$, $S_{\tau_{t,x}} \le \sup\{s \mid g(s) \le \varphi_{\tau_{t,x}+}\} = g^{-1}(\varphi_t - x)$ (the right limit is only needed for the case that $\tau_{t,x} = 0$, which is possible if $x = \varphi_t - \inf_{0<u\le t}\varphi_u$). By continuity of the paths of S, we conclude $\inf_{\tau_{t,x}\le u\le t} S_u = g^{-1}(\varphi_t - x)$.

This means, the purchasing price of the stock with label x is $S_{\tau_{t,x}} = g^{-1}(\varphi_t - x)$, and up to time t, the price does not fall below it. Now, let $x > \varphi_t - \inf_{0<u\le t}\varphi_u$. One has $\tau_{t,x} = 0$ which yields the assertion.

If $g' < 0$, one still has that $S_{\tau_{t,x}} = g^{-1}(\varphi_t - x)$ (of course, with g^{-1} defined appropriately). But now, $\inf_{\tau_{t,x}\le u\le t} S_u = g^{-1}(\sup_{\tau_{t,x}<u\le t}\varphi_u)$. As the infimum can be attained anywhere between $\tau_{t,x}$ and t, this implies that $F(t,\cdot)$ cannot be a direct function of $(S_t, \inf_{0\le u\le t} S_u)$.

From (2.46), it follows that

$$\begin{aligned} \int_0^{\varphi_t} F(t,x)\,dx =& (\varphi_t - \inf_{0<u\le t}\varphi_u)S_t - \int_0^{\varphi_t - \inf_{0<u\le t}\varphi_u} g^{-1}(\varphi_t - x)\,dx \\ &+ \inf_{0<u\le t}\varphi_u (S_t - \inf_{0\le u\le t} S_u) \\ =& \varphi_t S_t - \inf_{0<u\le t}\varphi_u \inf_{0\le u\le t} S_u - \int_{\inf_{0<u\le t}\varphi_u}^{\varphi_t} g^{-1}(x)\,dx. \end{aligned} \quad (2.47)$$

Using that $g' = 0$ on $(S_u, g^{-1}(\varphi_u))$, integration by parts yields

$$\begin{aligned} \int_{\inf_{0<u\le t}\varphi_u}^{\varphi_t} g^{-1}(x)\,dx =& \int_{g^{-1}(\inf_{0<u\le t}\varphi_u)}^{g^{-1}(\varphi_t)} y g'(y)\,dy \\ =& \int_{\inf_{0\le u\le t} S_u}^{S_t} y g'(y)\,dy \\ =& y g(y)\Big|_{\inf_{0\le u\le t} S_u}^{S_t} - \int_{\inf_{0\le u\le t} S_u}^{S_t} g(y)\,dy. \end{aligned} \quad (2.48)$$

Let G be an antiderivative of g, i.e., $G' = g$. Putting (2.47) and (2.48) together, we arrive at

$$\int_0^{\varphi_t} F(t,x)\,dx = G(S_t) - G\left(\inf_{0\le u\le t} S_u\right).$$

For the trading gains, one has by Itô's formula

$$g(S) \bullet S_t = G(S_t) - G(S_0) - \frac{1}{2}g'(S) \bullet [S,S]_t = G(S_t) - G(S_0) - \frac{1}{2}[g(S),S]_t, \tag{2.49}$$

which yields

$$\begin{aligned}\Pi_t =& \alpha \int_0^t \varphi\, dS - \alpha \int_0^{\varphi_t} F(t,x)\, dx \\ =& \alpha \left(\underbrace{G\left(\inf_{0\le u\le t} S_u\right)}_{\text{nonincreasing in } t} - G(S_0) - \frac{1}{2} \underbrace{[\varphi, S]_t}_{\text{nondecreasing in } t} \right).\end{aligned}$$

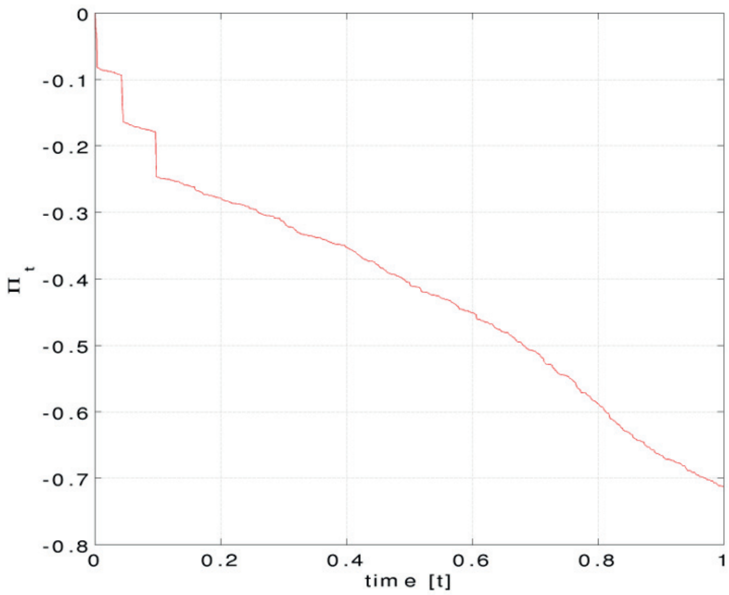
Remark 2.8.1. *First note that all tax payments are nonpositive (of course, only up to the liquidation of the portfolio). This is because trading gains are never realized if* $g' \geq 0$. *There are two components: payments triggered by wash sales when the stock price reaches its running infimum* $\inf_{0\le u\le t} S_t$*, and all the time, the taxes* $-0.5\alpha[\varphi, S] = -0.5\alpha g'(S) \bullet [S,S]$ *triggered by loss realizations from "recently" purchased stocks.*

To explain this phenomenon, consider an approximating sequence of Cox-Ross-Rubinstein type models with finite price grids $\{0, \sigma/\sqrt{n}, 2\sigma/\sqrt{n}, \ldots\}$*,* $n \in \mathbb{N}$*, and* $S^n_{(k+1)/n} - S^n_{k/n} = \pm\sigma/\sqrt{n}$ *each with probability* $1/2$*. First, we look at the case that at time* k/n *the stock price lies strictly above its minimum up to this time. Then, the investor holds exactly* $g(S^n_{k/n}) - g(S^n_{k/n} - \sigma/\sqrt{n})$ *shares with book profit zero. Namely, these shares were purchased after the last time* $\leq k/n$ *at which* S^n *jumps from* $S^n_{k/n} - \sigma/\sqrt{n}$ *to* $S^n_{k/n}$*. All other shares which are in the portfolio at time* k/n *were purchased earlier and have a higher book profit that cannot fall strictly below zero in the next period. Therefore, the tax payment at time* $(k+1)/n$ *is given by*

$$\begin{aligned}&-\alpha \left(g(S^n_{k/n}) - g\left(S^n_{k/n} - \frac{\sigma}{\sqrt{n}}\right)\right) (S_{(k+1)/n} - S_{k/n})^- \\ \approx& -\alpha \frac{g'(S^n_{k/n})\sigma^2}{n} 1_{\{S_{(k+1)/n} - S_{k/n} < 0\}},\end{aligned}$$

i.e., if the price goes up, there are no tax payments, and if it goes down the shares that have zero book profit before are sold. For $n \to \infty$*, by the law of large numbers, half of the price movements go down, and one arrives at the accumulated tax payments* $-0.5\alpha \int g'(S_t)\sigma^2\, dt$ *(note that in the limit the fraction of periods at which the stock price attains its running minimum vanishes). Then, the general case with nonconstant* $d[S,S]_t/dt$ *follows by stochastic time changes applied to the approximating price processes. If* $S^n_{k/n} = \min_{l\le k} S^n_{l/n}$*, all shares have book profit zero and after a further decrease they are wash-sold,* *which leads to the tax payment* $\alpha g(\min_{l\leq k} S^n_{l/n})(\min_{l\leq k+1} S^n_{l/n} - \min_{l\leq k} S^n_{l/n})$. *In the limit, the accumulated tax payments when the stock price coincides with its running minimum become* $\alpha\left(G(\inf_{0\leq u\leq \cdot} S_u) - G(S_0)\right)$, *where* $G' = g$.

In general, when building up a portfolio, an investor can generate negative tax payments, or at least off-set positive tax payments on dividends, by purchasing many new stocks and sell those stocks which go down. This is accompanied with higher book profits of the shares that go up. Thus, as time goes by, it gets increasingly more difficult to avoid tax payments.



An exemplary development of the accumulated tax-payments for the tax-efficient strategy described in (2.46) for a path of a stock $(S_t)_{t\in[0,1]}$ generated by a Geometric Brownian motion and the strategy given by $\varphi_t = S_t$.

Figure 2.6: Accumulated tax-payments for tax-efficient strategy

2.9 Conclusion

The first purpose of this chapter is to find a suitable set of continuous time trading strategies (specifying the number of identical shares that an investor holds in her portfolio) for which the payment flow of a linear tax on realized trading gains can be constructed. It turns out that this is the set of all adapted processes with left-continuous paths possessing finite right limits, i.e., the closure of elementary predictable processes w.r.t. the convergence "uniformly in probability". Then, the extension to trading strategies in different stocks is straightforward. From a theoretical point of view, it is appealing that tax payments can also be defined for strategies of infinite variation. This is not obvious at all because a reduction of the stock position leads to tax payments whereas an increase has no immediate effect. This property may suggest that a construction of the tax payment flow must be based on a decomposition into an increasing and a decreasing part of the investment strategy.

In the discrete time model of Dybvig and Koo [DK96], we prove that it is optimal to realize trading losses immediately and, when the total number of stocks has to be reduced, to sell shares with lower book profits / later purchasing times first. Based on this result, for elementary strategies in a continuous time model, we introduce an automatic loss realization when shares fall below their (individual) purchasing prices as well as a rule that dictates to sell shares with later purchasing time first when the stock position has to be reduced. Following this procedure, the tax payment flow is already determined by the stochastic process modeling the total number of shares in the portfolio. For the extension to nonelementary strategies, the representation of the book profits of the shares in the portfolio plays a key role (although all shares have the same price, their book profits differ because of different purchasing times).

Secondly, we prove that under the condition that the dividend policy has a neutral effect on the stochastic return process, for every investment strategy in a firm with dividends, there exists a strategy investing in an "identical" firm without dividends that leads to an almost surely higher or equal after-tax wealth.

Finally, we find out tax-efficient dynamic strategies. These try to defer tax payments as long as possible. Because profit-taking leads to early tax payments, a tax-efficient strategy reduces the position only after losses, i.e., there should be a positive dependence between the number of stocks in the portfolio and the stock price. If the position is a direct *nondecreasing* function of the stock price, the tax payment flow can be determined explicitly and is given, besides a local time component, by the tax rate times half the quadratic covariation of the strategy and the price process.

In the chapter, we consider the so-called *exact tax basis* which is economically the most reasonable one. For other tax bases, as the FIFO ("first-in-first-out") or the average of the purchasing prices, the main phenomena are similar, as, e.g., the suboptimality of dividends. But, the modeling is quite different. Especially, it is an open problem how to construct tax payment flows beyond strategies of finite variation.

2.10 A Motivation: The Discrete Time Model of Dybvig/Koo

In this section, we motivate the automatic loss realization as well as the rule to sell shares with lower book profits / shorter residence times first (based on this procedure, the tax payment flow was introduced for continuous time portfolio rebalancings in Section 2.2). For this, we prove that in the discrete time model of Dybvig and Koo [DK96], for all paths, this procedure leads to a higher or equal after-tax wealth than any other strategy (with the same total number of shares in the portfolio) if the riskless interest rate is nonnegative. Namely, the procedure minimizes the accumulated tax payments up to any time t (see Theorem 2.10.1). A similar assertion is already stated in [DK96] (see Properties 1 and 2 on page 6), but in less formal terms and, so far, a proof is only available for Property 1 in special cases (see Subsection 3.1 of Constantinides [Con83]). The idea is that investors always prefer tax payment obligations in the future to tax payments today.

Following the notation in [DK96], $N_{s,t}$ denotes the number of stocks that are bought at time $s \in \{0, \dots, T\}$, $T \in \mathbb{N}$, and kept in the portfolio at least after trading at time $t \in \{s, \dots, T\}$. Especially, $N_{t,t}$ is the number of shares purchased at time t, i.e., a position cannot be purchased and resold at the same time (on the other hand, a position can be sold and repurchased at the same time). One has the constraint

$$N_{t,t} \geq N_{t,t+1} \geq \dots \geq N_{t,T} \geq 0, \text{ for all } t \in \{0, \dots, T\}, \tag{2.50}$$

which contains a short-selling restriction. Following the standard notation in discrete time, we denote by

$$\varphi_{t+1} = \sum_{s=0}^{t} N_{s,t}, \quad t = 0, \dots, T \tag{2.51}$$

the number of stocks in the portfolio *after* trading at time t. Accumulated tax payments up to time u are given by

$$\Pi_u := \alpha \sum_{t=1}^{u} \sum_{s=0}^{t-1} (N_{s,t-1} - N_{s,t}) (S_t - S_s), \tag{2.52}$$

where $\sum_{t=u+1}^{u} \dots = 0$ throughout the section. With Π from (2.52), the self-financing condition is defined as in (2.28).

Of course, there are different strategies $N = (N_{s,t})_{s=0,1,\dots,T,\ t=s,s+1,\dots,T}$ that lead to the same number φ of risky assets. Given some nonnegative process φ, the rule of selling shares on which our model in Section 2.2 is based corresponds to the following strategy $\widetilde{N}$, constructed by (forward) induction in t: $\widetilde{N}_{0,0} = \varphi_1$

and, given $\widetilde{N}_{s,t-1}$, $s = 0, 1, \ldots, t-1$, $\widetilde{N}_{s,t}$ is defined as

$$\widetilde{N}_{s,t} = 1_{\{S_t \geq S_s\}} \left(\widetilde{N}_{s,t-1} - \left((\Delta\varphi_{t+1})^- - \sum_{j=s+1}^{t-1} \widetilde{N}_{j,t-1} \right)^+ \right)^+, \quad s \in \{0, \ldots, t-1\}, \tag{2.53}$$

$$\widetilde{N}_{t,t} = \Delta\varphi_{t+1} + \sum_{s=0}^{t-1} (\widetilde{N}_{s,t-1} - \widetilde{N}_{s,t}), \tag{2.54}$$

where $\varphi_{t+1} = \Delta\varphi_{t+1} - \varphi_t$. Following (2.53), the investor first reduces her total position by $(\Delta\varphi_{t+1})^-$, thereby selling the shares with the smallest residence time $t - s$. Then, remaining shares with negative book profits are sold. By (2.54), condition (2.51) is satisfied, and, by omitting the indicator functions in (2.53), one sees that $\widetilde{N}_{t,t} \geq 0$. Now, we can already formulate the main assertion of this section. In Subsection 2.10.1, the precise relation to the model introduced in Section 2.2 is established.

Theorem 2.10.1. *Let $(\varphi_t)_{t \in \{1,\ldots,T+1\}} \geq 0$ be a given position in the risky asset. Let $\widetilde{N}$ be the strategy defined in (2.53)/(2.54) and N be an arbitrary strategy satisfying (2.50)/(2.51). Then, for the corresponding accumulated tax payments, one has*

$$\widetilde{\Pi}_t \leq \Pi_t \quad \textit{for all } t \in \{0, \ldots, T\}. \tag{2.55}$$

From Theorem 2.10.1, it follows, as in Section 2.6, that the wealth process of $\widetilde{N}$ dominates the wealth process of N if the riskless interest rate is nonnegative. Namely, for both strategies, trading gains before taxes are given by $\varphi \bullet S_T := \sum_{u=1}^{T} \varphi_u (S_u - S_{u-1})$, but $\widetilde{N}$ defers tax payments to a larger extent.

Throughout the section, for t and ω fixed, $(k_0, k_1, \ldots, k_t)$ is a permutation of $(0, 1, \ldots, t)$ s.t.

$$S_{k_0} \geq S_{k_1} \geq \ldots \geq S_{k_t} \quad \text{and} \quad S_{k_i} > S_t \quad \forall i < j, \quad \text{where } k_j = t. \tag{2.56}$$

Then, for an arbitrary strategy N, the book profit function is defined as

$$F(t,x) := \sum_{i=0}^{t} (S_t - S_{k_i}) 1_{\left(\sum_{l=0}^{i-1} N_{k_l,t}, \sum_{l=0}^{i} N_{k_l,t}\right]}(x). \tag{2.57}$$

On $(0, \varphi_{t+1}]$, $F(t, \cdot)$ is obviously nondecreasing. Note that $F(t, \cdot)$ from (2.57) already contains the portfolio regroupings that take place at price S_t, i.e., it consists of φ_{t+1} shares (see Subsection 2.10.1 for the relation to the book profit function from Section 2.2). To prove Theorem 2.10.1, we need the following lemmata.

Lemma 2.10.2. *For every strategy N with corresponding number of stocks φ and book profit function F, one has*

$$\Pi_t = \alpha\varphi \bullet S_t - \alpha \int_0^{\varphi_{t+1}} F(t,x)dx, \quad t = 0, \ldots, T$$

(cf. Proposition 2.4.1).

Proof. We have

$$\begin{aligned}
&\alpha\varphi_t(S_t - S_{t-1}) - \alpha\left(\int_0^{\varphi_{t+1}} F(t,x)dx - \int_0^{\varphi_t} F(t-1,x)dx\right)\\
=&\alpha\left(\sum_{i=0}^{t-1} N_{i,t-1}(S_t - S_{t-1}) - \sum_{i=0}^{t-1}(N_{i,t}(S_t - S_i) - N_{i,t-1}(S_{t-1} - S_i))\right)\\
=&\alpha\sum_{i=0}^{t-1}(N_{i,t-1} - N_{i,t})(S_t - S_i)\\
=&\Pi_t - \Pi_{t-1}.
\end{aligned}$$

■

Lemma 2.10.3. *Let $\widetilde{F}$ be the book profit function of the strategy $\widetilde{N}$ from (2.53). Then, one has*

$$\widetilde{F}(t,x) = 1_{((\Delta\varphi_{t+1})^+,\varphi_{t+1}]}(x)\left(\widetilde{F}(t-1, x - \Delta\varphi_{t+1}) + S_t - S_{t-1}\right) \vee 0. \quad (2.58)$$

Proof. If the stock price falls below the purchasing price of a particular share, then, following (2.53), this share is definitely sold. Consequently, one has

$$S_{s_1} \le S_{s_2} \quad \text{for all } s_1 < s_2 \le t-1 \text{ with } \widetilde{N}_{s_1,t-1} > 0, \quad (2.59)$$

i.e., book profits are nondecreasing in the residence time $t - s$. Put differently, if one only considers points $s \in \{0, \ldots, t-1\}$ with $\widetilde{N}_{s,t-1} > 0$, the stock price is nondecreasing. Thus, in (2.53), the investor sells the shares with the lowest book profits (because they have the shortest residence times), and the strategy $\widetilde{N}$ given by (2.53)/(2.54) reads: $\widetilde{N}_{0,0} = \varphi_1$ and

$$\begin{aligned}
\widetilde{N}_{k_i,t} =&0, \quad \text{for } i \in \{0, \ldots, j-1\},\\
\widetilde{N}_{k_i,t} =&\left(\widetilde{N}_{k_i,t-1} - \left((\Delta\varphi_{t+1})^- - \sum_{\substack{l=0\\ l\neq j}}^{i-1}\widetilde{N}_{k_l,t-1}\right)^+\right)^+, \quad i \in \{j+1, \ldots, t\},\\
\widetilde{N}_{k_j,t} =&\widetilde{N}_{t,t} = \Delta\varphi_{t+1} + \sum_{\substack{l=0\\ l\neq j}}^{t-1}(\widetilde{N}_{k_l,t-1} - \widetilde{N}_{k_l,t}),
\end{aligned}$$

for $t \in \{1, \dots, T\}$, where j and the permutation $(k_0, \dots, k_t)$ are given by (2.56).

Case 1: $(\Delta\varphi_{t+1})^- \le \sum_{l=0}^{j-1} \widetilde{N}_{k_l,t-1}$
(note that this includes the case $\Delta\varphi_{t+1} > 0$).

One has $\widetilde{N}_{k_l,t} = \widetilde{N}_{k_l,t-1}$ for $l \ge j+1$ and arrives at

$$\sum_{l=0}^{i} \widetilde{N}_{k_l,t} = \begin{cases} 0, & i \in \{-1, 0, \dots, j-1\} \\ \sum_{\substack{l=0 \\ l \ne j}}^{i} \widetilde{N}_{k_l,t-1} + \Delta\varphi_{t+1}, & i \in \{j, \dots, t\} \end{cases} . \tag{2.60}$$

For $x \in (0, \varphi_{t+1}]$, one obtains

$$\begin{aligned} & \widetilde{F}(t,x) \\ = \ & \sum_{i=0}^{t} (S_t - S_{k_i}) 1_{\left(\sum_{l=0}^{i-1} \widetilde{N}_{k_l,t}, \sum_{l=0}^{i} \widetilde{N}_{k_l,t}\right]}(x) \\ \stackrel{(2.60)}{=} & \sum_{i=j+1}^{t} (S_{t-1} - S_{k_i} + S_t - S_{t-1}) 1_{\left(\sum_{\substack{l=0 \\ l \ne j}}^{i-1} \widetilde{N}_{k_l,t-1} + \Delta\varphi_{t+1}, \sum_{\substack{l=0 \\ l \ne j}}^{i-1} \widetilde{N}_{k_l,t-1} + \Delta\varphi_{t+1}\right]}(x) \\ = \ & 1_{((\Delta\varphi_{t+1})^+, \varphi_{t+1}]}(x) \left(\widetilde{F}(t-1, x - \Delta\varphi_{t+1}) + (S_t - S_{t-1}) \right) \vee 0, \end{aligned}$$

where the last equality holds by $(\Delta\varphi_{t+1})^+ \le \sum_{l=0}^{j-1} \widetilde{N}_{k_l,t-1} + \Delta\varphi_{t+1}$, $S_t - S_{k_i} \le 0$ for $i \le j$, and $S_t - S_{k_i} \ge 0$ for $i \ge j+1$.

Case 2: $(\Delta\varphi_{t+1})^- > \sum_{l=0}^{j-1} \widetilde{N}_{k_l,t-1}$
(i.e., after the reduction of the position by $(\Delta\varphi_{t+1})^-$, there are no shares with negative book profits).

Define

$$\widehat{m} := \min \left\{ i \mid (\Delta\varphi_{t+1})^- \le \sum_{\substack{l=0 \\ l \ne j}}^{i} \widetilde{N}_{k_l,t-1} \right\}.$$

We have $\widehat{m} \ge j+1$ and arrive at

$$\sum_{l=0}^{i} \widetilde{N}_{k_l,t} = \begin{cases} 0, & i \in \{-1, 0, \dots, \widehat{m}-1\} \\ \sum_{\substack{l=0 \\ l \ne j}}^{i} \widetilde{N}_{k_l,t-1} + \Delta\varphi_{t+1}, & i \in \{\widehat{m}, \dots, t\} \end{cases} . \tag{2.61}$$

For $x \in (0, \varphi_{t+1}]$, one obtains

$$\begin{aligned} & \widetilde{F}(t,x) \\ = \ & \sum_{i=0}^{t} (S_t - S_{k_i}) 1_{\left(\sum_{l=0}^{i-1} \widetilde{N}_{k_l,t}, \sum_{l=0}^{i} \widetilde{N}_{k_l,t}\right]}(x) \\ \stackrel{(2.61)}{=} & \sum_{i=\widehat{m}}^{t} (S_{t-1} - S_{k_i} + S_t - S_{t-1}) 1_{\left(\sum_{\substack{l=0 \\ l \ne j}}^{i-1} \widetilde{N}_{k_l,t-1} + \Delta\varphi_{t+1}, \sum_{\substack{l=0 \\ l \ne j}}^{i} \widetilde{N}_{k_l,t-1} + \Delta\varphi_{t+1}\right]}(x) \end{aligned}$$

$$= \widetilde{F}(t-1, x-\Delta\varphi_{t+1}) - (S_t - S_{t-1}),$$

where the last equality holds by $x - \Delta\varphi_{t+1} = x + (\Delta\varphi_{t+1})^- > \sum_{\substack{l=0 \\ l\neq j}}^{\widehat{m}-1} \widetilde{N}_{k_l,t-1}$ for $x > 0$. ■

Proof of Theorem 2.10.1. Let N and $\widetilde{N}$ be as in the theorem with corresponding book profit functions F and $\widetilde{F}$, respectively, as defined in (2.57). Let us first show that for $x \in (0, \varphi_{t+1}]$,

$$F(t,x) \leq 1_{((\Delta\varphi_{t+1})^+, \varphi_{t+1}]}(x) \left(F(t-1, x-\Delta\varphi_{t+1}) + S_t - S_{t-1}\right) \vee 0. \quad (2.62)$$

For $x \in (0, \varphi_{t+1}]$, let $i \in \{0, \ldots, t\}$ s.t. $x \in \left(\sum_{l=0}^{i-1} N_{k_l,t}, \sum_{l=0}^{i} N_{k_l,t}\right]$. If $i \leq j$ (cf. (2.56)), one has $F(t,x) = S_t - S_{k_i} \leq 0$, and (2.62) holds. Thus, it remains to consider the case $i \geq j+1$. For this, we have

$$\begin{aligned} x > \sum_{l=0}^{i-1} N_{k_l,t} = N_{t,t} + \sum_{\substack{l=0 \\ l\neq j}}^{i-1} N_{k_l,t} &= \sum_{\substack{l=0 \\ l\neq j}}^{i-1} N_{k_l,t-1} + N_{t,t} + \sum_{\substack{l=0 \\ l\neq j}}^{i-1} (N_{k_l,t} - N_{k_l,t-1}) \\ &\overset{(2.50)}{\geq} \sum_{\substack{l=0 \\ l\neq j}}^{i-1} N_{k_l,t-1} + N_{t,t} + \sum_{\substack{l=0 \\ l\neq j}}^{t} (N_{k_l,t} - N_{k_l,t-1}) \\ &\overset{(2.51)}{=} \sum_{\substack{l=0 \\ l\neq j}}^{i-1} N_{k_l,t-1} + \Delta\varphi_{t+1}. \end{aligned}$$

By monotonicity of F, this implies $F(t-1, x-\Delta\varphi_{t+1}) \geq S_{t-1} - S_{k_i} = F(t,x) - (S_t - S_{t-1})$ and together with $x > N_{t,t} \geq (\Delta\varphi_{t+1})^+$, we arrive at (2.62).

With Lemma 2.10.3 and inequality (2.62), it follows by induction w.r.t. t that $F(t,x) \leq \widetilde{F}(t,x)$ for all $t = 1, \ldots, T$ and $x \in (0, \varphi_{t+1}]$ (note that $F(0,x) = \widetilde{F}(0,x) = 0$). By Lemma 2.10.2, the assertion follows. ■

2.10.1 Relation to the Model from Section 2.2

It remains to prove that the discrete time version of our model introduced in Section 2.2 does indeed coincide with the model of Dybvig/Koo with $N = \widetilde{N}$. Let $(\varphi_t)_{t=1,\ldots,T+1}$ be a discrete time predictable process, i.e., φ_t is $\mathcal{F}_{t-1}-$ measurable. By $\widetilde{F}$, we denote the corresponding discrete time book profit function in the sense of (2.57) for $N = \widetilde{N}$. By F, we denote the continuous time book profit function in the sense of (2.7) for the piecewise constant strategy $\sum_{n=1}^{T} \varphi_n 1_{(n-1,n]} \in \mathbb{L}$ and the stock price process $S = \sum_{n=0}^{T} S_n 1_{[n,n+1)}$. This is the standard embedding of a discrete time market model into a continuous time framework. Let us show that

$$\widetilde{F}(t,x) = F(t+, x), \quad t = 0, 1, \ldots, T-1.$$

This means that $\widetilde{F}(t,\cdot)$ already contains the portfolio regroupings that take place at price S_t (note that in a discrete time model, there can only be *one* change at time t, whereas in continuous time, there can be a change between $t-$ and t, and between t and $t+$).

For the piecewise constant process $\sum_{n=1}^{T} \varphi_n 1_{(n-1,n]}$, the right limit of the purchasing time (2.3) reads

$$\tau_{t+,x} = \lim_{s>t,\ s\to t} \tau_{s,x} = \max\{u \in \{0,1,\dots,t\} \mid \varphi_u \le \varphi_{t+1} - x\}, \quad x \in [0, \varphi_{t+1}],$$

with the convention from Section 2.2 that $\varphi_0 = 0$ (note that the increment $\varphi_{u+1} - \varphi_u$ is purchased at price S_u). One has the implications $\tau_{t+,x} < t \quad \Rightarrow \tau_{(t-1)+,x-\Delta\varphi_{t+1}} = \tau_{t+,x}$ and $\tau_{t+,x} = t \quad \Leftrightarrow \quad x \le (\Delta\varphi_{t+1})^+$. This implies

$$\begin{aligned}
&S_t - \min_{\tau_{t+,x} \le u \le t} S_u \\
=&\left(S_t - S_{t-1} + S_{t-1} - \min_{\tau_{t+,x} \le u \le t-1} S_u\right) \vee 0 \\
=&1_{((\Delta\varphi_{t+1})^+, \varphi_{t+1}]}(x) \left(S_t - S_{t-1} + S_{t-1} - \min_{\tau_{(t-1)+,x-\Delta\varphi_{t+1}} \le u \le t-1} S_u\right) \vee 0
\end{aligned}$$

(with $\min \emptyset := \infty$), i.e., $F(t+,x) = S_t - \min_{\tau_{t+,x} \le u \le t} S_u$ satisfies the recursion (2.58), and thus, it coincides with $\widetilde{F}(t,x)$. By Lemma 2.10.2 and Proposition 2.4.1, this implies that the tax payment process defined in (2.52), with $N = \widetilde{N}$, coincides with the right limit of the tax payment process from Definition 2.2.8. ■

2.11 Modeling of Tax Bases in Mathematical Literature

In this section, we want to compare the modeling of different tax bases in mathematical literature and lighten the differences between our model from Section 2.2. Hereby, we focus on the first-in-first-out taxation priority rule, appearing in [JKT99; JKT00], and the average purchase price as tax basis, appearing in [BST07; BST10].

In each of the following sections, we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.11.1 “First-in-First-out“ Priority Rule

In the following paragraph, we follow Jouini et al. [JKT99; JKT00] who consider a first-in-first-out priority rule, according to which any stock sold at some time t should be the oldest one in the portfolio. We consider a market consisting of only one semimartingale price process $(S_t)_{t\in[0,T]}$ (in the following called stock).

Let $\Delta := \{(t,u) \in \mathbb{R} \times \mathbb{R} : 0 \le u \le t \le T\}$. Fix some $(t,u) \in \Delta$. For each monetary unit invested at time u and sold out at time t, denote by $\Phi(t,u)$ the after-tax amount received at time t, i.e.,

$$\Phi(t,u) = (1-\alpha)\frac{S_t}{S_u} + \alpha, \tag{2.63}$$

and the amount of tax paid by the investor is given by

$$\frac{S_t}{S_u} - \Phi(t,u).$$

By L^1_+, denote the set of all nonnegative Lebesgue-integrable functions in the interval $[0,T]$. Let (x,y) be a pair of L^1_+ functions. By x_t and y_t, we denote the investment rate and the disinvestment rate, respectively, in units of the stock at time t. Then, $\int_0^t x_s ds$ and $\int_0^t y_s ds$ is the cumulated number of assets purchased and sold out, respectively, up to time t. The pair (x,y) is called trading strategy if

$$\int_0^t y_s ds \le \int_0^t x_s ds, \quad 0 \le t \le T$$

holds (no short selling constraint).

Remark 2.11.1. *Alternatively, for defining this trading strategy, consider a continuous process φ of finite variation such that $\varphi = \varphi^+ - \varphi^-$, where $\varphi^+ \left(= \int_0^\cdot x_s ds\right)$ is the increasing and $\varphi^- \left(= \int_0^\cdot y_s ds\right)$ is the decreasing part of φ.*

Define

$$\theta_t := \sup\left\{ s \in [0,t] : \int_0^s x_u du \le \int_0^t y_u du \right\} \tag{2.64}$$

which is the purchasing date of the last asset sold out from the portfolio if $\int_0^t y_u du > 0$.

Set $\varphi_t := \int_0^t x_s ds - \int_0^t y_s ds$, which is the total number of stocks in the portfolio. In the same way as in Section 2.2, we want to sort φ_t stocks by the time spending in the portfolio and label them by $\widetilde{x}$: the larger $\widetilde{x}$ the longer the residence time in the portfolio.

Define

$$\widetilde{\theta}_{t,\widetilde{x}} := \begin{cases} \sup \widetilde{M}_{t,\widetilde{x}}, & \text{if } \widetilde{M}_{t,\widetilde{x}} \ne \emptyset \\ t, & \text{otherwise} \end{cases}, \tag{2.65}$$

where

$$\widetilde{M}_{t,\widetilde{x}} := \left\{ s \le t : \int_s^t x_u du = \widetilde{x} \right\}. \tag{2.66}$$

$\widetilde{\theta}_{t,\widetilde{x}}$ is the purchase time of the stock in the portfolio which has label $\widetilde{x}$ at time t. One can check that

$$\widetilde{\theta}_{t,\widetilde{x}} = \theta_t \quad \text{for} \quad \widetilde{x} = \varphi_t = \int_0^t x_u du - \int_0^t y_u du. \tag{2.67}$$

With the definition of $\widetilde{\theta}_{t,\widetilde{x}}$, we are now able (analogously to our model in Section 2.2) to define

$$F(t, \widetilde{x}) := (S_t - S_{\widetilde{\theta}_{t,\widetilde{x}}})1_{[0,\varphi_t]}(\widetilde{x})$$

as the *book profit function.*

Similar to our model, we can define an accumulated tax-payment process Π:

$$\Pi_t := \int_0^t \alpha \left(S_u - S_{\theta_u}\right) y_u du \overset{(2.63)}{=} \int_0^t (S_u - \Phi(u, \theta_u) S_{\theta_u}) y_u du,$$

where by $\alpha \in (0, 1)$, we denote the tax-rate. The accumulated tax-payment process can alternatively be expressed using the book profit function F:

$$\Pi_t = \int_0^t \alpha \left(S_u - S_{\theta_u}\right) y_u du \overset{(2.67)}{=} \int_0^t \alpha \left(S_u - S_{\widetilde{\theta}_{u,\varphi_u}}\right) y_u du = \alpha \int_0^t F(u, \varphi_u) y_u du.$$

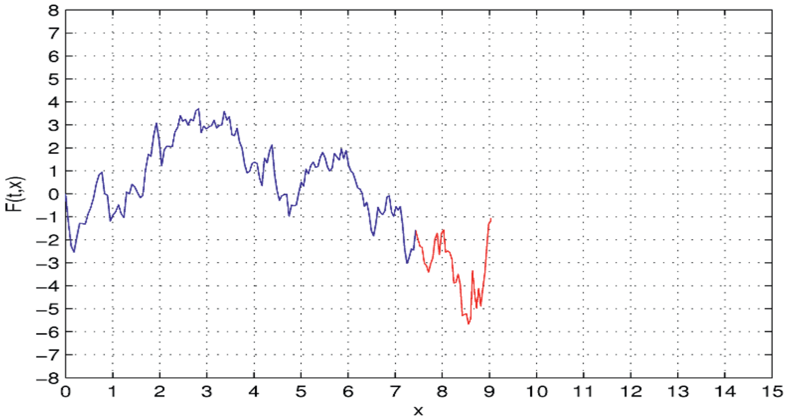
Remark 2.11.2. *The book profit function F is defined in such a way that newly purchased stocks are added up from the left at $x = 0$. The purchasing of new stocks shifts the stocks in the portfolio to the right. Sold stocks are removed from the right boundary of F, which are the oldest stocks in the portfolio, cf. Figure 2.7.*

2.11.2 Average Purchase Price

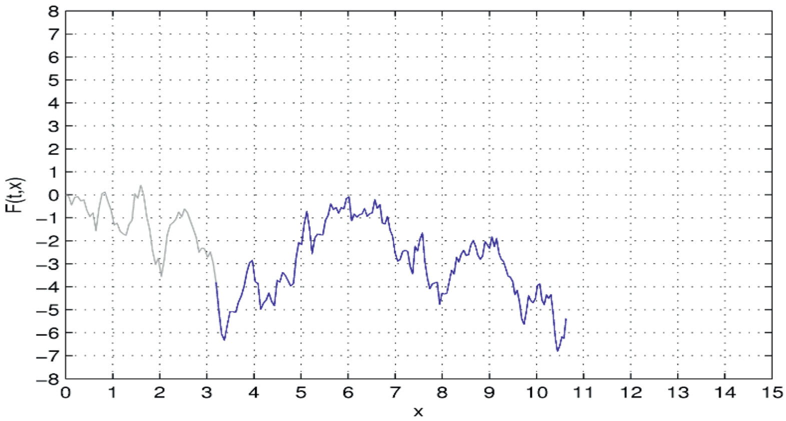
In the following paragraph, we follow Ben Tahar et al. [BST07; BST10] who consider the average purchase price as tax basis. We consider a market consisting of a continuous price process $(S_t^1)_{t\in[0,T]}$ with $\inf_{u\in[0,T]} S_u^1 > 0$ and a riskless asset $(S_t^0)_{t\in[0,T]}$ with $S_t^0 = \exp(rt)$.

With this kind of taxation, the purchase price of single stocks in the portfolio is not important to calculate the tax payments. The tax basis is the weighted average price of the shares purchased by the investor. Selling stocks does not change the tax basis, but sales are relevant from the date of the next purchase, i.e., the tax basis is only affected by the number of sold shares but not by the sale price. The tax payments are then calculated by comparison of the current stock price to the tax-basis. We will show off in a short example how the tax-basis average purchase price is calculated.

Example 2.11.3. *At t_1, an investor buys* 100 *stocks at a price of* \$10. *The tax basis is* \$10.



Book profit function $x \mapsto F(t_1, x)$ at time t_1. The red colored pieces are sold up to time t_2 and the blue colored pieces are still in the portfolio up to time t_2.



Book profit function $x \mapsto F(t_2, x)$ at time t_2. The red colored pieces from the first picture have been sold. The grey colored pieces have been bought between t_1 and t_2.

Figure 2.7: Exemplary Development of $x \mapsto F(t, x)$ between times t_1 and t_2 for FIFO-priority rule

At t_2, additional 50 stocks are bought at a price of \$13. The tax basis is $(\$10 \cdot 100 + \$13 \cdot 50)/150 = \$11$
At t_3, 100 stocks are sold. The tax basis does not change.
At t_4, 50 stocks are bought at a price of \$15. The tax basis is $(\$11 \cdot 50 + \$15 \cdot 50)/100 = \$13$.

The example illustrates that the tax basis cannot be modeled without memorizing the number of stocks in the portfolio.

We denote by $(X_t)_{t\in[0,T]}$ the money invested in the riskless asset (bank account position) and $(Y_t)_{t\in[0,T]}$ the money invested in the stock (stock account position). Trading strategies are described by money transfers between the two investment opportunities. We define a trading strategy as a pair (L, M), where

$(L_t)_{t\in[0,T]}$, $(M_t)_{t\in[0,T]}$ are adapted, right-continuous, increasing processes with $L_{0-} = M_{0-} = 0$.
The amount of money transferred from the bond account to the stock account at time t is given by dL_t. The amount of money transferred from the stock account to the bank account at time t is given by $Y_{t-}dM_t$, i.e., dM_t is the relative reduction of the stock position at time t. To define the tax basis, [BST07; BST10] express the reduction of the stock position by its relative reduction. This guarantees the existence of SDE (2.68) for every pair (L, M) of nondecreasing processes.

By $(K_t)_{t\in[0,T]}$, with the dynamics given by

$$dK_t = dL_t - K_{t-}dM_t, \tag{2.68}$$

we denote the money invested in the stock evaluated at its tax basis price.
The tax-basis is then defined by

$$B_t := K_t \frac{S_t^1}{Y_t} 1_{\{Y_t>0\}}, \quad t \geq 0,$$

where the dynamics of the stock position $(Y_t)_{t\in[0,T]}$ is given by

$$dY_t = \frac{Y_t}{S_t^1} dS_t^1 - Y_{t-}dM_t + dL_t. \tag{2.69}$$

Note that there exists a solution to SDE (2.69) because S^1 is continuous and $\inf_{u\in[0,T]} S_u^1 > 0$.

Remark 2.11.4. *Instead of using a strategy pair (L, M), one could start with a right-continuous, finite variation process $(\varphi_t)_{t\in[0,T]}$ such that $\varphi_t = \varphi_t^+ - \varphi_t^-$ and define K as solution of the SDE*

$$dK_t = S_t^1 d\varphi_t^+ - \frac{K_{t-}}{\varphi_{t-}} 1_{\{\varphi_{t-}>0\}} d\varphi_t^-, \tag{2.70}$$

of course, provided that the term $\frac{K_{t-}}{\varphi_{t-}} 1_{\{\varphi_{t-}>0\}} d\varphi_t^-$ exists.

Intuitively, this should be the right dynamics for defining the average purchase price as tax basis. The increment ΔK is determined by the purchases of $\Delta\varphi_t^+$ stocks at the price S_t^1, reduced by the sale of $\Delta\varphi_t^-$ stocks calculated at the average purchase price B_{t-} with $B_t := \frac{K_t}{\varphi_t} 1_{\{\varphi_t>0\}}$.

Defining the tax basis this way yields the problem that it is not possible to say something about the existence of a solution to (2.70). Introducing wash sales in this model, i.e., selling all stocks if $S_t^1 < B_t$ and repurchasing them at price S_t^1, guarantees the existence of a solution to (2.70). To show this, one needs to go into the theory of stochastic differential equations with reflection at the boundary. Since a more detailed view would break the mold, we will not carry *this out here any further.*

On the other hand, if a solution to (2.70) is guaranteed, one can start with φ *and derive the dynamics of* L, M *by setting*

$$dM_t := \frac{1}{\varphi_{t-}} 1_{\{\varphi_{t-}>0\}} d\varphi_t^-, \quad dL_t := S_t^1 d\varphi_t^+.$$

This, again, leads to (2.68). But, a reverse procedure, i.e., deriving φ *from a given strategy pair* (L, M)*, seems not to be possible.*

The accumulated tax-payment process for stock sales is given by

$$\Pi_t = \int_0^t \alpha(S_s^1 - B_{s-}) \frac{Y_{s-}}{S_s^1} dM_s.$$

The bank account is given by

$$\begin{aligned} dX_t =& r(1-\alpha)X_t dt - dL_t + Y_{t-}dM_t - \alpha(S_t^1 - B_{t-})\frac{Y_{t-}dM_t}{S_t^1} \\ =& r(1-\alpha)X_t dt - dL_t + Y_{t-}dM_t - d\Pi_t, \end{aligned}$$

which includes the convention that gains in the bank account are continuously taxed.

Chapter 3

Optimal Selling Time of a Stock under Capital Gains Taxes

In this chapter, we analyze a typical and analytically quite tractable investment decision problem under capital gains taxes. In our model, an investor tries to maximize her trading profits within a finite period of time by exchanging one asset for another one only once. This yields an optimal stopping problem, where different tax bases coincide. We look at an investor owning a stock where the price is modeled as stochastic process in the Black-Scholes market with lower expected return than the other asset, given by a riskless bank account. Capital gains in the stock are only taxed when realized. In the case that bond earnings are continuously taxed, we show the existence of a lower, time-depending and continuous stopping boundary which is decreasing in stock's volatility. In the case of a deterministic stock price process, we derive an explicit formula of the stopping boundary. In this chapter, standard techniques from the theory of optimal stopping are used, especially an approach that turns the problem with a terminal payoff to one with a running payoff, see Peskir and Shiryaev [PS06].
Furthermore, we investigate the case, where bond earnings are taxed at the end of maturity. In this case, the stopping boundary needs not to be monotone anymore. So, we are not able to show the boundary's continuity and the smooth-fit condition (with the help of standard methods). Finally, we show that under some restrictions to the drift rate μ, the boundary in the case of a maturity taxed bond dominates the boundary in the case of a continuously taxed bond.

3.1 Formulation of the Stopping Problem and Main Results

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $T \in \mathbb{R}_+$, generated by a one-dimensional standard Brownian motion $(B_t)_{t\in[0,T]}$. The investment opportunities consist of a bank account with continuously compounded fixed interest rate $r > 0$ and a stock whose price process $(X_t)_{t\in[0,T]}$ solves the SDE

$$dX_t = \mu X_t dt + \sigma X_t dB_t, \quad t \geq 0, \tag{3.1}$$

with $X_0 = x > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. At time 0, the investor holds one risky stock. It was purchased sometime in the past at price $P_0 > 0$. This means that already at time 0, the stock possesses the book profit $X_0 - P_0$. The economically interesting case is $X_0 - P_0 > 0$, but this need not be assumed. The investor can sell the stock at any time up to the end of the time horizon T. At the selling time t, the investor has to pay the capital gains taxes $\alpha(X_t - P_0)$, where $\alpha \in [0, 1)$ is the given tax rate, i.e., if $X_t < P_0$, the investor gets a tax credit. Then, the remaining wealth $X_t - \alpha(X_t - P_0) = (1 - \alpha)X_t + \alpha P_0$ is invested in the riskless bank account. At maturity T, the portfolio is liquidated anyway. As the bank account pays a continuous compounded interest rate, we assume that taxes also charge the account continuously. This corresponds to the taxation of a continuous dividend flow and leads to the after-tax interest rate $(1 - \alpha)r$. Thus, the investor's wealth at maturity, when selling the stock at time $t \in [0, T]$ at price $\widetilde{x}$, is

$$G(t, \widetilde{x}) := \left[(1 - \alpha)\widetilde{x} + \alpha P_0\right] e^{r(1-\alpha)(T-t)}, \quad (t, \widetilde{x}) \in [0, T] \times \mathbb{R}_+. \tag{3.2}$$

Maximizing investor's expected wealth at maturity leads to the optimal stopping problem

$$V(x) := \sup_{\tau\in\mathcal{T}_{[0,T]}} \mathbb{E}\left[G(\tau, X_\tau)\right], \tag{3.3}$$

where $X_0 = x$, and by $\mathcal{T}_{[0,T]}$, we denote the class of $(\mathcal{F}_s)_{s\in[0,T]}$–stopping times taking values in $[0, T]$. The assumption that the second investment opportunity is a riskless bank account rather than another risky asset makes the payoff function a bit more tractable and is, given that the investor is risk neutral and cannot change her position again before T, not very restrictive.

Because of the strong Markov property of $(X_t)_{t\in[0,T]}$, we can define the value function associated with problem (3.3) by

$$V(t, x) := \sup_{\tau\in\mathcal{T}_{[0,T-t]}} \mathbb{E}\left[G\left(t + \tau, X^{t,x}_{t+\tau}\right)\right], \tag{3.4}$$

where $(X^{t,x}_s)_{s\in[t,T]}$ is the unique solution of (3.1) with initial condition $X^{t,x}_t = x$ and $\mathcal{T}_{[0,T-t]}$ denotes the set of $(\mathcal{F}_{t+s})_{s\in[0,T-t]}$–stopping times taking values in $[0, T - t]$. Note that $V(x) = V(0, x)$ as $X^{0,x} = X$ with $X_0 = x$. By setting $\tau = 0$ in (3.4), it is clear that

$$V(t, x) \geq G(t, x) \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}_+. \tag{3.5}$$

In addition, we have

$$V(T, x) = G(T, x) \quad \text{for } x \in \mathbb{R}_+. \tag{3.6}$$

The continuation region is defined by

$$\mathcal{C} := \{(t,x) \in [0,T) \times \mathbb{R}_+ : V(t,x) > G(t,x)\}$$

and the stopping region by

$$\mathcal{S} := \{(t,x) \in [0,T) \times \mathbb{R}_+ : V(t,x) = G(t,x)\}.$$

Proposition 3.1.1. *$(t,x) \mapsto V(t,x)$ is continuous on $[0,T]\times\mathbb{R}_+$. In addition, for any $(t,x) \in [0,T] \times \mathbb{R}_+$, the stopping time*

$$\tau_{t,x} := \inf\left\{s \in [0, T-t) : \left(t+s, X^{t,x}_{t+s}\right) \in \mathcal{S}\right\} \wedge (T-t) \tag{3.7}$$

maximizes (3.4).

Of course, the proof follows more or less directly from the standard theory. Thus, it is only briefly sketched in Section 3.3. The following theorem states the main results of this chapter and is also proven in Section 3.3.

Theorem 3.1.2. *Consider problem (3.4) with stopping region $\mathcal{S}$. Let $r \geq 0$ and $\alpha \in [0,1)$. Then,*

(a) there exists a continuous, increasing boundary $b : [0,T) \to \mathbb{R}_+$ such that the stopping region is given by

$$\mathcal{S} = \begin{cases} [0,T) \times \mathbb{R}_+ & \text{if } \mu \leq (1-\alpha)r \\ \{(t,x) \in [0,T) \times \mathbb{R}_+ : x \leq b(t)\} & \text{if } \mu > (1-\alpha)r, \end{cases} \tag{3.8}$$

where for all $t \in [0,T)$, the equivalence $\alpha > 0 \Leftrightarrow b(t) > 0$ holds.
The boundary satisfies the terminal condition

$$\lim_{t \uparrow T} b(t) = \frac{r\alpha P_0}{\mu - r(1-\alpha)} =: f.$$

(b) if $\alpha > 0$ and $\mu > (1-\alpha)r$, the value function satisfies the smooth fit condition at the boundary, i.e.,

$$\partial_x V(t,x) = \partial_x G(t,x) = (1-\alpha)e^{r(1-\alpha)(T-t)} \quad \text{at } x = b(t). \tag{3.9}$$

Remark 3.1.3. *We are primarily interested in the case $\mu < r$, i.e., an investor who is not subject to taxation would immediately sell the stock. Then, Theorem 3.1.2 tells us that the solution is nontrivial if $\mu \in ((1-\alpha)r, r)$. Choosing the midpoint of the interval for μ and reasonable values for σ, T, and α, numerical calculations show that the boundary is far above the purchasing price P_0, see Figure 3.1. This means that the investor always sells the stock with positive book profits. Then, it is evident that there is no incentive to buy the stock back at any later time and to repeat the game (namely, a renewed investment starts with zero book profit). This justifies the modeling of the decision* *problem as a simple optimal stopping problem.*

On the other hand, if the stock is sold with negative book profits, the modeling is justified when wash-sales are forbidden, as, e.g., in the U.S. This means that the investor may sell the stock to realize the trading loss, but then she is not allowed to buy back the stock and has to take the other investment opportunity. Under the ban on wash sales, the investor may switch to the bank account even in the case $\mu \geq r$, just in order to realize losses prematurely, which leads to a nontrivial solution of the stopping problem (see Theorem 3.1.2).

Proposition 3.1.4.

(i) The value function is increasing in the volatility of the stock, i.e., $V^{\sigma_1}(t,x) \leq V^{\sigma_2}(t,x)$ for all $0 \leq \sigma_1 \leq \sigma_2$, $t \in [0,T]$, and $x \in \mathbb{R}_+$. Consequently, $b^{\sigma_2}(t) \leq b^{\sigma_1}(t)$ for all $\mu > (1-\alpha)r$.

(ii) For $\sigma = 0$, the exercise boundary reads

$$b(t) = \frac{\alpha P_0 \left(e^{r(1-\alpha)(T-t)} - 1\right)}{(1-\alpha)\left(e^{\mu(T-t)} - e^{r(1-\alpha)(T-t)}\right)}. \tag{3.10}$$

and the optimal stopping time (3.7) is given by

$$\tau_{t,x} = \begin{cases} 0 & , x \leq b(t) \\ T-t & , x > b(t) \end{cases} .$$

Proposition 3.1.4 (for a proof, see Section 3.3) makes sense from an economic point of view. For a more volatile asset, the option to time the tax payments has a higher value for investors. *This means that capital gains taxes can even motivate investors to take more risk.* Of course, the extent of this effect depends on the riskless interest rate r and vanishes for $r = 0$.

Seifried [Sei10] solves the utility maximization problem for terminal wealth with $r = 0$ (i.e., it can be assumed that taxes are paid at maturity), but there are *no tax credits*. In the model of [Sei10], there can appear two opposite effects. Roughly speaking, if the drift of the stock is low, the tax may prevent the investor to buy the risky stock at all because, without negative taxes on losses, the expected after-tax gain becomes negative (literally, this only holds for buy-and-hold strategies in the stock). On the other hand, if the expected return is high enough, the investor may buy even more risky stocks than in the same situation without taxes. To make the latter plausible, consider a one-period binary model for the stock and a utility function that is linear around the initial wealth and satiable at some higher level of wealth. For the optimal stock position, the investor's terminal wealth coincides with the saturation point if the stock price goes up. This means that, in the case with taxes, the investor buys even more risky stocks to offset the part of the gains that she has to pay to the government. [Sei10] derives that for the Black-Scholes model with CRRA investors and realistic tax rates, the overall strategy effect of taxes is negligible (see Figure 8 therein).

Remark 3.1.5 (Value of the tax-timing option)**.** *Assume that in our model, the book profit $x - P_0$ is taxed at time 0, and later gains on the stock are taxed immediately when they occur. Thereby, we assume that tax payments are financed by reducing the stock position, and tax rebates are reinvested in stocks. Then, the wealth in stocks satisfies the SDE*

$$dX_t = (1-\alpha)X_t\,(\mu\,dt + \sigma\,dB_t)\,, \quad t \geq 0 \quad \text{with } X_0 = (1-\alpha)x + \alpha P_0$$

and thus $E[X_T] = [(1-\alpha)x + \alpha P_0]\,e^{\mu(1-\alpha)T}$. *The optimal stopping problem degenerates: for $\mu \leq r$, it is optimal to sell the stock at time 0, and for $\mu \geq r$, it is optimal to sell at time T.*

Thus, one may interpret

$$e^{-r(1-\alpha)T}\left(V(0,x) - [(1-\alpha)x + \alpha P_0]\,e^{(1-\alpha)\max(\mu,r)T}\right)$$

as the time−0 value of the timing option *of the stock holder, i.e., the value of the right of the investor to influence the timing of the tax payments. By Proposition 3.1.4, this value is increasing in the volatility of the stock. This is in line with the results in Constantinides [Con83], where in a complete market model, including a market for short sell contracts, the price of the timing option, of course differently defined (see Equation (21) therein), is also increasing in the volatility of the stock (see Table III).*

3.2 Method of Solution

To solve problem (3.4), we make use of a standard method in the optimal stopping theory, see, e.g., Peskir and Shiryaev [PS06], where the terminal payoff is turned into a running payoff. Namely, thanks to the smoothness property of the payoff function G, we can apply Itô's formula to obtain the following decomposition of the payoff process $(G(t+s, X^{t,x}_{t+s}))_{s\geq 0}$:

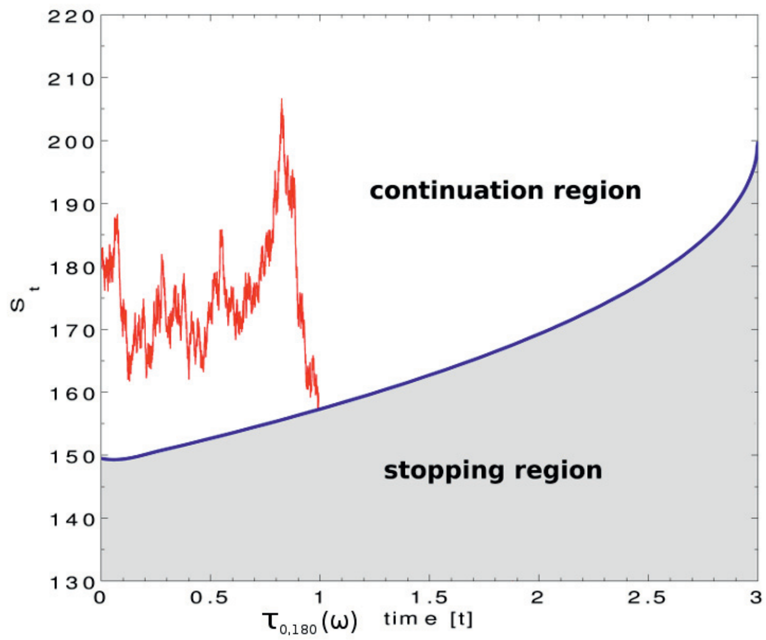
$$G\big(t+s, X^{t,x}_{t+s}\big) = G(t,x) + \int_0^s F\big(t+u, X^{t,x}_{t+u}\big)du + \mathcal{M}_s, \tag{3.11}$$

where $\mathcal{M}_s = \sigma \int_0^s X^{t,x}_{t+u}\partial_x G(t+u, X^{t,x}_{t+u})dB_u$ is a square integrable $(\mathcal{F}_{t+s})_{s\in[0,T-t]}-$martingale with zero expectation, and $F(t,x)$ is given by

$$F(t,x) = e^{r(T-t)(1-\alpha)}(1-\alpha)\left(-r\alpha P_0 + x\big[\mu - r(1-\alpha)\big]\right). \tag{3.12}$$

By (3.11), V defined in (3.4) can be written as

$$V(t,x) = G(t,x) + \sup_{\tau\in\mathcal{T}_{[0,T-t]}} \mathbb{E}\left[\int_0^\tau F(t+u, X^{t,x}_{t+u})du\right]. \tag{3.13}$$



Plot of the optimal stopping boundary in problem (3.4) with time horizon $T = 3$ years, when the stock's rate of return μ is smaller, but close to the bond's interest rate r. The other parameters are $\alpha = 0.3$, $\sigma = 0.25$, $\mu = 0.255$, $r = 0.03$, $P_0 = 100$ and $x = 180$. The boundary lies far above the purchasing price P_0, i.e., the stock is sold with positive book profits.

Figure 3.1: Plot of the stopping boundary in the optimization problem (3.4)

We have

$$F > (<)\ 0 \text{ on the set } \{(t,x) \in [0,T] \times \mathbb{R}_+ : x > (<)\ f\}, \tag{3.14}$$

with

$$f := \frac{r\alpha P_0}{\mu - r(1-\alpha)}. \tag{3.15}$$

This means that the sign of $F(t,x)$ does not depend on t.

Remark 3.2.1. *For (t,x) with $x > f$, the stopping time $\tau := \inf\{s \in [0, T - t] : X^{t,x}_{t+s} \leq f\} \wedge (T-t)$ is strictly positive, and we conclude from (3.13) that $V(t,x) \geq G(t,x) + \mathbb{E}\{\int_0^\tau F(t+u, X^{t,x}_{t+u})du\} > G(t,x)$, and thus $(t,x) \in \mathcal{C}$.*

We have $f \geq 0$ and

$$\mu > (1-\alpha)r \quad \Longrightarrow \partial_x F(t,x) = e^{r(T-t)(1-\alpha)}(1-\alpha)\big(\mu - r(1-\alpha)\big) > 0 \tag{3.16}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$. Furthermore,

$$\partial_t F(t,x) = -r(1-\alpha)^2 e^{r(T-t)(1-\alpha)}\left(-r\alpha P_0 + x\big[\mu - r(1-\alpha)\big]\right) > 0 \tag{3.17}$$

for $x < f$, and

$$\partial_t F(t,x) < 0 \qquad \text{for } x > f. \tag{3.18}$$

3.3 Proofs

3.3.1 Proof of Proposition 3.1.1

For all $x \le y$, one has

$$\begin{aligned} 0 \le & V(t,y) - V(t,x) \\ \le & \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \left\{ \mathbb{E}\left[G\big(t+\tau, X^{t,y}_{t+\tau}\big)\right] - \mathbb{E}\left[G\big(t+\tau, X^{t,x}_{t+\tau}\big)\right] \right\} \\ \le & (1-\alpha)e^{(\mu+r)T}\mathbb{E}\left[\sup_{0\le s\le T-t} \exp\left(\sigma B_s - \frac{1}{2}\sigma^2 s\right)\right](y-x) \\ \le & C(y-x) \end{aligned} \tag{3.19}$$

for some constant $C \in \mathbb{R}_+$ that does not depend on t, x, y. Thus, to establish joint continuity in (t,x), it remains to show that $t \mapsto V(t,x)$ is continuous. Let $s \le t$. One has

$$V(t,x) - V(s,x) \le \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \mathbb{E}\left[G(t+\tau, X^{t,x}_{t+\tau}) - G(s+\tau, X^{t,x}_{t+\tau})\right] \le 0,$$

where for the first inequality, we use that the processes $(X^{t,x}_{t+u})_{u\in[0,T-t]}$ and $(X^{s,x}_{s+u})_{u\in[0,T-t]}$ coincide in distribution.

To obtain an estimation in the other direction, one also has to find an upper bound for the increments of the payoff process $u \mapsto G(u, X^{s,x}_u)$ between $s+T-t$ and T because the remaining time to maturity is smaller for the problem started in t. By the monotonicity of $x \mapsto F(u,x)$, one has

$$\begin{aligned} V(s,x) - V(t,x) \le & \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \mathbb{E}\left[G(s+\tau, X^{s,x}_{t+\tau}) - G(t+\tau, X^{s,x}_{t+\tau})\right] \\ & + \mathbb{E}\left[\int_{T-(t-s)}^{T} F\left(u, \sup_{v\in[s,T]} X^{s,x}_v\right) \vee 0 \, du\right]. \end{aligned}$$

The first term can be estimated by

$$\left(e^{-r(1-\alpha)s} - e^{-r(1-\alpha)t}\right) e^{r(1-\alpha)T}\mathbb{E}\left[(1-\alpha)e^{|\mu|T} \sup_{v\in[0,T]} x \exp\left(\sigma B_v - \frac{1}{2}\sigma^2 v\right) + \alpha P_0\right]$$

and the second term by

$$(t-s)\left(C_1 + C_2\mathbb{E}\left[\sup_{0\le v\le T} x\exp\left(\sigma B_v - \frac{1}{2}\sigma^2 v\right)\right]\right)$$

for some constants $C_1, C_2 \in \mathbb{R}_+$. Altogether, it follows that $t \mapsto V(t,x)$ is continuous for any fixed $x \in \mathbb{R}_+$.

Then, by Theorem 2.4 of [PS06], it follows that (3.7) maximizes (3.4). ■

3.3.2 Proof of Theorem 3.1.2

We distinguish three cases. The first two cases are trivial, whereas the third case is the interesting one, where we show the existence of a continuous, increasing, positive boundary such that the stock is sold at the first time its price is smaller or equal to this boundary.

Case 1: $\mu \leq (1-\alpha)r$

From (3.12), we see that $F \leq 0$. Then, from (3.13), it follows that $\tau_{t,x} = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}_+$, i.e., the investor sells the stock immediately and invests the proceeds in the bank account.

Case 2: $\mu > (1-\alpha)r$ and $\alpha = 0$

One has $F(t,x) > 0$ for all $(t,x) \in [0,T] \times \mathbb{R}_+ \setminus \{0\}$. Then, again from (3.13), it follows that $\tau_{t,x} = T - t$ for all $(t,x) \in [0,T] \times \mathbb{R}_+ \setminus \{0\}$, i.e., the investor never sells the stock prematurely and thus $\mathcal{S} = [0,T) \times \{0\}$.

Case 3: $\mu > (1-\alpha)r$ and $\alpha > 0$

Step 1: Let us show that for every $t \in [0,T], x, y \in \mathbb{R}_+$ with $x \leq y$, the implication

$$V(t,y) = G(t,y) \quad \Longrightarrow \quad V(t,x) = G(t,x) \tag{3.20}$$

holds. Together with the closedness of the stopping region, (3.20) implies that $\mathcal{S}$ is of the form given in (3.8) with boundary $b(t) = \inf\{x \in \mathbb{R}_+ : V(t,x) > G(t,x)\}$.

By (3.13), for any $t \in [0,T)$ and $x \leq y$, we have

$$\begin{aligned}
&\big(V(t,y) - G(t,y)\big) - \big(V(t,x) - G(t,x)\big) \\
\geq &\mathbb{E}\left[\int_0^{\tau_{t,x}} F\left(t+u, X^{t,y}_{t+u}\right) du\right] - \mathbb{E}\left[\int_0^{\tau_{t,x}} F\left(t+u, X^{t,x}_{t+u}\right) du\right] \qquad (3.21)\\
= &\mathbb{E}\left[\int_0^{\tau_{t,x}} F\left(t+u, \frac{y}{x} X^{t,x}_{t+u}\right) - F\left(t+u, X^{t,x}_{t+u}\right) du\right] \\
\geq &0,
\end{aligned}$$

where the last inequality holds by (3.16). As $V \geq G$, this implies (3.20).

Step 2: Let us now show that $t \mapsto b(t)$ in increasing. For $x \in \mathbb{R}_+$ and $s \leq t$, one has

$$\begin{aligned}
V(t,x) - G(t,x) = &e^{-r(1-\alpha)(t-s)} \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \mathbb{E}\left[\int_0^\tau F(s+u, X^{t,x}_{t+u}) du\right] \\
= &e^{-r(1-\alpha)(t-s)} \sup_\tau \mathbb{E}\left[\int_0^\tau F(s+u, X^{s,x}_{s+u}) du\right] \qquad (3.22)
\end{aligned}$$

$$
\begin{aligned}
&\le e^{-r(1-\alpha)(t-s)} \sup_{\tau \in \mathcal{T}_{[0,T-s]}} \mathbb{E}\left[\int_0^\tau F(s+u, X_{s+u}^{s,x})du\right] \\
&= e^{-r(1-\alpha)(t-s)} \left(V(s,x) - G(s,x)\right),
\end{aligned}
$$

where the second supremum is taken over all $(\mathcal{F}_{s+u})_{u\in[0,T-t]}$ −stopping times taking values in $[0, T-t]$. Since $V - G \ge 0$, (3.22) yields the implication

$$
V(s,x) - G(s,x) = 0 \Rightarrow V(t,x) - G(t,x) = 0, \tag{3.23}
$$

which induces that $t \mapsto b(t)$ is increasing.

Step 3: Let us show that $b(t) > 0$ for all $t \in [0, T)$. At first, suppose there exists $t^* \in (0, T)$ such that $b(t^*) = 0$ (i.e., $t^* \ne 0$). As $t \mapsto b(t)$ is increasing, one has $b(u) = 0$ for all $u \in [0, t^*]$. As $X^{0,x}$ cannot reach 0, we have $\tau_{0,x} \ge t^*$, which yields

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau_{0,x}} F(u, X_u^{0,x})du\right] \\
\le& \mathbb{E}\left[\int_0^{t^*} F\left(u, X_u^{0,x}\right) 1_{\{X_u^{0,x} \le f/2\}} du\right] + \mathbb{E}\left[\int_0^{T} 0 \vee F\left(u, X_u^{0,x}\right) 1_{\{X_u^{0,x} > f/2\}} du\right] \\
\le& t^* \mathbb{P}\left(\sup_{0\le u\le T} X_u^{0,x} \le f/2\right) F(T, f/2) + T e^{rT} [\mu - r(1-\alpha)] \mathbb{E}\left[\sup_{0\le u \le T} X_u^{0,x}\right].
\end{aligned}
$$

For the first inequality, we use $F(u, \widetilde{x}) < 0$ for $\widetilde{x} \le f/2$. For the second one, we use that F is increasing in x and for $x \le f$, increasing in t.

We have that $F(T, f/2) < 0$. In addition, $\mathbb{P}\left(\sup_{0\le u\le T} X_u^{0,x} \le f/2\right) \to 1$ and $\mathbb{E}\left(\sup_{0\le u\le T} X_u^{0,x}\right) \to 0$ for $x \to 0$. This yields

$$
\mathbb{E}\left[\int_0^{\tau_{0,x}} F(u, X_u^{0,x})du\right] < 0 \quad \text{for } x \text{ small enough,}
$$

which is a contradiction to the optimality of $\tau_{0,x}$. Thus $b(t^*) > 0$ for all $t^* \in (0, T)$. $b(0) > 0$ follows analogously by extending problem (3.4) to the interval $[-1, T]$. ■

Thus, the theorem is now proven, besides the smooth-fit condition, the continuity of the exercise boundary and its terminal condition. These assertions need some more preparation provided by the following lemmata.

Continuity of the Optimal Stopping Times

The following two lemmata show that the optimal stopping times are close together for stock price processes started in a neighborhood of state and time.

Lemma 3.3.1. *Let $\tau_{t,x}$ be defined as in (3.7). Fix $a > 0$. Then, for all $\varepsilon > 0$, there exists $\widetilde{\delta} > 0$ such that*

$$\sup_{x \in [a,\infty)} \mathbb{P}(\{\tau_{t,x+\delta} - \tau_{t,x} > \varepsilon\}) < \varepsilon$$

for all $\delta \in (0, \widetilde{\delta})$.

Proof. For all $\varepsilon, \delta > 0$, one has

$$\mathbb{P}\left(\{\tau_{t,x+\delta} - \tau_{t,x} > \varepsilon\}\right) = \mathbb{P}\left(\{\tau_{t,x+\delta} - \tau_{t,x} > \varepsilon\} \cap \{\tau_{t,x} \leq T - t - \varepsilon\}\right).$$

To compare the optimal stopping times for different state variables, we use $X^{t,y} = yX^{t,1}$ for all $y \in \mathbb{R}_+$. Let $(B_u)_{u \geq 0}$ be a standard $\mathbb{P}-$Brownian motion. One gets

$$\begin{aligned}
&\mathbb{P}\left(\{\tau_{t,x+\delta} - \tau_{t,x} > \varepsilon\} \cap \{\tau_{t,x} \leq T - t - \varepsilon\}\right) \\
\leq &\mathbb{P}\left(\left\{\min_{u \in [0,\varepsilon]} X^{t,x+\delta}_{t+\tau_{t,x}+u} > b(t+\tau_{t,x})\right\} \cap \{\tau_{t,x} \leq T - t - \varepsilon\}\right) \\
\leq &\mathbb{P}\left(\left\{\min_{u \in [0,\varepsilon]} (x+\delta) X^{t,1}_{t+\tau_{t,x}+u} > x X^{t,1}_{t+\tau_{t,x}}\right\} \cap \{\tau_{t,x} \leq T - t - \varepsilon\}\right) \\
\leq &\mathbb{P}\left(\left\{\min_{u \in [0,\varepsilon]} \left\{\exp\left(\left(\mu - \frac{\sigma^2}{2}\right) u + \sigma B_u\right)\right\} > \frac{x}{x+\delta}\right\}\right) \\
\leq &\Phi\left(\frac{-\ln\left(\frac{a}{a+\delta}\right) + (\mu - \sigma^2/2)\varepsilon}{\sigma\sqrt{\varepsilon}}\right) \\
&- e^{2(\mu-\sigma^2/2)\ln\left(\frac{a}{a+\delta}\right)\sigma^{-2}} \Phi\left(\frac{\ln\left(\frac{a}{a+\delta}\right) + (\mu - \sigma^2/2)\varepsilon}{\sigma\sqrt{\varepsilon}}\right) \quad \forall x \in [a, \infty), \qquad (3.24)
\end{aligned}$$

where the first inequality holds because $t \mapsto b(t)$ is increasing, the second one holds by $xX^{t,1}_{t+\tau_{t,x}} \leq b(t + \tau_{t,x})$, and the third one follows by the strong Markov property of $X^{t,1}$. For $\varepsilon > 0$ fixed, the right-hand side of (3.24) converges to 0 as $\delta \downarrow 0$. ■

Lemma 3.3.2. *Let $\tau_{t,x}$ be defined as in (3.7). Fix $a, b \in \mathbb{R}_+$ with $0 < a < b$. Then, for all $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sup_{x \in [a,b]} \mathbb{P}(\{|\tau_{t_2,x} - \tau_{t_1,x}| > \varepsilon\}) < \varepsilon$$

for all t_1, t_2 with $0 \leq t_1 < t_2 \leq T$ and $t_2 - t_1 < \delta$.

Proof. Similar to the proof of Lemma 3.3.1, for all $t_2 \in (0, T]$, $\varepsilon > 0$, and $u \in (0, \varepsilon)$, one has

$$\mathbb{P}\left(\{\tau_{t_2-u,x} - \tau_{t_2,x} > \varepsilon\}\right) = \mathbb{P}\left(\{\tau_{t_2-u,x} - \tau_{t_2,x} > \varepsilon\} \cap \{\tau_{t_2,x} \leq T - t_2 + u - \varepsilon\}\right).$$

To compare the optimal stopping times, we write $X^{t_2,x}$ in terms of the process $X^{t_1,x}$. Namely, by construction, one has

$$X^{t_2,x}_{t_2+u} = x\frac{X^{t_1,x}_{t_2+u}}{X^{t_1,x}_{t_2}}, \quad u \geq 0,$$

where $t_2 \geq t_1$. In addition, the following argument uses the fact that $X^{t_1,x}_{t_2} \approx x$ for $t_2 - t_1$ small. Consequently, at the first time $X^{t_2,x}$ hits the boundary, the process $X^{t_1,x}$ is not far away, and one can argue as in Lemma 3.3.1. Let $(B_u)_{u\geq 0}$ be a standard $\mathbb{P}-$Brownian motion. Then, for $t_2-t_1 \in (0,\varepsilon/2)$ and $\delta > 0$, one gets

$$\begin{aligned}
&\mathbb{P}\left(\{\tau_{t_1,x} - \tau_{t_2,x} > \varepsilon\} \cap \{\tau_{t_2,x} \leq T - t_1 - \varepsilon\}\right) \\
\leq &\mathbb{P}\left(\left\{\min_{u\in[0,\varepsilon-(t_2-t_1)]} X^{t_1,x}_{t_2+\tau_{t_2,x}+u} > b(t_2+\tau_{t_2,x})\right\} \cap \{\tau_{t_2,x} \leq T - t_1 - \varepsilon\}\right) \\
\leq &\mathbb{P}\left(\left\{\min_{u\in[0,\varepsilon-(t_2-t_1)]} \frac{X^{t_1,x}_{t_2+\tau_{t_2,x}+u}}{X^{t_1,x}_{t_2+\tau_{t_2,x}}} > \frac{x}{X^{t_1,x}_{t_2}}\right\} \cap \{\tau_{t_2,x} \leq T - t_1 - \varepsilon\}\right) \\
\leq &\mathbb{P}\left(\left\{\min_{u\in[0,\varepsilon/2]} \frac{X^{t_1,x}_{t_2+\tau_{t_2,x}+u}}{X^{t_1,x}_{t_2+\tau_{t_2,x}}} > \frac{x}{x+\delta}\right\} \cap \{\tau_{t_2,x} \leq T - t_1 - \varepsilon\}\right) \\
&+ \mathbb{P}\left(\left\{X^{t_1,x}_{t_2} > x+\delta\right\}\right) \\
\leq &\mathbb{P}\left(\left\{\min_{u\in[0,\varepsilon/2]} \left\{\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)u + \sigma B_u\right)\right\} > \frac{a}{a+\delta}\right\}\right) \\
&+ \mathbb{P}\left(\left\{\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t_2-t_1) + \sigma B_{t_2-t_1}\right) > \frac{b+\delta}{b}\right\}\right) \quad \forall x \in [a,b],
\end{aligned} \tag{3.25}$$

where the first inequality holds because $t \mapsto b(t)$ is increasing, the second one holds by $X^{t_2,x}_{t_2+\tau_{t_2,x}} \leq b(t_2+\tau_{t_2,x})$, and for the fourth one, we use the strong Markov property of $X^{t_1,x}$.

As in Lemma 3.3.1, one can choose $\delta > 0$ small enough such that

$$\mathbb{P}\left(\min_{u\in[0,\varepsilon/2]} \left\{\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)u + \sigma B_u\right)\right\} > \frac{a}{a+\delta}\right) < \frac{\varepsilon}{2}. \tag{3.26}$$

Because $B_{t_2-t_1}$ converges stochastically to 0 for $t_2 - t_1 \downarrow 0$, there exists $\delta > 0$ such that

$$\mathbb{P}\left(\left\{\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t_2-t_1) + \sigma B_{t_2-t_1}\right) > \frac{b+\delta}{b}\right\}\right) < \frac{\varepsilon}{2} \tag{3.27}$$

for all t_1, t_2 with $t_2 - t_1 < \delta$. So, it remains to show

$$\mathbb{P}\left(\tau_{t_2,x} - \tau_{t_1,x} > \varepsilon\right) < \varepsilon, \quad \text{for } t_2 \geq t_1 \text{ and } t_2 - t_1 \text{ small enough.} \tag{3.28}$$

To estimate this probability, we renew X at $t_1 + \tau_{t_1,x}$, where it is sufficient to consider the set $\{t_1 + \tau_{t_1,x} \geq t_2\}$. Then, we estimate $X_{t_2}^{t_1,x}$ from below and not from above as in (3.25). But, the calculations are completely analog to the estimation of $\mathbb{P}\left(\tau_{t_1,x} - \tau_{t_2,x} > \varepsilon\right)$, and we are done. ■

Smooth-fit Condition

Next, we show (3.9), i.e., the value function V joints the payoff function G smoothly at the boundary. For $t \in [0, T)$ and $x = b(t)$, one has

$$\frac{V(t,x+\varepsilon)-V(t,x)}{\varepsilon} \geq \frac{G(t,x+\varepsilon)-G(t,x)}{\varepsilon} = \partial_x G(t,x) = (1-\alpha)e^{r(1-\alpha)(T-t)} \tag{3.29}$$

for all $\varepsilon > 0$. On the other hand, one has

$$\begin{aligned} & V(t,x+\varepsilon) - V(t,x) \\ \leq & E\left[G\left(t+\tau_{t,x+\varepsilon}, X_{t+\tau_{t,x+\varepsilon}}^{t,x+\varepsilon}\right)\right] - E\left[G\left(t+\tau_{t,x+\varepsilon}, X_{t+\tau_{t,x+\varepsilon}}^{t,x}\right)\right] \\ = & \varepsilon E\left[X_{t+\tau_{t,x+\varepsilon}}^{t,1} \partial_x G\left(t+\tau_{t,x+\varepsilon}, x\right)\right] \\ \leq & \varepsilon \partial_x G(t,x) E\left[X_{t+\tau_{t,x+\varepsilon}}^{t,1}\right], \end{aligned} \tag{3.30}$$

where the equality holds by the affine linearity of G in x, and the second inequality holds as $t \mapsto \partial_x G(t,x)$ is decreasing. By Lemma 3.3.1 and uniformly integrability, $E\left[X_{t+\tau_{t,x+\varepsilon}}^{t,1}\right]$ converges to 1 for $\varepsilon \downarrow 0$. Thus, (3.29) and (3.30) establish the smooth-fit condition (3.9). ■

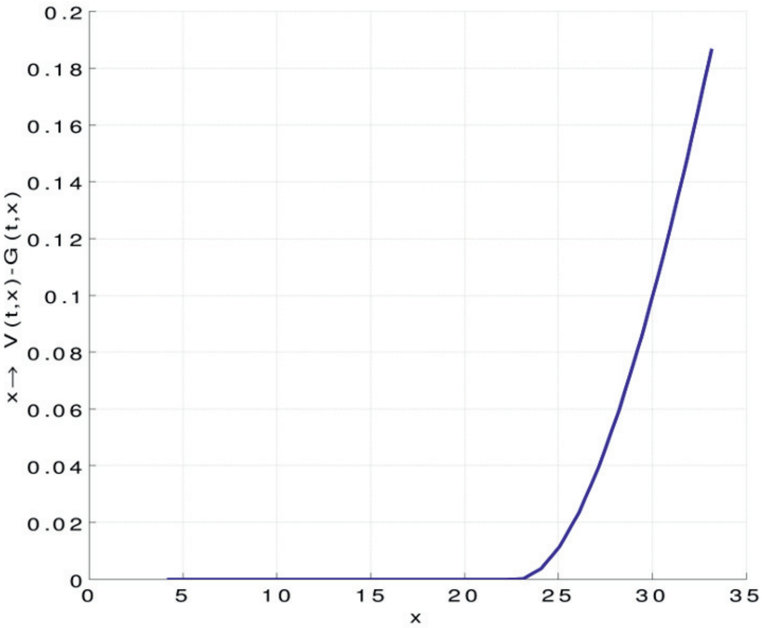
Continuity of the Boundary

Proposition 3.3.3. *The boundary $t \mapsto b(t)$ is right-continuous on $[0, T)$.*

Proof. Fix $t \in [0, T)$ and consider a sequence $t_n \downarrow t$ for $n \to \infty$. As $t \mapsto b(t)$ is increasing, $b(t+) := \lim_{s \downarrow t} b(s)$ exists. Since $(t_n, b(t_n)) \in \mathcal{S}$ for all $n \geq 1$, and V and G are continuous (Proposition 3.1.1), one gets $V(t, b(t+)) = G(t, b(t+))$, i.e., $(t, b(t+)) \in \mathcal{S}$. This results $b(t+) \leq b(t)$. As $t \mapsto b(t)$ is increasing on $[0, T)$, the claim is proven. ■

Proposition 3.3.4. *The boundary $t \mapsto b(t)$ is left-continuous on $(0, T)$.*

First, note that by monotonicity, the limit $b(t-) := \lim_{s \uparrow t} b(s)$ exists and $b(t-) \leq b(t)$. The proof of the proposition is divided into three steps. Under the assumption that a jump of the boundary occurs at some time t, in the first two steps, we find an upper bound for the t-derivative and the x-derivative of V for points in time in a left neighborhood of t and prices between $b(t-)$ and $b(t)$. Then, we use these upper bounds to argue with the PDE which is satisfied by V in $\mathcal{C}$ to see that $V_{xx} > 0$ is bounded away from 0. Roughly



This figure shows the proven smooth fit-condition (3.9)

Figure 3.2: Evolution of $x \mapsto V(t,x) - G(t,x)$ near the boundary.

speaking, the contribution of V_{xx} to the PDE is bounded away from zero by minus the drift rate F of the payoff function (cf. Step 3 of the proof). In a neighborhood of the stopping region, this drift rate is strictly negative. Since $\{t\} \times [b(t-), b(t)]$ is part of the stopping region, where $V = G$, this turns out to be a contradiction to the linearity of G in x if $b(t) > b(t-)$.

This line of argument has already been applied to quite diverse payoff functions, see [PS06].

Proof. Suppose that the stopping boundary b has a jump at t, i.e., $b(t) > b(t-)$.

Step 1 *(upper bound for the $t-$derivative)*

Let $\delta \in (0,t)$, $\varepsilon \in (0, t-\delta)$, and $x \in (b(t-\delta), b(t)]$. Define the stopping time

$$\sigma := \inf\left\{u \geq 0 \ : \ X^{t-\delta-\varepsilon,x}_{t-\delta-\varepsilon+u} \leq b(t-\delta+u)\right\} \wedge (T-(t-\delta)) \in \mathcal{T}_{[0,T-(t-\delta)]}, \tag{3.31}$$

which applies the optimal stopping rule for the problem started in $t-\delta$ to the problem started in $t-\delta-\varepsilon$. By construction, $\left(\sigma, X^{t-\delta-\varepsilon,x}_{t-\delta-\varepsilon+\sigma}\right)$ possesses the same distribution as $\left(\tau_{t-\delta,x}, X^{t-\delta,x}_{t-\delta+\tau_{t-\delta,x}}\right)$. Because σ is in general sub-optimal for the problem started in $t-\delta-\varepsilon$, one gets

$$\begin{aligned}
&\frac{V(t-\delta,x)-V(t-\delta-\varepsilon,x)}{\varepsilon}\\
\leq&\frac{1}{\varepsilon}\mathbb{E}\left[G(t-\delta+\tau_{t-\delta,x},X^{t-\delta,x}_{t-\delta+\tau_{t-\delta,x}})\right]-\frac{1}{\varepsilon}\mathbb{E}\left[G(t-\delta-\varepsilon+\sigma,X^{t-\delta-\varepsilon,x}_{t-\delta-\varepsilon+\sigma})\right]\\
=&\mathbb{E}\left[\left((1-\alpha)X^{t-\delta,x}_{t-\delta+\tau_{t-\delta,x}}+\alpha P_0\right)e^{r(1-\alpha)(T-t+\delta-\tau_{t-\delta,x})}\right]\frac{\left(1-e^{r(1-\alpha)\varepsilon}\right)}{\varepsilon}. \quad (3.32)
\end{aligned}$$

By the classic theory for parabolic equations, see, e.g., Friedman [Fri64], Chapter 3 of Shiryaev [Shi07] (Theorem 15 in Chapter 3), one knows that $V\in\mathcal{C}^{1,2}$ in the continuation region. Therefore, $\partial_t V(t-\delta,x)$ exists for all $x>b(t-\delta)$ and

$$(V(t-\delta,x)-V(t-\delta-\varepsilon,x))/\varepsilon\to\partial_t V(t-\delta,x),\quad \varepsilon\downarrow 0.$$

Together with (3.32), this implies

$$\partial_t V(t-\delta,x)\leq -r(1-\alpha)\mathbb{E}\left[\left((1-\alpha)X^{t-\delta,x}_{t-\delta+\tau_{t-\delta,x}}+\alpha P_0\right)e^{r(1-\alpha)(T-(t-\delta)-\tau_{t-\delta,x})}\right]$$

On the other hand, (t,x) lies in the stopping region for all $x\leq b(t)$ and thus $\tau_{t,x}=0$. Since $b(t-)>0$, one can apply Lemma 3.3.2, and therefore, $\tau_{t-\delta,x}\to 0$ in probability for $\delta\downarrow 0$, where the convergence holds uniformly in $x\in[b(t-)/2,b(t)]$. By uniformly integrability, one gets

$$\begin{aligned}
&\limsup_{\delta\downarrow 0}\sup_{x\in(b(t-\delta),b(t)]}\left[\partial_t V(t-\delta,x)-\partial_t G(t,x)\right]\\
\leq&\sup_{x\in(b(t-),b(t)]}\left[-\left((1-\alpha)x+\alpha P_0\right)e^{r(1-\alpha)(T-t)}r(1-\alpha)-\partial_t G(t,x)\right]\\
=&0 \quad (3.33)
\end{aligned}$$

Step 2 *(upper bound for the $x-$derivative)*

Let $\delta\in(0,t)$, $x\in(b(t-\delta),b(t)]$, and $\varepsilon\in(0,x-b(t-\delta))$. The arguments are similar to Step 1, but more convenient to write down because $\tau_{t-\delta,x}$ is already an admissible stopping time for the problem started in $(t-\delta,x-\varepsilon)$ and need not be transformed as in (3.31). One gets

$$\begin{aligned}
&\frac{V(t-\delta,x)-V(t-\delta,x-\varepsilon)}{\varepsilon}\\
\leq&\frac{1}{\varepsilon}\left[\mathbb{E}\left[G\left(t-\delta+\tau_{t-\delta,x},X^{t-\delta,x}_{t-\delta+\tau_{t-\delta,x}}\right)\right]-\mathbb{E}\left[G\left(t-\delta+\tau_{t-\delta,x},X^{t-\delta,x-\varepsilon}_{t-\delta+\tau_{t-\delta,x}}\right)\right]\right]\\
=&(1-\alpha)\mathbb{E}\left[X^{t-\delta,1}_{t-\delta+\tau_{t-\delta,x}}e^{r(1-\alpha)(T-(t-\delta)-\tau_{t-\delta,x})}\right], \quad (3.34)
\end{aligned}$$

Again, by $V\in\mathcal{C}^{1,2}$ in the continuation region, one gets that $\partial_x V(t-\delta,x)$ exists for all $x>b(t-\delta)$ and $(V(t-\delta,x)-V(t-\delta-\varepsilon,x))/\varepsilon\to\partial_x V(t-\delta,x)$ for $\varepsilon\downarrow 0$. Together with (3.34), one gets

$$\partial_x V(t-\delta,x)\leq(1-\alpha)\mathbb{E}\left[X^{t-\delta,1}_{t-\delta+\tau_{t-\delta,x}}e^{r(1-\alpha)(T-(t-\delta)-\tau_{t-\delta,x})}\right].$$

Again by $\tau_{t,x} = 0$ and $b(t-) > 0$, an application of Lemma 3.3.2 yields

$$\limsup_{\delta \downarrow 0} \sup_{x \in (b(t-\delta), b(t)]} \partial_x V(t-\delta, x) \leq (1-\alpha)e^{r(1-\alpha)(T-t)}. \tag{3.35}$$

The RHS is $\partial_x G(t, x)$ which does not depend on x.

Step 3 *(Conclusion of the left-continuity)*

Now, we want to lead the assumption $b(t) > b(t-)$ to a contradiction.

Let $x^* := (b(t-) + b(t))/2$. By Remark 3.2.1, one has $b(t) \leq f$ and thus, by (3.14), it follows that $F(t, x^*) < 0$. By Step 1 and Step 2, there exists $\delta > 0$ such that for all $s \in [t-\delta, t]$ and $x \in [b(s), x^*]$, one has

$$\begin{aligned}
\mu x \partial_x V(s,x) + \partial_t V(s,x) \leq& \mu x \partial_x G(s,x) + \partial_t G(s,x) - \frac{F(t,x^*)}{3} \\
=& F(s,x) - \frac{F(t,x^*)}{3} \\
\leq& \frac{2}{3} F(t,x^*) - \frac{F(t,x^*)}{3} \\
=& \frac{F(t,x^*)}{3},
\end{aligned}$$

where the second inequality holds by the continuity of F and its monotonicity in x. Again, by Theorem 15 in Chapter 3 of [Shi07], we know that the value function V solves the PDE

$$\partial_t V + \mu x \partial_x V + \frac{\sigma^2}{2} x^2 \partial_{xx} V = 0$$

in the continuation region $\mathcal{C}$. Thus, we have

$$\partial_{xx} V(s,x) \geq -\frac{2F(t,x^*)}{3\sigma^2 x^2} \geq -\frac{2F(t,x^*)}{3\sigma^2 x^{*2}} =: C > 0$$

for all $s \in [t-\delta, t),\ x \in (b(s), x^*]$.
By $V(s, b(s)) = G(s, b(s))$, $\partial_x V(s, b(s)) = \partial_x G(s, b(s))$ (smooth-fit condition), $\partial_{xx} G = 0$, and the Newton-Leibniz formula, it follows that

$$V(s,x^*) - G(s,x^*) = \int_{b(s)}^{x} \int_{b(s)}^{u} \partial_{xx} V(s,v) - \partial_{xx} G(s,v) dv\, du \geq \frac{C(x^* - b(s))^2}{2} \tag{3.36}$$

As this holds for all $s \in [t-\delta, t)$ and $V - G$ is continuous, one concludes that

$$V(t,x^*) - G(t,x^*) \geq \frac{C(x^* - b(t-))^2}{2} > 0,$$

which is a contradiction to the fact that (t, x^*) lies in the stopping region. So, it can be concluded that $b(t-) = b(t)$, and the continuity of the boundary is established. ■

Terminal Condition of the Boundary

By Remark 3.2.1, $b(T-) = \lim_{t\uparrow T} b(t)$ cannot exceed the boundary f, above which the drift rate of the payoff process is positive. It remains to exclude that $b(T-) < f$. But, this is done with the same arguments as in the proof of the left-continuity of the boundary, using the fact that $V(T,x) = G(T,x)$ for all $x \in [b(T-), f]$.

3.3.3 Proof of Proposition 3.1.4

***Proof of** (i)*

Let $0 \le \sigma_1 \le \sigma_2$ and w.l.o.g. $t = 0$. For two independent standard Brownian motions B and $\widetilde{B}$, the process

$$X_s = x \exp\left(\mu s + \sigma_1 B_s - \frac{\sigma_1^2}{2} s + \sqrt{\sigma_2^2 - \sigma_1^2}\,\widetilde{B}_s - \frac{\sigma_2^2 - \sigma_1^2}{2} s\right), \quad s \ge 0$$

possesses the same law as the stock price from (3.1) with $\sigma = \sigma_2$. In addition, X is Markov w.r.t. the filtration $(\mathcal{F}_s^{B,\widetilde{B}})_{s\in[0,T]}$ which is generated by B and $\widetilde{B}$. This implies that V from (3.3) with standard deviation σ_2 coincides with the value of the problem

$$\sup_\tau E\left[G(\tau, X_\tau)\right], \tag{3.37}$$

where the supremum is taken over all $(\mathcal{F}_s^{B,\widetilde{B}})_{s\in[0,T]}$–stopping times τ. Now consider the artificial optimal stopping problem where the second Brownian motion $\widetilde{B}$ that enters into the stock price is not observable to the maximizer. This corresponds to the restriction to $(\mathcal{F}_s^{B})_{s\in[0,T]}$–stopping times. Of course, the latter supremum is at least as high as the previous one. On the other hand, for an $(\mathcal{F}_s^{B})_{s\in[0,T]}$–stopping time τ, we have

$$\begin{aligned}
&E\left[G(\tau, X_\tau)\right] \\
=&E\Bigg[\left((1-\alpha)x\exp\left(\mu\tau + \sigma_1 B_\tau - \frac{\sigma_1^2}{2}\tau + \sqrt{\sigma_2^2-\sigma_1^2}\,\widetilde{B}_\tau - \frac{(\sigma_2^2-\sigma_1^2)\tau}{2}\right) + \alpha P_0\right) \\
&\qquad \times e^{r(1-\alpha)(T-\tau)}\Bigg] \\
=&(1-\alpha)E\Bigg[e^{r(1-\alpha)(T-\tau)} x \exp\left(\mu\tau + \sigma_1 B_\tau - \frac{\sigma_1^2}{2}\tau\right) \\
&\qquad\qquad \times E\left[\exp\left(\sqrt{\sigma_2^2-\sigma_1^2}\,\widetilde{B}_\tau - \frac{\sigma_2^2-\sigma_1^2}{2}\tau\right) \mid \mathcal{F}_T^B\right]\Bigg] \\
&+ \alpha P_0 E\left[e^{r(1-\alpha)(T-\tau)}\right] \\
=&E\left[(1-\alpha)e^{r(1-\alpha)(T-\tau)} x \exp\left(\mu\tau + \sigma_1 B_\tau - \frac{\sigma_1^2}{2}\tau\right) + \alpha P_0 e^{r(1-\alpha)(T-\tau)}\right],
\end{aligned}$$

where the conditional expectation is 1 because τ is $\mathcal{F}_T^B-$measurable whereas $\widetilde{B}$ is independent of $\mathcal{F}_T^B$. It follows that the value of problem (3.37) restricted to all $(\mathcal{F}_s^B)_{s\in[0,T]}-$stopping times coincides with V from (3.3) with smaller standard deviation σ_1. Thus, one has $V^{\sigma_1} \leq V^{\sigma_2}$. Note that the only property of the payoff function G we use is its affine linearity in X_τ.

Proof of *(ii)*

For $\sigma = 0$,the optimal stopping problem (3.4) reads

$$V(t,x) = \sup_{u\in[0,T]} G(t+u, xe^{\mu u}) = \sup_{u\in[0,T]} \left((1-\alpha)xe^{\mu u} + \alpha P_0\right) e^{r(1-\alpha)(T-t-u)}.$$

By simple algebra, one calculates that $d^2/du^2 G(t+u, xe^{\mu u}) > 0$. So, we conclude that for fixed $x \in \mathbb{R}_+$, the maximum of $u \mapsto G(t+u, xe^{\mu u})$ is either attained at $u = 0$ or at $u = T-t$.
For $b(t)$ given by (3.10), one has $G(t, b(t)) = G(T, b(t)e^{\mu(T-t)})$, i.e., the investor is indifferent between stopping at t and T. It implies that $(t, b(t))$ lies in the stopping region. On the other hand, by $\partial_x G(t,x) = (1-\alpha)e^{(1-\alpha)r(T-t)} < e^{\mu(T-t)}\partial_x G(T.x)$ for all $x \in \mathbb{R}_+$, one has that (t,x) lies in the continuation region for all $x > b(t)$. This implies that $b(t)$ is indeed the optimal exercise boundary given in (3.8).

It remains to show that $\tau_{t,x} = T-t$ for all $x > b(t)$. As $\sup_{u\in[0,T-t]} G(t+u, xe^{\mu u})$ is not attained at any $u \in (0, T-t)$, $X^{t,x}$ cannot hit the boundary if it starts above it. ■

3.4 Optimal Stopping when Bond Earnings are Taxed at Maturity

Instead of taxing the gains in the bond continuously, one can consider a bond where the earnings are taxed at the end of maturity when the portfolio is liquidated.

Under the same assumptions and similar notations as in Section 3.1, the payoff function reads

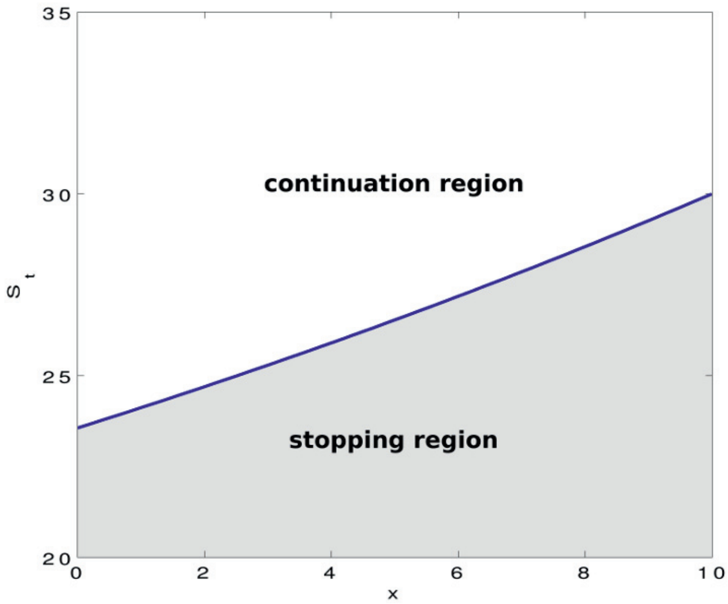
$$\widetilde{G}(t,x) = \big[(1-\alpha)x + \alpha P_0\big]\big[1 + (1-\alpha)(e^{r(T-t)} - 1)\big], \tag{3.38}$$

for $(t,x) \in [0,T] \times \mathbb{R}_+$.
Then, the optimal stopping problem becomes

$$\widetilde{V}(x) = \sup_{\tau\in\mathcal{T}_{[0,T]}} \mathbb{E}\left[\widetilde{G}(\tau, X_\tau)\right], \tag{3.39}$$

where $X_0 = x$.
The value function associated with problem (3.39) is given by

$$\widetilde{V}(t,x) := \sup_{\tau\in\mathcal{T}_{[0,T-t]}} \mathbb{E}\big[\widetilde{G}(t+\tau, X_{t+\tau}^{t,x})\big]. \tag{3.40}$$



The optimal stopping boundary in problem (3.4), when the stock price process is deterministic. The parameters in this figure are $\sigma = 0, \mu = 0.05, r = 0.06, P_0 = 10, T = 10, \alpha = 25\%$.

Figure 3.3: Deterministic stock price process in problem (3.4)

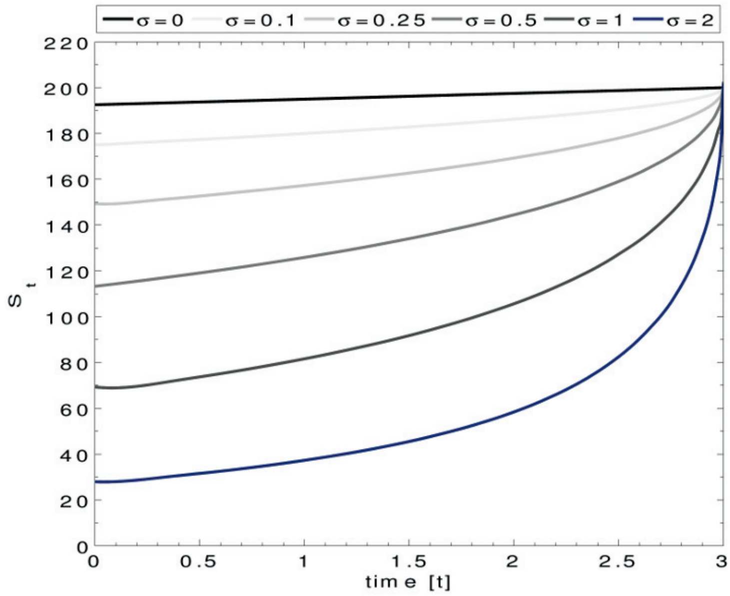
Equally to problem (3.4), one sees that $\widetilde{V}(t,x) \geq \widetilde{G}(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}_+$ and $\widetilde{V}(T,x) = \widetilde{G}(T,x)$ for any $x \in \mathbb{R}_+$. We denote by $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{S}}$ the continuation and the stopping region, respectively. With the standard methods, one can show (likely to Section 3.1) the existence of an optimal stopping time $\tau \in \mathcal{T}_{[0,T-t]}$ which solves (3.40) and is given by

$$\widetilde{\tau}_{t,x} := \inf\{s \in [0, T-t] : (t+s, X^{t,x}_{t+s}) \in \widetilde{\mathcal{S}}\}. \tag{3.41}$$

Similar as in problem (3.4), one can now show the existence of an optimal stopping boundary. But, trying to proof continuity of the optimal stopping boundary for (3.40), one faces the problem that the boundary needs not to be monotone, c.f. Figure 3.5. An explanation for this fact is given in the subsequent part. As we can see in Section 3.3.2, the monotonicity of the boundary is essential for proving continuity.

To explain the lacking monotonicity of the stopping boundary, again, we first turn the terminal payoff into a running payoff with the help of Itô's formula. Then, the value function reads

$$\widetilde{V}(t,x) = \widetilde{G}(t,x) + \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \mathbb{E}\left[\int_0^\tau \widetilde{F}(t+u, X^{t,x}_{t+u}) du\right], \tag{3.42}$$



The figure shows off the evolution of the optimal stopping boundary, when σ converges to 0. The boundary then converges to the optimal stopping boundary in the deterministic case. The parameters in this figure are $\mu = 0.044, r = 0.05, P_0 = 30, T = 10, \alpha = 25\%$.

Figure 3.4: Evolution of stopping boundary with varying σ

with

$$\widetilde{F}(t,x) = -r\alpha P_0(1-\alpha)e^{r(T-t)} + (1-\alpha)x\big[\mu\alpha + (\mu - r)(1-\alpha)e^{r(T-t)}\big]. \tag{3.43}$$

Note that $\widetilde{F}(t,x) > (<)\ 0$ on the set $\{(t,x) \in [0,T] \times \mathbb{R}_+ : x > (<)\ f(t)\}$, with

$$\widetilde{f}(t) = \frac{r\alpha P_0}{\mu\alpha e^{-r(T-t)} + (\mu - r)(1-\alpha)}. \tag{3.44}$$

It is also clear to see that $\widetilde{f}(t) \geq 0$ for all $t \in [0,T]$,

$$\partial_x \widetilde{F}(t,x) = (1-\alpha)\big(\mu\alpha + (\mu - r)(1-\alpha)e^{r(T-t)}\big) > 0, \tag{3.45}$$

and

$$\partial_t \widetilde{F}(t,x) = -r(1-\alpha)^2(\mu - r)xe^{r(T-t)} + r^2\alpha P_0(1-\alpha)e^{r(T-t)} > 0 \tag{3.46}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$ under the following condition:

$$\frac{(1-\alpha)r}{\alpha e^{-rT} + (1-\alpha)} < \mu. \tag{3.47}$$

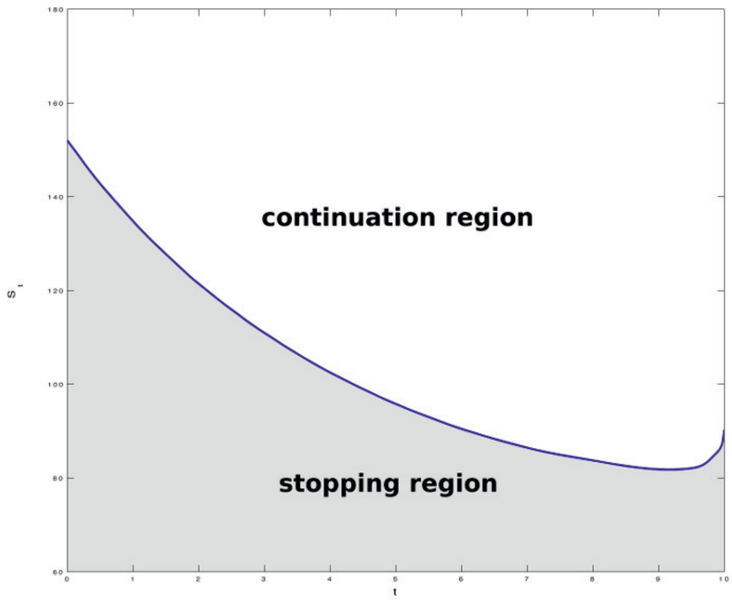


Figure 3.5: Exemplary plot of a nonmonotone stopping boundary in the optimization problem (3.40)

So, the drift rate $t \mapsto F(t,x)$ of G is positive for $F(t,\cdot) > f(t)$ and negative otherwise. In Figure 3.6, one can see that $t \mapsto \widetilde{f}(t)$ (red function) is monotonically decreasing in time. Furthermore, the set of stopping times becomes smaller with elapsing time. These two effects work against each other, so that the stopping boundary needs not to be monotonically increasing or decreasing. The time dependence of $\widetilde{f}$ is the main difference to the optimal stopping problem (3.4), where f is constant over time, and so ensures the monotonicity of the stopping boundary $t \mapsto b(t)$.

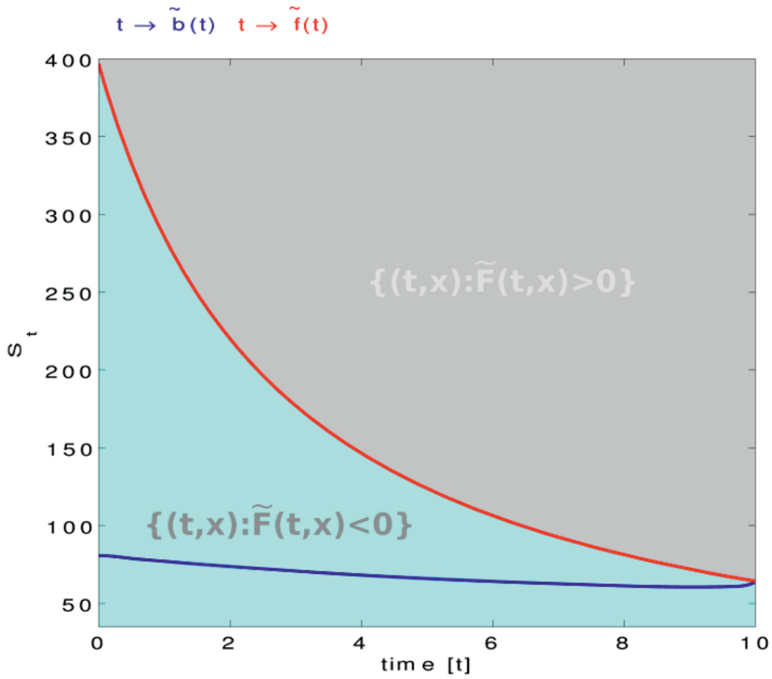
The existence of an optimal stopping boundary in (3.40) under condition (3.47) can be shown in the same way as in (3.4). In the following, we show that the stopping boundary $\widetilde{b}$ of problem (3.40) dominates the stopping boundary b of problem (3.4) if

$$\frac{(1-\alpha)r}{\alpha e^{-rT} + (1-\alpha)} < \mu < r.$$

To prove this, we first need the following lemmata to characterize the shape of the stopping region $\widetilde{\mathcal{S}}$.

Lemma 3.4.1. *Consider problem (3.40) with stopping region* $\widetilde{\mathcal{S}}$. *Let* $r \geq 0$ *and* $\alpha \in [0,1)$. *Then, there exists a boundary* $\widetilde{b} : [0,T) \mapsto \mathbb{R}_+$ *such that the stopping region is given by*

$$\widetilde{\mathcal{S}} = \{(t,x) \in [0,T) \times \mathbb{R}_+ : x \leq \widetilde{b}(t)\}. \tag{3.48}$$



The blue function is an example of an nonmonotone stopping boundary of the optimal stopping problem (3.40). In this example, the red function shows $t \mapsto \widetilde{f}(t)$, where above the drift rate $t \mapsto \widetilde{F}(t,x)$ of $\widetilde{G}$ is positive, and negative below. In this figure, one can also observe the presumption $\lim \widetilde{b}(t) \to \widetilde{f}(T)$ for $t \uparrow T$.

Figure 3.6: Plot of the optimal stopping boundary together with $t \mapsto \widetilde{f}(t)$

under condition (3.47), and

$$\widetilde{\mathcal{S}} = [0,T) \times \mathbb{R}_+ \tag{3.49}$$

if $\mu \leq r(1-\alpha)$.

Proof. ***Case 1:*** $\mu > \frac{(1-\alpha)r}{\alpha e^{-rT}+(1-\alpha)}$

Let us show that for every $t \in [0,T], x,y \in \mathbb{R}_+$ with $x \leq y$, the implication

$$\widetilde{V}(t,y) = \widetilde{G}(t,y) \quad \Longrightarrow \quad \widetilde{V}(t,x) = \widetilde{G}(t,x) \tag{3.50}$$

holds. Together with the closedness of the stopping region, (3.50) implies that $\widetilde{\mathcal{S}}$ is of the form given in (3.48) with boundary $\widetilde{b}(t) = \inf\{x \in \mathbb{R}_+ : \widetilde{V}(t,x) > \widetilde{G}(t,x)\}$.

By (3.13), for any $t \in [0,T)$ and $x \leq y$, we have

$$\begin{aligned}&\left(\widetilde{V}(t,y)-\widetilde{G}(t,y)\right)-\left(\widetilde{V}(t,x)-\widetilde{G}(t,x)\right)\\ \geq&\mathbb{E}\left[\int_0^{\tau_{t,x}}\widetilde{F}\left(t+u,X_{t+u}^{t,y}\right)du\right]-\mathbb{E}\left[\int_0^{\tau_{t,x}}\widetilde{F}\left(t+u,X_{t+u}^{t,x}\right)du\right]\\ =&\mathbb{E}\left[\int_0^{\tau_{t,x}}F\left(t+u,\frac{y}{x}X_{t+u}^{t,x}\right)-\widetilde{F}\left(t+u,X_{t+u}^{t,x}\right)du\right]\\ \geq&0,\end{aligned} \tag{3.51}$$

where the last inequality holds by (3.45). By $\widetilde{V} \geq \widetilde{G}$, this implies (3.50).

Case 2: $\mu \leq r(1-\alpha)$

From (3.43), we see that $F \leq 0$. Then, from (3.42), it follows that $\widetilde{\tau}_{t,x} = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}_+$, i.e., the investor sells the stock immediately and invests the proceeds in the bank account. ■

Remark 3.4.2. *Lemma 3.4.1 only treats the cases* $\mu < r(1-\alpha)$ *and* $\mu > \frac{(1-\alpha)r}{\alpha e^{-rT}+(1-\alpha)}$. *In the case* $(1-\alpha)r < \mu \leq \frac{(1-\alpha)r}{\alpha e^{-rT}+(1-\alpha)}$, *we face the problem that* $t \mapsto f(t)$ *has a pole at some point* t^*. *So, we cannot say anything about the stopping region for* $t = t^*$ *with the methods used in this chapter so far. But, one can easily check with the methods used in the proof of Lemma 3.4.1 that the stopping region can be characterized on the set* $[0,T)\setminus\{t^*\}$. *With this, one can check that there exists a stopping boundary* $\widetilde{\widetilde{b}} : (t^*,T) \mapsto \mathbb{R}_+$ *such that*

$$\widetilde{\mathcal{S}}/\left\{[0,T)\setminus\{t^*\}\right\} = \begin{cases}[0,t^*)\times\{0\}, & \text{for } t<t^*\\ \{(t,x)\in(t^*,T)\times\mathbb{R}_+ : x \leq \widetilde{\widetilde{b}}(t)\}, & \text{for } t>t^*\end{cases}.$$

Proposition 3.4.3. *Let* b *be the stopping boundary in optimization problem (3.4) and* $\widetilde{b}$ *be the stopping boundary in optimization problem (3.40). Then,* $b(t) \leq \widetilde{b}(t)$ *for all* $t \in [0,T)$ *if*

$$\frac{(1-\alpha)r}{\alpha e^{-rT}+(1-\alpha)} < \mu < r.$$

Proof. We first show that $F(t,x)$ dominates $\widetilde{F}(t,x)$ for all $(t,x) \in [0,T]\times\mathbb{R}_+$. Therefore, note that

$$F(T,x) = \widetilde{F}(T,x) \tag{3.52}$$

for all $x \in \mathbb{R}_+$. One calculates

$$\partial_t \widetilde{F}(t,x) = \big(r^2\alpha P_0(1-\alpha)\underbrace{-(1-\alpha)^2xr(\mu-r)}_{\geq 0 \text{ as } \mu<r}\big)e^{r(T-t)}$$

and

$$\partial_t F(t,x) = \big(r^2\alpha P_0(1-\alpha)^2 \underbrace{-(1-\alpha)^2 xr(\mu - r(1-\alpha))}_{<0 \text{ as } \mu>r(1-\alpha)}\big)e^{r(T-t)(1-\alpha)}.$$

Due to $(1-\alpha) < 1$ and the positivity of $\partial_t \widetilde{F}(t,x)$ (see (3.46)), one has $\partial_t(\widetilde{F}(t,x) - F(t,x)) \geq 0$ for all $x \in \mathbb{R}$. Together with (3.52), we arrive at

$$F(t,x) \geq \widetilde{F}(t,x) \tag{3.53}$$

for all $(t,x) \in [0,T] \times \mathbb{R}_+$.
(3.53) implies

$$\begin{aligned} V(t,x) - G(t,x) =& E\left(\int_0^{\tau_{t,x}} F(t+u, xH_u)du\right) \\ \geq& E\left(\int_0^{\widetilde{\tau}_{t,x}} F(t+u, xH_u)du\right) \\ \geq& E\left(\int_0^{\widetilde{\tau}_{t,x}} \widetilde{F}(t+u, xH_u)du\right) \\ =& \widetilde{V}(t,x) - \widetilde{G}(t,x). \end{aligned} \tag{3.54}$$

Because of (3.54) and $\widetilde{V}(t,x) \geq \widetilde{G}(t,x)$, we have the following implication:

$$V(t,x) - G(t,x) = 0 \Rightarrow \widetilde{V}(t,x) - \widetilde{G}(t,x) = 0 \quad \forall (t,x) \in [0,T) \times \mathbb{R}_+,$$

which yields $b(t) \leq \widetilde{b}(t)$ for all $t \in [0,T)$. ∎

Similar to the case of a continuously taxed bond, we can determine the optimal stopping boundary in the case $\sigma = 0$ explicitly.

Proposition 3.4.4. *Consider problem (3.40) where the stock price process X is given by equation (3.1) with $\sigma = 0$. If*

$$\mu > \frac{\ln\left[1 + (1-\alpha)\left(e^{rT} - 1\right)\right]}{T} =: A,$$

the optimal stopping boundary $\widetilde{b}$ is given by

$$\widetilde{b}(t) = \frac{\alpha P_0 (1-\alpha)\left(e^{r(T-t)} - 1\right)}{(1-\alpha)\left(e^{\mu(T-t)} - 1 - (1-\alpha)\left(e^{r(T-t)} - 1\right)\right)} \quad \forall t \in [0,T). \tag{3.55}$$

and the optimal stopping time (3.41) is given by

$$\widetilde{\tau}_{t,x} = \begin{cases} 0 & , x \leq \widetilde{b}(t) \\ T-t & , x > \widetilde{b}(t) \end{cases} .$$

Proof. In the case $\sigma = 0$, problem (3.40) reads

$$\begin{aligned}\widetilde{V}(t,x) &= \sup_{u\in[0,T]} \widetilde{G}(t+u, xe^{\mu u}) \\ &= \sup_{u\in[0,T]} \left((1-\alpha)xe^{\mu u} + \alpha P_0\right)\left(1 + (1-\alpha)\left(e^{r(T-t-u)} - 1\right)\right).\end{aligned}$$

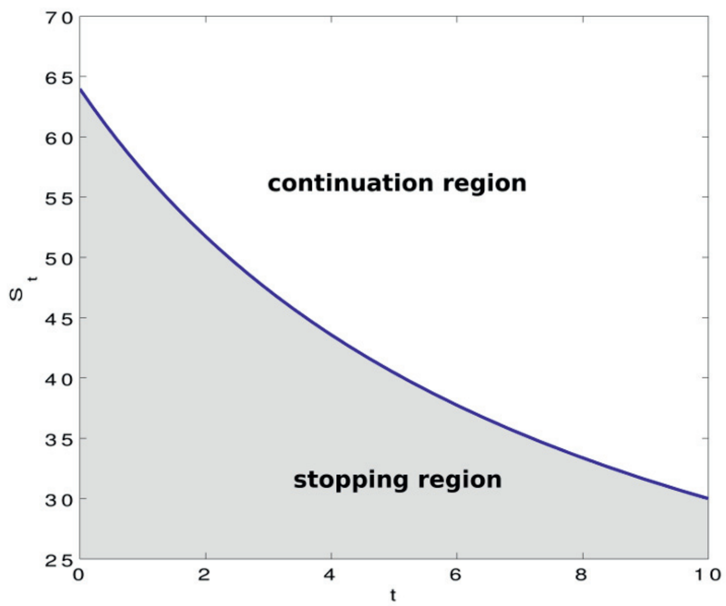
By simple algebra, one can calculate $d^2/du^2\widetilde{G}(t+u, xe^{\mu u}) > 0$. So, we conclude that for fixed $x \in \mathbb{R}_+$, the maximum of $u \mapsto \widetilde{G}\,(t+u, xe^{\mu u})$ is either attained at $u = 0$ or at $u = T - t$. For $\widetilde{b}(t)$ given by (3.55), one gets $\widetilde{G}(t, \widetilde{b}(t)) = \widetilde{G}(T, \widetilde{b}(t)e^{\mu(T-t)})$, i.e., the investor is indifferent between stopping at t and T. It implies that $(t, \widetilde{b}(t))$ lies in the stopping region. On the other hand, by $\partial_x G(t,x) = (1-\alpha)(1 + (1-\alpha)\left(e^{r(T-t)} - 1\right) < (1-\alpha)e^{\mu(T-t)} = \partial_x G(T.x)$ for all $x \in \mathbb{R}_+$ and $\mu > A$, one observes that (t, x) lies in the continuation region for all $x > \widetilde{b}(t)$. This implies that $\widetilde{b}(t)$ is indeed the optimal exercise boundary given in (3.48).

It remains to show $\widetilde{\tau}_{t,x} = T - t$ for all $x > \widetilde{b}(t)$. As $\sup_{u\in[0,T-t]} \widetilde{G}(t+u, xe^{\mu u})$ is not attained at any $u \in (0, T-t)$, $X^{t,x}$ cannot hit the boundary if it starts above it. ■

Remark 3.4.5. *Observing the formula for the stopping boundary, we see the effect of different choices of μ. $\mu > r$ implies a monotonically increasing stopping boundary (see Figure 3.7), whereas $\mu < r$ implies a monotonically decreasing stopping boundary (Figure 3.8). The case $\mu > r$ is less interesting from an economical point of view. In this case, $\widetilde{b}(t) < P_0$ for all $t \in [0, T)$ and $\sigma \geq 0$ which induces that an investor is better off with wash sales, and therefore, a single sale of the stock seems to be no reasonable strategy in the case of a positive interest rate.*

3.5 Conclusion

In this chapter, we consider a very simple optimal stopping problem, where a risk neutral investor owning one stock has to decide when to sell this stock and invest the proceeds into a bond up to maturity under capital gains taxes. In the case of no taxes, the solution to the problem is trivial; either she converts the stock immediately if the stock's rate of return is lower or equal to the bond's rate of return, and she holds the stock up to maturity otherwise. Considering capital gains taxes, there exists a trivial solution if and only if the stock's rate of return is lower than or equal to the bond's rate of return after taxes. In this case, the stock is immediately sold. Reversely, if the stock's rate of return higher than the bond's rate of return after taxes, we show the existence of an optimal stopping boundary. The investor, who is subject to paying capital gains taxes, sells her stocks and invests the proceeds into a bond



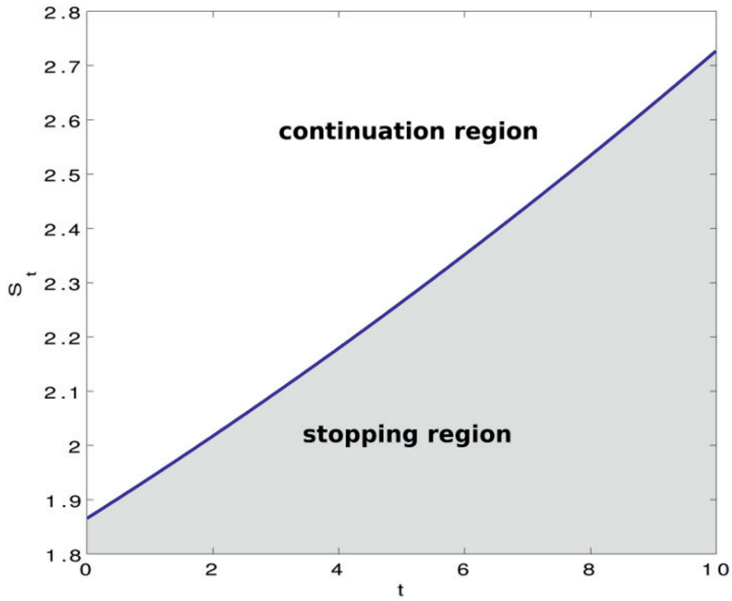
The optimal stopping boundary in problem (3.40), when the stock price process is deterministic and $\mu < r$. The parameters in this figure are $\sigma = 0, \mu = 0.05, r = 0.06, P_0 = 10, T = 10, \alpha = 25\%$.

Figure 3.7: Deterministic stock price process in problem (3.40), $\mu < r$

when the stock price is lower or equal to this boundary. The used methods in this chapter rely on techniques presented, e.g., in [PS06], where the *terminal* payoff is turned into a *running* payoff with the help of Itô's formula. Furthermore, we show that the boundary is decreasing in stock's volatility which is interesting from an economic point of view. For a more volatile asset, the option to time the tax payments has a higher value for investors. This means that capital gains taxes can even motivate investors to take more risk. In the case of a deterministic stock price process, we could even analytically calculate the stopping boundary.

In the case that the capital gains in the bond are continuously taxed, we could show that the stopping boundary is a continuous, increasing, time-dependent function, and at the stopping boundary, the state derivative of the value function equals the state derivative of the payoff function. In the case that the capital gains in the bond are taxed at the end of the period, we focus on the case that $\mu > \frac{(1-\alpha)r}{\alpha e^{-rT}+(1-\alpha)}$. Thereby, we could numerically see that the stopping boundary is in general not monotone. This mainly relies on the two facts that

- $t \mapsto \inf_{x\in\mathbb{R}_+}\{(t,x) : F(t,x) > 0\}$ is monotonically decreasing,
- the set of stopping times becomes smaller with elapsing time.



The optimal stopping boundary in problem (3.40), when the stock price process is deterministic and $\mu > r$. The parameters in this figure are $\sigma = 0, \mu = 0.10, r = 0.06, P_0 = 10, T = 10, \alpha = 25\%$.

Figure 3.8: Deterministic stock price process in problem (3.40), $\mu > r$

These two effects work against each other so that the stopping boundary is in general not monotonically increasing or decreasing.
At least, we were able to show that under some restrictions to the drift rate μ, the stopping boundary in this problem dominates the stopping boundary in the problem where capital gains are taxed continuously.

3.6 Outlook and Further Research

In this section, we give possible extensions of our model.

The $\widetilde{V}$ Problem

We have seen that we cannot say much about the optimal stopping boundary when capital gains in the bond are taxed at the end of the period. The resulting lack of monotonicity of the boundary yields that we cannot use the standard methods for showing properties like continuity or smooth-fit.

From numerical calculations, one can make two crucial observations for $\mu < r$:

- If $\sigma > 0$ is large enough, the boundary is monotonically increasing.

- If the boundary is not monotone, there exists a turning point where the derivative of the boundary changes its sign.

Therefore, two questions arise:

- Can we find $K \in \mathbb{R}_+$ such that the boundary is monotonically increasing for all $\sigma > K$?
- Is it possible to determine the changing point of the derivative, supposedly depending on μ, r, σ and T?

Finding answers to these questions would make it possible to solve the problem with the standard methods, used in this chapter.

Furthermore, we observe that

- the stopping boundary seems to be monotonically increasing if $\mu > r$.

Standard methods are again applicable if this monotonicity can be proven.

Appendix A

Appendix for Chapter 2

In this part of the appendix, we list results which are important or essential for statements in the second chapter. The results are listed by their order of appearance.

As we use the properties of regulated functions in Theorem 2.2.12, we will state the completeness property with respect to the supremum norm here.

Definition A.0.1 (regulated function). *Let X be a Banach space with norm $||\cdot||_X$. A function $f:[a,b]\to X$ is called regulated if $f(t+):=\lim_{\tau\downarrow t} f(\tau)$ and $f(s-):=\lim_{\tau\uparrow s} f(\tau)$ exist in X for every $t\in[a,b[$ and $s\in]a,b]$.*

Theorem A.0.2 (see [Hön75], Theorem I.3.6). *The space of all regulated functions mapping from $[a,b]$ to X is a Banach space when endowed with the supremum norm.*

Theorem A.0.3 (Yoeurp-Yor Formula (see [Jas03])). *Given a filtered probability space $(\Omega,\mathcal{F},\mathbb{F},P)$ satisfying the usual conditions, a càdlàg process C and a semimartingale Z satisfying $Z_0=0$ and $\Delta Z_s\neq -1\ \forall s\in[0,\infty)P-a.s.$, the linear SDE*

$$X = C + \int_{0+} X_- dZ$$

has a unique solution

$$X = C - \mathcal{E}(Z)\int_{0+} C_- d(\mathcal{E}(Z)^{-1}), \tag{A.1}$$

where

$$\mathcal{E}(Z)_t = \exp\left\{Z_t - \frac{1}{2}[Z,Z]^c_t\right\}\prod_{0<s\leq t}(1+\Delta Z_s)\exp(-\Delta Z_s)$$

is the stochastic exponential which is the solution to the homogeneous linear SDE

$$X = 1 + \int_{0+} X_- dZ.$$

If C is a semimartingal, (A.1) is indistinguishable from

$$X = \mathcal{E}(Z)\left\{C_0 + \int_{0+} \mathcal{E}(Z)_-^{-1} d\widetilde{C}\right\}$$

with

$$\widetilde{C} = C - [C, Z]^c - \sum_{0<s\leq \cdot} \frac{\Delta C_s \Delta Z_s}{1 + \Delta Z_s}.$$

A version of the Gronwall inequality, used in (2.42), is given in [Pro04] as an exercise. As this exercise is not proven in the book, we will give a short proof here. The proof mainly relies on standard iterating techniques to prove Gronwall inequalities.

Lemma A.0.4 (Gronwall inequality)**.** *Let $(A_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$ be càdlàg processes, where $(C_t)_{t\geq 0}$ is increasing with $C_0 \geq 0$. Suppose*

$$0 \leq A_t \leq \alpha + \int_0^t A_{s-} dC_s \text{ for } t \geq 0. \tag{A.2}$$

Then,

$$A_t \leq \alpha \exp(C_t) \text{ for each } t \geq 0. \tag{A.3}$$

Proof. Iterating inequality (A.2), we get

$$\begin{aligned}
A_t \leq & \alpha + \alpha \int_0^t A_{s_1-} dC_{s_1} \\
\leq & \alpha + \alpha(C_t - C_0) + \int_0^t \int_0^{s_1-} A_{s_2-} dC_{s_2} dC_{s_1} \\
\leq & \alpha + \alpha(C_t - C_0) + \alpha \int_0^t C_{s_1-} - C_0 dC_{s_1} + \int_0^t \int_0^{s_1-} \int_0^{s_2-} A_{s_3-} dC_{s_3} dC_{s_2} dC_{s_1} \\
\leq & \alpha + \alpha C_t + \alpha \int_0^t C_{s_1-} dC_{s_1} + \int_0^t \int_0^{s_1-} \int_0^{s_2-} A_{s_3-} dC_{s_3} dC_{s_2} dC_{s_1},
\end{aligned}$$

where we use that $C_0 \geq 0$.
Since C is increasing, it is thus of finite variation. Therefore, repeated application of Itô's formula yields

$$\begin{aligned}
e^{C_t} = & e^{C_0} + \int_0^t e^{C_{s-}} dC_s + \sum_{s\leq t} \left(e^{C_s} - e^{C_{s-}} - e^{C_{s-}} \Delta C_s\right) \\
\geq & 1 + \int_0^t e^{C_{s-}} dC_s \\
\geq & 1 + C_t + \int_0^t \int_0^{s_1-} e^{C_{s_2-}} dC_{s_2} dC_{s_1} \\
\geq & 1 + C_t + \int_0^t C_{s_1-} dC_{s_1} + \int_0^t \int_0^{s_1-} \int_0^{s_2-} e^{C_{s_3-}} dC_{s_3} dC_{s_2} dC_{s_1}.
\end{aligned}$$

Set $K_t := \sup_{u\in[0,t]} A_u$. Then, we have

$$\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} A_{s_n-} dC_{s_n} \cdots dC_{s_1} \leq K_t \left(\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} dC_{s_n} \cdots dC_{s_1} \right).$$

By induction, one can then show

$$\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} dC_{s_n} \cdots dC_{s_1} \leq \frac{C_t^n}{n!}. \tag{A.4}$$

As the RHS of (A.4) converges to 0 for fixed t, continuing the above iteration yields the result. ■

Remark A.0.5. *As C is a càdlàg increasing process, the integral in inequality (A.2) is well-defined as Riemann-Stieltjes integral.*

Appendix B

Appendix for Chapter 3

In this part of the appendix, we list results which are important or essential for statements in the third chapter. The results are again listed by their order of appearance.

The approach to determine the boundary numerically relies on the so called Markovian approach, which can be found in [PS06]. We will revisit some important aspects of this approach in relation to our problem. We start with approximating the Geometric Brownian Motion on the time interval $[0,T]$ by a binomial tree considering the grid points $0, \frac{T}{N}, 2\frac{T}{N}, \ldots, (N-1)\frac{T}{N}, T$. The approximating time-discrete process $(X_n^N)_{n\in\{0,\ldots,N\}}$ is modeled as a Cox-Ross-Rubinstein binomial tree, where the probability of an upward and a downward jump, respectively, is given by

$$\mathbb{P}\left(\frac{X_{n+1}^N}{X_n^N} = u\right) = \frac{e^{\mu T/N} - d}{u-d}, \quad \mathbb{P}\left(X_{n+1}^N / X_n^N = d\right) = \frac{u - e^{\mu T/N}}{u-d}, \qquad \text{(B.1)}$$

with $u = \beta + \sqrt{\beta^2 - 1}, d = \beta - \sqrt{\beta^2 - 1}$ and $\beta = 0.5\left(e^{-\mu T/N} + e^{(\mu+\sigma^2)T/N}\right)$. It is known (cf. [CRR79]) that the given process converges to a geometric $\mathbb{P}-$ Brownian motion with percentage drift μ and percentage volatility σ.

Set $G^N(n,x) := G\left(\frac{n}{N}T, x\right)$. The discretized problem (3.3) then reads

$$V^N(x) = \sup_{\tau\in\mathcal{T}_{\{0,\ldots,N\}}} \mathbb{E}\left[G^N\left(\tau, X_n^N\right)\right],$$

with $\mathcal{T}_{\{0,\ldots,N\}}$ being the set of all $\{0,\ldots,N\}-$valued stopping times, and the value function is given by

$$V^N(n,x) = \sup_{\tau\in\mathcal{T}_{\{0,\ldots,N-n\}}} \mathbb{E}\left[G\left(n+\tau, xH_\tau^{N-n}\right)\right] \qquad \text{(B.2)}$$

with $\mathcal{T}_{\{0,\ldots,N-n\}}$ being the set of all $\{0,\ldots,N-n\}-$valued stopping times, and $(H_m^{N-n})_{m\in\{0,\ldots,N-n\}}$ is a Cox-Ross-Rubinstein process with $H_0^{N-n} = 1$, where the probability of an upward or a downward jump, respectively, is given as in

(B.1).

We introduce the continuation region $\mathcal{C}^N = \{(n,x) \in \{0,\dots,N\} \times \mathbb{R}_+ : V^N(n,x) > G^N(n,x)\}$ and the stopping set $\mathcal{S}^N = \{(n,x) \in \{0,\dots,N\} \times \mathbb{R}_+ : V^N(n,x) = G^N(n,x)\}$. To determine the value function, we use the Wald-Bellman equation, which is the simplest form of a dynamic programming equation.

Theorem B.0.6 (see [PS06], p. 17, Theorem 1.9)**.** *Consider the optimal stopping problem (B.2). Then, the value function V^N satisfies the* ***Wald-Bellman equations***

$$V^N(n,x) = \max\left\{G^N(n,x), TV^N(n,x)\right\}$$

for $n = N-1,\dots,1,0$ where $TV^N(N-1,x) = \mathbb{E}[G^N(N, xH_1^1)]$, $TV^N(n,x) = \mathbb{E}[V^N(n+1, xH_1^1)]$ for $n \in \{0,\dots,N-2\}$ and $x \in \mathbb{R}_+$.

With the help of Theorem B.0.6, we can recursively determine the value function (B.2) for every discretized time point. In our calculations, the state space is discretized by the states calculated from the CRR-model.

Deutsche Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Modellierung von Kapitalertragsteuern in stetiger Zeit und dem Einfluss von Kapitalertragsteuern auf optimale Investitionsentscheidungen.
Für die Berechnung der zu zahlenden Kapitalertragsteuer sind drei Größen relevant: der Verkaufspreis, der Steuersatz und die Steuerbasis. Die zu zahlenden Steuern ergeben sich mithilfe der Formel

$$\textit{Steuersatz} \times (\textit{Verkaufspreis} - \textit{Steuerbasis}).$$

Da Kapitalertragsteuern nur gezahlt werden müssen, wenn Gewinne/Verluste realisiert werden, hat die Investorin die Möglichkeit, Steuerzahlungen auf spätere Zeitpunkte zu verschieben. Hier tritt ein Optimierungsproblem mit der Frage auf, wann eine Realisierung von Kapitalgewinnen, beziehungsweise Kapitalverlusten, optimal ist.
Diverse wissenschaftliche Arbeiten haben sich mit diesem Optimierungsproblem beschäftigt. Eine wesentliche Frage in diesem Kontext ist die Modellierung der sogenannten Steuerbasis. Im einfachsten Fall entspricht die Steuerbasis dem Kaufpreis der verkauften Aktie. Hat eine Investorin in diesem Fall beispielsweise zwei BMW Aktien im Portfolio, welche zu unterschiedlichen Zeitpunkten zu unterschiedlichen Preisen gekauft wurden, so tritt beim Verkauf einer BMW-Aktie die Frage auf, welche der beiden Einkaufspreise für die Berechnung der zu zahlenden Kapitalertragsteuer relevant ist. Hat eine Investorin die Wahl zu entscheiden, welche der Aktien relevant für die Versteuerung ist, so nennt man dies die *exakte Steuerbasis*. Diese wird im zeitdiskreten Modell von Dybvig und Koo [DK96] verwendet und bildet im Folgenden die Grundlage für die Modellierung der Steuerbasis in stetiger Zeit.
Andere verwendete Steuerbasen sind unter anderem:

- die *FIFO*-Prioritätsregel, bei der die Aktien mit der längsten Verweildauer im Portfolio zuerst verkauft werden,
- die *LIFO*-Prioritätsregel, bei der die Aktien mit der kürzesten Verweildauer im Portfolio zuerst verkauft werden,
- der *durchschnittliche Einkaufspreis* aller Aktien des gleichen Typs.

Obwohl Kapitalertragsteuern einen großen Einfluss auf das Gesamtvermögen einer Investorin haben (in Deutschland betragen diese 25%), gibt es doch nur sehr wenig Literatur, die sich damit beschäftigt. Die meisten Arbeiten behandeln andere Marktfriktionen, wie beispielsweise Transaktionskosten. Finanzmathematische Arbeiten zum Thema Kapitalertragsteuern sind unter anderem [DK96], [DU05] (diskrete Zeit), [BST07; BST10], [JKT99; JKT00], [CP99] und [BCP07] (stetige Zeit).

Einen großen Einfluss haben Kapitalertragsteuern auch bei Investitionsentscheidungen. Das Erwartungsnutzenoptimierungsproblem mit Steuern ist aber in voller Allgemeinheit sehr unhandlich und analytisch nicht lösbar. Besonders für die exakte Steuerbasis und die FIFO-Prioritätsregel ist ein Optimierungsproblem schwer handhabbar, da die Buchgewinne hierbei eine unendlich dimensionale Zustandsvariable sein können. Deshalb schauen wir uns spezielle Optimierungsprobleme mit weniger Handelsmöglichkeiten für den Investor an.

Modellierung von Kapitalertragsteuern für Strategien mit unendlicher Variation

In Kapitel 2 der Arbeit beschäftigen wir uns mit der Modellierung von Kapitalertragsteuern für Strategien wie sie im Allgemeinen in der Finanzmathematik auftreten. Dabei wird unter schwachen Bedingungen gezeigt, dass Dividendenzahlungen einen negativen Einfluss auf das Vermögen einer Investorin nach Steuerzahlungen haben, wenn der risikolose Zinssatz positiv ist.

Wir betrachten einen endlichen Zeithorizont mit Laufzeitende $T \in \mathbb{R}_+$ und einen filtrierten Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, der die üblichen Voraussetzungen erfüllt.
Da für eine Investorin, die mit endlich vielen Aktien handelt, die gesamten Steuerzahlungen nur die Summe der Steuerzahlungen für die einzelnen Aktientypen sind, reicht es aus, nur ein einziges risikobehaftetes Wertpapier (Aktie) zu betrachten. Der Preisprozess des Wertpapiers mit nichtnegativer Dividende, sei durch das Semimartingal $(S_t)_{t\in[0,T]}$ gegeben. Die Dividende wird durch den nichtfallenden adaptierten càdlàg Prozess $(D_t)_{t\in[0,T]}$ modelliert. Positive wie auch negative Kapitalerträge werden mit $\alpha \in (0,1)$ versteuert. Dividenden werden dabei immer sofort versteuert, während Handelsgewinne nur versteuert werden, wenn sie realisiert werden.

Ziel von Kapitel 2 ist die Modellierung der *exakten Steuerbasis* in stetiger Zeit. Diese Modellierung ist motiviert durch das zeitdiskrete Modell von Dybvig und Koo [DK96].

Definition 1 (Dybvig/Koo-Strategie). *Mit $(N_{s,t})_{s=0,1,\ldots,T,t=s,s+1,\ldots,T}$ bezeichne man die Anzahl der Aktien, die man zum Zeitpunkt s gekauft hat und die mindestens bis zum Zeitpunkt t im Portfolio sind, d.h. $N_{t,t} \geq N_{t,t+1} \geq \ldots \geq N_{t,T} \geq 0$ für alle $t \in \{0,\ldots,T\}$. Wir nennen N dann eine **Strategie**.*

Durch die Bedingung $N_{t,T} \geq 0$ in Definition 1 werden Leerverkäufe ausgeschlossen.
Die akkumulierten Steuerzahlungen bis zum Zeitpunkt u sind nun gegeben durch

$$\Pi_u := \alpha \sum_{t=1}^{u} \sum_{s=0}^{t-1} \left(N_{s,t-1} - N_{s,t}\right)\left(S_t - S_s\right). \tag{1}$$

Verkauft man die Aktien mit der kürzesten Verweildauer im Portfolio zuerst und realisiert man Verluste automatisch durch Washsales, so führt dies für die Investorin im Modell von Dybvig/Koo zu einem gleichen oder höheren Vermögen nach Steuern, wenn der risikolose Zinssatz nichtnegativ ist.

Theorem 1. *Sei $(\varphi_t)_{\{t=0,1,\ldots,T+1\}} \geq 0$ eine gegebene Position an Aktien. Die Strategie $\widetilde{N}$ sei induktiv gegeben durch:*
$\widetilde{N}_{0,0} = \varphi_1$ und, gegeben $\widetilde{N}_{s,t-1}$, $s = 0, 1, \ldots, t-1$, ist $\widetilde{N}_{s,t}$ definiert durch

$$\widetilde{N}_{s,t} = 1_{\{S_t \geq S_s\}} \left(\widetilde{N}_{s,t-1} - \left((\Delta\varphi_{t+1})^- - \sum_{j=s+1}^{t-1} \widetilde{N}_{j,t-1} \right)^+ \right)^+, \quad s \in \{0, \ldots, t-1\},$$

$$\widetilde{N}_{t,t} = \Delta\varphi_{t+1} + \sum_{s=0}^{t-1} (\widetilde{N}_{s,t-1} - \widetilde{N}_{s,t}).$$

Sei nun N eine beliebige Strategie, sodass $\varphi_{t+1} = \sum_{s=0}^{t} N_{s,t}$.
Dann sind die akkumulierten Steuerzahlungen $\widetilde{\Pi}$ für Strategie $\widetilde{N}$ stets kleiner oder gleich den akkumulierten Steuerzahlungen Π für Strategie N, d.h.

$$\widetilde{\Pi}_t \leq \Pi_t, \quad \textit{für alle } t \in \{0, 1, \ldots, T\}.$$

Damit hat (im Falle eines positiven Zinssatzes) eine Investorin, die Strategie $\widetilde{N}$ verfolgt, ein mindestens so großes Vermögen wie eine Investorin mit Strategie N. Somit lässt sich die doppelt-indizierte Strategie N auf eine eindimensionale Strategie φ reduzieren.

Sei $\varphi \in \mathbb{L}$ mit $\varphi_0 = 0$ und $\varphi \geq 0$. Die Einschränkung von Leerverkäufen soll ausschließen, dass eine Investorin gleichzeitig eine Long- und eine Shortposition der gleichen Aktie hält, da dies im Falle einer linearen Versteuerung der Kapitalerträge eine Arbitragemöglichkeit bietet. Die Annahme $\varphi_0 = 0$ hat notationelle Gründe; sie schließt nicht aus, dass die Investorin kurz nach dem Zeitpunkt 0 Aktien kauft, d.h. $\varphi_{0+} > 0$.
Motiviert durch das Modell von Dybvig/Koo, verwenden wir für die Modellierung der *exakten Steuerbasis* in stetiger Zeit die folgenden Konventionen:

(i) automatische Verlustrealisierung durch Washsales

(ii) Aktien mit der kürzesten Verweildauer im Portfolio werden zuerst verkauft.

Bedingung (ii) ist dabei unter (i) äquivalent zu der Bedingung, dass *Aktien mit den kleinsten Buchgewinnen im Portfolio zuerst verkauft werden.*

Wir beginnen nun damit, die unrealisierten Buchgewinne pro Aktie zu beschreiben, wenn die Investorin die Strategie φ verfolgt.

Definition 2 (Buchgewinne). *Sei $\varphi \in \mathbb{L}$ mit $\varphi_0 = 0$ und $\varphi \geq 0$. Für $t \in [0,T], x \in \mathbb{R}_+$, definiere*

$$\tau_{t,x} := \begin{cases} \sup M_{t,x} & \textit{falls } M_{t,x} \neq \emptyset \\ t & \textit{sonst} \end{cases}$$

mit $M_{t,x} := \{u \in \mathbb{R}_+ \mid (u \leq t \text{ und } x - \varphi_t + \varphi_u \leq 0) \text{ oder } (u < t \text{ und } x - \varphi_t + \varphi_{u+} \leq 0)\}$. Wir nennen $F : \Omega \times [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ mit

$$F_\omega(t,x) := S_t(\omega) - \inf_{\tau_{t,x}(\omega) \leq u \leq t} S_u(\omega) \tag{2}$$

*die **Buchgewinnfunktion**.*

F lässt sich wie folgt interpretieren. Alle Aktien mit Preisprozess S im Portfolio werden nach der Höhe ihrer Buchgewinne in aufsteigender Reihenfolge sortiert und mit der Variable x etikettiert. $x \mapsto F_\omega(t,x)$ bildet dann jedes infinitesimale Aktienstück des Portfolios auf seinen Buchgewinn zum Zeitpunkt t ab. Dies ist ein auf dem Papier vorhandener Buchgewinn, der noch nicht realisiert wurde. $\tau_{t,x}$ ist der Kaufzeitpunkt der Aktie, die zum Zeitpunkt t das Etikett x hat. Diese Aktie ist seit dem Zeitpunkt $\tau_{t,x}$ im Portfolio und wurde bis zum Zeitpunkt t, abgesehen von durchgeführten Washsales, nicht verkauft. Die Definition von F berücksichtigt bereits, dass Verluste direkt durch Washsales realisiert werden, d.h. eine Aktie mit negativem Buchgewinn wird verkauft und sofort zurückgekauft, ohne dass sich die Strategie ändert. Die Aktien mit den niedrigsten Buchgewinnen werden dabei zuerst verkauft.
Aufgrund der durchgeführten Washsales und der Tatsache, dass Aktien beim Kauf keinen Buchgewinn haben, hat eine Aktie mit längerer Portfolioverweildauer einen höheren Buchgewinn. Die Handelsgewinne lassen sich schreiben als:

$$S_t - S_{\tau_{t,x}} = \underbrace{F(t,x)}_{\text{unrealisierte Buchgewinne}} + \underbrace{\inf_{\tau_{t,x} \leq u \leq t} S_u - S_{\tau_{t,x}}}_{\text{realisierte Verluste}} .$$

Während die Buchgewinnfunktion in (2) direkt für alle $\varphi \in \mathbb{L}$ definiert ist, kann der akkumulierte Steuerzahlungsprozess nicht direkt für $\varphi \in \mathbb{L}$ definiert werden, da Aktienzukäufe und -verkäufe die Steuerzahlungen unterschiedlich beeinflussen. Wir definieren daher die Steuerzahlungsfunktion zunächst für Elementarstrategien.

Definition 3 (Akkumulierte Steuerzahlungsfunktion für Elementarstrategien). *Sei φ eine nichtnegative Elementarstrategie, sodass $\varphi = \sum_{i=1}^k H_{i-1} 1_{]\!]\kappa_{i-1},\kappa_i]\!]}$,* *wobei* $0 = \kappa_0 \leq \kappa_1 \leq \ldots \leq \kappa_k = T$ *Stoppzeiten sind und* H_{i-1} $\mathcal{F}_{\kappa_{i-1}}$*−messbar ist. Sei* τ *bzw.* F *wie in Definition 2. Wir definieren*

$$\begin{aligned}\Pi_t(\varphi) :=& \alpha \sum_{i=1}^{k} 1_{(\kappa_{i-1}<t)} \int_0^{(H_{i-1}-H_{i-2})^-} F(\kappa_{i-1}, x)dx \\ &+ \alpha \sum_{i=1}^{k} 1_{(\kappa_{i-1}<t)} \int_0^{\varphi_t} \left(F(\kappa_{i-1}+, x) + \inf_{\kappa_{i-1} \leq u \leq t \wedge \kappa_i} (S_u - S_{\kappa_{i-1}}) \right) \wedge 0 \; dx \\ &+ \alpha \int_0^t \varphi_u dD_u,\end{aligned}$$

als den Steuerzahlungsprozess *der Elementarstrategie* φ*, wobei* $H_{-1} := 0$*.*

Der erste Term auf der rechten Seite entspricht der Steuerzahlung durch "reale" Verkäufe. Aktienverkäufe haben, wie man an der Definition sieht, im Gegensatz zu Aktienzukäufen einen direkten Einfluss auf die Steuerzahlung. Dieser nichtlineare Effekt hat zur Folge, dass Definition 3 nicht für Strategien unendlicher Variation benutzt werden kann. Der zweite Term modelliert die akkumulierten Steuergutschriften durch Washsales. Der letzte Term sind Steuerzahlungen infolge von Dividendenausschüttungen.
Das folgende Theorem zeigt nun, dass diese Definition auf alle linksstetigen adaptierten Prozesse fortgesetzt werden kann. Diese Fortsetzung beruht unter anderem darauf, dass S ein Semimartingal ist.

Theorem 2. *Die Abbildung* $\varphi \mapsto \Pi(\varphi)$ *aus Definition 3 ist im folgenden Sinne stetig:*
Sei S *ein Semimartingal und sei* $(\varphi^n)_{n\in\mathbb{N}}$ *eine up-Cauchyfolge von nichtnegativen* ***Elementarstrategien****. Dann ist die Folge von Steuerzahlungsprozessen* $(\Pi(\varphi^n))_{n\in\mathbb{N}}$ *eine up-Cauchyfolge.*

Da die Menge von Elementarstrategien dicht in $\mathbb{L}$ ist (bezüglich der gleichmäßigen Konvergenz in Wahrscheinlichkeit), können wir den Steuerzahlungsprozess für Strategien aus $\mathbb{L}$ mithilfe von Theorem 2 als up-Grenzwert einer Folge von Steuerzahlungsprozessen für Elementarstrategien definieren.

Definition 4. *Sei* $\varphi \in \mathbb{L}$ *und* $(\varphi^n)_{n\in\mathbb{N}}$ *eine Folge von Elementarstrategien. Der* ***akkumulierte Steuerzahlungsprozess*** *ist definiert als*

$$\Pi(\varphi) := up - \lim_{n\to\infty} \Pi(\varphi^n) \text{ für } \varphi^n \overset{up}{\to} \varphi.$$

Würde man Elementarstrategien betrachten, die nur punktweise gegen eine linksstetige Strategie konvergieren, ließe sich leicht ein Gegenbeispiel finden, sodass die Folge der Steuerzahlungsprozesse nicht konvergiert.

In diesem Modell wird nun der Einfluss verschiedener Dividendenstrategien auf das Gesamtvermögen einer Investorin miteinander verglichen. Dazu trifft man zunächst einige Annahmen. Dividendenzahlungen müssen direkt versteuert werden. Mit $(D_t)_{t\in[0,T]}$ wird der akkumulierte Dividendenzahlungsprozess modelliert. Dieser sei ein nichtfallender stochastischer Prozess mit càdlàg Pfaden.

Definition 5 (Aktienpreisprozess). *Sei $s_0 \in \mathbb{R}_+$. Gegeben ein Semimartingal R mit $\Delta R \geq -1$, wählt eine Firma einen Dividendenprozess D. Der Aktienpreisprozess ist dann definiert als die eindeutige Lösung der stochastischen Differentialgleichung (SDE)*

$$S^D = s_0 + S^D_- \bullet R - D. \tag{3}$$

Die Lösung der SDE ist ein verallgemeinertes stochastisches Exponential. Wir nennen D dabei zulässig, wenn $S^D \geq 0$, d.h. wir betrachten nur Dividendenzahlungen, für die der Aktienpreis nicht negativ wird.
R ist nach dieser Definition der *Ertragsprozess*, welcher unabhängig von der Wahl von D ist.

Theorem 3 (Einfluss von Dividenden auf Steuerzahlungen). *Für jede Strategie φ^D im Modell mit Dividenden existiert eine Strategie φ^0 im Modell ohne Dividenden, sodass*

$$\int_0^t \varphi_s^0 \, dS_s^0 = \int_0^t \varphi_s^D \, dS_s^D + \int_0^t \varphi_s^D \, dD_s, \quad \forall t \in [0,T], \ P - f.s.$$

und

$$\Pi_t^0 \leq \Pi_t^D, \quad \forall t \in [0,T].$$

Wählt man also die Strategie im Modell ohne Dividenden derart, dass die gleiche Geldmenge in beiden Modellen in die Aktie investiert ist, dann stimmen die Handelsgewinne in beiden Modellen überein, und zu jedem Zeitpunkt sind die akkumulierten Steuerzahlungen im Modell ohne Dividenden geringer als im Modell mit Dividenden.
Verkauft man in beiden Modellen alle Aktien zum Laufzeitende, so stimmen die akkumulierten Steuerzahlungen überein. Nach Theorem 3 treten die anfallenden Steuern im Modell mit Dividenden früher auf. Im Falle eines positiven Zinssatzes kann so das überschüssige Vermögen (aufgrund von geringer Steuerzahlungen) in einen risikolosen Bond investiert werden, sodass somit das Vermögen einer Investorin im Modell ohne Dividenden stets größer ist.

Optimale Handelsstrategie in einem Markt bestehend aus einer Aktie und einem risikolosen Bond unter Berücksichtigung einer Kapitalertragsteuer

In Kapitel 3 wird der Effekt von Steuerzahlungen auf optimale Investitionsentscheidungen in einem einfachen Modell untersucht.
Im Fall ohne Steuern ist das Problem trivial: es ist optimal die Aktie bis zum Laufzeitende zu halten, wenn ihre Driftrate μ größer ist als der stetige Zinssatz r des Bonds; es ist optimal die Aktie sofort zu verkaufen, wenn $\mu \leq r$. Im Fall mit Steuern gibt es die gleiche triviale Lösung für den Fall, dass $\mu \leq r(1-\alpha)$ ist. Im Fall $(1-\alpha)r > \mu$ ist die Lösung des Problems allerdings

nicht mehr trivial. Hier lässt sich zeigen, dass es optimal ist, die Aktie zu verkaufen und das Kapital in den Bond zu investieren, wenn der Preisprozess eine bestimmte untere, zeitabhängige Schranke trifft bzw. unterschreitet. Im Folgenden betrachten wir daher im Wesentlichen den Fall, dass die Rendite des risikolosen Bonds größer ist als die erwartete Rendite der Aktie. Unsere Resultate sind inspiriert von den optimalen Stoppproblemen aus [SXZ08] und [DTP09].

Wir betrachten einen filtrierten Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Auf diesem Wahrscheinlichkeitsraum sei mit $(B_t)_{t\geq 0}$, $\mathbb{P}(B_0 = 0)$ eine $\mathbb{P}-$Brownsche Bewegung definiert. Die Investitionsmöglichkeiten bestehen aus einem risikolosen Bond mit stetigem Zinssatz $r > 0$ und einer Aktie, deren Dynamik gegeben sei durch

$$dX_t = X_t\left(\mu dt + \sigma dB_t\right), \quad t \geq 0, \tag{4}$$

mit $X_0 > 0$, wobei $\mu, \sigma > 0$ die Wachstumsrate und die Volatilität der Aktie modelliert.

Da das Vermögen im Bond stetig verzinst wird, nehmen wir zunächst an, dass die Erträge im Bond ebenfalls stetig versteuert werden.

Wenn die Aktie zur Zeit $t \in [0, T]$ zum Preis $x \in \mathbb{R}_+$ verkauft und dieser Ertrag in den risikolosen Bond investiert wird, ist das Vermögen der Investorin zum Laufzeitende gegeben durch

$$G(t, x) = \big[(1-\alpha)x + \alpha P_0\big]e^{r(1-\alpha)(T-t)}, \tag{5}$$

für $(t, x) \in [0, T] \times \mathbb{R}_+$, wobei $P_0 > 0$ der Kaufpreis der Aktie ist.

Ziel der Investorin ist es, das erwartete Vermögen in T zu maximieren. Daraus ergibt sich ein optimales Stoppproblem:

$$V(x) := \sup_{\tau\in\mathcal{T}_{[0,T]}} E(G(\tau, X_\tau)), \tag{6}$$

wobei $\mathcal{T}_{[0,T]}$ die Menge der $(\mathcal{F}_s)_{s\in[0,T]}-$ messbaren Stoppzeiten ist, die Werte aus dem Zeitintervall $[0, T]$ annehmen.

Wegen der starken Markoveigenschaft von $(X_t)_{t\in[0,T]}$ definieren wir die Wertefunktion zu Problem (6) als

$$V(t, x) := \sup_{\tau\in\mathcal{T}_{[0,T-t]}} \mathbb{E}\big\{G(t+\tau, X^{t,x}_{t+\tau})\big\}, \tag{7}$$

wobei $(X^{t,x}_s)_{s\in[t,T]}$ die eindeutige Lösung der Gleichung (4) mit Anfangsbedingung $X^{t,x}_t = x$ ist und $\mathcal{T}_{[0,T-t]}$ die Menge der $(\mathcal{F}_{t+s})_{s\in[0,T-t]}-$messbaren Stoppzeiten bezeichnet, die Werte aus dem Zeitintervall $[0, T-t]$ annehmen. Man beachte, dass $V(x) = V(0, x)$ gilt, da $X^{0,x} = X$ mit $X_0 = x$.

Setzt man $\tau = 0$ in (7), so erhält man

$$V(t, x) \geq G(t, x), \quad \text{für } (t, x) \in [0, T] \times \mathbb{R}_+, \tag{8}$$

und zusätzlich für die Randbedingung, dass

$$V(T,x) = G(T,x) \quad \text{für } x \in \mathbb{R}_+ \tag{9}$$

Die Fortsetzungsregion ist durch

$$\mathcal{C} := \{(t,x) \in [0,T) \times \mathbb{R}_+ : V(t,x) > G(t,x)\}$$

und die Stoppregion durch

$$\mathcal{S} := \{(t,x) \in [0,T) \times \mathbb{R}_+ : V(t,x) = G(t,x)\}$$

definiert. Man zeigt zunächst die Stetigkeit von V. Mithilfe der Stetigkeit, lässt sich mittels der Standardtheorie (vgl. [PS06]) zeigen, dass eine optimale Stoppzeit $\tau \in \mathcal{T}_{[0,T-t]}$ existiert, welche (7) maximiert.

Proposition 1. *$(t,x) \mapsto V(t,x)$ ist stetig auf $[0,T] \times \mathbb{R}_+$. Zusätzlich wird (7) durch die Stoppzeit*

$$\tau_{t,x} := \inf\{s \in [0,T-t] : (t+s, X^{t,x}_{t+s}) \in \mathcal{S}\}. \tag{10}$$

für jedes Paar $(t,x) \in [0,T] \times \mathbb{R}_+$ maximiert.

Die Idee zur Lösung des optimalen Stoppproblems (7) beruht darauf, die Auszahlung zum Endzeitpunkt als Akkumulation einer laufenden Auszahlung zu schreiben. Man stelle hierbei fest, dass diese Idee eine Standardmethode der optimalen Stopptheorie ist, vgl. hierzu [PS06].
Aufgrund der Glattheitseigenschaft der Auszahlungsfunktion G können wir die Itôformel anwenden, um die folgende Zerlegung für $(G(s,X_s))_{s\geq t}$ zu erhalten:

$$G\big(t+s, X^{t,x}_{t+s}\big) = G(t,x) + \int_0^s F\big(t+u, X^{t,x}_{t+u}\big)du + \mathcal{M}_s,$$

wobei $\mathcal{M}_s = \int_0^s \partial_x G(t+u, X^{t,x}_{t+u})dB_u$ ein quadratintegrierbares $(\mathcal{F}_{t+s})_{s\in[0,T-t]}-$ messbares $\mathbb{P}-$Martingal ist. $F(t,x)$ ist die Driftrate von $G(t,x)$ und ist gegeben durch

$$F(t,x) = e^{r(T-t)(1-\alpha)}(1-\alpha)\left(-r\alpha P_0 + x\big[\mu - r(1-\alpha)\big]\right).$$

Mithilfe dieses Ansatzes lässt sich die Wertefunktion folgendermaßen umschreiben:

$$V(t,x) = G(t,x) + \sup_{\tau\in\mathcal{T}_{[0,T-t]}} \mathbb{E}\Big\{\int_0^\tau F(t+u,, X^{t,x}_{t+u})du\Big\}.$$

Auf der Menge $\{(t,x) \in [0,T] \times \mathbb{R}_+ : x > (<)\ f\}$ mit

$$f := \frac{r\alpha P_0}{\mu - r(1-\alpha)},$$

haben wir $F(t,x) > (<)\ 0$. Somit gilt $\mathcal{S} \subset \{(t,x) \in [0,T] \times \mathbb{R}_+ : x \leq \ f\}$, beziehungsweise $\{(t,x) \in [0,T] \times \mathbb{R}_+ : x > \ f\} \subset \mathcal{C}$.

Mithilfe dieser Umformulierung des optimalen Stoppproblems lässt sich nun die Existenz einer unteren, stetigen, nichtfallenden, zeitabhängigen Stoppgrenze zeigen. Außerdem zeigen wir, dass die $x-$Ableitung der Wertefunktion V mit der $x-$Ableitung der Auszahlungsfunktion G an dieser Stoppgrenze übereinstimmt (Smooth-Fit).

Theorem 4. *Man betrachte das Problem (7) mit Stoppregion $\mathcal{S}$. Sei $r \geq 0$ und $\alpha \in [0,1)$. Dann*

(a) existiert eine stetige, nichtfallende Stoppgrenze $b : [0,T) \to \mathbb{R}_+$, sodass die optimale Stoppregion gegeben ist durch

$$\mathcal{S} = \begin{cases} [0,T) \times \mathbb{R}_+ & \text{if } \mu \leq (1-\alpha)r \\ \{(t,x) \in [0,T] \times \mathbb{R}_+ : x \leq b(t)\} & \text{if } (1-\alpha)r < \mu, \end{cases} \tag{11}$$

wobei für alle $t \in [0,T)$, die folgende Äquivalenz gilt: $\alpha > 0 \Leftrightarrow b(t) > 0$. Die Stoppgrenze erfüllt die Endbedingung

$$\lim_{t \uparrow T} b(t) = \frac{r\alpha P_0}{\mu - r(1-\alpha)} =: f.$$

(b) erfüllt die Wertefunktion die "Smooth-Fit"-Bedingung an der Stoppgrenze für $\alpha > 0$ und $\mu > (1-\alpha)r$, d.h.

$$\partial_x V(t,x) = \partial_x G(t,x) = (1-\alpha)e^{r(1-\alpha)(T-t)} \quad \text{für } x = b(t). \tag{12}$$

Um die Aussage zu beweisen, unterscheidet man die 3 Fälle $\mu \leq r(1-\alpha)$, $\mu > r(1-\alpha)$ und $\alpha = 0$, $\mu > r(1-\alpha)$ und $\alpha > 0$. In den ersten beiden Fällen sind die Lösungen jeweils trivial. So stoppt man im ersten Fall sofort und im zweiten Fall im Endzeitpunkt. Im nicht-trivialen dritten Fall lässt sich leicht die Existenz der echt positiven, nichtfallenden Stoppgrenze b zeigen. Um die Stetigkeit der Stoppgrenze zu zeigen, nutzt man die Stetigkeit der Wertefunktion V und der Auszahlungsfunktion G aus. Die Rechtsstetigkeit folgt dabei sofort aus der Monotonie der Stoppgrenze. Um die Linksstetigkeit zu zeigen, nehme man an, dass die Stoppgrenze zu einem Zeitpunkt t^* einen Sprung hat. Wir beginnen damit eine obere Schranke für die $t-$Ableitung und die $x-$Ableitung von V in einer linken Umgebung von t^* und Preisen zwischen $b(t^*-)$ und $b(t^*)$ zu finden. Mithilfe der Newton-Leibniz Formel lässt sich $V - G$ als Integral über die zweite Ableitung von $V - G$ darstellen. Wir nutzen aus, dass V in $\mathcal{C}$ die partielle Differentialgleichung

$$\partial_t V + \mu x \partial_x V + \frac{\sigma^2}{2} x^2 \partial_{xx} V = 0 \tag{13}$$

bezüglich der Randwertbedingung (9) erfüllt. Da $\partial_{xx}G = 0$ ist und sich $\partial_t V$ und $\partial_x V$ in der Nähe des Sprungs so gegeneinander abschätzen lassen, dass $V_{xx} - G_{xx} > C, C > 0$ (und damit auch $V - G > C$) führt dies, wegen der Stetigkeit von V und G, zu einem Widerspruch, da $V = G$ in der Stoppregion $\mathcal{S}$ gelten muss.

Statt einer stetigen Versteuerung im Bond, kann man alternativ annehmen, dass die Kapitalerträge wie bei der Aktie erst versteuert werden, sobald diese realisiert sind. In diesem Fall erhält man die folgende Auszahlungsfunktion

$$\widetilde{G}(t,x) = \big[(1-\alpha)x + \alpha P_0\big]\big[1 + (1-\alpha)(e^{r(T-t)} - 1)\big] \tag{14}$$

für $(t,x) \in [0,T] \times \mathbb{R}_+$. Der Unterschied zwischen den Auszahlungsfunktionen G und $\widetilde{G}$ beruht also darauf, dass die Erträge des Bonds für (5) stetig und für (14) am Laufzeitende versteuert werden. Das modifizierte optimale Stoppproblem für die Investorin ist nun

$$\widetilde{V}(t,x) = \sup_{\tau \in \mathcal{T}_{[0,T-t]}} \mathbb{E}\big\{\widetilde{G}(t+\tau, X^{t,x}_{t+\tau})\big\}, \tag{15}$$

wobei $(X^{t,x}_s)_{s\in[t,T]}$ Lösung der Gleichung (4) mit Anfangsbedingung $X^{t,x}_t = x$ ist.
Die Eigenschaften (8), (9) und (13) gelten für (15) in der gleichen Weise wie für (7). Die Stopp- und die Fortsetzungsregion sind ebenfalls gleich definiert und werden mit $\widetilde{\mathcal{S}}$ und $\widetilde{\mathcal{C}}$ bezeichnet. Ebenso lässt sich zeigen, dass die optimale Stoppzeit gegeben ist durch

$$\widetilde{\tau}_{t,x} := \inf\{s \in [0, T-t] : (t+s, X^{t,x}_{t+s}) \in \widetilde{\mathcal{S}}\}.$$

Die Existenz der Stoppgrenze $\widetilde{b}$ für Problem (15) lässt sich wie für Problem (7) zeigen. Man stellt jedoch leider fest, dass die Stoppgrenze in (15) nicht monoton sein muss. Somit ist es mit den im Problem (7) verwendeten Standardmethoden nicht möglich, bestimmte Eigenschaften wie Stetigkeit von $\widetilde{b}$ nachzuweisen.
Ein Vergleich der beiden Stoppgrenzen für (15) und (7) zeigt dann, dass die Stoppgrenze für (15) die Stoppgrenze für (7) unter gewissen Voraussetzungen an μ dominiert.

Proposition 2. *Sei b die Stoppgrenze im Optimierungsproblem (7) und $\widetilde{b}$ die Stoppgrenze im Optimierungsproblem (15). Dann wird b durch $\widetilde{b}$ dominiert, d.h. $b \le \widetilde{b}$ falls*

$$\frac{(1-\alpha)r}{\alpha e^{-rT} + (1-\alpha)} < \mu < r.$$

Außerdem lässt sich die Monotonie der Wertefunktion in σ für beide Stoppprobleme nachweisen, was eine umgekehrte Monotonie der Stoppgrenze in σ nach sich zieht. Im Fall $\sigma = 0$ lässt sich die Stoppgrenze sogar explizit angeben.

Proposition 3. *Die Wertefunktion ist wachsend in der Volatilität der Aktie, d.h.* $V^{\sigma_1}(t,x) \leq V^{\sigma_2}(t,x)$ *für alle* $0 \leq \sigma_1 \leq \sigma_2$, $t \in [0,T]$ *und* $x \in \mathbb{R}_+$. *Damit folgt* $b^{\sigma_2}(t) \leq b^{\sigma_1}(t)$. *Im Fall* $\sigma = 0$ *ist die Stoppgrenze für (7) gegeben durch*

$$b(t) = \frac{\alpha P_0 \left(e^{r(1-\alpha)(T-t)} - 1\right)}{(1-\alpha)\left(e^{\mu(T-t)} - e^{r(1-\alpha)(T-t)}\right)}$$

und für (15) gegeben durch

$$\widetilde{b}(t) = \frac{\alpha P_0 (1-\alpha)\left(e^{r(T-t)} - 1\right)}{(1-\alpha)\left(e^{\mu(T-t)} - 1 - (1-\alpha)\left(e^{r(T-t)} - 1\right)\right)}.$$

Proposition 3 macht auch aus ökonomischer Sicht Sinn. Die Option der Investorin selbst zu bestimmen, wann die Steuern gezahlt werden, hat für die Investorin einen höheren Wert je volatiler das Wertpapier ist. Dies kann die Investorin motivieren, mehr Risiko auf sich zu nehmen. Das Ausmaß des Effekts hängt dabei vom risikolosen Zinssatz r ab und verschwindet für $r = 0$.

Bereits veröffentlichte Inhalte

Die Inhalte dieser Dissertation sind in den Arbeiten [KU14; KSU14] veröffentlicht bzw. zur Veröffentlichung eingereicht.

List of Figures

Bibliography

[Asm03] S. Asmussen. *Applied Probability and Queues*. Vol. 51. Springer, 2003.

[AL03] M. Avellaneda and M.D. Lipkin. “A Market-induced Mechanism for Stock Pinning”. In: *Quantitative Finance* 3.6 (2003), pp. 417–425.

[Bau+14] E.J. Baurdoux, N. Chen, B.A. Surya, and K. Yamazaki. “Double Optimal Stopping of a Brownian Bridge”. Preprint, available at `http://arxiv.org/abs/1409.2226`. 2014.

[BSV11] M. Beiglboeck, W. Schachermayer, and B. Veliyev. “A Direct Proof of the Bichteler-Dellacherie Theorem and Connections to Arbitrage”. In: *Annals of Probability* 39 (2011), pp. 2424–2440.

[BST07] I. Ben Tahar, M. Soner, and N. Touzi. “The Dynamic Programming Equation for the Problem of Optimal Investment under Capital Gains Taxes”. In: *SIAM Journal on Control and Optimization* 46.5 (2007), pp. 1779–1801.

[BST10] I. Ben Tahar, M. Soner, and N. Touzi. “Merton Problem with Taxes: Characterisation, Computation and Approximation”. In: *SIAM Journal on Financial Mathematics* 1.1 (2010), pp. 366–395.

[Ber91] B.D. Bernheim. “Tax Policy and the Dividend Puzzle”. In: *RAND Journal of Economics* 22 (1991), pp. 455–476.

[Bla76] F. Black. “The dividend puzzle”. In: *Journal of Portfolio Management* (1976), pp. 5–8.

[BCP07] C. Buescu, A. Cadenillas, and S.R. Pliska. “A Note on the Effects of Taxes on Optimal Investment”. In: *Mathematical Finance* 17.4 (2007), pp. 477–485.

[CP99] A. Cadenillas and S. Pliska. “Optimal trading of a security when there are taxes and transaction costs”. In: *Finance Stoch.* 3 (1999), pp. 137–165.

[Con83] G. M. Constantinides. “Capital Market Equilibrium with Personal Taxes”. In: *Econometrica* 51.2 (1983), pp. 611–636.

[CRR79] J.C. Cox, S.A. Ross, and M. Rubinstein. "Option Pricing: a Simplified Approach". In: *Journal of financial Economics* 7.3 (1979), pp. 229–263.

[DZ12] M. Dai and Y. Zhong. "Optimal Stock Selling/Buying Strategy with Reference to the Ultimate Average". In: *Mathematical Finance* 22.1 (2012), pp. 165–184.

[DU05] V. DeMiguel and R. Uppal. "Portfolio Investment with the Exact Tax Basis via Nonlinear Programming". In: *Management Science* 51.2 (2005), pp. 277–290.

[DTP09] J. Du Toit and G. Peskir. "Selling a Stock at the Ultimate Maximum". In: *The Annals of Applied Probability* 19.3 (2009), pp. 983–1014.

[DK96] P. Dybvig and H.K. Koo. "Investment with Taxes". Washington University, Working Paper. 1996.

[Fri64] A. Friedman. *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Florida, 1964.

[Hön75] C.S. Hönig. *Volterra Stieltjes-Integral Equations: Functional Analytic Methods, Linear Constraints*. North-Holland Publishing Company, 1975.

[Jac79] J. Jacod. *Calcul Stochastique et Problemes de Martingales*. Vol. 714. Springer, 1979.

[JS03] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. second. Vol. 288. Fundamental Principles of Mathematical Sciences. Springer, Berlin, 2003.

[Jas03] S. Jaschke. "A Note on the Inhomogeneous Linear Stochastic Differential Equation". In: *Insurance: Mathematics and Economics* 32 (2003), pp. 461–464.

[JKT99] E. Jouini, P.F. Koehl, and N. Touzi. "Optimal Investment with Taxes: an Optimal Control Problem with Endogenous Delay". In: *Nonlinear Analysis* 37.4 (1999), pp. 31–56.

[JKT00] E. Jouini, P.F. Koehl, and N. Touzi. "Optimal Investment with Taxes: an Existence Result". In: *Journal of Mathematical Economics* 33 (2000), pp. 373–388.

[Kat91] G.S. Katzenberger. "Solutions of a Stochastic Differential Equation Forced Onto a Manifold by a Large Drift". In: *Annals of Probability* 19 (1991), pp. 1587–1628.

[KSU14] C. Kühn, B.A. Surya, and B. Ulbricht. "Optimal Selling Time of a Stock under Capital Gains Taxes". Preprint, available at `http://arxiv.org/abs/1501.00026`. 2014.

[KU14] C. Kühn and B. Ulbricht. "Modeling Capital Gains Taxes for Trading Strategies of Infinite Variation". Preprint, available at http://arxiv.org/abs/1309.7368. 2014.

[Lea+00] R.C. Lease, K. John, A. Kalay, U. Loewenstein, and O.H. Sarig. *Dividend Policy: its Impact on Firm Value.* Harvard Business School Press, Boston, 2000.

[Mer71] R.C. Merton. "Optimum Consumption and Portfolio Rules in a continuous-time Model". In: *Journal of Economic Theory* 3 (1971), pp. 373–413.

[MM61] M.H. Miller and F. Modigliani. "Dividend Policy, Growth, and the Valuation of Shares". In: *The Journal of Business* 34.4 (1961), pp. 411–433.

[PS06] G. Peskir and A. Shiryaev. *Optimal Stopping and Free-boundary Problems.* Springer, New York, 2006.

[Pro04] P.E. Protter. *Stochastic Integration and Differential Equations.* Second. Springer, 2004.

[Sei10] F.T. Seifried. "Optimal Investment with Deferred Capital Gains Taxes". In: *Mathematical Methods of Operations Research* 71.1 (2010), pp. 181–199.

[Shi07] A. Shiryaev. *Optimal Stopping Rules.* Vol. 8. Springer, New York, 2007.

[SXZ08] A. Shiryaev, Z. Xu, and X.Y. Zhou. "Thou Shalt Buy and Hold". In: *Quantitative Finance* 8.8 (2008), pp. 765–776.

[YY77] Ch. Yoeurp and M. Yor. "Espace Orthogonal à une Semimartingale: Applications". Unpublished. 1977.

[Yor79] M. Yor. "Sur le Balayage des Semimartingales Continues". In: *Séminaire de Probabilités, XIII, Lecture Notes in Mathematics, Springer, 721* (1979), pp. 453–471.