Symmetry via maximum principles

for nonlocal nonlinear boundary value problems

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Contents

1 Introduction

Motivation In the qualitative analysis of solutions of partial differential equations, many interesting questions are related to the shape of solutions. In particular, the symmetries of a given solution are of interest. One of the first more general results in this direction was given in 1979 by Gidas, Ni and Nirenberg [47, Theorem 1]: *Let $B \subset \mathbb{R}^N$ be a ball of radius $R > 0$ centered in 0 and let $u \in C^2(\overline{B})$ be a strictly positive solution of*

$$\begin{cases} -\Delta u = f(u) & \text{in } B; \\ \quad u = 0 & \text{on } \partial B, \end{cases} \tag{1.1}$$

where f is continuously differentiable. Then u is radially symmetric and strictly decreasing in the radial direction. The main tool in proving this symmetry and monotonicity result is the moving plane method. This method, which goes back to Alexandrov's work on constant mean curvature surfaces in 1962 (see [1]), was introduced in 1971 by Serrin (see [65]) in the context of partial differential equations to analyze an overdetermined problem.
The situation is more complicated in the case of parabolic equations of the type

$$\begin{cases} \partial_t u = \Delta u + f(t,u) & \text{in } (0,\infty) \times B; \\ \quad u = 0 & \text{on } (0,\infty) \times \partial B, \end{cases} \tag{1.2}$$

where $f : (0,\infty) \times [0,\infty) \to \mathbb{R}$ is continuously differentiable w.r.t. u and continuous in t. It is clear that (1.2) may have positive solutions without any spatial symmetries at finite times $t > 0$. Thus symmetry in finite time cannot be expected. Therefore, to analyze the asymptotic shape of a given solution u, it is natural to study its ω-limit set

$$\omega(u) := \{z \in C(B) \ : \ \exists (t_k)_{k\in\mathbb{N}} \subset [0,\infty),\, t_k \to \infty \text{ for } k \to \infty, \text{ such that } \lim_{k\to\infty} \|u(t_k,\cdot) - z\|_\infty = 0\}.$$

Global boundedness and equicontinuity of a solution ensure that $\omega(u)$ is nonempty. Here we will use that for a global (in time) solution u the functions $u(\tau + \cdot, \cdot)$, $\tau \geq 1$ are equicontinuous on $[0,1] \times B$, i.e.

$$\lim_{h\to 0} \sup_{\substack{\tau \geq 1 \\ x,\tilde{x} \in \overline{B},\, t,\tilde{t} \in [\tau,\tau+1], \\ |x-\tilde{x}|,|t-\tilde{t}|<h}} |u(t,x) - u(\tilde{t},\tilde{x})| = 0. \tag{1.3}$$

If f does not depend on t, then (1.2) admits a Lyapunov functional, and thus it is easy to see that all elements of $\omega(u)$ are solutions of (1.1). Hence, as a consequence of the above-mentioned result in [47], global bounded and equicontinuous solutions of (1.2) are *asymptotically symmetric*. However, if f depends on t, then in general $\lim_{t\to\infty} u(t,\cdot)$ does not exist and even if there are convergent subsequences, their limits will in general not solve an elliptic equation of type (1.1). Under the assumption that u is a global bounded classical nonnegative solution of (1.2), which satisfies (1.3) and that there is at least one strictly positive element in $\omega(u)$, Poláčik proves in [56] that all elements in $\omega(u)$ are radially symmetric and strictly decreasing in the radial direction. This is a special case of a more general result for parabolic equations of second order. For a general survey and further details we refer to [57].

Reaction-diffusion equations like (1.2) have a broad relevance to many fields as they describe the large scale behavior of an observed object. However, from a small scale observation in comparison to the large scale, there are situations in which diffusion does not describe appropriately the behavior of the observed object. On small scale, a diffusion process describes more or less a random movement in any direction where this movement happens in a continuous way. But there are examples where this spreading occurs in a non-continuous way. One of these examples comes from biology: In 2006, Brockmann, Hufnagel and Geisel [15] studied the spreading of diseases. They constructed a model for the travel behavior of money in the United States and related it to the transmission of a disease. Interestingly this is not described by a continuous diffusion (or local diffusion) but rather by a process with jumps. As a result, the process is not localized since the observed object – in this case money – is able to move in a short time very far from its origin. The spreading thus occurs in a discontinuous way. We will call this spreading a *nonlocal diffusion* (or *discontinuous diffusion*).
From the stochastic point of view nonlocal diffusions can be described by a certain class of Markov processes called Hunt processes which is a broader class of processes than the class of diffusion processes. By the Markov property it is possible to use the transition probability of the process to describe a semigroup which acts on bounded measurable functions (see [3, Chapter 3]). In case of Hunt processes, these semigroups enable us to define a bilinear form that is associated to the process and for nonlocal diffusion processes this leads to nonlocal bilinear forms (see e.g. [44, 45]).

Presentation of the main results The focus of the present work will be on equations related to purely nonlocal bilinear forms in divergence form. To be precise, we will consider bilinear forms such as

$$\mathscr{J}(u,v) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))(v(x)-v(y))J(x-y)\,dxdy, \tag{1.4}$$

where $J : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ is a measurable function with the following properties:

(A1) J is even, nonnegative, and it satisfies

$$\int_{\mathbb{R}^N} J(y)\,dy = \infty \qquad \text{and} \qquad \int_{\mathbb{R}^N} \min\{1,|y|^2\}J(y)\,dy < \infty.$$

We will call J a kernel function. From the stochastic point of view, J is related to the intensity of the jumps of the associated process (see [44, 45]). Corresponding to such a bilinear form there is an operator I which is given in the following way: Let $u \in C_c^2(\mathbb{R}^N)$, then for $x \in \mathbb{R}^N$ we have

$$Iu(x) = P.V.\int_{\mathbb{R}^N} (u(x)-u(y))J(x-y)\,dy := \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} (u(x)-u(y))J(x-y)\,dy. \tag{1.5}$$

Note that the value of Iu in a point $x \in \mathbb{R}^N$ depends on all values of u in $\mathbb{R}^N$. Thus a boundary value problem related to I is not well-posed under classical Dirichlet boundary conditions. To set up a well-defined problem *nonlocal* Dirichlet boundary conditions are used, i.e. we consider for $\Omega \subset \mathbb{R}^N$, $N \geq 1$ open:

$$\begin{cases} Iu = f(x,u) & \text{in } \Omega; \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.6}$$

and similarly for the time dependent problem

$$\begin{cases} \partial_t u + Iu = f(t,x,u) & \text{in } (0,\infty) \times \Omega; \\ u = 0 & \text{on } [0,\infty) \times (\mathbb{R}^N \setminus \Omega). \end{cases} \tag{1.7}$$

In the special case where $J(z) = c_{N,s}|z|^{-N-2s}$, with $c_{N,s} = s(1-s)4^s\pi^{-N/2}\Gamma(s+N/2)\Gamma^{-1}(2-s)$ for some $s \in (0,1)$ this operator is the so called fractional Laplacian $(-\Delta)^s$. This operator appears in particular in the model of Brockmann et al. [15] with $s \approx 0.6$. Aside from biology, other models using the fractional Laplacian appear in some competing systems (see e.g. [73, 74] and the references therein) or in the study of porous media (see e.g. [31] and the references therein). Moreover, nonlocal diffusions also appear in quantum physics (see e.g. [54, 77] and the references therein) and in finance (see e.g. [3, 69] and the references therein). For other applications, see also [24, 75] and the references therein.

The aim of this work is to study symmetry properties of solutions of equations of type (1.6) and (1.7). Here we use a weak notion of solution (see Definition 2.8 and 6.1 below). At first glance the nonlocal structure leads to additional difficulties since the equation cannot be localized anymore. In particular, when using the moving plane method we will not be able to deal with the usual notion of supersolutions. This is due to the fact that the difference between the solution and its reflection about a hyperplane is antisymmetric. If u is antisymmetric about a hyperplane T the values of u of both sides of the hyperplane will contribute to the value of Iu and cannot be ignored. We will thus prove various versions of maximum principles for antisymmetric supersolutions of linear equations. Moreover, we will extend a recent result from Felsinger and Kaßmann [40] to the antisymmetric case, i.e. we prove a weak parabolic Harnack inequality for *antisymmetric supersolutions* of linear nonlocal problems, where the kernel functions J of the operator is comparable in some sense to the kernel function of the fractional Laplacian (see assumption (A2) below).
These tools enable us to apply the moving plane method in the nonlocal setting with bilinear forms and – under suitable monotonicity and symmetry assumptions on J, f and Ω – extend classical symmetry results to the nonlocal case. To state our main results, we need to introduce further assumptions on the kernel function J, the underlying open set Ω and the nonlinearity f.

(A2) There is $s \in (0,1)$ and $c \geq 1$ such that

$$c^{-1}|y|^{-N-2s} \leq J(y) \leq c|y|^{-N-2s} \qquad \text{for all } y \in \mathbb{R}^N.$$

(A3) J is strictly monotone in $|x_1|$ in the sense that for all $s,t \in \mathbb{R}$ with $|s| < |t|$ we have

$$\underset{z' \in B_r^{N-1}(0)}{\operatorname{essinf}} \big(J(s,z') - J(t,z')\big) > 0 \qquad \text{for all } r > 0.$$

We will assume in addition that Ω fulfills

(A4) $\Omega \subset \mathbb{R}^N$ is an open bounded set which is Steiner symmetric in x_1, i.e. for every $x \in \Omega$ and $s \in [-1,1]$ we have $(sx_1, x_2, \ldots, x_N) \in \Omega$.

(A5) For every $\lambda > 0$, the set $\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\}$ has at most finitely many connected components.

The nonlinearity is a function $f : [0,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$, which satisfies

(A6) f is continuous. Moreover, for every $K > 0$ there is $L = L(K) > 0$ such that

$$\sup_{x\in\Omega, t>0} |f(t,x,u) - f(t,x,v)| \leq L|u-v| \qquad \text{for } u,v \in [-K,K].$$

(A7) f is symmetric in x_1 and monotone in $|x_1|$, i.e. for every $t \in [0,\infty)$, $u \in \mathbb{R}$, $x \in \Omega$ and $s \in [-1,1]$ we have $f(t, sx_1, x_2, \ldots, x_N, u) \geq f(t,x,u)$.

Our first result is devoted to the time-dependent problem (1.7).

Theorem 1.1. *Let (A1)–(A4), (A6), (A7) be satisfied, and let u be a nonnegative global solution of (1.7) satisfying the following conditions:*

(A8) There is $c_u \in \mathbb{R}$ such that $\|u(t)\|_{L^\infty} \leq c_u$ for every $t > 0$.

(A9) The functions $u(\tau + \cdot, \cdot)$, $\tau \geq 1$ are equicontinuous on $[0,1] \times \Omega$, i.e. u satisfies (1.3) with Ω in place of B.

Suppose in addition that (A5) holds <u>or</u> that $z \not\equiv 0$ for every $z \in \omega(u)$.
Then u is asymptotically symmetric in x_1, i.e. for all $z \in \omega(u)$ we have $z(-x_1, x') = z(x_1, x')$ for all $(x_1, x') \in \Omega$.
Moreover, for every $z \in \omega(u)$ we have the following alternative: Either $z \equiv 0$ on Ω, or z is strictly decreasing in $|x_1|$ and therefore strictly positive in Ω.

Theorem 1.1 is a special case of Theorem 10.1 below. Moreover, in Section 8 we will discuss a specific example where Theorem 1.1 applies.
In the case where f does not depend on t, Theorem 1.1 immediately yields the corresponding symmetry and monotonicity property for equilibrium solutions. In fact, we can derive these properties for solutions of (1.6) under much weaker assumptions, as our second main result shows.

Theorem 1.2. *(see Theorem 4.1 below) Let (A1), (A3), (A4) be satisfied, and assume that the nonlinearity f has the following properties.*

(A10) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$ is a Carathéodory function such that for every $K > 0$ there is $L = L(K) > 0$ with

$$\sup_{x\in\Omega} |f(x,u) - f(x,v)| \leq L|u-v| \qquad \textit{for } u,v \in [-K,K].$$

(A11) *f as in (A10) is symmetric in x_1 and monotone in $|x_1|$, i.e. for every $u \in \mathbb{R}$, $x \in \Omega$ and $s \in [-1,1]$ we have $f(sx_1, x_2, \ldots, x_N, u) \geq f(x,u)$.*

Then every nonnegative bounded solution u of (1.6) with $\mathscr{J}(u,u) < \infty$ is symmetric in x_1. Moreover, either $u \equiv 0$ in $\mathbb{R}^N$, or

$$\operatorname*{essinf}_K u > 0 \qquad \text{for every compact set } K \subset \Omega.$$

Moreover, in the latter case, u is strictly decreasing in $|x_1|$, i.e. for every $\lambda \in \left(0, \sup_{x\in\Omega} x_1\right)$ and every compact set $K \subset \{x \in \Omega : x_1 > \lambda\}$ we have

$$\operatorname*{essinf}_{(x_1,x')\in K} \Big[u(2\lambda - x_1, x') - u(x_1, x')\Big] > 0.$$

This result has already been published in [50, Theorem 1.1]. We point out two key differences which distinguish the above Theorems from their classical counterparts in the local case. First, we do not need the underlying set Ω to be connected, which is obviously a necessary requirement in the local case. Second, we do not need to assume the strict positivity of a solution; this property follows a posteriori if the solution is nontrivial. Such a conclusion is also not available in the local case. In particular, as shown by Poláčik and Terracini [58], there exist planar Steiner symmetric smooth domains Ω and symmetric nonlinearities such that the corresponding Dirichlet problem of the type (1.1) on Ω admits nontrivial nonnegative solutions with interior zeros.
It is still an open question, whether a solution of (1.6) is continuous under these assumptions on J, this is why we need to use essential infima in Theorem 1.2. If we assume J to be also radial, then the continuity follows from recent results by Kaßmann and Mimica (see [53]).
Finally, in Section 10 below, we will also give symmetry results for globally bounded time-periodic positive solutions of (1.7) for the case $\Omega = \mathbb{R}^N$.

Let us compare the above Theorems with related results in the literature. The following works consider positive solutions for problems of type (1.6) in the case $I = (-\Delta)^s$. One of the first symmetry results was given by Birkner, Lòpez-Mimbela and Wakolbinger in [9] in a ball using a stochastic setting and assuming additionally that f is independent of x and monotone in u. Chen, Li and Ou [27] analyze (1.6) with the fractional Laplacian in $\mathbb{R}^N$ and with $f(u) = u^{\frac{N+2s}{N-2s}}$ via the inverse operator of the fractional Laplacian. They apply the moving plane method for a related integral equation and show that such solutions have to be radially symmetric up to translation. In [28] they generalize this application to equations of type (1.6), where Ω is a ball but with a more general right-hand side than in [9]. Their symmetry result relies strongly on the explicit representation of the Green function for $(-\Delta)^s$ in the ball. Felmer, Quaas and Tan apply the moving plane method in [37] to prove radial symmetry for classical positive solutions of (1.6) in $\mathbb{R}^N$ under some growth conditions on f in u and in [38] the authors consider classical positive solutions of (1.6), where Ω is either a ball in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. Barrios, Montoro and Sciunzi [5] prove symmetry results where the nonlinearity $f(\cdot,u)$ is allowed to have a singularity in 0 of order $2s$.

The observation that the statements are also true if one only assumes the solution to be bounded and nonnegative was first made in [49, Corollary 1.2].

Up to now, the only asymptotic symmetry result for equations of type (1.7) in the literature is contained in [49], where we prove Theorem 1.1 in the special case $I = (-\Delta)^s$. As mentioned before, the proofs of Theorems 1.1 and 1.2 rely on different versions of maximum principles and Harnack inequalities, in particular variants dealing with antisymmetric functions which enter in stationary or time-dependent variants of a moving plane type method inspired in particular by Poláčik [56]. To a large extent, the proof of Theorem 1.1 follows the approach we have already developed in [49]. However, one step of the proof in [49] requires a subsolution estimate obtained by identifying $(-\Delta)^s$ as a Dirichlet-to-Neumann map via the extension considered in [22]. Such an identification is not available in the general case. We present here an alternative method which is based only on the nonlocal structure of the bilinear form. The proof of Theorem 1.2, as presented in Section 4 below and also in [50], only uses weak and strong forms of maximum principles and does not require a Harnack type inequality.
We emphasize that in this work we only use properties of the bilinear form associated to the kernel function as stated in (1.4). We do not need Green functions or an extension problem for our analysis.
In [65], Serrin considered the overdetermined problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega; \\ \partial_\eta u = c & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $c \in \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary and $\eta : \partial\Omega \to S^1$ is the outer normal unit vector field on $\partial\Omega$. By a variant of the moving plane method which relies on the Hopf boundary point lemma and a corner point lemma, he proved that this problem only admits a solution if Ω is a ball. In [36], we derive the following fractional version of Serrin's result.

Theorem 1.3. *(see Theorem 5.9 below) Let $s \in (0,1)$ and $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be an open bounded set such that $\partial\Omega$ is C^2 and assume that there is a solution $u \in C^s(\mathbb{R}^N)$ of*

$$(-\Delta)^s u = 1 \quad \text{in } \Omega, \qquad u \equiv 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.$$

If there is a negative real number c such that

$$\lim_{t\to 0} \frac{-u(x_0 - t\eta(x_0))}{t^s} \equiv c \quad \text{for all } x_0 \in \partial\Omega,$$

then Ω is a ball.

We point out that here, as in Theorems 1.1 and 1.2 above, Ω does not need to be connected a priori. In [30] Theorem 1.3 is proven for the special case $N = 2$ and $s = \frac{1}{2}$. While the approach in [30] relies on the extension in [22], our approach to the overdetermined fractional problem is based purely on the corresponding nonlocal bilinear form and the availability of explicit comparison functions given by Dyda in [33]. In particular, we derive fractional counterparts both of the Hopf lemma and of Serrin's corner point lemma (see [36] or Subsection 5.1).

Related problems We note that there are nonlocal operators I which are not given by a bilinear form of the type (1.4). Two main examples are the operator $(-\Delta)^{s(\cdot)}$, where $s:\mathbb{R}^N\to(0,1)$ is a Lipschitz function (see e.g. [7, 6]) or the operator $(-\Delta|_D)^s$ for a bounded Lipschitz set $D\subset\mathbb{R}^N$ (see e.g. [19, 25]), where $(-\Delta|_D)^s$ denotes the spectral theoretic s-th power of the Dirichlet Laplacian on D. To the authors' knowledge the question of similar symmetry results as above for equations involving $(-\Delta)^{s(\cdot)}$ remains open. In [25] the authors prove a symmetry result for equations involving $(-\Delta|_D)^s$ for the stationary problem in the case where Ω is a ball.

Outline This work is organized in the following way. After this introduction we give some basic notations used in this report. In Section 2 we introduce the class of nonlocal operators that are of interest to us and define suitable spaces on which we analyze them. Several basic results for these spaces and some basic properties of these nonlocal operators are given. Section 3 is devoted to different variants of maximum principles and we also state some applications of these variants. Using the results of the previous sections we prove Theorem 1.2 and a generalization in Section 4. As an example we consider the fractional Laplacian in Section 5. As part of the analysis for the fractional Laplacian, we prove the fractional Hopf Lemma and use it to investigate an overdetermined problem which involves the fractional Laplacian. There we give the proof to Theorem 1.3. We also discuss operators which are related to the fractional Laplacian and show that they have a similar regularizing effect. In Section 6 we introduce a nonlocal evolutionary problem and prove a weak time dependent maximum principle. Moreover, we develop an L^2-theory for the nonlocal Cauchy problem with initial data in L^2. The purpose of Section 7 is to state the weak time dependent Harnack inequality as proven in [40] for the time dependent problem and show that it implies interior Hölder regularity if the right-hand side is bounded. We also discuss the ω-limit set for global (in time) solutions. Using results of the previous sections we present in Section 8 a setting such that the nonlinear evolutionary problem is locally solvable in the space of continuous functions. We also discuss the existence of global (in time) solutions. In Section 9 we prove the main results on time dependent antisymmetric supersolutions. These results constitute the main ingredients to apply the moving plane method in the time dependent case. Section 10 is dedicated to state and prove our main symmetry results for the time dependent case. In particular, the results presented in this section imply Theorem 1.1.

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1.1 Notation

The following notation is used. For any $x \in \mathbb{R}^N$ we put $|x| = |x|_2 = \sqrt{\sum_{i=1}^N |x_i|^2}$. If $D, U \subset \mathbb{R}^N$ are subsets, the notation $D \subset\subset U$ means that $\overline{D}$ is compact and contained in the interior of U. Moreover, we set

$$\operatorname{dist}(D,U) := \inf(|x-y| \, : \, x \in D, \, y \in U\},$$

so this notation does *not* stand for the usual Hausdorff distance. If $D = \{x\}$ is a singleton, we simply write $\operatorname{dist}(x,U)$ in place of $\operatorname{dist}(\{x\},U)$.
For $U \subset \mathbb{R}^N$ and $r > 0$ the set

$$B_r(U) := B_{r,N}(U) := \{x \in \mathbb{R}^N \; : \; \operatorname{dist}(x,U) < r\}$$

is an open neighborhood of U. In particular, for $x \in \mathbb{R}^N$ and $r > 0$ we set $B_r(x) = B_r(\{x\})$. We denote the Euclidean unit sphere by $S^1 := \partial B_1(0)$ and

$$\omega_N := |B_1(0)| = |B_1^N(0)| = \frac{\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}$$

will denote the volume of the N-dimensional unit ball. Here and throughout this work $r \mapsto \Gamma(r) := \int_0^\infty t^{r-1} e^{-t} \, dt$ denotes the Gamma function on $(0,\infty)$.
For any subset $M \subset \mathbb{R}^N$ we denote by $1_M : \mathbb{R}^N \to \mathbb{R}$ the characteristic function of M and by $\operatorname{diam}(M)$ the diameter of M. If M is measurable, then $|M|$ denotes the Lebesgue measure of M. Moreover, we let $\operatorname{inrad}(M)$ denote the supremum of all $r > 0$ such that *every* connected component of M contains a ball $B_r(x_0)$ with $x_0 \in M$. This notation – taken from [56] – differs slightly from the usual one but is very convenient in our setting.
If $T \subset \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ are subsets and $u : T \times \Omega \to \mathbb{R}$, $(t,x) \mapsto u(t,x)$ is a function, we frequently write $u(t)$ in place of $u(t,\cdot) : \Omega \to \mathbb{R}$ for $t \in T$. If $M \subset \mathbb{R}^N$ resp. $M \subset \mathbb{R}^{N+1}$ is a subset and $w : M \to \mathbb{R}$ is a function we set $w^+ = \max\{w,0\}$ resp. $w^- = -\min\{w,0\}$ as the positive and negative part of w, respectively.
If M is measurable and $w \in L^1(M)$, we put

$$[w]_{L^1(M)} := \frac{1}{|M|} \int\limits_M w(x) \, dx, \qquad [w]_{L^1(M)} := \frac{1}{|M|} \int\limits_M w(t,x) \, dt dx, \qquad \text{respectively,}$$

to denote the mean of w over M.
Finally, when we call an interval $T \subset \mathbb{R}$ a *time interval*, we always assume that it consists of more than one point.

Occasionally we will state main assumptions on sets or functions which we will need throughout this work. For the readers convenience these are also listed in order of appearance in the appendix starting on p. I.

2 A general class of nonlocal operators

We will study the following nonlocal and semilinear Dirichlet problem in an open set $\Omega \subset \mathbb{R}^N$:

$$(P) \qquad \begin{cases} Iu = f(x,u) & \text{in } \Omega; \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Here I is a nonlocal linear operator and the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function[1]. We will add further assumptions on this nonlinearity as we need them.
Problem (P) has been studied extensively with $I = (-\Delta)^s$, the fractional Laplacian of order $s \in (0,1)$. In this case special properties of the fractional Laplacian have been used to study existence, regularity and symmetry of solutions to (P). In particular, some approaches rely on available Green function represention associated with $(-\Delta)^s$ (see e.g. [11, 12, 9, 13, 28, 27] or Section 5), whereas other techniques are based on a representation of $(-\Delta)^s$ as a Dirichlet-to-Neumann map (see e.g [22, 18, 43]). These useful features of the fractional Laplacian are closely linked to its isotropy and its scaling laws. However, in the modeling of anisotropic diffusion phenomena and of processes which do not exhibit similar properties, it is necessary to study more general nonlocal operators I. In this spirit general classes of nonlocal operators have been considered e.g. in [4, 40, 41, 66, 52, 50, 53, 16, 20, 23].

In the present work we consider (P) for a class of nonlocal operators I which includes the fractional Laplacian but also more general operators which may be anisotropic and may have varying order. More precisely, the class of operators I we study is related to nonnegative nonlocal bilinear forms of the type

$$\mathscr{J}(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x)-u(y))(v(x)-v(y))J(x,y)\, dxdy. \tag{2.1}$$

Here $J : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,x) \,:\, x \in \mathbb{R}^N\} \to [0,\infty)$ is a measurable *(kernel) function* which fulfills the following conditions:

(J1)$_a$ $\quad J(x,y) = J(y,x) \quad$ for $x,y \in \mathbb{R}^N$, $x \neq y$.

(J1)$_b$ $\quad \sup\limits_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1, |x-y|^2\} J(x,y)\, dy < \infty.$

(J1)$_c$ There is a measurable function $j : \mathbb{R}^N \to [0,\infty)$ with $|\{j > 0\}| > 0$,
$j(z) = j(-z)$, $z \in \mathbb{R}^N \setminus \{0\}$ and $J(x,y) \geq j(x-y)$ for $x,y \in \mathbb{R}^N$, $x \neq y$.

(J1)$_d$ The function j in (J1)$_c$ satisfies additionally $\int_{\mathbb{R}^N} j(y)\, dy = \infty$.

[1] $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function if $x \mapsto f(x,\cdot)$ is continuous, $u \mapsto f(\cdot,u)$ is measurable and for each compact set $K \subset \mathbb{R}$ we have that $x \mapsto \sup\{|f(x,u)| \,:\, u \in K\}$ is Lebesgue integrable on Ω

Special attention will be given to functions J which satisfy

$$(\mathrm{J1})_e \qquad J \text{ depends only on the difference } x-y.$$

If J satisfies $(\mathrm{J1})_e$ we usually write $J(x-y)$ instead of $J(x,y)$, so in this case we assume that J is a function $\mathbb{R}^N \setminus \{0\} \to \mathbb{R}$. Since we will analyze problem (P) with the moving plane method we will have to deal with equations with kernel functions of the form $\bar{J}(x,y) = J(x-y) - J(x-\bar{y})$, where $\bar{y}$ is the reflection of y at some hyperplane. Later on we will show that this function can be associated to a kernel function J' which satisfies in particular $(\mathrm{J1})_c$ but not $(\mathrm{J1})_e$.

In the remainder of this section, we assume in most cases that J satisfies (J1), i.e. $(\mathrm{J1})_a$-$(\mathrm{J1})_d$. If we say J satisfies $(\mathrm{J1})_{diff}$, we mean that $J : \mathbb{R}^N \setminus \{0\} \to [0,\infty)$ satisfies $(\mathrm{J1})_a$-$(\mathrm{J1})_e$ where we write $J(x-y)$ instead of $J(x,y)$. We will frequently need one of the following assumptions on the lower bound j of J given by $(\mathrm{J1})_c$:

$(\mathrm{J}_+)_{r_0}$ The function j in $(\mathrm{J1})_c$ satisfies $\operatorname*{essinf}_{B_{r_0}(0)} j > 0$ w.r.t. some $r_0 > 0$.

(J_+) The function j in $(\mathrm{J1})_c$ satisfies $\operatorname*{essinf}_{B_r(0)} j > 0$ for all $r > 0$.

Remark 2.1. Note that $(\mathrm{J}_+)_{r_0}$ implies that for all $r \in (0, r_0]$ the following holds: For all $M \subset \mathbb{R}^N$ with $|M| > 0$ and diam $M < \frac{r}{2}$ we have

$$\inf_{x \in B_{\frac{r}{2}}(M)} \int_M J(x,y)\, dy \geq \inf_{x \in B_{\frac{r}{2}}(M)} \int_M j(x-y)\, dy > 0. \tag{2.2}$$

Before we start stating some basic properties on the quadratic form $\mathscr{J}$ let as mention some examples which satisfy $(\mathrm{J1})_{diff}$ and which are different from the fractional Laplacian. Note that the bilinear form associated to the fractional Laplacian is given by a function which corresponds up to a constant to $J_s(y) = |y|^{-N-2s}$, $s \in (0,1)$, $y \in \mathbb{R}^N \setminus \{0\}$.

Example 2.2. Let $\alpha, \beta \in (0,2)$, $c \geq 1$ and consider a measurable map $k : (0,\infty) \to (0,\infty)$ such that

$$\frac{\rho^{-N}}{c} \leq k(\rho) \leq c\rho^{-N-\alpha} \quad \text{for } \rho \leq 1 \qquad \text{and} \qquad k(\rho) \leq c\rho^{-N-\beta} \quad \text{for } \rho > 1.$$

Suppose moreover that k is strictly decreasing on $(0,\infty)$, and let $|\cdot|_\sharp$ denote an arbitrary norm on $\mathbb{R}^N$. Then the function

$$J : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}, \qquad J(z) = k(|z|_\sharp)$$

satisfies $(\mathrm{J1})_{diff}$ and (J_+). The class defined here also includes operators of order varying between 0 and $\alpha \in (0,2)$. In particular, zero order operators are admissible. Moreover, the choice of non-euclidean norms $|\cdot|_\sharp$ leads to anisotropic operators. In particular, for $1 \leq p < \infty$, the norm

$$|x|_\sharp = |x|_p := \Big(\sum_{i=1}^N |x_i|^p\Big)^{1/p} \qquad \text{for } x \in \mathbb{R}^N \tag{2.3}$$

has the required properties. For $r > 0$ the measurable map $J'(z) = J(z)1_{B_r(0)}(z)$ fulfills $(J1)_{diff}$ and $(J_+)_r$ but not (J_+). The corresponding bilinear form then models short range nonlocal interactions.

For an open set $\Omega \subset \mathbb{R}^N$, we consider the space

$$\mathscr{D}^J(\Omega) := \{u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))^2 J(x,y)\, dxdy < \infty$$
$$\text{and } u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega\}. \quad (2.4)$$

Then the quadratic form $\mathscr{J}$ is well-defined on $\mathscr{D}^J(\Omega)$ by (2.1). In the following, we identify $L^2(\Omega)$ with the space of functions $u \in L^2(\mathbb{R}^N)$ with $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$, and we define

$$\Lambda_{1,J}(\Omega) := \inf_{u \in \mathscr{D}^J(\Omega)} \frac{\mathscr{J}(u,u)}{\|u\|^2_{L^2(\Omega)}} \geq 0. \quad (2.5)$$

The following Poincaré-Friedrichs type inequality has been derived in [41, Lemma 2.7].

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume J satisfies $(J1)_a$, $(J1)_b$ and $(J1)_c$. Then we have $\Lambda_{1,J}(\Omega) > 0$, which implies that $\mathscr{D}^J(\Omega) \subset L^2(\Omega)$ and*

$$\mathscr{J}(u,u) \geq \Lambda_{1,J}(\Omega)\|u\|^2_{L^2(\Omega)} > 0 \qquad \text{for all } u \in \mathscr{D}^J(\Omega) \setminus \{0\}. \quad (2.6)$$

In particular $\mathscr{J}$ is a scalar product on $\mathscr{D}^J(\Omega)$.

The following is a special case of [41, Lemma 2.3], we include the proof for the readers convenience.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume that J satisfies $(J1)_a$, $(J1)_b$ and that $\Lambda_{1,J}(\Omega) > 0$. Then $\mathscr{D}^J(\Omega)$ is a Hilbert space with the scalar product $\mathscr{J}$.*

Proof. Let $(u_n)_n \subset \mathscr{D}^J(\Omega)$ be a Cauchy sequence. Since (2.6) is satisfied and $L^2(\Omega)$ is complete, we have that $u_n \to u \in L^2(\Omega)$ for a function $u \in L^2(\Omega)$. Hence there exists a subsequence such that $u_{n_k} \to u$ a.e. in Ω as $k \to \infty$. By Fatou's Lemma, we therefore have that

$$\mathscr{J}(u,u) \leq \liminf_{k\to\infty} \mathscr{J}(u_{n_k}, u_{n_k}) \leq \sup_{k\in\mathbb{N}} \mathscr{J}(u_{n_k}, u_{n_k}) < \infty,$$

so that $u \in \mathscr{D}^J(\Omega)$. Applying Fatou's Lemma again, we find that

$$\mathscr{J}(u_{n_k}-u, u_{n_k}-u) \leq \liminf_{j\to\infty} \mathscr{J}(u_{n_k}-u_{n_j}, u_{n_k}-u_{u_j}) \leq \sup_{j\geq k} \mathscr{J}(u_{n_k}-u_{n_j}, u_{n_k}-u_{u_j}) \quad \text{for } k \in \mathbb{N}.$$

Since $(u_n)_n$ is a Cauchy sequence with respect to the scalar product $\mathscr{J}$, it thus follows that $\lim_{k\to\infty} u_{n_k} = u$ and therefore also $\lim_{n\to\infty} u_n = u$ in $\mathscr{D}^J(\Omega)$. This shows the completeness of $\mathscr{D}^J(\Omega)$. □

Proposition 2.5. *Assume that J satisfies $(J1)_a$, $(J1)_b$. Then the following holds:*

(i) $\mathscr{C}_c^{0,1}(\mathbb{R}^N) \subset \mathscr{D}^J(\mathbb{R}^N)$.

(ii) Assume additionally (J1)$_e$ and let $v \in \mathscr{C}_c^2(\mathbb{R}^N)$*. Then the principle value integral*

$$[Iv](x) := P.V. \int_{\mathbb{R}^N} (v(x)-v(y))J(x-y)\,dy = \lim_{\varepsilon \to 0} \int_{|x-y|\geq\varepsilon} (v(x)-v(y))J(x-y)\,dy \tag{2.7}$$

exists for every $x \in \mathbb{R}^N$*. Moreover, there is* $K = K(N,J) > 0$ *such that* $Iv \in L^\infty(\mathbb{R}^N)$ *with*[2]

$$\|Iv\|_{L^\infty(\mathbb{R}^N)} \leq K\|v\|_{C^2(\mathbb{R}^N)},$$

and for every bounded open set $\Omega \subset \mathbb{R}^N$ *and every* $u \in \mathscr{D}^J(\Omega)$ *we have*

$$\mathscr{J}(u,v) = \int_{\mathbb{R}^N} u(x)[Iv](x)\,dx.$$

Proof. (i) Let $u \in \mathscr{C}_c^{0,1}(\mathbb{R}^N)$, and let $K > 0$, $R > 2$ be such that $\mathrm{supp}(u) \subset B_{R-2}(0)$,

$$|u(x)| \leq K \quad \text{and} \quad |u(x)-u(y)| \leq K|x-y| \qquad \text{for } x,y \in \mathbb{R}^N,\ x \neq y.$$

Then, as a consequence of (J1)$_a$ and (J1)$_b$,

$$\begin{aligned} 2\mathscr{J}(u,u) &= \int_{B_R(0)}\int_{B_R(0)} (u(x)-u(y))^2 J(x,y)\,dxdy + 2\int_{B_R(0)} u^2(x) \int_{\mathbb{R}^N\setminus B_R(0)} J(x,y)\,dydx \\ &\leq K^2 \int_{B_R(0)}\int_{B_R(0)} |x-y|^2 J(x,y)\,dxdy + 2K^2 \int_{B_{R-2}(0)}\int_{\mathbb{R}^N\setminus B_R(0)} J(x,y)\,dydx \\ &\leq 2K^2|B_R(0)|\Big(\sup_{x\in\mathbb{R}^N}\int_{B_{2R}(0)} |x-y|^2 J(x,y)\,dy + \sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N\setminus B_1(0)} J(x,y)\,dy\Big) < \infty \end{aligned}$$

and thus $u \in \mathscr{D}^J(\mathbb{R}^N)$.
(ii) Since $v \in \mathscr{C}_c^2(\mathbb{R}^N)$ we have

$$|2v(x)-v(x+z)-v(x-z)| \leq \|v\|_{C^2(\mathbb{R}^N)}|z|^2 \qquad \text{for all } x,z \in \mathbb{R}^N. \tag{2.8}$$

Put $h(x,y) := (v(x)-v(y))J(x-y)$ for $x,y \in \mathbb{R}^N$, $x \neq y$. For every $x \in \mathbb{R}^N$, $\varepsilon > 0$ we then have, since J is even by (J1)$_e$,

$$\begin{aligned} \int_{\varepsilon\leq|y-x|} h(x,y)\,dy &= \int_{\varepsilon\leq|z|} [v(x)-v(x+z)]J(z)\,dz = \int_{\varepsilon\leq|z|} [v(x)-v(x-z)]J(z)\,dz \\ &= \frac{1}{2}\int_{\varepsilon\leq|z|} [2v(x)-v(x+z)-v(x-z)]J(z)\,dz. \end{aligned}$$

[2]Here we use for $v \in C^2(\mathbb{R}^N)$: $\|v\|_{C^2(\mathbb{R}^N)} = \sup_{x\in\mathbb{R}^N}\sum_{|\alpha|\leq 2}|\partial^\alpha v(x)|$

By $(J1)_b$, (2.8) and Lebesgue's theorem we thus conclude the existence of the limit

$$\lim_{\varepsilon\to 0}\int_{\varepsilon\le|y-x|} h(x,y)\,dy = \frac{1}{2}\int_{\mathbb{R}^N}[2v(x)-v(x+z)-v(x-z)]J(z)\,dz.$$

Moreover we have for $x\in\mathbb{R}^N$ and $\varepsilon\in(0,1)$

$$\begin{aligned}\int_{|y-x|\ge\varepsilon} h(x,y)\,dy &\le 2\|v\|_{L^\infty(\mathbb{R}^N)}\int_{\mathbb{R}^N\setminus B_1(0)} J(z)\,dz + \frac{\|v\|_{C^2(\mathbb{R}^N)}}{2}\int_{B_1(0)}|z|^2J(z)\,dz\\ &\le 2\|v\|_{C^2(\mathbb{R}^N)}\int_{\mathbb{R}^N}\min\{1,|z|^2\}J(z)\;dz,\end{aligned} \tag{2.9}$$

where the right hand side is finite by $(J1)_b$. In particular, $[Iv](x)$ is well-defined by (2.7), and $\big|[Iv](x)\big|\le K\|v\|_{C^2(\mathbb{R}^N)}$ for $x\in\mathbb{R}^N$ with $K := 2\int_{\mathbb{R}^N}\min\{1,|z|^2\}J(z)\;dz$. In particular, $Iv\in L^\infty(\mathbb{R}^N)$. Next let $\Omega\subset\mathbb{R}^N$ be open and bounded and $u\in\mathscr{D}^J(\Omega)$, so that also $u\in L^2(\Omega)$. Then we have, by (2.9) and Lebesgue's Theorem,

$$\begin{aligned}\mathscr{J}(u,v) &= \frac{1}{2}\lim_{\varepsilon\to 0}\int_{|x-y|\ge\varepsilon}(u(x)-u(y))h(x,y)\,dx\,dy\\ &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}u(x)\int_{|y-x|\ge\varepsilon}h(x,y)\,dy dx = \int_{\mathbb{R}^N}u(x)\Big[\lim_{\varepsilon\to 0}\int_{|y-x|\ge\varepsilon}h(x,y)\,dy\Big]dx = \int_{\mathbb{R}^N}u(x)[Iv](x)\,dx.\end{aligned}$$

The proof is finished. □

Corollary 2.6. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set and assume (J1). Then $\mathscr{J}$ is a closed quadratic form with dense form domain $\mathscr{D}^J(\Omega)$ in $L^2(\Omega)$. Consequently, $\mathscr{J}$ is the quadratic form of a unique self-adjoint operator I in $L^2(\Omega)$ with domain* $\mathrm{dom}(I)\subset\mathscr{D}^J(\Omega)$*. Moreover, if $(J1)_{diff}$ is satisfied, then $\mathscr{C}^2_c(\Omega)$ is contained in* $\mathrm{dom}(I)$*, and for every $v\in\mathscr{C}^2_c(\Omega)$ the function $Iv\in L^2(\Omega)$ is a.e. given by (2.7).*

Proof. Since $\mathscr{C}^{0,1}_c(\Omega)\subset L^2(\Omega)$ is dense, $\mathscr{D}^J(\Omega)$ is a dense subset of $L^2(\Omega)$ by Proposition 2.5(i). Moreover, the quadratic form $\mathscr{J}$ is closed in $L^2(\Omega)$ as a consequence of (2.3) and Lemma 2.4. Hence $\mathscr{J}$ is the quadratic form of a unique self-adjoint operator I in $L^2(\Omega)$ (see e.g. [59, Theorem VIII.15, pp. 278]). Moreover, for every $v\in\mathscr{C}^2_c(\Omega)$, $u\in\mathscr{D}^J(\Omega)$ we have $|\mathscr{J}(u,v)|\le\|Iv\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}$ by Proposition 2.5(ii). Consequently, v is contained in the domain of I and satisfies $\mathscr{J}(u,v)=\int_{\mathbb{R}^N}u(x)[Iv](x)\,dx$ for every $u\in\mathscr{D}^J(\Omega)$. From Proposition 2.5(ii) it then follows that Iv is a.e. given by (2.7). □

Remark 2.7.

(i) We note that for a general kernel function J we do not have a simple representation of the operator I associated to $\mathscr{J}$.

(ii) Strictly speaking Corollary 2.6 only gives the existence of a unique self-adjoint operator I in the case where Ω is an open bounded set, but in any case we will also use the existence of I in unbounded sets. If $\Omega \subset \mathbb{R}^N$ is an arbitrary open set, then the existence of I as stated in Corollary 2.6 follows with the Norm $\left(\|u\|^2_{L^2(\Omega)} + \mathscr{J}(u,u)\right)^{1/2}$ for $u \in \mathscr{D}^J(\Omega) \cap L^2(\Omega)$.

One may study solutions u of (P) in strong sense, requiring that u is contained in the domain of the operator I. However, it is more natural to consider the weaker notion of solutions given by the quadratic form $\mathscr{J}$ itself. We define for $\Omega \subset \mathbb{R}^N$ open:

Definition 2.8. We call a function $u \in \mathscr{D}^J(\Omega)$ *a solution of* (P) *in* Ω if the integral $\int_\Omega f(x,u(x))\varphi(x)\,dx$ exists for all $\varphi \in \mathscr{D}^J(\Omega)$ with compact support in $\mathbb{R}^N$ and

$$\mathscr{J}(u,\varphi) = \int_\Omega f(x,u(x))\varphi(x)\,dx \qquad \text{for all } \varphi \in \mathscr{D}^J(\Omega) \text{ with compact support in } \mathbb{R}^N.$$

By Riesz representation theorem we immediately get

Corollary 2.9. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume J satisfies (J1)$_a$, (J1)$_b$, and that $\Lambda_{1,J}(\Omega) > 0$. Then for every $g \in L^2(\Omega)$ there is a unique $u \in \mathscr{D}^J(\Omega)$ with*

$$\mathscr{J}(u,\varphi) = \int_\Omega g(x)\varphi(x)\,dx \qquad \textit{for all } \varphi \in \mathscr{D}^J(\Omega).$$

Next we wish to extend the definition of $\mathscr{J}(v,\varphi)$ to more general pairs of functions (v,φ).

Definition 2.10. Let $U' \subset \mathbb{R}^N$ be an open set. We define $\mathscr{V}^J(U')$ as the space of all functions $v \in L^2(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$[3] such that

$$\rho(v,U') := \int_{U'}\int_{U'} (v(x)-v(y))^2 J(x,y)\,dxdy < \infty. \tag{2.10}$$

Note that $\mathscr{D}^J(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \subset \mathscr{V}^J(U')$ for any measurable subset $U' \subset \mathbb{R}^N$, and thus $\mathscr{D}^J(U) \subset \mathscr{V}^J(U')$ for any open bounded set $U \subset \mathbb{R}^N$ by Proposition 2.3 if J satisfies (J1)$_a$ – (J1)$_c$.

Lemma 2.11. *Assume that J satisfies (J1)$_a$, (J1)$_b$. Let $U' \subset \mathbb{R}^N$ be an open set and $v,\varphi \in \mathscr{V}^J(U')$. Moreover, suppose that $\varphi \equiv 0$ on $\mathbb{R}^N \setminus U$ for some bounded subset $U \subset\subset U'$. Then*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)||\varphi(x)-\varphi(y)|J(x,y)\,dxdy < \infty, \tag{2.11}$$

and thus

$$\mathscr{J}(v,\varphi) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (v(x)-v(y))(\varphi(x)-\varphi(y))J(x,y)\,dxdy$$

is well-defined.

[3] Here use for $L^2(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ the norm $\|v\|_{2,\infty} := \inf\{\|v_1\|_{L^2(\mathbb{R}^N)} + \|v_2\|_{L^\infty(\mathbb{R}^N)} \;:\; v = v_1 + v_2\;,\; v_1 \in L^2(\mathbb{R}^N),\, v_2 \in L^\infty(\mathbb{R}^N)\}$.

Proof. Since J satisfies $(\mathrm{J1})_b$, we have $K := \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N\setminus B_r(x)} J(x,y)\,dy < \infty$ where we fix $r := \min\{1, \mathrm{dist}(U, \mathbb{R}^N\setminus U')\} > 0$. Since $\operatorname{supp}\varphi \subset U$ we have $\varphi \in L^2(\mathbb{R}^N)$ and thus we have for $v \in \mathscr{V}^J(U')$ given by $v = v_1 + v_2$, with $v_1 \in L^2(\mathbb{R}^N)$ and $v_2 \in L^\infty(\mathbb{R}^N)$

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |v(x)-v(y)||\varphi(x)-\varphi(y)|J(x,y)\,dxdy\\
&\quad= \int_{U'}\int_{U'} |v(x)-v(y)||\varphi(x)-\varphi(y)|J(x,y)\ dxdy + 2\int_{U}\int_{\mathbb{R}^N\setminus U'} |v(x)-v(y)||\varphi(x)|J(x,y)\ dydx\\
&\quad\le \frac{1}{2}\big(\rho(v,U')+\rho(\varphi,U')\big) + 2\int_{U}\int_{\mathbb{R}^N\setminus U'} (|v_1(x)|+|v_1(y)|)|\varphi(x)|J(x,y)\ dydx\\
&\qquad\qquad + 4\|v_2\|_{L^\infty(\mathbb{R}^N)}\int_{U}\int_{\mathbb{R}^N\setminus U'} |\varphi_1(x)|J(x,y)\ dydx\\
&\quad\le \frac{1}{2}\big(\rho(v,U')+\rho(\varphi,U')\big) + \int_{U}\int_{\mathbb{R}^N\setminus U'} (|v_1(x)|^2+|v_1(y)|^2+2|\varphi(x)|^2)J(x,y)\ dydx\\
&\qquad\qquad + 4K\|v_2\|_{L^\infty(\mathbb{R}^N)}|U|^{\frac{1}{2}}\|\varphi_1\|^2_{L^2(\mathbb{R}^N)}\\
&\quad\le \frac{1}{2}\big(\rho(v,U')+\rho(\varphi,U')\big) + 2K\Big(\|v_1\|^2_{L^2(\mathbb{R}^N)} + \|\varphi\|^2_{L^2(\mathbb{R}^N)} + 2\|v_2\|_{L^\infty(\mathbb{R}^N)}|U|^{\frac{1}{2}}\|\varphi\|^2_{L^2(\mathbb{R}^N)}\Big) < \infty.
\end{aligned}
$$

□

Corollary 2.12. *Assume that J satisfies $(J1)_a$, $(J1)_b$, let $U' \subset \mathbb{R}^N$ be an open set, and let $u \in \mathscr{V}^J(U')$. If there is $U \subset\subset U'$ such that $u \equiv 0$ on $\mathbb{R}^N \setminus U$, then $u \in \mathscr{D}^J(U')$.*

Proof. This follows immediately from Lemma 2.11. □

Lemma 2.13. *Assume J satisfies $(J1)_a$, $(J1)_b$. If $U' \subset \mathbb{R}^N$ is open and $v \in \mathscr{V}^J(U')$, then $v^\pm \in \mathscr{V}^J(U')$ and $\rho(v^\pm, U') \le \rho(v, U')$.*

Proof. We have $v^\pm \in L^2(\mathbb{R}^N)+L^\infty(\mathbb{R}^N)$ since $v \in L^2(\mathbb{R}^N)+L^\infty(\mathbb{R}^N)$. Moreover, $v^+(x)v^-(x) = 0$ for $x \in \mathbb{R}^N$ and thus

$$
\begin{aligned}
\rho(v,U') &= \rho(v^+,U') + \rho(v^-,U') - 2\int_{U'}\int_{U'} (v^+(x)-v^+(y))(v^-(x)-v^-(y))J(x,y)\ dxdy\\
&= \rho(v^+,U') + \rho(v^-,U') + 2\int_{U'}\int_{U'} [v^+(x)v^-(y)+v^+(y)v^-(x)]J(x,y)\ dxdy\\
&\ge \rho(v^+,U') + \rho(v^-,U').
\end{aligned}
$$

The claim follows. □

Lemma 2.14. *Let $U' \subset \mathbb{R}^N$ be an open set and assume $(J1)_a$, $(J1)_b$. Then for any open set $U \subset\subset U'$, $\varphi \in \mathscr{C}_c^{0,1}(U)$, $0 \le \varphi \le 1$ there is $C = C(U', U, \varphi, J) > 0$ such that*

$$\rho(\varphi u, U) \le C\Big(\rho(u,U') + \|u\|_{L^2(U')}\Big) \qquad \text{for all } u \in \mathscr{V}^J(U') \cap L^2(U').$$

Proof. Denote by L the Lipschitz constant of φ. Since $U \subset\subset U'$ there is $\varepsilon > 0$ such that $B_\varepsilon(U) \subset U'$. Due to $(J1)_a$ and $(J1)_b$ we may fix

$$k := \sup_{x\in B_\varepsilon(U)} \int_{B_\varepsilon(U)} |x-y|^2 J(x,y)\, dxdy + \sup_{x\in U} \int_{U'\setminus B_\varepsilon(U)} J(x,y)\, dy + \sup_{x\in U'\setminus B_\varepsilon(U)} \int_U J(x,y)\, dy < \infty.$$

Then following the proof idea of [32, Lemma 5.3] we have

$$\begin{aligned}
\rho(\varphi u, U) &\le \int_{U'}\int_{U'} (\varphi(x)u(x) - \varphi(x)u(y) + \varphi(x)u(y) - \varphi(y)u(y))^2 J(x,y)\, dxdy \\
&\le 2\int_{U'}\int_{U'} \Big[(\varphi(x)u(x) - \varphi(x)u(y))^2 + (\varphi(x)u(y) - \varphi(y)u(y))^2\Big] J(x,y)\, dxdy \\
&\le 2\rho(u,U') + 2\int_{B_\varepsilon(U)}\int_{B_\varepsilon(U)} (\varphi(x)-\varphi(y))^2 u(y)^2 J(x,y)\, dxdy \\
&\quad + 2\int_{U'\setminus B_\varepsilon(U)}\int_{B_\varepsilon(U)} \varphi(x)^2 u(y)^2 J(x,y)\, dxdy + 2\int_{B_\varepsilon(U)}\int_{U'\setminus B_\varepsilon(U)} \varphi(y)^2 u(y)^2 J(x,y)\, dxdy \\
&\le 2\rho(u,U') + 2L^2 \int_{B_\varepsilon(U)} u(y)^2 \int_{B_\varepsilon(U)} |x-y|^2 J(x,y)\, dxdy \\
&\quad + 2\int_{U'\setminus B_\varepsilon(U)} u(y)^2 \int_U J(x,y)\, dxdy + 2\int_U u(y)^2 \int_{U'\setminus B_\varepsilon(U)} J(x,y)\, dxdy \\
&\le 2\rho(u,U') + 2k(L^2+2)\|u\|_{L^2(U')}.
\end{aligned}$$

□

Remark 2.15. Let $\Omega \subset \mathbb{R}^N$ be an open set. Then by a combination of Corollary 2.12 and Lemma 2.14 we have $\varphi u \in \mathscr{D}^J(\Omega)$ for any $u \in \mathscr{V}^J(\Omega) \cap L^2(\Omega)$ and $\varphi \in \mathscr{C}_c^{0,1}(\Omega)$ with $0 \le \varphi \le 1$.

Definition 2.16. Let $\Omega \subset \mathbb{R}^N$ be an open set, and let $f \in L^2(\Omega) + L^\infty(\Omega)$

1. We call function $u \in L^2(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ a *supersolution of the equation*

$$Iu = f \qquad \text{in } \Omega \tag{2.12}$$

   if $u \in \mathscr{V}^J(U')$ for some open subset $U' \subset \mathbb{R}^N$ with $\Omega \subset U'$ and $\operatorname{dist}(\Omega, \mathbb{R}^N \setminus U') > 0$ and if

$$\mathscr{J}(u,\varphi) \ge \int_\Omega f(x)\varphi(x)\, dx \quad \text{for all } \varphi \in \mathscr{D}^J(\Omega),\ \varphi \ge 0 \text{ with compact support in } \mathbb{R}^N.$$

2. We call function $u \in L^2(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ a *supersolution of the problem*

$$Iu = f(x) \qquad \text{in } \Omega, \qquad u \equiv 0 \quad \text{on } \mathbb{R}^N \setminus \Omega \tag{2.13}$$

   if u is a supersolution of (2.12) and $u \ge 0$ on $\mathbb{R}^N \setminus \Omega$.

3. We call u a *subsolution* of (2.12) resp. (2.13) if $-u$ is a supersolution of (2.12), (2.13), respectively.

In the following section, we will need lower bounds for $\Lambda_{1,J}(\Omega)$ in the case where $|\Omega|$ is small. For this we set

$$\Lambda_{1,J}(r) := \inf\{\Lambda_{1,J}(\Omega) \,:\, \Omega \subset U \text{ open}, |\Omega| = r\} \qquad \text{for } r > 0.$$

Lemma 2.17. *Assume J satisfies (J1). Then*

$$\Lambda_{1,J}(r) \to \infty \qquad \text{as} \qquad r \to 0.$$

Proof. Let j be given by $(J1)_c$, i.e. we have $J(x,y) \geq j(x-y)$ for all $x,y \in \mathbb{R}^N$, $x \neq y$. We set

$$j_c := \{z \in \mathbb{R}^N \setminus \{0\} \,:\, j(z) \geq c\} \qquad \text{and} \qquad j^c := \{z \in \mathbb{R}^N \setminus \{0\} \,:\, j(z) < c\}$$

for $c \in [0,\infty]$. We also consider the decreasing rearrangement $d : (0,\infty) \to [0,\infty]$ of j given by $d(r) = \sup\{c \geq 0 \,:\, |j_c| \geq r\}$. We first note that

$$|j_{d(r)}| \geq r \qquad \text{for every } r > 0 \tag{2.14}$$

Indeed, this is obvious if $d(r) = 0$, since $j_0 = \mathbb{R}^N \setminus \{0\}$. If $d(r) > 0$, we have $|j_c| \geq r$ for every $c < d(r)$ by definition, whereas $|j_c| < \infty$ for every $c > 0$ as a consequence of the fact that $j \in L^1(\mathbb{R}^N \setminus B_1(0))$ by $(J1)_b$. Consequently, since $j_{d(r)} = \bigcap\limits_{c<d(r)} j_c$, we have $|j_{d(r)}| = \inf\limits_{c<d(r)} |j_c| \geq r$.

Next we claim that

$$\Lambda_{1,J}(r) \geq \int_{j^{d(r)}} j(z)\,dz \qquad \text{for } r > 0. \tag{2.15}$$

Indeed, let $r > 0$ and $\Omega \subset \mathbb{R}^N$ be measurable with $|\Omega| = r$. For $u \in \mathscr{D}^J(\Omega)$ we have

$$\begin{aligned}
\mathscr{J}(u,u) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))^2 J(x,y)\; dxdy \\
&\geq \frac{1}{2}\int_{\Omega}\int_{\Omega} (u(x)-u(y))^2 j(x-y)\; dxdy + \int_{\Omega} u^2(x) \int_{\mathbb{R}^N\setminus\Omega} j(x-y)\; dy\; dx \\
&\geq \inf_{x\in\Omega}\Big(\int_{\mathbb{R}^N\setminus\Omega_x} j(y)\; dy\Big)\|u\|^2_{L^2(\Omega)}
\end{aligned} \tag{2.16}$$

with $\Omega_x := x + \Omega$. Let $d := d(r)$. Since $|j_d| \geq r = |\Omega|$ by (2.14), we have $|j_d \setminus \Omega_x| \geq |\Omega_x \setminus j_d|$ and thus, for every $x \in \Omega$,

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus\Omega_x} j(y)\; dy &= \int_{\mathbb{R}^N\setminus j_d} j(y)\; dy + \int_{j_d\setminus\Omega_x} j(y)\; dy - \int_{\Omega_x\setminus j_d} j(y)\; dy \\
&\geq \int_{j^d} j(y)\; dy + \Big(|j_d \setminus \Omega_x| - |\Omega_x \setminus j_d|\Big)d \geq \int_{j^d} j(y)\; dy.
\end{aligned}$$

Combining this with (2.16), we obtain (2.15), as claimed. By $(J1)_d$, the decreasing rearrangement of j satisfies $d(r) \to \infty$ as $r \to 0$ and

$$\int_{j^{d(r)}} j(y)\,dy \to \infty \qquad \text{as } r \to 0.$$

Together with (2.15), this shows the claim. □

3 Maximum principles for nonlocal operators

Let $\mathscr{J}$ be the quadratic form as in (2.1) given by a function J which fulfills (J1) or $(J1)_{diff}$. Furthermore let I be the operator associated to the quadratic form $\mathscr{J}$. Finally for an open set $\Omega \subset \mathbb{R}^N$ we consider $\mathscr{D}^J(\Omega)$ as defined in (2.4), $\mathscr{V}^J(\Omega)$ as defined in Definition 2.10 and a supersolution as defined in Definition 2.16.

We will need the following results

Lemma 3.1. *Let J satisfy $(J1)_a$, $(J1)_b$. Let $U' \subset \mathbb{R}^N$ be an open set, and let $v \in \mathscr{V}^J(U')$ be a function such that $v \geq 0$ on $\mathbb{R}^N \setminus U$ for some open bounded set $U \subset\subset U'$. Then $v^- \in \mathscr{D}^J(U)$ and*

$$\mathscr{J}(v^-, v^-) \leq -\mathscr{J}(v, v^-) \tag{3.1}$$

Proof. By Lemma 2.13 we have $v^- \in \mathscr{V}^J(U')$. Moreover, Lemma 2.11 implies $|\mathscr{J}(v, v^-)| < \infty$ and since $v^- \equiv 0$ on $\mathbb{R}^N \setminus U$, it thus follows that $v^- \in \mathscr{D}^J(U)$ by Corollary 2.12. To show (3.1), we first note that

$$[v^- + v]v^- = v^+ v^- \equiv 0 \qquad \text{on } \mathbb{R}^N$$

and therefore

$$[v^-(x) - v^-(y)]^2 + [v(x) - v(y)][v^-(x) - v^-(y)] = -\Big(v^-(x)v^+(y) + v^-(y)v^+(x)\Big)$$

for $x, y \in \mathbb{R}^N$. Thus, due to the fact that $v^- \equiv 0$ on $\mathbb{R}^N \setminus U$, we find that

$$\mathscr{J}(v^-, v^-) + \mathscr{J}(v, v^-) = -\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big(v^-(x)v^+(y) + v^-(y)v^+(x)\Big) J(x,y)\,dydx \leq 0,$$

Hence (3.1) is true. □

For the next result we will need another definition.

Definition 3.2. Let $U \subset \mathbb{R}^N$ be an open set, and let J satisfy (J1). We let $\mathscr{D}^J_\infty(U)$ denote the space of all functions $f \in \mathscr{D}^J(U)$ such that there exists a constant $C = C(f) > 0$ with

$$\mathscr{J}(f, \varphi) \leq C \int_U |\varphi(x)|\,dx \qquad \text{for all } \varphi \in \mathscr{D}^J(U) \text{ with compact support in } \mathbb{R}^N.$$

Moreover, for $f \in \mathscr{D}^J_\infty(U)$ we put

$$\|f\|_{\mathscr{D}^J_\infty(U)} := \inf\Big\{k \geq 0 \;:\; \mathscr{J}(f,\varphi) \leq k \int_U |\varphi(x)|\,dx \quad \text{for all } \varphi \in \mathscr{D}^J(U) \text{ with compact support in } \mathbb{R}^N\Big\}$$

Remark 3.3. We emphasize that for $U \subset \mathbb{R}^N$ open we have that $\|\cdot\|_{\mathscr{D}^J_\infty(U)}$ is a semi norm and for $f \in \mathscr{D}^J_\infty(U)$ we have

$$\mathscr{J}(f,\varphi) \leq \|f\|_{\mathscr{D}^J_\infty(U)}\|\varphi\|_{L^1(U)} \qquad \text{for all } \varphi \in \mathscr{D}^J(U) \text{ with compact support in } \mathbb{R}^N.$$

If, in addition, J satisfies $(\mathrm{J1})_{diff}$, then $\mathscr{C}^2_c(U) \subset \mathscr{D}^J_\infty(U)$ as a consequence of Proposition 2.5 and there is $K = K(N,J) > 0$ such that

$$\|f\|_{\mathscr{D}^J_\infty(U)} \leq K\|f\|_{C^2(\mathbb{R}^N)} \qquad \text{for all } f \in \mathscr{C}^2_c(U).$$

In the following Lemma, we will need assumptions (J_+) resp. $(\mathrm{J}_+)_{r_0}$ (see p. 2).

Lemma 3.4.

(i) *Let J satisfy (J1), (J_+), and let $k > 0$. Moreover, let $U \subset \mathbb{R}^N$ be an open bounded set, $x_0 \in U$ and $f \in \mathscr{D}^J_\infty(U)$. Finally let $M \subset\subset \mathbb{R}^N \setminus \{x_0\}$ be measurable with $|M| > 0$, and let $0 < r < \frac{1}{4}\min\{\mathrm{dist}(x_0,M), \mathrm{dist}(x_0,\mathbb{R}^N \setminus U\}$. Then there is $a > 0$ such that $w = f + a1_M$ is a subsolution of*

$$Iw = -k \qquad \text{in } B_{2r}(x_0), \tag{3.2}$$

i.e.

$$\mathscr{J}(w,\varphi) \leq -k \int_{B_{2r}(x_0)} \varphi(x)\,dx \qquad \text{for all } \varphi \in \mathscr{D}^J(B_{2r}(x_0)) \text{ with } \varphi \geq 0.$$

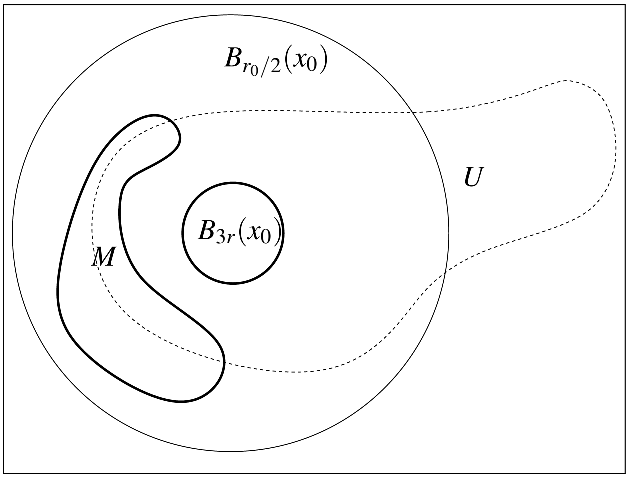
(ii) *Let J satisfy (J1), $(J_+)_{r_0}$ for some $r_0 > 0$, and let $k > 0$. Moreover, let $U \subset \mathbb{R}^N$ be an open bounded set, $x_0 \in U$ and $f \in \mathscr{D}^J_\infty(U)$. Finally let $M \subset\subset B_{r_0/2}(x_0) \setminus \{x_0\}$ be measurable with $|M| > 0$, and let $0 < r < \frac{1}{4}\min\{r_0, \mathrm{dist}(x_0,M), \mathrm{dist}(x_0,\mathbb{R}^N \setminus U\}$. Then there is $a > 0$ such that $w = f + a1_M$ is a subsolution of (3.2).*

Proof. We only prove (ii); the proof of (i) is similar but simpler. Fix r, M and f as stated and let $w = f + a1_M$. Put $U_0 := B_{2r}(x_0)$ and $U_0' := B_{3r}(x_0)$. Note that the function w satisfies

$$w \equiv 0 \quad \text{on } \mathbb{R}^N \setminus (U_0 \cup M), \qquad w \equiv a \quad \text{on } M.$$

We claim that $w \in \mathscr{V}^J(U_0')$. Since U is bounded we have $f \in \mathscr{D}^J_\infty(U) \subset \mathscr{D}^J(U) \subset L^2(U)$ and thus $f \in \mathscr{V}^J(U_0')$, whereas $1_M \in \mathscr{V}^J(U_0')$ since M is bounded and $\mathrm{dist}(M,U_0') > 0$.
Next, let $\varphi \in \mathscr{D}^J(U_0)$, $\varphi \geq 0$. Since $U_0 \subset U$ and $f \in \mathscr{D}^J_\infty(U)$ by assumption, we have

$$\mathscr{J}(f,\varphi) \leq \|f\|_{\mathscr{D}^J_\infty(U)} \int_{U_0} \varphi(x)\,dx \tag{3.3}$$



with $C = C(f) > 0$ independent of φ. Thus we have

$$\mathscr{J}(w,\varphi) = \mathscr{J}(f,\varphi) + a\mathscr{J}(1_M,\varphi) \leq \|f\|_{\mathscr{D}^J_\infty(U)} \int_{U_0} \varphi(x)\,dx - a\int_{U_0} \varphi(x) \int_M J(x,y)\,dydx$$
$$\leq \left(\|f\|_{\mathscr{D}^J_\infty(U)} - a \inf_{x\in U_0} \int_M J(x,y)\,dy \right) \int_{U_0} \varphi(x)\,dx \leq C_a \int_{U_0} \varphi(x)\,dx$$

with

$$C_a := \|f\|_{\mathscr{D}^J_\infty(U)} - a \inf_{x\in U_0} \int_M j(x-y)\,dy,$$

where $(J1)_c$ was used in the latter inequality. Since J satisfies $(\mathrm{J}_+)_{r_0}$ and $\overline{U}_0, M \subset\subset B_{r_0/2}(x_0)$ we have

$$\inf_{x\in U_0} \int_M j(x-y)\,dy > 0.$$

Thus we may fix $a > 0$ sufficiently large such that $C_a \leq -k$. Hence

$$\mathscr{J}(w,\varphi) \leq -k \int_{U_0} \varphi(x)\,dx$$

as claimed. □

Proposition 3.5 (Weak maximum principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume that J satisfies (J1)$_a$, (J1)$_b$ and $\Lambda_{1,J}(\Omega) > 0$. Let $c, g \in L^\infty(\Omega)$ with $\|c^+\|_{L^\infty(\Omega)} < \Lambda_{1,J}(\Omega)$. Then every supersolution u of $Iu = c(x)u + g$ in Ω, $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$ satisfies*

$$\|u^-\|_{L^2(\Omega)} \leq \frac{\|g^-\|_{L^2(\Omega)}}{\Lambda_{1,J}(\Omega) - \|c^+\|_{L^\infty(\Omega)}}.$$

In particular, if $g \geq 0$ a.e. in Ω, then $u \geq 0$ a.e. in $\mathbb{R}^N$.

Proof. By Lemma 3.1 we have that $u^- \in \mathscr{D}^J(\Omega)$ and $\mathscr{J}(u^-,u^-) \leq -\mathscr{J}(u,u^-)$. Consequently,

$$\begin{aligned}\Lambda_{1,J}(\Omega)\|u^-\|^2_{L^2(\Omega)} &\leq \mathscr{J}(u^-,u^-) \leq -\mathscr{J}(u,u^-) \leq -\int_\Omega c(x)u(x)u^-(x)\,dx - \int_\Omega g(x)u^-(x)\,dx \\ &\leq \int_\Omega c(x)(u^-)^2(x)\,dx + \int_\Omega g^-(x)u^-(x)\,dx \\ &\leq \|c^+\|_{L^\infty(\Omega)}\|u^-\|^2_{L^2(\Omega)} + \|g^-\|_{L^2(\Omega)}\|u^-\|_{L^2(\Omega)}.\end{aligned}$$

Since $\|c^+\|_{L^\infty(\Omega)} < \Lambda_{1,J}(\Omega)$ by assumption, we conclude

$$\|u^-\|_{L^2(\Omega)} \leq \frac{\|g^-\|_{L^2(\Omega)}}{\Lambda_{1,J}(\Omega) - \|c^+\|_{L^\infty(\Omega)}}.$$

□

Remark 3.6. We note that a combination of Proposition 3.5 and Lemma 2.17 gives rise to a small volume maximum principle if J satisfies the assumptions of Lemma 2.17, see also Proposition 9.10 below.

Lemma 3.7. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume that J satisfies (J1)$_a$, (J1)$_b$ and $\Lambda_{1,J}(\Omega) > 0$. Moreover, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$ be a Carathéodory function such that there is some $\lambda \in (0,\Lambda_{1,J}(\Omega))$ with*

$$f(x,u) - f(x,\tilde{u}) \leq \lambda(u-\tilde{u}) \qquad \text{for all } x \in \Omega \text{ and } u,\tilde{u} \in \mathbb{R}.$$

Then there is at most one solution $u \in \mathscr{D}^J(\Omega)$ of $Iu = f(x,u)$ in Ω with $u \equiv 0$ on $\mathbb{R}^N \setminus \Omega$.

Proof. Let u_1 be another solution with $u_1 \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. Then $w := u - u_1 \in \mathscr{D}^J(\Omega)$ solves $Iw = c(x)w$ in Ω, $w = 0$ in $\mathbb{R}^N \setminus \Omega$ with

$$c \in L^\infty(\Omega), \qquad c(x) = \begin{cases} \dfrac{f(x,u(x)) - f(x,u_1(x))}{u(x)-u_1(x)} & \text{if } u(x) \neq u_1(x), \\ 0 & \text{if } u(x) = u_1(x), \end{cases}$$

By assumption, we have $\|c^+\|_{L^\infty(\Omega)} \leq \lambda < \Lambda_{1,J}(\Omega)$, and thus Proposition 3.5 yields $\pm w \geq 0$ in $\mathbb{R}^N$. Thus $u_1 = u$ a.e. in $\mathbb{R}^N$. □

Lemma 3.8. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and denote $r_0 := \mathrm{diam}(\Omega)$. Assume J satisfies (J1), $(J_+)_{2r_0}$. Furthermore let $g \in L^\infty(\Omega)$ and let $u \in \mathscr{D}^J(\Omega)$ be the unique solution of $Iu = g$ in Ω, $u = 0$ in $\mathbb{R}^N \setminus \Omega$ given by Corollary 2.9. Then there is a constant $C = C(N,J,\Omega) > 0$ such that*

$$\|u\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\Omega)}.$$

Proof. Denote

$$C_1 := \inf_{x\in\Omega} \int_{B_{r_0}(\Omega)\setminus B_{r_0/2}(\Omega)} j(x-y)\,dy.$$

Then by $(J_+)_{2r_0}$ we have $C_1 > 0$. Moreover, for $\varphi \in \mathscr{D}^J(\Omega)$, $\varphi \geq 0$ we have

$$\mathscr{J}(1_{B_{r_0/2}(\Omega)}, \varphi) = \int_{\mathbb{R}^N\setminus B_{r_0/2}(\Omega)} \int_{\Omega} \varphi(x)J(x,y)\,dxdy \geq C_1 \int_{\Omega} \varphi(x)\,dx \geq \frac{C_1}{\|g\|_{L^\infty(\Omega)}} \int_{\Omega} \varphi(x)g(x)\,dx.$$

Thus $f:\mathbb{R}^N \to \mathbb{R}$, $f(x) = C_1^{-1}\|g\|_{L^\infty(\Omega)} 1_{B_{r_0/2}(\Omega)}(x)$ is a supersolution of $If = g$ in Ω, $f \equiv 0$ on $\mathbb{R}^N \setminus \Omega$. Hence by Proposition 3.5 we have

$$u \leq f = C_1^{-1}\|g\|_{L^\infty(\Omega)} \quad \text{in } \Omega.$$

A similar argument with $-f$ in place of f gives $u \geq -C_1^{-1}\|g\|_{L^\infty(\Omega)}$ in Ω. Thus the statement follows with $C = C_1^{-1}$. □

Proposition 3.9 (Strong maximum principle (Variant 1))**.** *Assume that J satisfies (J1), (J_+). Moreover, let $\Omega \subset \mathbb{R}^N$ be an open set and assume that $\mathscr{C}_c^2(\Omega) \subset \mathscr{D}_\infty^J(\Omega)$. Furthermore, let $c \in L^\infty(\Omega)$, and let u be a supersolution of $Iu = c(x)u$ in Ω such that $u \geq 0$ a.e. in $\mathbb{R}^N$. Then either $u \equiv 0$ a.e. in $\mathbb{R}^N$, or*

$$\operatorname*{essinf}_K u > 0 \qquad \textit{for every compact subset } K \subset \Omega.$$

Proof. We assume that $u \not\equiv 0$ in $\mathbb{R}^N$. For given $x_0 \in \Omega$, it then suffices to show that $\operatorname*{essinf}_{B_r(x_0)} u > 0$ for $r > 0$ sufficiently small. Since $u \not\equiv 0$ in $\mathbb{R}^N$ there exists a bounded set $M \subset \mathbb{R}^N$ of positive measure with $x_0 \notin \overline{M}$ and such that

$$\delta := \operatorname*{essinf}_M u > 0. \tag{3.4}$$

By Lemma 2.17 we may fix $0 < r < \frac{1}{4}\operatorname{dist}(x_0, M \cup (\mathbb{R}^N \setminus \Omega))$ such that $\Lambda_{1,J}(B_{2r}(x_0)) > \|c\|_{L^\infty(\Omega)}$. Fix a function $f \in \mathscr{C}_c^2(\Omega)$ such that $0 \leq f \leq 1$ and

$$f(x) := \begin{cases} 1, & \text{for } |x-x_0| \leq r, \\ 0, & \text{for } |x-x_0| \geq 2r \end{cases}$$

Let $U_0 := B_{2r}(x_0)$ and $U_0' := B_{3r}(x_0)$. By Lemma 3.4, there exists $a > 0$ such that the function

$$w \in \mathscr{V}^J(U_0'), \qquad w(x) := f(x) + a1_M(x)$$

satisfies

$$\mathscr{J}(w,\varphi) \leq -\|c\|_{L^\infty(U_0)} \int_{U_0} \varphi(x)\,dx \leq \int_{U_0} c(x)w(x)\varphi(x)\,dx \tag{3.5}$$

for $\varphi \in \mathscr{D}^J(U_0), \varphi \geq 0$. Note that the function w satisfies

$$w \equiv 0 \quad \text{on } \mathbb{R}^N \setminus (U_0 \cup M), \qquad w \equiv a \quad \text{on } M, \tag{3.6}$$

We now consider the function $\tilde{v} := u - \frac{\delta}{a} w \in \mathscr{V}^J(U_0')$, which by (3.4) and (3.6) satisfies $\tilde{v} \geq 0$ on $\mathbb{R}^N \setminus U_0$. Hence, by assumption and (3.5), $\tilde{v}$ is a supersolution of the problem

$$I\tilde{v} = c(x)\tilde{v} \quad \text{in } U_0, \qquad \tilde{v} \equiv 0 \quad \text{on } \mathbb{R}^N \setminus U_0 \tag{3.7}$$

Since $\|c\|_{L^\infty(U_0)} < \Lambda_{1,J}(U_0)$, Proposition 3.5 implies that $\tilde{v} \geq 0$ a.e. in U_0, so that $u \geq \frac{\delta}{a} w = \frac{\delta}{a} > 0$ a.e. in $B_r(x_0)$. This ends the proof. □

We note that in Proposition 3.9 we did not assume Ω to be connected using (J_+). We may weaken assumption (J_+) on J by assuming connectedness of Ω and thus get the strong maximum principle for domains:

Proposition 3.10 (Strong maximum principle (Variant 2))**.** *Assume that J satisfies (J1), $(J_+)_{r_0}$ for some $r_0 > 0$. Moreover, let Ω be any domain in $\mathbb{R}^N$ and assume that $\mathscr{C}_c^2(\Omega) \subset \mathscr{D}_\infty^J(\Omega)$. Furthermore let $c \in L^\infty(\Omega)$ and let u be a supersolution of $Iu = c(x)u$ in Ω such that $u \geq 0$ a.e. in $\mathbb{R}^N$.*
Then either $u \equiv 0$ a.e. in $B_{r_0/2}(\Omega)$, or

$$\operatorname*{essinf}_K u > 0 \qquad \textit{for every compact subset } K \subset \Omega.$$

Proof. Let W denote the set of points $y \in \Omega$ such that $\operatorname*{essinf}_{B_r(y)} u > 0$ for $r > 0$ sufficiently small. Note that we have from $(J_+)_{r_0}$

$$\int_{\mathbb{R}^N \setminus B_r(0)} j(y)\, dy > 0 \qquad \text{for all } r \in (0, r_0].$$

Moreover, $\Lambda_{1,J}(B_r(z)) > 0$ for any $z \in \mathbb{R}^N$ and $r \in (0, r_0]$. Furthermore, we have $\Lambda_{1,J}(B_r(z)) \to \infty$ for $r \to 0$ by Lemma 2.17. We claim the following:

$$\text{If } x_0 \in \Omega \text{ is such that } u \not\equiv 0 \text{ in } B_{\frac{r_0}{2}}(x_0), \text{ then } x_0 \in W. \tag{3.8}$$

To prove this, let $x_0 \in \Omega$ be such that $u \not\equiv 0$ in $B_{\frac{r_0}{2}}(x_0)$. Then there exists a bounded set $M \subset B_{\frac{r_0}{2}}(x_0)$ of positive measure with $x_0 \notin \overline{M}$ and such that

$$\delta := \inf_M u > 0 \tag{3.9}$$

By Lemma 2.17, we may fix $0 < r < \frac{1}{4} \min\{r_0\,, \operatorname{dist}(x_0, M \cup (\mathbb{R}^N \setminus \Omega))\}$ such that $\Lambda_{1,J}(B_{2r}(x_0)) > \|c\|_{L^\infty(\Omega)}$ and $U_0' := B_{3r}(x_0) \subset \Omega$. Put $U_0 := B_{2r}(x_0)$ and, since $U_0' \subset \Omega$, let $f \in \mathscr{C}_c^2(U_0) \subset \mathscr{D}_\infty^J(U_0)$ be given as in the proof of Proposition 3.9. Then for $a > 0$ we have $w = f + a 1_M \in \mathscr{V}^J(U_0')$, where by Lemma 3.4 we may choose a large such that we have

$$\mathscr{J}(w, \varphi) \leq -\|c\|_{L^\infty(U_0)} \int_{U_0} w(x)\varphi(x)\, dx \leq \int_{U_0} c(x) w(x) \varphi(x)\, dx. \tag{3.10}$$

Since w satisfies

$$w \equiv 0 \quad \text{on } \mathbb{R}^N \setminus (U_0 \cup M), \qquad w \equiv a \quad \text{on } M, \tag{3.11}$$

we may proceed precisely as in the proof of Proposition 3.9 to prove that $u \geq \frac{\delta}{a} > 0$ a.e. in $B_r(x_0)$ for $a > 0$ sufficiently large, so that $x_0 \in W$. Hence (3.8) is true.
From (3.8) it immediately follows that W is both open and closed in Ω. Moreover, if $u \not\equiv 0$ on $\{x \in \mathbb{R}^N : \text{dist}(x,\Omega) < \frac{r_0}{2}\}$, then W is nonempty and therefore $W = \Omega$ by the connectedness of Ω. This ends the proof. □

Remark 3.11. We emphasize that in Proposition 3.9 and Proposition 3.10 we do not need Ω to be bounded. In particular these results hold for $\Omega = \mathbb{R}^N$.

Lemma 3.12. *Let $r_0 > \rho > 0$ and $x_0 \in \mathbb{R}^N$ be given and assume J satisfies (J1), $(J_+)_{2r_0}$ and that $\mathscr{C}_c^2(B_\rho(x_0)) \subset \mathscr{D}_\infty^J(B_\rho(x_0))$. Then there is $\delta_0 > 0$ such that for all $\delta \in (0,\delta_0]$ the following is true: The unique solution $u_\delta \in \mathscr{D}^J(B_\rho(x_0))$ of*

$$\begin{cases} Iu_\delta = 1_{B_{3\rho/4}(x_0)} - \delta 1_{B_\rho(x_0)\setminus B_{3\rho/4}(x_0)} & \text{in } B_\rho(x_0) \\ u_\delta \equiv 0 & \text{on } \mathbb{R}^N \setminus B_\rho(x_0) \end{cases} \tag{3.12}$$

satisfies $\operatorname*{essinf}_K u_\delta > 0$ *for all $K \subset\subset B_\rho(x_0)$.*

Proof. In the following we put $B_r := B_r(x_0)$ for any $r > 0$. By Corollary 2.9 , for any $\delta > 0$ there is a unique solution $u_\delta \in \mathscr{D}^J(B_\rho)$ of (3.12), and $u_\delta \in L^\infty(\mathbb{R}^N)$ by Lemma 3.8. Fix $f \in \mathscr{C}_c^2(B_{\rho/2}) \subset \mathscr{D}_\infty^J(B_\rho)$, $0 \leq f \leq 1$ such that $f \equiv 1$ on $B_{\rho/4}$. Then with $C_f = \|f\|_{\mathscr{D}_\infty^J(B_\rho)} + 1$ we have

$$\mathscr{J}(f,\varphi) \leq C_f \int_{B_\rho} \varphi(x)\,dx \quad \text{for every } \varphi \in \mathscr{D}^J(B_\rho),\ \varphi \geq 0.$$

Moreover, we have for any $\varphi \in \mathscr{D}^J(B_\rho \setminus B_{5\rho/8})$, $\varphi \geq 0$:

$$\begin{aligned} \mathscr{J}(f,\varphi) &= -\int_{B_{\rho/2}} f(x) \int_{B_\rho\setminus B_{5\rho/8}} \varphi(y)J(x,y)\,dy\,dx \leq -\int_{B_\rho\setminus B_{5\rho/8}} \varphi(y) \int_{B_{\rho/4}} J(x,y)\,dx\,dy \\ &\leq -\frac{1}{2}\inf_{y\in B_\rho}\left(\int_{B_{\rho/8}} J(x,y)\,dx\right)\int_{B_\rho\setminus B_{5\rho/8}} \varphi(y)\,dy. \end{aligned}$$

By assumption $(J_+)_{2r_0}$ and Remark 2.1, we have

$$\delta_0 := \frac{1}{2C_f}\inf_{y\in B_\rho}\int_{B_{\rho/8}} J(x,y)\,dx > 0, \tag{3.13}$$

Next, let $\varphi \in \mathscr{D}^J(B_\rho)$, $\varphi \geq 0$ and fix $\psi \in \mathscr{C}_c^2(B_{3\rho/4})$ with $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on $B_{5\rho/8}$. Note that by Lemma 2.14 and Remark 2.15 it follows that $\psi\varphi, (1-\psi)\varphi \in \mathscr{D}^J(B_\rho)$ and thus

$$\mathscr{J}(f,\varphi) = \mathscr{J}(f,\psi\varphi) + \mathscr{J}(f,(1-\psi)\varphi)$$

$$
\begin{aligned}
&\leq C_f \int\limits_{B_{3\rho/4}} \psi(x)\varphi(x)\,dx - \int\limits_{B_\rho\setminus B_{5\rho/8}} (1-\psi(y))\varphi(y) \int\limits_{B_{\rho/4}} J(x,y)\,dx\,dy \\
&\leq C_f \int\limits_{B_{3\rho/4}} \varphi(x)\,dx - 2C_f\delta_0 \int\limits_{B_\rho\setminus B_{5\rho/8}} (1-\psi(y))\varphi(y)\,dy \\
&\leq C_f \int\limits_{B_{3\rho/4}} \varphi(x)\,dx - 2C_f\delta_0 \int\limits_{B_\rho\setminus B_{3\rho/4}} \varphi(y)\,dy
\end{aligned}
$$

Thus f is a subsolution of

$$
If = C_f\big(1_{B_{3\rho/4}} - 2\delta_0 1_{B_\rho\setminus B_{3\rho/4}}\big) \quad \text{in } B_\rho, \quad f=0 \quad \text{on } \mathbb{R}^N\setminus B_\rho. \tag{3.14}
$$

Consequently, for any $\delta\in(0,\delta_0]$, the function $\tilde u = u_\delta - \frac{1}{C_f}f$ is a supersolution of

$$
I\tilde u = 0 \quad \text{in } B_\rho, \qquad \tilde u = 0 \quad \text{on } \mathbb{R}^N\setminus B_\rho.
$$

Thus Proposition 3.5 gives

$$
u_\delta \geq \frac{1}{C_f} f \geq 0 \quad \text{a.e. in } B_\rho. \tag{3.15}
$$

In particular,

$$
\operatorname*{essinf}_{B_{\rho/4}} u_\delta \geq \frac{1}{C_f} > 0. \tag{3.16}
$$

It remains to show that for any $\delta\in(0,\delta_0]$ and $K\subset\subset B_\rho\setminus B_{\rho/8}$ we have

$$
\operatorname*{essinf}_{K} u_\delta > 0. \tag{3.17}
$$

Fix $r_1, r_2\in\mathbb{R}$ with $\frac{3\rho}{16} < r_1 < \frac{\rho}{4} < r_2 < \rho$ and denote $M := \overline{B_{r_2}}\setminus B_{r_1}$ and $U := B_\rho\setminus B_{3\rho/16}$. Let $g\in\mathscr{C}_c^2(U)$ with $0\leq g\leq 1$ and $g\equiv 1$ in M. Fix $C_g = \|g\|_{\mathscr{D}^J_\infty(B_\rho)}+1$. Then $\mathscr{J}(g,\varphi)\leq C_g\int\limits_U\varphi(x)\,dx$ for all $\varphi\in\mathscr{D}^J(U)$, $\varphi\geq 0$. Next consider $w = \frac{\delta_0}{C_g}g + \frac{1}{C_f}1_{B_{\rho/8}}\in\mathscr{V}^J(U)$. Note that we have $u_\delta\geq w$ in $\mathbb{R}^N\setminus U$ by (3.15). Moreover, for $\varphi\in\mathscr{D}^J(U)$, $\varphi\geq 0$ we have

$$
\begin{aligned}
\mathscr{J}(w,\varphi) &= \frac{\delta_0}{C_g}\mathscr{J}(g,\varphi) - \frac{1}{C_f}\int\limits_U \varphi(y)\int\limits_{B_{\rho/8}} J(x,y)\,dx\,dy \\
&\leq (\delta_0 - 2\delta_0)\int\limits_U \varphi(y)\,dy = -\delta_0\int\limits_U\varphi(y)\,dy.
\end{aligned}
$$

Thus $v = u_\delta - w\in\mathscr{V}^J(U)$ is a supersolution of $Iv=0$ in U, $v=0$ in $\mathbb{R}^N\setminus U$. Proposition 3.5 gives $v\geq 0$ a.e. in U and thus $u_\delta\geq\frac{\delta_0}{C_g}g = \frac{\delta_0}{C_g}$ a.e. in M. Combining this with (3.16) we have

$$
\operatorname*{essinf}_{B_{r_2}} u_\delta \geq \min\left\{\frac{1}{C_f}, \frac{\delta_0}{C_g}\right\} \tag{3.18}
$$

Hence (3.17) holds since r_2 were chosen arbitrarily, and the proof is finished. □

Remark 3.13. The proof given above allows to derive more information on possible choices of δ_0 and on estimates for u_δ in Lemma 3.12. We will need this information in Section 9 below. For this fix $\xi \in C^2([0,\infty))$ with $\xi \equiv 1$ on $[0,\frac{7}{8}]$, $\xi \equiv 0$ on $[1,\infty)$ and $0 \le \xi \le 1$ in $\mathbb{R}$. Let $x_0 \in \mathbb{R}^N$ and for $r > 0$ denote $\xi_r : \mathbb{R}^N \to [0,\infty)$, $\xi_r(x) = \xi(|x-x_0|/r)$. Let $r_0 > \rho > 0$ be given and assume J satisfies (J1), $(\mathrm{J}_+)_{2r_0}$ and that $\mathscr{C}_c^2(B_\rho(x_0)) \subset \mathscr{D}_\infty^J(B_\rho(x_0))$. Then the following is true:

(i) The constant δ_0 in Lemma 3.12 can be chosen as

$$\delta_0 = \frac{1}{2+2\|\xi_{\rho/2}\|_{\mathscr{D}_\infty^J(B_\rho(x_0))}} \inf_{y \in B_\rho(x_0)} \int\limits_{B_{\rho/8}(x_0)} J(x,y)\,dx > 0,$$

since $f := \xi_{\rho/2} \in \mathscr{C}_c^2(B_{\rho/2}(x_0))$ is a suitable choice in the proof (see (3.13)).

(ii) For any $\delta \in (0,\delta_0)$ the solution u_δ of (3.12) satisfies

$$\operatorname*{essinf}_{B_{3\rho/4}(x_0)} u_\delta \ge \min\left\{ \frac{1}{1+\|\xi_{\rho/2}\|_{\mathscr{D}_\infty^J(B_\rho)}}, \frac{\delta_0}{1+\|\xi_\rho - \xi_{3\rho/14}\|_{\mathscr{D}_\infty^J(B_\rho)}} \right\}$$

This follows from (3.18) by choosing $g = \xi_\rho - \xi_{3\rho/14} \in \mathscr{C}_c^2(B_\rho(x_0) \setminus B_{\rho/8}(x_0))$.

(iii) We have

$$\|u_\delta\|_{L^\infty(B_\rho(x_0))} \le \max\{\delta,1\} \left(\inf_{x \in B_\rho(x_0)} \int\limits_{B_{2\rho}(x_0)\setminus B_{3\rho/2}(x_0)} j(x-y)\,dy \right)^{-1}$$

for any $\delta \in (0,\delta_0)$ by Lemma 3.8.

Next we will extend the result of Proposition 3.5 to equations of the form

$$Iu = c(x)u \qquad \text{in } \mathbb{R}^N. \tag{3.19}$$

Proposition 3.14. *Assume that J satisfies (J1), $(J_+)_{r_0}$, and let $c \in L^\infty_{loc}(\mathbb{R}^N)$ with $c \le 0$. Then every supersolution u of (3.19) with $\liminf\limits_{|x|\to\infty} u(x) \ge 0$ satisfies either $u \equiv 0$ a.e. in $\mathbb{R}^N$, or*

$$\operatorname*{essinf}_K u > 0 \qquad \textit{for every compact subset } K \subset \mathbb{R}^N.$$

Proof. We only need to show that we have $u \ge 0$ in $\mathbb{R}^N$ since then an application of Proposition 3.10 finishes the proof. Since we have $\liminf\limits_{|x|\to\infty} u(x) \ge 0$ there is for every $\varepsilon > 0$ a radius $R > 0$ such that $u_\varepsilon(x) := u(x) + \varepsilon \ge 0$ on $\mathbb{R}^N \setminus B_R(0)$. Note that $u_\varepsilon \in \mathscr{V}^J(\mathbb{R}^N)$ for any $\varepsilon > 0$ and we have for $\varphi \in \mathscr{D}^J(B_R(0))$, $\varphi \ge 0$

$$\mathscr{J}(u_\varepsilon,\varphi) = \mathscr{J}(u,\varphi) \ge \int\limits_{B_R(0)} c(x)u(x)\varphi(x)\,dx \ge \int\limits_{B_R(0)} c(x)u_\varepsilon(x)\varphi(x)\,dx,$$

since $c \leq 0$. It follows that for every $\varepsilon > 0$ we have that u_ε is a supersolution of $Iu_\varepsilon = c(x)u_\varepsilon$ in $B_R(0)$, $u \equiv 0$ on $\mathbb{R}^N \setminus B_R(0)$. Since $c^+ \equiv 0 < \Lambda_{1,J}(B_R(0))$ for all $R > 0$ Proposition 3.5 gives $u_\varepsilon \geq 0$ in $B_R(0)$. Sending $\varepsilon \to 0$ we reach $u \geq 0$ as required. □

Lemma 3.15. *Assume that J satisfies (J1), $(J_+)_{r_0}$. Moreover, let $f \in C^1(\mathbb{R}^N \times \mathbb{R})$ with $\partial_u f(x,u) \leq 0$ for every $x \in \mathbb{R}^N$, $u \in \mathbb{R}$. Then there is at most one solution $u \in \mathscr{D}^J(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ of $Iu = f(x,u)$ in $\mathbb{R}^N$ with $\lim\limits_{|x|\to\infty} u(x) = 0$.*

Proof. Let u_1 be another solution with $\lim\limits_{|x|\to\infty} u_1(x) = 0$. Then $w := u - u_1 \in \mathscr{D}^J(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ solves $Iw = c(x)w$ in Ω, $\lim\limits_{|x|\to\infty} w(x) = 0$ with

$$c(x) = \int_0^1 \partial_u f(x, u_1(x) - t(u_1(x) - u(x)))\, dt.$$

By assumption we have $c \leq 0$ and $c \in L^\infty_{loc}(\mathbb{R}^N)$. Thus Proposition 3.14 yields $\pm w \geq 0$ in $\mathbb{R}^N$. Thus $u_1 = u$ a.e. in $\mathbb{R}^N$. □

4 Symmetry results for a general class of nonlocal problems

This section is devoted to prove our first main symmetry result. To state this result, we recall the following geometric assumptions on J and the set Ω which were already stated in the introduction. Note that we consider only kernel functions which satisfy (J1)$_{diff}$ in this part.

(D1) $\Omega \subset \mathbb{R}^N$ is an open bounded set which is Steiner symmetric in x_1, i.e. for every $x \in \Omega$ and $s \in [-1,1]$ we have $(sx_1, x_2, \dots, x_N) \in \Omega$.

(J2) The function J satisfies (J1)$_e$ and is strictly monotone in $|x_1|$, in the sense that for all $s,t \in \mathbb{R}$ with $|s| < |t|$ we have

$$\operatorname*{essinf}_{z' \in B_r^{N-1}(0)} \big(J(s,z') - J(t,z')\big) > 0 \qquad \text{for all } r > 0.$$

Note that (J2) in particular implies that J is positive on $\mathbb{R}^N \setminus \{0\}$, i.e. (J2) implies (J$_+$). The following is one of our main symmetry results (see also [50]).

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ satisfy (D1), let J satisfy (J1)$_{diff}$ and (J2), and assume that the nonlinearity f has the following properties.*

(F) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$ is a Carathéodory function such that for every $K > 0$ there exists $L = L(K) > 0$ with

$$\sup_{x \in \Omega} |f(x,u) - f(x,v)| \le L|u-v| \quad \textit{for } u,v \in [-K,K].$$

(F_{symm}) f is symmetric in x_1 and monotone in $|x_1|$, i.e. for every $u \in \mathbb{R}$, $x \in \Omega$ and $s \in [-1,1]$ we have $f(sx_1, x_2, \dots, x_N, u) \ge f(x,u)$.

Then every nonnegative solution $u \in L^\infty(\Omega) \cap \mathscr{D}^J(\Omega)$ of (P) (see p. 1) is symmetric in x_1. Moreover, either $u \equiv 0$ in $\mathbb{R}^N$, or u is strictly decreasing in $|x_1|$ in the sense given in Theorem 1.2 and therefore satisfies

$$\operatorname*{essinf}_K u > 0 \qquad \textit{for every compact set } K \subset \Omega. \tag{4.1}$$

As a direct consequence of Theorem 4.1 we have the following.

Corollary 4.2. *Let $J(z) = k(|z|_p)$, where k is as in Remark 2.2, $1 \le p < \infty$ and $|\cdot|_p$ is given in (2.3).*

(i) Let $\Omega \subset \mathbb{R}^N$ be Steiner symmetric in $x_1, \dots, x_N$, i.e. for every $x \in \Omega$, $j = 1, \dots, N$ and $s \in [0,2]$ we have $x - sx_j e_j \in \Omega$[4]. Moreover, let f fulfill (F) and be symmetric and monotone in $x_1, \dots, x_N$, i.e. for every $u \in \mathbb{R}$, $x \in \Omega$, $j = 1, \dots, N$ and $s \in [0,2]$ we have $f(x - sx_j e_j, u) \ge f(x,u)$. Then every nonnegative solution $u \in L^\infty(\Omega) \cap \mathscr{D}^J(\Omega)$ of (P) is symmetric in $x_1, \dots, x_N$. Moreover, either $u \equiv 0$ in $\mathbb{R}^N$, or u is strictly decreasing in $|x_1|, \dots, |x_N|$ and therefore satisfies (4.1).

[4]Here $e_j \in \mathbb{R}^N$ denotes the j-th unit vector for $j = 1, \dots, N$.

(ii) If $p = 2$, $\Omega \subset \mathbb{R}^N$ is a ball centered in 0 and f fulfills (F), (F_{symm}) and is radial in x i.e. $f(x,u) = f(|x|e_1,u)$ for $x \in \Omega$, then every nonnegative solution $u \in L^\infty(\Omega) \cap \mathscr{D}^J(\Omega)$ of (P) is radially symmetric. Moreover, either $u \equiv 0$ in $\mathbb{R}^N$, or u is strictly decreasing in $|x|$ and therefore satisfies (4.1).

Note that we do not assume Ω to be connected in Theorem 4.1. The positivity property (4.1) can be seen as a consequence of the long range nonlocal interaction enforced by (J2). Note that (J2) is not satisfied for kernels of the form

$$z \mapsto J(z) = 1_{B_r(0)}|z|^{-N-2s} \qquad \text{with } s \in (0,1),\, r > 0. \tag{4.2}$$

It is therefore natural to ask whether a result similar to Theorem 4.1 also holds for kernels of the type (4.2) which vanish outside a compact set and therefore model short range nonlocal interaction. We have the following result in this case for *a.e. positive solutions of (P) in Ω.*

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^N$ satisfy (D1), and let J satisfy $(J1)_{diff}$ and*

(J2)' For all $z' \in \mathbb{R}^{N-1}$, $s,t \in \mathbb{R}$ with $|s| < |t|$ we have $J(s,z') \geq J(t,z')$. Moreover, there is $r_0 > 0$ such that

$$\operatorname*{essinf}_{z' \in B_{r_0}^{N-1}(0)} \big(J(s,z') - J(t,z')\big) > 0 \qquad \text{for all } s,t \in \mathbb{R} \text{ with } |s| < |t| \leq r_0.$$

Furthermore, assume that the nonlinearity f satisfies (F) and (F_{symm}). Then every a.e. positive solution $u \in L^\infty(\Omega) \cap \mathscr{D}^J(\Omega)$ of (P) is symmetric in x_1 and strictly decreasing in $|x_1|$ on Ω. Consequently, it satisfies (4.1).

Note that the kernel class given by (4.2) satisfies $(J1)_{diff}$ and (J2)'.

Remark 4.4. Suppose that $J(z) = k(|z|_2)$, where k is as in Remark 2.2, and let $u \in L^\infty(\Omega) \cap \mathscr{D}^J(\Omega)$ be a solution of (P) such that the function $x \mapsto f(x,u(x))$ is bounded in Ω. Then it follows from [53, Theorem 2] that a solution satisfies $u \in C(\Omega)$. In the general case where J merely satisfies $(J1)_{diff}$, it is open if solutions $u \in L^\infty(\Omega) \cap \mathscr{D}^J(\Omega)$ of (P) are continuous.

The proofs of Theorem 4.1 and 4.3 can be found in Subsection 4.2. We will need some preliminary results for these proofs so that we can apply the moving plane method.

Remark 4.5. Suppose that (J2) is satisfied. Then, for every fixed $z' \in \mathbb{R}^N$, the function $t \mapsto J(t,z')$ is strictly decreasing in $|t|$ and therefore coincides a.e. on $\mathbb{R}$ with the function $t \mapsto \tilde{J}(t,z') := \lim_{s \to t^-} J(s,z')$. Hence J and the function $\tilde{J}$ differ only on a set of measure zero in $\mathbb{R}^N$. Replacing J by $\tilde{J}$ if necessary, we may therefore deduce from (J2) the symmetry property

$$J(-t,z') = J(t,z') \qquad \text{for every } z' \in \mathbb{R}^{N-1}, t \in \mathbb{R}. \tag{4.3}$$

This will be used in the following subsections.

4.1 A linear problem via a reflection

In the following, we consider a fixed open affine half space $H \subset \mathbb{R}^N$, and we let $Q : \mathbb{R}^N \to \mathbb{R}^N$ denote the reflection at ∂H. For the sake of brevity, we sometimes write $\bar{x}$ in place of $Q(x)$ for $x \in \mathbb{R}^N$. A function $v : \mathbb{R}^N \to \mathbb{R}^N$ is called antisymmetric (with respect to Q) if $v(\bar{x}) = -v(x)$ for $x \in \mathbb{R}^N$. As before, we consider an even measurable map $J : \mathbb{R}^N \setminus \{0\} \to [0,\infty)$ satisfying (J1)$_{diff}$. We also assume the following symmetry and monotonicity assumptions on J:

$$J(\bar{x}-\bar{y}) = J(x-y) \qquad \text{for a.e. } x,y \in \mathbb{R}^N, x \neq y; \tag{4.4}$$

$$J(x-y) \geq J(x-\bar{y}) \qquad \text{for a.e. } x,y \in H, x \neq y. \tag{4.5}$$

Remark 4.6. If (J1)$_{diff}$, (J2) and (4.3) are satisfied and

$$H = \{x \in \mathbb{R}^N : x_1 > \lambda\} \qquad \text{or} \qquad H = \{x \in \mathbb{R}^N : x_1 < -\lambda\}$$

for some $\lambda \in \mathbb{R}$, then (4.4) and (4.5) hold. In this case, J even satisfies the following strict variant of (4.5): Denote $H_b := \{x \in H : \operatorname{dist}(x,\partial H) > b\}$ for $b \geq 0$ then we have for all $r_0 > 0$

$$\operatorname*{essinf}_{\substack{x,y\in H_b\\ |x-y|\leq \min\{b,r_0\}}} (J(x-y) - J(x-\bar{y})) > 0 \qquad \text{for all } b > 0. \tag{4.6}$$

We will need this property in Proposition 4.11 below.

Lemma 4.7. *Let J satisfy (J1)$_{diff}$, (4.4) and (4.5). Moreover, let $U' \subset \mathbb{R}^N$ be an open set with $Q(U') = U'$, and let $v \in \mathscr{V}^J(U')$ be an antisymmetric function such that there is $\kappa \geq 0$ with $v \geq -\kappa$ on $H \setminus U$ for some open bounded set $U \subset H \cap U'$ with* $\operatorname{dist}(U, \mathbb{R}^N \setminus U') > 0$. *Then the function $w := 1_H (v+\kappa)^-$ is contained in $\mathscr{D}^J(U)$ and satisfies*

$$\mathscr{J}(w,w) \leq -\mathscr{J}(v,w) \tag{4.7}$$

Proof. Since v is antisymmetric we have by (4.4), the symmetry of U' and (4.5)

$$\begin{aligned}
\rho(v,U') &= \int_{U'\cap H}\int_{U'\cap H} (v(x)-v(y))^2 J(x-y)\,dxdy \\
&\quad + \int_{U'\setminus H}\int_{U'\setminus H} (v(x)-v(y))^2 J(x-y)\,dxdy + 2\int_{U'\setminus H}\int_{U'\cap H} (v(x)-v(y))^2 J(x-y)\,dxdy \\
&= 2\int_{U'\cap H}\int_{U'\cap H} \Big[(v(x)-v(y))^2 J(x-y) + (v(x)+v(y))^2 J(x-\bar{y})\Big]\,dxdy \\
&\geq \int_{U'\cap H}\int_{U'\cap H} \Big[(v(x)-v(y))^2 J(x-y) + [(v(x)-v(y))^2 + (v(x)+v(y))^2] J(x-\bar{y})\Big]\,dxdy \\
&\geq \int_{U'\cap H}\int_{U'\cap H} \Big[(v(x)-v(y))^2 J(x-y) + 2v^2(x) J(x-\bar{y})\Big]\,dxdy \\
&= \int_{U'}\int_{U'} (1_H v(x) - 1_H v(y))^2 J(x-y)\,dxdy = \rho(1_H v, U')
\end{aligned} \tag{4.8}$$

Since $\rho(1_H v, U') = \rho(1_H v + \kappa, U')$ we have $1_H v + \kappa \in \mathscr{V}^J(U')$. And thus, since $v \geq -\kappa$ in $H \setminus U$ and $\kappa \geq 0$, we have $(1_H v + \kappa)^- = 1_H(v+\kappa)^-$. Hence by Lemma 2.13 $w \in \mathscr{V}^J(U')$. Since $w \equiv 0$ in $\mathbb{R}^N \setminus U$ and $\mathrm{dist}(U, \mathbb{R}^N \setminus U') > 0$, the right hand side of (4.7) is well defined and finite by Lemma 2.11. To show (4.7) we first note that with $\tilde{v} = v + \kappa$ we have

$$[w + \tilde{v}]w = [1_H \tilde{v}^{\,+} + 1_{\mathbb{R}^N \setminus H} \tilde{v}] 1_H \tilde{v}^{\,-} \equiv 0 \qquad \text{on } \mathbb{R}^N$$

and therefore

$$[w(x) - w(y)]^2 + [v(x) - v(y)][w(x) - w(y)] = -\Big(w(x)[w(y) + \tilde{v}(y)] + w(y)[w(x) + \tilde{v}(x)] \Big)$$

for $x, y \in \mathbb{R}^N$. Using this identity in the following together with the antisymmetry of v, the symmetry properties of J and the fact that $w \equiv 0$ on $\mathbb{R}^N \setminus H$, we find that

$$\begin{aligned}
\mathscr{J}(w,w) + \mathscr{J}(v,w) &= \mathscr{J}(w,w) + \mathscr{J}(\tilde{v},w) \\
&= -\int_H \int_{\mathbb{R}^N} w(x)[w(y) + \tilde{v}(y)] J(x-y)\, dy dx && (4.9) \\
&= -\int_H \int_{\mathbb{R}^N} w(x)[1_H(y) \tilde{v}^{\,+}(y) + 1_{\mathbb{R}^N \setminus H} \tilde{v}(y)] J(x-y)\, dy dx \\
&= -\int_H \int_H w(x)[\tilde{v}^{\,+}(y) J(x-y) + (-v(y) + \kappa) J(x - \bar{y})]\, dy dx \\
&= -\int_H \int_H w(x)[\tilde{v}^{\,+}(y) J(x-y) + (-\tilde{v}(y) + 2\kappa) J(x - \bar{y})]\, dy dx \\
&\leq -\int_H \int_H w(x)[\tilde{v}^{\,+}(y) \left(J(x-y) - J(x-\bar{y})\right) + 2\kappa J(x - \bar{y})]\, dy dx \leq 0, && (4.10)
\end{aligned}$$

where in the last step we used $J(x-y) \geq J(x-\bar{y}) \geq 0$ for $x, y \in H$. Hence (4.7) is true, and in particular we have $\mathscr{J}(w,w) < \infty$. Since $w \equiv 0$ on $\mathbb{R}^N \setminus U$, it thus follows that $w \in \mathscr{D}^J(U)$. □

In order to implement the moving plane method, we have to deal with antisymmetric supersolutions of a class of linear problems. The following notion is slightly more general than the one introduced in [50, Definition 3.3].

Definition 4.8. Let $U \subset H$ be an open bounded set and let $c \in L^\infty(U)$. We call an antisymmetric function $v : \mathbb{R}^N \to \mathbb{R}^N$ an *antisymmetric supersolution of the problem*

$$Iv = c(x)v \quad \text{in } U, \qquad v \equiv 0 \quad \text{on } H \setminus U \tag{4.11}$$

if $v \in \mathscr{V}^J(U')$ for some open bounded set $U' \subset \mathbb{R}^N$ with $Q(U') = U'$ and $\overline{U} \subset U'$, $v \geq 0$ on $H \setminus U$ and

$$\mathscr{J}(v, \varphi) \geq \int_U c(x) v(x) \varphi(x)\, dx \qquad \text{for all } \varphi \in \mathscr{D}^J(U),\ \varphi \geq 0. \tag{4.12}$$

Remark 4.9. Assume $(J1)_{diff}$ and (4.4), and let $\Omega \subset \mathbb{R}^N$ be an open bounded set such that $Q(\Omega \cap H) \subset \Omega$. Furthermore, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying (F) and such that

$$f(\bar{x}, \tau) \geq f(x, \tau) \qquad \text{for every } \tau \in \mathbb{R}, x \in H \cap \Omega. \tag{4.13}$$

If $u \in \mathscr{D}^J(\Omega)$ is a nonnegative solution of (P), then $v := u \circ Q - u$ is an antisymmetric supersolution of (4.11) with $U := \Omega \cap H$ and $c \in L^\infty(U)$ defined by

$$c(x) = \begin{cases} \dfrac{f(x, u(\bar{x})) - f(x, u(x))}{v(x)} & \text{if } v(x) \neq 0; \\ 0 & \text{if } v(x) = 0. \end{cases}$$

Indeed, since $u \in \mathscr{D}^J(\Omega)$, we have $v \in \mathscr{D}^J(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and thus $v \in \mathscr{V}^J(U')$ for any open set $U' \subset \mathbb{R}^N$. Moreover, $v \geq 0$ on $H \setminus U$ since u is nonnegative and $u \equiv 0$ on $H \setminus U$. Furthermore, if $\varphi \in \mathscr{D}^J(U)$, then $\varphi \circ Q - \varphi \in \mathscr{D}^J(\Omega)$ by the symmetry properties of J and since $Q(U) \subset \Omega$. If, in addition, $\varphi \geq 0$, then we have, using (4.4),

$$\begin{aligned} \mathscr{J}(v, \varphi) = \mathscr{J}(u \circ Q - u, \varphi) = \mathscr{J}(u, \varphi \circ Q - \varphi) &= \int_\Omega f(x, u(x))[\varphi(Q(x)) - \varphi(x)]\, dx \\ &= \int_{Q(U)} f(x, u(x))\varphi(Q(x))\, dx - \int_U f(x, u(x))\varphi(x)\, dx \\ &= \int_U [f(\bar{x}, u(\bar{x})) - f(x, u(x))]\varphi(x)\, dx \geq \int_U c(x)v(x)\varphi(x)\, dx. \end{aligned}$$

Here (4.13) was used in the last step. The boundedness of c follows from (F).

We now establish a weak maximum principle for antisymmetric supersolutions of (4.11).

Proposition 4.10. *Assume that J satisfies $(J1)_{diff}$, (4.4) and (4.5), and let $U \subset H$ be an open bounded set. Let $c \in L^\infty(U)$ with $\|c^+\|_{L^\infty(U)} < \Lambda_{1,J}(U)$, where $\Lambda_{1,J}(U)$ is given in (2.5). Then every antisymmetric supersolution v of (4.11) in U satisfies $v \geq 0$ a.e. in H.*

Proof. By Lemma 4.7 we have that $w := 1_H v^- \in \mathscr{D}^J(U)$ and $\mathscr{J}(w, w) \leq -\mathscr{J}(v, w)$. Consequently,

$$\begin{aligned} \Lambda_{1,J}(U)\|w\|^2_{L^2(U)} \leq \mathscr{J}(w, w) \leq -\mathscr{J}(v, w) \leq -\int_U c(x)v(x)w(x)\, dx &= \int_U c(x)w^2(x)\, dx \\ &\leq \|c^+\|_{L^\infty(U)}\|w\|^2_{L^2(U)}. \end{aligned}$$

Since $\|c^+\|_{L^\infty(U)} < \Lambda_{1,J}(U)$ by assumption, we conclude that $\|w\|_{L^2(U)} = 0$ and hence $v \geq 0$ a.e. in H. □

A combination of Proposition 4.10 with Lemma 2.17 immediately gives rise to an "antisymmetric" small volume maximum principle which generalizes the available variants for the fractional Laplacian, see [36, Proposition 3.3 and Corollary 3.4] and [64, Lemma 5.1]. We omit the details since Proposition 4.10 is sufficient for our purposes. Next, we prove a strong maximum principle for antisymmetric supersolutions which requires the strict inequality (4.6).

Proposition 4.11. *Assume that J satisfies (J1)$_{diff}$, (4.4) and (4.6). Moreover, let $U \subset H$ be an open bounded set and $c \in L^\infty(U)$. Furthermore, let v be an antisymmetric supersolution of (4.11) such that $v \geq 0$ a.e. in H. Then either $v \equiv 0$ a.e. in $\mathbb{R}^N$, or*

$$\operatorname*{essinf}_K v > 0 \qquad \textit{for every compact subset } K \subset U.$$

Proof. We assume that $v \not\equiv 0$ in $\mathbb{R}^N$. For given $x_0 \in U$, it then suffices to show that $\operatorname*{essinf}_{B_r(x_0)} v > 0$ for $r > 0$ sufficiently small. Since $v \not\equiv 0$ in $\mathbb{R}^N$ and v is antisymmetric with $v \geq 0$ in H, there exists a bounded set $M \subset H$ of positive measure with $x_0 \notin \overline{M}$ and such that

$$\delta := \inf_M v > 0. \tag{4.14}$$

By Lemma 2.17, we may fix $0 < r < \frac{1}{4}\mathrm{dist}(x_0, [\mathbb{R}^N \setminus U] \cup M)$ such that $\Lambda_{1,J}(B_{2r}(x_0)) > \|c\|_{L^\infty(U)}$. Next, we fix a function $f \in \mathscr{C}_c^2(\mathbb{R}^N)$ such that $0 \leq f \leq 1$ on $\mathbb{R}^N$ and

$$f(x) := \begin{cases} 1, & \text{for } |x - x_0| \leq r, \\ 0, & \text{for } |x - x_0| \geq 2r. \end{cases}$$

Moreover, we define

$$w : \mathbb{R}^N \to \mathbb{R}, \qquad w(x) := f(x) - f(\bar{x}) + a\big[1_M(x) - 1_M(\bar{x})\big],$$

where $a > 0$ will be fixed later. We also put $U_0 := B_{2r}(x_0)$ and $U_0' := B_{3r}(x_0) \cup Q(B_{3r}(x_0))$. Note that the function w is antisymmetric and satisfies

$$w \equiv 0 \quad \text{on } H \setminus (U_0 \cup M), \qquad w \equiv a \quad \text{on } M. \tag{4.15}$$

We claim that $w \in \mathscr{V}^J(U_0')$. Indeed, by Proposition 2.5(i) we have $f - f \circ Q \in \mathscr{D}^J(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \subset \mathscr{V}^J(U_0')$, whereas $1_M - 1_{Q(M)} \in \mathscr{V}^J(U_0')$ since $\mathrm{dist}(M \cup Q(M), U_0') > 0$ and M is bounded. Next, let $\varphi \in \mathscr{D}^J(U_0)$, $\varphi \geq 0$. By Proposition 2.5(ii) we have with $C = \|f\|_{\mathscr{D}^J_\infty(U_0)}$

$$\mathscr{J}(f, \varphi) \leq C \int_{U_0} \varphi(x)\, dx. \tag{4.16}$$

Since

$$f(\bar{x})\varphi(x) = 1_M(x)\varphi(x) = 1_{Q(M)}(x)\varphi(x) = 0 \qquad \text{for every } x \in \mathbb{R}^N,$$

we have

$$\begin{aligned}
\mathscr{J}(w,\varphi) &= \mathscr{J}(f,\varphi) - \mathscr{J}(f \circ Q, \varphi) + a\big[\mathscr{J}(1_M, \varphi) - \mathscr{J}(1_{Q(M)}, \varphi)\big] \\
&\leq C \int_{U_0} \varphi(x)\, dx + \int_{U_0} \int_{Q(U_0)} \varphi(x) f(\bar{y}) J(x-y)\, dy dx \\
&\quad - a\Big[\int_{U_0} \int_M \varphi(x) J(x-y)\, dy dx - \int_{U_0} \int_{Q(M)} \varphi(x) J(x-y)\, dy dx\Big]
\end{aligned}$$

$$\leq \Big(C + \sup_{x\in U_0} \int\limits_{Q(U_0)} J(x-y)\,dy\Big) \int\limits_{U_0} \varphi(x)\,dx - a \int\limits_{U_0} \varphi(x) \int\limits_{M} [J(x-y) - J(x-\bar{y})]\,dydx$$
$$\leq C_a \int\limits_{U_0} \varphi(x)\,dx$$

with

$$C_a := C + \sup_{x\in U_0} \int\limits_{Q(U_0)} J(x-y)\,dy - a \inf_{x\in U_0} \int\limits_{M} (J(x-y) - J(x-\bar{y}))\,dy \in \mathbb{R}$$

Since $\overline{U}_0 \subset H$, (4.6) implies that

$$\inf_{x\in U_0} \int\limits_{M} (J(x-y) - J(x,\bar{y}))\,dy > 0$$

Consequently, we may fix $a > 0$ sufficiently large such that $C_a \leq -\|c\|_{L^\infty(U_0)}$. Since $0 \leq w \leq 1$ in U_0, we then have

$$\mathscr{J}(w,\varphi) \leq -\|c\|_{L^\infty(U_0)} \int\limits_{U_0} \varphi(x)\,dx \leq \int\limits_{U_0} c(x)w(x)\varphi(x)\,dx. \tag{4.17}$$

We now consider the function $\tilde{v} := v - \frac{\delta}{a} w \in \mathscr{V}^J(U_0')$, which by (4.14) and (4.15) satisfies $\tilde{v} \geq 0$ on $H \setminus U_0$. Hence, by assumption and (4.17), $\tilde{v}$ is an antisymmetric supersolution of the problem

$$I\tilde{v} = c(x)\tilde{v} \quad \text{in } U_0, \qquad \tilde{v} \equiv 0 \quad \text{on } H \setminus U_0 \tag{4.18}$$

Since $\|c\|_{L^\infty(U_0)} < \Lambda_{1,J}(U_0)$, Proposition 4.10 implies that $\tilde{v} \geq 0$ a.e. in U_0, so that $v \geq \frac{\delta}{a} w = \frac{\delta}{a} > 0$ a.e. in $B_r(x_0)$. This ends the proof. □

Remark 4.12. We note that Proposition 4.11 could also be proved by applying Proposition 3.9 to an associated problem related to the difference kernel $(x,y) \mapsto J(x-y) - J(x-\bar{y})$. However, we believe that the above proof is more direct and intuitive. In contrast, in the context of time dependent problems, we will be forced to study associated problems related to the difference kernel, see Section 9 below.

Next we derive a variant of Proposition 4.11 which only relies on the following *local* strict monotonicity condition:

$$\text{There exists } r_0 > 0 \text{ such that } \operatorname*{essinf}_{\substack{x,y\in H_b \\ |x-y|\leq \min\{b,r_0\}}} (J(x-y) - J(x-\bar{y})) > 0 \text{ for all } b > 0, \tag{4.19}$$

where $H_b = \{x \in H \,:\, \operatorname{dist}(x,H) > b\}$ for $b \geq 0$ as above.

Proposition 4.13. *Assume that J satisfies (J1)$_{diff}$, (4.4), (4.5) and (4.19). Moreover, let $U \subset H$ be a subdomain and $c \in L^\infty(U)$. Furthermore, let v be an antisymmetric supersolution of (4.11) such that $v \geq 0$ a.e. in H.*
Then either $v \equiv 0$ a.e. in a neighborhood of $\overline{U}$, or

$$\operatorname*{essinf}_{K} v > 0 \qquad \textit{for every compact subset } K \subset U.$$

We stress that, in contrast to Proposition 4.11, we require connectedness of U here.

Proof. Let W denote the set of points $y \in U$ such that $\operatorname*{essinf}_{B_r(y)} v > 0$ for $r > 0$ sufficiently small, and let $r_0 > 0$ be as in (4.19). We claim the following.

$$\text{If } x_0 \in U \text{ is such that } v \not\equiv 0 \text{ in } B_{\frac{r_0}{2}}(x_0), \text{ then } x_0 \in W. \tag{4.20}$$

To prove this, let $x_0 \in U$ be such that $v \not\equiv 0$ in $B_{\frac{r_0}{2}}(x_0)$. Then there exists a bounded set $M \subset H \cap B_{\frac{r_0}{2}}(x_0)$ of positive measure with $x_0 \notin \overline{M}$ and such that

$$\delta := \inf_M v > 0 \tag{4.21}$$

By Lemma 2.17, we may fix $0 < r < \frac{1}{4}\min\{r_0\,, \operatorname{dist}(x_0, [\mathbb{R}^N \setminus U] \cup M)\}$ such that $\Lambda_{1,J}(B_{2r}(x_0)) > \|c\|_{L^\infty(U)}$. Next, we put $U_0 := B_{2r}(x_0)$ and $U_0' := B_{3r}(x_0) \cup Q(B_{3r}(x_0))$. Moreover, we define the functions $f \in \mathscr{C}_c^2(\mathbb{R}^N)$ and $w \in \mathscr{V}^J(U_0')$, depending on $a > 0$, as in the proof of Proposition 4.11. As noted there, w is antisymmetric and satisfies

$$w \equiv 0 \quad \text{on } H \setminus (U_0 \cup M), \qquad w \equiv a \quad \text{on } M. \tag{4.22}$$

As in the proof of Proposition 4.11, we also see that

$$\mathscr{J}(w, \varphi) \le C_a \int_{U_0} \varphi(x)\,dx \qquad \text{for all } \varphi \in \mathscr{D}(U_0), \varphi \ge 0$$

with

$$C_a := \|f\|_{\mathscr{D}^J_\infty(U_0)} + \sup_{x \in U_0} \int_{Q(U_0)} J(x-y)\,dy - a \inf_{x \in \overline{U}_0} \int_M (J(x-y) - J(x-\bar{y}))\,dy$$

Since $\overline{U}_0 \subset H \cap B_{\frac{r_0}{2}}(x_0)$ and $M \subset H \cap B_{\frac{r_0}{2}}(x_0)$, (4.19) and the continuity of the function $x \mapsto \int_M (J(x-y) - J(x-\bar{y}))\,dy$ on $\overline{U}_0$ imply that

$$\inf_{x \in \overline{U}_0} \int_M (J(x-y) - J(x,\bar{y}))\,dy > 0$$

Hence we may proceed precisely as in the proof of Proposition 4.11 to prove that $v \ge \frac{\delta}{a} > 0$ a.e. in $B_r(x_0)$ for $a > 0$ sufficiently large, so that $x_0 \in W$. Hence (4.20) is true.
From (4.20) it immediately follows that W is both open and closed in U. Moreover, if $v \not\equiv 0$ in $\{x \in H \,:\, \operatorname{dist}(x, U) < \frac{r_0}{2}\}$, then W is nonempty and therefore $W = U$ by the connectedness of U. This ends the proof. □

4.2 The moving plane argument

Proof of Theorem 4.1

We assume that $J : \mathbb{R}^N \setminus \{0\} \to [0, \infty)$ satisfies (J1)$_{diff}$ and (J2), $\Omega \subset \mathbb{R}^N$ satisfies (D1) and the nonlinearity f satisfies (F) and (F_{symm}). Moreover, we let $u \in L^\infty(\Omega) \cap \mathscr{D}^J(\Omega)$ be a nonnegative solution of (P). For $\lambda \in \mathbb{R}$, we consider the open affine half space

$$H_\lambda := \begin{cases} \{x \in \mathbb{R}^N \;:\; x_1 > \lambda\} & \text{if } \lambda \geq 0; \\ \{x \in \mathbb{R}^N \;:\; x_1 < \lambda\} & \text{if } \lambda < 0. \end{cases}$$

Moreover, we let $Q_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ denote the reflection at ∂H_λ, i.e. $Q_\lambda(x) = (2\lambda - x_1, x')$. By Remark 4.5, we may assume without loss of generality that (4.3) holds. As noted in Remark 4.6, J therefore satisfies the symmetry and monotonicity conditions (4.4) and (4.6) with H replaced by H_λ for $\lambda \in \mathbb{R}$. Let $\ell := \sup\limits_{x\in\Omega} x_1$. Setting $\Omega_\lambda := H_\lambda \cap \Omega$ for $\lambda \in \mathbb{R}$, we note that $Q_\lambda(\Omega_\lambda) \subset \Omega$ for all $\lambda \in (-\ell, \ell)$ and $Q_0(\Omega) = \Omega$ as a consequence of assumption (D1). Then for all $\lambda \in (-\ell, \ell)$, Remark 4.9 implies that $v_\lambda := u \circ Q_\lambda - u \in \mathscr{D}^J(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ is an antisymmetric supersolution of the problem

$$Iv = c_\lambda(x)v \quad \text{in } \Omega_\lambda, \qquad v \equiv 0 \quad \text{on } H_\lambda \setminus \Omega_\lambda \tag{4.23}$$

with

$$c_\lambda \in L^\infty(\Omega_\lambda) \quad \text{given by} \quad c_\lambda(x) = \begin{cases} \dfrac{f(x, u(Q_\lambda(x))) - f(x, u(x))}{v_\lambda(x)}, & v_\lambda(x) \neq 0; \\ 0, & v_\lambda(x) = 0. \end{cases}$$

Note that, as a consequence of (F) and since $u \in L^\infty(\Omega)$, we have

$$c_\infty := \sup_{\lambda \in (-\ell, \ell)} \|c_\lambda\|_{L^\infty(\Omega_\lambda)} < \infty.$$

We now consider the statement

$$(S_\lambda) \qquad \operatorname*{essinf}_K v_\lambda > 0 \qquad \text{for every compact subset } K \subset \Omega_\lambda.$$

Assuming that $u \not\equiv 0$ from now on, we will show (S_λ) for all $\lambda \in (0, \ell)$. Since $|\Omega_\lambda| \to 0$ as $\lambda \to \ell$, Lemma 2.17 implies that there exists $\varepsilon \in (0, \ell)$ such that $\Lambda_{1,J}(\Omega_\lambda) > c_\infty$ for all $\lambda \in [\varepsilon, \ell)$. Applying Proposition 4.10 we thus find that

$$v_\lambda \geq 0 \quad \text{a.e. in } H_\lambda \qquad \text{for all } \lambda \in [\varepsilon, \ell). \tag{4.24}$$

We now show

Claim 1: If $v_\lambda \geq 0$ a.e. in H_λ for some $\lambda \in (0, \ell)$, then (S_λ) holds.

To prove this, by Proposition 4.11 it suffices to show that $v_\lambda \not\equiv 0$ in $\mathbb{R}^N$. If, arguing by contradiction, $v_\lambda \equiv 0$ in $\mathbb{R}^N$, then ∂H_λ is a symmetry hyperplane of u. Since $\lambda \in (0, \ell)$ and $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, we then have $u \equiv 0$ in the nonempty set $\Omega_{-\ell+2\lambda}$. Setting $\lambda' = -\ell + \lambda$, we thus infer that $v_{\lambda'} \equiv 0$ in $\Omega_{\lambda'}$. Consequently, $v_{\lambda'} \equiv 0$ in $\mathbb{R}^N$ by Proposition 4.11. Thus u has the two different parallel symmetry hyperplanes ∂H_λ and $\partial H_{\lambda'}$. Since u vanishes outside a bounded set, this implies that $u \equiv 0$, which is a contradiction. Thus Claim 1 is proved.

Next we show

Claim 2: If (S_λ) holds for some $\lambda \in (0, \ell)$, then there is $\delta \in (0, \lambda)$ such that (S_μ) holds for all

$\mu \in (\lambda - \delta, \lambda)$.

To prove this, suppose that (S_λ) holds for some $\lambda \in (0,\ell)$. Using Lemma 2.17, we fix $s \in (0, |\Omega_\lambda|)$ such that $\Lambda_{1,J}(s) > c_\infty$, which implies that $\Lambda_{1,J}(U) > c_\infty$ for all open sets $U \subset \mathbb{R}^N$ with $|U| \le s$. Since Ω is bounded, we may also fix $\delta_0 > 0$ such that

$$|\Omega_\mu \setminus \Omega_{\mu+\delta_0}| < s/2 \qquad \text{for all } \mu \ge 0.$$

By Lusin's Theorem, there exists a compact subset $K \subset \Omega$ such that $|\Omega \setminus K| < s/4$ and such that the restriction $u|_K$ is continuous. For $\mu \ge 0$, we now consider the compact set

$$K_\mu := \overline{\Omega}_{\mu+\delta_0} \cap K \cap Q_\mu(K) \subset K \cap \Omega_\mu$$

and the open set $U_\mu := \Omega_\mu \setminus K_\mu$. Note that

$$|U_\mu| \le |\Omega_\mu \setminus \Omega_{\mu+\delta_0}| + |\Omega_\mu \setminus K| + |\Omega_\mu \setminus Q_\mu(K)| \le \frac{s}{2} + 2|\Omega \setminus K| < s \qquad \text{for } \mu \ge 0. \tag{4.25}$$

As a consequence, for $0 \le \mu \le \lambda$ we have $|K_\mu| > |\Omega_\mu| - s \ge |\Omega_\lambda| - s > 0$ and thus $K_\mu \ne \varnothing$. Property (S_λ) and the continuity of $u|_K$ imply that $\min\limits_{K_\lambda} v_\lambda > 0$. Thus, again by the continuity of $u|_K$, there exists $\delta \in (0, \min\{\lambda, \delta_0\})$ such that

$$\min_{K_\mu} v_\mu > 0 \qquad \text{for all } \mu \in [\lambda - \delta, \lambda].$$

Consequently, for $\mu \in (\lambda - \delta, \lambda)$, the function v_μ is an antisymmetric supersolution of the problem

$$Iv = c_\mu(x)v \quad \text{in } U_\mu, \qquad v \equiv 0 \quad \text{on } H_\mu \setminus U_\mu,$$

whereas $\Lambda_{1,J}(U_\mu) > c_\infty$ by (4.25) and the choice of s. Hence $v_\mu \ge 0$ in H_μ by Proposition 4.10, and thus (S_μ) holds by Claim 1. This proves Claim 2.

To finish the proof, we consider

$$\lambda_0 := \inf\{\tilde\lambda \in (0,\ell) \;:\; (S_\lambda) \text{ holds for all } \lambda \in (\tilde\lambda, \ell)\} \quad \in [0,\ell).$$

We then have $v_{\lambda_0} \ge 0$ in H_{λ_0}. Hence Claim 1 and Claim 2 imply that $\lambda_0 = 0$. Since the procedure can be repeated in the same way starting from $-\ell$, we find that $v_0 \equiv 0$. Hence the function u has the asserted symmetry and monotonicity properties.

It remains to show (4.1). So let $K \subset \Omega$ be compact. Replacing K by $K \cup Q_0(K)$ if necessary, we may assume that K is symmetric with respect to Q_0. Let $K' := \{x \in K \;:\; x_1 \le 0\}$. Since for $\lambda > 0$ sufficiently small $Q_\lambda(K')$ is a compact subset of Ω_λ, the property (S_λ) and the symmetry of u then imply that

$$\operatorname*{essinf}_K u = \operatorname*{essinf}_{K'} u \ge \operatorname*{essinf}_{Q_\lambda(K')} v_\lambda > 0,$$

as claimed in (4.1).

Proof of Theorem 4.3

Throughout the remainder of this section, we assume that $J:\mathbb{R}^N\setminus\{0\}\to[0,\infty)$ is even and satisfies $(\mathrm{J1})_{diff}$ and (J2)', $\Omega\subset\mathbb{R}^N$ satisfies (D1) and the nonlinearity f satisfies (F) and (F_{symm}). Moreover, we let $u\in L^\infty(\Omega)\cap\mathscr{D}^J(\Omega)$ denote an a.e. positive solution of (P). For $\lambda\in\mathbb{R}$, we let H_λ, Q_λ, Ω_λ, c_λ and v_λ be defined as in Section 4.2, and again we put $\ell:=\sup_{x\in\Omega} x_1$. As a consequence of $(\mathrm{J1})_{diff}$ and (J2)', we may assume that J satisfies (4.4) (4.5) and (4.19) with H replaced by H_λ for $\lambda\in\mathbb{R}$ (the argument of Remark 4.6 still applies). As in Section 4.2, we then consider the statement

$$(S_\lambda)\qquad \operatorname*{essinf}_K v_\lambda>0\qquad \text{for every compact subset } K\subset\Omega_\lambda.$$

We wish to show (S_λ) for all $\lambda\in(0,\ell)$. As in the previous proof, we find $\varepsilon\in(0,\ell)$ such that

$$v_\lambda\geq 0\quad \text{a.e. in } H_\lambda\qquad \text{for all } \lambda\in[\varepsilon,\ell). \tag{4.26}$$

We now show

Claim 1: If $v_\lambda\geq 0$ a.e. in H_λ for some $\lambda\in(0,\ell)$, then (S_λ) holds.

To prove this, we argue by contradiction. If (S_λ) does not hold, then, by Proposition 4.13, there exists a connected component Ω' of Ω_λ and a neighborhood N of $\overline{\Omega'}$ such that $v_\lambda\equiv 0$ in N. However, since $\lambda\in(0,\ell)$, the set $\tilde N:=Q_\lambda(N\setminus\Omega)\cap\Omega$ has positive measure and $v_\lambda\equiv 0$ in $\tilde N$ by the antisymmetry of v_λ. However, $v\equiv -u$ on $\tilde N$, so $u\equiv 0$ a.e. on $\tilde N$, contrary to the assumption that $u>0$ a.e. in Ω. Thus Claim 1 is proved.

Precisely as in the proof of Theorem 4.1 we may now show

Claim 2: If (S_λ) holds for some $\lambda\in(0,\ell)$, then there is $\delta\in(0,\lambda)$ such that (S_μ) holds for all $\mu\in(\lambda-\delta,\lambda)$.

Moreover, based on (4.26), Claim 1 and Claim 2, we may now finish the proof of Theorem 4.3 precisely as in the end of the proof of Theorem 4.1.

5 The fractional Laplacian

In this section we focus on the bilinear form corresponding to the fractional Laplacian. Fix $s \in (0,1)$ and $J_s(y) := c_{N,s}|y|^{-N-2s}$ for $y\mathbb{R}^N$, $y \neq 0$, where

$$c_{N,s} := s(1-s)\pi^{-N/2}4^s \frac{\Gamma(\frac{N}{2}+s)}{\Gamma(2-s)}.$$

Then for an open set $\Omega \subset \mathbb{R}^N$ define

$$\mathscr{H}_0^s(\Omega) := \{u \in L^2(\mathbb{R}^N) \ : \ \mathscr{J}_s(u,u) < \infty \text{ and } u = 0 \text{ on } \mathbb{R}^N \setminus \Omega\},$$

where we put

$$\mathscr{J}_s(u,v) := \frac{c_{N,s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dxdy$$

We note that for $\Omega \subset \mathbb{R}^N$ bounded we have that $\mathscr{H}_0^s(\Omega)$ coincides with $\mathscr{D}^{J_s}(\Omega)$ (for more information on the space $\mathscr{H}_0^s(\Omega)$ and other *fractional* function spaces we refer to [48, 32]). The next Lemma contains some well known properties of the bilinear form $\mathscr{J}_s$. We include a proof for the readers convenience (see e.g. [32, 49]).

Lemma 5.1. *Let $s \in (0,1)$, $N \in \mathbb{N}$ and J_s be defined as above. Then the following assertions hold.*

(i) For any $A \subset \mathbb{R}^N$ measurable and any $x \in \mathbb{R}^N$ we have

$$\kappa_{s,A} := \int_{\mathbb{R}^N \setminus A} J_s(x-y)\, dy \geq K_{N,s}|A|^{-\frac{2s}{N}},$$

where $K_{N,s} := c_{N,s}\frac{N}{2s}\omega_N^{1+2s/N}$. In particular, for $\Omega \subset \mathbb{R}^N$ open and bounded we have (cf. Proposition 2.3 and Lemma 2.17)

$$\mathscr{J}_s(u,u) \geq K_{N,s}|\Omega|^{-2s/N}\|u\|^2_{L^2(\Omega)} \qquad \text{for all } u \in \mathscr{H}_0^s(\Omega).$$

(ii) Let H be a halfspace, then

$$\kappa_{s,H} := \int_{\mathbb{R}^N \setminus H} J_s(x-y)\, dy = \frac{4^s\Gamma(\frac{1}{2}+s)}{\sqrt{\pi}\,\Gamma(1-s)}[\operatorname{dist}(x,\partial H)]^{-2s}.$$

(iii) Let H be a halfspace and define $\bar{J}_s(x,y) := J_s(x-y) - J_s(x-\bar{y})$, where $x \mapsto \bar{x}$ is the reflection about the hyperplane ∂H, then

$$\bar{J}_s(x,y) \geq (1-5^{-N/2-s})J_s(x,y) \qquad \text{for } x,y \in H \text{ with } |x-y| \leq \min\{\operatorname{dist}(x,\partial H), \operatorname{dist}(y,\partial H)\}.$$

Proof. To see (i) and following the proof of Lemma 2.17 note that

$$\int_{\mathbb{R}^N\setminus A} J_s(x-y)\,dy \geq \int_{\mathbb{R}^N\setminus B_\rho(0)} J_s(y)\,dy,$$

where ρ is choosen such that $|B_\rho(0)| = |A|$, i.e. $\rho = |A|^N \omega_N^{-N}$. The last integral then gives

$$\int_{\mathbb{R}^N\setminus B_\rho(0)} J_s(y)\,dy = N\omega_N \int_\rho^\infty r^{-1-2s}\,dr = \frac{N\omega_N}{2s}\rho^{-2s} = \frac{N}{2s}\omega_N^{1+2s/N}|A|^{-\frac{2s}{N}}.$$

To proof (ii) we will use polar coordinates.

$$\begin{aligned}
\int_{\mathbb{R}^N\setminus H} \frac{c_{N,s}}{|x-y|^{N+2s}}\,dy &= c_{N,s}\int_{x_1}^\infty \int_{\mathbb{R}^{N-1}} (y_1^2+|y'|^2)\frac{-N/s-s}{}dy'dy_1\\
&= c_{N,s}(N-1)\omega_{N-1}\int_{x_1}^\infty\int_0^\infty (y_1^2+r^2)\frac{-N/s-s^{N-2}}{r}\,drdy_1\\
&= \frac{c_{N,s}(N-1)\pi^{\frac{N-1}{2}}\Gamma(1/2+s)\Gamma(\frac{N-1}{2})}{2\Gamma(\frac{N+1}{2})\Gamma(N/2+s)}2\int_{x_1}^\infty y_1^{-2s-1}\,dy_1\\
&= \frac{c_{N,s}\pi^{\frac{N-1}{2}}\Gamma(1/2+s)}{\Gamma(N/2+s)s}x_1^{-2s} = \frac{4^s\Gamma(1/2+s)}{\sqrt{\pi}\Gamma(1-s)}\operatorname{dist}(x,\partial H)^{-2s}.
\end{aligned}$$

Finally to see (iii), let $d>0$ and $x,y\in H$ with $|x-y|\leq d\leq \min\{\operatorname{dist}(x,\partial H),\operatorname{dist}(y,\partial H)\}$. Then $|x-\bar{y}|^2 \geq |x-y|^2+4d^2$ and therefore

$$\frac{|x-y|^2}{|x-\bar{y}|^2} \leq \frac{|x-y|^2}{|x-y|^2+4d^2} \leq \frac{1}{5},$$

which implies that

$$\frac{\bar{J}_s(x,y)|x-y|^{N+2s}}{c_{N,s}} = \left(1-\left(\frac{|x-y|^2}{|x-\bar{y}|^2}\right)^{\frac{N+2s}{2}}\right) \geq 1-5^{-N/2-s}$$

as claimed. □

Let $s\in(0,1)$. The *fractional Laplacian* $(-\Delta)^s$ is usually defined via the Fourier transform $\mathscr{F}$: For $u\in C_c^2(\mathbb{R}^N)$, we set

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}(|\cdot|^{2s}\mathscr{F}(u))(x) = \tilde{c}_{N,s}^{-1}\ P.V.\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy$$

with $\tilde{c}_{N,s} = \int_{\mathbb{R}^N}\frac{1-\cos(\xi_1)}{|\xi|^{N+2s}}\,d\xi$, where the last equality follows from [32, Proposition 3.3]. The following identity is well known to experts, but it is hard to find a simple and direct proof in the literature. The proof we give here is similar to but somewhat simpler than the proof given in [39, 16] which rely on properties of (modified) Bessel functions.

Lemma 5.2. *For any $s \in (0,1)$, $N \in \mathbb{N}$ we have $\tilde{c}_{N,s} = \frac{1}{c_{N,s}}$. Consequently (cf. Proposition 2.5(ii)),*

$$(-\Delta)^s u(x) = P.V. \int_{\mathbb{R}^N} (u(x) - u(y)) J_s(x-y)\, dy \qquad \textit{for any } u \in C_c^2(\mathbb{R}^N).$$

Proof. Let $s \in (0,1)$. We will start by calculating the constant $\tilde{c}_{N,s}$ in the case $N = 1$. Thus we have to show

$$\int_{-\infty}^{\infty} \frac{1-\cos(x)}{|x|^{1+2s}}\, dx = \frac{\sqrt{\pi}}{s4^s} \frac{\Gamma(1-s)}{\Gamma(s+\frac{1}{2})} \left(= c_{1,s}^{-1} \right). \tag{5.1}$$

Note that the left-hand side can can be transformed, using properties of trigonometric functions and a substitution, to

$$\int_{-\infty}^{\infty} \frac{1-\cos(x)}{|x|^{1+2s}}\, dx = 2\int_0^{\infty} \frac{2\sin^2(x/2)}{x^{1+2s}}\, dx = 4^{1-s} \int_0^{\infty} \frac{\sin^2(t)}{t^{1+2s}}\, dt.$$

Furthermore using the identities (see e.g. [61, Chapter 2, pp.35+pp.41])

$$\Gamma(1-z) = \frac{\pi}{\Gamma(z)\sin(\pi z)} \quad \text{and} \quad \Gamma(2z) = \frac{\Gamma(z)\Gamma(z+1/2)}{2^{1-2z}\sqrt{\pi}},$$

also the right-hand side can be transformed into

$$\begin{aligned} \frac{\sqrt{\pi}}{s4^s} \frac{\Gamma(1-s)}{\Gamma(s+\frac{1}{2})} &= \frac{4^{-s}\sqrt{\pi}}{\sin(\pi s) s\Gamma(s)\Gamma(s+1/2)} = \frac{\sqrt{\pi}}{\sin(\pi s) 2s\Gamma(2s)} \\ &= \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(2s+1)} = \frac{B(s,1-s)}{\Gamma(2s+1)}, \end{aligned}$$

where B is for $z, w > 0$ the Beta-function, which fulfills the following identities (see [61, Chapter 5 pp.61+ eq.(1) on pp.62])

$$B(w,z) := \int_0^1 t^{w-1}(1-t)^{z-1}\, dt = \int_0^{\infty} \frac{k^{w-1}}{(k+1)^{w+z}}\, dk = \frac{\Gamma(w)\Gamma(z)}{\Gamma(w+z)} = B(z,w). \tag{5.2}$$

With these transformations of the left- and right-hand side of 5.1 it is thus enough to prove

$$\Gamma(2s+1)4^{1-s} \int_0^{\infty} \frac{\sin^2(t)}{t^{1+2s}}\, dt = B(s,1-s).$$

This equation holds since we are able to interchange the order of integration, i.e. we have via substitution and partial integration

$$\Gamma(2s+1)4^{1-s} \int_0^{\infty} \frac{\sin^2(t)}{t^{1+2s}}\, dt = 4^{1-s} \int_0^{\infty} e^{-k} k^{2s} \int_0^{\infty} \frac{\sin^2(t)}{t^{1+2s}}\, dt\, dk = 4^{1-s} \int_0^{\infty} e^{-k} \int_0^{\infty} \frac{\sin^2(kp)}{p^{1+2s}}\, dp\, dk$$

$$= 4^{1-s} \int_0^\infty p^{-1-2s} \int_0^\infty e^{-k} \sin^2(kp)\, dk\, dp = 4^{1-s} \int_0^\infty p^{-2-2s} \int_0^\infty e^{-\frac{\tilde{k}}{p}} \sin^2(\tilde{k})\, d\tilde{k}\, dp$$

Note that we have $\int_0^\infty e^{-\frac{\tilde{k}}{p}} \sin^2(\tilde{k})\, d\tilde{k}$ is the Laplace transform $\mathscr{L}$ of $\sin^2$ at $\frac{1}{p}$, i.e. we have (see e.g. [34, 4.7 (3), p. 150])

$$\mathscr{L}(\sin^2)(\frac{1}{p}) = \int_0^\infty e^{-\frac{\tilde{k}}{p}} \sin^2(\tilde{k})\, d\tilde{k} = \frac{2}{\frac{1}{p}\left(4+\frac{1}{p^2}\right)} = \frac{2p^3}{4p^2+1}.$$

Thus

$$\Gamma(2s+1)4^{1-s} \int_0^\infty \frac{\sin^2(t)}{t^{1+2s}}\, dt = 4^{1-s} \int_0^\infty \frac{2p^{1-2s}}{4p^2+1}\, dp = \int_0^\infty \frac{\tau^{-s}}{\tau+1}\, d\tau = B(s,1-s).$$

We will now turn on the proof for general $N \geq 2$. By a transformation we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{1-\cos(x_1)}{|x|^{N+2s}}\, dx &= 2\int_0^\infty (1-\cos(x_1)) \int_{\mathbb{R}^{N-1}} \left(x_1^2 + |x'|_2^2\right)^{-s-N/2} dx'\, dx_1 \\
&= \frac{4\pi^{(N-1)/2}}{\Gamma((N-1)/2)} \int_0^\infty (1-\cos(x_1)) \int_0^\infty r^{N-2} \left(x_1^2+r^2\right)^{-s-N/2} dr\, dx_1 \\
&= \frac{4\pi^{(N-1)/2}}{\Gamma((N-1)/2)} \left(\int_0^\infty \frac{1-\cos(x_1)}{x_1^{1+2s}}\, dx_1 \right) \left(\int_0^\infty \frac{y^{N-2}}{(y^2+1)^{s+N/2}}\, dy \right) \\
&= \frac{2\pi^{(N-1)/2}}{\Gamma((N-1)/2)} \left(\int_0^\infty \frac{1-\cos(x_1)}{x_1^{1+2s}}\, dx_1 \right) \left(\frac{\Gamma((N-1)/2)\Gamma(s+1/2)}{\Gamma\left(\frac{N}{2}+s\right)} \right) \\
&= \frac{\pi^{N/2}\, \Gamma(1-s)}{s4^s\, \Gamma\left(\frac{N}{2}+s\right)} = c_{N,s}^{-1},
\end{aligned}$$

where we used equation (5.1) and the fact that (5.2) gives

$$\int_0^\infty \frac{y^{N-2}}{(y^2+1)^{s+N/2}}\, dy = \frac{1}{2} B\left(\frac{N-1}{2}, s+1/2\right) = \frac{\Gamma((N-1)/2)\Gamma(s+1/2)}{2\,\Gamma\left(\frac{N}{2}+s\right)}.$$

□

5.1 Boundary regularity and Hopf's Lemma

In the following denote $\delta(x) = \delta_\Omega(x) := \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega)$ for any $\Omega \subset \mathbb{R}^N$. For the fractional Laplacian the optimal regularity result is given in [62, Theorem 1.1 + Theorem 1.2]:

Theorem 5.3 ([62], Theorem 1.1 + Theorem 1.2)**.** *Let $s \in (0,1)$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, which fulfills an exterior ball condition*[5]*. Then for every $g \in L^\infty(\Omega)$ the unique solution u of*

$$\begin{cases} (-\Delta)^s u = g & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.3}$$

satisfies $u \in C^s(\overline{\Omega})$.
If, in addition, $\partial\Omega$ is of class $C^{1,1}$, then there is $a \in (0, \min\{s, 1-s\})$ such that

$$u/\delta^s \in C^a(\overline{\Omega}).$$

We note that the boundary behavior of the fractional Laplacian is different to $(-\Delta)$. A variant of Hopf's Lemma is shown in [9, Lemma 4.3]. There the authors show that if u is a continuous supersolution of $(-\Delta)^s u = 0$ in some open ball $B \subset \mathbb{R}^N$ such that $u \equiv 0$ on $\mathbb{R}^N \setminus B$, then the outer normal derivative of u satisfies

$$\partial_\eta u(x_1) = -\infty.$$

In the following we will prove a variant of Hopf's Lemma which is related to the following kind of *outer normal derivative of order s* for $s \in (0,1)$:

Definition 5.4. Let $\Omega \subset \mathbb{R}^N$ open with $\partial\Omega$ of class C^1 and let $\eta : \partial\Omega \to S^1$ denote the outer normal unit vector field on $\partial\Omega$. Then for $u \in C^s(\overline{\Omega})$ and $x \in \partial\Omega$ we call

$$\left(\partial_\eta\right)_s u(x) := \lim_{t \to 0^+} \frac{u(x) - u(x - t\eta)}{t^s}.$$

the *outer normal derivative of order s of u in x*.

Example 5.5. Let $R > 0$ and $x_0 \in \mathbb{R}^N$ and put $B := B_R(x_0)$. The unique solution $\psi_B \in \mathscr{H}_0^s(B)$ of $(-\Delta)^s u = 1$ in B, $\psi_B \equiv 0$ on $\mathbb{R}^N \setminus B$ is given by

$$\psi_B(x) = \gamma_{N,s} \left((R^2 - |x - x_0|^2)^+ \right)^s \quad \text{for } x \in \mathbb{R}^N \text{ with} \quad \gamma_{N,s} = \frac{4^{-s}\Gamma(\frac{N}{2})}{\Gamma(\frac{N+2s}{2})\Gamma(1+s)},$$

see e.g. [10, Corollary 4], [12, pp. 319 eq. (5.4)] or [33]. For $z \in \partial B$ we have, with $\eta(z) = \frac{z - x_0}{|z - x_0|}$,

$$\left(\partial_\eta\right)_s \psi_B(z) = -\gamma_{N,s} \lim_{t \to 0^+} \frac{\left(R^2 - (R-t)^2\right)^s}{t^s} = -\gamma_{N,s} \lim_{t \to 0^+} \left(\frac{2Rt - t^2}{t} \right)^s = -\gamma_{N,s}(2R)^s.$$

Proposition 5.6 (Fractional Hopf lemma)**.**
Let $B \subset \mathbb{R}^N$ be a ball with radius R and $c_0 \geq 0$. Furthermore, let $c \in L^\infty(B)$ with $\|c^-\|_{L^\infty(B)} \leq c_0$. Let u be a supersolution of $(-\Delta)^s u = c(x)u$ in B with $u \geq 0$ in $\mathbb{R}^N$. Moreover let $K \subset\subset$

[5] Ω fulfills an *exterior ball condition*, if for every $x \in \partial\Omega$ there exits a ball $B \subset (\mathbb{R}^N \setminus \Omega)$ such that $\partial B \cap \partial\Omega = \{x\}$. E.g. every convex set Ω satisfies an exterior ball condition; if Ω is bounded and $\partial\Omega$ is of class C^2 then Ω fulfills an exterior ball condition where the radius of the balls can be chosen uniformly.

$\mathbb{R}^N \setminus \overline{B}$ *be a set of positive measure and suppose that* $\operatorname*{ess\,inf}_K u > 0$*. Then there is a constant* $d = d(N, s, c_0, R, K, \operatorname{dist}(K, B), \operatorname*{ess\,inf}_K u) > 0$ *such that*

$$u(x) \geq d\delta_B^s(x) \quad \textit{for a.e. } x \in B.$$

In particular, if $u \in C(\overline{B})$ *and* $u(x_0) = 0$ *for some* $x_0 \in \partial B$*, then we have*

$$-\liminf_{t \to 0^+} \frac{u(x_0 - t\eta(x_0))}{t^s} < 0,$$

where $\eta(x_0) \in S^1$ *is the outer unit normal of* B *at* x_0*.*

Proof. Consider the (barrier) function $w = \psi_B + a1_K \in \mathscr{V}^{J_s}(\tilde{B})$, where $\tilde{B} \subset \mathbb{R}^N$ is such that $B \subset\subset \tilde{B}$ and $\operatorname{dist}(K, \tilde{B}) > 0$, and $\psi_B \in \mathscr{H}_0^s(B)$ is the solution of $(-\Delta)^s u = 1$ in B, $u \equiv 0$ on $\mathbb{R}^N \setminus B$ as in Example 5.5 and

$$a := \left(\sup_{x \in B,\ y \in K} |x-y|^{N+2s} \right) \left(\frac{1 + \gamma_{N,s} R^{2s} c_0}{c_{N,s} |K| \gamma_{N,s} R^{2s}} \right) + 1.$$

Then for $\varphi \in \mathscr{H}_0^s(B)$, $\varphi \geq 0$ we have

$$\begin{aligned}
\mathscr{J}_s(w, \varphi) &= \int_B \varphi(x)\, dx - a \int_B \varphi(x) \int_K \frac{c_{N,s}}{|x-y|^{N+2s}}\, dy\, dx \\
&\leq -c_0 \gamma_{N,s} R^{2s} \int_B \varphi(x)\, dx \leq \int_B c(x) w(x) \varphi(x)\, dx
\end{aligned}$$

since $w = \psi_B \geq 0$ in B and $\|\psi_B\|_{L^\infty(\mathbb{R}^N)} = \gamma_{N,s} R^{2s}$. Denote $\varepsilon := \operatorname*{ess\,inf}_K u$. Then $v := u - \frac{\varepsilon}{a} w$ is a supersolution of $(-\Delta)^s v = c(x) v$ in B, $v \equiv 0$ on $\mathbb{R}^N \setminus B$. Thus we have by Proposition 3.5

$$u \geq \frac{\varepsilon}{a} w \geq \frac{\varepsilon \gamma_{N,s} R^s}{a} \delta_B^s \qquad \text{a.e. in } B.$$

In particular, if $u \in C(\overline{B})$ and $u(x_0) = 0$ for some $x_0 \in \partial B$, we have

$$-\liminf_{t \to 0^+} \frac{u(x_0 - t\eta(x_0))}{t^s} \leq -\frac{\varepsilon}{a} \lim_{t \to 0^+} \frac{\psi_B(x_0 - t\eta(x_0))}{t^s} < 0.$$

□

Next, we prove a variant of Proposition 5.6 for antisymmetric supersolutions (see [36, Proposition 3.3]).

Proposition 5.7 (Antisymmetric fractional Hopf lemma)**.** *Let* $H \subset \mathbb{R}^N$ *and consider a ball* $B \subset\subset H$ *of radius* $R > 0$*. Furthermore, let* $c_0 \geq 0$ *and* $c \in L^\infty(B)$ *with* $\|c^-\|_{L^\infty(B)} \leq c_0$*. Let* u *be an antisymmetric supersolution of*

$$(-\Delta)^s u = c(x) u \quad \textit{in } B \tag{5.4}$$

such that $u \geq 0$ *on* H. *Moreover, let* $K \subset H$ *be a set of positive measure such that* $K \subset\subset H \setminus \overline{B}$, *and suppose that* $\operatorname*{ess\,inf}_K u > 0$.
Then there is a constant $d = d(N, s, c_0, R, K, \operatorname{dist}(K,B), \operatorname{dist}(B, \mathbb{R}^N \setminus H), \operatorname*{ess\,inf}_K u) > 0$ *such that*

$$u(x) \geq d\delta_B^s(x) \quad \textit{for a.e. } x \in B.$$

In particular, if $u \in C(\overline{B})$ *and* $u(x_0) = 0$ *for some* $x_0 \in \partial B_1$, *then we have*

$$-\liminf_{t \to 0^+} \frac{u(x_0 - t\eta(x_0))}{t^s} < 0.$$

Proof. For $\alpha > 0$, consider the barrier

$$w(x) := \psi_B(x) + \alpha 1_K(x) - \psi_{Q(B)}(x) - \alpha 1_{Q(K)}(x),$$

where $Q : \mathbb{R}^N \to \mathbb{R}^N$, $x \mapsto \bar{x}$ is the reflection at ∂H. Let $\varphi \in \mathscr{H}_0^s(B)$, $\varphi \geq 0$ be arbitrary. Then we have

$$\begin{aligned}
\mathscr{J}_s(w,\varphi) &= \int_B \varphi(x)\,dx - \alpha c_{N,s} \int_B \varphi(x) \int_K \frac{c_{N,s}}{|x-y|^{N+2s}}\,dy\,dx \\
&\quad + \alpha c_{N,s} \int_B \int_{Q(K)} \frac{\varphi(x)}{|x-y|^{N+2s}}\,dydx + \alpha c_{N,s} \int_B \varphi(x) \int_{Q(K)} \frac{c_{N,s}}{|x-y|^{N+2s}}\,dy\,dx \\
&\leq \int_B \varphi(x)\,dx - \alpha c_{N,s} \int_B \varphi(x) \int_K \left(\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-\bar{y}|^{N+2s}} \right) dy\,dx \\
&\quad + c_{N,s}\gamma_{N,s} R^{2s} \int_B \varphi(x) \int_{Q(B)} \frac{1}{|x-y|^{N+2s}}\,dy\,dx \\
&\leq C_\alpha \int_{B_1} \varphi(x)\,dx,
\end{aligned}$$

where

$$\begin{aligned}
C_\alpha := \Bigg(1 &+ c_{N,s}\gamma_{N,s} \sup_{x \in B} \int_B \varphi(x) \int_{Q(B)} \frac{1}{|x-y|^{N+2s}}\,dy \\
&- \alpha \inf_{x \in B} c_{N,s} \int_K \left(\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-\bar{y}|^{N+2s}} \right) dy \Bigg).
\end{aligned}$$

Since $\operatorname{dist}(B, \mathbb{R}^N \setminus H) > 0$ it follows that $\sup_{x \in B} \int_B \varphi(x) \int_{Q(B)} \frac{1}{|x-y|^{N+2s}}\,dy < \infty$. Since moreover $\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-\bar{y}|^{N+2s}} > 0$ for $x \in B \subset H$, $y \in H$, we may choose α sufficiently large such that $C_\alpha = -c_0\gamma_{N,s}R^{2s}$. With this choice of α we have that w is a subsolution of $(-\Delta)^s w = c(x)w$ in B. Let $\varepsilon := \operatorname*{ess\,inf}_K u$, then $v = u - \frac{\varepsilon}{\alpha} w$ is an antisymmetric supersolution of $(-\Delta)^s v = c(x)v$ in B with $v \geq 0$ on $H \setminus B$. By Proposition 4.10 we have

$$u \geq \frac{\varepsilon}{\alpha} w \geq \frac{\varepsilon \gamma_{N,s} R^s}{\alpha} \delta_B^s \qquad \text{a.e. in } B.$$

In particular, if $u \in C(\overline{B})$ and $u(x_0) = 0$ for some $x_0 \in \partial B$, we have

$$-\liminf_{t \to 0^+} \frac{u(x_0 - t\eta(x_0))}{t^s} \leq -\frac{\varepsilon}{\alpha} \lim_{t \to 0^+} \frac{\psi_B(x_0 - t\eta(x_0))}{t^s} < 0.$$

□

Next, we establish a fractional variant of Serrin's corner boundary point lemma (see [65, Lemma 1]). More precisely, we have the following (see [36, Lemma 4.4]):

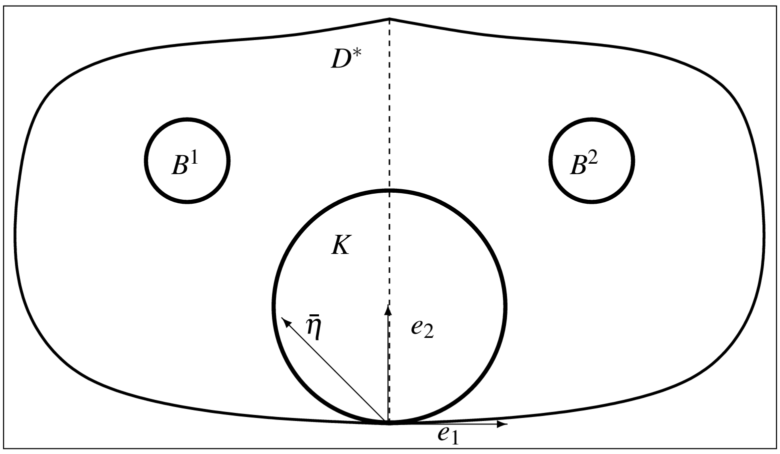
Lemma 5.8 (Fractional corner point lemma)**.**
Let $D \subset \mathbb{R}^N$, $N \geq 2$ be an open bounded set with C^2 boundary such that the origin $0 \in \partial D$. Assume furthermore that the hyperplane $\{x_1 = 0\}$ is orthogonal to ∂D at 0, that D is symmetric about $\{x_1 = 0\}$ and that the inner unit normal $-\eta(0)$ coincides with the second coordinate vector e_2. Let $D^* := D \cap \{x_1 < 0\}$. Let $c \in L^\infty(D^*)$, and let w be an antisymmetric supersolution of $(-\Delta)^s w = c(x)w$ in D^* such that $w \geq 0$ in $\{x_1 < 0\}$ and $\operatorname{essinf} w > 0$ *on every compact subset of D^*. Then letting $\bar{\eta} = (-1, 1, 0 \ldots, 0)$, there exists $C, t_0 > 0$ such that*

$$w(t\bar{\eta}) \geq Ct^{1+s} \qquad \forall t \in (0, t_0).$$

Proof. Let $R > 0$ small so that $B := B_R(Re_2) \subset D$ and $\partial B_R(Re_2) \cap \overline{D} = \{0\}$. Put

$$K = B_R(Re_2) \cap \{x_1 < 0\}.$$

Define $B^2 = B_R(4R\eta)$ and $B^1 = B_R(4R\bar{\eta})$, where $\eta = e_2 + e_1$ and $\bar{\eta} = e_2 - e_1$. From now on we will consider R small such that $B^1 \cup B^2 \subset\subset D$.



As before we use

$$\psi_B(x) = \left((R^2 - |x - Re_2|^2)^+\right)^s,$$

and for $a > 0$ (to be chosen later), we consider the (barrier) function

$$h(x) = -x_1 [\psi_R(x) + a(1_{B^1}(x) + 1_{B^2}(x))].$$

Note that $h(x) = -h(\bar{x})$ and $h \in C^{1,s}(B)$. Using [33, Theorem 1 + Table 3, pp.549], together with a scaling and translation,

$$|(-\Delta)^s(x_1\psi_R(x))| = |C_{N,s}R^{-1}x_1| \le C_1|x_1| \qquad \forall x \in K, \tag{5.5}$$

where here and in the following $C_1, C_2, \dots$ denote positive constants (possibly depending on R, N, s but not on a). Now we put for $x \in K$:

$$I(x) := (-\Delta)^s\tilde{h}(x) \qquad \text{with } \tilde{h}(y) := -y_1(1_{B^1}(y) + 1_{B^2}(y)) \quad \text{for } y \in \mathbb{R}^N$$

Then for $x \in K$ we have

$$\begin{aligned} I(x) &= -P.V. \int_{\mathbb{R}^N} \frac{-y_1(1_{B^1}(y) + 1_{B^2}(y))}{|x-y|^{N+2s}}\, dy = \int_{B^1} y_1 \left(|x-y|^{-N-2s} - |x-\bar{y}|^{-N-2s}\right) dy \\ &= \int_{B^1} \frac{y_1}{|x-y|^{N+2s}} \left(1 - \left(\frac{|x-y|}{|x-\bar{y}|}\right)^{N+2s}\right) dy \\ &= \int_{B^1} \frac{y_1}{|x-y|^{N+2s}} \left(1 - \left(\frac{|x-y|^2}{|x-y|^2 + 4x_1y_1}\right)^{\frac{N+2s}{2}}\right) dy. \end{aligned}$$

Observe that by construction,

$$R < |x-y| \le 7R \qquad \text{for all } x \in K \text{ and } y \in B^1.$$

Using this together with the facts that $y_1 \le -3R$ for $y \in B^1$ and that the map $\tau \mapsto 1 - \left(\frac{\tau}{\tau+d}\right)^k$ is strictly monotone decreasing in τ for all $d, k > 0$, we therefore get

$$\begin{aligned} I(x) &\le \int_{B^1} y_1|x-y|^{-N-2s} \left(1 - \left(\frac{49R^2}{49R^2 + 4x_1y_1}\right)^{\frac{N+2s}{2}}\right) dy \\ &\le -3R(7R)^{-N-2s} \int_{B^1} \left(1 - \left(\frac{49R^2}{49R^2 + 4x_1y_1}\right)^{\frac{N+2s}{2}}\right) dy \\ &= -C_2 \int_{B^1} \left(1 - \left(1 - \frac{4x_1y_1/(49R^2)}{1 + 4x_1y_1/(49R^2)}\right)^{\frac{N+2s}{2}}\right) dy \qquad \text{for all } x \in K. \end{aligned}$$

Since $N \ge 2$ and thus $(1-t)^{\frac{N+2s}{2}} \le 1 - t$ for $t \in (0,1)$, we get

$$I(x) \le -C_2 x_1 \int_{B^1} \frac{4y_1/(49R^2)}{1 + 4x_1y_1/(49R^2)} dy \le -C_3|x_1| \qquad \text{for all } x \in K,$$

where we have used again the fact that $|y_1| \geq 3R$ and $|x_1| \leq 5R$ for $y \in B^1$, $x \in K$. Combining this with (5.5), we get

$$(-\Delta)^s h(x) - c(x)h(x) \leq (C_1 + \|c\psi_B\|_{L^\infty(D^*)} - aC_3)|x_1| = (C_4 - aC_3)|x_1| \quad \text{for all } x \in K.$$

Hence we can choose a so that $(-\Delta)^s h - c(x)h \leq 0$ in K. Since by assumption we also have $w \geq \varepsilon h$ in B^1 for some $\varepsilon > 0$, we get $w - \varepsilon h \geq 0$ in $\{x_1 < 0\} \setminus K$. We then deduce from Proposition 4.10 that $w \geq \varepsilon h$ in D^*. Now since for $t > 0$ small we have

$$h(t\bar{\eta}) = t^{1+s}(2R - 4t^2),$$

the proof follows immediately because $t\bar{\eta} \in D^*$ for $t > 0$ small. □

5.1.1 An overdetermined problem involving the fractional Laplacian

This part is devoted to an overdetermined problem involving the fractional Laplacian. For $s = 1$ the results go back to Serrin in 1971 [65]. We will extend this to the case of $s \in (0,1)$. This part follows closely [36]. We note that recently also the overdetermined problem for the fractional Laplacian in exterior sets has been studied in a similar way (see [70]). We refer also to the following very recent studies on constant nonlocal mean curvature in [17, 29].

Theorem 5.9. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open bounded set such that $\partial\Omega$ is C^2 and let $u \in \mathscr{H}_0^s(\Omega)$ be the unique solution of*

$$(-\Delta)^s u = 1 \quad \text{in } \Omega, \qquad u \equiv 0 \quad \text{on } \mathbb{R}^N \setminus \Omega.$$

If there is $c \in \mathbb{R}$ with

$$(\partial_\eta)_s u \equiv c \quad \text{on } \partial\Omega,$$

then Ω is a ball.

Remark 5.10. We note that by Theorem 5.3 we have $u \in C^s(\mathbb{R}^N)$ since $\partial\Omega$ is C^2, thus $(\partial_\eta)_s u$ is well-defined on $\partial\Omega$. In addition, a solution $u \in \mathscr{H}_0^s(\Omega)$ to $(-\Delta)^s u = 1$ in Ω is strictly positive in Ω by Proposition 4.11 and since $\partial\Omega$ is C^2, Proposition 5.6 implies that a solution u as above may only exist if $c < 0$.

Proof of Theorem 5.9. Let $e \in S^1$ be fixed and consider $T_\lambda := \{x \in \mathbb{R}^N \,:\, x \cdot e = \lambda\}$ as a hyperplane in $\mathbb{R}^N$, which we will move by continuously varying λ. Since Ω is bounded, denote $l := \max_{x \in \Omega} x \cdot e$, so that $T_\lambda \cap \Omega = \emptyset$ for $\lambda \geq l$. Denote $H_\lambda := \{x \in \mathbb{R}^N \,:\, x \cdot e > \lambda\}$ and define $\Omega_\lambda := \Omega \cap H_\lambda$. Let $Q_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection about T_λ as described before and denote $\Omega'_\lambda := Q_\lambda(\Omega_\lambda)$, i.e. the reflection of Ω_λ about T_λ. Since $\partial\Omega$ is C^2 we have for $\lambda < l$ but close to l that $\Omega'_\lambda \subset \Omega$. Next let $\eta : \partial\Omega \to S^1$ be the outer normal unit vector field of Ω and put

$$\lambda_0 := \inf\{\lambda \in \mathbb{R} \,:\, \Omega'_\mu \subset \Omega \text{ for all } \mu \geq \lambda, \text{ and } \eta(x) > 0 \text{ for all } x \in \partial\Omega \cap H_\lambda\}. \tag{5.6}$$

We note that we have $\Omega'_{\lambda_0} \subset \Omega \setminus \overline{H_{\lambda_0}}$ and one of the following is true (see [42, Section 5.2]):

$$\begin{aligned} &\text{Case 1: There is a point } P_0 \in \partial\Omega \cap \overline{\Omega'_\lambda} \setminus T_\lambda. \\ &\text{Case 2: } T_\lambda \text{ is orthogonal to } \partial\Omega \text{ at some point } P_0 \in \partial\Omega \cap T_\lambda. \end{aligned} \tag{5.7}$$

This is because $\partial\Omega$ is C^2 and Ω is bounded.
For simplicity, we put $T = T_{\lambda_0}$ and $H = H_{\lambda_0}$. Our aim is to prove that in Case 1 or 2, Ω must be symmetric with respect to the plane T and convex in direction e.

To prove that in both cases above we obtain symmetry, we let Q be the reflection about T as described before. Then define $\bar{x} := Q(x)$ and consider the function

$$v(x) := u(x) - u(\bar{x}) \quad \text{for } x \in \mathbb{R}^N.$$

Since $U := \Omega'_{\lambda_0} \subset \Omega$ we have that v satisfies

$$(-\Delta)^s v = 0 \text{ in } U$$

and

$$\begin{aligned} v &\geq 0 && \text{on } H' \setminus U; \\ v(\bar{x}) &= -v(x) && \text{for all } x \in \mathbb{R}^N. \end{aligned}$$

Here $H' := \mathbb{R}^N \setminus H$. Thus we have that v is an antisymmetric supersolution of $(-\Delta)^s v = 0$ on U with $v \geq 0$ on H' by Proposition 4.10. Proposition 4.11) then implies $v \equiv 0$ on $\mathbb{R}^N$ or $v > 0$ in U. In the following, we argue by contradiction and assume that $v > 0$ in U.

We then first consider *Case 1*, i.e. we assume that there is some point $P_0 \in \partial\Omega \cap \overline{U} \setminus T$.
Note that we have $P_0 \in \partial\Omega \cap \partial U$ due to the choice of λ_0, and we have $v(P_0) = 0$ since $u(P_0) = 0 = u(\bar{P}_0)$. Since moreover $v > 0$ in U, Proposition 5.7 gives that $(\partial_\eta)_s v(P_0) < 0$, where η is the common outer normal of ∂U and $\partial\Omega$ at P_0. But since $(\partial_\eta)_s u(P_0) = c = (\partial_\eta)_s u(\bar{P}_0)$ we must have $(\partial_\eta)_s v(P_0) = 0$ which is a contradiction.

Next we consider *Case 2*, i.e. we assume that T is orthogonal to $\partial\Omega$ at a point $P_0 \in T \cap \partial\Omega$. Up to translation and rotations, we may assume that $P_0 = 0$, $e = e_1$, e_2 is the interior normal of $\partial\Omega$ at 0, and $\nabla^2 \delta_\Omega(0)$ is diagonal. Without loss of generality, we may also assume that $\lambda_0 = 0$. We then claim that

$$v(t\bar{\eta}) = o(t^{1+s}) \qquad \text{as } t \to 0, \tag{5.8}$$

where $\bar{\eta} = (-1, 1, 0, \ldots, 0)$. Indeed, thanks to Theorem 5.3, we can write for $x \in \overline{\Omega}$:

$$u(x) = \delta^s(x)\psi(x),$$

where $\psi \in C^{0,a}(\overline{\Omega})$ for some $a \in (0,1)$ (recall that $\delta = \delta_\Omega$ is the distance function to $\partial\Omega$). It is clear from our hypothesis that

$$\psi(x) = -c \quad \forall x \in \partial\Omega. \tag{5.9}$$

Put $\bar{u}(x) = u(\bar{x}) = u(-x_1, x_2, \ldots, x_N)$, $\bar{\delta}(x) = \delta(\bar{x})$ and $\bar{\psi}(x) = \psi(\bar{x})$. By continuity, we have

$$\psi(t\bar{\eta}) = -c + o(1) = \bar{\psi}(t\bar{\eta}), \quad \text{as } t \to 0.$$

Then we have

$$v(t\bar{\eta}) = u(t\bar{\eta}) - \bar{u}(t\bar{\eta}) = [\delta^s(t\bar{\eta}) - \bar{\delta}^s(t\bar{\eta})](c + o(1)), \quad \text{as } t \to 0. \tag{5.10}$$

By Taylor expansion, we have

$$\delta(t\bar{\eta}) = \delta(0) + \nabla\delta(0)\cdot(t\bar{\eta}) + \frac{1}{2}\nabla^2\delta(0)[(t\bar{\eta})]\cdot(t\bar{\eta}) + o(t^2), \quad \text{as } t \to 0$$

and

$$\bar{\delta}(t\bar{\eta}) = \delta(0) + \nabla\bar{\delta}(0)\cdot(t\bar{\eta}) + \frac{1}{2}\nabla^2\bar{\delta}(0)[(t\bar{\eta})]\cdot(t\bar{\eta}) + o(t^2), \quad \text{as } t \to 0.$$

Moreover, since $e_2 = \nabla\delta(0)$ is the normal direction, $\partial_{x_i}\delta(0) = 0$ for all $i \neq 2$. Therefore

$$\nabla\delta(0)\cdot\bar{\eta} = \nabla\bar{\delta}(0)\cdot\bar{\eta} = e_2\cdot\bar{\eta} = 1.$$

Since $\nabla^2\delta(0)$ is diagonal we have

$$\nabla^2\delta(0)[\bar{\eta}]\cdot\bar{\eta} = \nabla^2\bar{\delta}(0)[\bar{\eta}]\cdot\bar{\eta} = \nabla^2\delta(0)[e_2]\cdot e_2 + \nabla^2\delta(0)[e_1]\cdot e_1.$$

By a Taylor expansion of $a \mapsto a^s$ at $a = 1$ it follows that

$$\delta^s(t\bar{\eta}) = t^s\left(1 + \frac{s}{2}\nabla^2\delta(0)[\bar{\eta}]\cdot(t\bar{\eta}) + o(t)\right), \quad \text{as } t \to 0$$

and

$$\bar{\delta}^s(t\bar{\eta}) = t^s\left(1 + \frac{s}{2}\nabla^2\delta(0)[\bar{\eta}]\cdot(t\bar{\eta}) + o(t)\right), \quad \text{as } t \to 0.$$

We then conclude that

$$\delta^s(t\bar{\eta}) - \bar{\delta}^s(t\bar{\eta}) = o(t^{1+s}), \quad \text{as } t \to 0.$$

This together with (5.10) proves (5.8).
Since 5.8 contradicts Lemma 5.8, we then conclude that also in Case 2 we have $v \equiv 0$, as claimed.
In conclusion, we have proved that for all $e \in S^1$ there is a hyperplane T^e perpendicular to e and such that Ω is symmetric with respect to T^e and convex in direction e. In particular, considering hyperplanes T^{e_i} corresponding to the coordinate vectors e_i, we have that Ω is symmetric with respect to T^{e_i} for $i = 1, \dots, N$ and convex in all coordinate directions. Consequently, Ω is also symmetric with respect to reflection at the unique intersection point z_0 of $T^{e_1}, \dots, T^{e_N}$, i.e. we have $z_0 + x \in \Omega$ if and only if $z_0 - x \in \Omega$. It is then easy to see that $z_0 \in T^e$ for all $e \in S^1$, and this implies that Ω is a ball centered at z_0. □

We now turn to some variants of Theorem 5.9.

Theorem 5.11. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open, bounded set with C^2 boundary. Furthermore, let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and assume that there is a solution $u \in C^s(\mathbb{R}^N) \cap \mathscr{H}_0^s(\Omega)$ of*

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega; \\ u \equiv 0 & \text{on } \mathbb{R}^N \setminus \Omega; \end{cases} \tag{5.11}$$

which is nonnegative and nontrivial in Ω. If there is $c \in \mathbb{R}$ such that $(\partial_\eta)_s u = c$ on $\partial\Omega$, then Ω is a ball and $u > 0$ in Ω.

Remark 5.12. We note that Theorem 5.9 is a special case of Theorem 5.11. However, in Theorem 5.11 we do not have a priori that the solution u must be positive. Thus the argument which leads to the cases 1 and 2 in the proof of Theorem 5.11 has to be slightly different.

Proof. Let $e \in S^1$ and consider λ_0 as defined in (5.6) and $U := \Omega'_{\lambda_0} \subset \Omega$ as before. We define $v_{\lambda_0}(x) := u(x) - u(\bar{x})$ for all $x \in \mathbb{R}^N$, where we use the notation as in the proof of Theorem 5.9, i.e. $\bar{x} := Q_\lambda(x)$ and Q_λ is the reflection about $T = T_\lambda$. Then v_{λ_0} is an antisymmetric solution of

$$(-\Delta)^s v_{\lambda_0} = c_f(x) v_{\lambda_0} \qquad \text{in } U,$$

where

$$c_f(x) := \begin{cases} \dfrac{f(u(x)) - f(u(\bar{x}))}{u(x) - u(\bar{x})}, & \text{if } u(x) \neq u(\bar{x}); \\ 0, & \text{if } u(x) = u(\bar{x}). \end{cases}$$

Let L_f be the Lipschitz constant of f for the interval $[0, \|u\|_{L^\infty(\mathbb{R}^N)}]$. Then we have $\|c\|_{L^\infty(U)} \leq L_f$. Here, we cannot directly apply the maximum principle to get $v_{\lambda_0} \geq 0$ in H' as in the previous section because L_f might be large. However, by using the moving plane method, we can prove that

$$v_{\lambda_0} \geq 0 \qquad \text{on } H'; \tag{5.12}$$

To see this, we note that for $\lambda \in (\lambda_0, l)$ but close to l we have $L_f \leq \lambda_{1,J_s}(\Omega'_\lambda)$ so that $u(x) - u(Q_{\lambda,e}(x)) \geq 0$ in Ω'_λ by Proposition 4.10. Now by Proposition 4.11

$$(S_\lambda) \quad v_\lambda(x) := u(x) - u(Q_\lambda(x)) > 0 \quad \text{for all } x \in \Omega'_\lambda$$

as u is nontrivial. We let

$$\tilde{\lambda} := \inf\{\lambda > \lambda_0 \; : \; (S_\mu) \text{ holds for all } \lambda > \mu\}.$$

Our aim is to prove that $\tilde{\lambda} = \lambda_0$. Assume by contradiction that $\tilde{\lambda} > \lambda_0$. Then by continuity and Proposition 4.11 we have that $(S_{\tilde{\lambda}})$ holds. Since $\tilde{\lambda} > \lambda_0$, there is by continuity $\varepsilon > 0$ such that $\Omega'_{\tilde{\lambda}-\varepsilon} \subset \Omega$. Choose an open set $K \subset \Omega'_{\tilde{\lambda}-\varepsilon}$ such that $\{v \leq 0\} \cap \Omega'_{\tilde{\lambda}-\varepsilon} \subset K$ and we may assume that $|K|$ is small by making ε possibly smaller. Proposition 4.10 and Proposition 4.11 then can be applied to K giving $v_{\tilde{\lambda}-\varepsilon} > 0$ in K (as before) and thus $S_{\tilde{\lambda}-\varepsilon}$ holds in contradiction to the choice of $\tilde{\lambda}$. Thus $\tilde{\lambda} = \lambda_0$. Hence (5.12) is proved. We have now that v is an antisymmetric supersolution of $(-\Delta)^s v = c_f(x)v$ on U with $v_{\lambda_0} \geq 0$ in H'. Arguing now as in the proof of Theorem 5.9 we obtain $v_{\lambda_0} \equiv 0$ on $\mathbb{R}^N$. Since e was chosen arbitrarily we conclude as in the proof of Theorem 5.9 that $\text{supp}(u)$ is a ball.
To finish the prove, we need to show $\overline{\Omega} = \text{supp}(u)$. Assume by contradiction that $\text{supp}(u) \neq \overline{\Omega}$. Then there is a ball $B \subset\subset \Omega \setminus \text{supp}(u)$, such that $u = 0$ in B. Consider the hyperplane T separating B and $\text{supp}(u)$ – this is possible since B and $\text{supp}(u)$ are balls. It is clear that $u \equiv 0$ on the halfspace H with boundary T containing B. Let $e \in S^1$ be perpendicular to T and contained in H. Now by moving the planes T_λ as above w.r.t. this e, we get, for every $\lambda \in (\lambda_0, l)$

$$u(x) > u(Q_\lambda(x)) \geq 0 \quad \text{for all } x \in \Omega'_{\lambda,e} \,.$$

This, in particular, implies that $u > 0$ in $\Omega \cap H_{\lambda_0}$ which is impossible. We then conclude that Ω must be a ball. □

Theorem 5.13. *Let $c \in \mathbb{R}$ and $\Omega \subset \mathbb{R}$ be a bounded open set. Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and assume that there is a function $u \in C(\overline{\Omega})$, which is nonnegative and nontrivial in Ω and satisfies*

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \Omega; \\ u \equiv 0 & \text{on } \mathbb{R} \setminus \Omega; \\ (\partial_\eta)_s u \equiv c & \text{on } \partial\Omega. \end{cases} \tag{5.13}$$

Then $\Omega = (a,b)$ for some $a,b \in \mathbb{R}$, $a < b$ and $u > 0$ in Ω.

Proof. Assume that Ω has at least two different connected components $(\tilde{a},\tilde{b})$ and (a,b) with $a < b < \tilde{a} < \tilde{b}$. Note that as in the case $N \geq 2$ we can move points from the right up to $\lambda_0 = (\tilde{a}+\tilde{b})/2$, so that $v(x) := u(x) - u(\bar{x})$ solves

$$(-\Delta)^s v \geq -c_f(x)v \qquad \text{in } (a,\lambda_0)$$

and $v(x) \geq 0$ for $x < \lambda_0$ by arguing as in the proof of Theorem 5.11. Note that only interior touching can occur. Hence by Hopf's Lemma we obtain $v \equiv 0$ on $\mathbb{R}$, but this gives $u \equiv 0$ on $\mathbb{R} \setminus (\tilde{a},\tilde{b})$. Next moving from the left up to $\lambda_0 = (\tilde{a}+\tilde{b})/2$ implies, as previously, $u \equiv 0$ in (a,b). Therefore $u \equiv 0$ in $\mathbb{R}$ leading to a contradiction. The positivity of u finally follows as in Theorem 5.11 by the monotonicity which is a byproduct of the moving plane method. □

5.2 Some additional estimates for nonlocal operators of order $s \in (0,1)$

In this section we add some estimates and a generalization of Proposition 5.6 to a more general class of nonlocal operators of order $s \in (0,1)$. We fix $s \in (0,1)$ and introduce the following assumptions for a measurable function J satisfying $(J1)_a$ and $(J1)_b$:

(JL_s) There is $r_0 > 0$ and $c > 0$ such that

$$J(x,y) \geq c|x-y|^{-N-2s} \qquad \text{for a.e. } x,y \in \mathbb{R}^N \text{ with } x \neq y \text{ and } |x-y| \leq r_0.$$

(JU_s) There is $r_0 > 0$ and $c > 0$ such that

$$J(x,y) \leq c|x-y|^{-N-2s} \qquad \text{for a.e. } x,y \in \mathbb{R}^N \text{ with } x \neq y \text{ and } |x-y| \leq r_0.$$

(J_s) There is $r_0, k > 0$ such that the map $\mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,x) : x \in \mathbb{R}^N\} \to \mathbb{R}$, $(x,y) \mapsto J(x,y) - k|x-y|^{-N-2s}$ is bounded in $\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : |x-y| < r_0\}$.

Lemma 5.14.

(i) If J satisfies (JL_s), then $\mathscr{D}^J(\Omega) \subset \mathscr{H}^s_0(\Omega)$ for every open bounded set $\Omega \subset \mathbb{R}^N$.

(ii) If J satisfies (JU_s), then $\mathscr{D}^J(\Omega) \supset \mathscr{H}^s_0(\Omega)$ for every open bounded set $\Omega \subset \mathbb{R}^N$.

Proof. To see (i), let $u \in \mathscr{D}^J(\Omega)$, then $u \in L^2(\Omega)$, since Ω is bounded, and thus

$$\begin{aligned}\mathscr{J}_s(u,u) &\leq \frac{c}{c_{N,s}}\mathscr{J}(u,u) + \frac{c_{N,s}}{2}\int\limits_{|x-y|>r_0}\frac{(u(x)-u(y))^2}{|x-y|^{N+2s}}\,d(x,y)\\ &\leq \frac{c}{c_{N,s}}\mathscr{J}(u,u) + 2r_0^{-N-2s}c_{N,s}\|u\|^2_{L^2(\Omega)} \leq (\frac{c}{c_{N,s}} + 2\Lambda_{1,J}(\Omega)^{-1}r_0^{-N-2s}c_{N,s})\mathscr{J}(u,u) < \infty.\end{aligned}$$

To see (ii), first of all note that by $(J1)_b$ there is $K > 0$ such that

$$\sup_{x\in\mathbb{R}^N}\int\limits_{|x-y|>r_0} J(x,y)\,dy \leq K.$$

Next let $u \in \mathscr{H}^s_0(\Omega)$, then $u \in L^2(\Omega)$, since Ω is bounded, and thus

$$\begin{aligned}\mathscr{J}(u,u) &\leq \frac{c}{c_{N,s}}\mathscr{J}_s(u,u) + \frac{1}{2}\int\limits_{|x-y|>r_0}(u(x)-u(y))^2 J(x,y)\,d(x,y)\\ &\leq \frac{c}{c_{N,s}}\mathscr{J}_s(u,u) + \int\limits_{\mathbb{R}^N} u^2(x)\int\limits_{|x-y|>r_0} J(x,y)\,dy\,dx + \int\limits_{|x-y|>r_0} u(x)J^{\frac12}(x,y)u(y)J^{\frac12}(x,y)\,d(x,y)\\ &\leq \left(\frac{c}{c_{N,s}} + \frac{K}{\Lambda_{1,J_s}(\Omega)}\right)\mathscr{J}_s(u,u) + K^2\int\limits_{\mathbb{R}^N} u^2(x)\,dx\\ &\leq \left(\frac{c}{c_{N,s}} + \frac{K}{\Lambda_{1,J_s}(\Omega)} + \frac{K^2}{\Lambda_{1,J_s}(\Omega)}\right)\mathscr{J}_s(u,u).\end{aligned}$$

□

Remark 5.15. Note that if (JL_s) and (JU_s) are satisfied then we have $\mathscr{D}^J(\Omega) = \mathscr{H}^s_0(\Omega)$ for every bounded open set Ω. Moreover, if J satisfies (JL_s) for some $r_0 > 0$, then $(J1)_d$ and $(J_+)_{r_0}$ are satisfied.
Obviously, (JL_s) and (JU_s) are satisfied if and only if there is $r_0 > 0$, $0 < c_1 \leq c_2 < \infty$ such that

$$c_1|x-y|^{-N-2s} \leq J(x,y) \leq c_2|x-y|^{-N-2s} \qquad \text{for a.e. } x,y \in \mathbb{R}^N \text{ with } 0 < |x-y| < r_0. \quad (5.14)$$

Finally, we note that (J_s) implies the existence of some $r_0 > 0$ such that (5.14) holds for some $c_1,\ c_2 > 0$, i.e., (J_s) implies (JL_s) and (JU_s).

Lemma 5.16. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume J satisfies $(J1)_{diff}$ and (J_s). Then for any $u \in \mathscr{H}^s_0(\Omega) \cap L^\infty(\Omega)$ we have $Iu \in L^\infty(\Omega)$ if and only if $(-\Delta)^s u \in L^\infty(\Omega)$.*

Proof. Note that since J fulfills (J_s), there is $k, r_0, C > 0$ such that for a.e. $x \in \mathbb{R}^N$ and a.e. $y \in B_{r_0}(x)$ we have

$$\left|J(x-y) - \frac{kc_{N,s}}{|x-y|^{N+2s}}\right| \leq C.$$

Thus for a.e. $x \in \Omega$ we have

$$|k(-\Delta)^s u(x) - Iu(x)| \leq C \int_{B_{r_0}(x)} |u(x)-u(y)|\, dy + \int_{\mathbb{R}^N \setminus B_{r_0}(x)} |u(x)-u(y)| \cdot \left| J(x-y) - \frac{kc_{N,s}}{|x-y|^{N+2s}} \right| dy$$
$$\leq 2\|u\|_{L^\infty(\Omega)} \left(CNr_0^N \omega_N + \left(\int_{\mathbb{R}^N \setminus B_{r_0}(0)} J(y)\, dy + \frac{kc_{N,s} N \omega_N r_0^{-2s}}{2s} \right) \right) < \infty.$$

□

With this we immediately get

Corollary 5.17. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Moreover, let $g \in L^\infty(\Omega)$. Furthermore assume (J1)$_{diff}$, (J$_s$) and (J$_+$)$_{2\mathrm{diam}(\Omega)}$. If Ω fulfills an exterior ball condition, then every solution u of*

$$\begin{cases} Iu = g & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{5.15}$$

fulfills $u \in C^s(\overline{\Omega})$.
If additional $\partial\Omega$ is of class $C^{1,1}$, then there is $a \in (0, \min\{s,(1-s)\})$ such that

$$u/\delta^s \in C^a(\overline{\Omega}),$$

where $\delta := \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega)$.

Proof. Since $g \in L^\infty(\Omega)$ and since (J$_+$)$_{2\mathrm{diam}(\Omega)}$ is satisfied, we have $u \in L^\infty(\Omega)$ by Lemma 3.8. Thus by Lemma 5.16 $(-\Delta)^s u = \tilde{g}$ in Ω for some $\tilde{g} \in L^\infty(\Omega)$. Thus with Theorem 5.3 the proof is finished. □

Definition 5.18. Let $M \subset \mathbb{R}^N$ an open bounded set such that ∂M is of class C^2, and let J satisfy for some (J1)$_{diff}$, (J$_s$) and (J$_+$). In the following, we let $\Psi_{M,J} \in \mathscr{H}_0^s(M)$ denote the unique solution of $I\Psi_{M,J} = 1$ in M, $\Psi_{M,J} \equiv 0$ on $\mathbb{R}^N \setminus M$ given by Corollary 2.9.

Remark 5.19. If M and J are as in Definition 5.18, then Corollary 5.17 gives that $\Psi_{M,J} \in C^s(\overline{M})$. Moreover, by Proposition 3.5 and Proposition 3.10 we have $\Psi_{M,J} > 0$ in M. Hence there is a constant $C > 0$ such that

$$0 < \Psi_{M,J}(x) \leq C \left(\mathrm{dist}(x, \mathbb{R}^N \setminus M)\right)^s \qquad \text{for } x \in M.$$

Lemma 5.20. *If J fulfills (J1)$_{diff}$, (J$_s$) and (J$_+$), then for all $x_0 \in \mathbb{R}^N$ and $\rho > 0$ the function $\psi_{B_\rho(x_0),J} \in \mathscr{H}_0^s(B_\rho(x_0))$ satisfies*

$$C_1 (\rho - |x-x_0|)^s \leq \psi_{B_\rho(x_0),J}(x) \leq C_2 (\rho - |x-x_0|)^s \qquad \text{for } x \in B_\rho(x_0)$$

with constants $C_1, C_2 > 0$ depending only on N, J and ρ.

Proof. Since J satisfies $(\mathrm{J1})_{diff}$, it suffices to consider $x_0 = 0$. Recall that the unique solution $\psi \in \mathscr{H}_0^s(B_\rho(0))$ of

$$(-\Delta)^s \psi = 1 \qquad \text{in } B_\rho(x_0), \qquad \psi \equiv 0 \qquad \text{on } \mathbb{R}^N \setminus B_\rho(0),$$

is given by

$$\psi(x) = \gamma_{N,s} \left[\left(\rho^2 - |x_0 - x|^2 \right)^+ \right]^s.$$

Due to Lemma 5.16 there is $C > 0$ such that

$$I\psi \leq C + (-\Delta)^s \psi = C + 1 \qquad \text{in } B_\rho(x_0).$$

Thus we have by the weak maximum principle given in Proposition 3.5:

$$\psi_{B_\rho(0),J} \geq \frac{1}{1+C} \psi \qquad \text{a.e. in } B_\rho(x_0).$$

The upper bound then follows from Corollary 5.17. □

By exactly the same argument as in the proof of Proposition 5.6 based on the comparison function $\psi_{B,J}$ in place of ψ_B and Lemma 5.19, we now get the following.

Corollary 5.21. *Let $B \subset \mathbb{R}^N$ be a ball with radius R. Furthermore, let $c_0 \geq 0$, $c \in L^\infty(B)$ with $\|c^-\|_{L^\infty(B)} \leq c_0$ and assume J satisfies $(\mathrm{J1})_{diff}$, (J_s) and (J_+). Let u be a supersolution of $Iu = c(x)u$ in B with $u \geq 0$ on $\mathbb{R}^N$. Moreover, assume there is $K \subset\subset \mathbb{R}^N \setminus \overline{B}$ with $|K| > 0$ and $\operatorname*{essinf}_K u > 0$. Then there is a constant $d = d(N, s, c_0, R, K, \mathrm{dist}(K, B), \operatorname*{essinf}_K u) > 0$ such that*

$$u(x) \geq d\delta_B^s(x) \quad \text{for a.e. } x \in B.$$

In particular, if $u \in C(\overline{B})$ and $u(x_0) = 0$ for some $x_0 \in \partial B$, then we have

$$-\liminf_{t \to 0^+} \frac{u(x_0 - t\eta(x_0))}{t^s} < 0,$$

where $\eta(x_0) \in S^1$ is the outer unit normal of B at x_0.

Remark 5.22. Let $M \subset \mathbb{R}^N$ an open bounded set such that ∂M is of class C^2, and let J satisfy for some $(\mathrm{J1})_{diff}$, (J_s) and (J_+). A combination of Remark 5.19 with Corollary 5.21 gives constants $d, C > 0$ such that

$$d \left(\mathrm{dist}(x, \mathbb{R}^N \setminus M) \right)^s \leq \Psi_{M,J}(x) \leq C \left(\mathrm{dist}(x, \mathbb{R}^N \setminus M) \right)^s \qquad \text{for } x \in M.$$

Finally we discuss the existence of a bounded first eigenfunction of the operator I in $L^2(\Omega)$. Combining [67, Proposition 4] and [63, Theorem 1.1] we have that this is true if $I = (-\Delta)^s$, $s \in (0,1)$. As already mentioned in [67], the proof provided in [67, Proposition 4] carries over to the situation of more general kernels of possibly varying order.

Theorem 5.23. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Assume that J satisfies (J1), (JL$_s$) and that $\mathscr{C}_c^2(\Omega) \subset \mathscr{D}_\infty^J(\Omega)$. Then*

$$\Lambda_{1,J}(\Omega) = \min_{u \in \mathscr{D}^J(\Omega)} \frac{\mathscr{J}(u,u)}{\|u\|_{L^2(\Omega)}^2} > 0$$

and there exists a normalized first eigenfunction $\varphi_1 \in \mathscr{D}^J(\Omega) \cap L^\infty(\Omega)$ of the associated self-adjoint operator I given in Corollary 2.6 which satisfies

$$\begin{cases} I\varphi_1 = \Lambda_{1,J}(\Omega)\varphi_1, & \text{in } \Omega; \\ \varphi_1 = 0 & \text{on } \mathbb{R}^N \setminus \Omega; \\ \|\varphi_1\|_{L^2(\Omega)} = 1. \end{cases} \tag{5.16}$$

Moreover, φ_1 is unique up to sign and can be chosen such that

$$\operatorname*{essinf}_K \varphi_1 > 0 \qquad \text{for all } K \subset\subset \Omega; \tag{5.17}$$

Furthermore, for any $\lambda > 0$ there is a constant $C = C(N,J,\Omega,\lambda) > 0$ such that any solution $v \in \mathscr{D}^J(\Omega)$ of

$$Iv = \lambda v \quad \text{in } \Omega \tag{5.18}$$

fulfills

$$\|v\|_{L^\infty(\Omega)} \le C\|v\|_{L^2(\Omega)}. \tag{5.19}$$

Proof. Due to (JL$_s$) we have $\Lambda_{1,J}(\Omega) > 0$ by Proposition 2.3. Let $(u_n)_{n\in\mathbb{N}} \subset \mathscr{D}^J(\Omega)$ be a minimizing sequence of $u \mapsto \mathscr{J}(u,u)$ subject to the constraint $\|u_n\|_{L^2(\Omega)} = 1$ for all $n \in \mathbb{N}$. Since

$$\begin{aligned} \mathscr{J}(u,u) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)-u(y))^2 J(x,y)\,dxdy \ge \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (|u(x)|-|u(y)|)^2 J(x,y)\,dxdy \\ &= \mathscr{J}(|u|,|u|) \qquad \text{for all } u \in \mathscr{D}^J(\Omega), \end{aligned}$$

we may assume $u_n \ge 0$ for all $n \in \mathbb{N}$. Note that by Lemma 5.14 and [32, Theorem 7.1] we have

$$\mathscr{D}^J(\Omega) \hookrightarrow \mathscr{H}_0^s(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega).$$

Since $\sup_{n\in\mathbb{N}} \mathscr{J}(u_n,u_n) < \infty$, there is a subsequence $(u_{n_k})_k$ such that $u_{n_k} \to \varphi_1$ in $L^2(\Omega)$ as $k \to \infty$. As in the proof of Proposition 2.4 we conclude that $\varphi_1 \in \mathscr{D}^J(\Omega)$ and that

$$\mathscr{J}(\varphi_1,\varphi_1) \le \liminf_{k\to\infty} \mathscr{J}(u_{n_k},u_{n_k}) = \Lambda_{1,J}(\Omega).$$

Since moreover $\|\varphi\|_{L^2(\Omega)}^2 = 1$, it follows that $\mathscr{J}(\varphi_1,\varphi_1) = \Lambda_{1,J}(\Omega)$. Moreover, $\varphi_1 \ge 0$ in $\mathbb{R}^N$. To show that φ_1 solves

$$I\varphi_1 = \Lambda_{1,J}(\Omega)\varphi_1 \quad \text{in } \Omega, \qquad \varphi_1 = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \tag{5.20}$$

we let $\psi \in \mathscr{D}^J(\Omega)$ and consider the function

$$g:\mathbb{R}\to\mathbb{R},\quad g(t):=\mathscr{J}(\varphi_1+t\psi,\varphi_1+t\psi)-\Lambda_{1,J}(\Omega)\|\varphi_1+t\psi\|^2_{L^2(\Omega)}.$$

Then

$$g'(t)=\mathscr{J}(\varphi_1,\psi)+2t\,\mathscr{J}(\psi,\psi)-\Lambda_{1,J}(\Omega)\int\limits_{\Omega}\varphi_1(x)\psi(x)\,dx-2t\Lambda_{1,J}(\Omega)\int\limits_{\Omega}\psi^2(x)\,dx,$$

for $t\in\mathbb{R}$. Since g has a local minimum at $t=0$ we have

$$0=g'(0)=\mathscr{J}(\varphi_1,\psi)-\Lambda_{1,J}(\Omega)\int\limits_{\Omega}\varphi_1(x)\psi(x)\,dx.$$

Hence φ_1 solves (5.20). Since $\varphi_1\geq 0$ in $\mathbb{R}^N$ and $\|\varphi_1\|_{L^2(\Omega)}=1$ we have $\operatorname*{ess\,inf}_K \varphi>0$ for all compact sets $K\subset\Omega$ by Proposition 3.10. Here we used the assumption $\mathscr{C}^2_c(\Omega)\subset\mathscr{D}^J_\infty(\Omega)$.
Next we show the uniqueness of φ_1. If $\psi\in\mathscr{D}^J(\Omega)$ is any function satisfying (5.16), then ψ is also a minimizer of $u\mapsto\mathscr{J}(u,u)$ subject to the constraint $\|u\|_{L^2(\Omega)}=1$, and the same holds for $|\psi|$ by the estimate above. Hence $|\psi|$ is a solution of (5.20) and thus $\operatorname*{ess\,inf}_K|\psi|>0$ for all compact sets $K\subset\Omega$ by Proposition 3.10. Now suppose by contradiction that the eigenspace of I corresponding to $\Lambda_{1,J}(\Omega)$ has dimension greater than one. Let $K\subset\Omega$ be a compact subset of positive measure. Then we find $\psi\in\mathscr{D}^J(\Omega)$ satisfying (5.16) and such that $\int_K\psi\,dx=0$. This contradicts the property that $\operatorname*{ess\,inf}_K|\psi|>0$, and the uniqueness follows.
The fact that $\varphi_1\in L^\infty(\Omega)$ follows from (5.19). The proof of (5.19) can be done as for the case $I=(-\Delta)^s$. For the readers convenience we included the proof in the Appendix, p. 108. □

6 A class of nonlocal evolution equations

We now turn to the following time dependent problem for an open set $\Omega \subset \mathbb{R}^N$ and a time interval $T \subset [0,\infty)$:

$$(P_T) \qquad \begin{cases} \partial_t u + Iu = f(t,x,u) & \text{in } T \times \Omega; \\ u \equiv 0 & \text{on } T \times (\mathbb{R}^N \setminus \Omega) \end{cases}$$

Here $f : T \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.[6] and I is the operator associated to a measurable kernel function $J : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,x) : x \in \mathbb{R}^N\} \to [0,\infty)$ as in Section 2. We suppose that J satisfies (J1), and we let $\mathscr{J}$, $\mathscr{D}^J(\Omega)$ and $\mathscr{V}^J(\Omega)$ be defined as in Section 2. As in the time independent case we will work with the weak formulation of problem (P_T).

Definition 6.1.

1. We call a function $u \in C(T, \mathscr{D}^J(\Omega)) \cap C^1(T, L^2(\Omega))$ *a solution of* (P_T) *in* $T \times \Omega$ if for all $\varphi \in \mathscr{D}^J(\Omega)$ with compact support in $\mathbb{R}^N$ and $t \in T$ the integral $\int_\Omega f(t,x,u(x))\varphi(x)\, dx$ exists and
$$\mathscr{J}(u(t),\varphi) = \int_\Omega (f(x,u(t,x)) - \partial_t u(t,x))\varphi(x)\, dx.$$
If, in addition, $T = [0,\infty)$, then we also call u a *global solution*.

2. We call a function $u \in C(T, \mathscr{V}^J(\mathbb{R}^N)) \cap C^1(T, L^2(\Omega))$ *a supersolution of the equation* $\partial_t u + Iu = f(t,x,u)$ *in* $T \times \Omega$ if for all $\varphi \in \mathscr{D}^J(\Omega)$, $\varphi \geq 0$ with compact support in $\mathbb{R}^N$ and $t \in T$ the integral $\int_\Omega f(t,x,u(x))\varphi(x)\, dx$ exists and
$$\mathscr{J}(u(t),\varphi) \geq \int_\Omega (f(t,x,u(t,x)) - \partial_t u(t,x))\varphi(x)\, dx.$$

3. We call u *a supersolution u of the problem* (P_T) *in* Ω if u is a supersolution of $\partial_t u + Iu = f(t,x,u)$ in $T \times \Omega$ satisfying $u \geq 0$ on $T \times (\mathbb{R}^N \setminus \Omega)$.

4. We call u a *subsolution* of $\partial_t u + Iu = f(t,x,u)$, resp. (P_T) in $T \times \Omega$ if $-u$ is a supersolution of $\partial_t u + Iu = f(t,x,u)$, resp. (P_T) in $T \times \Omega$

6.1 Time dependent maximum principles

For $\Omega \subset \mathbb{R}^N$ open and bounded, $c \in L^\infty(T \times \Omega)$ we analyze supersolutions u of

$$\partial_t u + Iu = c(t,x)u \qquad \text{in } T \times \Omega, \qquad u \equiv 0 \qquad \text{on } T \times (\mathbb{R}^N \setminus \Omega). \tag{6.1}$$

Proposition 6.2. *Assume that J satisfies (J1), let Ω be an open bounded set, let $T = [t_0, T_0) \subset \mathbb{R}$ be a time interval and let $c \in L^\infty(T \times \Omega)$. Moreover, let u be a supersolution u of (6.1) in $T \times \Omega$.*

[6] $f(\cdot,\cdot,u)$ is measurable for every (t,x), $f(t,x,\cdot)$ is continuous and for each compact set $K \subset \mathbb{R}$ we have that $(t,x) \mapsto \sup\{|f(t,x,u)| : u \in K\}$ are Lebesgue integrable on $T \times \Omega$.

(i) We have

$$\|u^-(t)\|_{L^2(\Omega)} \le \exp\left[\left(\|c^+\|_{L^\infty(T\times\Omega)} - \Lambda_{1,J}(\Omega)\right)(t-t_0)\right]\|u^-(t_0)\|_{L^2(\Omega)} \tag{6.2}$$

for $t \in T$, where $\Lambda_{1,J}(\Omega)$ is given in (2.5). In particular, if $\|c^+\|_{L^\infty(T\times\Omega)} < \Lambda_{1,J}(\Omega)$ and $T_0 = \infty$, we have $\|u^-(t)\|_{L^2(\Omega)} \to 0$ for $t \to \infty$.

(ii) If additionally we have $u^-(t_0) \in L^\infty(\Omega)$, then

$$\|u^-(t)\|_{L^\infty(\Omega)} \le \exp\left(\left(\|c^+\|_{L^\infty(T\times\Omega)} - \Lambda_{1,J}(\Omega)\right)(t-t_0)\right)\|u^-(t_0)\|_{L^\infty(\Omega)} \tag{6.3}$$

for $t \in T$. In particular, if $\|c^+\|_{L^\infty(T\times\Omega)} < \Lambda_{1,J}(\Omega)$ and $T_0 = \infty$, then $u^-(t) \to 0$ uniformly in Ω for $t \to \infty$.

Proof. Without restriction, we may assume $t_0 = 0$. Put $c_\infty := \|c^+\|_{L^\infty(T\times\Omega)}$ and let $\varepsilon > 0$. Note that $v(t) = e^{(\Lambda_{1,J}(\Omega)-c_\infty)t}u(t)$ is a supersolution of

$$\partial_t v + Iv = (\tilde{c}(t,x) + \Lambda_{1,J}(\Omega))v \text{ in } T\times\Omega \tag{6.4}$$

with $\tilde{c}(t,x) = c(t,x) - c_\infty \le 0$. By Lemma 2.13 and Lemma 3.1 we have

$$\varphi(t) := (v(t)+d)^- \in \mathscr{D}^J(\Omega) \quad \text{for } t \in T \text{ and } d \ge 0.$$

Note that we have $v(t,x)\varphi(t,x) \le 0$ in Ω. Testing equation (6.4) with $\varphi(t)$ for $t \in T$ gives, again by Lemma 3.1,

$$\begin{aligned}
\Lambda_{1,J}(\Omega)\|\varphi(t)\|^2_{L^2(\Omega)} &\le \mathscr{J}(\varphi(t),\varphi(t)) \le -\mathscr{J}(v(t)+d,\varphi(t)) = -\mathscr{J}(v(t),\varphi(t)) \\
&\le \int_\Omega \partial_t v(t,x)\varphi(t,x)\,dx - \int_\Omega \tilde{c}(t,x)v(t,x)\varphi(t,x)\,dx - \Lambda_{1,J}(\Omega)\int_\Omega v(t,x)\varphi(t,x)\,dx \\
&\le -\frac{1}{2}\frac{d}{dt}\int_\Omega \varphi^2(t,x)\,dx + \Lambda_{1,J}(\Omega)\int_\Omega v^-(t,x)\varphi(t,x)\,dx.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dt}\|\varphi(t)\|^2_{L^2(\Omega)} &\le 2\Lambda_{1,J}(\Omega)\int_\Omega (v^-(t,x) - \varphi(t,x))\varphi(t,x)\,dx \\
&= 2\Lambda_{1,J}(\Omega)d\int_\Omega \varphi(t,x)\,dx, \quad \text{for } t \in T.
\end{aligned} \tag{6.5}$$

Thus with $d = 0$ we have $\frac{d}{dt}\|\varphi(t)\|^2_{L^2(\Omega)} \le 0$. Since in this case $\varphi(t) = v^-(t)$ for $t \in T$, we have $\|v^-(t)\|_{L^2(\Omega)} \le \|u^-(0)\|_{L^2(\Omega)}$ since $v^-(0) = u^-(0)$. This gives (6.2). To prove (6.3) assume additionally $u^-(0) \in L^\infty(\Omega)$ and put $d = \|u^-(0)\|_{L^\infty(\Omega)}$. Note that $w := v + d$ is a supersolution of

$$\partial_t w + Iw = \tilde{c}(t,x)w \text{ in } T\times\Omega, \tag{6.6}$$

since $\tilde{c} \leq 0$ in $T \times \Omega$. Thus from (6.2) we get for $t \in T$:

$$\|\varphi(t)\|_{L^2(\Omega)} = \|w^-(t)\|_{L^2(\Omega)} \leq e^{-\Lambda_{1,J}(\Omega)t}\|w^-(0)\|_{L^2(\Omega)} = e^{-\Lambda_{1,J}(\Omega)t}\|\varphi(0)\|_{L^2(\Omega)}.$$

Since $\varphi(0) \equiv 0$ by the choice of d we get $\varphi(t) = 0$ for all $t \in T$ and thus $v(t) \geq -d$ in Ω. This proves (6.3).

□

We note that if $(J1)_d$ is satisfied, a combination of Proposition 6.2 with Lemma 2.17 gives rise to a time-dependent small volume maximum principle (see also Proposition 9.10 below).

Corollary 6.3. *Assume that J satisfies (J1), let Ω be an open bounded set and let $T = [t_0, T_0) \subset \mathbb{R}$ be a time interval. Let $c \in L^\infty(T \times \Omega)$ and $u_0 \in L^\infty(\Omega)$. For $T_0 < \infty$ we have that every solution u of (6.1) in $T \times \Omega$ with $u(t_0) = u_0$ in Ω fulfills $u \in L^\infty(T \times \Omega)$. If $T_0 = \infty$ and, in addition, $\|c\|_{L^\infty(T\times\Omega)} < \Lambda_{1,J}(\Omega)$, then $u \in L^\infty(T \times \Omega)$ and $\lim\limits_{t\to\infty} \|u(t)\|_{L^\infty(\Omega)} = 0$.*

Proof. Since any solution is also a supersolution we may apply Proposition 6.2 to get for all $t \in T$, $t < \infty$

$$\|u^-(t)\|_{L^\infty(\Omega)} \leq \exp\left(\left(\|c\|_{L^\infty(T\times\Omega)} - \Lambda_{1,J}(\Omega)\right)(t - t_0)\right)\|u_0^-\|_{L^\infty(\Omega)} < \infty.$$

Since u is a solution of (6.1) in $T \times \Omega$ we also have that $-u$ is a solution of (6.1) in $T \times \Omega$ with initial condition $-u(t_0) = -u_0$ and thus a supersolution of (6.1) in $T \times \Omega$. Since $(-u)^- = u^+$ we get by Proposition 6.2 for any $t \in T$ with $t < \infty$

$$\|u^+(t)\|_{L^\infty(\Omega)} \leq \exp\left(\left(\|c\|_{L^\infty(T\times\Omega)} - \Lambda_{1,J}(\Omega)\right)(t - t_0)\right)\|u_0^+\|_{L^\infty(\Omega)} < \infty.$$

Combining these we get for $T_0 < \infty$:

$$\|u\|_{L^\infty(T\times\Omega)} \leq \exp\left(\left(\|c\|_{L^\infty(T\times\Omega)} - \Lambda_{1,J}(\Omega)\right)(T_0 - t_0)\right)\|u_0\|_{L^\infty(\Omega)}.$$

This proves the first part. The last part follows immediately by the assumption $\|c\|_{L^\infty(T\times\Omega)} < \Lambda_{1,J}(\Omega)$. □

6.2 The Cauchy problem with initial data in L^2

Assuming (J1) we will show in this part that we can apply the standard semigroup approach to the operator I given by the nonlocal bilinear form $\mathscr{J}$ (see Corollary 2.6). At several occasions we will use different results from standard semigroup theory which we will state as we need them and refer to [26].

We will first discuss dissipativity and m-dissipativity of the operator I in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is an open bounded set. In Section 8 below we will state further regularity results by discussing dissipativity and m-dissipativity in $L^\infty(\Omega)$ and $\mathscr{C}_0(\Omega)$[7]. The following definitions will be useful.

Definition 6.4. Let X be a Banach space with norm $\|\cdot\|_X$ and let $B : \mathrm{dom}(B) \subset X \to X$ be a densely defined operator with domain $\mathrm{dom}(B)$, i.e. $\overline{\mathrm{dom}(B)}^X = X$.

[7] Here we use $\mathscr{C}_0(\Omega) = \{u \in C(\mathbb{R}^N) \, : \, u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega\}$

1. The operator B is called *dissipative* (in X), if for all $\lambda > 0$ and all $x \in \operatorname{dom}(B)$ we have $\|x - \lambda Bx\|_X \geq \|x\|_X$.
2. A dissipative operator B is called *m-dissipative* (in X), if for all $\lambda > 0$ and all $f \in X$ there is $x \in \operatorname{dom}(B)$ such that $x - \lambda Bx = f$ in X.

If, in addition, X is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_X$ we say that B is negative semi definite, if $\langle Bx, x \rangle_X \leq 0$ for all $x \in \operatorname{dom}(B)$.

We recall some classical results, which we will apply below to the operator I introduced in Section 2, Proposition 2.6.

Theorem 6.5. *(Lax-Milgram) (see e.g. [26, Theorem 1.1.5, pp. 1]) Let X be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear functional. Assume that there exist two constants $c_2 < \infty$ and $c_1 > 0$ such that*

1. $|a(u,v)| \leq c_2 \|u\| \|v\|$ for all $(u,v) \in X \times X$ (continuity);

2. $a(u,u) \geq c_1 \|u\|^2$ for all $u \in X$ (coerciveness).

Then for every $f \in X^*$ (the dual of X), there exists a unique $u \in X$ such that $a(u,v) = \langle f, v \rangle$ for all $v \in X$.

Remark 6.6. Note that if H is another Hilbert-space such that $X \subset H$ is dense and the imbedding is continuous, then we also have by identification that for all $f \in H$ there is a unique $u \in X$ such that $a(u,v) = \langle f, v \rangle$ for all $v \in X$, if $a(\cdot,\cdot)$ fulfills the conditions of Theorem 6.5 on X.

Proposition 6.7. *(see [26, Proposition 2.4.2, pp. 22]) Let X be a Hilbert space and let $B : \operatorname{dom}(B) \to X$ be given with dense domain. Then B is dissipative if and only if B is negative semi definite.*

Corollary 6.8. *(see [26, Corollary 2.4.10, pp. 24]) Let X be a Hilbert space and let $B : \operatorname{dom}(B) \to X$ be a densely defined negative semi definite operator such that $\operatorname{Graph}(B) \subset \operatorname{Graph}(B^*)$, where $\operatorname{Graph}(B)$ and $\operatorname{Graph}(B^*)$ denote the graphs of B and B^* resp., and B^* is the adjoint of B. Then B is m-dissipative if and only if B is self-adjoint.*

Theorem 6.9. *(Hille-Yosida-Phillips) (see e.g. [26, Theorem 3.1.1, pp. 33])*
Let X be a Banach space. If B is m-dissipative in X, then B generates a contraction semigroup $S(t) : X \to X$, $t \geq 0$, i.e.

$$\begin{aligned} S(0) &= id(X); \\ S(t)S(s) &= S(t+s) \quad \text{for all } t,s \geq 0; \\ S(t) &\in \mathscr{L}(X) \quad \text{and } \|S(t)\| \leq 1 \text{ for all } t \geq 0. \end{aligned}$$

In addition, $u(t) := S(t)x$, $x \in \operatorname{dom}(B)$ is the unique solution of the Cauchy problem

$$\begin{cases} u \in C([0,\infty), \operatorname{dom}(B)) \cap C^1([0,\infty), X); \\ \partial_t u(t) = Bu(t) \quad \text{for all } t > 0; \\ u(0) = x. \end{cases}$$

Finally, $S(t)Bx = BS(t)x$ for all $x \in \operatorname{dom}(B)$ and $t \geq 0$.

Theorem 6.10. *(see e.g. [26, Theorem 3.2.1]) Let X be a Hilbert space with scalar product $\langle \cdot,\cdot\rangle_X$ and induced norm $\|\cdot\|_X$. Moreover, let B be a self-adjoint and negative semi definite operator in X. For $x \in X$ let $u(t) = S(t)x$, where $S(t)$ is the semigroup generated by B as consequence of Theorem 6.9 and Corollary 6.8. Then u is the unique solution of the Cauchy problem*

$$\begin{cases} u \in C([0,\infty),X)\cap C((0,\infty),\mathrm{dom}(B))\cap C^1((0,\infty),X); \\ \partial_t u(t) = Bu(t) \qquad \text{for all } t>0; \\ u(0) = x. \end{cases}$$

In addition, we have

$$\|Bu(t)\|_X \le \frac{1}{t\sqrt{2}}\|x\|_X \text{ and } -\langle Bu(t),u(t)\rangle_X \le \frac{1}{2t}\|x\|_X^2 \tag{6.7}$$

Finally, for all $x \in \mathrm{dom}(B)$ we have

$$\|Bu(t)\|_X^2 \le -\frac{1}{2t}\langle Bx,x\rangle_X. \tag{6.8}$$

Let $\Omega \subset \mathbb{R}^N$ be an open bounded set. Assuming (J1), we now consider the operator I in $L^2(\Omega)$ associated with J , see Corollary 2.6. Moreover, we put $B = -I$. We recall from Corollary 2.9 that for every $g \in L^2(\Omega)$ there is a unique $u \in \mathrm{dom}(B) = \mathrm{dom}(I) \subset \mathscr{D}^J(\Omega)$ such that

$$\mathscr{J}(u,\varphi) = \int_\Omega g(x)\varphi(x)\,dx \qquad \text{for all } \varphi \in \mathscr{D}^J(\Omega),$$

i.e. we have $-Bu = g$ in $L^2(\Omega)$.

Proposition 6.11. *Let B be as above and let $\Omega \subset \mathbb{R}^N$ open and bounded. Then the operator B with domain* $\mathrm{dom}(B)$ *is densely defined, negative semi definite, m-dissipative and self-adjoint on $L^2(\Omega)$.*

Proof. By Corollary 2.6 and Proposition 2.3 we have that B is self-adjoint and negative semi definite. In particular, B is densely defined. By Corollary 2.9 we have that B is m-dissipative. □

Corollary 6.12. *Let B be as above and let $\Omega \subset \mathbb{R}^N$ open and bounded. Then there is a contraction semigroup $S_B(t) : L^2(\Omega) \to L^2(\Omega)$, $t \ge 0$ associated to B such that for any $\varphi \in L^2(\Omega)$ we have that $u(t) := S_B(t)\varphi$, $t \ge 0$ is the unique solution of*

$$\begin{cases} u \in C([0,\infty),L^2(\Omega))\cap C((0,\infty)),\mathrm{dom}(B))\cap C^1((0,\infty),L^2(\Omega)) \\ \partial_t u(t) = Bu(t) \qquad \text{for } t>0, \\ u(0) = \varphi. \end{cases} \tag{6.9}$$

In addition, we have the following bounds for any $t > 0$:

$$\|Iu(t)\|_{L^2(\Omega)} \le \frac{1}{t\sqrt{2}}\|\varphi\|_{L^2(\Omega)} \quad \text{and} \quad \mathscr{J}(u(t),u(t)) \le \frac{1}{2t}\|\varphi\|_{L^2(\Omega)}^2. \tag{6.10}$$

Moreover, if $\varphi \in \mathrm{dom}(B)$*, then also for any* $t > 0$

$$\|Iu(t)\|^2_{L^2(\Omega)} \leq \frac{1}{2t} \mathscr{J}(\varphi, \varphi). \tag{6.11}$$

Furthermore, if $\varphi \in L^\infty(\Omega)$*, then also* $S_B(t)\varphi \in L^\infty(\Omega)$ *for* $t \geq 0$ *and* $S_B(t)\varphi \to 0$ *uniformly for* $t \to \infty$*.*

Proof. All assertions but the last are due to Theorem 6.10. The last assertion follows from Corollary 6.3. □

7 Weak time dependent Harnack inequality

In this part we want to discuss several versions of Harnack inequalities if J fulfills (JU$_s$) and (JL$_s$). We will also show below that the weak parabolic Harnack inequality implies interior Hölder regularity if the right-hand side is bounded.
We introduce the notation for parabolic cylinders. For $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^N$, $r, \vartheta > 0$ we put

$$Q(r,\vartheta,t_0,x_0) := (t_0,t_0+8\vartheta) \times B_{2r}(x_0),$$

$$Q^-(r,\vartheta,t_0,x_0) := (t_0,t_0+\vartheta) \times B_r(x_0) \text{ and}$$

$$Q^+(r,\vartheta,t_0,x_0) := (t_0+7\vartheta,t_0+8\vartheta) \times B_r(x_0)$$

We will investigate supersolutions of the following problem:

$$(L) \qquad \begin{cases} \partial_t v + Iv = c(t,x)v + g(t,x) & \text{in } T \times U; \\ \qquad v \equiv 0 & \text{on } T \times (\mathbb{R}^N \setminus U), \end{cases}$$

where I is associated to a kernel function J which satisfies (J1), (JU$_s$) and (JL$_s$), c, g are functions in $L^\infty(T \times U)$ and $U \subset \mathbb{R}^N$ is a bounded domain.
We have the following scaling property as proven in [40, Lemma 2.5] and [40, Remark after Theorem 1.2]:

Lemma 7.1. *Let $U \subset \mathbb{R}^N$ be an open set and fix $B_{r_0}(x_0) \subset U$ for $r_0 > 0$ and $x_0 \in U$. Assume J satisfies (J1), (JU$_s$) and (JL$_s$) with r_0 and $s \in (0,1)$. Let v be a supersolution of (P_T) in $T \times U$. Then $\tilde{v}(t,x) = v(t,r_0x+x_0)$ satisfies the following inequality for every $t \in T$ and every $\varphi \in \mathscr{D}^J(B_1(0))$, $\varphi \geq 0$:*

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\tilde{v}(t,x)-\tilde{v}(t,y))(\varphi(x)-\varphi(y))\tilde{J}(x,y)\,dxdy \geq \int_{B_1(0)} (\tilde{f}(t,x,\tilde{v})-\partial_t\tilde{v})\varphi(x)\,dx$$

with

$$\tilde{f}(t,x,u) = f(t,r_0x+x_0,u) \qquad \text{and} \qquad \tilde{J}(x,y) = r_0^N J(r_0x+x_0,r_0y+x_0).$$

In particular, $\tilde{J}$ is a measurable function which fulfills again (J1), (JU$_s$) and (JL$_s$) such that

$$c_1|x-y|^{-N-2s} \leq \tilde{J}(x,y) \leq c_2|x-y|^{-N-2s} \qquad \text{for all } (x-y) \in B_{1/r_0}(0), \tag{7.1}$$

where c_1, c_2 are the same as for J in (5.14).

Due to this and Remark 5.15 it is evident that the following is a mere reformulation of a special case of [40, Theorem 1.1]. We note that the notion of supersolution considered in [40] is weaker than the one considered here. Indeed, for the result to hold, it is enough to assume $v \in H^s_{loc}(U) \cap L^\infty(\mathbb{R}^N)$.

Theorem 7.2. *Let $r_0 \in (0,1]$, $s \in (0,1)$ and $\vartheta > 0$ be given. Then for any even function J which fulfills (J1), (JU$_s$) and (JL$_s$) with r_0 there are constants $C_i = C_i(N, c_\infty, r_0, \vartheta, k) > 0$, $i = 1,2$ such that for any $(t_0,x_0) \in \mathbb{R}^{N+1}$, any $\vartheta > 0$, any $g \in L^\infty(Q(r_0,\vartheta,t_0,x_0))$ and any bounded supersolution v of (L) in $Q(r_0,\vartheta,t_0,x_0) \subset [0,\infty)\times\mathbb{R}^N$ which is nonnegative on $(t_0,t_0+8\vartheta)\times\mathbb{R}^N$ we have*

$$\inf_{(t,x)\in Q^+(r_0,\vartheta,t_0,x_0)} v \geq C_1[v]_{L^1(Q^-(r_0,\vartheta,t_0,x_0))} - C_2\|g\|_{L^\infty(Q(r_0,\vartheta,t_0,x_0))}. \tag{7.2}$$

By an argument based on building chains of cylinders, we may deduce the following Harnack inequality for general pairs of domains. A similar argument has been detailed in [56, Appendix], but we need to argue slightly differently since the triples of parabolic cylinders in Theorem 7.2 have a smaller overlap than the ones considered in [56]. The following result is as in [49, Corollary 2.8]. We include the proof here for completeness.

Corollary 7.3. *Let $r_0 \in (0,1]$, $s \in (0,1)$, $R, \tau, \varepsilon > 0$ and $C_1, C_2 > 0$ be given. Then there exist a positive constants $C_i = C_i(N,s,r_0,C_1,C_2,R,\varepsilon,\tau) > 0$, $i = 1,2$ with the following property: Assume (J1), (JU$_s$) and (JL$_s$) with r_0 and let $D \subset\subset U \subset \mathbb{R}^N$ be a pair of domains such that $\operatorname{dist}(D,\partial U) \geq 2r_0$, $|D| \geq \varepsilon$ and $\operatorname{diam}(D) \leq R$. Moreover, let $g \in L^\infty(T\times U)$ and a bounded supersolution v of (L) on $T\times U$ be given such that v is nonnegative in $T\times\mathbb{R}^N$, where $T = [t_0, t_0+4\tau]$ for some $t_0 \in \mathbb{R}$. Then we have*

$$\inf_{(t,x)\in T_+\times D} v(t,x) \geq C_1[v]_{L^1(T_-\times D)} - C_2\|g\|_{L^\infty(T\times U)}, \tag{7.3}$$

where $T_+ = [t_0+3\tau, t_0+4\tau]$ and $T_- = [t_0+\tau, t_0+2\tau]$.

Proof. We first note that there exists $n = n(N,R,r_0) \in \mathbb{N}$ and $\mu = \mu(N,R,r_0) > 0$ such that the following holds:
For every subset $D \subset \mathbb{R}^N$ with $\operatorname{diam} D \leq R$ there exists a subset $S_D \subset D$ of $n+1$ points such that D is covered by the balls $B_{r_0}(x)$, $x \in S_D$, and for every two points $x_*, x^* \in S_D$ there exists a finite sequence $x_j \in S_D$, $j = 0,\dots,n$ such that

$$x_0 = x_*, \quad x_n = x^* \qquad \text{and} \qquad |B_{r_0}(x_j)\cap B_{r_0}(x_{j+1})| \geq \mu \quad \text{for } j = 0,\dots,n-1. \tag{7.4}$$

We now fix $D \subset\subset U \subset H$ as in the assertion, and we fix n, μ and a set S_D with the property above. Next, we put $\vartheta = \frac{\tau}{7}\min\{\frac{1}{17}, \frac{1}{n+3}\}$, and we claim the following:
For given $t_* \in [t_0+\tau, t_0+2\tau]$ and $t^* \in [t_0+3\tau, t_0+4\tau]$ there exists a finite sequence $t_* = s_0 < \dots < s_m = t^* - 8\vartheta$ such that

$$s_j + 7\vartheta \leq s_{j+1} \leq s_j + \frac{15}{2}\vartheta \qquad \text{for } j = 0,\dots,m-1 \tag{7.5}$$

and

$$\max\{14, n\} \leq m \leq \max\{51, 3(n+3)\} \tag{7.6}$$

Indeed, let $m \in \mathbb{N}$ and $\sigma \in [0, 7\vartheta)$ be such that $t_* + 7m\vartheta + \sigma = t^* - 8\vartheta$. The definition of ϑ and the restrictions on t_*, t^* then force (7.6), and (7.5) holds with $s_j := t_* + j\Big(7\vartheta + \frac{\sigma}{m}\Big)$ for $j = 0,\dots,m$. Next, we fix $t_* \in [t_0+\tau, t_0+2\tau]$, $x_* \in S_D$ such that

$$\|v\|_{L^1(Q_-(r_0,\vartheta,t_*,x_*))} = \max\Big\{\|v\|_{L^1(Q_-(r_0,\vartheta,t,x))} \,:\, x \in S_D,\, t_0+\tau \leq t \leq t_0+2\tau\Big\}.$$

Since the cylinders

$$Q_-(r_0,\vartheta,t_0+\tau+l\vartheta,x), \qquad l\in\mathbb{N}\cup\{0\},\, l\le\frac{\tau}{\vartheta},\, x\in S_D$$

cover $[t_0+\tau,t_0+2\tau]\times D$, we have

$$\begin{aligned}
[v]_{L^1([t_0+\tau,t_0+2\tau]\times D)} &= \frac{1}{\tau|D|}\|v\|_{L^1([t_0+\tau,t_0+2\tau]\times D))} \le \frac{(n+1)(\frac{\tau}{\vartheta}+1)}{\tau\varepsilon}\|v\|_{L^1(Q_-(r_0,\vartheta,t_*,x_*))}\\
&= \frac{(n+1)(\frac{1}{\vartheta}+\frac{1}{\tau})}{\varepsilon}7\vartheta|B_{r_0}(0)|\,[v]_{L^1(Q_-(r_0,\vartheta,t_*,x_*))}\\
&\le \kappa_1[v]_{L^1(Q_-(r_0,\vartheta,t_*,x_*))} \qquad \text{with} \quad \kappa_1 := \frac{14(n+1)|B_{r_0}(0)|}{|\varepsilon|} \qquad (7.7)
\end{aligned}$$

We now consider $t^*\in[t_0+3\tau,t_0+4\tau]$, $x\in D$ arbitrary. Then we choose $x^*\in S_D$ such that $x\in B_{r_0}(x^*)$, and we choose s_j, $j=0,\dots,m$ with the properties (7.5) and (7.6). Moreover, we fix a sequence of points $x_j\in S_D$, $j=0,\dots,m$ such that (7.4) holds with m in place of n. This may be done, since $m\ge n$, by repeating some of the points in the chain if necessary. We now define

$$Q_j := Q(r_0,\vartheta,s_j,x_j) \quad \text{and} \quad Q_j^\pm := Q^\pm(r_0,\vartheta,s_j,x_j) \qquad \text{for } j=0,\dots,m.$$

We note that, by (7.4) and (7.5), we have

$$|Q_j^+\cap Q_{j+1}^-|\ge\frac{\mu\vartheta}{2} \qquad \text{for } j=0,\dots,m-1.$$

Hence we may estimate, using Theorem 7.2 and the fact that $Q_j\subset T\times U$ for $j=0,\dots,m$,

$$\begin{aligned}
c_1[v]_{L^1(Q_j^-)} &\le \inf_{Q_j^+} v + c_2\|g\|_{L^\infty(Q_j)} \le [v]_{L^1(Q_j^+\cap Q_{j+1}^-)} + c_2\|g\|_{L^\infty(T\times U)}\\
&\le \frac{|Q_{j+1}^-|}{|Q_j^+\cap Q_{j+1}^-|}[v]_{L^1(Q_{j+1}^-)} + c_2\|g\|_{L^\infty(T\times U)} \le \frac{2|B_{r_0}(0)|}{\mu}[v]_{L^1(Q_{j+1}^-)} + c_2\|g\|_{L^\infty(T\times U)}.
\end{aligned}$$

Iterating this estimate m times and using Theorem 7.2 once more, we obtain

$$\begin{aligned}
[v]_{L^1(Q_0^-)} &\le \Big(\frac{2|B_{r_0}(0)|}{c_1\mu}\Big)^m[v]_{L^1(Q_m^-)} + \frac{c_2}{c_1}\sum_{k=0}^{m-1}\Big(\frac{2|B_{r_0}(0)|}{c_1\mu}\Big)^k\|g\|_{L^\infty(T\times U)}\\
&\le \Big(\frac{2|B_{r_0}(0)|}{\mu}\Big)^m c_1^{-(m+1)}\inf_{Q_m^+} v + \frac{c_2}{c_1}\sum_{k=0}^{m}\Big(\frac{2|B_{r_0}(0)|}{c_1\mu}\Big)^k\|g\|_{L^\infty(T\times U)}.
\end{aligned}$$

Hence, since $(t^*,x^*)\in Q_m^+$, we conclude by (7.7) that

$$v(t^*,x^*)\ge\inf_{Q_m^+} v\ge \hat{c}_1[v]_{L^1(Q_0^-} - \tilde{c}_2\|g\|_{L^\infty(T\times U)} \ge \frac{\hat{c}_1}{\kappa_1}[v]_{L^1([t_0+\tau,t_0+2\tau]\times D)} - \tilde{c}_2\|g\|_{L^\infty(T\times U)}$$

with

$$\hat{c}_1 = \Big(\frac{2|B_{r_0}(0)|}{\mu}\Big)^{-m}c_1^{m+1} \qquad \text{and} \qquad \tilde{c}_2 = \hat{c}_1\,\frac{c_2}{c_1}\sum_{k=0}^{m}\Big(\frac{2|B_{r_0}(0)|}{c_1\mu}\Big)^k$$

Hence the claim follows with $\tilde{c}_1=\frac{\hat{c}_1}{\kappa_1}$ and $\tilde{c}_2$ as above. Note that $\tilde{c}_1$ and $\tilde{c}_2$ only depend, via n, m, μ, c_1, c_2, κ_1, on N, s and the given quantities $r_0,R,\varepsilon,\tau,C_1,C_2$. □

7.1 Interior Hölder regularity

In this part we will shortly discuss how we reach interior Hölder regularity if the right hand side is bounded. For the case of the fractional Laplacian, this was done in [49, Appendix]. The generalization to the case where J satisfies (JU$_s$) and (JL$_s$) is very similar. The interior regularity will be deduced from the Harnack inequality of Felsinger and Kassmann [40]. More precisely, we will use the following rescaled variant of a special case of [40, Corollary 5.2]. We refer also to [53, Theorem 2] for interior regularity in the time independent case.

Proposition 7.4. *(cf. [40, Corollary 5.2]) Assume J satisfies (J1), (JU$_s$) and (JL$_s$) for some $s \in (0,1)$ and let*

$$D_{\ominus} := (-2^{2s+1}, -2^{2s+1}+1) \times B_1(0) \qquad \text{and} \qquad D_{\oplus} := (-1,0) \times B_1(0).$$

There exists $\varepsilon_0, \delta > 0$ such that for every nonnegative supersolution

$$w : (-2^{2s+1}, 0) \times \mathbb{R}^N \to \mathbb{R}$$

of the equation

$$\partial_t w + Iw = -\varepsilon_0 \qquad \text{in } (-2^{2s+1}, 0) \times B_4(0)$$

with the property that

$$|D_{\ominus} \cap \{w \geq 1\}| \geq \frac{1}{2}|D_{\ominus}| \tag{7.8}$$

we have $w \geq \delta$ a.e. on $D_{\oplus}$. Here δ and ε_0 depend only on s, N, and the constants r_0, c_1 and c_2 given by (JU$_s$) and (JL$_s$) (cf. (5.14).

The following is similar to [49, Corollary 4.4]. We will follow closely the proof presented there.

Corollary 7.5. *Let $r_0 \in (0,1]$, $R > r_0$, $c_u > 0$ and $f_\infty > 0$. Assume (J1)$_{diff}$, (JU$_s$) and (JL$_s$) for some $s \in (0,1)$ with r_0 and let $T := (t_0 - r_0^{2s}, t_0)$ for some $t_0 \in \mathbb{R}$. Then there exist constants $a \in (0,1)$ and $C > 0$ depending on $N, s, f_\infty, c_u, r_0, R$ with the following property:*
If $x_0 \in \mathbb{R}^N$ and $u \in C(T, \mathscr{D}^J(B_R(x_0)) \cap L^\infty(\mathbb{R}^N)) \cap C^1(T, L^2(B_{r_0}(x_0)))$ is a function satisfying $\|u\|_{L^\infty(T\times\mathbb{R}^N)} \leq c_u$ and

$$\partial_t u + Iu = f(t,x) \qquad \text{in } T \times B_{r_0}(x_0),$$

with some $f \in L^\infty(T \times B_{r_0}(x_0))$ with $\|f\|_{L^\infty(T\times B_{r_0}(x_0))} \leq f_\infty$, then we have

$$\operatorname*{osc}_{Q(r)} u \leq Cr^a \qquad \text{for } r \in (0, r_0], \text{ where } Q(r) := (t_0 - r^{2s}, t_0) \times B_r(x_0). \tag{7.9}$$

Proof. Without loss of generality, we may assume that $t_0 = 0$ and $x_0 = 0$. Moreover, we may assume by normalization that $c_u = \frac{1}{4}$. In this case we will prove (7.9) with $C = 1$ for some suitable $a \in (0,1)$. Suppose by contradiction that the statement is false. Then there exist, for every $k \in \mathbb{N}$, functions $f_k \in L^\infty(T \times B_{r_0}(0))$ with $\|f_k\|_{L^\infty(T\times B_{r_0}(0))} \leq f_\infty$ and $u_k \in C(T, \mathscr{D}^J(B_R(0)) \cap L^\infty(\mathbb{R}^N)) \cap C^1(T, L^2(B_{r_0}(0)))$ with

$$\|u_k\|_{L^\infty(T\times\mathbb{R}^N)} \leq \frac{1}{4}$$

solving

$$\partial_t u_k + I u_k = f_k(t,x) \qquad \text{in } T \times B_{r_0}(0)$$

as well as $a_k \in (0,1)$ and $r_k \in (0,r_0]$ such that $a_k \to 0$ as $k \to \infty$ and

$$\underset{Q(r_k)}{\operatorname{osc}} u_k \geq r_k^{a_k} \qquad \text{for every } k \in \mathbb{N}.$$

Passing to a subsequence, we may assume $a_k < s$ for $k \in \mathbb{N}$, $(a_k)_k$ is monotone decreasing and

$$\underset{T\times\mathbb{R}^N}{\operatorname{osc}} u_k \leq 2\|u_k\|_{L^\infty(T\times\mathbb{R}^N)} \leq \frac{1}{2} \leq r_0^{a_k} \qquad \text{for every } k \in \mathbb{N}.$$

By making $r_k \in (0,r_0]$ larger if necessary, we may therefore assume that

$$\underset{Q(r_k)}{\operatorname{osc}} u_k = r_k^{a_k} \qquad \text{for every } k \in \mathbb{N}$$

and

$$\underset{Q(r)}{\operatorname{osc}} u_k \leq r^{a_k} \qquad \text{for } r \in [r_k, r_0] \text{ and } k \in \mathbb{N}.$$

Since also $\underset{Q(r_k)}{\operatorname{osc}} u_k \leq \frac{1}{2}$ for every $k \in \mathbb{N}$, we conclude that $r_k \to 0$ as $k \to \infty$. We next define for $k \in \mathbb{N}$: $T_k := (-(\frac{r_0}{r_k})^{2s}, 0)$,

$$v_k : T_k \times \mathbb{R}^N \to \mathbb{R}, \qquad v_k(t,x) = 2r_k^{-a_k} u_k(r_k^{2s}t, r_k x) \quad \text{and}$$
$$J_k : \mathbb{R}^N \setminus \{0\} \to [0,\infty), \qquad J_k(z) = r_k^{N+2s} J(r_k z).$$

Note that since J satisfies (J1)$_{diff}$, (JU$_s$) and (JL$_s$) with r_0 and s we have that J_k satisfies (J1)$_{diff}$, (JU$_s$) and (JL$_s$) with

$$c_1|z|^{-N-2s} \leq J_k(z) \leq c_2|z|^{-N-2s} \quad \text{for a.e. } z \in \mathbb{R}^N \text{ with } 0 < |z| < \frac{r_0}{r_k},$$

where c_1 and c_2 are the same as for the bounds for J. Let $\mathscr{J}_k$ and I_k be the bilinear form and associated operator w.r.t. the kernel function J_k. Note that $\varphi \in \mathscr{D}^J(B_R(0))$ if and only if $\varphi(r_k\cdot) \in \mathscr{D}^{J_k}(B_{R/r_k}(0))$ since

$$\begin{aligned}\mathscr{J}_k(\varphi(r_k\cdot),\varphi(r_k\cdot)) &= \frac{r_k^{N+2s}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\varphi(r_k x) - \varphi(r_k y))^2 J(r_k(x-y))\, dxdy \\ &= \frac{r_k^{2s-N}}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\varphi(x)-\varphi(y))^2 J(x-y)\, dxdy = r_k^{2s-N} \mathscr{J}(\varphi,\varphi).\end{aligned}$$

In particular, $v_k(t) \in \mathscr{D}^{J_k}(B_{R/r_k}(0))$ for $t \in T_k$ and for $\varphi \in \mathscr{D}^{J_k}(B_{r_0/r_k}(0))$ we have

$$\mathscr{J}_k(v_k(t),\varphi) = 2r_k^{2s-N-a_k} \mathscr{J}(u(r_k^{2s}t), \varphi(\frac{\cdot}{r_k})) = 2r_k^{2s-N-a_k} \int_{B_{r_0}(0)} (f(r_k^{2s}t,x) - \partial_t u(r_k^{2s}t,x))\varphi(\frac{x}{r_k})\, dx$$

$$= \int\limits_{B_{r_0/r_k}(0)} (2r_k^{2s-a_k} f(r_k^{2s}t, r_k x) - 2r_k^{2s-a_k} \partial_t u(r_k^{2s}t, r_k x))\varphi(x)\, dx$$

$$= \int\limits_{B_{r_0/r_k}(0)} (\tilde{f}_k(t,x) - \partial_t v_k(t,x))\varphi(x)\, dx$$

with

$$\tilde{f}_k(t,x) = 2r_k^{2s-a_k} f_k(r_k^{2s}t, r_k x).$$

Thus v_k is a solution of

$$\partial_t v_k + I_k v_k = \tilde{f}_k(t,x) \qquad \text{in } D_k := T_k \times B_{\frac{r_0}{r_k}}(0)$$

Without loss of generality, we may assume that $\frac{r_0}{r_k} \geq \max\{2^{1+\frac{1}{2s}}, 5\}$ for every $k \in \mathbb{N}$, so that $(-2^{2s+1}, 0) \times B_5(0) \subset D_k$ for every $k \in \mathbb{N}$. Moreover, we have $\operatorname*{osc}_{Q(1)} v_k = 2$,

$$\operatorname*{osc}_{Q(r)} v_k \leq 2r^{a_k} \qquad \text{for } r \in [1, \frac{r_0}{r_k}],\, k \in \mathbb{N} \tag{7.10}$$

and

$$\operatorname*{osc}_{T_k \times \mathbb{R}^N} v_k \leq 2\left(\frac{r_0}{r_k}\right)^{a_k} \qquad \text{for } k \in \mathbb{N}. \tag{7.11}$$

By adding a constant $c_k \in \mathbb{R}$ to v_k if necessary, we may assume that

$$\sup_{Q(1)} v_k = 1 \qquad \text{and} \qquad \inf_{Q(1)} v_k = -1. \tag{7.12}$$

Note that $v_k(t,x) = c_k$ for $x \in \mathbb{R}^N \setminus B_{R/r_k}(0)$, $t \in T_k$. Moreover, since $\|u_k\|_{L^\infty(T \times \mathbb{R}^N)} \leq \frac{1}{4}$ we have

$$|c_k| \leq 2r_k^{-a_k} \operatorname*{osc}_{T \times \mathbb{R}^N} u_k \leq r_k^{-a_k}.$$

Let $D_\ominus$ and $D_\oplus$ be defined as in Proposition 7.4. Replacing v_k by $-v_k$ and $\tilde{f}_k$ by $-\tilde{f}_k$ if necessary, we may assume

$$|D_\ominus \cap \{v_k \geq 0\}| \geq \frac{1}{2}|D_\ominus|.$$

Note that by (7.10), (7.11) and (7.12) we have

$$v_k(t,x) \geq \min\{-1, 1 - 2|x|^{a_k}\} \qquad \text{for } x \in \mathbb{R}^N,\ t \in (-2^{2s+1}, 0).$$

We now consider

$$w_k : T_k \times \mathbb{R}^N \to \mathbb{R}, \qquad w_k(t,x) := v_k(t,x) + 2 \cdot 5^{a_k} - 1.$$

Then

$$w_k(t,x) \geq \min\bigl\{0, 2(5^{a_k} - |x|^{a_k})\bigr\} \qquad \text{for } x \in \mathbb{R}^N,\ t \in (-2^{2s+1}, 0). \tag{7.13}$$

In particular, we have $w_k \geq 0$ in $(-2^{4s+1},0)\times B_5(0)$. Moreover, for all k we have

$$w_k \equiv c_k + 2\cdot 5^{a_k} - 1 \qquad \text{in } (-2^{4s+1},0)\times \big(\mathbb{R}^N \setminus B_{R/r_k}(0)\big),$$

and thus $w_k \geq c_k \geq -r_k^{-a_k}$ in $(-2^{4s+1},0)\times \mathbb{R}^N \setminus B_{R/r_k}(0)$. Consequently, for k large such that $5r_k \leq \frac{r_0}{16}$, $t\in(-2^{4s+1},0)$ and $x\in B_4(0)$ we have

$$\begin{aligned}
|[I_k w_k^-(t)](x)| &= \left| r_k^{N+2s} \int_{\mathbb{R}^N\setminus B_5(0)} w_k^-(t,y) J(r_k(x-y))\,dy \right| \\
&\leq 2r_k^{N+2s} \int_{B_{R/r_k}(0)\setminus B_5(0)} (|y|^{a_k} - 5^{a_k}) J(r_k(x-y))\,dy + r_k^{N+2s-a_k} \int_{\mathbb{R}^N\setminus B_{R/r_k}(0)} J(r_k(x-y))\,dy \\
&= 2r_k^{2s-a_k} \int_{B_R(0)\setminus B_{5r_k}(0)} (|y|^{a_k} - (r_k5)^{a_k}) J(r_kx-y)\,dy + r_k^{2s-a_k} \int_{\mathbb{R}^N\setminus B_R(r_kx)} J(z)\,dz \\
&\leq 2r_k^{2s-a_k} R^{a_k} \int_{B_R(0)\setminus B_{r_0/2}(0)} J(r_kx-y)\,dy + 2r_k^{2s-a_k} \int_{B_{r_0/2}(0)\setminus B_{5r_k}(0)} (|y|^{a_k} - (r_k5)^{a_k}) J(r_kx-y)\,dy \\
&\qquad + r_k^{2s-a_k} \int_{\mathbb{R}^N\setminus B_{R-r_0}(0)} J(z)\,dz \\
&\leq 2r_k^{2s-a_k} R^{s} \int_{B_R(0)\setminus B_{r_0/4}(0)} J(z)\,dz + r_k^{2s-a_k} \int_{\mathbb{R}^N\setminus B_{R-r_0}(0)} J(z)\,dz \\
&\qquad + 2r_k^{2s-a_k} \int_{B_{r_0/2}(0)\setminus B_{5r_k}(0)} (|y|^{a_k} - (r_k5)^{a_k}) J(r_kx-y)\,dy, \qquad (7.14)
\end{aligned}$$

using in the last inequality $|y-r_kx| \geq |y| - r_k|x| \geq |y| - \frac{r_0}{4}$. Moreover, since for $y\in B_{r_0/2}(0)$ we have $|r_kx-y| \leq 4r_k + \frac{r_0}{2} < r_0$ we have for the third summand in (7.14) by $(\text{JU})_s$:

$$\begin{aligned}
2r_k^{2s-a_k} \int_{B_{r_0/2}(0)\setminus B_{5r_k}(0)} (|y|^{a_k} - (r_k5)^{a_k}) J(r_kx-y)\,dy &\leq 2r_k^{2s-a_k} c_2 \int_{B_{r_0/2}(0)\setminus B_{5r_k}(0)} \frac{|y|^{a_k} - (r_k5)^{a_k}}{|r_kx-y|^{N+2s}}\,dy \\
&= 2r_k^{2s+N} c_2 \int_{B_{r_0/(2r_k)}(0)\setminus B_5(0)} \frac{|y|^{a_k} - 5^{a_k}}{|r_k(x-y)|^{N+2s}}\,dy \leq 2c_2 \int_{\mathbb{R}^N\setminus B_5(0)} \frac{|y|^{a_k} - 5^{a_k}}{|x-y|^{N+2s}}\,dy \qquad (7.15)
\end{aligned}$$

Recall, that $a_k, r_k \to 0$ for $k\to\infty$. Combining (7.14) and (7.15) with $(\text{J1})_b$ we conclude that for some $c = c(N,J,R,r_0) > 0$ we have

$$\limsup_{k\to\infty} \big|[I_k w_k^-(t)](x)\big| \leq \limsup_{k\to\infty} \left(r_k^s c + 2c_2 \int_{\mathbb{R}^N\setminus B_5(0)} \frac{|y|^{a_k} - 5^{a_k}}{|x-y|^{N+2s}}\,dy \right)$$

$$= 2c_2 \limsup_{k\to\infty} \int_{\mathbb{R}^N\setminus B_5(0)} \frac{|y|^{a_k} - 5^{a_k}}{|x-y|^{N+2s}}\, dy = 0,$$

by applying Lebesgue's theorem – here we use that for $x \in B_4(0)$ the function $y \mapsto \frac{|y|^s}{|x-y|^{N+2s}}$ is integrable over $\mathbb{R}^N \setminus B_5(0)$. Hence

$$\lim_{k\to\infty} \|I_k w_k^-\|_{L^\infty((-2^{2s+1},0)\times B_4(0))} = 0. \tag{7.16}$$

We next note that the function w_k^+ is a nonnegative solution of

$$\partial_t w_k^+ + I_k w_k^+ = g_k \qquad \text{in } (-2^{2s+1},0)\times B_4(0) \text{ for every } k\in\mathbb{N}$$

with $g_k := \tilde{f}_k + I_k w_k^-$, whereas $\|g_k\|_{L^\infty((-2^{2s+1},0)\times B_4(0))} \to 0$ as $k\to\infty$ as a consequence of (7.16) and the fact that

$$\|\tilde{f}_k\|_{L^\infty((-2^{2s+1},0)\times B_4(0))} \le 2r_k^{2s-a_k} f_\infty.$$

Consequently, there exists $k_0 \in \mathbb{N}$ such that $\|g_k\|_{L^\infty((-2^{2s+1},0)\times B_4(0))} \le \varepsilon_0$, where ε_0 is given by Proposition 7.4 depending on N, s, c_1, c_2 and not on k since we have $\frac{r_0}{r_k} \ge 5$. On the other hand, since $D_\oplus = Q(1)$, we infer from (7.12) that

$$\inf_{D_\oplus} w_k^+ = \inf_{D_\oplus} w_k = 2\cdot 5^{a_k} - 2 \to 0 \qquad \text{as } k\to\infty.$$

This contradicts Proposition 7.4, applied to $w = w_k^+$. The proof is thus finished. □

7.2 Equicontinuity and the ω-limit set

In the following we fix an open bounded set $\Omega \subset \mathbb{R}^N$. We will next discuss equicontinuity of solutions of the problem

$$(P_T) \qquad \begin{cases} \partial_t u + Iu = f(t,x,u) & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times(\mathbb{R}^N\setminus\Omega), \end{cases}$$

where $f : [0,\infty)\times\Omega\times\mathbb{R} \to \mathbb{R}$ and $u : (0,\infty)\times\mathbb{R}^N \to \mathbb{R}$ satisfy the following conditions:

(F1) $f : [0,\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous. Moreover, for every $K>0$ there exists $L = L(K) > 0$ such that $\sup_{x\in\Omega,\, t>0} |f(t,x,u) - f(t,x,v)| \le L|u-v|$ for $u,v \in [-K,K]$

(U1) There is $c_u \in \mathbb{R}^+$ such that $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \le c_u$ for every $t>0$.

We say that a function $u : (0,\infty)\times\mathbb{R}^N \to \mathbb{R}$ is *eventually equicontinuous on* Ω if there exists $t_0 > 0$ such that the functions $u(\tau+\cdot,\cdot) : [0,1]\times\Omega\to\mathbb{R}$, $\tau \ge t_0$ are equicontinuous, i.e.

$$\lim_{h\to 0} \sup_{\substack{\tau\ge t_0 \\ x,\tilde{x}\in\overline{\Omega},\, t,\tilde{t}\in[\tau,\tau+1], \\ |x-\tilde{x}|,|t-\tilde{t}|<h}} |u(t,x) - u(\tilde{t},\tilde{x})| = 0.$$

The next proposition is similar to [49, Proposition 4.1] (in the context of local parabolic boundary value problems of second order we refer to [56, Proposition 2.7]). We will generalize the results to kernel functions J which satisfy $(J1)_{diff}$, (J_s) and (J_+) for some $s \in (0,1)$.

Proposition 7.6. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, and suppose that the nonlinearity f satisfies (F1). Suppose furthermore that $f(\cdot,\cdot,0)$ is bounded on $(0,\infty)\times\Omega$. Assume that J fulfills $(J1)_{diff}$ and (J_s) for some $s \in (0,1)$. Then for any solution u of (P_T) satisfying (U1) we have:*

(i) For any set $G \subset\subset \Omega$ there exist $a > 0$ such that

$$\sup_{\substack{\tau \geq 1 \\ t,\tilde{t} \in [\tau,\tau+1], t \neq \tilde{t} \\ x,\tilde{x} \in \overline{G}, x \neq \tilde{x}}} \frac{|u(t,x)-u(\tilde{t},\tilde{x})|}{(|x-\tilde{x}|+|t-\tilde{t}|^s)^a} < \infty. \tag{7.17}$$

(ii) If, in addition, Ω fulfills a uniform exterior ball condition[8]*, J satisfies (J_+) and for some $t_0 > 0$, $C_1 > 0$, we have*

$$|u(t_0,x)| \leq C_1 \operatorname{dist}(x,\partial\Omega)^s \qquad \text{for all } x \in \Omega, \tag{7.18}$$

then

$$\sup_{t \geq t_0, x \in \Omega} \frac{|u(t,x)|}{\operatorname{dist}(x,\partial\Omega)^s} < \infty. \tag{7.19}$$

In particular, u is eventually equicontinuous on Ω.

Proof. (i) Let $T := (0,\infty)$. Note that u is a solution of

$$\partial_t u(t,x) + Iu(t,x) = c(t,x)u + f(t,x,0) \qquad \text{in } T \times \Omega$$

with

$$c(t,x) = \begin{cases} \dfrac{f(t,x,u(t,x)) - f(t,x,0)}{u(t,x)}, & u(t,x) \neq 0; \\ 0, & u(t,x) = 0. \end{cases}$$

Moreover, c is bounded on $T \times \Omega$ by (F1), and $(t,x) \mapsto f(t,x,0)$ is bounded on $T \times \Omega$ by assumption. Thus (U1) implies that $(t,x) \mapsto f(t,x,u)$ is a bounded function on $T \times \Omega$. By Corollary 7.5 we conclude that (7.17) holds.
(ii) We use the (barrier) function from Definition 5.18 for an annulus M. Since Ω satisfies the exterior ball condition, there exists for every $x_0 \in \partial\Omega$ a point $y \in \mathbb{R}^N \setminus \Omega$ and $\rho > 0$ such that $\overline{B_\rho(y)} \cap \overline{\Omega} = \{x_0\}$. Consider the function

$$\Psi_{M,J} \in \mathscr{D}^J \quad \text{with} \quad M = B_{\operatorname{diam}(\Omega)+\rho}(y) \setminus B_\rho(y).$$

[8] We say that a set $\Omega \subset \mathbb{R}^N$ satisfies a *uniform* exterior ball condition, if there is $\rho > 0$ such that for all $x \in \partial\Omega$ there is $y \in \mathbb{R}^N \setminus \Omega$ with $\overline{B_\rho(y)} \cap \overline{\Omega} = \{x\}$

Note that we have

$$\begin{cases} I\Psi_{M,J} = 1 \quad \text{in } M; \\ 0 < \Psi_{M,J}(x) \le C(\text{dist}(x, B_\rho(y)))^s \quad \text{for } x \in M; \\ \Psi_{M,J} \equiv 0 \quad \text{in } \mathbb{R}^N \setminus M; \end{cases} \tag{7.20}$$

and we have $\Omega \subset M$.
Here, using (7.20) and the assumptions (F1), (U1) and (7.18), we may choose $\lambda > 0$ sufficiently large so that

$$\begin{cases} \lambda I\Psi_{M,J} \ge \sup_{t \ge t_0, x \in \Omega} f(t,x,u(t,x)) & \text{in } M, \\ \lambda \Psi_{M,J}(x) \ge u(t_0,x) & \text{for } x \in \Omega. \end{cases} \tag{7.21}$$

Let $w(t,x) = \lambda \Psi_{M,J}(x) - u(t,x)$ for $t \ge t_0$, $x \in \mathbb{R}^N$. Then w is a supersolution of $\partial_t w + Iw = 0$ in $[t_0,\infty) \times \Omega$, and $w(t_0)$ is nonnegative on $\mathbb{R}^N$. Hence, by Proposition 6.2 we have $w(t,x) \ge 0$ for $x \in \Omega$, $t \ge t_0$ and therefore

$$u(t,x) \le \lambda \Psi_{M,J}(x) \le \lambda C(\text{dist}(x, B_\rho(y)))^s \qquad \text{for } x \in \Omega,\ t \ge t_0.$$

Since the parameter λ can be chosen uniformly with respect to the ρ-balls touching Ω from outside, we find – using also the boundedness of u on $[t_0,\infty) \times \Omega$ – a constant $C' > 0$ such that

$$u(t,x) \le C' \text{dist}(x, \partial\Omega)^s \qquad \text{for } x \in \Omega,\ t \ge t_0. \tag{7.22}$$

Repeating the same argument with $-u$ in place of u, we find a constant $C'' > 0$ such that

$$u(t,x) \ge -C'' \text{dist}(x, \partial\Omega)^s \qquad \text{for } x \in \Omega,\ t \ge t_0. \tag{7.23}$$

Combining (7.22) and (7.23), we obtain (7.19), as claimed. Now the eventual equicontinuity of u follows easily by combining (7.17) and (7.19). □

Definition 7.7. Let $\Omega \subset \mathbb{R}^N$ be an open set.

1. We set $\mathscr{C}_0(\Omega) := \{u \in C(\mathbb{R}^N) \ : \ u \equiv 0 \ \text{ on } \mathbb{R}^N \setminus \Omega\}$.
2. If $u : (0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ is a function such that, for some $t_0 > 0$ we have $u(t) \in \mathscr{C}_0(\Omega)$ for $t \ge t_0$, we define the ω-limit set $\omega(u)$ of u as the set of all $z \in \mathscr{C}_0(\Omega)$ such that there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset [0,\infty)$ with $t_k \to \infty$ and $\|u(t_k) - z\|_\infty \to 0$ as $k \to \infty$.

Note that under the assumptions of Proposition 7.6 (ii) the set $\{u(t) \ : \ t \ge t_0\} \subset \mathscr{C}_0(\Omega)$ is relatively compact for some $t_0 > 0$, and hence $\omega(u)$ is nonempty.

8 Local and global existence in the space of continuous functions

In the following we analyze solutions to the Cauchy problem stated in Section 6 with continuous initial data. We will also discuss existence results for global solutions.

8.1 Continuous solutions

Using the results from before we want to show that the operator

$$Au := -Iu, \quad \text{for } u \in \text{dom}(A) := \{u \in \text{dom}(I) \cap \mathscr{C}_0(\Omega) \ : \ Iu \in C_0(\Omega)\}$$

is m-dissipative on $\mathscr{C}_0(\Omega)$ [9], where $\Omega \subset \mathbb{R}^N$ is an open bounded set. To do so, we will first consider the operator

$$B_\infty u := -Iu, \quad \text{for } u \in \text{dom}(B_\infty) := \{u \in \text{dom}(I) \cap L^\infty(\Omega) \ : \ Iu \in L^\infty(\Omega)\}.$$

Lemma 8.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set and assume (J1)$_{diff}$. Then the operator B_∞ is m-dissipative on $L^\infty(\Omega)$.*

Proof. Recall the definition of the operator B for Proposition 6.11 and note $\text{dom}(B_\infty) \subset \text{dom}(B) = \text{dom}(I) \subset \mathscr{D}^J(\Omega) \subset L^2(\Omega)$. By Proposition 6.11 we have that B is m-dissipative and thus there is for all $\lambda > 0$ and any $f \in L^\infty(\Omega) \subset L^2(\Omega)$ a function $u \in \text{dom}(I)$ with

$$u - \lambda Bu = f \qquad \text{in } L^2(\Omega).$$

Denote $M := \|f\|_{L^\infty(\Omega)}$, then trivially we have $(u-M) - \lambda B(u-M) = f - M$ in $L^2(\Omega)$. Denote $v := (u-M)^+ \in \mathscr{D}^J(\Omega)$. Testing the equation with v gives

$$\int_\Omega v^2(x)\, dx + \lambda \mathscr{J}(v,v) = \int_\Omega (f-M)(x)v(x)\, dx \le 0,$$

resulting in $v = 0$ and thus $u \le M$. Similarly we obtain $u \ge -M$ and thus $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le M$. Finally we also get

$$-\lambda Bu = f - u \in L^\infty(\Omega),$$

proving that $u \in \text{dom}(B_\infty)$. We conclude that for all $\lambda > 0$ and $u \in \text{dom}(B_\infty)$ we have

$$\|u\|_{L^\infty(\Omega)} \le \|u - \lambda B_\infty u\|_{L^\infty(\Omega)},$$

i.e. B_∞ is dissipative. The fact that B_∞ is m-dissipative now follows from this inequality and the beginning of the proof. □

Proposition 8.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary which satisfies a uniform exterior ball condition. Assume (J1)$_{diff}$, (J_s) and (J_+). Then the operator A is m-dissipative on $\mathscr{C}_0(\Omega)$.*

[9] Recall $\mathscr{C}_0(\Omega) = \{u \in C(\mathbb{R}^N) \ : \ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \Omega)\}$.

Proof. Since $\mathrm{dom}(A) \subset \mathrm{dom}(B_\infty)$ we have that A is dissipative. Moreover, for all $f \in \mathscr{C}_0(\Omega) \subset L^\infty(\Omega)$ and all $\lambda > 0$ there is $u \in \mathrm{dom}(B_\infty)$ such that

$$u - \lambda B_\infty u = f \qquad \text{in } L^\infty(\Omega).$$

Note that by the assumptions we may apply Corollary 5.17 to get $u \in \mathscr{C}_0(\Omega)$. This gives $u \in \mathrm{dom}(A)$ proving the claim. □

8.2 Local existence

With the results from the previous sections we want to apply again the standard theory to prove local solvability of the original problem (P_T) stated in section 6. We will again follow [26] and therefore we will state several results that follow immediately by the previous paragraphs. In this whole section let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary which satisfies a uniform exterior ball condition. Moreover, let $J : \mathbb{R}^N \setminus \{0\} \to [0,\infty)$ be a kernel function which satisfies $(\mathrm{J1})_{diff}$, (J_s) and $(\mathrm{J}_+)_{2\mathrm{diam}(\Omega)}$. Note that with these assumptions we have $S_A(t)\varphi = S_B(t)\varphi$ for any $t > 0$ and $\varphi \in \mathscr{C}_0(\Omega)$ (for $S_B(\cdot)$ see Corollary 6.12).

Lemma 8.3. *(see e.g. [26, Lemma 4.1.1]) Let $T > 0$ be a constant, $g \in C([0,T],\mathscr{C}_0(\Omega))$ and $\varphi \in \mathscr{C}_0(\Omega)$. Then we have that any solution u of*

$$(P_{inhom}) \qquad \begin{cases} u \in C([0,T],\mathscr{C}_0(\Omega)) \cap C((0,T],\mathscr{D}^J(\Omega)) \cap C^1((0,T),\mathscr{C}_0(\Omega)) \\ \partial_t u(t) = Au(t) + g(t) \qquad \text{for } t > 0, \\ u(0) = \varphi, \end{cases}$$

can be represented for any $t \in [0,T]$ as

$$u(t) = S_A(t)\varphi + \int_0^t S_A(t-\tau)g(\tau)\, d\tau.$$

We will now turn to the general semilinear mild problem

$$(P_{mild}) \qquad \begin{cases} u \in C([0,T],\mathscr{C}_0(\Omega)) \\ u(t) = S_A(t)\varphi + \displaystyle\int_0^t S_A(t-\tau)F(\tau,u(\tau))\, d\tau \qquad \text{for } t \in [0,T], \end{cases}$$

for $\varphi \in \mathscr{C}_0(\Omega)$ and $F : [0,T] \times \mathscr{C}_0(\Omega) \to \mathscr{C}_0(\Omega)$.

Proposition 8.4. *(cf. [26, Theorem 4.3.4]) Suppose that $F : [0,\infty) \times \mathscr{C}_0(\Omega) \to \mathscr{C}_0(\Omega)$ is continuous and $F(t,\cdot) : \mathscr{C}_0(\Omega) \to \mathscr{C}_0(\Omega)$ is Lipschitz continuous on bounded subsets of $\mathscr{C}_0(\Omega)$ uniformly with respect to $t \in [0,\infty)$. Then there exists a function $T : \mathscr{C}_0(\Omega) \to (0,\infty]$ with the following property. For every $\varphi \in \mathscr{C}_0(\Omega)$ there exists a unique $u \in C([0,T(\varphi)),\mathscr{C}_0(\Omega))$ such that u is the unique solution of (P_{mild}) in $C([0,T],\mathscr{C}_0(\Omega))$ for all $0 < T < T(\varphi)$. Moreover, either*

$$T(\varphi) = \infty \qquad \text{or} \qquad T(\varphi) < \infty \text{ and } \lim_{t \to T(\varphi)} \|u(t)\|_\infty = \infty. \tag{8.1}$$

For nonlinear operators F not depending on t this proposition is stated in a more general framework in [26, Theorem 4.3.4]. The proof given there however extends, without significant changes, to t-dependent nonlinear operators as considered above.

In the following, we consider the special case of a substitution operator

$$F : [0,\infty) \times \mathscr{C}_0(\Omega) \to \mathscr{C}_0(\Omega) \qquad \text{given by} \quad [F(t,w)](x) = f(t,x,w(x)), \tag{8.2}$$

where $f : [0,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition (F1) (see p. 63) on $[0,\infty) \times \Omega \times \mathbb{R}$ and fulfills

$$f(\cdot,\cdot,0) \equiv 0 \qquad \text{on } [0,\infty) \times \Omega. \tag{8.3}$$

Since $\mathscr{C}_0(\Omega)$ is endowed with the norm $\|\cdot\|_\infty$, it follows immediately from (F1), (8.3) that F is continuous and $F(t,\cdot)$ is Lipschitz continuous in u on bounded subsets of $\mathscr{C}_0(\Omega)$ uniformly with respect to $t \in [0,\infty)$. We also have the following.

Lemma 8.5. *For all $t > 0$, $F(t,\cdot)$ maps $\mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ into $\mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$. Moreover, if $M \subset \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ is bounded with respect to $\|\cdot\|_\infty$, there exists $L = L(M) > 0$ such that*

$$\mathscr{J}(F(t,u),F(t,u)) \le L\mathscr{J}(u,u) \qquad \textit{for all } u \in M,\ t > 0. \tag{8.4}$$

Proof. Let $M \subset \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ be bounded with respect to $\|\cdot\|_\infty$. As a consequence of (F1), there exists $L_0 = L_0(M)$ such that

$$|f(t,x,u(x)) - f(t,x,u(y))| \le L_0|x-y| \qquad \text{for all } t > 0,\ x \in \Omega \text{ and } u \in M.$$

Consequently, for $u \in M$ we have

$$\begin{aligned}\mathscr{J}(F(t,u),F(t,u)) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} [f(t,x,u(x)) - f(t,x,u(y))]^2 J(x-y)\,dx\,dy \\ &\le \frac{L_0^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(x)) - u(y))^2)J(x-y)\,dx\,dy = L_0^2\,\mathscr{J}(u,u).\end{aligned}$$

Since $F(t,u) \in \mathscr{C}_0(\Omega) \subset L^2(\Omega)$ by the remarks preceding the lemma, we conclude that $F(t,\cdot)$ maps $\mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ into $\mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$, as claimed. Moreover, it follows that (8.4) holds with $L = L_0^2$. □

In the following we are interested in solutions of (P_T) with $f : [0,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying (F1) and (8.3). For this, we consider, for $\varphi \in \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$, the problem

$$(P_{semi}) \qquad \begin{cases} u \in C([0,T],\mathscr{C}_0(\Omega))) \cap C((0,T],\mathscr{D}^J(\Omega))) \cap C^1((0,T),L^2(\Omega)) \\ \partial_t u(t) = Au(t) + F(t,u(t)) \qquad \text{for } t \in (0,T), \\ u(0) = \varphi, \end{cases}$$

where F is the substitution operator as given in (8.2). Following closely the lines of [26, Proposition 5.1.1] we have

Lemma 8.6. *Suppose that F is defined by (8.2), where $f : [0,\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$ satisfies (F1) and (8.3). Let $\varphi\in\mathscr{C}_0(\Omega)\cap\mathscr{D}^J(\Omega)$ and $T\in(0,\infty)$. Then a function $u\in C([0,T],\mathscr{C}_0(\Omega))$ solves (P_{semi}) if and only if u solves (P_{mild}).*

Proof. We will follow closely the proof of [26, Proposition 5.1.1] where the semilinear heat equation is considered. Assume u solves (P_{semi}), let $t\in(0,T]$ and let $\varepsilon\in(0,t]$. Set $v(\tau)=u(\varepsilon+\tau)$, for $\tau\in[0,t-\varepsilon]$. Clearly v satisfies

$$\partial_t v(\tau)=Au(\tau)+F(\tau,u(\tau))\quad\text{for }\tau\in[0,t-\varepsilon].$$

Moreover, $v(0)=u(\varepsilon)\in\mathscr{D}^J(\Omega)$. Hence Lemma 8.3 gives

$$\begin{aligned}u(\tau+\varepsilon)=v(\tau)&=S_A(\tau)u(\varepsilon)+\int_0^\tau S_A(\tau-\sigma)F(\sigma,v(\sigma))\,d\sigma\\&=S_A(\tau)u(\varepsilon)+\int_0^\tau S_A(\tau-\sigma)F(\sigma,u(\sigma+\varepsilon))\,d\sigma\quad\tau\in[0,t-\varepsilon].\end{aligned}$$

Since $u\in C([0,T],\mathscr{C}_0(\Omega))$ by assumption we have

$$S_A(\cdot)u(\varepsilon)\to S_A(\cdot)\varphi\quad\text{and}\quad F(\cdot,u(\cdot+\varepsilon))\to F(\cdot,u(\cdot))\quad\text{for }\varepsilon\to0^+\text{ uniformly on }[0,t].$$

Thus

$$\begin{aligned}u(t)&=\lim_{\tau\to t^-}\lim_{\varepsilon\to0^+}S_A(\tau)u(\varepsilon)+\int_0^\tau S_A(\tau-\sigma)F(\sigma,u(\sigma+\varepsilon))\,d\sigma\\&=S_A\varphi+\int_0^t S_A(t-\sigma)F(\sigma,u(\sigma))\,d\sigma.\end{aligned}$$

Next assume $u\in C([0,T],\mathscr{C}_0(\Omega))$ solves (P_{mild}). In the following let $C_1,C_2,\ldots$ stand for constants (depending possibly on u and T). We will first show

$$u\in C((0,T],\mathscr{D}^J(\Omega)).\tag{8.5}$$

For this fix $t\in(0,T]$. Since $\varphi\in\mathscr{C}_0(\Omega)$, we have $S_A(\tau)\varphi\in\mathscr{D}^J(\Omega)$ for any $\tau\in(0,t]$ by (6.10). Moreover, since $F:[0,\infty)\times\mathscr{C}_0(\Omega)\to\mathscr{C}_0(\Omega)$ is Lipschitz continuous in the second variable with $F(t,0)=0$ for all $t\geq0$, the map

$$[0,t)\to\mathscr{D}^J(\Omega),\qquad\tau\mapsto S_A(t-\tau)F(\tau,u(\tau)),$$

is well-defined with

$$\|S_A(t-\tau)F(\tau,u(\tau))\|_{\mathscr{D}^J(\Omega)}\leq C_1\left(1+\frac{1}{\sqrt{2(t-\tau)}}\right)\quad\text{for }\tau\in(0,t].\tag{8.6}$$

Using again (6.10) and (8.6) we conclude that $u(t) \in \mathscr{D}^J(\Omega)$ with

$$\|u(t)\|_{\mathscr{D}^J(\Omega)} \le C_2 \left(1 + \frac{1}{\sqrt{t}}\right) + C_1 \int_0^t \left(1 + \frac{1}{\sqrt{2(t-\tau)}}\right) d\tau \le C_3 \left(1 + \frac{1}{\sqrt{t}}\right).$$

Hence by Lemma 8.5 we also have $F(t, u(t)) \in \mathscr{D}^J(\Omega)$ with

$$\|F(t, u(t))\|_{\mathscr{D}^J(\Omega)} \le C_4 \left(1 + \frac{1}{\sqrt{t}}\right). \tag{8.7}$$

From this we conclude that the map $t \mapsto F(t, u(t))$ is weakly continuous from $(0, T]$ into $\mathscr{D}^J(\Omega)$. Hence we have $u \in C((0, T], \mathscr{D}^J(\Omega))$ as claimed in (8.5). Next we show

$$Iu(t) \in L^2(\Omega) \qquad \text{for } t \in (0, T]. \tag{8.8}$$

Indeed, by (6.11) and (8.7) the map $\tau \mapsto IS_A(t-\tau)F(\tau, u(\tau))$ is weakly continuous on $(0, t)$ to $L^2(\Omega)$ with

$$\begin{aligned} \|IS_A(t-\tau)F(\tau, u(\tau))\|_{L^2(\Omega)} &\le \left(\frac{1}{2(t-\tau)}\right)^{\frac{1}{2}} \sqrt{\mathscr{J}(F(\tau, u(\tau)), F(\tau, u(\tau)))} \\ &\le C_6 \left(\frac{1}{\sqrt{(t-\tau)}}\left(1 + \frac{1}{\sqrt{\tau}}\right)\right) \qquad \text{for every } \tau \in (0, t). \end{aligned} \tag{8.9}$$

Hence

$$\int_0^t \|IS_A(t-\tau)F(\tau, u(\tau))\|_{L^2(\Omega)}\, d\tau \le C_6 \int_0^t \left(\frac{1}{\sqrt{(t-\tau)}}\left(1 + \frac{1}{\sqrt{\tau}}\right)\right) d\tau < \infty$$

Combining this with (6.10) we have

$$Iu(t) = IS_A(t)\varphi + \int_0^t IS_A(t-\tau)F(\tau, u(\tau))\, d\tau \in L^2(\Omega) \qquad \text{for } t \in (0, T].$$

as claimed in (8.8). Finally, we claim

$$u \in C((0, T), \operatorname{dom}(B)), \tag{8.10}$$

where B is as in Proposition 6.11. Recall that the map $(0, T] \to L^2(\Omega)$, $t \mapsto IS_A(t)\varphi$ is continuous. Denote for $t \in (0, T]$

$$\tilde{u}(t) := \int_0^t S_A(t-\tau)F(\tau, u(\tau))\, d\tau.$$

Fix $t \in (0, T]$, $h \in \mathbb{R}$ with $0 < t + h \le T$, then we have with (8.9) and (6.11)

$$\|I(\tilde{u}(t+h) - \tilde{u}(t))\|_{L^2(\Omega)} = \left\| \int_0^{t+h} IS_A(t+h-\tau)F(\tau, u(\tau))\, d\tau \right.$$

$$
\begin{aligned}
&\qquad\qquad - \int_0^t IS_A(t-\tau)F(\tau,u(\tau))\,d\tau\Bigg\|_{L^2(\Omega)} \\
&= \Bigg\| \int_0^h IS_A(h-\tau)F(\tau+h,u(\tau+h))\,d\tau \\
&\qquad\qquad + \int_0^t IS_A(t-\tau)(S_A(h)-\mathrm{id}(L^2(\Omega)))F(\tau,u(\tau))\,d\tau\Bigg\|_{L^2(\Omega)} \\
&\le C_6 \int_0^h \frac{1}{\sqrt{(h-\tau)}}\left(1+\frac{1}{\sqrt{\tau+h}}\right)d\tau \\
&\qquad\qquad + \int_0^t \frac{1}{\sqrt{2(t-\tau)}}\sqrt{\|(S_A(h)-\mathrm{id}(L^2(\Omega)))F(\tau,u(\tau))\|_{\mathscr{D}^J(\Omega)}}\,d\tau.
\end{aligned}
\tag{8.11}
$$

Clearly, the first integral in (8.11) converges to 0 for $h \to 0$. Moreover, since $F(\tau,u(\tau)) \in \mathscr{D}^J(\Omega)$ for all $\tau \in (0,t)$ and since $(S_A(h)-\mathrm{id}(L^2(\Omega)))F(\tau,u(\tau)) \to 0$ for $h \to 0$ pointwise for every $\tau \in (0,t)$ we conclude

$$\lim_{h\to 0} \|I(\tilde{u}(t+h)-\tilde{u}(t))\|_{L^2(\Omega)} = 0,$$

using Lebesgue's theorem for the second integral in (8.11) and the fact that the integrand in the second integral is dominated by (8.9). This shows (8.10). Combining (8.5) with (8.10) we conclude

$$u \in C([0,T],\mathscr{C}_0(\Omega))) \cap C((0,T],\mathscr{D}^J(\Omega))) \cap C((0,T),\mathrm{dom}(B)).$$

Hence for any $\varepsilon > 0$ and $v(\tau) := u(\varepsilon+\tau)$ for $\tau \in [0,t-\varepsilon]$ we have that v satisfies

$$v(\tau) = S_B(\tau)u(\varepsilon) + \int_0^\tau S_B(\tau-\sigma)F(\sigma,v(\sigma))\,d\sigma \qquad \text{for } \tau \in [0,t-\varepsilon].$$

By an application of [26, Corollary 4.1.8] it follows that v solves (P_{semi}) on $[0,t-\varepsilon]$. Considering the limit $\varepsilon \to 0$, we conclude that u solves (P_{semi}), as claimed. □

8.3 Global existence

As developed in [26, Section 5.3] we will discuss under which circumstances we will have a global solution of problem (P_{mild}), which is bounded in $L^\infty((0,\infty)\times\Omega)$. In the following let J satisfy for some $s \in (0,1)$, $(J1)_{diff}$, (J_s) and (J_+). Let f be a function satisfying (F1) and (8.3) and let F be the substitution operator as described in (8.2). Moreover, let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary, which satisfies a uniform exterior ball condition. Note that then the conclusions of Proposition 8.4 hold. By Lemma 8.6 and Proposition 8.4 there is for any $\varphi \in \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ a constant $T(\varphi) > 0$ such that for any $T < T(\varphi)$ there is a unique solution $u \in C([0,T], \mathscr{C}_0(\Omega))) \cap C((0,T], \mathscr{D}^J(\Omega)) \cap C^1((0,T), L^2(\Omega))$ of

$$\begin{cases} \partial_t u(t) = Au(t) + F(t,u(t)) & \text{for } t \in [0,T], \\ u(0) = \varphi. \end{cases} \tag{8.12}$$

Moreover, this u also satisfies $u(t) = S_A(t)\varphi + \int_0^t S_A(t-\tau)F(\tau,u(\tau))\, d\tau$ for $t \in [0,T]$. The following Proposition gives a sufficient condition such that $T(\varphi) = \infty$:

Proposition 8.7. *Assume, there is $C > 0$ such that f satisfies*

$$|f(t,x,u)| \le C|u|, \qquad \text{for all } t > 0,\ x \in \Omega \text{ and } u \in \mathbb{R}. \tag{8.13}$$

Then for all $\varphi \in \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ we have $T(\varphi) = \infty$. Moreover, if $C \le \Lambda_{1,J}(\Omega)$ then

$$\|u(t)\|_{L^\infty(\Omega)} \le \|\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } t \ge 0.$$

Proof. Let $\varphi \in \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ be given and let u be the unique solution of (8.12) given by Proposition 8.4. Then by (8.13) we have that u satisfies, in weak sense,

$$C\operatorname{sign}(u)u \ge \partial_t u + Iu \ge -C\operatorname{sign}(u)u \qquad \text{in } (0,T(\varphi)) \times \Omega. \tag{8.14}$$

Since $\|u(0)\|_{L^\infty(\Omega)} = \|\varphi\|_{L^\infty(\Omega)} < \infty$ we have by Proposition 6.2 (see also Corollary 6.3)

$$\|u(t)\|_{L^\infty(\Omega)} \le e^{(C-\Lambda_{1,J}(\Omega))t}\|\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (0,T(\varphi)).$$

This implies $T(\varphi) = \infty$. The second assertion follows immediately. □

Proposition 8.8. *Let $a,b : [0,\infty) \times \Omega \to \mathbb{R}$ be continuous bounded functions such that $a \ge b$ in $[0,\infty) \times \Omega$ and fix $p \ge 2$. Let $f : [0,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$, $f(t,x,u) = a(t,x)u - b(t,x)u^p$. Then for every $\varphi \in \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ with $0 \le \varphi \le 1$ there is a unique global solution u of (8.12). Moreover, we have*

$$0 \le u \le 1 \qquad \text{in } [0,\infty) \times \Omega. \tag{8.15}$$

Proof. Let $\varphi \in \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ be given. Note that f satisfies (F1) and (8.3) and thus there is by Proposition 8.4 a unique solution u of (8.12) in $(0,T(\varphi)) \times \Omega$. Note that $v \equiv 0 \in \mathscr{V}^J(\Omega)$ satisfies $\partial_t v + Iv = f(t,x,0) = 0$ in $[0,\infty) \times \Omega$. Since $\varphi \ge 0$, we have thus by Proposition 6.2 $u(t) \ge 0$ in $[0,T(\varphi)) \times \Omega$. Moreover, we have that $w \equiv 1 \in \mathscr{V}^J(\Omega)$ is a supersolution of $\partial_t w + Iw = f(t,x,1) = 0$ in $[0,\infty) \times \Omega$ since $a \ge b$ in $(0,\infty) \times \Omega$. Since $1 - \varphi \ge 0$, we have $w(t) - u(t) \ge 0$ in Ω for $t \in [0,T(\varphi))$ by Proposition 6.2. We conclude $T(\varphi) = \infty$ and (8.15). □

Remark 8.9. Let $\varphi \in \mathscr{C}_0(\Omega) \cap \mathscr{D}^J(\Omega)$ with $\sup\limits_{x\in\Omega} \frac{|\varphi(x)|}{\operatorname{dist}(x,\mathbb{R}^N\setminus\Omega)^s} < \infty$.

1. If, in addition, f satisfies (8.13) in Proposition 8.7 with $C = \Lambda_{1,J}(\Omega)$, then the unique global solution u is eventually equicontinuous on Ω by Proposition 7.6.
2. If, in addition, $0 \le \varphi \le 1$, then the unique global solution u given by Proposition 8.8 is eventually equicontinuous on Ω by Proposition 7.6.

In particular, in both cases $\omega(u)$ is nonempty.

9 Antisymmetric supersolutions for time dependent equations

In this Section we will give the main tools for the proof of asymptotic symmetry for problem (P_T) with $T = [0,\infty)$. Since we will use the moving plane method, we will deal with antisymmetric functions and antisymmetric sub- and supersolutions of corresponding linear problems as in the time independent case. In the following, we will use the notation of Subsection 4.1 for a fixed open half space $H \subset \mathbb{R}^N$ concerning the moving plane method, in particular we will use that the kernel function J satisfies

$$J(\bar{x}-\bar{y}) = J(x-y) \qquad \text{for a.e. } x,y \in \mathbb{R}^N, x \neq y; \tag{9.1}$$

$$J(x-y) \geq J(x-\bar{y}) \qquad \text{for a.e. } x,y \in H, x \neq y. \tag{9.2}$$

Moreover, we will put for $b \geq 0$

$$H_b := \{x \in H \ : \ \text{dist}(x,\partial H) > b\} \tag{9.3}$$

and we will also need for some statements the following strict variant of (9.2) (cf. (4.6), (4.19):

$$\text{There exists } r_0 > 0 \text{ such that } \operatorname*{essinf}_{\substack{x,y\in H_b \\ |x-y|\leq \min\{b,r_0\}}} (J(x-y) - J(x-\bar{y})) > 0 \text{ for all } b > 0. \tag{9.4}$$

We have the following additional definition for the time dependent case:

Definition 9.1. Let $U \subset H$ be an open bounded set, where $H \subset \mathbb{R}^N$ is an open half space, let $T \subset \mathbb{R}$ be a time interval and let $c \in L^\infty(T \times U)$. We denote by $Q : \mathbb{R}^N \to \mathbb{R}^N$, $x \mapsto Q(x) = \bar{x}$ the reflection at ∂H as usual. Furthermore, let $v : T \times \mathbb{R}^N \to \mathbb{R}^N$ be antisymmetric w.r.t. H, i.e. we have $v(t,\bar{x}) = -v(t,x)$ for all $x \in \mathbb{R}^N$, $t \in T$.

1. We call v an *antisymmetric supersolution of the equation*

$$\partial_t v + Iv = c(t,x)v \qquad \text{in } T \times U \tag{9.5}$$

   if

   - $v \in C(T, \mathcal{V}^J(U')) \cap C^1(T, L^2(U))$ for some open set $U' \subset \mathbb{R}^N$ with $Q(U') = U'$ and $\overline{U} \subset U'$,
   - for all $t \in T$ and $\varphi \in \mathscr{D}^J(U)$, $\varphi \geq 0$ with compact support in $\mathbb{R}^N$ we have

$$\mathscr{J}(v(t),\varphi) \geq \int_U (c(t,x)v(t,x) - \partial_t v(t,x))\, \varphi(x)\, dx. \tag{9.6}$$

2. We call v an *antisymmetric supersolution of the problem*

$$\partial_t v + Iv = c(t,x)v \quad \text{in } T \times U, \qquad v \equiv 0 \quad \text{on } T \times (H \setminus U) \tag{9.7}$$

   if v is an antisymmetric supersolution of (9.5) and if $v \geq 0$ on $T \times (H \setminus U)$.

3. We call v an *antisymmetric subsolution* of (9.5) resp. (9.7) if $-v$ is an antisymmetric supersolution of (9.5), (9.7) resp.

Remark 9.2. If v is an antisymmetric supersolution of (9.7) we occasionally say that v satisfies (in the weak sense)

$$(P_a) \qquad \begin{cases} \partial_t v + Iv \geq c(t,x)v & \text{in } T \times U, \\ \quad v \geq 0 & \text{on } T \times (H \setminus U), \\ v \circ Q = -v \text{ in } T \times \mathbb{R}^N. \end{cases}$$

The following result connects problems (P_T) and (P_a):

Lemma 9.3. *Let H be an open half space and let $\Omega \subset \mathbb{R}^N$ be an open set such that $Q(\Omega) \subset \Omega$. Let J satisfy $(J1)_a$, $(J1)_b$, (9.1) and (9.2). Furthermore, assume the following conditions on the nonlinearity f:*

(F1) *$f : (0,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous. Moreover, for every $K > 0$ there exists $L = L(K) > 0$ such that*

$$\sup_{\substack{x\in\Omega,\\ t\geq 0}} |f(t,x,u) - f(t,x,v)| \leq L|u-v| \qquad \textit{for } u,v \in [-K,K].$$

$(F2)_H$ *f is symmetric and monotone w.r.t. H, i.e. for every $t \in [0,\infty)$, $u \in \mathbb{R}$, $x \in H \cap \Omega$ we have $f(t,\bar{x},u) \geq f(t,x,u)$.*

If u is a bounded nonnegative solution of (P_T) on $T \times \Omega$, then $v := u \circ Q - u$ satisfies

$$(P') \qquad \begin{cases} \partial_t v + Iv \geq c(t,x)v & \textit{in } T \times (H \cup \Omega), \\ \quad u \geq 0 & \textit{on } T \times (H \setminus \Omega), \\ v \circ Q = -v \textit{ for all } (t,x) \in T \times \mathbb{R}^N, \end{cases}$$

where

$$c(t,x) = \begin{cases} \dfrac{f(t,x,u(t,\bar{x})) - f(t,x,u(t,x))}{v(t,x)}, & v(t,x) \neq 0; \\ 0, & v(t,x) = 0 \end{cases}$$

is bounded on $T \times (H \cap \Omega)$.

Proof. We first note that since $u(t) \in \mathscr{D}^J(\Omega) \cap L^\infty(\mathbb{R}^N)$ we have that $v(t) \in \mathscr{D}^J(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \subset \mathscr{V}^J(\mathbb{R}^N)$ for all $t \in T$. The boundary conditions then follow from the boundary conditions of u. Fix $\varphi \in \mathscr{D}^J(\Omega \cap H)$, then

$$\begin{aligned} \mathscr{J}(v(t),\varphi) &= \mathscr{J}(u(t)\circ Q - u(t),\varphi) = \mathscr{J}(u(t),\varphi\circ Q - \varphi) \\ &= \int_\Omega (f(t,x,u(x)) - \partial_t u(t,x))[\varphi(Q(x)) - \varphi(x)]\,dx \\ &= \int_{Q(\Omega\cap H)} (f(t,x,u(t,x)) - \partial_t u(t,x))\varphi(Q(x))\,dx - \int_{\Omega\cap H} (f(t,x,u(t,x)) - \partial_t u(t,x))\varphi(x)\,dx \end{aligned}$$

$$= \int_{\Omega\cap H} [f(t,Q(x),u(t,Q(x))) - f(t,x,u(t,x)) - \partial_t(u(t,Q(x)) - u(t,x))]\varphi(x)\,dx$$
$$\geq \int_U [c(t,x)v(t,x) - \partial_t v(t,x)]\varphi(x)\,dx,$$

where (F2)$_H$ was used in the last step. The boundedness of c then follows from (F1). □

Next, for a kernel function J satisfying (J1)$_a$ and (J1)$_b$ and a measurable subset $A \subset \mathbb{R}^N$ we set

$$\kappa_{J,A}(x) := \int_{\mathbb{R}^N \setminus A} J(x,y)\,dy \in [0,\infty]. \tag{9.8}$$

The following two Lemmas will be helpful when dealing with antisymmetric solutions (see also [49, Lemma 2.3 and 2.10]).

Lemma 9.4. *Let $H \subset \mathbb{R}^N$ be an open half space and let J satisfy (J1)$_{diff}$, (9.1) and (9.2). Then for any $\varphi \in \mathscr{D}^J(H)$ and every antisymmetric $v \in \mathscr{D}^J(\mathbb{R}^N)$ we have*

$$\mathscr{J}(v,\varphi) = \frac{1}{2}\int_H\int_H (v(x)-v(y))(\varphi(x)-\varphi(y))\bar{J}(x,y)dx\,dy + 2\int_H \kappa_{J,H}(x)v(x)\varphi(x)\,dx \tag{9.9}$$

with

$$\bar{J}(x,y) := J(x-y) - J(x-\bar{y})$$

for $x,y \in H$, where $\bar{y}$ is the reflection of y at ∂H.
If, in addition, (9.4) holds for some $r_0 > 0$, then J satisfies $(J_+)_{r_0}$ and there is a constant $d \in (0,1)$ depending only on J and H such that

$$\bar{J}(x,y) \geq dJ(x-y) \text{ for a.e. } x,y \in H \text{ with } 0 < |x-y| \leq \min\{\operatorname{dist}(x,\partial H), \operatorname{dist}(y,\partial H), r_0\}. \tag{9.10}$$

Proof. For $\varphi \in \mathscr{D}^J(H)$ and an antisymmetric $v \in \mathscr{D}^J(\mathbb{R}^N)$ we have

$$\mathscr{J}(v,\varphi) = \frac{1}{2}\Big(\int_H\int_H (v(x)-v(y))(\varphi(x)-\varphi(y))J(x-y)\,dx\,dy + \int_H\int_{\mathbb{R}^N\setminus H} \dots dx\,dy + \int_H\int_{\mathbb{R}^N\setminus H} \dots dx\,dy\Big)$$
$$= \frac{1}{2}\int_H\int_H \Big[(v(x)-v(y))(\varphi(x)-\varphi(y))J(x-y) - (v(\bar{x})-v(y))\varphi(y)J(x-\bar{y}) + (v(x)-v(\bar{y}))\varphi(x)J(x-\bar{y})\Big]\,dx\,dy$$
$$= \frac{1}{2}\int_H\int_H (v(x)-v(y))(\varphi(x)-\varphi(y))J(x-y)\,dx\,dy + \int_H\int_H (v(x)+v(y))\varphi(y)J(x-\bar{y})\,dx\,dy$$

$$= \frac{1}{2}\int_H\int_H (v(x)-v(y))(\varphi(x)-\varphi(y))\bar{J}(x,y)\;dx\,dy + 2\int_H\int_H v(y)\varphi(y)J(x-\bar{y})\;dx\,dy$$
$$= \frac{1}{2}\int_H\int_H (v(x)-v(y))(\varphi(x)-\varphi(y))\bar{J}(x,y)dx\,dy + 2\int_H \kappa_{J,H}(x)v(x)\varphi(x)\;dx$$

with $\bar{J}$ and $\kappa_{J,H}$ as defined above as claimed.
Next assume J satisfies additionally (9.4) for some $r_0 > 0$. Then, choosing $x \in H$ with $\operatorname{dist}(x,\partial H) > 2r_0$ and $b = r_0$, we find that $B_r(0) \subset -x + H_b$ and thus

$$0 < \operatorname*{essinf}_{y\in B_r(x)\cap H_b} J(x-y) = \operatorname*{essinf}_{z\in B_r(0)\cap(-x+H_b)} J(z) = \operatorname*{essinf}_{z\in B_r(0)} J(z).$$

Thus $(J_+)_{r_0}$ holds. To see (9.10), fix

$$f : H\times H \to [0,1], \qquad f(x,y) := \begin{cases} \left(1 - \dfrac{J(x-\bar{y})}{J(x-y)}\right) & \text{if } 0 < J(x-y) < \infty; \\ 1 & \text{if } J(x-y) = \infty \text{ or } J(x-y) = 0. \end{cases}$$

Note that f is well-defined by (9.2) and (9.4). Moreover, by (9.1) we have $f(x,y) = f(y,x)$ and

$$\bar{J}(x,y) = J(x-y) - J(x-\bar{y}) = f(x,y)J(x-y) \qquad \text{for all } x,y \in H.$$

Let $m \in (0,r_0]$ and denote $M := \{(x,y) \in H\times H \;:\; 0 < |x-y| \le m \le \operatorname{dist}(x,\partial H), \operatorname{dist}(y,\partial H)\}$, then

$$\operatorname*{essinf}_{M} f > 0$$

by (9.4). Hence (9.10) holds. □

Lemma 9.5. *Let $r_0 \ge b > 0$ be given, and put H_b as in (9.3), where $H \subset \mathbb{R}^N$ is an open half space. Furthermore, let J satisfy $(J1)_{diff}$, (9.1), (9.2) and (9.4) w.r.t. r_0. Then there exists a measurable map $J' : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,x) \;:\; x \in \mathbb{R}^N\} \to [0,\infty)$ which fulfills (J1) such that for any antisymmetric $v \in \mathscr{D}^J(\mathbb{R}^N)$ and any $\varphi \in \mathscr{D}^J(H_b)$ with compact support we have*

$$\mathscr{J}(v,\varphi) = \mathscr{J}'(\tilde{v},\varphi) + 2\int_{H_b} \kappa_{J,H}(x)\tilde{v}(x)\varphi(x)\;dx \tag{9.11}$$

with $\kappa_{J,H}(x)$ as given in (9.8), $\tilde{v} = v1_H \in \mathscr{D}^J(H)$ and where $\mathscr{J}'$ is the bilinear form induced by J'. Moreover, there exists a constant $d > 0$ depending on J, b and H such that J' fulfills

$$d\,J(x-y) \le J'(x,y) \le J(x-y) \qquad \text{for a.e. } x,y \in \mathbb{R}^N \text{ with } 0 < |x-y| \le b/2, \tag{9.12}$$

so that J' satisfies $(J_+)_{b/2}$.

Proof. We may assume without loss of generality that $H = \{x \in \mathbb{R}^N \;:\; x_1 > 0\}$. For simplicity, we write $\bar{x} = Q(x) = (-x_1, x_2, \dots, x_N)$ for $x \in \mathbb{R}^N$. We consider $\bar{J}(x,y)$ as defined in Lemma 9.4. Denote $\triangle := \{(x,x) \;:\; x \in \mathbb{R}^N\}$. Obviously we have

$$0 \le \bar{J}(x,y) \le J(x-y) \qquad \text{for } x,y \in \mathbb{R}^N,\; x \ne y. \tag{9.13}$$

whereas, by Lemma 9.4 there is $d > 0$ such that

$$J(x,y) \geq dJ(x-y) \quad \text{for a.e. } x,y \in H \text{ with } 0 < |x-y| \leq \frac{b}{2} \text{ and } \min\{x_1,y_1\} \geq \frac{b}{2}. \tag{9.14}$$

To define J' with the asserted properties, we set

$$g : \mathbb{R}^N \times \mathbb{R}^N \setminus \triangle \to \mathbb{R}, \qquad g(x,y) := \begin{cases} \bar{J}(x,y) & (x,y) \in H \times H \setminus \triangle, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\tau : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, \qquad \tau(x,y) := \begin{cases} \min\{b-x_1, b-y_1\}, & \text{if } \min\{b-x_1, b-y_1\} \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we set $J'(x,y) := g(x+\tau(x,y)e_1, y+\tau(x,y)e_1)$ for $x,y \in \mathbb{R}^N$, $x \neq y$. Then J' satisfies $(J1)_a$ and $(J1)_b$ which follows directly by construction and (9.13). To see the lower bound, we note that if $0 < |x-y| \leq \frac{b}{2}$ then also $|\tilde{x}-\tilde{y}| \leq \frac{b}{2}$, where $\tilde{x} = x+\tau(x,y)e_1$ and $\tilde{y} = y+\tau(x,y)e_1$. Furthermore, we have that $\max\{x_1,y_1\} \geq b$ and therefore $\min\{x_1,y_1\} \geq \frac{b}{2}$. Consequently,

$$J'(x,y) = g(\tilde{x},\tilde{y}) \geq dJ(\tilde{x}-\tilde{y}) = dJ(x-y)$$

by (9.14). It remains to show (9.11): So let $v \in \mathscr{D}^J(\mathbb{R}^N)$ be antisymmetric, and let $\varphi \in \mathscr{D}^J(H_b)$ with compact support. In the following we denote $\tilde{v} = 1_H v$. Then Lemma 9.4 gives

$$\mathscr{J}(v,\varphi) = \frac{1}{2}\int_H\int_H (\tilde{v}(x)-\tilde{v}(y))(\varphi(x)-\varphi(y))\bar{J}(x,y)\,dx\,dy + 2\int_{H_b} \kappa_{J,H}(x)\tilde{v}(x)\varphi(x)\,dx, \tag{9.15}$$

whereas, since $\varphi \equiv 0$ on $\mathbb{R}^N \setminus H_b$,

$$\int_H\int_H (\tilde{v}(x)-\tilde{v}(y))(\varphi(x)-\varphi(y))\bar{J}(x,y)\,dx\,dy = \int_H\int_{H_b} \ldots dx\,dy + \int_{H_b}\int_{H\setminus H_b} \ldots dx\,dy. \tag{9.16}$$

If $x \in H_b$, then for $y \in H$ we have $\tau(x,y) = 0$ and thus $\bar{J}(x,y) = g(x,y) = J'(x,y)$, while for $y \in \mathbb{R}^N \setminus H$ we have that $J'(x,y) = 0$. Hence we can rewrite the first integral of the RHS of (9) as

$$\int_H\int_{H_b} (\tilde{v}(x)-\tilde{v}(y))(\varphi(x)-\varphi(y))\bar{J}(x,y)\,dx\,dy$$
$$= \int_{\mathbb{R}^N}\int_{H_b} (\tilde{v}(x)-\tilde{v}(y))(\varphi(x)-\varphi(y))J'(x,y)\,dx\,dy$$

Similarly, if $y \in H_b$, then for $x \in H \setminus H_b$ we have $\tau(x,y) = 0$ and thus $\bar{J}(x,y) = g(x,y) = J'(x,y)$, while for $x \in \mathbb{R}^N \setminus H$ we have $J'(x,y) = 0$. Hence we may rewrite the second integral of the RHS of (9) as

$$\int_{H_b}\int_{H\setminus H_b} (\tilde{v}(x)-\tilde{v}(y))(\varphi(x)-\varphi(y))\bar{J}(x,y)\,dx\,dy$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N \setminus H_b} (\tilde{v}(x) - \tilde{v}(y))(\varphi(x) - \varphi(y))J'(x,y)\, dx\, dy,$$

where the last equality follows again since $\varphi = 0$ on $\mathbb{R}^N \setminus H_b$. Combining these identities, we get

$$\int_H \int_H (\tilde{v}(x) - \tilde{v}(y))(\varphi(x) - \varphi(y))\bar{J}(x,y)\, dxdy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\tilde{v}(x) - \tilde{v}(y))(\varphi(x) - \varphi(y))J'(x,y)\, dxdy,$$

and together with (9.15) it follows that

$$\mathscr{J}(v,\varphi) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\tilde{v}(x) - \tilde{v}(y))(\varphi(x) - \varphi(y))J'(x,y)\, dy + 2 \int_{H_b} \kappa_{J,H}(x)\tilde{v}(x)\varphi(x)\, dx,$$

as claimed. □

Remark 9.6. We note that in the situation of Lemma 9.5 we have $\mathscr{D}^J(U) = \mathscr{D}^{J'}(U)$ for any open bounded set $U \subset H_b$. To see this, denote $\bar{x}$ as the reflection at ∂H and note that for any $u \in L^2(\mathbb{R}^N)$ with $\operatorname{supp} u \subset U$ we have

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (u(x) - u(y))(u(\bar{x}) - u(\bar{y})J(x-y)\, dxdy \right| \le 2 \int_U \int_U |u(x)u(y)| J(x - \bar{y})\, dxdy$$
$$\le \int_U u^2(x)\kappa_{J,H}(x)\, dx < \infty$$

since U is bounded away from ∂H and thus $\sup_{x\in U} \kappa_{J,H}(x) < \infty$. Next, consider $u \in \mathscr{D}^J(U) \subset L^2(U)$ and put $\tilde{u}(x) = u(x) - u(\bar{x})$ for $x \in \mathbb{R}^N$. Then $\tilde{u} \in \mathscr{D}^J(\mathbb{R}^N)$ is antisymmetric and Lemma 9.5 implies

$$\mathscr{J}(u,u) - \mathscr{J}(u, u(\bar{\cdot})) = \mathscr{J}(u,\tilde{u}) = \mathscr{J}'(u,u) + 2 \int_U \kappa_{J,H}(x)u^2(x)\, dx.$$

Hence $u \in \mathscr{D}^{J'}(U)$. If, conversely, $u \in \mathscr{D}^{J'}(U)$, we may define $\tilde{u}$ in the same way and we also have $\|u\|_{L^2(U)} < \infty$ since J' satisfies (J1). Hence by Lemma 9.5

$$\mathscr{J}(u,u) \le |\mathscr{J}(u, u(\bar{\cdot}))| + \mathscr{J}'(u,u) + 2 \int_U \kappa_{J,H}(x)u^2(x)\, dx < \infty,$$

and thus $u \in \mathscr{D}^J(U)$.

Lemma 9.7. *Let $r_0 \ge b > 0$ be given, and put $H_b := \{x \in H : \operatorname{dist}(x, \partial H) > b\}$, where $H \subset \mathbb{R}^N$ is an open half space. Furthermore, let J satisfy $(J1)_{diff}$, (9.1), (9.2) and (9.4) w.r.t. r_0. Let J' be the kernel function given by Lemma 9.5 satisfying (J1) and let $\mathscr{J}'$ be the bilinear form induced by J'. Then for every $v \in \mathscr{C}^2_c(H_{2b})$ there is a constant $C = C(N,J,H,b) > 0$ such that*

$$\mathscr{J}'(v,\varphi) \le \|v\|_{C^2(\mathbb{R}^N)} C \int_{H_{2b}} |\varphi(x)|\, dx \qquad \text{for every } \varphi \in \mathscr{D}^{J'}(H_{2b}) \text{ with compact support in } \mathbb{R}^N.$$

In particular, $\mathscr{C}_c^2(H_{2b}) \subset \mathscr{D}_\infty^{J'}(H_{2b})$.

Proof. Let $v \in \mathscr{C}_c^2(H_{2b})$. Since $\mathscr{C}_c^2(H_{2b}) \subset \mathscr{D}_\infty^J(\mathbb{R}^N) \subset \mathscr{D}^J(\mathbb{R}^N)$ and since v has compact support we have that $v \in \mathscr{D}^{J'}(H_{2b})$ by Remark 9.6. Next let $\varphi \in \mathscr{D}^{J'}(H_{2b})$ with compact support and denote $\tilde{\varphi}(x) = \varphi(x) - \varphi(\bar{x})$, $x \in \mathbb{R}^N$. Using Remark 9.6 and Remark 3.3 we have for some $K = K(N,J) > 0$

$$\mathscr{J}(v,\tilde{\varphi}) \leq K\|v\|_{C^2(\mathbb{R}^N)} \int_{H_{2b}} |\varphi(x)|\, dx.$$

Hence

$$\mathscr{J}'(v,\varphi) \leq \left(K\|v\|_{C^2(\mathbb{R}^N)} + 2\|v\|_{L^\infty(\mathbb{R}^N)} \sup_{x\in H_b} \kappa_{J,H}(x)\right) \int_{H_{2b}} |\varphi(x)|\, dx.$$

Thus the claim follows with $C = K + 2\sup_{x\in H_b} \kappa_{J,H}(x) < \infty$. □

9.1 Time dependent maximum principles for antisymmetric supersolutions

Proposition 9.8. *Let H be a half space and let $U \subset H$ be an open bounded set. Assume J satisfies (J1)$_{diff}$, (4.4) and (4.5). If v satisfies (P_a) in $T \times U$ with $T = [t_0, T_0)$ and $c \in L^\infty(T \times U)$, then*

$$\|v^-(t)\|_{L^2(H)} \leq \exp\left[(\|c^+\|_{L^\infty(T\times U)} - \Lambda_{1,J}(U))(t-t_0)\right] \|v^-(t_0)\|_{L^2(H)} \qquad \textit{for all } t \in T. \quad (9.17)$$

Proof. Without restriction let $t_0 = 0$ and put $c_\infty := \|c^+\|_{L^\infty(T\times U)}$. Then $t \mapsto u(t) = e^{(\Lambda_{1,J}(U)-c_\infty)t}v(t)$ is an antisymmetric supersolution of

$$\partial_t u + Iu = (\tilde{c}(t,x) + \Lambda_{1,J}(U))u \quad \text{in } T\times U, \qquad u \equiv 0 \quad \text{on } T\times(H\setminus U), \qquad (9.18)$$

where $\tilde{c}(t,x) := c(t,x) - c_\infty \leq 0$ for $t,x \in T\times U$. For $t \in T$ put $\varphi(t) = u^-(t)$, then $\varphi(t) \in \mathscr{D}^J(U)$ by Lemma 4.7 and $\varphi(t)u(t) = -\varphi^2(t)$ for $t \in T$. Thus testing (9.18) with $\varphi(t)$ gives

$$\begin{aligned}\mathscr{J}(u(t),\varphi(t)) &\geq \int_U \tilde{c}(t,x)u(t,x)\varphi(t,x) + \Lambda_{1,J}(U)u(t,x)\varphi(t,x) - \partial_t u(t,x)\varphi(t,x)\, dx \\ &\geq -\Lambda_{1,J}(U)\|\varphi(t)\|_{L^2(U)}^2 + \frac{1}{2}\frac{d}{dt}\|\varphi(t)\|_{L^2(U)}^2 \qquad \text{for } t \in T.\end{aligned}$$

Since by Lemma 4.7 we have

$$-\mathscr{J}(\varphi(t),\varphi(t)) \geq \mathscr{J}(u(t),\varphi(t)) \qquad \text{for } t \in T$$

we get

$$\frac{d}{dt}\|\varphi(t)\|_{L^2(U)}^2 \leq \Lambda_{1,J}(U)\|\varphi(t)\|_{L^2(U)}^2 - \mathscr{J}(\varphi(t),\varphi(t)) \leq 0.$$

This gives $\|\varphi(t)\|_{L^2(U)} \leq \|\varphi(0)\|_{L^2(U)} = \|v^-(0)\|_{L^2(U)}$ finishing the proof. □

Corollary 9.9. *Let w satisfy (P_a) on $T \times U$, where $U \subset H$ is an open bounded set and $T := [t_0, T_0)$ for some $0 \le t_0 < T_0 \le \infty$ and assume J satisfies (J1)$_{diff}$, (9.1) and (9.2). If $w(t_0,x) \ge 0$ for a.e. $x \in H$ and $c \in L^\infty(U)$, then also $w(t,x) \ge 0$ for all $t \in (t_0, T_0)$ and a.e. $x \in H$.*

Proposition 9.10. *Let H be a half space and assume J satisfies (J1)$_{diff}$, (9.1) and (9.2). Then for every $c_\infty > 0$, $k \in \mathbb{R}$ there exists $\delta = \delta(N, J, c_\infty, k)$ such that for any open bounded subset $U \subset H$ with $|U| \le \delta$, any time interval $T := [t_0, T_0)$, any $c \in L^\infty(T \times \Omega)$ with $\|c^+\|_{L^\infty(T\times U)} \le c_\infty$, and any antisymmetric supersolution v of*

$$\partial_t v + Iv = c(t,x)v \quad \text{in } T \times U, \qquad v \equiv 0 \quad \text{on } T \times (H \setminus U)$$

with $v^-(t_0) \in L^\infty(H)$ we have

$$\|v^-(t)\|_{L^\infty(H)} \le e^{-k(t-t_0)} \|v^-(t_0)\|_{L^\infty(H)} \qquad \text{for all } t \in T. \tag{9.19}$$

Proof. Without loss of generality, we may assume that $t_0 = 0$. By Lemma 2.17, we may fix $\delta > 0$ such that for any measurable set $U \subset \mathbb{R}^N$ with $|U| \le \delta$ and any $x \in \mathbb{R}^N$ we have

$$\kappa_{J,U}(x) = \int_{\mathbb{R}^N \setminus U} J(x-y)\, dy \ge k + c_\infty. \tag{9.20}$$

Next let $U \subset H$ with $|U| < \delta$ and let v be given as stated. Let $d := \|v^-(0)\|_{L^\infty(U)}$, and define $u(t,x) := e^{kt} v(t,x)$ for $t \in T$, $x \in \mathbb{R}^N$. Then u is an antisymmetric supersolution of

$$\partial_t u + Iu = \tilde{c}(t,x)u \quad \text{on } T \times U, \qquad u \equiv 0 \quad \text{on } T \times (H \setminus U)$$

with $\tilde{c}(t,x) = c(t,x) + k$. To finish the proof, we need to show that

$$u(t,x) \ge -d \qquad \text{for a.e. } x \in H \text{ and } t \in T. \tag{9.21}$$

For $t \in T$ consider $\varphi(t) = (u(t) + d)^- 1_H \in \mathscr{D}^J(U)$ (using Lemma 4.7). We then have

$$\begin{aligned} \mathscr{J}(u(t), \varphi(t)) &\ge \int_U (\tilde{c}(t,x)u(t,x) - \partial_t u(t,x))\varphi(t,x)\, dx \\ &\ge \int_U (c_\infty + k)u(t,x)\varphi(t,x)\, dx + \frac{1}{2}\frac{d}{dt}\int_U \varphi(t,x)^2\, dx. \end{aligned} \tag{9.22}$$

Note that by Lemma 4.7 we have $u^- 1_H \in \mathscr{D}^J(U)$. Since for $(t,x) \in T \times \mathbb{R}^N$ we have

$$\begin{aligned} &(u(t,x) - u(t,y))(\varphi(t,x) - \varphi(t,y)) + (u^-(t,x)1_H(x) - u^-(t,y)1_H(y))(\varphi(t,x) - \varphi(t,y)) \\ &\qquad = -\left[\varphi(t,x)(u(t,y) + u^-(t,y)1_H(y)) + \varphi(t,y)(u(t,x) + u^-(t,x)1_H(x))\right] \end{aligned}$$

we get using (9.2)

$$\mathscr{J}(u(t), \varphi(t)) + \mathscr{J}(u^-(t)1_H, \varphi) = -\int_{\mathbb{R}^N} \varphi(t,y) \int_{\mathbb{R}^N} \big(u(t,x) + u^-(t,x)1_H(x)\big) J(x-y)\, dxdy$$

$$= -\int_H \varphi(t,y) \int_H u^+(t,x)J(x-y) - u(t,x)J(\bar{x}-y)\, dx \leq 0 \qquad \text{for } t \in T. \qquad (9.23)$$

Next we need to estimate $\mathscr{J}(u^-(t)1_H, \varphi(t))$. Put for $t \in T$

$A_1(t) := \{x \in H \,:\, u(t) \leq -d\}$ and $A_2(t) := (\mathbb{R}^N \setminus H) \cup \{x \in H \,:\, u(t) > -d\} = \mathbb{R}^N \setminus A_1(t)$.

Note that for $t \in T$ we have $A_1(t) \subset U$ and $\mathbb{R}^N \setminus U \subset A_2(t)$. Then for $t \in T$

$$\begin{aligned}
\mathscr{J}(u^-(t)1_H, \varphi(t)) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u(t,x)-u(t,y))(\varphi(t,x)-\varphi(t,y))J(x-y)\, dxdy \\
&= \frac{1}{2}\int_{A_1(t)}\int_{A_1(t)} (u^-(t,x)1_H(x) - u^-(t,y)1_H(y))(\varphi(t,x)-\varphi(t,y))J(x-y)\, dxdy \\
&\qquad + \int_{A_1(t)} \varphi(t,x) \int_{A_2(t)} (u^-(t,x)1_H(x) - u^-(t,y)1_H(y))J(x-y)\, dydx \\
&\geq \frac{1}{2}\int_{A_1(t)}\int_{A_1(t)} (\varphi(t,x)-\varphi(t,y))^2 J(x-y)\, dxdy \\
&\qquad + \int_{\mathbb{R}^N} \varphi(t,x) \int_{A_2(t)} (d - u^-(t,y)1_H(y))J(x-y)\, dydx \\
&\geq \int_{\mathbb{R}^N} \varphi(t,x) \int_{A_2(t)} (d - u^-(t,y)1_H(y))J(x-y)\, dydx \geq \int_{\mathbb{R}^N} \varphi(t,x) d\kappa_{J,U}(x)dx.
\end{aligned}$$

Combining this with (9.23) and (9.22), we get for $t \in T$

$$\begin{aligned}
\frac{d}{dt}\|\varphi(t)\|^2_{L^2(U)} &\leq -2(k+c_\infty)\int_U u(t,x)\varphi(t,x)\, dx - 2\mathscr{J}(u^-(t)1_H, \varphi(t)) \\
&\leq 2(k+c_\infty)\int_U \varphi^2(t,x)\, dx + 2d(k+c_\infty)\int_U \varphi(t,x)\, dx - 2\int_{\mathbb{R}^N} \varphi(t,x) d\kappa_{J,U}(x)dx \\
&\leq 2(k+c_\infty)\int_U \varphi^2(t,x)\, dx.
\end{aligned}$$

We conclude $\|\varphi(t)\|_{L^2(U)} \leq e^{(k+c_\infty)t}\|\varphi(0)\|_{L^2(U)}$. Since $\varphi(0) \equiv 0$, we conclude that $\varphi(t) \equiv 0$ for $t \in T$. This shows (9.21), as required. □

9.2 Time dependent Harnack inequality for antisymmetric supersolutions

As before, we consider a fixed open half space $H \subset \mathbb{R}^N$. In this subsection we deduce a weak Harnack inequality for bounded antisymmetric supersolutions of the equation

$$\partial_t v + Iv = c(t,x)v \qquad \text{in } T \times U. \qquad (9.24)$$

Here $U \subset H$ is open and bounded, J satisfies $(J1)_{diff}$, (JL_s), (JU_s) for some $s \in (0,1)$ and (9.1) – (9.4).

Theorem 9.11. *Let $r_0 \in (0,1]$, $c_\infty, R, \tau, \varepsilon > 0$ be given and let $H \subset \mathbb{R}^N$ be an open half space. Let J satisfy (J1)$_{diff}$, (JL$_s$) and (JU$_s$) for some $s \in (0,1)$ with r_0 in (5.14). Moreover, assume J satisfies (9.1), (9.2) and (9.4). Then there exist positive constants $K_i = K_i(N,J,r_0,c_\infty,\varepsilon,R,\tau,H) > 0$, $i = 1,2$ with the following property: If $D \subset\subset U \subset H$ is a pair of domains with $|U| < \infty$, $\operatorname{dist}(D,\partial U) \geq 4r_0$, $\operatorname{diam}(D) \leq R$, $|D| \geq \varepsilon$, and v is a bounded antisymmetric supersolution of (9.24) in $T \times U$ with $T = [t_0, t_0 + 4\tau]$ for some $t_0 \in \mathbb{R}$ and $c \in L^\infty(T \times U)$, $\|c\|_{L^\infty(T\times U)} \leq c_\infty$, then*

$$\inf_{(t,x)\in T_+\times D} v(t,x) \geq K_1[v^+]_{L^1(T_-\times D)} - K_2\|v^-\|_{L^\infty(T\times H)}, \tag{9.25}$$

where $T_+ = [t_0 + 3\tau, t_0 + 4\tau]$ and $T_- = [t_0 + \tau, t_0 + 2\tau]$.

We will follow closely the proof of [49, Theorem 2.9].

Proof. Put $b = 2r_0$, $U_0 = \{x \in U : \operatorname{dist}(x,D) < b\} \subset\subset U$, and let J' be the kernel function given by Lemma 9.5 for this choice of b. Note that J' satisfies (JL$_s$) and (JU$_s$) for s with $b/2$ in (5.14). Let v be a bounded antisymmetric supersolution of (9.24) on $T \times U$, and consider $\tilde{v}(t) = 1_H v(t)$ for $t \in T$. Since $U_0 \subset H_b$, Lemma 9.5 implies that

$$\mathscr{J}'(\tilde{v}(t),\varphi) \geq \int_{U_0} \Big([c(t,x) - 2\kappa_{J,H}(x)]\tilde{v}(t,x) - \partial_t\tilde{v}(t,x)\Big)\varphi(x)\,dx$$
$$\text{for any } \varphi \in \mathscr{D}^J(U_0),\ \varphi \geq 0 \text{ and } t \in T,$$

where $0 \leq \kappa_{J,H}(x) < \infty$ for $x \in H_b$ by Lemma 9.4. Let

$$d := 2(\sup_{x\in H_b}\kappa_{J,H}(x) + c_\infty), \quad \sigma := \|v^-\|_{L^\infty(T\times H)} \quad \text{and} \quad w(t,x) := e^{d(t-t_0)}[\tilde{v}(t,x) + \sigma]$$

for $t \in T, x \in \mathbb{R}^N$. Observe that $w(t) \geq 0$ on $\mathbb{R}^N$ for all $t \in T$. Moreover, for any $t \in T$ and any nonnegative $\varphi \in \mathscr{D}^J(U_0)$ we have

$$\begin{aligned}
\mathscr{J}'(w(t),\varphi) &= e^{d(t-t_0)}\mathscr{J}'(\tilde{v}(t),\varphi)\\
&\geq \int_{U_0}\Big([d + c(t,x) - 2\kappa_{J,H}(x)]w(t,x) - \partial_t w(t,x) - e^{d(t-t_0)}\sigma[c(t,x) - 2\kappa_{J,H}(x)]\Big)\varphi(x)\,dx\\
&\geq \int_{U_0}\Big(e^{d(t-t_0)}\sigma[2\kappa_{J,H}(x) - c(t,x)] - \partial_t w(t,x)\Big)\varphi(x)\,dx.
\end{aligned}$$

Hence w is a nonnegative supersolution of (L) (see pp. 56) on $T \times U_0$ with $g(t,x) = e^{d(t-t_0)}\sigma[2\kappa_{J,H}(x) - c(t,x)]$. Applying Corollary 7.3 with U_0 in place of U (noting that $\operatorname{dist}(D,\partial U_0) = b = 2r_0$) and using the properties of J' given by Lemma 9.5, we find $c_i = c_i(N,J,r_0,R,\varepsilon,\tau,H) > 0$ such that

$$\inf_{T_+\times D} w(t,x) \geq c_1[w]_{L^1(T_-\times D)} - c_2\|g\|_{L^\infty(T\times U_0)}$$

We note that $[w]_{L^1(T_-\times D)} \geq [v+\sigma]_{L^1(T_-\times D)} \geq [v^+]_{L^1(T_-\times D)}$ and

$$\inf_{T_+\times D} w \leq e^{4\tau d}\Big(\inf_{T_+\times D} v + \sigma\Big),$$

so that

$$\inf_{T_+\times D} v \geq c_1 e^{-4\tau d}[v^+]_{L^1(T_+\times D)} - e^{-4\tau d} c_2 \|g\|_{L^\infty(T\times U_0)} - \sigma$$

Noting furthermore that $\|g\|_{L^\infty(T\times U_0)} \leq e^{4\tau d}\sigma d$, we conclude that

$$\inf_{T_+\times D} v \geq c_1 e^{-4\tau d}[v^+]_{L^1(T_+\times D)} - (c_2 d + 1)\sigma.$$

Hence the assertion follows with $K_1 = c_1 e^{-4\tau d}$ and $K_2 = c_2 d + 1$. Note that both constants only depend on N, J, r_0, $c_\infty, \varepsilon, R, H$ and τ. □

9.3 A subsolution estimate

In this part we want to show that if an antisymmetric supersolution is not too negative everywhere and positive in some ball at initial time, then we can bound the positive part in a smaller ball from below with an exponential decay. This bound will be derived by comparison with a suitable subsolution of an auxiliary problem.

Proposition 9.12. *Let $r_0 > 0$, $\rho \in (0, r_0/8)$ and $c_\infty > 0$ be given, and let $H \subset \mathbb{R}^N$ be an open half space. Furthermore, let J satisfy (J1)$_{diff}$, (9.1), (9.2) and (9.4) w.r.t. r_0. Then there are constants $C, q, p > 0$ with the following property:*
Given $x_0 \in H$ with $\text{dist}(x_0, \mathbb{R}^N \setminus H) \geq 8\rho$, $k_1, k_2 > 0$ with $k_1 \geq qk_2$, a time interval $T = (t_0, T_0)$ with $t_0, T_0 \in \mathbb{R}$, $t_0 < T_0$, a function $c \in L^\infty(T\times B_{2\rho}(x_0))$ with $\|c\|_{L^\infty(T\times B_{2\rho}(x_0))} \leq c_\infty$ and an antisymmetric supersolution v of

$$\partial_t v + Iv = c(t,x)v \qquad \text{in } T\times B_{2\rho}(x_0) \tag{9.26}$$

which satisfies

$$\begin{cases} v(t_0) \geq k_1 & \text{in } B_{2\rho}(x_0), \\ v \geq 0 & \text{on } T\times B_{3\rho}(x_0), \\ v(t,x) \geq -k_2 e^{-C(t-t_0)} & \text{for } (t,x) \in T\times H, \end{cases} \tag{9.27}$$

we have

$$v(t,x) \geq k_1 p e^{-C(t-t_0)} \qquad \text{for every } t \in T \text{ and a.e. } x \in B_\rho(x_0). \tag{9.28}$$

Proof. Let J', $\mathscr{J}'$ and I' be given as in Lemma 9.5 and Lemma 9.7 w.r.t. $b := 8\rho$. Let $x \in H$ with $\text{dist}(x, \mathbb{R}^N \setminus H) \geq 8\rho$, and let $\xi \in C^2([0,\infty))$ be a function with $0 \leq \xi \leq 1$, $\xi \equiv 1$ on $[0, \frac{7}{8}]$ and $\xi \equiv 0$ on $[1,\infty)$. Moreover, for $r > 0$ consider the function $\xi_r = \xi((\cdot - x)/r)$, and let

$$\delta_0(x) = \frac{1}{2 + 2\|\xi_\rho\|_{\mathscr{D}^{J'}_\infty(B_\rho(x))}} \inf_{y\in B_{2\rho}(x)} \int_{B_{\rho/4}(x)} J'(z,y)\, dz$$

By Remark 3.13 (i), Lemma 3.12 applies with 2ρ in place of ρ, J' in place of J and $\delta_0 = \delta_0(x)$ as above. Furthermore, by Lemma 9.7 there is $C_1 = C_1(J, H, \rho) > 0$ with

$$\|\xi_\rho\|_{\mathscr{D}^{J'}_\infty(B_\rho(x))} \leq C_1 \|\xi_\rho\|_{C^2(\mathbb{R}^N)} \leq \frac{C_1}{\rho^2}\|\xi\|_{C^2([0,\infty))}$$

Moreover, since $\text{dist}(x,\mathbb{R}^N\setminus H)\geq 8\rho=b$, Lemma 9.5 implies that there is $d=d(J,\rho,H)>0$ with

$$J'(z,y)\geq dJ(z-y)\qquad \text{for a.e. } z,y\in B_{4\rho}(x).$$

Then, since J satisfies $(\mathrm{J}_+)_{8\rho}$ by Lemma 9.4, we have

$$\delta_0(x)\geq \frac{d\rho^2}{2\rho^2+2C_1\|\xi\|_{C^2([0,\infty))}}\inf_{y\in B_{2\rho}(x)}\int\limits_{B_{\rho/4}(x)} J(z-y)\,dz\geq \frac{d\rho^{2+N}\omega_N \operatorname*{essinf}_{z\in B_{4\rho}(0)} J(z)}{4^N(2\rho^2+2C_1\|\xi\|_{C^2([0,\infty))})}>0.$$

Denote

$$\delta:=\min\left\{1\,,\frac{d\rho^{2+N}\omega_N \operatorname*{essinf}_{z\in B_{4\rho}(0)} J(z)}{4^N(2\rho^2+2C_1\|\xi\|_{C^2([0,\infty))})}\right\}.$$

Next let $x_0\in H$ with $\text{dist}(x_0,\mathbb{R}^N\setminus H)\geq 8\rho$ and denote $B_r:=B_r(x_0)$ for any $r>0$. Let $f\in L^\infty(\mathbb{R}^N)\cap\mathscr{D}^{J'}(B_{2\rho})$ be the unique solution of

$$I'f=1_{B_{3\rho/2}}-\delta 1_{B_{2\rho}\setminus B_{3\rho/2}}\quad \text{in } B_{2\rho},\qquad f\equiv 0\quad \text{in } \mathbb{R}^N\setminus B_{2\rho}$$

given by Lemma 3.12. Note that $f\geq 0$ in $\mathbb{R}^N$. Moreover, by Remark 3.13 (ii) and Lemma 9.7 we can pick

$$p:=\min\left\{\frac{\rho^2}{\rho^2+C_1\|\xi\|_{C^2([0,\infty))}},\frac{\delta}{1+C_1\|\xi_{2\rho}-\xi_{3\rho/7}\|_{C^2(\mathbb{R}^N)}}\right\}>0$$

independent of x_0 such that

$$\operatorname*{essinf}_{B_{3\rho/2}} f\geq p>0.$$

Put $f_\infty:=1+\frac{1}{d}\left(\inf\limits_{x\in B_{2\rho}(0)}\int\limits_{B_{4\rho}(0)\setminus B_{3\rho}(0)} J(x-y)\,dy\right)^{-1}$ and note that since $\delta\leq 1$ we have $\|f\|_{L^\infty(\mathbb{R}^N)}\leq f_\infty$ by Remark 3.13 (iii). Put

$$d(t,x):=c(t,x)-2\kappa_{J,H}(x)\qquad \text{for } (t,x)\in T\times B_{2\rho}.$$

Note that $d\in L^\infty(T\times B_{2\rho})$, since $\text{dist}(B_{2\rho},\mathbb{R}^N\setminus H)\geq\rho$, and that

$$d_\infty:=c_\infty+2\sup_{x\in H_\rho}\int\limits_{\mathbb{R}^N\setminus H} J(x-y)\,dy\geq \|d^-\|_{L^\infty(T\times B_{2\rho})}.$$

Moreover denote

$$C:=d_\infty+\frac{2}{p},\qquad q:=\frac{f_\infty}{\delta}\sup_{x\in B_{2\rho}}\int\limits_{\mathbb{R}^N\setminus B_{3\rho}} J(x-y)\,dy=\frac{f_\infty}{\delta}\sup_{x\in B_{2\rho}(0)}\int\limits_{\mathbb{R}^N\setminus B_{3\rho}(0)} J(x-y)\,dy.$$

We emphasize that all these quantities do only depend on J, H, ρ and c_∞ but not on x_0. Now let $k_1, k_2 > 0$ be given with $k_1 \geq qk_2$, and put $\psi(t,x) = e^{-Ct}(\frac{k_1}{f_\infty} f - k_2 1_{\mathbb{R}^N \setminus B_{3\rho}})$ for $(t,x) \in T \times \mathbb{R}^N$. Here and in the following, we assume without loss that $t_0 = 0$. Then $\psi(t) \in \mathscr{V}^J(B_{5\rho/2})$ for all $t \in T$. We show that ψ is a subsolution of the equation

$$\partial_t \psi + I'\psi = d(t,x)\psi \qquad \text{in } T \times B_{2\rho}. \tag{9.29}$$

For this we fix $\varphi \in \mathscr{D}^J(B_{2\rho})$, $\varphi \geq 0$ and $t \in T$. Then

$$\begin{aligned}
&\mathscr{J}'(\psi(t),\varphi) + \int_{B_{2\rho}} (\partial_t \psi(t,x) - d(t,x)\psi(t,x))\,\varphi(x)\,dx \\
&= e^{-Ct}\Bigg(\frac{k_1}{f_\infty}\mathscr{J}'(f,\varphi) - k_2 \mathscr{J}'(1_{\mathbb{R}^N\setminus B_{3\rho}},\varphi) \\
&\qquad - \int_{B_{2\rho}} (C + d(t,x))\left(\frac{k_1}{f_\infty} f(x) - k_2 1_{\mathbb{R}^N\setminus B_{3\rho}}(x)\right)\varphi(x)\,dx\Bigg) \\
&= e^{-Ct}\int_{B_{2\rho}}\Bigg(\frac{k_1}{f_\infty}\left(1_{B_{3\rho/2}}(x) - \delta 1_{B_{2\rho}\setminus B_{3\rho/2}}(x)\right) \\
&\qquad + k_2 \int_{\mathbb{R}^N\setminus B_{3\rho}} J'(x,y)\,dy - (C + d(t,x))\frac{k_1}{f_\infty} f(x)\Bigg)\varphi(x)\,dx \\
&\leq e^{-Ct}\int_{B_{2\rho}}\left(\frac{k_1}{f_\infty}\left(1_{B_{3\rho/2}}(x) - \delta 1_{B_{2\rho}\setminus B_{3\rho/2}}(x)\right) + \frac{k_2 q\delta}{f_\infty} - \frac{2k_1}{pf_\infty} f(x)\right)\varphi(x)\,dx \\
&\leq \frac{k_1}{f_\infty} e^{-Ct}\left(\int_{B_{3\rho/2}} (1 + \delta - \frac{2}{p} f(x))\varphi(x)\,dx - \int_{B_{2\rho}\setminus B_{3\rho/2}} \frac{2}{p} f(x)\varphi(x)\,dx\right) \\
&\leq \frac{k_1}{f_\infty} e^{-Ct}\int_{B_{3\rho/2}} (1 + \delta - 2)\varphi(x)\,dx \leq 0.
\end{aligned}$$

Hence ψ is a subsolution of (9.29). Next, let v be an antisymmetric supersolution of (9.26) satisfying (9.27), and let $w(t) = 1_H v(t)$ for $t \in T$. Then

$$\begin{cases} w(0) \geq k_1 & \text{in } B_{2\rho}, \\ w(t,x) \geq -k_2 e^{-Ct} 1_{\mathbb{R}^N\setminus B_{3\rho}} & \text{for } (t,x) \in T \times \mathbb{R}^N. \end{cases} \tag{9.30}$$

Moreover, by Lemma 9.5, w is a supersolution of (9.29), and thus $w - \psi$ is also a supersolution of (9.29). Furthermore, we have $w - \psi \geq 0$ on $T \times (\mathbb{R}^N \setminus B_{2\rho})$ and $w(0) - \psi(0) \geq 0$ on $\mathbb{R}^N$ by the construction of ψ and (9.30). Proposition 6.2 gives $w \geq \psi$ a.e. in $T \times \mathbb{R}^N$ and, in particular,

$$v(t,x) = w(t,x) \geq pk_1 e^{-Ct} \qquad \text{for every } t \in T \text{ and a.e. } x \in B_\rho.$$

□

9.4 An estimate for the long-time behavior

We have developed several estimates for antisymmetric supersolutions in the previous subsections. Next we combine these to estimate the long-time behavior of supersolutions which are positive in a large enough subset. The following is a nonlocal pendant of [56, Theorem 3.7] (for $I=(-\Delta)^s$, $s\in(0,1)$ see also [49, Theorem 2.15]).
We recall from Section 1.1 that, for a subset $D\subset\mathbb{R}^N$, $\operatorname{inrad}(D)$ denote the supremum of all $r>0$ such that every connected component of D contains a ball $B_r(x_0)$ with $x_0\in D$. Note that $\operatorname{inrad}(D)\geq\rho>0$ implies that every connected component of D has at least measure $|B_\rho(0)|$, so in this case D has only finitely many components if it has finite measure.

Theorem 9.13. *Let $H\subset\mathbb{R}^N$ be an open half space, $r_0>0$, $\rho\in(0,r_0/8)$ and $c_\infty>0$ be given and assume J satisfies (J1)$_{diff}$ and for some $s\in(0,1)$ assume J satisfies (JL$_s$), (JU$_s$), (9.1), (9.2) and (9.4) with this r_0. Moreover, let $C,p,q>0$ be as in Proposition 9.12 and let $\delta>0$ be such that the conclusions of Proposition 9.10 hold with $\gamma:=C+1$ in place of k.*
Then for any $\tau,R>0$ there exists $\mu>0$ such that the following holds:
If $D\subset\subset U\subset H$ are open sets with $|U|<\infty$, $\operatorname{inrad}(D)>3\rho$, $\operatorname{diam}(D)\leq R$, $|U\setminus\overline{D}|<\delta$ and $\operatorname{dist}(D,\partial U)>4r_0$ and if v is any continuous function and satisfying (P_a) on $[t_0,\infty)\times U$ for some $t_0\in\mathbb{R}$ with $\|c\|_{L^\infty([t_0,\infty)\times U)}\leq c_\infty$ such that v is nonnegative on $[t_0,t_0+8\tau]\times\overline{D}$ and

$$\|v^-(t_0)\|_{L^\infty(U\setminus\overline{D})}\leq\mu[v]_{L^1((t_0+\tau,t_0+2\tau)\times D_*)}\quad\textit{for each connected component } D_* \textit{ of } D, \tag{9.31}$$

then:

(i) $v(t,x)>0$ *for all* $(t,x)\in[t_0,\infty)\times\overline{D}$

(ii) $\|v^-(t)\|_{L^\infty(U)}\to 0$ *for* $t\to\infty$.

Proof. We let $r_0,\rho,\gamma,\delta,\tau,R,H$ and J be given with the properties stated in the theorem. Let K_1,K_2 – depending on these quantities – be given as in Theorem 9.11, and let $C,p,q>0$ – depending on these quantities – be given as in Proposition 9.12. We fix $\mu>0$ sufficiently small such that

$$\Big(\frac{K_1}{\mu}-K_2\Big)>q\qquad\text{and}\qquad K_1|B_\rho(0)|p\Big(\frac{K_1}{\mu}-K_2\Big)-K_2>0,. \tag{9.32}$$

Next, we consider $D\subset\subset U\subset H$ and an antisymmetric supersolution v of (P') on $[t_0,\infty)\times U$ with the properties stated in the theorem, which implies in particular that $|B_{3\rho}(0)|\leq|D_*|\leq(2R)^N$ for every connected component D_* of D. We put $k_2=\|v^-(t_0)\|_{L^\infty(U\setminus\overline{D})}$ and

$$T_0:=\sup\{t\geq t_0+8\tau\,:\,v>0\text{ in }[t_0,t)\times\overline{D}\},$$

so that $t_0+8\tau\leq T_0\leq\infty$ by assumption. Applying Proposition 9.10, we get

$$\|v^-(t)\|_{L^\infty(U)}=\|v^-(t)\|_{L^\infty(U\setminus\overline{D})}\leq k_2e^{-(\gamma+1)(t-t_0)}\qquad\text{for all } t\in[t_0,T_0). \tag{9.33}$$

To prove (i), we suppose by contradiction that $T_0<\infty$. Then there exists a connected component D_* of D and $x_*\in\overline{D}_*$ such that

$$v>0\text{ in }[t_0,T_0)\times\overline{D}_*\qquad\text{and}\qquad v(T_0,x_*)=0. \tag{9.34}$$

Let U_* be the connected component of U with $D_* \subset U_*$. Since $v \geq 0$ on $[t_0, t_0 + 8\tau) \times \overline{D_*}$, we have, by Theorem 9.11, (9.31) and Proposition 9.10,

$$\begin{aligned}
&\inf_{[t_0+3\tau,t_0+4\tau]\times\overline{D_*}} v \geq K_1[v^+]_{L^1([t_0+\tau,t_0+2\tau]\times D_*)} - K_2\|v^-\|_{L^\infty([t_0,t_0+4\tau]\times U_*)} \\
&\geq K_1[v]_{L^1([t_0+\tau,t_0+2\tau]\times D_*)} - K_2\|v^-\|_{L^\infty([t_0,t_0+4\tau]\times[U\setminus\overline{D}])} \\
&\geq \frac{K_1}{\mu}\|v^-(t_0)\|_{L^\infty(U\setminus\overline{D})} - K_2\|v^-(t_0)\|_{L^\infty(U\setminus\overline{D})} = \Big(\frac{K_1}{\mu} - K_2\Big)k_2 =: k_1.
\end{aligned} \tag{9.35}$$

We fix $x_0 \in D_*$ such that $B_{3\rho}(x_0) \subset D_*$, which is possible by assumption. Since $k_1 \geq qk_2$ by (9.32), the estimates (9.35) and (9.33) allow us to apply Proposition 9.12 with $t_0 + 4\tau$ in place of t_0, which yields

$$v(t,x) \geq pk_1 e^{-\gamma(t-t_0-4\tau)} \qquad \text{for every } x \in B_\rho(x_0),\, t \in [t_0 + 4\tau, T_0]. \tag{9.36}$$

With the help of Theorem 9.11, (9.33) and (9.36), we find that

$$\begin{aligned}
v(T_0, x_*) &\geq K_1[v]_{L^1([T_0-3\tau,T_0-2\tau]\times D_*)} - K_2\|v^-\|_{L^\infty([T_0-4\tau,T_0]\times U_*)} \\
&\geq K_1|B_\rho(0)|k_1 p e^{-\gamma(T_0-4\tau-t_0)} - K_2 k_2 e^{-(\gamma+1)(T_0-4\tau-t_0)} \\
&\geq k_2 e^{-\gamma(T_0-4\tau-t_0)}\Big[K_1|B_\rho(0)|p\Big(\frac{K_1}{\mu} - K_2\Big) - K_2\Big] > 0,
\end{aligned}$$

by our choice of μ in (9.32), contradicting (9.34). We conclude that $T_0 = \infty$. In particular, (i) holds, and (ii) follows since, by (9.33),

$$\|v^-(t)\|_{L^\infty(U)} = \|v^-(t)\|_{L^\infty(U\setminus\overline{D})} \leq k_2 e^{-(\gamma+1)(t-t_0)} \qquad \text{for all } t \in [t_0, \infty). \tag{9.37}$$

□

Remark 9.14. We note that the constants K_1, K_2 in Theorem 9.11, C, p, q in Proposition 9.12 and μ in Theorem 9.13 depend on H only through the function

$$(0,\infty) \to \mathbb{R}, \qquad b \mapsto \operatorname*{essinf}_{\substack{x,y \in H_b \\ |x-y| \leq r_0}} \big(J(x-y) - J(x-\bar{y})\big).$$

As a consequence, if $e \in S^1$ is fixed, these constants can be chosen uniformly in $\lambda \in \mathbb{R}$ for the family of half spaces $H_\lambda := \{x \in \mathbb{R}^N : x \cdot e > \lambda\}$.

10 Asymptotic symmetry

In this part we show our main results on asymptotic symmetry. We will follow ideas which were developed by Poláčik for the case of parabolic second order nonlinear evolution equations (see [57] and the references in there). Since in the bounded and the unbounded case the argumentation is different we will split the results in two parts. The first subsection will deal with the case were $\Omega \subset \mathbb{R}^N$ is an open bounded set, and in Subsection 10.2 below we will treat the case $\Omega = \mathbb{R}^N$. For the case $I = (-\Delta)^s$ the asymptotic symmetry in open bounded sets was treated in [49].

10.1 The bounded case

In this whole part we will consider the following problem

$$(P_T) \qquad \begin{cases} \partial_t u + Iu = f(t,x,u) & \text{in } (0,\infty)\times\Omega; \\ u = 0 & \text{on } (0,\infty)\times(\mathbb{R}^N\setminus\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is open and bounded and I is given with respect to J which satisfies (J1)$_{diff}$, (J2), (JL$_s$), (JU$_s$) for some $s \in (0,1)$ and $r_0 > 0$. We will additionally assume that Ω fulfills

(D1) $\Omega \subset \mathbb{R}^N$ is an open bounded set which is Steiner symmetric in x_1, i.e. for every $x \in \Omega$ and $s \in [-1,1]$ we have $(sx_1, x_2, \dots, x_N) \in \Omega$.

(D2) For every $\lambda > 0$, the set $\Omega_\lambda := \{x \in \Omega : x_1 > \lambda\}$ has at most finitely many connected components.

and the nonlinearity $f : [0,\infty)\times\Omega\times\mathbb{R} \to \mathbb{R}$ fulfills

(F1) $f : [0,\infty)\times\Omega\times\mathbb{R} \to \mathbb{R}$ is continuous. Moreover, for every $K > 0$ there exists $L = L(K) > 0$ such that $\sup_{x\in\Omega, t>0} |f(t,x,u) - f(t,x,v)| \leq L|u-v|$ for $u, v \in [-K,K]$.

(F2) f is symmetric in x_1 and monotone in $|x_1|$, i.e., for every $t \in [0,\infty)$, $u \in \mathbb{R}$, $x \in \Omega$ and $s \in [-1,1]$ we have $f(t, sx_1, x_2, \dots, x_N, u) \geq f(t,x,u)$.

We note that assumption (D2) is not needed for all of our results. In the following, we consider solutions $u \in C((0,\infty), \mathscr{D}^J(\Omega)\cap C(\Omega)) \cap C^1((0,\infty), L^2(\Omega))$ of problem (P_T) as defined in Definition 6.1.

Theorem 10.1. *Let J satisfy (J1)$_{diff}$, (J2), (JL$_s$), (JU$_s$) for some $s \in (0,1)$ and $r_0 > 0$. Furthermore, let (D1), (F1), (F2) be satisfied, and let u be a nonnegative global solution of (P_T) satisfying the following conditions:*

(U1) There is $c_u > 0$ such that $\|u(t)\|_{L^\infty} \leq c_u$ for every $t > 0$.

(U2) u is eventually equicontinuous on Ω (see p. 64).

Suppose in addition that (D2) holds or that $z \not\equiv 0$ for every $z \in \omega(u)$.
Then u is asymptotically symmetric in x_1, i.e., for all $z \in \omega(u)$ we have $z(-x_1,x') = z(x_1,x')$ for all $(x_1,x') \in \Omega$.
Moreover, for every $z \in \omega(u)$ we have the following alternative: Either $z \equiv 0$ on Ω, or z is strictly decreasing in $|x_1|$ and therefore strictly positive in Ω.

Remark 10.2. We have shown in Section 8 that assumptions (U1) and (U2) are satisfied in specific examples where the kernel function J fulfills additionally (J_s). More precisely, in Remark 8.9 we considered situations in which Theorem 10.1 can be applied. We note that these examples include nonlocal operators which are more general than the fractional Laplacian. For the fractional Laplacian similar examples are given in [49].

We immediately deduce the following corollary for time-periodic solutions.

Corollary 10.3. *Suppose that $f : (0,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies (F1), (F2) and is periodic in t, i.e. there is $\tau > 0$ such that $f(t+\tau,x,u) = f(t,x,u)$ for all t,x,u. Suppose furthermore that u is a nontrivial nonnegative τ-periodic solution of (P_T), i.e., $u(t+\tau,x) = u(t,x)$ for all $x \in \Omega, t \in (0,\infty)$. Suppose finally that either (D2) holds or that $u(t) \not\equiv 0$ on Ω for all t. Then $u(t)$ is symmetric in x_1 and strictly decreasing in $|x_1|$ for all times $t \in (0,\infty)$.*

Proof of Theorem 10.1

Throughout the proof we will use the notation for the reflection at a hyperplane as introduced in Sections 4 and 9 and the various estimates for antisymmetric supersolutions which arise during the moving plane method. We may follow the main lines of the moving plane method as developed by Poláčik in [56], but we note that some steps in the argument – in particular the proofs of Lemma 10.5 and Proposition 10.8 below – differ from [56]. This is due to the fact that, contrary to [56], we do not a priori assume the existence of an element $\varphi \in \omega(u)$ with $\varphi > 0$. For theses steps of the proof we will follow [49]. For $\lambda \in \mathbb{R}$, we use the notations

$$\Omega_\lambda = \{x \in \Omega \,:\, x_1 > \lambda\}, \quad H_\lambda := \{x \in \mathbb{R}^N \,:\, x_1 > \lambda\}, \quad T_\lambda = \partial H_\lambda \quad \text{and} \quad \Gamma_\lambda = T_\lambda \cap \Omega.$$

Moreover, we let $Q_\lambda : \mathbb{R}^N \to \mathbb{R}^N$ denote the reflection at T_λ given by $Q_\lambda(x) = (2\lambda - x_1, x_2, \dots, x_N)$. For a function $z : \mathbb{R}^N \to \mathbb{R}$, we put

$$z^\lambda = z \circ Q_\lambda \,:\, \mathbb{R}^N \to \mathbb{R}$$

and

$$V_\lambda z : \mathbb{R}^N \to \mathbb{R}, \qquad V_\lambda z(x) = z^\lambda(x) - z(x).$$

We now assume the hypotheses $(J1)_{diff}$, (J2), (JL_s), (JU_s) for some $s \in (0,1)$ and $r_0 > 0$ on J, (D1) on Ω and (F1), (F2) on the nonlinearity f. Let u be a nonnegative global solution of (P_T) satisfying (U1) and (U2). Without restriction we may assume that u satisfies the equicontinuity property with $t_0 = 1$, i.e.

$$\lim_{h \to 0} \sup_{\substack{\tau \geq 1 \\ x,\tilde{x} \in \overline{\Omega},\, t,\tilde{t} \in [\tau,\tau+1], \\ |x-\tilde{x}|,|t-\tilde{t}| < h}} |u(t,x) - u(\tilde{t},\tilde{x})| = 0.$$

We set

$$l := \max\{x_1 \,:\, (x_1,x') \in \Omega \quad \text{for some } x' \in \mathbb{R}^{N-1}\},$$

and we fix $\lambda \in [0,l)$ for the moment. As discussed in Lemma 9.3, the function $v := V_\lambda u$ is an antisymmetric supersolution of the problem

$$\partial_t v + Iv = c_\lambda(t,x)v \tag{10.1}$$

in $(0,\infty)\times\Omega_\lambda$ with

$$c_\lambda(x,t) = \begin{cases} \dfrac{f(t,x,u^\lambda(x)) - f(t,x,u(x))}{u^\lambda(x)-u(x)}, & u^\lambda(t,x) \neq u(t,x); \\ 0, & u^\lambda(t,x) = u(t,x). \end{cases}$$

Here the term antisymmetric supersolution refers to the notion defined in the beginning of Section 9 with respect to the half space $H = H_\lambda$. Indeed, for $\lambda \in [0,l)$ and this choice of H, we have that $H\cap\Omega \neq \emptyset$, $Q_\lambda(\Omega_\lambda) \subset \Omega$ and $f(t,Q_\lambda(x),u) \geq f(t,x,u)$ for all $t \in (0,\infty)$, $x \in \Omega_\lambda$ and $u \in \mathbb{R}$ are fulfilled as a consequence of assumptions (D1) and (F2). Moreover, as a consequence of (F1) and (U1), there exists $c_\infty > 0$ such that

$$\|c_\lambda\|_{L^\infty((0,\infty)\times\Omega_\lambda)} \leq c_\infty \qquad \text{for every } \lambda \in [0,l).$$

In the following, we fix c_∞ with this property. We also note that $[V_\lambda u](t) \in \mathscr{D}^J(\mathbb{R}^N)\cap C(\mathbb{R}^N)$ for all $t \in (0,\infty)$. Furthermore, by (J2) we have that J satisfies (9.1), (9.2) and (9.4) for H_λ, $\lambda \geq 0$. For $\lambda \in [0,l)$, we now consider the following statement:

$$(S_\lambda) \qquad \|(V_\lambda u)^-(t)\|_{L^\infty(H_\lambda)} \to 0 \text{ as } t \to \infty.$$

Our aim is to show via the method of moving planes that (S_λ) holds for every $\lambda \in [0,l)$. We need the following lemmas.

Lemma 10.4. *There is $\delta > 0$ such that for each $\lambda \in [0,l)$ the following statement holds. If K is a closed subset of Ω_λ with $|\Omega_\lambda \setminus K| < \delta$ and there is $t_0 \geq 0$ such that $V_\lambda u(t) \geq 0$ on K for all $t \geq t_0$, then*

$$\|(V_\lambda u)^-(t)\|_{L^\infty(H_\lambda)} \leq e^{-(t-t_0)}\|(V_\lambda u)^-(t_0)\|_{L^\infty(H_\lambda)}, \tag{10.2}$$

for all $t \geq t_0$. In particular (S_λ) holds if $\lambda < l$ is sufficiently close to l.

Proof. This follows immediately by applying Proposition 9.10 to $\gamma = 1$, $c_\infty > 0$ as fixed above, $H = H_\lambda$ and $U = \Omega_\lambda \setminus K$. Note that the number $\delta > 0$ given by Proposition 9.10 in this case does not depend on λ and K. The second statement of the lemma follows since $|\Omega_\lambda| < \delta$ if λ is close to l. □

We note that the first two assertions of the following Lemma do not need the boundedness of Ω.

Lemma 10.5. *Suppose $\lambda_0 \in [0,l)$ is such that (S_λ) holds for all $\lambda \in (\lambda_0,l)$. Then we have:*

(i) (S_{λ_0}) *holds.*

(ii) For each $z \in \omega(u)$ *and each* $\lambda \in [\lambda_0, l)$ *we have either* $V_\lambda z > 0$ *on* Ω_λ *or* $V_\lambda z \equiv 0$ *on* $\mathbb{R}^N$.

(iii) If $\lambda_0 > 0$*, then for each* $z \in \omega(u)$ *we have either* $V_{\lambda_0} z > 0$ *on* Ω_λ *or* $z \equiv 0$ *on* $\mathbb{R}^N$.

Proof. (i) Since the set $\{u(t) : t \geq 0\}$ is relatively compact in $\mathscr{C}_0(\Omega)$, the statement (S_λ) is equivalent to $V_\lambda z \geq 0$ on H_λ for all $z \in \omega(u)$. Hence (S_{λ_0}) holds by assumption and continuity of all $z \in \omega(u)$.
(ii) Note that by Definition of (S_{λ_0}) we have that (S_λ) holds for any $\lambda \geq \lambda_0$. Fix $\lambda \in [\lambda_0, l)$.
Step one: We first claim that on each connected component U of Ω_λ we either have $V_\lambda z > 0$ on U or $V_\lambda z \equiv 0$ on U. To prove this, we fix $z \in \omega(u)$ and a connected component U of Ω_λ such that $V_\lambda z \not\equiv 0$ on U. Since $V_\lambda z \geq 0$ and since z is continuous, there exists $x_0 \in U$ and $\rho > 0$ such that $B := B_\rho(x_0) \subset\subset \Omega_\lambda$ and $V_\lambda z > 0$ on $\overline{B}$. Since $z \in \omega(u)$, there exists a sequence of numbers $t_n > 0$, such that $t_n \to \infty$ and $u(t_n) \to z$ in $C(\overline{\Omega})$, hence also $V_\lambda u(t_n) \to V_\lambda z$ in $C(\overline{\Omega_\lambda})$ as $n \to \infty$. Consequently, there exists $\sigma > 0$ and $n_0 \in \mathbb{N}$ such that

$$V_\lambda u(t_n, x) > 2\sigma \qquad \text{for } x \in \overline{B},\, n > n_0.$$

By the equicontinuity property (U2), there exists $\tau > 0$ such that

$$V_\lambda u(t, x) > \sigma \qquad \text{for } x \in \overline{B},\, t \in [t_n - 4\tau, t_n],\, n > n_0. \tag{10.3}$$

Now fix a subdomain $D \subset\subset U$ such that $B \subset\subset D$. Applying Theorem 9.11 with $t_0 = t_n - 4\tau$ and using (10.3), we get

$$\begin{aligned}\inf_{x \in D} V_\lambda u(t_n, x) &\geq K_1 [(V_\lambda u)^+]_{L^1([t_n - 4\tau, t_n - 3\tau] \times D)} - K_2 \sup_{t \in T} \|(V_\lambda u)^-(t)\|_{L^\infty(U)} \\ &\geq K_1 \sigma \frac{|B|}{|D|} - K_2 \|v^-(t, \cdot)\|_{L^\infty(T \times H_\lambda)} \qquad \text{for } n > n_0\end{aligned}$$

with suitable constants $K_1, K_2 > 0$ independent of n. Since (S_λ) holds, we conclude that

$$\inf_{x \in D} V_\lambda z = \lim_{n \to \infty} \inf_{x \in D} V_\lambda u(t_n, x) \geq K_1 \sigma \frac{|B|}{|D|} > 0.$$

Since $D \subset\subset U$ was chosen arbitrarily, we conclude that $V_\lambda z > 0$ in U. This shows the claim.
Step two: Let $z \in \omega(u)$ be such that

$$U_z := \{x \in \Omega_\lambda \,:\, [V_\lambda z](x) = 0\}$$

is nonempty. To finish the proof of (ii), we need to show that $V_\lambda z \equiv 0$ on $\mathbb{R}^N$. We suppose by contradiction that this is false; then there exists a compact set $\mathscr{K} \subset H_\lambda \setminus \overline{U_z}$ of positive measure such that

$$\inf_{\mathscr{K}} V_\lambda z > 0 \tag{10.4}$$

By Step one above, U_z has a nonempty interior. Hence we may fix a nonnegative function $\varphi \in C_c^\infty(U_z)$, $\varphi \not\equiv 0$, and we set $D := \operatorname{supp} \varphi$. Moreover, we fix $\rho \in (0, r_0)$ with $\operatorname{dist}(D, \partial U) > 2\rho$, and we note that there exists $M > 0$ such that

$$\Big| \int\limits_{B_\rho(x)} (\varphi(x) - \varphi(y))J(x-y)\, dy \Big| < M \qquad \text{for all } x \in \mathbb{R}^N, \tag{10.5}$$

(see e.g. Proposition 2.5). In the following, we put $v = V_\lambda u$ and $H = H_\lambda$. Moreover, we consider $\bar{J}$ and $\kappa_{J,H}$ as defined in Lemma 9.4 for this choice of H. By Lemma 9.4 we have

$$\mathscr{J}(v(t), \varphi) = \frac{1}{2} \int\limits_H \int\limits_H (v(t,x) - v(t,y))(\varphi(x) - \varphi(y))\bar{J}(x,y)\, dxdy + 2\int\limits_H v(t,x)\kappa_{J,H}(x)\varphi(x)\, dx, \tag{10.6}$$

where

$$\int\limits_H v(t,x)\kappa_{J,H}(x)\varphi(x)\, dx \le K\|\varphi\|_{L^1(U_z)}\|v(t)\|_{L^\infty(U_z)}$$

with $K = K(J,N,\rho) < \infty$ by $(\mathrm{J}1)_{diff}$. To estimate the double integral on the right hand side of (10.6), we put

$$\mathscr{H}_1 := \{(x,y) \in H \times H : |x-y| \le \delta\},\ \mathscr{H}_2 := H \times H \setminus \mathscr{H}_1 \text{ and } D_\rho := \{x \in \mathbb{R}^N : \operatorname{dist}(x,D) \le \rho\}.$$

Then

$$\int\limits_{\mathscr{H}_1} (v(t,x) - v(t,y))(\varphi(x) - \varphi(y))\bar{J}(x,y)dxdy = \int\limits_{\substack{|x-y|\le\delta,\\ x,y\in D_\rho}} (v(t,x) - v(t,y))(\varphi(x) - \varphi(y))\bar{J}(x,y)dxdy$$

$$= \left(2\int\limits_{D_\rho} v(t,x) \int\limits_{B_\rho(x)} (\varphi(x) - \varphi(y))J(x-y)\, dydx \right.$$

$$\left. - \int\limits_{\substack{|x-y|\le\delta,\\ x,y\in D_\rho}} ((v(t,x) - v(t,y))(\varphi(x) - \varphi(y)))J(x - Q_{\lambda_0}(y))\, dxdy \right)$$

$$\le 2M\,|D_\rho|\, \|v(t)\|_{L^\infty(U_z)} + \frac{4c_2|D_\rho|^2}{(2\rho)^{N+2s}}\, \|\varphi\|_{L^\infty(U_z)}\, \|v(t)\|_{L^\infty(U_z)},$$

where we used the fact that $|x - Q_\lambda(y)| \ge 2\rho$ for every $x, y \in D_\rho$, $\rho < r_0$ and (JU_s). To estimate the integral over $\mathscr{H}_2$, we first note that by $(\mathrm{J}1)_{diff}$

$$\sup_{x\in H} \int\limits_{H\setminus B_\rho(x)} \bar{J}(x,y)\, dydx \le \sup_{x\in\mathbb{R}^N} \int\limits_{R^N\setminus B_\rho(x)} J(x-y)\, dy = \int\limits_{\mathbb{R}^N\setminus B_\rho(0)} J(z)\, dz =: J_{N,J} < \infty.$$

Hence

$$\int_{\mathscr{H}_2} (v(t,x)-v(t,y))(\varphi(x)-\varphi(y))\bar{J}(x,y)\,dxdy = 2\int_D \varphi(x) \int_{H\setminus B_\rho(x)} (v(t,x)-v(t,y))\bar{J}(x,y)\,dydx$$

$$= 2\int_D \varphi(x)\Bigg\{v(t,x)\int_{H\setminus B_\rho(x)} \bar{J}(x,y)\,dydx - \int_{H\setminus[B_\rho(x)\cup\mathscr{K}]} v(t,y)\bar{J}(x,y)\,dydx - \int_{\mathscr{K}} v(t,y)\bar{J}(x,y)\,dydx\Bigg\}$$

$$\leq 2J_{N,J}\|\varphi\|_{L^1(U_z)}\Big(\|v(t)\|_{L^\infty(U_z)} + \|v^-(t)\|_{L^\infty(H)}\Big) - dm(t)$$

where in the last step we have set

$$m(t) := \inf_{y\in\mathscr{K}} v(t,y) \qquad \text{and} \qquad d := \int_D \varphi(x)\int_{\mathscr{K}} \bar{J}(x,y)\,dydx > 0, \text{ since } \operatorname{dist}(D,\mathscr{K}) < 2r.$$

We now consider the function $t \mapsto h(t) = \int_{U_z} v(t,x)\varphi(x)\,dx$ for $t > 0$. Combining the estimates above and using (10.1), we get

$$h'(t) = \int_{\Omega_\lambda} \partial_t v(t,x)\varphi(x)\,dx \geq \int_D c_\lambda(t,x)v(t,x)\varphi(x)\,dx - \mathscr{J}(v(t),\varphi) \tag{10.7}$$

$$\geq -c_\infty\|\varphi\|_{L^1(U_z)}\|v(t)\|_{L^\infty(U_z)} - \mathscr{J}(v(t),\varphi) \geq -C_1\|v(t)\|_{L^\infty(U_z)} - C_2\|v^-(t)\|_{L^\infty(H)} + m(t)d$$

with

$$C_1 := \|\varphi\|_{L^1(U_z)}\big[c_\infty + 2J_{N,J}\big] + 2M|D_\rho| + \frac{c_2 4|D_\rho|^2}{(2\rho)^{N+2s}}\|\varphi\|_{L^\infty(U_z)} \quad \text{and}$$

$$C_2 := 2J_{N,J}\|\varphi\|_{L^1(U_z)}.$$

We now consider a sequence $(t_k)_k \subset (0,\infty)$ such that $t_k \to \infty$ and $u(t_k) \to z$ in $L^\infty(\Omega)$ as $k \to \infty$, which yields in particular that $h(t_k) \to 0$ as $k \to \infty$. Using (10.4) and the equicontinuity property $(U2)$, we find $\delta > 0$ and $k_0 \in \mathbb{N}$ such that

$$m_* := \inf\{m(t) \,:\, t \in [t_k-\delta, t_k+\delta],\, k \geq k_0\} > 0.$$

Moreover, making $\delta > 0$ smaller and $k_0 \in \mathbb{N}$ larger if necessary, we may assume that

$$\|v(t)\|_{L^\infty(U_z)} \leq \|v(t)-v(t_k)\|_{L^\infty(U_z)} + \|v(t_k)\|_{L^\infty(U_z)} \leq \frac{m_* d}{4C_1} \qquad \text{for } t \in [t_k-\delta, t_k+\delta],\, k \geq k_0. \tag{10.8}$$

Moreover, using that $\|v^-(t)\|_{L^\infty(H)} \to 0$ as $t \to \infty$ as a consequence of (S_λ), we may again make $k_0 \in \mathbb{N}$ larger such that

$$\|v^-(t)\|_{L^\infty(H)} \leq \frac{m_* d}{4C_2} \qquad \text{for } t \in [t_k-\delta, t_k+\delta],\, k \geq k_0. \tag{10.9}$$

Combining (10.7), (10.8) and (10.9), we thus obtain

$$h'(t) \geq \frac{m_* d}{2} \qquad \text{for } t \in [t_k-\delta, t_k+\delta],\, k \geq k_0.$$

This implies that

$$\limsup_{k\to\infty} h(t_k - \delta) \le \lim_{k\to\infty}\Big(h(t_k) - \frac{\delta m_* d}{2}\Big) = -\frac{\delta m_* d}{2},$$

contradicting the fact that $\|v^-(t)\|_{L^\infty(U_z)} \to 0$ as $t \to \infty$ and thus $\liminf_{t\to\infty} h(t) \ge 0$. The proof of (ii) is finished.
(iii) Suppose that $\lambda_0 > 0$, and let $z \in \omega(u)$ such that $V_{\lambda_0} z \equiv 0$ on $\mathbb{R}^N$. In view of (ii), we need to show that $z \equiv 0$ on $\mathbb{R}^N$. For this we consider the reflected functions

$$\begin{aligned} &\tilde{u} : (0,\infty)\times\mathbb{R}^N \to \mathbb{R}, \quad && \tilde{u}(t,x) = u(t, Q_0(x)) \\ &\tilde{z} : \mathbb{R}^N \to \mathbb{R}, && \tilde{z}(x) = z(Q_0(x)). \end{aligned} \tag{10.10}$$

Since Ω and the nonlinearity f are symmetric in the x_1-variable, $\tilde{u}$ is also a solution of (P) satisfying the same hypotheses as u. Moreover, $\tilde{z} \in \omega(\tilde{u})$. Putting $\lambda_* := l - 2\lambda_0 \in (-l, l)$, it follows from $V_{\lambda_0} z \equiv 0$ on $\mathbb{R}^N$ that $\tilde{z} \equiv 0$ on Ω_{λ_*} and therefore

$$V_\lambda \tilde{z} \equiv 0 \quad \text{in } \Omega_\lambda \qquad \text{for every } \lambda \in (\frac{\lambda_* + l}{2}, l). \tag{10.11}$$

For $\lambda \in (\frac{\lambda_*+l}{2}, l)$ sufficiently close to l, it also follows from Lemma 10.4 that (S_λ) holds for $\tilde{u}$ in place of u, so that (10.11) and (ii) imply that

$$V_\lambda \tilde{z} \equiv 0 \text{ on } \mathbb{R}^N \text{ for } \lambda < l \text{ sufficiently close to } l. \tag{10.12}$$

From this we easily conclude that $\tilde{z} \equiv 0$ and therefore $z \equiv 0$ on $\mathbb{R}^N$, as claimed. □

Lemma 10.6. *Suppose $\lambda_0 \in (0,l)$ is such that (S_λ) holds for all $\lambda \in (\lambda_0, l)$. Suppose furthermore that one of the following conditions hold:*

(i) $z \not\equiv 0$ on Ω for all $z \in \omega(u)$.

(ii) Ω fulfills $(D2)$ and $V_{\lambda_0} z > 0$ on Ω_{λ_0} for some $z \in \omega(u)$.

Then there exists $\varepsilon > 0$ such that $(S)_\lambda$ holds for each $\lambda \in (\lambda_0 - \varepsilon, \lambda_0]$.

For the proof of this lemma the following observation is useful.

Lemma 10.7. *Let $M \subset C(\overline{\Omega})$ be a bounded and equicontinuous subset, and let*

$$K_\lambda(M) := \inf_{u\in M, x\in\Omega_\lambda} V_\lambda u(x) \qquad \textit{for } \lambda \in [0,l).$$

Then the map $\lambda \mapsto K_\lambda(M)$ is left continuous, i.e. for $\lambda_0 \in (0,l)$ we have $K_\lambda(M) \to K_{\lambda_0}(M)$ as $\lambda \to \lambda_0$, $\lambda < \lambda_0$.

Proof. Since $\Omega_{\lambda_0} \subset \Omega_\lambda$ for $\lambda < \lambda_0$ and $V_\lambda z \to V_{\lambda_0} z$ uniformly on Ω_{λ_0} for every $z \in M$, we have $\liminf_{\lambda \to \lambda_0^-} K_\lambda(M) \leq K_{\lambda_0}(M)$. Now suppose by contradiction that there exists sequences of numbers $\lambda_n \in (0, \lambda_0)$, of functions $u_n \in M$ and of points $x^n \in \Omega_{\lambda_n}$ such that

$$\lambda_n \to \lambda \quad \text{and} \quad V_{\lambda_n} u_n(x^n) \to c < K_{\lambda_0}(M) \qquad \text{for } n \to \infty.$$

By compactness and equicontinuity, we may assume that there exists $\bar{x} \in \overline{\Omega_{\lambda_0}}$ and $u \in \overline{M} \subset C(\overline{\Omega})$ such that

$$x^n \to \bar{x} \quad \text{and} \quad \|u_n - u\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } n \to \infty,$$

where $\overline{M}$ denotes the closure of M with respect to $\|\cdot\|_{L^\infty(\Omega)}$. Consequently,

$$Q_{\lambda_n}(x^n) = (2\lambda_n - x_1^n, x_2^n, \ldots, x_N^n) \to (2\lambda_0 - \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N) = Q_{\lambda_0}(\bar{x})$$

and therefore

$$u_n(x^n) \to u(\bar{x}) \quad \text{and} \quad u_n(Q_{\lambda_n}(x^n)) \to u(Q_{\lambda_0}(\bar{x})) \qquad \text{as } n \to \infty.$$

Hence

$$V_{\lambda_0} u(\bar{x}) = \lim_{n \to \infty} V_{\lambda_n} u_n(x_n) = c < K_{\lambda_0}(M)$$

On the other hand, since $u \in \overline{M}$ and $\bar{x} \in \overline{\Omega_{\lambda_0}}$, it is easy to see from the definition of $K_{\lambda_0}(M)$ that $V_{\lambda_0} u(\bar{x}) \geq K_{\lambda_0}$. Hence we arrived at a contradiction, and thus the proof is finished. □

Proof of Lemma 10.6. Case one: We first assume in addition that $z \not\equiv 0$ on Ω for all $z \in \omega(u)$. By Lemma 10.5 this implies that $V_{\lambda_0} z > 0$ in Ω_{λ_0} for all $z \in \omega(u)$. Let $\delta > 0$ be such that the conclusion of Lemma 10.4 holds, and let $K \subset \Omega_{\lambda_0}$ be a compact subset and $\varepsilon_1 \in (0, \lambda_0)$ be chosen such that

$$|\Omega_\lambda \setminus K| < \delta \qquad \text{for } \lambda \in (\lambda_0 - \varepsilon_1, \lambda_0]. \tag{10.13}$$

Since $V_{\lambda_0} z > 0$ in Ω_{λ_0} for all $z \in \omega(u)$ and $\omega(u)$ is a compact subset of $C(\overline{\Omega})$, we may choose $\varepsilon \in (0, \varepsilon_1)$ such that

$$\inf_{z \in \omega(u), x \in K} V_\lambda z(x) > 0 \qquad \text{for all } \lambda \in (\lambda_0 - \varepsilon, \lambda_0]. \tag{10.14}$$

Let $\lambda \in (\lambda_0 - \varepsilon, \lambda_0]$, then (10.14) implies that there exists $t_0 = t_0(\lambda)$ such that

$$V_\lambda u(t, x) \geq 0 \qquad \text{for } x \in K,\ t \geq t_0.$$

Hence $\|(V_\lambda u)^-(t)\|_{L^\infty(H_\lambda)} \to 0$ as $t \to \infty$ by Lemma 10.4. Thus (S_λ) holds for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0]$, as claimed.

Case two: We assume that (D2) holds, and that $V_{\lambda_0} z > 0$ on Ω_{λ_0} for some $z \in \omega(u)$. By (D2), the set Ω_{λ_0} has only finitely many connected components, and hence $\rho := \frac{1}{8} \min\{r_0, \mathrm{inrad}(\Omega_{\lambda_0})/4\} > 0$. Let $C_2 = C_2(N, J, \rho, c_\infty) > 0$, $C_1 = C_1(N, J, \rho, c_\infty) > 0$ be as in Proposition 9.12, and let $\delta > 0$

be such that the conclusions of Proposition 9.10 hold with C_2+1 in place of k.
Choose $D \subset\subset \Omega_{\lambda_0}$ such that D intersects each connected component of Ω_{λ_0},

$$|\Omega_{\lambda_0} \setminus \overline{D}| < \frac{\delta}{2}, \qquad \mathrm{inrad}(D) > 3\rho \tag{10.15}$$

and, moreover by making r_0 smaller and D larger if necessary, we may assume $r_0 = \frac{1}{4}\mathrm{dist}(\overline{D}, \partial\Omega_{\lambda_0})$. Fix $z \in \omega(u)$ such that $V_{\lambda_0} z > 0$ in Ω_{λ_0}, and let $t_n \to \infty$ be a sequence with $h(t_n) \to z$. Using the equicontinuity as in the proof of Lemma 10.5 we can find $r_1 > 0$, $\tau \in (0, \frac{1}{8})$ and n_0 such that

$$V_{\lambda_0} u(t,x) > 2r_1, \text{ for all } x \in \overline{D}, t \in [t_n - 8\tau, t_n], n > n_0. \tag{10.16}$$

Denote $R = \mathrm{diam}(D)$ and choose μ as in Theorem 9.13 for these parameter values but independent of λ. This is possible by Remark 9.14. We first fix $\varepsilon_1 > 0$ such that

$$|\Omega_\lambda \setminus \Omega_{\lambda_0}| < \frac{\delta}{2}, \text{ for } \lambda \in [\lambda_0 - \varepsilon_1, \lambda_0). \tag{10.17}$$

From the equicontinuity assumption (U2) we may deduce that

$$\sup_{n\in\mathbb{N}} \sup_{[t_n - 8\tau, t_n]\times D} |V_\lambda u - V_{\lambda_0} u| \to 0 \text{ as } \lambda \to \lambda_0. \tag{10.18}$$

This and (10.16) imply the existence of $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$V_\lambda u(t) > r_1, \text{ for all } x \in \overline{D}, t \in [t_n - 8\tau, t_n], n > n_0, \lambda \in [\lambda_0 - \varepsilon_2, \lambda_0]. \tag{10.19}$$

By (S_{λ_0}), we can find $n_1 > n_0$ such that for all $n > n_1$ we have

$$\|(V_{\lambda_0} u)^-(t_n - 8\tau)\|_{L^\infty(\Omega_{\lambda_0} \setminus \overline{D})} \le \frac{\mu r_1}{2}.$$

Using the equicontinuity of the functions $x \mapsto u(t_n - 8\tau, x)$, $n \in \mathbb{N}$ and Lemma 10.7, we may choose $\varepsilon \in (0, \varepsilon_2)$ such that

$$\|(V_\lambda u)^-(t_n - 8\tau)\|_{L^\infty(\Omega_\lambda \setminus \overline{D})} \le \mu r_1 \qquad \text{for } \lambda \in [\lambda_0 - \varepsilon, \lambda_0]. \tag{10.20}$$

We now fix $n \ge n_1$ and $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$, and we claim that the assumptions of Theorem 9.13 are satisfied with $t_0 = t_n - 8\tau$, $U = \Omega_\lambda$, D as above and $v = V_\lambda u$. Indeed, $\mathrm{dist}(\overline{D}, \partial U) \ge \mathrm{dist}(\overline{D}, \partial\Omega_{\lambda_0}) \ge 4r_0$ and $|\Omega_\lambda \setminus D| < \delta$ by (10.15) and (10.17). Moreover, $\mathrm{inrad}(D) > 3\rho$ and $\mathrm{diam}(D) \le R$ by our choice of D and the definition of R. Moreover, by (10.19), $V_\lambda u$ is nonnegative on $[t_n - 8\tau, t_n] \times \overline{D}$, and by (10.19) and (10.20) we have

$$\|(V_\lambda u)^-(t_n - 8\tau)\|_{L^\infty(U\setminus\overline{D})} \le \mu r_1 \le \mu [V_\lambda u]_{L^1([t_n - 7\tau, t_n - 6\tau]\times D_*)}.$$

for each connected component D_* of D. An application of Theorem 9.13(ii) with these parameters therefore yields that (S_λ) holds for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$. The proof is finished. □

The following Proposition evidently completes the *Proof of Theorem 10.1*.

Proposition 10.8. *Suppose that (D2) holds or that $z \not\equiv 0$ on Ω for all $z \in \omega(u)$. Then we have:*

(i) $V_0 z \equiv 0$ *on* $\mathbb{R}^N$ *for every* $z \in \omega(u)$.

(ii) For every $z \in \omega(u)$ *we have the following alternative. Either* $z \equiv 0$ *on* Ω, *or* z *is strictly decreasing in* $|x_1|$ *and therefore strictly positive in* Ω.

Proof. (i) We define

$$\lambda_0 := \inf\{\mu > 0 : (S_\lambda) \text{ holds for all } \lambda > \mu\},$$

and we first claim that $\lambda_0 = 0$. By Lemma 10.4 we have $\lambda_0 < l$. If $z \not\equiv 0$ on Ω for all $z \in \omega(u)$, then Lemma 10.6 immediately implies that $\lambda_0 = 0$. If $(D2)$ holds and we assume – on the contrary – $\lambda_0 > 0$, then Lemma 10.5(iii) and Lemma 10.6(ii) readily imply that $z \equiv 0$ on $\mathbb{R}^N$ for every $z \in \omega(u)$, which then also yields $\lambda_0 = 0$. Hence we conclude in both cases that $\lambda_0 = 0$, and therefore $(S_0)_0$ is true by Lemma 10.5(i). This implies that $V_0 z \geq 0$ on Ω_0 for every $z \in \omega(u)$. Since the analogous statement can also be shown for the reflected solution $\tilde{u}$ defined in (10.10, we also have that $V_0 z \leq 0$ on Ω_0 for every $z \in \omega(u)$. Hence for every $z \in \omega(u)$ we have $V_0 z \equiv 0$ on Ω_0 and thus also on $\mathbb{R}^N$, since $z \equiv 0$ on $\mathbb{R}^N \setminus \Omega$.
(iii) Let $z \in \omega(u)$ be given such that z is not strictly decreasing in $|x_1|$. Then there exists $\lambda > 0$ such that $V_\lambda z$ is not strictly positive in Ω_λ. By Lemma 10.5(ii), applied to λ in place of λ_0, we then have that $V_\lambda z \equiv 0$ on $\mathbb{R}^N$. By (ii), z therefore has two different parallel symmetry hyperplanes. This implies that $z \equiv 0$, since z vanishes outside a bounded subset of $\mathbb{R}^N$. □

10.2 The unbounded case

In this part we consider the case $\Omega = \mathbb{R}^N$, i.e. we consider the problem

$$(R) \qquad \begin{cases} \partial_t u + Iu = f(t,u) & \text{in } (0,\infty) \times \mathbb{R}^N; \\ \lim\limits_{|x|\to\infty} u(t,x) = 0 & \text{for all } t \in (0,\infty) =: T \end{cases}$$

Here we assume that the kernel function J corresponding to the nonlocal operator I (see Remark 2.7 (ii)) satisfies $(J1)_{diff}$, (JL_s), (JU_s) for some $s \in (0,1)$ and

(J3) The function $J : \mathbb{R}^N \setminus \{0\} \to [0,\infty)$ is radially symmetric and strictly monotone, i.e. there is a strictly decreasing function $k : (0,\infty) \to [0,\infty)$, such that

$$J(z) = k(|z|) \qquad \text{for all } z \in \mathbb{R}^N \setminus \{0\}.$$

We also assume that the nonlinearity f fulfills the following properties.

(F1)' $f \in C^1([0,\infty) \times \mathbb{R})$ and for every $K > 0$ there is $L = L(K) > 0$ such that

$$\sup_{t \geq 0} |\partial_u f(t,u)| \leq L \qquad \text{for all } u \in [-K,K].$$

(F2)' For all $t \in T$ we have $f(t,0) = 0$, and there exists $\delta > 0$ and $c_f > 0$ such that

$$\sup_{t \geq 0} \partial_u f(t,u) \leq -c_f \quad \text{for all } u \in [-\delta, \delta].$$

Theorem 10.9. *Assume the kernel function J satisfies (J1)$_{diff}$, (J3), (JL$_s$), (JU$_s$) for some $s \in (0,1)$, and let f satisfy (F1)' and (F2)'. Let u be a nonnegative continuous global solution of (R) in the sense of Definition 6.1 and such that $u(t) \in L^2(\mathbb{R}^N)$ for all $t > 0$. Moreover, assume that u satisfies*

(U1) There is $c_u > 0$ such that $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq c_u$ for every $t > 0$.

(U2) u is eventually equicontinuous on $\mathbb{R}^N$ (see p. 64).

(U3) $\sup_{t \geq 0} u(t,x) \to 0$ as $|x| \to \infty$.

Then either $\lim_{t \to \infty} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = 0$ <u>or</u> all $z \in \omega(u)$[10] satisfy $z > 0$ in $\mathbb{R}^N$.
Suppose, in addition, that f and u are τ-periodic in t for some $\tau > 0$, i.e. we have $f(t+\tau,u) = f(t,u)$ for all $t \geq 0$, $u \in \mathbb{R}$ and $u(t+\tau,x) = u(t,x)$ for all $t > 0$, $x \in \mathbb{R}^N$. Then there is $z_0 \in \mathbb{R}^N$ such that $u(t,\cdot - z_0)$ is radially symmetric and strictly decreasing in the radial direction for all times $t \in (0,\infty)$.

We immediately deduce the following corollary for time independent problems in $\mathbb{R}^N$.

Corollary 10.10. *Assume the kernel function J satisfies (J1)$_{diff}$, (J3), (JL$_s$), (JU$_s$) for some $s \in (0,1)$, and let $f \in C^1(\mathbb{R})$ with $f(0) = 0$ and $f'(0) < 0$. Let $u \in \mathscr{D}^J(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ be a bounded nonnegative continuous solution of*

$$Iu = f(u) \qquad \text{in } \mathbb{R}^N.$$

Then there is $z_0 \in \mathbb{R}^N$ such that $u(\cdot - z_0)$ is radially symmetric. Moreover, either $u \equiv 0$ on $\mathbb{R}^N$ or $u(\cdot - z_0)$ is strictly decreasing in the radial direction.

Remark 10.11. Theorem 10.9 is inspired by a related result of Poláčik [55] for the case of second order equations with I replaced by Δ in (R) (see also the survey [57]). In the proof of Theorem 10.9, we will follow again the main lines of the moving plane method as developed by Poláčik in [55], but some steps in the argument differ because of the nonlocal structure of I. We point out that Theorem 10.9 applies in the case where I is the fractional Laplacian but also in the case of more general nonlocal operators.
We emphasize that up to the authors knowledge symmetry results even in the time independent case for nonlocal equations in $\mathbb{R}^N$ are only known for $I = (-\Delta)^s$, $s \in (0,1)$. Radial symmetry was proven for positive solutions of equations of type $(-\Delta)^s u + u = f(u)$ in $\mathbb{R}^N$ in [37], where

[10] Denote $C_0(\mathbb{R}^N) := \{u \in C(\mathbb{R}^N) \,:\, u(x) \to 0 \text{ as } |x| \to \infty\}$. Here $\omega(u)$ is the set of all $z \in C_0(\mathbb{R}^N)$ such that there exists a sequence $(t_k)_{k \in \mathbb{N}} \subset [0,\infty)$ with $t_k \to \infty$ and $\|u(t_k,\cdot) - z\|_\infty \to 0$ as $k \to \infty$. Note that (U1)–(U3) imply that $\{u(t) \,:\, t \geq 0\}$ is a relatively compact set in $C_0(\mathbb{R}^N)$ and hence $\omega(u)$ is a nonempty compact subset of $C_0(\mathbb{R}^N)$.

$f \in C^1(\mathbb{R})$ is nonnegative, $f(0) = f'(0) = 0$, $u \mapsto \frac{f(u)}{u}$ is increasing and there is $\tau > 0$ with $\lim\limits_{u\to 0^+} \frac{f'(u)}{u^\tau} = 0$. For equations of type $(-\Delta)^s u = u^{\frac{N+2s}{N-2s}}$ in $\mathbb{R}^N$ with $N > 2s$ radial symmetry was proven in [27]. We note that both approaches rely strongly on Green function. For a direct approach for classical positive solutions we refer to [38]. Finally, in [43] the authors prove uniqueness of radial symmetric solutions for equations of type $(-\Delta)^s u + u = |u|^\alpha u$ in $\mathbb{R}^N$ using an extension method. The approach we present here does not need Green functions or any related extension problem and is only based on the structure of nonlocal bilinear forms as introduced in Section 2.

Proof of Theorem 10.9

For this whole section assume that for some $s \in (0,1)$ the kernel function J satisfies $(\mathrm{J1})_{diff}$, (JL_s), (JU_s) and (J3). Moreover, let f satisfy (F1)' and (F2)', and let $\delta, c_f > 0$ be given as stated in (F2)'. Let u be a nonnegative nontrivial continuous solution of (R) which satisfies (U1) – (U3). We may assume without loss that u satisfies the equicontinuity property with $t_0 = 1$, i.e.

$$\lim_{h\to 0} \sup_{\substack{\tau\geq 1 \\ x,\tilde{x}\in\mathbb{R}^N, t,\tilde{t}\in[\tau,\tau+1], \\ |x-\tilde{x}|,|t-\tilde{t}|<h}} |u(t,x) - u(\tilde{t},\tilde{x})| = 0. \tag{10.21}$$

Lemma 10.12. *We have that* 0 *is a stable solution of* (R) *in the following sense: If there is* $\tau \in T$ *with* $\|u(\tau)\|_{L^\infty(\mathbb{R}^N)} < \delta$*, then* $\lim\limits_{t\to\infty} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = 0$.

Proof. Let $\tau \in T$ be as stated. Denote by $\xi \in C^1([\tau,\infty))$ the unique solution of the initial value problem $\xi' = f(t,\xi)$ in $[\tau,\infty)$, $\xi(\tau) = \delta$. By (F2)' we have that ξ is well defined on $[\tau,\infty)$ and that $\lim\limits_{t\to\infty} \xi(t) = 0$. Put $v(t,x) = \xi(t) - u(t,x)$ for $t \in [\tau,\infty)$, $x \in \mathbb{R}^N$. Note that $v^-(t) \in \mathscr{D}^J(\mathbb{R}^N)$ has compact support in $\mathbb{R}^N$ for all $t \geq \tau$ by (U3), and $v^-(\tau) \equiv 0$ on $\mathbb{R}^N$. Moreover, for $t \geq \tau$ we have

$$\begin{aligned}
\mathscr{J}(v(t),v^-(t)) = -\mathscr{J}(u(t),v^-(t)) &= \int_{\mathbb{R}^N} (\partial_t u(t,x) - f(t,u(t,x)))v^-(t,x)\,dx \\
&= -\int_{\mathbb{R}^N} \partial_t v(t,x)v^-(t,x)\,dx + \int_{\mathbb{R}^N} (f(t,\xi(t)) - f(t,u(t,x)))v^-(t,x)\,dx \\
&= \int_{\mathbb{R}^N} (c(t,x)v(t,x) - \partial_t v(t,x))v^-(t,x)\,dx,
\end{aligned} \tag{10.22}$$

with

$$c(t,x) = \int_0^1 \partial_u f(t,u(t,x) + \sigma(\xi(t) - u(t,x)))\,d\sigma.$$

Note that $c \in L^\infty([\tau,\infty)\times\mathbb{R}^N)$ by (F1)' and (U1). Denote $c_\infty := \|c\|_{L^\infty([\tau,\infty)\times\mathbb{R}^N)}$. Then for $t \geq \tau$ we have by Lemma 4.7

$$0 \leq \mathscr{J}(v^-(t),v^-(t)) \leq -\mathscr{J}(v(t),v^-(t)) = \int_{\mathbb{R}^N} c(t,x)[v^-(t,x)]^2\,dx - \frac{1}{2}\partial_t \int_{\mathbb{R}^N} [v^-(t,x)]^2\,dx$$

$$\leq c_\infty \|v^-(t)\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\partial_t \|v^-(t)\|^2_{L^2(\mathbb{R}^N)}.$$

Thus

$$\|v^-(t)\|^2_{L^2(\mathbb{R}^N)} \leq e^{2c_\infty(t-\tau)} \|v^-(\tau)\|^2_{L^2(\mathbb{R}^N)} = 0 \qquad \text{for } t \geq \tau.$$

We conclude that $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq \xi(t)$ for $t \geq \tau$ and thus $\lim\limits_{t\to\infty} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = 0$. □

As a consequence of Lemma 10.12 we have that either $\lim\limits_{t\to\infty} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = 0$ or that

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \geq \delta \quad \text{for all } t \in T, \tag{10.23}$$

where δ is given by Lemma 10.12. In the following, we may assume that (10.23) holds, which implies that $0 \notin \omega(u)$. Moreover, we have

Lemma 10.13. *Given any ball B, there is a constant $\kappa(B) > 0$ such that*

$$u(t,x) \geq \kappa(B), \qquad \textit{for all } t \in (2,\infty),\ x \in \overline{B}.$$

Proof. Let $B \subset \mathbb{R}^N$ be any ball. By (10.23) we have $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \geq \delta$ for all $t \in T$. Moreover, similarly as in the proof of Lemma 10.12 we have that u is a solution of the linear equation

$$\partial_t u(t,x) + Iu(t,x) = c(t,x)u(t,x),$$

where $c(t,x) = \int\limits_0^1 \partial_u f(t, su(t,x))\, ds$. By (U3) we may fix domains $D_1, D_2 \subset\subset \mathbb{R}^N$ with $D_1 \subset\subset D_2$ and

$$u(t,x) \leq \frac{\delta}{2} \quad \text{in } T \times (\mathbb{R}^N \setminus D_1).$$

Making D_1 and D_2 larger, we may assume without loss that $B \subset D_1$. Next let $(x_n)_n \subset \overline{D_1}$ be a sequence with $u(\frac{n}{2}, x_n) \geq \delta$ for all $n \in \mathbb{N}$. Note that by the equicontinuity property (see (10.21)) there is $h > 0$ such that

$$\inf_{\substack{x \in B_h(x_n)\\ t \in [\frac{n}{2}, \frac{n}{2}+h]}} u(t,x) \geq \frac{\delta}{2} \qquad \text{for } n \geq 2.$$

Thus by with the weak Harnack inequality (Corollary 7.3) there is $C = C(J, D_1, D_2, u, f) > 0$ such that for all $n \in \mathbb{N}$, $n \geq 2$ we have

$$\begin{aligned} \inf_{(\frac{n+2}{2}, \frac{n+3}{2}) \times B} u \geq \inf_{(\frac{n+2}{2}, \frac{n+3}{2}) \times D_1} u &\geq C[u]_{L^1((\frac{n}{2}, \frac{n+1}{2}) \times D_1)} \\ &\geq \frac{2C}{|D_1|} \int\limits_{[\frac{n}{2}, \frac{n}{2}+h]} \int\limits_{B_h(x_n)} u(t,x)\, dx\, dt \geq \frac{C\delta h}{|D_1|} |B_h(0)|. \end{aligned}$$

Thus the claim follows with $\kappa(B) = \frac{C\delta h}{|D_1|} |B_h(0)|$. □

To finish the proof of Theorem 10.9 we will fix some $e \in S^1$ arbitrary and apply the moving plane method with respect to reflections at $H_\lambda := \{x \cdot e > \lambda\}$, $\lambda \in \mathbb{R}$. Denote for $\lambda \in \mathbb{R}$: $T_\lambda := \partial H_\lambda$, $Q_\lambda : \mathbb{R}^N \to \mathbb{R}^N$, $x \mapsto x^\lambda$, the reflection at T_λ. For any function $z : \mathbb{R}^N \to \mathbb{R}$ define by $z_\lambda(x) := z(Q_\lambda(x))$ the reflected function. Furthermore we will denote $V_\lambda z = z_\lambda - z$, the difference between z reflected and the original z.
By reflecting problem (R) we will get that u_λ solves for any $\lambda \in \mathbb{R}$ again (cf. Lemma 9.3)

$$(R) \qquad \begin{cases} \partial_t u_\lambda + I u_\lambda = f(t, u_\lambda) & \text{in } T \times \mathbb{R}^N, \\ \lim\limits_{|x|\to\infty} \sup\limits_{t \in T} u(t, x^\lambda) = 0, & \end{cases}$$

Put $v(t,x) := V_\lambda u(t,x) = u(t,x^\lambda) - u(t,x)$, then v satisfies

$$(R') \qquad \begin{cases} \partial_t v + I v = d(t,x) v(t,x) & \text{in } T \times H_\lambda, \\ \lim\limits_{|x|\to\infty} \sup\limits_{t \in T} v(t,x) = 0 & \\ v(t,x) = -v(t,x^\lambda) & \text{for all } (t,x) \in T \times \mathbb{R}^N, \end{cases}$$

where

$$d(t,x) := \begin{cases} \dfrac{f(t,u_\lambda) - f(t,u)}{u_\lambda - u}, & u_\lambda \neq u \\ 0, & u_\lambda = u. \end{cases}$$

Note that by our assumptions on (F1)' and (U1) there is $d_1 \geq 0$ such that

$$\|d\|_{L^\infty(T \times H_\lambda)} \leq d_1 \quad \text{for all } \lambda \in \mathbb{R}.$$

Furthermore by (F2)' and since $\lim\limits_{|x|\to\infty} \sup\limits_{t \in T} u(t,x) = 0$, we can pick $\rho > 0$ large enough such that for any $\lambda \in \mathbb{R}$

$$d(t,x) \leq -c_f \text{ for all } t \in T \text{ and } x \in \mathbb{R}^N \text{ such that } |x| \geq \rho \text{ and } |x^\lambda| \geq \rho. \tag{10.24}$$

Denote

$$G_\lambda := B_\rho(0) \cup Q_\lambda(B_\rho(0)). \tag{10.25}$$

Note that for λ large enough we have that

$$H_\lambda \cap G_\lambda = Q_\lambda(B_\rho(0)).$$

We will follow closely [55, Section 3]. Consider the statement

$$(S)_\lambda \qquad V_\lambda z \geq 0 \text{ in } H_\lambda \text{ for all } z \in \omega(u).$$

We will show the following three steps to prove the statement.

Step 1 $(S)_\lambda$ holds for λ sufficiently large.

Step 2 Define $\lambda_\infty := \inf\{\mu \;:\; (S)_\lambda \text{ holds for all } \lambda \geq \mu\}$, and prove $\lambda_\infty > -\infty$ and $V_{\lambda_\infty} z \equiv 0$ on $\mathbb{R}^N$ for *some* $z \in \omega(u)$.

Step 3 In the case where u and f are time-periodic, prove $V_{\lambda_\infty} z \equiv 0$ on $\mathbb{R}^N$ for <u>all</u> $z \in \omega(u)$.

Note that Step 1 to Step 3 imply that for all $e \in S^1$ there is a hyperplane T^e perpendicular to e and such that all elements in $\omega(u)$ are symmetric with respect to T^e and monotone in direction e. In particular, considering hyperplanes T^{e_i} corresponding to the coordinate vectors e_i, we have that all elements in $\omega(u)$ are symmetric with respect to T^{e_i} for $i = 1, \dots, N$ and monotone in all coordinate directions. Consequently, all elements in $\omega(u)$ are also symmetric with respect to reflection at the unique intersection point z_0 of $T^{e_1}, \dots, T^{e_N}$. It is then easy to see that there is $z_0 \in T^e$ for all $e \in S^1$, and this implies that all elements in $\omega(u)$ are radial up to translation about the same point z_0.
Note that in the case of a time-periodic solution u with periodicity τ, we have $\omega(u) = \{u(t) : t \in [0, \tau)\}$.
To prove Step 1 to Step 3 we will need the following Lemma.

Lemma 10.14. *There is $\delta_1 > 0$ and independent of λ with the following property: If there is $t_1 > 0$ and a domain $D_0 \subset H_\lambda$ such that*

$$D_0 \supset G_\lambda \cap \overline{H}_{\lambda+\delta_1},$$

and such that $v = V_\lambda u$ satisfies

$$v(t,x) > 0 \quad \textit{for every } t \geq t_1,\ x \in D_0,$$

then $(S)_\lambda$ holds.

Proof. Note that by (J3) we have

$$\inf_{x \in H_\lambda \setminus H_{\lambda+1}} \int_{(\mathbb{R}^N \setminus H_\lambda) \setminus B_{r_0}(x)} J(y-x)\, dy = \inf_{x \in H_0 \setminus H_1} \int_{(\mathbb{R}^N \setminus H_1) \setminus B_{r_0}(x)} J(y-x)\, dy$$
$$\geq \int_{(\mathbb{R}^N \setminus H_1) \setminus B_{r_0}(0)} J(y)\, dy > 0.$$

Moreover, we have

$$\int_{\mathbb{R}^N \setminus B_{r_0}(x)} |y-x|^{-N-2s}\, dy \leq \frac{N\omega_N}{2s} r_0^{-2s}.$$

Denote

$$\gamma := \frac{2s r_0^{2s}}{c_1 N \omega_N} \int_{(\mathbb{R}^N \setminus H_1) \setminus B_{r_0}(0)} J(y)\, dy,$$

then

$$\inf_{x \in H_\lambda \setminus H_{\lambda+1}} \int_{(\mathbb{R}^N \setminus H_\lambda) \setminus B_{r_0}(x)} J(y-x) - \gamma c_1 |y-x|^{-N-2s}\, dy \geq 0. \tag{10.26}$$

Thus with (10.26) and $(\mathrm{JL})_s$ we have for $x \in H_\lambda \setminus H_{\lambda+1}$

$$\begin{aligned}\kappa_{J,H_\lambda}(x) &\geq c_1\gamma \int\limits_{\mathbb{R}^N\setminus H_\lambda} |x-y|^{-N-2s}\,dy + \int\limits_{(\mathbb{R}^N\setminus H_\lambda)\setminus B_{r_0}(x)} J(y-x) - \gamma c_1 |y-x|^{-N-2s}\,dy \\ &\geq c_1\gamma \int\limits_{\mathbb{R}^N\setminus H_\lambda} |x-y|^{-N-2s}\,dy = \tilde{\gamma}[\mathrm{dist}(x,\partial H_\lambda)]^{-2s},\end{aligned}$$

where $\tilde{\gamma}$ is up to the factor $c_1\gamma$ given in Lemma 5.1 b). Thus we may fix $\delta_1 \in (0,1]$ such that

$$\begin{aligned}\inf_{x\in H_\lambda\setminus H_{\lambda+\delta_1}} \kappa_{J,H_\lambda}(x) &\geq \inf_{x\in H_\lambda\setminus H_{\lambda+\delta_1}} \tilde{\gamma}[\mathrm{dist}(x,\partial H_\lambda)]^{-2s} \\ &\geq \inf_{x\in H_0\setminus H_{\delta_1}} \tilde{\gamma}[\mathrm{dist}(x,\partial H_0)]^{-2s} > c_f + d_1. \qquad (10.27)\end{aligned}$$

Next let t_1 and D_0 be such that v satisfies the stated assumptions for this δ_1. Let $\varepsilon > 0$ and denote $\varphi_\varepsilon(t,x) = (v+\varepsilon)^-(t,x)1_{H_\lambda}(x)$ for $t \in T$, $x \in \mathbb{R}^N$. Note that by Lemma 4.7 we have $\varphi_\varepsilon(t) \in \mathscr{V}^J(\mathbb{R}^N)$ for $t \in T$. Since moreover $\lim\limits_{|x|\to\infty} v(t,x) = 0$ for all $t \in T$, we have that $\varphi_\varepsilon(t)$ has compact support for every $t \in T$. Thus testing (R') with $\varphi_\varepsilon(t)$, $t \in T$ we get

$$\begin{aligned}\mathscr{J}(v(t),\varphi_\varepsilon(t)) &= \frac{1}{2}\partial_t\|\varphi_\varepsilon(t)\|^2_{L^2(H_\lambda)} + \int\limits_{H_\lambda} d(t,x)v(t,x)\varphi_\varepsilon(t,x)\,dx \\ &= \frac{1}{2}\partial_t\|\varphi_\varepsilon(t)\|^2_{L^2(H_\lambda)} - \int\limits_{H_\lambda} d(t,x)\varphi_\varepsilon^2(t,x)\,dx - \varepsilon\int\limits_{H_\lambda} d(t,x)\varphi_\varepsilon(t,x)\,dx. \qquad (10.28)\end{aligned}$$

Next denote $A := (H_\lambda \cap G_\lambda) \setminus \overline{H_{\lambda+\delta_1} \cup D_0}$, and note that

$$\begin{aligned}&\{(t,x) \in [t_1,\infty)\times A \;:\; \varphi_\varepsilon(t,x) > 0 \text{ and } d(t,x) > 0\} \\ &\qquad = \{(t,x) \in [t_1,\infty)\times H_\lambda \;:\; \varphi_\varepsilon(t,x) > 0 \text{ and } d(t,x) > 0\}.\end{aligned}$$

Note that $d(t,x) < -c_f$ for $t \geq t_1$ and $x \in H_\lambda \setminus (A \cup D_0)$. Thus we have for $t \geq t_1$

$$\begin{aligned}\partial_t\|\varphi_\varepsilon(t)\|^2_{L^2(H_\lambda)} &\leq -2c_f\int\limits_{H_\lambda} \varphi_\varepsilon^2(t,x)\,dx + 2(c_f+d_1)\int\limits_A \varphi_\varepsilon^2(t,x)\,dx \\ &\qquad + \varepsilon 2d_1 \int\limits_A \varphi_\varepsilon(t,x)\,dx + 2\mathscr{J}(v(t),\varphi_\varepsilon(t)). \qquad (10.29)\end{aligned}$$

To estimate $\mathscr{J}(v(t),\varphi_\varepsilon(t))$ we will use the inequality (4.10) in the proof of Lemma 4.7 to get

$$\begin{aligned}\mathscr{J}(v(t),\varphi_\varepsilon(t)) &\leq -\mathscr{J}(\varphi_\varepsilon(t),\varphi_\varepsilon(t)) - 2\varepsilon\int\limits_{H_\lambda}\int\limits_{H_\lambda} \varphi_\varepsilon(t,x)J(x-Q(y))\,dydx \\ &\leq -\int\limits_{H_\lambda} \varphi_\varepsilon^2(t,x)\kappa_{J,H_\lambda}(x)\,dx - 2\varepsilon\int\limits_{H_\lambda} \kappa_{J,H_\lambda}(x)\varphi_\varepsilon(t,x)\,dx\end{aligned}$$

$$\leq -\int_A \varphi_\varepsilon^2(t,x)\kappa_{J,H_\lambda}(x)\,dx - 2\varepsilon\int_A \kappa_{J,H_\lambda}(x)\varphi_\varepsilon(t,x)\,dx$$
$$\leq -(c_f+d_1)\int_A \varphi_\varepsilon^2(t,x)\,dx - \varepsilon d_1\int_A \varphi_\varepsilon(t,x)\,dx. \tag{10.30}$$

Here we used that $\mathrm{supp}(\varphi_\varepsilon(t)) \subset H_\lambda$ for all $t \in T$. Combining (10.29) and (10.30) we have

$$\partial_t\|\varphi_\varepsilon(t)\|^2_{L^2(H_\lambda)} \leq -2c_f\int_{H_\lambda} \varphi_\varepsilon^2(t,x)\,dx \quad \text{for all } t \geq t_1.$$

This implies that $\|\varphi_\varepsilon(t)\|^2_{L^2(H_\lambda)} \to 0$ for $t \to \infty$ for all $\varepsilon > 0$. Hence $(S)_\lambda$ holds. □

Step 1: Large λ

Lemma 10.15. *There exists $\lambda_1 \in \mathbb{R}$ such that $(S)_\lambda$ holds for all $\lambda > \lambda_1$*

Proof. Note that for λ sufficiently large we have

$$H_\lambda \cap G_\lambda = Q_\lambda(B_\rho(0)).$$

Let $k(B_\rho(0))$ be given by Lemma 10.13. Then since $\lim\limits_{|x|\to\infty} \sup\limits_{t\geq 0} u(t,x) = 0$, we have for λ possibly larger by (U3)

$$u(t,y) \leq \frac{k(B_\rho(0))}{2} \text{ for any } t > 1 \text{ and } y \in Q_\lambda(B_\rho(0)),$$

since $\inf\{|x| \,:\, x \in Q_\lambda(B_\rho(0))\} \to \infty$ as $\lambda \to \infty$. Take $\lambda_1 < \infty$ as the first value such that the above holds and note that thus for any $\lambda > \lambda_1$ we have

$$u(t,x) - u(t,x^\lambda) \geq \frac{k(B_\rho(0)}{2} \text{ for any } t > 1 \text{ and } x \in B_\rho(0),$$

which is equivalent to

$$V_\lambda u(t,x) = u(t,x^\lambda) - u(t,x) \geq \frac{k(B_\rho(0))}{2} \text{ for any } t > 1 \text{ and } x \in Q_\lambda(B_\rho(0)).$$

An application of Lemma 10.14 with $D_0 = Q_\lambda(B_\rho(0))$ gives that $(S)_\lambda$ holds for any $\lambda > \lambda_1$. □

Step 2: $\lambda = \lambda_\infty$
We will fix λ_1 given by Lemma 10.15 and let

$$\lambda_\infty = \inf\{\mu \,:\, (S)_\lambda \text{ holds for all } \lambda \geq \mu\},$$

be defined as above.

Lemma 10.16. *The following statements hold:*

(i) $-\infty < \lambda_\infty \leq \lambda_1$.

(ii) *For each* $z \in \omega(u)$ *and any* $\lambda \geq \lambda_\infty$ *we have either* $V_\lambda z > 0$ *on* H_λ *or* $V_\lambda z \equiv 0$ *on* $\mathbb{R}^N$. *In particular,* $(S)_{\lambda_\infty}$ *holds.*

(iii) *There is* $\hat{z} \in \omega(u)$ *such that* $V_{\lambda_\infty}\hat{z} \equiv 0$

Proof. (i) simply follows by Lemma 10.13, since for $\lambda \to -\infty$ we have $|x^\lambda| \to \infty$ for any fixed $x \in \mathbb{R}^N$. Thus $V_\lambda(u(t,x)) = u(t,x^\lambda) - u(t,x) < 0$ for fixed $t \in T$ using Lemma 10.13, the decay property of u and λ sufficiently negative. Thus $(S)_\lambda$ does not hold for these λ's. The upper bound of λ_∞ follows from Step 1.
(ii) This follows from Lemma 10.5 noting that the proof does not need any boundedness assumption on the underlying set.
To prove (iii) one has to notice that by the compactness of $\omega(u)$ in $C_0(\mathbb{R}^N)$ the statement follows if for any bounded domain $D \subset\subset H_{\lambda_\infty}$ there is $z \in \omega(u)$ such that $V_{\lambda_\infty} z \equiv 0$ on D. Assume by contradiction that for a given domain $D \subset\subset H_{\lambda_\infty}$ we have for some $b > 0$

$$\|V_{\lambda_\infty} z\|_{L^\infty(D)} \geq 2b \quad \text{for any } z \in \omega(u).$$

Thus there is $t_1 > 0$ such that

$$\|V_{\lambda_\infty} u(t)\|_{L^\infty(D)} \geq b \qquad \text{for all } t \geq t_1.$$

For $\lambda \in (\lambda_\infty - \delta_1, \lambda_\infty]$ where δ_1 is as in Lemma 10.14 we replace D with D_λ such that $D_\lambda \supset G_\lambda \cap H_{\lambda+\delta_1} \cup D$. Notice that we have by (ii)

$$\lim_{t\to\infty} \| \left(V_{\lambda_\infty} u\right)^- (t)\|_{L^\infty(H_{\lambda_\infty})} = 0$$

Let $D_1 \subset\subset H_{\lambda_\infty}$ be a domain with $D_\lambda \subset\subset D_1$. By the weak Harnack inequality for antisymmetric functions (Theorem 9.11) there is $t_2 \geq t_1$ and $\kappa > 0$ such that

$$\inf_{x\in\overline{D_\lambda}} V_{\lambda_\infty} u(t,x) \geq \kappa b \qquad \text{for all } t \geq t_2.$$

By the equicontinuity property (10.21) we can choose $\tilde{\kappa}$ and a $\tilde{\lambda} \in (\lambda_\infty - \delta_1, \lambda_\infty)$ sufficiently close to λ_∞ such that

$$\inf_{x\in\overline{D_\lambda}} V_\lambda u(t,x) \geq \tilde{\kappa} b \quad \text{for } t \geq t_2 \text{ and any } \lambda \in (\tilde{\lambda}, \lambda_\infty].$$

Applying Lemma 10.14 we get that $(S)_\lambda$ holds for all $\lambda \in (\tilde{\lambda}, \lambda_\infty]$ which contradicts the definition of λ_∞. Thus (iii) follows. □

Lemma 10.17. *Each* $z \in \omega(u)$ *is strictly decreasing in the direction of* e *on* $\mathbb{R}^N \setminus H_{\lambda_\infty}$ *in the sense that for each* $x \in \partial H_{\lambda_\infty}$ *and* $z \in \omega(u)$ *the map* $r \mapsto z(x+re)$, $r > 0$ *is strictly decreasing.*

Proof. We claim

$$V_\lambda z > 0 \qquad \text{in } H_\lambda \text{ for all } z \in \omega(u),\ \lambda > \lambda_\infty. \tag{10.31}$$

Assume this is false, then by Lemma 10.16 (ii) we have $V_\lambda z \equiv 0$ on $\mathbb{R}^N$ for some $z \in \omega(u)$, $\lambda > \lambda_\infty$ and also $V_{\lambda_\infty} z \geq 0$ in H_{λ_∞}. Denote $\mu := \lambda - \lambda_\infty > 0$ and $P = Q_\lambda \circ Q_{\lambda_\infty} : \mathbb{R}^N \to \mathbb{R}^N$. Then

$$P^k(H_{\lambda_\infty}) = H_{\lambda_\infty + (1+2k)\mu} \tag{10.32}$$

Fix $x \in H_{\lambda_\infty} \setminus H_{\lambda_\infty + \mu}$ with $z(x) > 0$ (this is possible by (10.23) and Lemma 10.13). Then we have for each $k \in \mathbb{N}$:

$$z(x) \leq z(P^k(x)) \qquad \text{for all } k \in \mathbb{N}. \tag{10.33}$$

To see this, not that for each $y \in H_{\lambda_\infty}$ we have

$$z(y) \leq z(Q_{\lambda_\infty}(y)) = z(Q_\lambda Q_{\lambda_\infty}(y)) = z(P(y))$$

Thus (10.33) holds for $k = 1$. Next assume (10.33) holds for some $k \in \mathbb{N}$. Then, since $P^k(x) \in H_{\lambda_\infty}$ by (10.32), we have $z(P^k(x)) \leq z(P^{k+1}(x))$ with the same argument as above. Thus (10.33) holds for any $k \in \mathbb{N}$.

Note that we thus have $\liminf_{k\to\infty} z(P^k(x)) \geq z(x) > 0$. However, since $|P^k(x)| \to \infty$ for $k \to \infty$ this is a contradiction to the fact that $z \in C_0(\mathbb{R}^N)$ and thus $\lim_{|x|\to\infty} z(x) = 0$.

Hence (10.31) holds. Note that from (10.31) the statement follows easily. □

Step 3: $\lambda < \lambda_\infty, \lambda \approx \lambda_\infty$

In this part we want to prove that actually $V_{\lambda_\infty} z \equiv 0$ for all $z \in \omega(u)$. The idea is as in [55]. We will start with the moving plane method from $-\infty$ and we will reach with the same steps as before.

Lemma 10.18. *There exists $\lambda_\infty^- \in (-\infty, \lambda_\infty)$ with the following properties*

(i) For each $z \in \omega(u)$ and $\lambda \leq \lambda_\infty^-$ we have either $V_\lambda z < 0$ on H_λ or $V_\lambda z \equiv 0$ on $\mathbb{R}^N$.

(ii) There is $\bar{z} \in \omega(u)$ such that $V_{\lambda_\infty^-} \bar{z} \equiv 0$.

(iii) Each $z \in \omega(u)$ is strictly decreasing in the direction of $-e$ on $\mathbb{R}^N \setminus H_{\lambda_\infty^-}$.

Remark 10.19. Note that by (10.23) and Lemma 10.13 we have $z > 0$ in $\mathbb{R}^N$ for all $z \in \omega(u)$.

Next assume for some $\tau > 0$ that

$$f \text{ and } u \text{ are } \tau\text{-periodic in } t. \tag{10.34}$$

Note that (10.34) implies $\omega(u) = \{u(t) \ : \ t \in [0, \tau)\}$. The proof of Theorem 10.9 is done after we have shown

Lemma 10.20. *If, in addition, (10.34) is satisfied, then $\lambda_\infty^- = \lambda_\infty$.*

Proof. Assume by contradiction $\lambda_\infty^- < \lambda_\infty$. Let $\hat{z}$ and $\bar{z} \in \omega(u)$ be as in Lemma 10.16 and Lemma 10.18. Then we have for each $\lambda \in (\lambda_\infty^-, \lambda_\infty)$ that

$$V_\lambda \hat{z} < 0 \text{ in } H_\lambda \tag{10.35}$$
$$V_\lambda \bar{z} > 0 \text{ in } H_\lambda. \tag{10.36}$$

This follows since $V_{\lambda_\infty}\hat{z} \equiv 0$ and $V_{\lambda_\infty^-}\bar{z} \equiv 0$ in combination with Lemma 10.17 and Lemma 10.18 (iii). Since u is τ-periodic in t, there is $\bar{t}, \hat{t} \in [0,\tau)$ such that $\bar{z} \equiv u(\bar{t})$ and $\hat{z} \equiv u(\hat{t})$ on $\mathbb{R}^N$. Without restriction we may assume $\hat{t} > \bar{t}$ (by considering the time interval $[\bar{t}, \bar{t}+\tau)$ instead of $[0,\tau)$). Next fix $\lambda = \max\left\{\lambda_\infty - \delta_1, \frac{\lambda_\infty^- + \lambda_\infty}{2}\right\}$, where $\delta_1 > 0$ is given by Lemma 10.14. Then the function $v(t) = V_\lambda u(t)$, $t \geq 0$ satisfies

$$v(\bar{t}) > 0 \quad \text{in } H_\lambda \text{ and } \quad v(\hat{t}) < 0 \quad \text{in } H_\lambda. \tag{10.37}$$

and moreover v is a bounded continuous antisymmetric supersolution of

$$\partial_t v + Iv = d(t,x)v \quad \text{in } [\bar{t},\hat{t}] \times H_\lambda, \qquad \lim_{|x|\to\infty} \sup_{t\in[\bar{t},t^*]} v(t,x) = 0.$$

Fix $D_1 \subset\subset H_\lambda$ with $G_\lambda \cap H_{\lambda_\infty} \subset D_1$. Note that by (10.37) there is $h \in (0, \hat{t}-\bar{t})$ and $x^* \in \overline{D_1}$ such that $v(\bar{t}+h, x^*) = 0$. Let h be minimal with this property so that $v > 0$ in $[\bar{t}, \bar{t}+h) \times \overline{D_1}$. Let $\varepsilon > 0$ and denote $\varphi_\varepsilon(t,x) = (v+\varepsilon)^-(t,x)1_{H_\lambda}(x)$ for $t \in T$, $x \in \mathbb{R}^N$. Then with the same arguments as in Lemma 10.14 we get

$$\partial_t \|\varphi_\varepsilon(t)\|^2_{L^2(H_\lambda)} \leq -2c_f \|\varphi_\varepsilon(t)\|^2_{L^2(H_\lambda)} \qquad \text{for } t \in [\bar{t}, \bar{t}+h].$$

Since $\varphi_\varepsilon(\bar{t}) \equiv 0$ on H_λ for every $\varepsilon > 0$ and $v \in L^2(\mathbb{R}^N)$ we conclude $v \geq 0$ on $[\bar{t}, \bar{t}+h] \times H_\lambda$. Next fix $D_2 \subset\subset H_\lambda$ with $D_1 \subset\subset D_2$. Then by Theorem 9.11 with h, D_1 and D_2 there are constants K_1 and K_2 such that

$$\inf_{\substack{t\in[\bar{t}+\frac{3h}{4}, \bar{t}+h]\\ x\in D_1}} v(t,x) \geq K_1 [v]_{L^1([\bar{t}+\frac{h}{4}, \bar{t}+\frac{h}{2}]\times D_1)} - K_2 \|v^-\|_{L^\infty([\bar{t}+\frac{h}{4}, \bar{t}+\frac{h}{2}]\times H_\lambda)} \geq K_1 [v]_{L^1([\bar{t}, \bar{t}+\theta]\times D_1)} > 0.$$

By the choice of $h \in (0, \hat{t}-\bar{t})$ we have reached a contradiction. Thus we must have $\lambda_\infty = \lambda_\infty^-$.
□

11 Appendix

Theorem 11.1. *Assume (J1), (JL_s) and let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Then for any $\lambda > 0$ there is a constant $C = C(N, J, \Omega, \lambda) > 0$ such that any solution $v \in \mathscr{D}^J(\Omega)$ of*

$$Iv = \lambda v \quad \text{in } \Omega \tag{11.1}$$

fulfills

$$\|v\|_{L^\infty(\Omega)} \leq C\|v\|_{L^2(\Omega)}.$$

Proof. Fix $\alpha = 2s$, $\lambda > 0$ and let v be the solution of (11.1). We will follow closely the proof of [67, Proposition 4], where the statement was proven for the fractional Laplacian (but as mentioned there can be applied to a more general setting). Since $v \in L^2(\Omega)$ let $\delta > 0$ be a constant to be chosen later. By scaling we may assume

$$\|v\|^2_{L^2(\Omega)} = \delta.$$

We have

$$(v(x) - v(y))(v^+(x) - v^+(y)) \geq (v^+(x) - v^+(y))^2.$$

Let $C_k := 1 - 2^{-k}$, $v_k := e - C_k$, $w_k := v_k^+$ and $U_k := \|w_k\|^2_{L^2(\Omega)}$ for any $k \in \mathbb{N}$. Since w_k is constructed via cut of functions, we have $w_k \in \mathscr{D}^J(\Omega)$ and also

$$\lim_{k\to\infty} w_k = (v-1)^+ \in \mathscr{D}^J(\Omega).$$

With Lebesgue's Theorem we get

$$\lim_{k\to\infty} U_k = \int_\Omega \left((v(x) - 1)^+\right)^2 \, dx.$$

Moreover, since $C_{k+1} > C_k$ for all $k \in \mathbb{N}$, we have $w_{k+1} \leq w_k$ in $\mathbb{R}^N$.
Define $A_k := C_{k+1}/(C_{k+1} - C_k) = 2^{k+1} - 1$ for any $k \in \mathbb{N}$. And as in the proof of [67, Proposition 4] we then claim

$$v < A_k w_k \quad \text{on } \{w_{k+1} > 0\}.$$

Indeed, let $x \in \{w_{k+1} > 0\}$, then $v(x) - C_{k+1} > 0$ and so by the properties of C_k we have $v(x) > C_{k+1} > C_k$. Hence $w_k(x) = v_k(x) = v(x) - C_k$ and

$$\begin{aligned} A_k w_k(x) = A_k(v(x) - C_k) &= \frac{C_{k+1}}{C_{k+1} - C_k} v(x) - \frac{C_k C_{k+1}}{C_{k+1} - C_k} \\ &= v(x) + \frac{C_k}{C_{k+1} - C_k} (v(x) - C_{k+1}) > v(x). \end{aligned}$$

finally we have $v_{k+1}(x) - v_{k+1}(y) = v(x) - v(y)$ for any $x, y \in \mathbb{R}^N$, which gives for any $k \in \mathbb{N}$

$$\mathscr{J}(w_{k+1}, w_{k+1}) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left(v^+_{k+1}(x) - v^+_{k+1}(y)\right)^2 J(x, y) \, dxdy$$

$$\begin{aligned}
&\le \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(v_{k+1}^+(x)-v_{k+1}^+(y)\right)\left(v_{k+1}(x)-v_{k+1}(y)\right)J(x,y)\,dxdy\\
&= \mathscr{J}(v,w_{k+1}) = \lambda\int_{\Omega} v(x)w_{k+1}(x)\,dx = \lambda \int_{\{w_k>0\}} v(x)w_{k+1}(x)\,dx\\
&\le \lambda A_k \int_{\{w_k>0\}} w_k w_{k+1}(x)\,dx \le \lambda A_k U_k \le \lambda 2^{k+1}U_k.
\end{aligned}$$

To finish the proof we claim

$$\{w_{k+1}>0\}\subset\{w_k>2^{-(k+1)}\}.$$

Indeed, let $x\in\{w_{k+1}>0\}$, then $v(x)-C_{k+1}>0$ and thus

$$v_k(x)=v(x)-C_k>C_{k+1}-C_k=2^{-(k+1)}.$$

Hence $w_k(x)=v_k>2^{-(k+1)}$ proving the claim. As a consequence of this claim we have

$$\begin{aligned}
U_k = \|w_k\|^2_{L^2(\Omega)} &\ge \int_{\{w_k>2^{-(k+1)}\}} w_k^2(x)\,dx \ge 2^{-2(k+1)}|\{w_k\ge 2^{-(k+1)}\}|\\
&\ge 2^{-2(k+1)}|\{w_{k+1}>0\}|.
\end{aligned}$$

Hölder's inequality (with exponents $2N/(N-\alpha)$ and N/α) and the fractional Sobolev inequality w.r.t. $\alpha/2=s$ (see e.g. [32, Chapter 6], using $\mathscr{D}^J(\Omega)\subset\mathscr{H}_0^s(\Omega)$ be Lemma 5.14) we have that there is $\tilde{c}=\tilde{c}(N,\alpha,c_1,r,\Omega)>0$ such that

$$\begin{aligned}
U_{k+1} &\le \left(\int_{\Omega}|w_{k+1}(x)|^{2N/(N-\alpha)}\,dx\right)^{(N-\alpha)/N} |\{w_{k+1}>0\}|^{\alpha/N}\\
&\le \tilde{c}\mathscr{J}(w_{k+1},w_{k+1})|\{w_{k+1}>0\}|^{\alpha/N}.
\end{aligned}$$

With the above calculations we thus have

$$\begin{aligned}
U_{k+1} &\le \left(\tilde{c}\lambda 2^{k+1}U_k\right)\left(2^{2(k+1)}U_k\right)^{\alpha/N} = \tilde{c}\lambda 2^{k(1+2\alpha/N)+1+2\alpha/N}U_k^{1+\alpha/N}\\
&= \left(\tilde{c}\lambda 2^{d_1(k+1)}\right)U_k^{d_2} \le \left(1+\tilde{c}\lambda 2^{d_1+1}\right)^k U_k^{d_2} = d_3^k U_k^{d_2},
\end{aligned}$$

where we set $d_1:=1+2\alpha/N$, $d_2:=1+\alpha/N$ and $d_3=d_3(N,\alpha,c_1,\lambda,\Omega,r)=1+\tilde{c}\lambda 2^{d_1+1}>1$. We will now choose $\delta>0$ w.r.t. d_3 and d_2: Let $\delta>0$ be so small, such that

$$\delta^{d_2-1}<\frac{1}{d_3^{1/(d_2-1)}}.$$

Fix also

$$\eta\in\left(\delta^{d_2-1},\frac{1}{d_3^{1/(d_2-1)}}\right).$$

Notice that $d_3 > 1$ and $d_2 > 1$ imply $\eta \in (0,1)$ and moreover

$$\delta^{d_2-1} \leq \eta \quad \text{and} \quad d_3\eta^{d_2-1} \leq 1.$$

To finish the proof we will show via induction that we have for all $k \in \mathbb{N}$

$$U_k \leq \delta\eta^k. \tag{11.2}$$

Note that $U_0 = \|v^+\|^2_{L^2(\Omega)} \leq \|v\|^2_{L^2(\Omega)} = \delta$. Let (11.2) hold for some $k \in \mathbb{N}$, then the calculations from above give

$$U_{k+1} \leq d_3^k U_k^{d_2} \leq d_3^k \left(\eta^k \delta\right)^{d_2} = \delta(d_3\eta^{\beta-1})\delta^{\beta-1}\eta^k \leq \delta\eta^{k+1}.$$

Since (11.2) holds for all k we have

$$0 = \lim_{k\to\infty} U_k = \int_\Omega \left((v(x)-1)^+\right)^2 \, dx.$$

This gives $v(x) \leq 1$. By replacing v with $-v$ we get $\|v\|_{L^\infty(\Omega)} \leq 1$. Thus we may choose the constant C as $1/\delta$. Note that thus C only depends on N, α, c_1, λ, Ω and r. □

12 List of assumptions

In the following we present a list of assumptions in order of appearing with the page reference. $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$ will always denote an open set.

(J1) $J : \mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,x) \,:\, x \in \mathbb{R}^N\} \to [0,\infty)$ is measurable and satisfies (see p. 1)

- $(\mathrm{J1})_a$ $\quad J(x,y) = J(y,x) \qquad$ for $x,y \in \mathbb{R}^N$, $x \neq y$
- $(\mathrm{J1})_b$ $\quad \sup\limits_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \min\left\{1, |x-y|^2\right\} J(x,y)\, dy < \infty$
- $(\mathrm{J1})_c$ There is a measurable function $j : \mathbb{R}^N \to [0,\infty)$ with $|\{j > 0\}| > 0$, $j(z) = j(-z)$, $z \in \mathbb{R}^N \setminus \{0\}$ and $J(x,y) \geq j(x-y)$ for $x,y \in \mathbb{R}^N$, $x \neq y$
- $(\mathrm{J1})_d$ The function j as in $(\mathrm{J1})_c$ satisfies additionally $\int_{\mathbb{R}^N} j(y)\, dy = \infty$

$(\mathrm{J1})_{diff}$ $J : \mathbb{R}^N \setminus \{0\} \to [0,\infty)$ is measurable and satisfies (see p. 1)

- $(\mathrm{J1})_a$ $\quad J(z) = J(-z)$ for $z \in \mathbb{R}^N \setminus \{0\}$
- $(\mathrm{J1})_b$ $\quad \int_{\mathbb{R}^N} \min\left\{1, |z|^2\right\} J(z)\, dz < \infty \qquad (\mathrm{J1})_d \quad \inf\limits_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} J(z)\, dz = \infty$

$(\mathrm{J}_+)_{r_0}$ The function j as in $(\mathrm{J1})_c$ satisfies additionally $\operatorname{essinf}\limits_{B_{r_0}(0)} j > 0$ w.r.t. some $r_0 > 0$ (see p. 2).

(J_+) The function j as in $(\mathrm{J1})_c$ satisfies additionally $\operatorname{essinf}\limits_{B_{r_0}(0)} j > 0$ for all $r > 0$ (see p. 2).

(D1) $\Omega \subset \mathbb{R}^N$ is an open bounded set which is Steiner symmetric in x_1, i.e. for every $x \in \Omega$ and $s \in [-1,1]$ we have $(sx_1, x_2, \ldots, x_N) \in \Omega$ (see p. 20).

(J2) The function J satisfies $(\mathrm{J1})_e$ and is strictly monotone in $|x_1|$, in the sense that for all $s,t \in \mathbb{R}$ with $|s| < |t|$ we have (see p. 20)

$$\operatorname*{essinf}_{z' \in B_r^{N-1}(0)} \big(J(s,z') - J(t,z')\big) > 0 \qquad \text{for all } r > 0.$$

(F) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$ is a Carathéodory function such that for every $K > 0$ there exists $L = L(K) > 0$ with (see p. 20)

$$\sup_{x \in \Omega} |f(x,u) - f(x,v)| \leq L|u - v| \quad \text{ for } u,v \in [-K,K].$$

(F_{symm}) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$ is symmetric in x_1 and monotone in $|x_1|$, i.e. for every $u \in \mathbb{R}$, $x \in \Omega$ and $s \in [-1,1]$ we have $f(sx_1, x_2, \ldots, x_N, u) \geq f(x,u)$ (see p. 20).

(J2)' The function $J:\mathbb{R}^N\setminus\{0\}\to[0,\infty)$ satisfies for all $z'\in\mathbb{R}^{N-1}$, $s,t\in\mathbb{R}$ with $|s|\le|t|$ that we have $J(s,z')\ge J(t,z')$. Moreover, there is $r_0>0$ such that (see p. 21)

$$\operatorname*{essinf}_{z'\in B_{r_0}^{N-1}(0)}\big(J(s,z')-J(t,z')\big)>0 \qquad \text{for all } s,t\in\mathbb{R} \text{ with } |s|<|t|\le r_0.$$

(JL_s) The measurable function $J:\mathbb{R}^N\times\mathbb{R}^N\setminus\{(x,x)\ :\ x\in\mathbb{R}^N\}\to[0,\infty)$ satisfies that there is $r_0>0$ and $k>0$ such that (see p. 44)

$$J(x,y)\ge k|x-y|^{-N-2s} \qquad \text{for a.e. } x,y\in\mathbb{R}^N \text{ with } x\ne y \text{ and } |x-y|\le r_0.$$

(JU_s) The measurable function $J:\mathbb{R}^N\times\mathbb{R}^N\setminus\{(x,x)\ :\ x\in\mathbb{R}^N\}\to[0,\infty)$ satisfies that there is $r_0>0$ and $k>0$ such that (see p. 44)

$$J(x,y)\le k|x-y|^{-N-2s} \qquad \text{for a.e. } x,y\in\mathbb{R}^N \text{ with } x\ne y \text{ and } |x-y|\le r_0.$$

(J_s) There is $r_0,k>0$ such that the map $\mathbb{R}^N\times\mathbb{R}^N\setminus\{(x,x)\ :\ x\in\mathbb{R}^N\}\to\mathbb{R}$, $(x,y)\mapsto J(x,y)-k|x-y|^{-N-2s}$ is bounded in $\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\ :\ |x-y|<r_0\}$.

(F1) $f:[0,\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$ is continuous. Moreover, for every $K>0$ there exists $L=L(K)>0$ such that $\sup\limits_{x\in\Omega,t>0}|f(t,x,u)-f(t,x,v)|\le L|u-v|$ for $u,v\in[-K,K]$ (see p. 63).

(U1) There is $c_u>0$ such that the function $u:(0,\infty)\times\mathbb{R}^N\to\mathbb{R}$ satisfies $\|u(t)\|_{L^\infty}\le c_u$ for every $t>0$ (see p. 63).

$(F2)_H$ $f:[0,\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$ is symmetric and monotone w.r.t. H, i.e. for every $t\in[0,\infty)$, $u\in\mathbb{R}$, $x\in H\cap\Omega$ we have $f(t,\bar{x},u)\ge f(t,x,u)$ (see p. 74). Here $H\subset\mathbb{R}^N$ is a half space and $\bar{x}$ denotes the reflection of x at ∂H.

(D2) For every $\lambda>0$, the set $\Omega_\lambda:=\{x\in\Omega\ :\ x_1>\lambda\}$ has at most finitely many connected components (see p. 88).

(F2) $f:[0,\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$ is symmetric in x_1 and monotone in $|x_1|$, i.e., for every $t\in(0,\infty)$, $u\in\mathbb{R}$, $x\in\Omega$ and $s\in[-1,1]$ we have $f(t,sx_1,x_2,\dots,x_N,u)\ge f(t,x,u)$ (see p. 88).

(U2) The function $u:(0,\infty)\times\mathbb{R}^N\to\mathbb{R}$ is eventually equicontinuous in Ω in the sense given on p. 64.

(F1)' $f\in C^1([0,\infty)\times\mathbb{R})$ and for every $K>0$ there is $L=L(K)>0$ such that

$$\sup_{t\ge0}|\partial_u f(t,u)|\le L \qquad \text{for all } u\in[-K,K] \text{ (see p. 98).}$$

(F2)' For all $t\in T$ we have $f(t,0)=0$, and there exists $\delta>0$ and $c_f>0$ such that

$$\sup_{t\ge0}\partial_u f(t,u)\le -c_f \quad \text{for all } u\in[0,\delta) \text{ (see p. 98).}$$

(J3) The function $J : \mathbb{R}^N \setminus \{0\} \to [0,\infty)$ is radially symmetric and strictly monotone, i.e. there is a strictly decreasing function $k : (0,\infty) \to [0,\infty)$, such that $J(z) = k(|z|)$ for all $z \in \mathbb{R}^N \setminus \{0\}$ (see p. 97).

(U3) The function $u : (0,\infty) \times \mathbb{R}^N \to \mathbb{R}$ satisfies $\sup_{t \geq 0} u(t,x) \to 0$ as $|x| \to \infty$.

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