European Music Portfolio

Maths

European Music Portfolio (EMP) – Maths: 'Sounding Ways into Mathematics'

Teacher's Handbook

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1 Introduction

Music and mathematics share an odd character: many people believe that they are not good at one or the other (or both). However, 'I cannot sing' or 'I never understood mathematics' will probably not keep them from having successful careers, and nor will it change the opinions others have about them.

The project 'European Music Portfolio – Sounding Ways into Mathematics' (EMP-Maths) aims towards a different understanding with regards to this character. Everyone can sing and make music, and everyone can do mathematics. Both topics are integral parts of our life and society. What needs to be improved is our ability to give students opportunities to like them.

Combining mathematics and music for classroom activity is not something new. As a matter of fact, the number of published examples is continuously increasing. It is unfortunate that many researchers have focused only on the use of music to increase mathematical, or general, knowledge, and even intelligence. Peter Hilton clarifies this point with regards to both mathematics and music:

> […] mathematics, like music, is worth doing for its own sake [...]. This is not to deny the great usefulness of mathematics; this very usefulness, however, tends to conceal and disguise the cultural aspect of mathematics. The role of music suffers no such distortion, for it is clearly an art whose exercise enriches composer, performer, and audience; music does not need to be justified by its contribution to some other aspect of human existence. Nobody asks, after listening to a Beethoven symphony, 'What is the use of that?' Moreover, mathematics does not gain in utility by having its inherent worth ignored – on the contrary, an appreciation of mathematics and an understanding of its inherent quality and dynamic are necessary in order to be able to apply it effectively (Gullberg, 1997, p. xvii).

EMP-Maths addresses music and mathematics teachers equally, as well as everyone who is interested in exploring the world of mathematics and music.

This handbook has three main parts. First, it elaborates on the interconnectedness of mathematics and music. Beginning with creative steps into the topic, it highlights pattern recognition as a core skill to both topics and finally embraces common myths about music being mathematical, and mathematics being musical, respectively.

The second part focuses on the basics of learning and goes deeper into the question of why music and mathematics should be taught together without the pitfall of utilising one for the sake of the other. Co-construction, perception and action, as much as making experiences, are keywords taken into account.

The third part, which is the core of the handbook, is a compilation of activities that can be used in the classroom. Many activities and suggestions are already available. We aim to encourage everyone to use them. The ones in this handbook highlight a number of fields in both mathematics and music in order to cover major domains: singing, dancing, listening, problem solving, numbers, measurement, and so on. With this approach, we want to connect the project to core topics in the curricula of the participating countries: Germany, Greece, Romania, Slovakia, Spain, Switzerland and the United Kingdom. All examples are built on the concept of didactic design patterns.

This teacher's handbook presents activities with different mathematical and musical content in order to offer teachers resources, ideas and examples. These activities are designed to be expandable, adaptable to different contexts, and adjustable to the needs of each teacher and their students. Furthermore, these activities are not just planned to be carried out individually; a teaching unit could be used to make sense of them, or they could even be developed in connection with each other.

Apart from this teacher's handbook, the project provides a continuing professional development (CPD) course, a webpage (http://maths.emportfolio.eu) from which all materials can be downloaded, and an online collaboration platform. A general overview of related literature and research is available in separate documents.[1] Additional teacher booklets provide related materials and a brief overview of the theoretical background, and are the basis for the CPD courses. The project 'Sounding Ways into Mathematics' is related to the EMP-Languages project 'A Creative Way into Languages' (http://emportfolio.eu/emp/).

[1] See also the connected 'Literature Review' (Hilton, Saunders, Henley, & Henriksson, 2015) and 'State of the Art Paper' (Saunders, Hilton, and Welch, 2015).

2 Sounding ways – the interconnectedness of music and mathematics

2.1 Creative steps for teachers and students

Learning and teaching that combine different disciplines often create new approaches to solving problems and give new insights into materials for all involved. Established paths can be abandoned, especially those that are tainted with negative emotions, to make way for new and better ones.

The combination of the two disciplines of music and mathematics in the EMP-Maths project provides content and methods from two areas of studies to enrich the teaching and learning process. From this point, new combinations can be created with a selection of examples. This handbook aims to guide teachers through this creative task and simplify common processes.

The combination of two academic topics demands creativity. We mean creativity in the following sense:

- Selections have to be made from the content and methods available for both disciplines. These methods and content should support and enhance the students' musical and mathematical development. To be creative, according to Poincaré (1948), is to find a new combination (cf. Hümmer et al., 2011, pp. 178–179) of the given content.
- With these new combinations, there is no approved standard practice to realise the lesson itself. To be creative, according to Ervynck (1991), is to find new ways that "deviate from established and expectable attempts" (Hümmer et al., 2011, p. 179).
- These newly developed ways would not be creative if they were not adaptable (Sternberg & Lubart, 2000). Here, to be creative is "the ability to present an unexpected and inventive result that is arguably adaptive" (Hümmer et al., 2011, p. 179).

These aspects of (mathematical and musical) creativity can be adapted, on the one hand, to the creative process of the development of activities in the EMP-Maths project and, on the other, to the actions and thinking of all teachers and students who participate in the activities.

In general, several of a multitude of topics can be selected to be the two connected disciplines. Every connection creates new paths through the combination of mathematics and music. Certainly, these paths are not standardised. Simultaneously, new adaptive accesses to mathematical and musical topics emerge. This aspect has been added to the description of variations in the context of the developed activities.

The activities themselves frame the creative process for all participants. New paths provide new experiences for those students who would otherwise be sceptical towards mathematical or musical activities, and help to reduce this scepticism. Furthermore, different approaches help to overcome existing difficulties and give the involved actors space to gain experiences in the two disciplines music and mathematics.

2.2 Pattern recognition and pattern production

Pattern recognition is a basic human activity that is bound to awareness. Pattern recognition is, first of all, paying *attention* to the connecting pattern (Bateson, 2002, p. 16). Some theories claim that attention is rhythmically organised (Auhagen, 2008, p. 444). Attention to, and awareness of, connecting mechanisms can be observed in children very frequently, and they often include expressions of happiness: rope skipping, jumping muddy puddles, and making rhythmic noises with sticks on fences are happy childhood activities. The human capability for rhythmic synchronisation, as well as pattern recognition, begins in early childhood and seems to be encouraged by dandling babies on the knees (Fischinger & Kopiez, 2008, p. 459).

Humans have the capacity to follow rhythmic patterns from the first. Experiments with newborn babies prove this very fact in that they are able to differentiate between rhythmic and non-rhythmic clicks (Gembris, 1998, pp. 403f.). Even early on, while floating in the womb of the mother, their leg movements show patterns of tempo, which are in time with the mother's heartbeat (Gruhn, 2005, p. 126). These early rhythmic musical abilities have in common that the baby is capable of recognising patterns and tuning into them, or, as Björn Merker has put it, can "entrain to a repetitive beat" (Merker, 2000, p. 59). Later, entrainment is obvious in countless activities, mostly through play; for example, with a ball in groups, in increasingly complex activities – such as when accompanying rhythmic language patterns and rhymes with movement – and in singing activities.

> "The analysis of patterns and the description of their regularities and properties is one of the aims of mathematics, which Alan H. Schoenfeld (1992, p. 334) characterizes as '… a living subject which seeks to understand patterns that permeate both the world around us and the mind within us.' Keith Devlin goes as far as to describe mathematics as the science of patterns: 'It was only within the last twenty years or so that a definition of mathematics emerged on which most mathematicians now agree: mathematics is the science of patterns.' (Devlin, 2003, p. 3)" (Vogel, 2005, p. 445)

Another important aspect of pattern recognition is classification or *chunking* (Jourdain, 2001, p. 163). Chunks are small packages of information that we can handle as one unit.[2] Chunks are treated hierarchically. From small chunks, bigger ones are created. From those, further and bigger chunks are built, and so on. As a matter of fact, we create patterns in order to chunk. Listening to a constant sequence of similar tones leads to building groups of two or three (Auhagen, 2008, p. 439), and therefore building (rhythmical) patterns. Alikeness, nearness and similar behaviour are all features that enable mental pattern recognition. Not only can we recognise patterns, but we also construct them and give meaning to them.

For example, the significance of chunks for the interaction with patterns (Vogel, 2005, p. 446) gets important during the exploration of geometrically patterns. „During the exploration it is important that the base elements or units of the phenomenon are found" (ibid.). Only the

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[2] See also the Teacher's Handbook for EMP-Languages, pp. 21–24: http://emportfolio.eu/emp/images/stories/materials/EMP_Teachers_Handbook_Final_2012.pdf

identification of these base units enable the mathematical analysis of complexes ornaments and clarify the fascination of mathematics.

Composers use this capacity in order to write polyphonic pieces for monophonic instruments. They group tones in a way that means our ear and mind 'hear' two or more different voices. Pattern recognition is an important task for hearing sounds (Bharucha & Mencl, W. Einar, 1996). Recognising the sounds of instruments and octave equivalence is a pattern recognition task, as is our ability to categorise tones C, D, E, F, G, A and B as a major scale and recognise the same melody when it is played in different keys. This shows "that pitch and key can serve to gate spectra into pitch-invariant representations" (Bharucha & Mencl, W. Einar, 1996, p. 149). Bharucha et al. suggest that "western listeners seem to have a highly elaborated representation of keys and their relationships" (ibid., p. 148). Several studies show that this is also important for sight-singing ability (Fine, Berry, & Rosner, 2006; Waters, Underwood, & Findlay, 1997). This is especially the case with the ability to predict following tones in sequences; this ability is better when these tones are part of tonal melodies or well-known patterns.

The need for pattern recognition and synchronisation is rooted in nature. Small animals that hunt bigger ones synchronise their steps in order to catch them (Fischinger & Kopiez, 2008, p. 460), and chimpanzees synchronise their voices in order to increase the distance at which they can be heard (Merker, 2000).

The childhood games mentioned above, as well as activities such as rope skipping, jumping muddy puddles and dancing, are occasions to practise coordination and pattern recognition (Spychiger, 2015a).

Pattern recognition and grouping enable us to do things simultaneously: marching, rowing, clapping and playing symphonies. Doing things together (and letting others know about it) enforces the group, attracts females and keeps enemies away; this is true at a recreation place's campfire as well as deep in the jungle, where chimpanzees do exactly the same thing (Merker, 2000). When things are done in complete synchronisation, they are louder and more effective.

Grouping is also an important technique that can be used to memorise numbers. Humans' short-term memory is (on average) able to hold up to seven items. If we had to memorise the number 1685175017561791, we could group it as 1685, 1750, 1756 and 1791, which are the years of the births and deaths of J.S. Bach and W.A Mozart, respectively. If we do not find such a convenient example, groups of two or three numbers (e.g. to memorise telephone numbers) work best. This connects to the domain of rhythm, in which we tend to group events in twos or threes. Grouping activities are some of the examples included in the teacher's handbook. Pattern recognition is at the core of the shared characteristics in mathematical and musical activities.

In all kinds of human activities, people show how they are not only capable of recognising patterns, but also of creating and producing them. This takes us to the semiotic function circle model, which offers the integration of those two aspects in human behaviour, perception and action, as explained in the next chapter (figure 1).

2.3 Music and mathematics are sign systems that overlap and interact

Pythagoras was one of the first people to describe sound and pitch as mathematical relations, based on the system of overtones. This musical enlightenment added intellectual, human value to music, which had previously been part of the divine world. Since then, music has been seen as "as an academic discipline". Musical practice was split into the two professional fields of the "*musicus* and *cantor* […] in the first millennium A.D." in order to separate the emotional elements of music from the intellectual ones (Spychiger, 1995, p. 54). While the intellectual part of music, the part that can be mathematically explained, was included in the *Quadrivium* through the field of musicology, the emotional aspects of music were not granted the same value (which luckily is no longer the case in our days).

This relationship between music and mathematics led to the idea that music can be used for the enhancement of mathematical knowledge, academic achievement and intelligence in general (Kelstrom, 1998). Research studies were conducted with the goal of proving such effects of musical learning (Costa-Giomi, 2004; Smolej Fritz & Peklaj, 2011). With the so called Mozart-effect (Rauscher, Shaw, & Ky, 1995), stating that spatial intelligence can be increased by listening to the music of W.A. Mozart, these relationships seemed to be confirmed (Hilton, Saunders, Henley, & Henriksson, 2015, p. 18). However, further studies revealed that all these results regarding the benefits of musical education were no stronger than those achieved by regularly engaging in sports (Simpkins, Vest, & Becnel, 2010). The idea of long-term transfer effects of musical learning on other domains has been widely dropped in the meantime.

Pythagoras and music

The earliest reference to describing music with mathematical symbols comes from Pythagoras (Henning, 2009; Weber, 1991), who discovered the physical principles underlying western music. He used the monochord to do the first experiments and discovered that the relation of overtones is constant and relative to the length of the string. Additionally, the relations 2:3:4:5 of the first four overtones are also fundamental in geometry and were used in Egyptian pyramids and tombs (Weber, 1991, pp. 19–20).

But, apart from that, the harmonic system is much more complicated and developing a modern scale just out of these principles is almost impossible (Hindemith, 1940). Only the intervals octave, fifth, fourth and third are fundamental parts of the overtone system. Everything else is either theoretically constructed or taken through the years by enculturation, but cannot be easily explained from within the overtone system.

Numeric symbols in the work of J.S. Bach and others

Another important music–mathematics discussion concerns the symbolic use of numbers in renaissance and baroque music (Achermann, 2003; Egeler-Wittmann, 2004; Stoll, 2001a). One predominant technique is the transition of names through numbers into compositions. Every letter was connected to its position in the alphabet (a=1, b=2, etc.) and names calculated to sums. The resulting numbers have been used to determine the numbers of bars and/or notes per section. J.S. Bach, for example, had a special connection to the number 14 (Buchborn, 2004). Another example of this technique, explained by Stoll (Stoll, 2001a), saw the names of the people who financially supported the composer incorporated into a piece of music.

Although this shows the use of numbers, relations and basic mathematics, it also indicates composers' desire to transmit secret messages or signatures, maybe known only to themselves. This is a desire that they satisfied by applying mathematical thinking during the composition process.

Numbers, rows and symmetries – contemporary music and the need for formal structure

After the abandonment of the harmonic system and its formal consequences at the beginning of the 20th century, composers began looking for new systems to give music a distinct formal structure. Serialism, therefore, first started to organise all musical parameters (length, dynamics and pitch) around the 12 semitones. Still, this did not lead to a solution for the formal structure of complete works. Early serial compositions (e.g. Messiaen's *Mode de valeurs et d'intensités*, 1949) therefore seem to begin and end without any reason; they could go on forever. Thus, his system of irreversible rhythms demonstrated the new need for symmetry in music; in the first instance, Messiaen did not find a solution himself.

Pierre Boulez, Luigi Nono and Karlheinz Stockhausen later developed systems that used predefined numeric rows as basic principles for their compositions. They used a method to calculate tables of numbers that not only defined aspects of single notes (time, pitch, dynamic), but also aspects of the formal structure of the whole piece of music (total length, time signatures, number of bars) (Decroupet, 1995; Henning, 2009; Lehmann, 2009; Stoll, 2001b). Very popular at that time were the Fibonacci numbers and the Golden Ratio, which are related to each other. With these techniques, composers wanted to restore higher order symmetries and rules into their compositions, which were lost with the waiving of the harmonic system and its inherent symmetries. Another approach is shown by Tom Johnson, who 'counts' music (Nimczik, 2002).

The connection between mathematics and music, in this case, was not because music is especially close to mathematics, but because musical parameters can be transformed and organised using mathematical techniques, and vice versa. The sign systems underlying music (notation) and mathematics (numbers) are, at some point, compatible. Mathematical relations and symmetries have been used to determine musical structure.

Semiotic theories in mathematics and music

Early semiotic theories act on the assumption of a direct sign-object-relation. Music, however, was not considered to be a sign system, first of all because it does not have an object to signify. Only through a more pragmatic and philosophical semiotic approach, as introduced by Charles S. Peirce, it was possible to resolve many of these problems (Spychiger, 2001, p. 55).

Alfred Lang (1993) developed a semiotic model based on how a person relates to the world. It looks at sign systems as the basis of human perception and action in an ongoing way, as shown in the semiotic function circle (figure 1).[3] This approach denies the need for a distinction between subject and object, and instead distinguishes between processes that "take place *within* the person and […] *outside* of the person" (Spychiger, 2001, p. 57), using the terms 'presentant' (in the place of the object) and 'interpretant' (in the place of the subject). Musical mental processes, then, take place in a circular way; a *musical perception* ('IntrO', what comes in) leads to a *musical experience* ('IntrA', what takes place within the person) that can evoke *musical production* ('ExtrO', what goes out from the person into the world). "These musical actions then manifest themselves outside of a person: as *musical culture*" (which is 'ExtrA', ibid., p. 58). This point closes the circle, which then creates new opportunities of musical perception again (as the arrows show in figure 1).

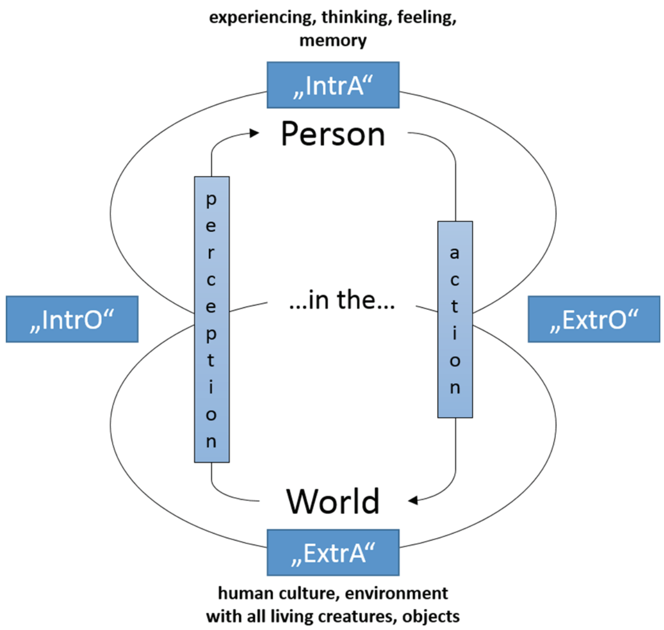


Figure 1: General psychological model of the person–world relationship. Semiotic function circle (according to Lang, 1993)

Understanding music as an independent sign system again makes it possible for us to compare this system with other systems, e.g. mathematical ones, without neglecting the independent reason for music. We can search for and find musical principles that can be explained mathematically. Music is full of symmetries, and notation is a system with mathematical accuracy.

The aggregation of multimodal sign systems (speech, music, numbers, visual signs, gestures and mimics) to a 'semiotic bundle' plays an important role within modern educational theories for teaching and learning (Arzarello, 2015; Gardner, 1983). With this approach, teaching processes and interaction in the classroom can be described much more precisely.

With music and mathematics being proper sign systems and the theory of semiotic bundles, interdisciplinary projects may come to a new meaning. Just as gestures and mimicry complement

[3] Summarised in Spychiger (2001, p. 56).

aural communication, mathematics can be used to explain music, and vice versa. To give it a simple name, we use the term *metaphor*, e.g. the mathematical principle of the lowest common multiple is a metaphoric description of polyrhythmic processes. Vice versa, overtones are metaphors for constant length ratios and fractions. Just like metaphors in lyrics, our metaphors do not represent the underlying principle exactly, but help us to understand the relationships and principles.

Although we do not believe that music is a mathematical system and vice versa, there are numerous connections between both worlds (Bamberger, 2010; Brüning, 2003; Christmann, 2011; Lorenz, 2003). With the concept of semiotic bundles, we want to develop creative learning environments (as shown in more detail in chapter 4.1) to bring together several semiotic systems.

3 Basics of learning

3.1 From task to construction

This chapter picks up two aspects of learning. The task symbolises the starting point of a learning process. Tasks can be characterised so "that they always refer to something which is missing" (Girmes, 2003, p. 6). In this way, the task will be turned into a source for learning, because the learners will need to close the identified gap. Of course, it is necessary to distinguish between "life-tasks" and "tasks in school" (cf. Girmes, 2003, p. 8). "Life-tasks" arise "in the meeting between human being and the world without somebody formulating a task for others ..." (ibid.). Tasks in school, so-called "learning tasks" (ibid., p. 10), are staged and designed professionally.

In the construction process of tasks, institutional framework conditions and the world view of the teacher become operative. The degrees of freedom of such tasks range from low to high. The degree of freedom refers to the leeway afforded to the learners while executing the task. If the procedures and the results are exactly defined, the leeway for the learners is very low. On the other hand, the degree of freedom of open tasks, which are embedded in learning environments, is often high. According to individual previous knowledge, the cognitive abilities, interests and motivations of the learners can be struck in different ways when processing the tasks. These different ways usually lead to different results, which lie in the range of possible results.

The concept of construction stands for the following learning process. This concept of learning emphasises the own activity of the individual person. The teacher gives suggestions, which are picked up by the learners to support the active and self-controlled knowledge construction. Additionally, including situational moments of concrete learning situations focuses on the importance of the interaction processes between learners and teachers (Gerstenmaier & Mandl, 1995; Greeno, 1989) in order to embrace institutional frameworks, socio-cultural and motivational, as well as the volitional prerequisites of the learners.

To deal with tasks in mathematics represents a central aspect of the common educational work of learners and teachers. In response to the diversity of the learners, tasks are currently arranged in a way that allows learners to choose different approaches, i.e. they can be processed on the respective level of learners and their mathematical or musical prerequisites. Often, after a phase of individual occupation with the task, the single approaches are discussed in larger groups. The activation of learners, in the sense of discovering mathematics or music, stands in the foreground.

Very often, the think-pair-share approach (cf. Barzel, Büchter, & Leuders, 2007, pp. 118–123) allows, in the first instance, an individual to analyse the task, uninfluenced by the ideas of other pupils. The pair phase is intended to be an exchange with a learning partner; the limited public nature of this phase offers space for unfinished thoughts. Only in the last step is the public classroom introduced. This is frequently carried out in the form of presentations, which are then discussed in the plenum. This way of dealing with tasks leads to individual knowledge constructions, which, in the pair and share phases, can be developed further discursively; this finally leads to co-constructive processes. The concept of co-construction refers to a shared knowledge construction achieved through social exchange (cf. Brandt & Höck, 2011).

Opposed to mathematical learning, musical learning often starts with group processes. In the group, characteristically musical learning through interaction is possible, e.g. "call and response" (Spychiger, 2015a, p. 57). Experiences with the efficiency of individual actions against the background of common actions are meaningful in music lessons. For example, an individual singing in a choir as part of a greater whole is able to achieve expression in common performances (Spychiger, 2015a, p. 53). Furthermore, imitation plays a significant role in the musical learning process, especially while teaching someone/learning to play an instrument.

Both learning processes, in mathematics as well as in music, are interweaving between the poles 'individual learning' and 'group learning' in a circular way to enhance problem solving skills. Altogether, mathematical and musical learning in a constructivist sense can be described as action-oriented, situated and social processes (cf. Reinmann-Rothmeier & Mandl, 2001; Spychiger, 2015a).

The learning tasks that are staged in the activities of the EMP project carry potential for construction and co-construction principles and take up the methodical approaches of mathematics and music.

3.2 Perception and action

Perception and action are central elements in the semiotic function circle, which describes person–world interaction: perception brings information into the person while, through the action, the person interacts with the world. Inside the person, perception creates knowledge and action creates culture in the world (see chapter 2.3, figure 1).

In music education, this unity was not always obvious. Especially well known is the discussion between Bennett Reimer and David Elliott (Spychiger, 1997). Bennett Reimer claimed that 'school music programs exist [only] to provide communities with a variety of social services' (Reimer, 1989, p. 24). As a consequence, and with the goal of facilitating aesthetic experience, he wanted to strengthen music perception within the curriculum.

David Elliott, on the other side, criticises the prevalence of classical music in the curriculum and corresponding teaching concepts, especially the lack of acceptance of affective elements (Elliott, 1987). Together with Christopher Small, Elliott supports music making – musicking – as a central element in the classroom (Elliott & Silverman, 2014; Small, 1998). With the semiotic function circle, Maria Spychiger shows the importance of both elements – action and perception – for music education (Spychiger, 1997), as has been shown in the general model for human life as a whole.

In the modern mathematics education, the interplay of perception and action also gets more and more important. Different levels of communication shall work together in semiotic bundles (Arzarello, 2015) and students use action and perception cycles to develop mathematical understanding.

A central element of mathematics is the *close look*. The identification of patterns and their translation into a sign system is a central mathematical task. Repetitions and, therefore, regularities can be found by observing the written symbols. These regularities are the basis for mathematical insights. During the lessons, students reconstruct this approach. Mathematical tasks serve as stimuli for activities performed on paper. With the analyses of these activities, based on perception, regularities are found and transformed into awareness.

The discovery of mathematical aspects in everyday phenomena works in the same way. A modelling process transfers central aspects of the real situation into a realistic model that contains the central structural elements of the real situation. This is the foundation of a mathematical model. Action forces children to discover mathematical regularities and structural principles in the classroom. Here, making mathematics would be the corresponding concept to musicking – making music. All the shown activities combine elements of action and perception to open minds and encourage emotions.

3.3 Making experiences

The activities developed in this project are meant to open the learning environments in which mathematical and musical experiences can be formed. Musical and mathematical content are merged. They should provide insights for both topics. Interdisciplinary learning environments frame this content differently and therefore allow experiences that are not possible in subject-oriented learning situations.

As John Dewey (1925; 1980/1934) understood it, "experience" is an "interactive, comprehensive event that contains not only cognitive but also affective, emotional and aesthetic components" (Neubert, 2008, pp. 234–235). We follow his approach in not putting cognition at the centre of learning, but instead "experience". First, before reflection and thinking, we are immersed in feelings, aesthetic perception, and current situational impressions (ibid., p. 235).

A sequence is taken intuitively out of the "stream of events" (Spychiger, 2015b, p. 111), and is turned into an experience from this emphasis. Real experience is a temporally limited unit with emotional quality, descriptive character and nameable content: "Those things, of which we say in recalling them, 'that was an experience' […] – a quarrel with one who was once an intimate, a catastrophe finally averted by a heir's breath […], that meal in a Paris restaurant […]" (Dewey, 1980/1934, p. 37). According to Dewey, experiences are additionally marked by their communicative character. Through interaction, people can participate in the experiences of others and potentially gain other perspectives on their own experiences (cf. Neubert, 2008, p. 238).

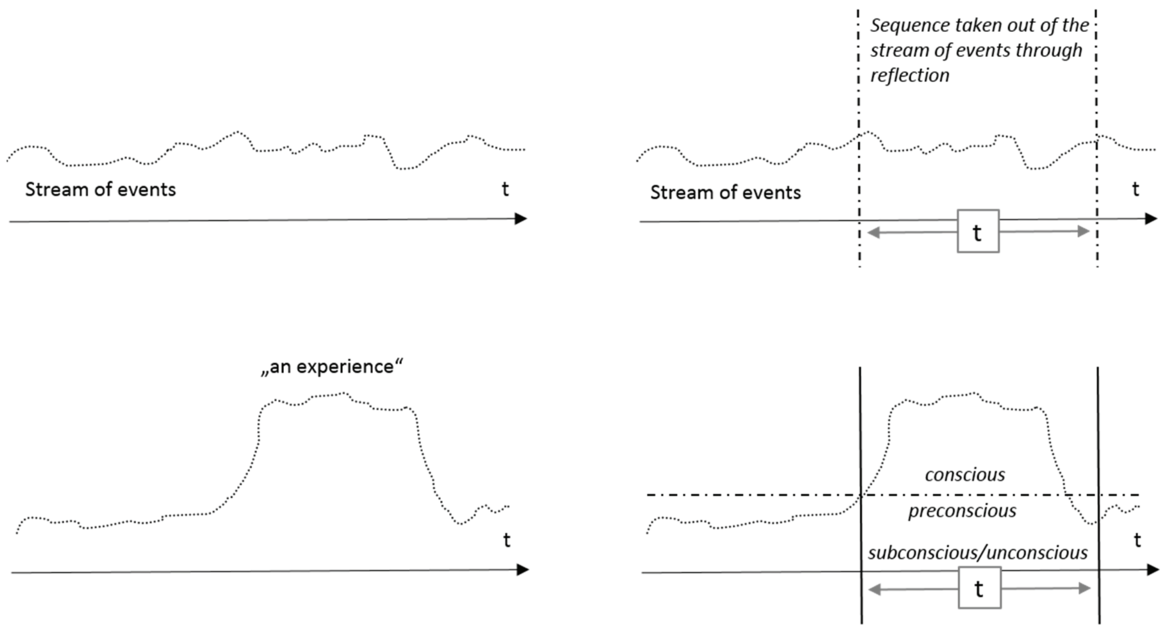


Figure 2: Experience (Spychiger, 2015b, p. 112)

Against the background of Dewey, the EMP-Maths activities provide learning environments where experiences are possible.

The EMP-Maths activities are developed to provoke entangled mathematical and musical events (see figure 2). The situational base for the participants to have new experiences with mathematics and music, or both, is created through the focus on a selection of singular events, for example through reflection and group discussions. This approach can help to change the unconscious image of mathematics and music through experiences in the EMP-Maths activities.

4 Educational aspects and structure of examples

4.1 Teaching and learning environments

The terms 'teaching' and 'learning environment' were developed at a time when alternatives to teacher-centred education were being developed. The search for new forms of teaching and learning is often bound to a change of attitude towards learning itself. Today, constructivist approaches shape our understanding of learning. The dominant idea of learning is that it is a process of situational construction of knowledge, which is embedded into context and culture (Greeno, 1989). Furthermore, it is assumed that learning is constructed between the learner and the teacher (Krummheuer, 2007, p. 62).

Learning within learning environments, which is seen as a construction of knowledge, is based on design principles. These principles find their expression in different constructivist instructional approaches. Examples of these approaches are the anchored instruction approach, the cognitive flexibility approach, and the cognitive apprenticeship approach. These approaches, which are from the 1990s, have one aspect in common: teachers design a 'learning-room' in which the learners are practically introduced to professional thinking and acting. These types of teaching and learning environments can be characterised in the following manner: "A learning environment is a place where people can draw upon resources to make sense of things and construct meaningful solutions to problems" (Wilson, 1996, p. 3). The definition for this type of constructivist learning environment is, according to Wilson (1996, p. 5):

> ... a place where learners may work together and support each other as they use a variety of tools and information resources in their guided permit of learning goals and problem-solving activities.

This definition shows clearly that teaching and learning environments create spaces for the learner and, at the same time, are designed by the teacher. Thus, the learning in these environments is still institutionalised, as it is planned ahead and specifically designed, but it generates creative spaces for the leaner to make contact with the material on their own.

> Thinking of instruction as an environment gives emphasis to the 'place' or 'space' where learning occurs. At a minimum, a learning environment contains the learner, a 'setting' or 'space' wherein the learner acts, using tools and devices, collecting and interpreting information, interacting perhaps with others, etc. (Wilson, 1996, p. 4).

Currently, the term 'learning-environment' often comes up together with the term 'to differentiate', especially in combination with 'natural differentiation'. It is important that the students/learners find their own ways to learn, their own pace of learning, and their own way of creating individual revelations. Lately, the term co-construction appears to be more and more important. With the term co-construction, the "individual design achievement" receives a "cultural character" (Brandt & Höck, 2011, p. 249).

In the field of mathematics, this is called a "substantial learning environment", which possesses the following attributes:

> mathematical substance with visible structures and patterns (professional framework); orientation towards central aspects; high cognitive activation potential; activity orientated towards mathematical contents and processes; initiation of independency of all learners; encouragement of individual ways of thinking and learning as well as the learners own form of presentation; access for everybody: mathematical activity should be possible on a basic level, use of the ability to connect prior knowledge; challenges for fast learners with demanding problems; facilitation of social exchange and mathematical communication (Hirt & Wälti, 2008, p. 14; translation by Peter Ludes).

These characterisations of learning environments can be transferred to the activities of the EMP-Maths project. They offer very high cognitive activation potential, which can be intensified by the physical experience. The focus stays clearly on the students' own activity. Mutual activity and experiences create rooms of discovery for learners, which integrate the individual learning progress with the interconnection of mathematics and music. In such rooms, which are open for pupils' ideas, new learning environments can be created. As Cslovjecsek and Linneweber (2011) show, learners become substantial collaborators in the process of teaching and learning.

4.2 The role of materials and space

Materials are assigned to many different mathematical learning processes. Materials serve as tools for the imagination, initiating thinking processes and making them explicit (cf. Hülswitt, 2003, p. 24). Materials visualise mathematical thoughts and help the learning process. The structures of mathematical objects, e.g. numbers, are materialised. Mental images can be built up by the activities with this mathematical material (Lorenz, 1993). Beside the concept of mental images, the interaction between learner and material can be focused (Fetzer, 2015). Musical learning is accompanied by sound and musical instruments, as well as visual elements and rhythms. In this way, the musical materials serve as part of the musical production.

According to Vygotsky, a material has the function of a mediator:

> Higher mental functions exist for some time in a distributed or 'shared' form, when learners and their mentors use new cultural tools jointly in the context of solving some tasks. After acquiring (in Vygotsky's terminology 'appropriate in') a variety of cultural tools, children become capable of using higher mental functions independently (Bodrova, E. & Leong, D.J., 2001, p. 9).

Materials, and especially the instructed actions associated with the materials, represent technical language, approaches and thinking in a functional, topic-related culture. Materials can, therefore, grant access to this topic-related world. At the same time, materials offer the opportunity to include the world of the learners (Vogel, 2014). Materials take the mediating function in mathematical learning, as well as in musical learning. Early education often starts with the playing materials of children (toys). Functions are assigned to this playing material in the process of mathematical or musical learning. A set of objects is turned into a representation of a number, the allocations of the table settings are regarded as functional relations, and the saucepan or the mug becomes a sound instrument.

Including the space in the creation of learning environments allows for consideration of the human body as a third dimension. The person experiences him/herself in the interaction with the surrounding space. The motion sequences and movements of the body can be interpreted mathematically (Vogel, 2008). The movements of the body, like clapping, can be means of musical production.

4.3 Structure of the examples

This teacher's handbook includes six examples to give an impression of the possibilities of combining mathematics and music in the classroom. The given structure follows a didactic design pattern. Design patterns were first developed by Alexander et al. (1977), and were then "adopted for the area of teaching and learning" (Vogel, 2014, p. 232). Design patterns describe repeating problems and provide generalised solutions for them (Vogel & Wippermann, 2011). This is realised through a formal structure to describe (didactic) situations (patterns) in an open yet standardised way. Examples have to go through several revisions before they reach their final state.

The following examples, presented in chapter five, are all structured in four main parts, of which the third, *Implementation*, describes the content of the activity.

Overview	Preparatory deliberations	Implementation	Variations
• Title • Topic • Keywords • Short description • Assignment to the collection of subjects of music and maths	• Prerequisites in maths • Prerequisites in music • Connections between math and musc	• Aims • Target group • Timescale • Standard approach • Materials, pictures, music	• Variations • Further approaches in music • Further approaches in maths

Figure 3: Continuous structure through all examples in section 5

Part I: Overview

This section provides general information on every example to make it easy to find appropriate activities for every purpose. The given keywords and a short description provide a quick insight into the activity. As many of the examples build on simple ideas, advanced classroom teachers may work with this overview and a brief look into section three. But never mind taking a look into the variations in any case, as we think that this is the most important part for further developments.

Connected with this handbook is a list of 'Key skills and core features' for both mathematics and music. Every activity is connected to this collection of subjects, as they are given in the various official documents of all partner countries.

Part II: Preparatory deliberations

The preparatory deliberations make sure that pupils have all the knowledge and skills needed for this activity. Some of them may be more important than others, but the activities are supposed to be fun and should be easily handled by the pupils without major difficulties halfway. Please make sure to take a close look at this section.

Part III: Implementation

The third section gives short instructions on how the activity *could* be implemented in school. The given standard approach provides a guide on how to start. It follows the idea of having a cheat slip.[4] It is no more than a brief introduction, and cannot replace a proper preparation of lessons and topics. In addition, the aims, target group and expected timescale give more detailed information that can be used to prepare the activity.

Part IV: Variations

The variations not only show different approaches to the given activity, but much more than that they want to be an eye opener into the world of transversal learning on the given topic of the activity. The activities given in this teacher's handbook are short and easy on purpose. Every activity can be seen as a gateway into a new universe of ideas.

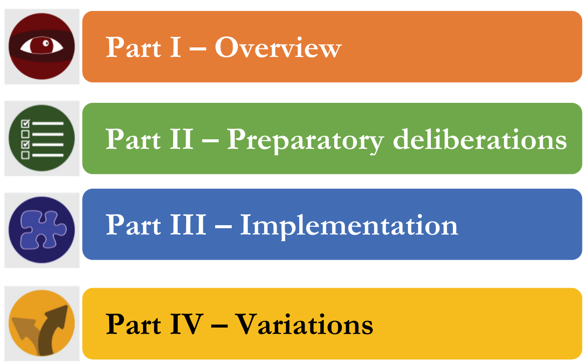
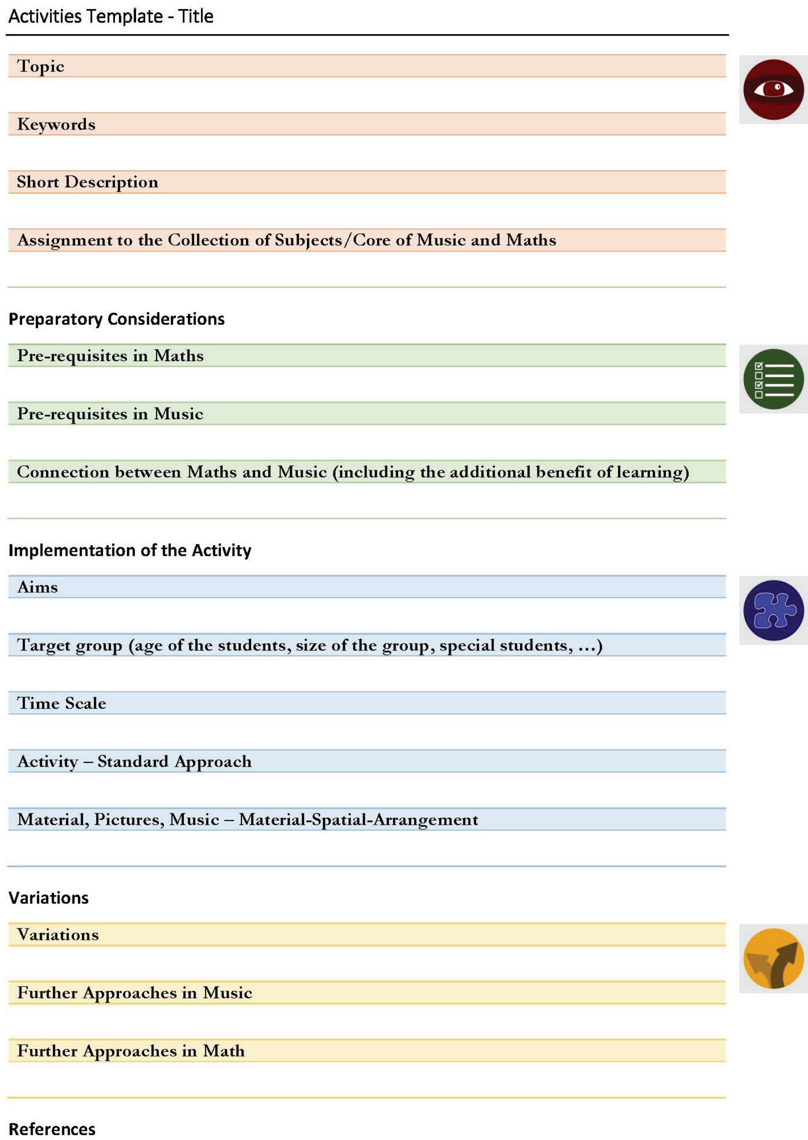


Figure 4: Structure of examples with icons

The examples presented in chapter 5 are displayed in the template shown in figure 5. The template uses icons for quick orientation within the parts: Part I, the overview, comes with an *eye*. Part II, the preparatory description of the prerequisites in mathematics and in music, uses the image of a *writing pad*. This part also collects background ideas with regards to the connection between mathematics and music, and is the most intellectual part of the presentation. The icon for part III shows a *puzzle piece*. This means that this activity – with its aims and characteristics – is a concrete contribution to the overall idea in the background of this approach to learning: the sounding ways into mathematics, or the mathematical ways into sound. The icon for part

[4] Liebetrau (2004, p. 9).

IV, finally, shows two *arrows* going in different directions. Under this paragraph, variations of the activity are given, so teachers have more than just one way to carry it out, and may, hopefully, be encouraged to find further ways by themselves.



Activities Template - Title

Topic

Keywords

Short Description

Assignment to the Collection of Subjects/Core of Music and Maths

Preparatory Considerations

Pre-requisites in Maths

Pre-requisites in Music

Connection between Maths and Music (including the additional benefit of learning)

Implementation of the Activity

Aims

Target group (age of the students, size of the group, special students, …)

Time Scale

Activity – Standard Approach

Material, Pictures, Music – Material-Spatial-Arrangement

Variations

Variations

Further Approaches in Music

Further Approaches in Math

References

Figure 5: Template of the examples

5 Examples

5.1 Sounding ways around school

Topic

'Sounding ways around school' is about soundscapes, their relations and their possible representations.

Keywords

Soundscapes (acoustic environment), listening, timelines, relations

Short description

In this activity, learners will listen to sounds from the school environment, allocate them to a timeline, and explore soundscapes on their own.

Assignment to the collection of subjects/the core of music and maths

Music: Appreciation of music and aural awareness through listening; differentiated perception of sounds; ability to describe sounds and noises according to various aspects; recognising the volatility of sounds and noises; graphic notation

Maths: Geometry (length, transformation); measurement (length); numbers (estimation and comparison); spatial orientation; temporal orientation; orders; relations (and/or, before, after, simultaneous, etc.); and the set theory

Preparatory considerations

Prerequisites in maths

Basic skills in spatial orientation and estimation of time and distances

Preprequisites in music

Basic skills in aural awareness of surrounding sounds

Connection between maths and music (including the additional benefits of learning)

Listening to a soundwalk and recognising the recorded sounds connects spatial orientation and the estimation of time and distances with aural awareness of the sounds of the environment.

The allocation of a sound/event to a certain time is related to distance/time allocation in maths. Making clusters according to different criteria (distance, source, duration, intensity) leads to some aspects of set theory.

Implementation of the activity

Aims

Improve the learner's listening skills and ability to describe sounds. Develop an understanding that sounds are often momentary and that perceptions and memories of sounds are subjective. Make proper use of the timeline, and group sounds in sets according to different criteria. Find orders (closest to furthest, loudest to softest, first to last, etc.).

Target group (age of the students, size of the group, special students, etc.)

Ages: 6–11 years (+), up to 30 students. Discussions can also happen in smaller groups.

Timescale

30 minutes for the standard approach

Activity – Standard approach

Preparation*:* The teacher records the noises of a soundwalk (for a definition, see the resources) in the surroundings of the school. (When wearing 'loud' shoes, floors and rooms will sound in combination with the many other sounds and noises of the surroundings.)

1. In the class, the students listen carefully to the recording. As well as listening, they write down or draw what they think they hear on the recording.
2. Collect answers on cards and discuss them with the class. Sort them in different ways (source, shape, distance, loudness, etc.) by making clusters and bringing them into relation with each other.
3. Allocate the sounds with the learners on a timeline, represented on the board or on the floor, with a line or on a string and clothes pegs. The discussion can start with the order of the sounds, and later on there can be a discussion of how much time there is between the different events.
4. Do the same soundwalk with the learners (this can also be done on another day).

Materials, pictures, music – Material spatial arrangement

Own recording, preferably of a soundwalk around the school (we strongly suggest that this walk is no longer than two minutes)

Recording devices (mobile phone apps, audio recorders, etc.) ('soundOscope' is our recommended app for mobile devices)

Variations

Variations

Make another recording (or have one made by the learners) and compare it to the first soundwalk. What is new, what is the same, and what has changed? Try to fit the new sounds and noises into the first timeline.

Groups of learners create/install a soundwalk near the school and expand and explore it according to the standard approach (e.g. under other time and weather conditions).

In higher grades, GPS navigation devices that track and later display a route (e.g. on an online map like Google Maps) can be used.

Share your soundwalk(s) with classes from other schools.

Further approaches in music

Combine the various sounds with a musical score and play them with instruments. Use individual sounds as samples to create a rhythm.

Invent a notation in order to describe sounds. Invent different, suitable signs for different sounds and their development.

Using the recorder, typical sounds can be recorded. Who knows the places/sounds near the school, in the neighbourhood, and in the city? Out of the material, it is then possible to draw up a quiz or an orientation game, possibly with the participation of other classes and/or parents.

Further approaches in maths

Learners draw maps of the soundwalks and compare them.

Prepare a map and divide it into clusters or spots that are connected by pathways. Learners then try to find a path on the map that allows them to cross every pathway just once. Alternatively, they find the shortest way to cross every spot on the school grounds. Afterwards, they make a recording of this path.

Measure the distances at which the fountain, the street or the school bell can (still) be heard under different conditions (weather, noise, time).

Collect and identify the sounds of a specific soundwalk over a longer period, and cluster them into sets. Some of them will be totally different, while others might overlap, i.e. a frequently used motorway and trucks are human inventions, whereas a stream is natural. Both the stream and the motorway noise are continuous if you don't move.

Record the same soundwalks while varying the pace with loud shoe soles. Stop, and then walk back.

References

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Examples of soundwalks on YouTube

5.2 Jump the rhythm: multiplicative relationships and metre

Topic

This activity uses physical embodiment, timbre and metre to encourage children to use pattern and rhythm to develop a deeper understanding of multiplicative relationships.

Keywords

Metre, rhythm, multiplicative relationships

Short description

By counting metre aloud in a circle, combined with elements of body percussion, children further develop their understanding of multiplicative relationships. Both musical metre and multiplicative relationships will be emphasised in this activity.

Assignment to the collection of subjects/the core of music and maths

Music: Pulse, metre and rhythm; practical music making

Maths: Reason mathematically and make connections; communicate mathematical ideas; numerical relationships – multiplication, estimation

Preparatory considerations

Prerequisites in maths

Addition, multiplication, patterns

Prerequisites in music

Physical coordination (clapping/stamping), pulse

Connection between maths and music (including the additional benefits of learning)

Multiplicative relationships and musical metre

Implementation of the activity

Aims

Children's understanding of multiplicative relationships and musical metre are developed through group focus and embodiment.

Target group (age of the students, size of the group, special students, etc.)

Ages: 7+ years, whole-class activity

Timescale

20+ minutes

Activity – Standard approach

- Stand in a circle with the children. Explain that each child will say only one number from 1 to 4 as you go round the class. Now, start with the child on your left and go round the class, counting 1, 2, 3, 4; continue until every child in the circle has said a number (1, 2, 3, 4, 1, 2, 3, 4, etc.). Repeat this process and get a rhythm going.
- Once the children have got the hang of this, you are going to add some body percussion. Ask the children who are number 1 to clap upon hearing their number, and ask the children who are number 4 to stamp. When you go round the class, do you end on number 4? Can the children explain why this happens?
- It is most likely that the round did not end on 4. If this is the case, ask the children to predict how many times they would have to go round in order to end on a 4? Try it and see what happens.
- Now, try the same activity with different numbers (e.g. 1, 2, 3, 4, 5 or 1, 2, 3). It is important that the children are encouraged to predict what will happen and why before trying out the activity. Were they right?
- What do the children notice about the different metres? Are there some metres that they prefer? Why is this?

Materials, pictures, music – Material spatial arrangement

Resources: No additional resources are required.

Other considerations: This activity should be carried out in a room where the children have space to stand in a circle and then work in pairs.

Variations

Variations

- You could ask the children to add in other elements of body percussion (e.g. slap, click) on numbers that are in between the first and last numbers.
- The class could be divided into two or more groups and the activity could then be repeated with each group. What do they notice this time? Was it harder or easier?
- Based on the first activity, child 2 and child 3 remain silent, but the group still need to keep time, so that the only sounds are for beats 1 and 4.

Further approaches in music

- Children could create their own element of body percussion.
- Children could use instruments instead of numbers and body percussion.
- For more of a challenge, children could include rests in their performances.

Further approaches in maths

- Activities based on multiples, factors, lowest common multiples and highest common factors
- Activities involving patterns and sequences
- Developing ideas of permutations and combinations

5.3 Clapping the lowest common multiple of 2, 3 and 5

Topic

'Clapping the lowest common multiple of 2, 3 and 5' is about using body percussion and different body timbres, superposition and connection in order to solve the following question: which is the lowest common multiple of the numbers 2, 3 and 5?

Keywords

Lowest common multiple, body percussion, body timbre

Short description

In this activity, students will learn three body percussion rhythm patterns, each one related to the numbers 2, 3 and 5. Then, they will play each pattern simultaneously, counting from 1 to 30 in order to find the lowest common multiple of numbers 2, 3 and 5. Listening to the different body timbres will let students discover not only the lowest common multiple, but also other multiples and the relations between these numbers.

Assignment to the collection of subjects/the core of music and maths

Music: Body percussion performance; ability to listen to the different body timbres; timbre recognition; follow the beat; rhythmic reading; rhythmic imitation; regular and rhythmic precision; and the ability to perform and listen to different sound plans at the same time

Maths: Use the reasoning and proof to find the lowest common multiple; order; relations; numbering; multiples (and divisors); and connections

Preparatory considerations

Prerequisites in maths

Basic skills in counting and numbering are required. It is not necessary to know the lowest common multiple; you can introduce it through the activity.

Prerequisites in music

Basic skills in rhythm patterns (reading and imitation) are required. It is not necessary to know body percussion; you can introduce it through the activity.

Connection between maths and music (including the additional benefits of learning)

Performing a regular rhythm pattern by counting numbers (1–30) while following a beat is related to mathematical precision, relation, numbering, order and time.

Dividing the classroom group into three lines, each playing different rhythm patterns, and listening to body timbres is also related to simultaneity and the inter-relations between numbers (multiples and divisors).

In this activity, children should try to solve a mathematical question by listening to the same timbre, so it is necessary to follow the same beat and have rhythm precision.

The sound simultaneity is connected with the lowest common multiple of some numbers.

Implementation of the activity

Aims

To improve the performing and rhythm skills of the learners; to listen to and recognise the same timbre in order to get to the solution of a problem or question; to find the lowest common multiple and other multiples of some numbers; to follow the beat and rhythm with precision; and to perform rhythm patterns using body percussion

Target group (age of the students, size of the group, special students, etc.)

Ages: 8–11 years (+), up to 30 students (use rhythm imitation instead of reading with special students)

Timescale

Two sessions to understand the whole of the activity. 30 minutes to find the lowest common multiple of two numbers (2/3; 2/5; 3/5) instead of three (2, 3, 5)

Activity – Standard approach

1. Start with the body percussion of number 2. The teacher introduces the children to a body percussion pattern of 30 beats (see materials) and explains the meaning of each symbol.
   If the level of the students is sufficient, it is possible to learn the body's percussion pattern by reading it. Make sure to follow the beat by counting the numbers (from 1 to 30). The multiples of 2 (2, 4, 6, 8, until 30) should coincide with the hand clapping. If the students cannot read the score, the teacher can teach it through imitation, so they can improve their rhythm memory. Once students learn the body percussion, they have to perform it while counting until 30 (following the beat).
2. Follow the same procedure with the body percussion of number 3 (see materials). Notice that now the hand clapping is based on multiples of number 3 (3, 6, 9, until 30).
3. The teacher divides the classroom group in two lines (face to face). One line performs the body percussion of number 2, and the other performs the body percussion of number 3. Each time there is a multiple of 2 and 3, the students will clap their hands at the same time. The first time that this happens, they find the lowest common multiple of numbers 2 and 3. At the end, we can list the common multiples we have found by listening to the same timbre (hand clapping).
4. The teacher can introduce the body percussion of number 5 (see materials) and try to find, following the same procedure, the lowest common multiple of 2, 3 and 5. Notice that in the body percussion for number 5 the multiples also coincide with the hand clapping. Facing two lines, students can follow the lowest common multiple of 2 and 5 (10) or 3 and 5 (15).
5. Finally, the teacher organises the students into three lines, two parallel and one perpendicular, and each line performs the body percussion of one number (2, 3, or 5). When all the students are clapping their hands at the same time, they will find the lowest common multiple of numbers 2, 3 and 5 (30). It is necessary to count until 30, following the beat in order to know in which number the three lines coincide.

6. The teacher can project the image of the three rhythm pattern overlapped (see materials) to show which numbers coincide with the hand clapping (so the common multiples of numbers 2, 3, and 5).

Materials, pictures, music – Material spatial arrangement

Clapping the Lowest Common Multiples of 2,3,5

Additional Materials

Variations

Variations

Variation #1: Instead of using body percussion, use the notes do (feet), mi (thighs), sol (hands) and do (chest). The concept of chords appears in this variation. The procedure would be the same; however, instead of using body percussion rhythm patterns, use easy melodic patterns with the note sol in the multiples of each number.

Every time that the note sol is sung, a multiple is found. While singing, it is not possible to say the numbers, so they can be written on the board and the teacher can point at them while following the beat, or a volunteer can do this or say them aloud while the rest of the students are singing the melodic pattern of one number (2, 3 or 5).

Variation #2: Using body percussion, the students are situated in a circle. They are doing a step towards the right side. Every time, a student performs the beat of a given rhythm pattern while saying aloud the number of the beat that it corresponds to. For instance, if they follow the rhythm pattern of number 2, they would realise that the students who said number 2, 4, 6 or 8 clapped their hands. So, they are the multiples of number 2. If we repeat the activity with the rhythm patterns of numbers 3 and 5 (starting with the same person every time), we could discover the common multiples of these numbers.

Further approaches in music

Students can create a more complicated body percussion rhythm pattern by changing the timbre of the multiples.

Students can create a melodic pattern for each number and write it down. Change the chord or change the note that corresponds to the multiples.

Change the parts of the body used in the body percussion.

Use instruments to perform each rhythm in order to have as many timbres as those used in the body percussion.

Further approaches in maths

Change the numbers and find their lowest common multiple and their other multiples.

If you are doing variation #2, try to build a perfect circle and talk about geometry.

5.4 Sounding numbers

Topic

The 'sounding numbers' activity is about the creation of different acoustic models of the natural numbers.

Keywords

Maths: Numbers, digits, the positional notation of a number in the decimal numeral system (advanced and shortened notation), decomposition of a number

Music: Rhythm, metre, metro-rhythm

Short description

In this activity, learners will invent different types of acoustic models for the natural numbers, and identify and write down *n*-digit natural numbers based on their acoustic representations.

Assignment to the collection of subjects/the core of music and maths

Music: Elements of music (pulse, rhythm); playing musical instruments and singing; rhythmic playing echo (imitation)

Maths: Numbers (natural numbers, place value); numeration; the positional notation of a number in the decimal numeral system

Preparatory considerations

Prerequisites in maths

Basic numeration skills – reading and writing natural numbers in the decimal numeral system, graphically representing *n*-digit numbers

Prerequisites in music

Knowing and understanding the principle of echos (body playing, children's rhythmical musical instruments)

Connection between maths and music (including the additional benefits of learning)

Listening to different types of sounds for units (tens, hundreds, etc.) and counting them to connect the abstract concepts of decimal positional numeral systems with the acoustic model of the number.

Creating and using sounds can help children to understand the basic rules of the decimal numeral system.

'Sounding numbers' includes elements of combinatorics.

'Sounding numbers' is connected with playing musical instruments.

Implementation of the activity

Aims

Develop an understanding that the natural numbers could be represented in different ways (written notation, graphical representations (symbols), manipulation with small objects, acoustic models). Improve learners' skills when it comes to transforming a written model of numbers into an acoustic one, and vice versa.

Target group (age of the students, size of the group, special students, etc.)

Ages: 7–9 years (+); two groups of four(+) students; or work in pairs

Timescale

20 minutes for the standard approach

Activity – Standard approach

- The teacher writes a 3-digit number in its decimal notation and in its graphical representation (e.g. 235, // --- +++++).
- The teacher then makes the sounding number using stamping (2x), slapping (3x) and clapping (5x). The next number is played and learners write it using digits or signs (graphical representation).
- Learners from each group invent sets of sounds in order to code an acoustic representation of the natural numbers (e.g. four 3-digit numbers). They can use different sounds (body playing, Orff instruments, spoons, etc.).
- The learners of the first group present (play) the numbers using their invented sounding code.
- The learners of the second group write the sounding numbers (or draw a graphical model of the numbers).
- Control of solution and discuss: what numbers were played (represented), and what types of coding were used?
- Discuss the advantages and disadvantages of different types of representation of natural numbers (graphical, auditory, decimal). Compare different representations of the numbers.

Materials, pictures, music – Material spatial arrangement

Paper, pen, table, Orff instruments

Learners sit at their desks and work in the two groups or in pairs.

Variations

Variations

- This activity can also be performed in pairs (cooperative education).
- Any subjects or musical instruments (sticks, triangles, drums, mugs, cans, and pellets) can be chosen for presenting the sounds.
- The signs of sounds will make learners play numbers using the sign models of numbers; for example, 235 and // --- +++++.
- The activity can also be conducted with a group of older students, depending on the selected line of numbers (for example, over 1,000, 10,000, etc.).
- There is space for creating different tasks and variants based on the abilities and skills of the target group. From the point of view of the target group, it is possible to flexibly adjust the tasks for any age group or line of numbers.

Further approaches in music

Invent different sounds for signs in graphical models of the natural numbers.

Invent a notation in order to write the number (units, tens, hundreds).

By using Orff instruments, make an acoustic model of numbers to create the rhythm.

Note values could represent the place value of digits in the decimal numeral system (e.g. quarter note = one, half note = ten, whole note = hundred).

Further approaches in maths

Through the regular application of the above-mentioned activity from the first year of secondary school, a new atypical model of the natural numbers is created, which is different from the concrete ones (abacus, cubes, subjects, graphic representation) that are usually used. During the process of realisation, it is necessary to develop one's own mental representation of a multi-digit number. A number of sounds are transformed into the symbol of a digit, which is kept in the memory and finally recorded using mathematical terminology. Conducting the above-mentioned activities develops higher cognitive processes and involves executive functions, especially the working memory and shifting.

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5.5 Angle dances

Topic

Different types of angles are expressed with different arm and leg positions in a choreographed dance.

Keywords

Angles, body movements, patterns

Short description

A choreographed dance is to be developed in the learning environment. The dancers should express different types of angles through different arm and leg positions. The choreography is to be developed with small cardboard figures; later, the dancers are supposed to act out their development with matching music. The different positions of the legs and the arms have to be connected in a continuous flow. Depending on the knowledge of the participating person, different types of angles can be introduced via the use of pictures in the learning environment.

Assignment to the collection of subjects/the core of music and maths

Communicate mathematical ideas using multiple representations; types of different angles; pattern recognition; connecting music with body movements; and physical responses to music

Preparatory considerations

Prerequisites in maths

- Knowledge of the different types of angles: Right angle, acute angle, obtuse angle, straight angle
- Basic elements of angles: Two legs bound the field of angle, two fields of angles emerge that add up to 360°

The angle dance's focus is on the arms and legs. These parts of the body are particularly agile on account of their joint structure. Very high agility can be found in the knees and elbows. With these, the upper and lower arm and the upper and lower leg can be arranged in such a way that geometric fields of angles appear.

Prerequisites in music

The dance is the centre of this learning environment. The expression of dance is created through the movement of the body and limbs (arms and legs).

Connection between maths and music (including the additional benefits of learning)

Music creates patterns of sound, which represent different moods and will be transformed into a geometric figure. An acute angle has a different power of appearance to an obtuse angle. This can be developed in the angle dance.

Implementation of the activity

Aims

- Identification of sound patterns in music -> moods
- Sound patterns shall be transformed into body movements (dance)
- Translation of moods (caused by music) into movements and therefore into geometric figures: in adequate types of angles

Target group (age of the students, size of the group, special students, etc.)

Possible with children, teens and young adults; appropriate music to be chosen depending on the age of the participants

Timescale

About three hours, including performance

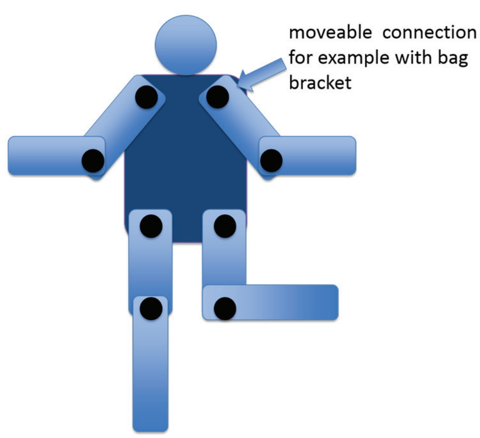
Activity – Standard approach

At the beginning of the learning experience, there has to be clarity on which angles are which (right angle, acute angle, obtuse angle, straight angle, etc.). Children unfamiliar with the angles can be helped through the use of pictures. After having cleared this issue, one has to consider how to create the different types of angles with different arm and leg movements. These considerations can be met with cardboard figures. Some types of angles offer different possibilities. The expression through the dance can lead to special tensions when the arm and body movements have a special character. At the same time, one has to consider whether the arm and body movements can actually be copied with real arms and legs.

After clearing the different possible positions, the choreography has to be created. For the choreography, one has to listen to the music that is presented and think about which sequences of angles match the music, showing the character of the music in a specific way.

Materials, pictures, music – Material spatial arrangement

It is advised that the choreography be planned at a table. Therefore, cardboard figures should be used. The cardboard figures have movable knees and elbows (see picture). With the figures, different positions can be considered dynamically. The final choreography can be documented with drawings.



Variations

Variations
Present different geometric figures in dances. Geometric figures can be presented by multiple persons. This way, the corners can be represented by single people and the edges by the connection of these people's hands and arms.
Further approaches in music
The quality of the music can be expressed with different geometric figures. For example, fast or high-pitched music could be expressed through triangles. Slower and more harmonious music could be represented with circles and regular polygons that move through the space. According to the music, different dances occur.
Further approaches in maths
A basic element for the translation and integration of geometric figures (fields of angles, plane figures) is the analysis of the central elements of these figures. This means the number and position of the edges and the corners. In this way, the central elements can be learned playfully and transferred into the different dances. A square could be built with four people who represent the corners, and, according to the positions of their arms, rectangles with right angles or parallelograms with different angles could be created.

5.6 Twinkle, Twinkle Little Star

Topic

Use singing to explore symmetry, pattern, time and reflection.

Keywords

Rhythm, reflection, motif, transformation and symmetry

Short description

Children will explore what happens when they transform music. They will also discover that there are different patterns, depending on whether they focus on the rhythm or the musical notes. This will help them to understand that if we can focus on the different aspects of a problem, there will be different solutions.

Assignment to the collection of subjects/the core of music and maths

Pulse and beat; practical music making; composition and improvisation using voice; appreciation of music; and aural awareness through listening and performance

Preparatory considerations

Prerequisites in maths

Patterns and sequences, as well as some experience of reflections

Prerequisites in music

Physical coordination (clapping/stamping), pulse, use of voice for singing, listening

Connection between maths and music (including the additional benefits of learning)

Patterns, sequences, and transformations

Implementation of the activity

Aims

Children will learn about symmetries, patterns and motifs in music and mathematics.

Target group (age of the students, size of the group, special students, etc.)

Ages: 8+ years. Whole-class and pair/group work

Timescale

20+ minutes

Activity – Standard approach

- Sing the song with the whole class a few times to make sure that the children become familiar with it. It may help to have the words on the board or on paper for the children to see. Ask the children if they notice any patterns or symmetries in the tune (rhythm patterns, melody patterns, form A-B-A).
- Draw the melody as lines, showing the ups and downs.
- Now, clap the rhythm with the children and ask what patterns they notice this time. Are they the same or different from the ones they noticed before?
- Next, ask the children to work in pairs or small groups. The children need to choose a motif from the song, using either the song, the tune or the rhythm. Ask the children to create their own notation to represent the motif. The children should then explore what happens when they reflect the motif and they should draw this reflection. The children may want to use mirrors to check that they have drawn their reflections correctly. Once this has been done, the children should practise singing or clapping their motif along with its reflection. It might be easier if the children try singing the tune without the words.

Materials, pictures, music – Material spatial arrangement

Resources: Mirrors, copies of the song

Other considerations: This activity should be carried out in a room where the children have space to stand in a circle. If there is a board, the children may not need copies of the song.

Twinkle, twinkle little star

Variations

Variations

There are many different songs that could be chosen as a starting point, but it is important that they are songs that are very familiar to the children and have a simple structure.

Further approaches in music

Use different versions of the song, as in:

- *A, B, C* (song);
- *Baa, Baa Black Sheep*;
- *A vous dirais je maman* (original version);
- Mozart variations of the song;
- Louis Armstrong's *What a Wonderful World* (inspired by the melody);
- Choose a theme and present a new version of the song. For example:

I came into school today
And I shouted "Let's go play!"
Saw my friends and off we went
Round the playground, through the fence
I came into school today
And I shouted "Let's go play!"

- Instruments could be used to explore different transformations.

Further approaches in maths

- Work could be developed using other transformations (rotations and translations). Can we do the same things with rhythms and scores as we can with words?
- The idea of using a motif and transforming it could also be explored using designs for wallpaper or wrapping paper. More traditional designs, such as those used in Islamic art and design, could also be explored.
- This activity could also lead into work on combinations and permutations, and this might support work on fractions.
- The ideas could be developed to include work on sequences.

6 Conclusions

With this handbook, we highlight the importance of music and mathematics in everyday life and strongly promote the equivalent significance of both topics in learning environments. Music and mathematics serve as equal partners in a modern interdisciplinary teaching approach. We believe that, with the aid of the activities set out in this handbook and on the project's website, teachers will be able to work with students and develop new ideas, not only about mathematics and music, but also about other possible combinations, as has been shown in the language project.

The major conclusion to emerge from didactically combining mathematical and musical learning is that more and more ideas come up when focusing on the shared aspects of the two sign systems and human intelligence (according to Gardner, 1983). In summary, there are sounding ways into mathematics, just as there are mathematical ways into sound.

Finally, we want to encourage everyone to join the project by taking part in a CPD course, collaborating with their peers through our online platform (http://maths.emportfolio.eu) and sharing their own activities.

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