MATHEMATICS OF THE QUANTUM ZENO EFFECT

ANDREAS U. SCHMIDT

ABSTRACT. We present an overview of the mathematics underlying the quantum Zeno effect. Classical, functional analytic results are put into perspective and compared with more recent ones. This yields some new insights into mathematical preconditions entailing the Zeno paradox, in particular a simplified proof of Misra's and Sudarshan's theorem. We emphasise the complex-analytic structures associated to the issue of existence of the Zeno dynamics. On grounds of the assembled material, we reason about possible future mathematical developments pertaining to the Zeno paradox and its counterpart, the anti–Zeno paradox, both of which seem to be close to complete characterisations.

1. INTRODUCTION

The Zeno effect consists in the impediment of a quantum system's evolution by frequent measurements performed on it. Apart from the generic quantum phenomenon of entanglement, it is probably the most striking difference separating the classical from the quantum world, and an example for the sometimes counterintuitive features of the latter. In the most concise manifestation of the Zeno effect, a decaying state of a quantum system, say an excited state of an atom, is conserved and prevented from decay simply by 'looking at it', i.e., observing the presence of the undecayed state. That this observation can clearly, in the quantum formalism, also be done by 'doing nothing' through a negative-result experiment *not* observing the decay products, see [44], adds the same scent of magic (or 'spookiness') to this effect that adheres to the Einstein–Podolsky–Rosen paradox. The effect's name was aptly coined after the classical argument of Zeno that was meant to prove the impossibility of any real motion ('a watched arrow never flies').

It is no wonder that such a baffling phenomenon has by now also entered popular science texts [66]. The interest of the mathematical, theoretical, and experimental physics communities in the effect — which was formerly seen as a mere curiosity, even possibly due only to a 'wrong' interpretation of quantum theory — was rekindled by the seminal work of Misra and Sudarshan [51], which also drew the present author into the subject, and inspired his recent works [63, 64].

In the present survey, we report on the mathematical side of the story, and put classical and more recent results into perspective. More specifically, we concern ourselves with the stricter version of the effect presented by the **Zeno**

Date: 21st July 2003, revised 5th October 2004 including corrections from the published corrigendum to [64].

1991 *Mathematics Subject Classification*. 46L60, 47D03, 81P15, 81R15, 82B10.

Key words and phrases. Quantum Zeno Effect, anti–Zeno effect, measurement, Trotter's product formula, degenerate semigroup, operator algebra, modular automorphism group, quantum statistical mechanics, KMS-state, return to equilibrium, Gibbs state.

PACS classification. 03.65.Xp, 03.65Db, 05.30.–d, 02.30.Tb.

The author thanks Máté Matolcsi, Roman Shvidkoy, as well as Harald Atmanspacher, Werner Ehm, and Tilmann Gneiting for making their recent works available to him.

paradox [51]. In its mathematical formulation using the projection postulate of the standard interpretation of quantum mechanics to model the single measurements by a projection E, the question that arises is that of the existence of the limit

$$W(t) = \lim_{n\to\infty} \big[EU(t/n)E\big]^n,$$

where $U(t)$ is the original quantum dynamics — which need in fact *not* necessarily be unitary as suggested by the use of the letter U. That is, does this limit exist in an appropriate topology on the operators on the quantum mechanical Hilbert space of the system? If so, is it a 'sensible' quantum evolution by itself, i.e., is it continuous and satisfies a (semi)group law? In the paradigmatic case where E is a rank one projection onto a single, decaying, initial state ψ_i, written $|\psi_i\rangle\langle\psi_i|$ in Dirac's notation, the paradox amounts to a complete impediment of the decay, if ψ_i is observed with infinite frequency. Therefore, the Zeno paradox has been nicknamed 'a watched pot never boils' effect or 'watchdog' effect by some [65].

This extreme manifestation of Zeno's effect is of conceptual interest, although it is not a proper physical phenomenon since the limit in question is not attainable due to the finite duration of real world measurements, respectively, interactions involving a *finite* amount of energy. These principal upper bounds on the frequency of measurements that can be exerted on a quantum system are essentially a consequence of the time/energy uncertainty principle [55, 58, 37]. Nevertheless, the paradox is interesting both from a mathematical as well as from a physical viewpoint, for not only is the occurrence of the paradox, or rather sufficient conditions for it, also indicative for the Zeno effect at finite measurement frequency. But it is also very helpful for the study of the **Zeno subspace**, i.e., the subspace of the full Hilbert space of the system to which the dynamics is confined in the **Zeno limit** of infinitely frequent measurement. The general picture that has evolved over the past few years [28, 24, 27, 63, 64], is that the emergent **Zeno dynamics** on the Zeno subspace is (more or less, depending on the model considered) a free quantum dynamics, amended by specific boundary conditions. These interesting, and sometimes deep, physical structures are in fact best studied in the firm framework provided by the Zeno limit.

The present survey addresses physicists who might not be aware of the functional analytic structures underlying Zeno's paradox, as well as mathematicians who might not know of this special physical application of these structures. We hope to incite the interest of the two cultures in the subject, and to promote the mutual transfer of knowledge about it. The main sources of the paper are the classical results by Kato [42], and Misra and Sudarshan [51] which are put into perspective with the recent ones by Matolcsi and Shvidkoy [48, 49, 46], Exner and Ichinose [14], Atmanspacher *et al.* [5], and of the present author [63, 64].

Yet, many interesting ramifications of Zeno effect and paradox are neglected, in particular, we do not delve into the vast history of the subject, which dates back to the 1930ies and is associated with the names of Turing and von Neumann. The reader interested in this part of the story, as well as in the relation of Zeno's effect to the controversies surrounding the interpretation of quantum theory, is deferred to [35, 52, 36, 65, 32], and their extensive lists of references. Recent experiments confirming the Zeno effect [67], and its counterpart the anti-Zeno effect [30] (see Section 4.2) are exciting, but also left out, as well as the possible practical applications of the effects, and many theoretical treatments of model cases, some of which also yield proposals for experiments [1, 19, 21, 10, 13, 18, 26, 64, 53]. A subject which would also be relevant to the mathematical side of the matter, but does only receive marginal consideration in Section 3.3, is the rather different manifestation of the Zeno effect caused by a *continuous* measurement performed on a system coupled to a measurement device with a coupling strength approaching infinity. Mathematical models for this, and the question of equivalence of the different realisations of the Zeno paradox, have been treated elsewhere [7, 22, 23, 24, 27]. Finally, counterexamples, i.e., the rather exceptional cases in which the Zeno paradox does not emerge [3, 48, 49], are not expanded on.

The outline of the paper is as follows. In Section 2 we gather mathematical results which are fundamental for the examination of the Zeno paradox, in the sense that they do not depend on specific information about the physical system considered, respectively, its mathematical model. These results appear in the form of convergence theorems in the case of semigroups in Section 2.1, and existence theorems for the Zeno dynamics in the quantum mechanical case of unitary groups treated by Misra's and Sudarshan's Theorem. We show in Section 2.2 how the proof of the latter can be considerably shortened by using the former results. As a third case we present in Section 2.3 our corresponding, recent result in the framework of modular flows of von Neumann algebras. We highlight the crucial differences between the former and the latter two cases, hinging on analyticity domains. This points to a possible way to achieve sharper characterisations of the Zeno paradox by use of Payley–Wiener type arguments, for which some preliminary thoughts are sketched in Section 2.4.

Section 3 presents a collection of more specific conditions for the occurrence of the Zeno paradox, and results about its physical consequences, formulated in operator theoretical frameworks. The *asymptotic Zeno condition* introduced in [64], and which is treated in Section 3.1, is efficient in that it can be easily tested in concrete cases and enables the use of perturbation theory for that purpose. This condition yields a mathematical formulation of the Zeno paradox in the operator algebraic framework of quantum statistical mechanics [8], presented in Section 3.2. As a consequence, we obtain the proper paradigmatic manifestation of the Zeno effect in quantum statistical mechanics — the prevention of return to equilibrium. We further show in Section 3.3, that in many benign cases it is possible to identify the Zeno subspace as well as the generator of the Zeno dynamics acting on it. This enables the construction of an important subclass of equilibrium states for the Zeno dynamics. Note that examples for the Zeno effect in quantum statistical mechanics, more specifically quantum spin systems, and the XY-model of a one-dimensional spin chain, are also treated in [64], but not reproduced here. Finally in Section 3.4, we contrast our own results with recent ones by Exner and Ichinose, which give a very sharp, abstract condition on the generator of the Zeno dynamics that ensures existence of the Zeno limit.

Finally, Section 4 complements the foregoing abstract presentation with some more physically-minded considerations. In particular, we explain in Section 4.1 how the reasons for the occurrence of the Zeno effect and the Zeno paradox can be traced back to fundamental properties of the Hilbert space and its geometry, which entail the quadratic short-time behaviour of quantum probabilities. On a quite different note, and in opposition to the genericity of the quantum Zeno phenomenon stressed throughout the previous sections, Section 4.2 presents the existence of the opposite phenomenon — the *anti-Zeno* effect. Under certain conditions, there can be a region of measurement frequencies at which the quantum evolution is spurred rather than impeded. In the extreme case, even an anti–Zeno *paradox*, i.e., induction of instantaneous decay, seems conceivable. We present two remarkable, recent results about necessary *and* sufficient conditions for Zeno and anti–Zeno paradox, based on the asymptotics of the state's energy distribution, in Section 4.3.

Some conclusions and open questions are noted in Section 5.

2. Mathematical Foundations

2.1. **Analytic Semigroups.** One important case in which the Zeno dynamics is sure to exist presents itself within the theory of analytic semigroups. Many fundamental results in that field are concerned with **Trotter's product formula**

$$\mathrm{e}^{t(A+B)} = \lim_{n\to\infty}\left[\mathrm{e}^{\frac{t}{n}A}\mathrm{e}^{\frac{t}{n}B}\right]^n,$$

which holds, e.g., if A is the generator of a contractive semigroup and B a dissipative operator on a Banach space $\mathcal{B}$, see [8, Corollary 3.1.31], and [12] for an extensive treatment of product formulae. If B is replaced by a projection E, one speaks of a **degenerate product formula** of the form

$$S(t) = \lim_{n\to\infty}\left[\mathrm{e}^{\frac{t}{n}A}E\right]^n,$$

and wants to determine conditions under which this limit exists and defines a strongly continuous semigroup on the invariant subspace $S(0)\mathcal{B} \subset E\mathcal{B}$ (note that $S(0)$ is a projection in this case), a so called **degenerate semigroup**.

The one result in that vein which we want to present here is based on Kato's work on non-densely defined sesquilinear forms [42]. It says, essentially, that the Zeno dynamics *always* exists for semigroups which are analytic in a sector in $\mathbb{C}$ with positive opening angle.

Theorem 2.1 ([3, Theorem 4])**.** *Let $-A$ be the generator of a semigroup (e^{-zA}), $z \in \Sigma(\theta)$ on a Hilbert space $\mathcal{H}$, which is holomorphic in an open sector $\Sigma(\theta) = \{z \in \mathbb{C} \mid z \neq 0, |\arg z| < \theta\}$ for some $\theta \in (0, \pi/2]$. Assume $\|\mathrm{e}^{-zA}\| \leq 1$ for all $z \in \Sigma(\theta)$, and let E be an orthogonal projection. Then*

$$S(t)\phi = \lim_{n\to\infty}\left[\mathrm{e}^{-\frac{t}{n}A}E\right]^n\phi$$

exists for all $\phi \in \mathcal{H}$, $t \geq 0$, and defines a degenerate semigroup $(S(t))_{t\geq 0}$.

The generality of this result is quite remarkable. It makes no assumptions on the projection and also does not depend on other technical details, for instance whether or not $\mathcal{H}$ is separable. The strength of the assumption lies exclusively in the analyticity domain. What this result already shows is that the Zeno effect is by no means restricted to *unitary* evolutions. The following lucid explanation of its proof is taken almost literally from [3], see also [49].

Let $a\colon D(a) \times D(a) \to \mathbb{C}$ be a sesquilinear form with a subspace $D(a) \subset \mathcal{H}$ as domain. Assume that a is **semibounded**, i.e.,

$$\exists\lambda \in \mathbb{R}\colon \forall 0 \neq \phi \in D(a)\colon \quad \|\phi\|_a^2 \stackrel{\text{def}}{=} \operatorname{Re} a(\phi, \phi) + \lambda(\phi, \phi)_{\mathcal{H}} > 0,$$

where $(\cdot, \cdot)_{\mathcal{H}}$ is the scalar product of $\mathcal{H}$, and moreover that it is **sectorial**, i.e.,

$$\exists M > 0\colon \quad |\operatorname{Im} a(\phi, \phi)| \leq M\big(\operatorname{Re} a(\phi, \phi) + \lambda(\phi, \phi)_{\mathcal{H}}\big),$$

and **closed**, i.e., $(D(a), \|\cdot\|_a)$ is complete. On the closure $K = \overline{D(a)}$, define the operator A associated with a by

$$\begin{aligned} D(A) &= \{\phi \in D(a) \mid \exists \psi \in K\colon\ a(\phi, \chi) = (\psi, \chi)_{\mathcal{H}}\ \forall \chi \in D(a)\}, \\ A\phi &= \psi. \end{aligned}$$

Then $-A$ generates a C_0-semigroup $(\mathrm{e}^{-tA})_{t\geq 0}$ on K. If $P(K)$ denotes the orthogonal projection onto K, then we can define the operators e^{-ta} on $\mathcal{H}$ by $\mathrm{e}^{-tA}P(K)$, and they constitute a degenerate semigroup on $\mathcal{H}$. If b is a second semibounded, closed, sesquilinear form, then their sum $a + b$ defined by

$$\begin{aligned} D(a+b) &= D(a) \cap D(b), \\ (a+b)(\phi, \psi) &= a(\phi, \psi) + b(\phi, \psi), \end{aligned}$$

is again semibounded and closed. In this case holds the following product formula.

Theorem 2.2 ([42, Theorem and Addendum]). *For $\phi \in \mathcal{H}$ holds*

$$\mathrm{e}^{-t(a+b)}\phi = \lim_{n\to\infty} \left[\mathrm{e}^{-\frac{t}{n}a}\mathrm{e}^{-\frac{t}{n}b}\right]^n \phi$$

for all $t > 0$.

To make contact with Theorem 2.1, we note that under the conditions stated there, there exists a closed, semibounded, sesquilinear form a associated with the generator A, see [41, Section VI.2.1, Theorem 2.7]. In particular, this form is sectorial in the sense above, by Theorem 1.2 of [4]. On the other hand, the projection E defines the form b by

$$\begin{aligned} D(b) &= E\mathcal{H}, \\ b(\phi, \psi) &\equiv 0 \quad \text{on } E\mathcal{H}, \end{aligned}$$

entailing $\mathrm{e}^{-tb} = E$ for all $t \geq 0$. Thus

$$S(t)\phi = \mathrm{e}^{-t(a+b)}\phi = \lim_{n\to\infty} \left[\mathrm{e}^{-\frac{t}{n}a}E\right]^n \phi, \quad \phi \in \mathcal{H},$$

is the semigroup of the conclusion of Theorem 2.1.

Well, on this level the whole issue of the Zeno paradox might seem resolved. But this is not quite the case, as is shown by the counterexamples constructed in [3] in the broader context of positive semigroups on Banach spaces, and in [48, 49] even for unitary semigroups on Hilbert spaces. That, and why, things are a bit more involved in the Hilbert space case, will be seen in the following.

2.2. **Unitary Groups.** It is not without irony that the above fundamental results on degenerate semigroups appeared in [42], one year after the publication of the seminal article of Misra and Sudarshan [51] — one of the main sources of inspiration for everyone who is today interested in the Zeno effect. For, as we will show now, the analytic semigroup results can be used to simplify the original proof of Misra's and Sudarshan's existence theorem for Zeno dynamics significantly.

Theorem 2.3. *Let $U(t) = \mathrm{e}^{\mathrm{i}tH}$ be a unitary group of operators on a Hilbert space $\mathcal{H}$, with nonnegative, self-adjoint generator H, and E an orthogonal projection. Assume that the limits*

$$T(t) = \operatorname*{s-lim}_{n\to\infty} \big[EU(t/n)E\big]^n$$

exist for all $t \in \mathbb{R}$, are weakly continuous in t, and satisfy the initial condition

$$\operatorname*{w-lim}_{t\to 0} T(t) = E.$$

Then, $T(t)$ is a degenerate group of unitaries on $E\mathcal{H}$.

Proof. Since H is nonnegative, $U(t)$ extends to a holomorphic, operator-valued function $U(z)$ in the upper half-plane $\mathbb{H}_+ = \{z \mid \operatorname{Im} z > 0\}$ with boundary value $U(t)$. Theorem 2.1 entails the existence of $T(z) = \lim_{n\to\infty}[EU(z/n)E]^n$ on every ray $\{z = re^{i\theta} \mid \theta \in (0,\pi), r > 0\}$, and therefore in whole $\mathbb{H}_+$. In particular, $(W(is))_{s\geq 0}$ is a degenerate, analytic semigroup, and long known structure theorems for these [34] ensure the existence of a positive operator B and a projection G such that $W(is) = e^{-sB}G = Ge^{-sB}$. Thus, for all $\phi \in \mathcal{H}$ holds $W(is)\phi = Ge^{-sB}G\phi$, and the identity theorem for vector-valued, holomorphic functions [34, Theorem 3.11.15] ensures the identity $W(z)\phi = Ge^{izB}G\phi$ for all $z \in \mathbb{H}_+$. In turn, this yields the semigroup property $W(z_1 + z_2) = W(z_1)W(z_2)$ for all z_1, $z_2 \in \mathbb{H}_+$, since e^{izB} commutes with G. We obtain the integral representation

$$\begin{aligned}
W(z) &= \operatorname*{s-lim}_{n\to\infty}\big[EU(z/n)E\big]^n \\
&= \operatorname*{s-lim}_{n\to\infty} \frac{(z+i)^2}{2\pi i}\int_{-\infty}^{\infty} \frac{\big[EU(t/n)E\big]^n}{(t+i)^2(t-z)}dt \quad \text{(Cauchy's formula)} \\
&= \frac{(z+i)^2}{2\pi i}\int_{-\infty}^{\infty} \frac{\operatorname*{s-lim}_{n\to\infty}\big[EU(t/n)E\big]^n}{(t+i)^2(t-z)}dt \quad \begin{pmatrix}\text{uniform boundedness}\\ \text{of the integrand}\end{pmatrix} \\
&= \frac{(z+i)^2}{2\pi i}\int_{-\infty}^{\infty} \frac{T(t)}{(t+i)^2(t-z)}dt, \quad \text{(convergence assumption)}
\end{aligned}$$

for $\operatorname{Im} z > 0$. On the other hand

$$0 = \frac{(z+i)^2}{2\pi i}\int_{-\infty}^{\infty} \frac{T(t)}{(t+i)^2(t-z)}dt,$$

for $\operatorname{Im} z < 0$, both of which we now use to carry the semigroup property forward from the complex domain $\mathbb{H}_+$ to the boundary $\mathbb{R}$. Namely writing explicitly $z = s \pm i\eta$, $\eta > 0$, and adding the two integral representations above yields

$$(\phi, W(s+i\eta)\psi) = \frac{(s+i+i\eta)^2}{\pi}\int_{\mathbb{R}} \frac{(\phi, T(t)\psi)}{(t+i)^2}\frac{\eta}{(t-s)^2+\eta^2}dt,$$

for any two vectors ϕ, $\psi \in \mathcal{H}$. The right hand side is nothing but the Poisson transformation of the function $(\phi, T(s)\psi)$, modified by the convergence factor $(t+i)^{-2}$, and it reproduces this function s-almost everywhere as $\eta \to 0_+$. But, since $T(t)$ is weakly continuous, we obtain

$$\operatorname*{w-lim}_{\eta\to 0_+} W(t+i\eta) = T(t), \quad \text{for all } t \in \mathbb{R}.$$

In turn, $T(t) = Ge^{itB}G$ for all t, yielding in particular the strong continuity of $T(t)$. Thus $T(t)T(t)^* = Ge^{itB}e^{-itB}G = G$ for all t, which together with the initial condition implies $G = E$. This finally shows $T(t) = Ee^{itB}E$ for all $t \in \mathbb{R}$. Since the explicit form of the limit in n entails $T(-t) = T(t)^*$, we conclude that $(T(t))_{t\in\mathbb{R}}$ is a strongly continuous group of operators which are unitary on $E\mathcal{H}$, and the proof is completed. $\square$

2.3. **Modular Automorphisms of von Neumann Algebras.** The last, and technically most involved, abstract result on Zeno dynamics is about modular flows of von Neumann algebras. This might seem of purely mathematical interest, were it not for the close connection between these flows and the dynamical flows of systems in quantum statistical mechanics, associated with KMS states, which in turn are the paradigm for thermal equilibrium (we refer to [8] for the background). But also from our present viewpoint it is interesting to compare the case of modular flows with the other two presented above, since here the generator of the original dynamics is not semibounded. It rather is such that the negative spectral degrees of freedom are exponentially damped in the following sense, cf. [33, Section V.2.1]. To fix notation, we let $\mathcal{A}$ be a von Neumann algebra with faithful, normal state ω, represented on the Hilbert space $\mathcal{H} = \overline{\mathcal{A}\Omega}$ with cyclic and separating vector Ω associated with ω. Let Δ be the modular operator of $(\mathcal{A}, \Omega)$, generating the modular flow $U(t) = \Delta^{\mathrm{i}t}$. Write $\Delta = \mathrm{e}^{-K}$, $U(t) = \mathrm{e}^{-\mathrm{i}tK}$. Let $E_\kappa^{(-)}$, $\kappa > 0$, be the spectral projection of K for the interval $[-\infty, -\kappa]$. Then holds

$$\|E_\kappa^{(-)} A\Omega\| \leq \mathrm{e}^{-\kappa/2}\|A\|, \quad \text{for all } A \in \mathcal{A},$$

and therefore the vectors in $\mathcal{A}\Omega$ are in the domain of Δ^α for $0 \leq \alpha \leq 1/2$.

This difference to the quantum mechanical case entails that the original evolution is no longer analytically extensible to the whole upper half-plane, but only to a strip of positive width (which lies, due to a notorious change of sign, in the *lower* half-plane). For us this means in particular that the analyticity domain contains no open sector, and we cannot use Theorem 2.1 to infer the existence of the Zeno dynamics within it. Therefore, we have to use the independent, original strategy of [51] and adapt it accordingly. A second, more technical point is that we now have to handle the generically unbounded operators in question with some additional care. For the reader's convenience, and to ease comparison with Section 2.2, we reproduce the results and proofs of [63] rather completely. The tenets of their application to quantum statistical mechanics are described in Section 3.2 below.

Theorem 2.4. *Let $E \in \mathcal{A}$ be a projection. Set $\mathcal{A}_E \stackrel{\mathrm{def}}{=} E\mathcal{A}E$, and define a subspace of $\mathcal{H}$ by $\mathcal{H}_E \stackrel{\mathrm{def}}{=} \overline{\mathcal{A}_E\Omega} \subset E\mathcal{H}$. Assume:*

i) *For all $t \in \mathbb{R}$, the strong operator limits*

$$W(t) \stackrel{\mathrm{def}}{=} \operatorname*{s-lim}_{n\to\infty}\bigl[E\Delta^{\mathrm{i}t/n}E\bigr]^n$$

exist, are weakly continuous in t, and satisfy the initial condition

$$\operatorname*{w-lim}_{t\to 0} W(t) = E.$$

ii) *For all $t \in \mathbb{R}$, the following limits exist:*

$$W(t - \mathrm{i}/2) \stackrel{\mathrm{def}}{=} \operatorname*{s-lim}_{n\to\infty}\bigl[E\Delta^{\mathrm{i}(t-\mathrm{i}/2)/n}E\bigr]^n,$$

where the convergence is strong on the common, dense domain $\mathcal{A}\Omega$.

Then the $W(t)$ form a strongly continuous group of unitary operators on $\mathcal{H}_E$. The group $W(t)$ induces an automorphism group τ^E of $\mathcal{A}_E$ by

$$\tau^E\colon \mathcal{A}_E \ni A_E \longmapsto \tau_t^E(A_E) \stackrel{\mathrm{def}}{=} W(t)A_E W(-t) = W(t)A_E W(t)^*,$$

such that $(\mathcal{A}_E, \tau^E)$ is a W^*-dynamical system. The vectors $W(z)A_E\Omega$, $A_E \in \mathcal{A}_E$, are holomorphic in the strip $0 < -\operatorname{Im} z < 1/2$ and continuous on its boundary.

Notice that $\mathcal{A}_E$ is a von Neumann subalgebra of $\mathcal{A}$, see [39, Corollary 5.5.7], for which Ω is cyclic for $\mathcal{H}_E$, and separating. Thus, Ω induces a faithful representation of $\mathcal{A}_E$ on the closed Hilbert subspace $\mathcal{H}_E$, and thus all notions above are well-defined. The remainder of this section contains the proof of the above theorem, split into several lemmas. Set $S \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid -1/2 < \operatorname{Im} z < 0\}$. Define operator-valued functions

$$F_n(z) \stackrel{\text{def}}{=} \left[E\Delta^{\mathrm{i}z/n}E\right]^n, \quad \text{for } z \in \overline{S},\ n \in \mathbb{N}.$$

The $F_n(z)$ are operators whose domains of definition contain the common, dense domain $\mathcal{A}\Omega$. They depend holomorphically on z in the sense that the vector-valued functions $F_n(z)A\Omega$ are holomorphic on S and continuous on $\overline{S}$ for every $A \in \mathcal{A}$. For this and the following lemma see [8, Section 2.5, Section 5.3, and Theorem 5.4.4].

Lemma 2.5. *For $z \in \overline{S}$ and $\psi \in D(\Delta^{|\operatorname{Im} z|})$ holds the estimate*

$$\|F_n(z)\psi\| \leq \|\psi\|,$$

for all $n \in \mathbb{N}$.

Proof. Define vector-valued functions $f_k^{\psi,n}(z) \stackrel{\text{def}}{=} \left[E\Delta^{\mathrm{i}z/n}E\right]^k\psi$. These are well-defined for $z \in \overline{S}$, $\psi \in D(\Delta^{|\operatorname{Im} z|})$ and all $k \leq n$, since for such ψ, z we have $\left[E\Delta^{\mathrm{i}z/n}E\right]^{k-1} \in D(E\Delta^{\mathrm{i}z/n}E)$. Approximate $f_{k-1}^{\psi,n}(z)$ by elements of the form $A_l\Omega$, $A_l \in \mathcal{A}$. Then for any $B \in \mathcal{A}$ holds

$$\begin{aligned}
|\langle B\Omega\,, E\Delta^{\mathrm{i}z/n}EA_l\Omega\rangle| &= |\langle \Omega\,, B^*E\Delta^{\mathrm{i}z/n}EA_l\Delta^{-\mathrm{i}z/n}\Omega\rangle| \\
&= \left|\omega(B^*E\sigma_{z/n}(EA_l))\right| \\
&\leq \|B^*E\Omega\|\|EA_l\Omega\| \\
&\leq \|B\|\|A_l\Omega\|.
\end{aligned}$$

Here, ω is the state on $\mathcal{A}$ associated with the cyclic and separating vector Ω (we always identify elements of $\mathcal{A}$ with their representations on $\mathcal{H}$), and σ denotes the modular group. The first estimate above follows explicitly from the corresponding property of σ, see [8, Proposition 5.3.7] (the connection between faithful states of von Neumann algebras and KMS states given by Takesaki's Theorem [8, Theorem 5.3.10] is used here and in the following). This means $\left\|E\Delta^{\mathrm{i}z/n}EA_l\Omega\right\| \leq \|A_l\Omega\|$, and since $A_l\Omega \longrightarrow f_{k-1}^{\psi,n}(z)$ in the norm of $\mathcal{H}$, it follows $\|f_k^{\psi,n}(z)\| \leq \left\|f_{k-1}^{\psi,n}(z)\right\|$. Since this holds for all $k = 1, \ldots, n$, we see

$$\|F_n(z)\psi\| = \|f_n^{\psi,n}(z)\| \leq \ldots \leq \|f_1^{\psi,n}(z)\| \leq \|\psi\|,$$

as desired. □

The estimate proved above also yields that the F_n are closeable. We will denote their closures by the same symbols in the following.

Lemma 2.6. *For $z \in S$ holds the representation*

$$F_n(z)A\Omega = \frac{(z+\mathrm{i})^2}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{F_n(t-\mathrm{i}/2)A\Omega}{(t+\mathrm{i}/2)^2(t-\mathrm{i}/2-z)} - \frac{F_n(t)A\Omega}{(t+\mathrm{i})^2(t-z)}\,\mathrm{d}t. \tag{1}$$

where the integrals are taken in the sense of Bochner. One further has

$$0 = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{F_n(t-\mathrm{i}/2)A\Omega}{(t+\mathrm{i}/2)^2(t-\mathrm{i}/2-z)} - \frac{F_n(t)A\Omega}{(t+\mathrm{i})^2(t-z)}\,\mathrm{d}t, \tag{2}$$

for $z \notin \overline{S}$.

Proof. By Cauchy's theorem for vector-valued functions [34, Theorem 3.11.3], we can write

$$\frac{F_n(z)A\Omega}{(z+\mathrm{i})^2} = \frac{1}{2\pi\mathrm{i}} \oint \frac{F_n(\zeta)A\Omega}{(\zeta+\mathrm{i})^2(\zeta-z)}\,\mathrm{d}\zeta,$$

where the integral runs over a closed, positively oriented contour in S, which encloses z. We choose this contour to be the boundary of the rectangle determined by the points $\{R-\mathrm{i}\varepsilon, -R-\mathrm{i}\varepsilon, -R-\mathrm{i}(1/2-\varepsilon), R-\mathrm{i}(1/2-\varepsilon)\}$ for $R>0$, $1/4>\varepsilon>0$. By Lemma 2.5, the norms of the integrals over the paths parallel to the real line stay bounded as $R\to\infty$, while those of the integrals parallel to the imaginary axis vanish. Thus

$$\frac{F_n(z)A\Omega}{(z+\mathrm{i})^2} = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{F_n(t-\mathrm{i}(1/2-\varepsilon))A\Omega}{(t+\mathrm{i}(1/2+\varepsilon))^2(t-\mathrm{i}(1/2-\varepsilon)-z)} - \frac{F_n(t)A\Omega}{(t+\mathrm{i}(1-\varepsilon))^2(t-\mathrm{i}\varepsilon-z)}\,\mathrm{d}t.$$

For $0<\varepsilon_0<\min\{|\mathrm{Im}\,z|, |1/2-\mathrm{Im}\,z|\}$ and all ε such that $0\le\varepsilon\le\varepsilon_0$, the integrand is bounded in norm by $\|A\| \,/\, \big[(1+t^2)\min\{|\mathrm{Im}\,z-\varepsilon_0|, |\mathrm{Im}\,z-(1/2-\varepsilon_0)|\}\big]$. Since moreover, in the strong sense and pointwise in t, $\lim_{\varepsilon\to0} F_n(t-\mathrm{i}\varepsilon)A\Omega = F_n(t)A\Omega$, and $\lim_{\varepsilon\to0} F_n(t-\mathrm{i}(1/2-\varepsilon))A\Omega = F_n(t-\mathrm{i}/2)A\Omega$, the conditions for the application of the vector-valued Lebesgue theorem on dominated convergence [34, Theorem 3.7.9] are given, and the desired representation follows in the limit $\varepsilon\to0$. The vanishing of the second integral follows analogously. □

Lemma 2.7. *The strong limits $F(z) \stackrel{\mathrm{def}}{=} \text{s-lim}_{n\to\infty} F_n(z)$, $z\in S$, are closeable operators with common, dense domain $\mathcal{A}\Omega$ (we denote their closures by the same symbols). The integral representation*

$$F(z)A\Omega = \frac{(z+\mathrm{i})^2}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{W(t-\mathrm{i}/2)A\Omega}{(t+\mathrm{i}/2)^2(t-\mathrm{i}/2-z)} - \frac{W(t)A\Omega}{(t+\mathrm{i})^2(t-z)}\,\mathrm{d}t \tag{3}$$

holds good, and the functions $F(z)A\Omega$ are holomorphic on S, for all $A\in\mathcal{A}\Omega$. There exists a projection G and a positive operator Γ such that $\Gamma = G\Gamma = \Gamma G$, and $\Gamma^{4iz} = F(z)$ for all $z\in S$.

Proof. Using Lemma 2.5, we see that the norm of the integrand in (1) is uniformly bounded in n by $2\|A\| \,/\, \big[(1+t^2)\min\{|\mathrm{Im}\,z|, 1/2-|\mathrm{Im}\,z|\}\big]$, which is integrable in t. Furthermore, $F_n(t)A\Omega$ and $F_n(t-\mathrm{i}/2)A\Omega$ converge in norm to $W(t)A\Omega$ and $W(t-\mathrm{i}/2)A\Omega$, respectively, by assumptions i) and ii) of Theorem 2.4. Thus, we can again apply Lebesgue's theorem on dominated convergence to infer the existence of the limits $\lim_{n\to\infty} F_n(z)A\Omega$ for all $A\in\mathcal{A}$. This defines linear operators on the common, dense domain $\mathcal{A}\Omega$. Again, by the estimate of Lemma 2.5, we have $F(z)A_n\Omega\to0$ if $\|A_n\|\to0$, and therefore the $F(z)$ are closeable. The validity of Equation (3) is then clear. Since the bound noted above is uniform in n, and all the functions $F_n(z)A\Omega$ are holomorphic in S, we can apply the Stieltjes–Vitali theorem [34, Theorem 3.14.1] to deduce the stated holomorphy of $F(z)A\Omega$. We now consider the operators $F(-\mathrm{i}s)$, $0<s<1/2$. Using the same properties of Δ, E, one sees that these operators are self-adjoint, and in fact, positive: Namely, the limits are densely defined, symmetric and closeable operators, and an analytic vector for $\Delta^{1/2}$ is also analytic for $F(-\mathrm{i}s)$, $0<s<1/2$. Thus the $F(-\mathrm{i}s)$ possess a common, dense set of analytic vectors. Under these circumstances, the $F(-\mathrm{i}s)$ are essentially self-adjoint, and we denote their unique, self-adjoint extension by the same symbol. We now follow [51] to show that the functional equation $F(-\mathrm{i}(s+t)) = F(-\mathrm{i}s)F(-\mathrm{i}t)$ holds for s, $t>0$ such that $s+t<1/2$. To this end, consider first the case that s and t are rationally related, i.e., there exist $p, q \in \mathbb{N}$ such that

$$\frac{s+t}{r(p+q)} = \frac{s}{rp} = \frac{t}{rq}, \quad \text{for all } r \in \mathbb{N}.$$

Then

$$\left[E\Delta^{\frac{s+t}{r(p+q)}}E\right]^{r(p+q)} A\Omega = \left[E\Delta^{\frac{s}{rp}}E\right]^{rp} \left[E\Delta^{\frac{t}{rq}}E\right]^{rq} A\Omega, \quad A \in \mathcal{A},$$

from which the claim follows in the limit $r \to \infty$. The general case follows since $F(-\mathrm{i}s)A\Omega$ is holomorphic and therefore also strongly continuous in s for all $A \in \mathcal{A}$. Now set $\Gamma = F(-\mathrm{i}/4)$. By the spectral calculus for unbounded operators [39, Section 5.6], the positive powers Γ^σ exist for $0 < \sigma \leq 1$, and are positive operators with domain containing the common, dense domain $\mathcal{A}\Omega$. They satisfy the functional equation $\Gamma^{\sigma+\tau} = \Gamma^\sigma \,\hat{\cdot}\, \Gamma^\tau$ for σ, $\tau > 0$ such that $\sigma + \tau \leq 1$, and where $\hat{\cdot}$ denotes the closure of the operator product. The solution to this functional equation with initial condition $\Gamma = F(-\mathrm{i}/4)$ is unique and thus it follows $\Gamma^\sigma = F(-\mathrm{i}\sigma/4)$, since the operators F satisfy the same functional equation, and all operators in question depend continuously on σ, in the strong sense when applied to the common core $\mathcal{A}\Omega$. For $1/4 \leq s < 1/2$ we have $F(-\mathrm{i}s) = F(-\mathrm{i}/4)F(-\mathrm{i}(s-1/4)) = \Gamma F(-\mathrm{i}(s-1/4)) = \Gamma\Gamma^{4s-1} = \Gamma^{4s}$, which finally shows the identity $F(-\mathrm{i}s) = \Gamma^{4s}$ for $0 < s < 1/2$. Now, for every $A \in \mathcal{A}$, $\Gamma^{4iz}A\Omega$ extends to a holomorphic function on S which coincides with $F(z)A\Omega$ on the segment $\{-\mathrm{i}s \mid 0 < s < 1/2\}$ as we have just seen. The identity theorem for vector-valued, holomorphic functions [34, Theorem 3.11.5] then implies $\Gamma^{4iz}A\Omega = F(z)A\Omega$, $z \in S$ and all $A \in \mathcal{A}$. Thus $\Gamma^{4iz} = F(z)$ holds on S as an identity of densely defined, closed operators. Setting $G = P([0,\infty))$, where P is the spectral resolution of the identity for Γ, we see that we can write $\Gamma = G\Gamma = \Gamma G$, concluding this proof. □

Lemma 2.8. *It holds $G = E$, and $W(t) = E\Gamma^{4it}E$, for all t.*

Proof. Using (3) we can write, adding a zero contribution to that integral representation,

$$\begin{aligned}\langle B\Omega\,, F(t-\mathrm{i}\eta)A\Omega\rangle &= \\ \frac{(t+\mathrm{i}-\mathrm{i}\eta)^2}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}s &\left\{ \frac{\langle B\Omega\,, W(s-\mathrm{i}/2)A\Omega\rangle}{(s+\mathrm{i}/2)^2(s-t-\mathrm{i}/2+\mathrm{i}\eta)} - \frac{\langle B\Omega\,, W(s)A\Omega\rangle}{(s+\mathrm{i})^2(s-t+\mathrm{i}\eta)} - \right. \\ &\left. - \frac{\langle B\Omega\,, W(s-\mathrm{i}/2)A\Omega\rangle}{(s+\mathrm{i}/2)^2(s-t-\mathrm{i}/2-\mathrm{i}\eta)} + \frac{\langle B\Omega\,, W(s)A\Omega\rangle}{(s+\mathrm{i})^2(s-t-\mathrm{i}\eta)} \right\},\end{aligned}$$

where the integral over the last two terms is zero, as can be seen from (2) and the same arguments that were used to derive (3). This yields

$$= \frac{(t+\mathrm{i}-\mathrm{i}\eta)^2}{\pi} \int_{-\infty}^{\infty} \frac{\eta\cdot\langle B\Omega\,, W(s-\mathrm{i}/2)A\Omega\rangle}{(s+\mathrm{i}/2)^2((s-t-\mathrm{i}/2)^2+\eta^2)} - \frac{\eta\cdot\langle B\Omega\,, W(s)A\Omega\rangle}{(s+\mathrm{i})^2((s-t)^2+\eta^2)}\,\mathrm{d}s.$$

As $\eta \to 0_+$, the first term under the integral vanishes, while the second reproduces the integrable function $\langle A\Omega\,, W(t)B\Omega\rangle/(t+\mathrm{i})^2$ as the boundary value of its Poisson transformation. Thus we have seen

$$\lim_{\eta\to 0_+} \langle A\Omega\,, F(t-\mathrm{i}\eta)B\Omega\rangle = \langle A\Omega\,, W(t)B\Omega\rangle,$$

for given A, $B \in \mathcal{A}$, and almost all $t \in \mathbb{R}$. Since the integral is uniformly bounded in η, the boundary value of this Poisson transformation is continuous in t, see, e.g., [9, Section 5.4]. The same holds for $\langle A\Omega, W(t)B\Omega\rangle$ by assumption i) of Theorem 2.4, and therefore the limiting identity at $\eta = 0$ follows for all t. On the other hand, since $G\Gamma^{4it}G$ is strongly continuous in t, we have $\lim_{\eta\to 0_+} \langle A\Omega, F(t - i\eta)B\Omega\rangle = \langle A\Omega, G\Gamma^{4it}GB\Omega\rangle$ for all t. Thus, the identity of bounded operators $W(t) = G\Gamma^{4it}G$ holds for all t. By assumption we have $\text{w-lim}_{t\to 0} W(t) = E$, thus $W(s)W^*(s) = G\Gamma^{4is}\Gamma^{-4is}G = G$ implies $G = E$. □

Lemma 2.9. *The action* $\tau_t^E\colon \mathcal{A}_E \ni A_E \longmapsto \tau_t^E(A_E) = \Gamma^{4it}A_E\Gamma^{-4it}$ *is a strongly continuous group of automorphisms of* $\mathcal{A}_E$.

Proof. For $A_E = EAE \in \mathcal{A}_E$ we have $E\Delta^{\mathrm{it}/n}EA_EE\Delta^{-\mathrm{it}/n}E = E\sigma_{t/n}(A_E)E$, where σ is the modular group of $(\mathcal{A}, \Omega)$, and this shows $F_n(t)A_EF_n(-t) \in \mathcal{A}_E$ for all n. Since $\mathcal{A}_E$ is weakly closed and $F_n(t)A_EF_n(-t)$ converges strongly, and therefore also weakly, by assumption i) of Theorem 2.4, it converges to an element of $\mathcal{A}_E$. Since $\|F_n(t)A_EF_n(-t)\| \leq \|A_E\|$ for all n, the limit mapping is continuous on $\mathcal{A}_E$. By Lemma 2.8, it equals τ_t^E, as defined above, for all t. Since Γ^{4it} is a strongly continuous group of unitary operators on $E\mathcal{H}$, the assertion follows. □

Proof of Theorem 2.4. We note first, that $W(-t) = W(t)^*$ can be seen by direct methods as in [51]. Secondly, since τ^E is an automorphism group of $\mathcal{A}_E$, it follows by definition of $\mathcal{H}_E$, that the $W(t)$ leave that subspace invariant and thus form a unitary group on it. The stated analyticity properties of W are contained in the conclusions of Lemmas 2.7 and 2.8. □

2.4. **Preliminary Conclusions and Remarks.** Let us pause for a moment to put the results compiled so far into perspective. Reconsidering the assumptions and conclusions of Theorems 2.3 and 2.4, one might wonder what has actually been proved. In fact, the only additional information gained is the group property of the boundary values T, respectively, W. That is, the theorems ensure the existence of the *Zeno dynamics* if it is already known that the *Zeno effect* occurs, and persists in the Zeno limit. Both results therefore have a quite different status from Theorem 2.1. Hence the results of Sections 2.2 and 2.3 have to be complemented by convergence conditions that can be efficiently tested in models. This will be the subject of the remainder of the present paper. Yet, we use the occasion to present some general thoughts in that direction.

We have seen in Section 2.1 that the occurrence of the Zeno effect does not hinge on unitary evolutions, as has been noted by various authors in different contexts [56, 65]. Rather, for the existence of Zeno dynamics, i.e., the infinitely frequent measurement limit, analyticity properties of the original evolution seem to be of the essence. Three fundamentally different cases can be distinguished, and the pertinent analyticity domains are sketched in Figure 1. In

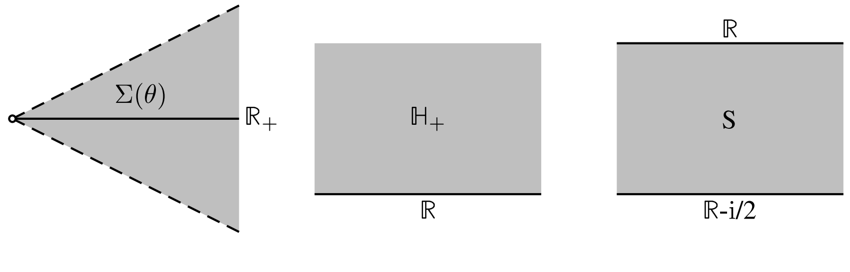

FIGURE 1. Analyticity domains relevant for Zeno dynamics

the simplest case of open sectors, shown on the left hand side, convergence of the Zeno dynamics is virtually unconditional. This was used in Section 2.2 to conclude that the Zeno dynamics exists in the upper half-plane $\mathbb{H}_+$, to simplify the proof of Misra's and Sudarshan's Theorem. In the third case of von Neumann Algebras, the sectorial result could not be used at all since the domain of analyticity does not contain an open sector, and we were bound to use the original 'bootstrap' strategy for the proof. But what renders these latter two cases more difficult is obviously the necessity to extend the limiting dynamics from the interior of the analyticity domain to its boundary. The general technique that is always applied here is the usage of reproducing kernels like the Cauchy–Hilbert or Poisson kernel to render a function as the boundary value of some complex integral transform of itself [9]. The difference described above between the first and the other two cases is also the reason why the latter do not yield explicit conditions for the existence of Zeno dynamics, but have to assume convergence *a priori*.

In the theory of generalised functions, many conditions are known as to when and in which sense a boundary value of an analytic function can be taken, and constitutes a continuous function (smooth function, tempered distribution, hyperfunction). These conditions hinge on the asymptotic growth of the analytic function as the boundary of the domain is approached — for continuity, it simply has to stay bounded. In turn, such growth conditions can be derived if the analytic function is considered as the Fourier–Laplace transform of some function on the real axis, and the properties of the boundary values then depend essentially on the growth or decay of the latter at infinity. Characterisations of regularity of functions by the growth of their (inverse) Fourier–Laplace transform are known as Payley–Wiener Theorems, see [59, 61, 43, 62], the simplest instance of which is the well known Riemann–Lebesgue Lemma, stating that the Fourier transforms of continuous functions vanishing at real infinity form exactly the class of integrable functions [61, Theorem 7.5].

Now, a unitary quantum evolution $U(t)$ generated by a Hamiltonian H is the Fourier–Laplace transform of the Hamiltonian's spectral density $P(\lambda)$,

$$U(z) = \mathrm{e}^{\mathrm{i}zH} = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}z\lambda} \mathrm{d}P(\lambda).$$

The growth conditions on P were what allowed us to extend U to the upper half-plane in the quantum mechanical case — P supported in $\mathbb{R}_+ \cup \{0\}$ since H is semibounded — respectively, to a strip in the von Neumann case, where P is exponentially damped in one direction. One should note that the semiboundedness of the Hamiltonian in the quantum mechanical case is *not* a necessary precondition for the Zeno effect, as was previously thought [51]. This is clearly shown by Theorem 2.4.

In conclusion, it would hence seem tempting to formulate growth conditions on some spectral density which would ensure the existence of Zeno dynamics, all the more since the Payley–Wiener Theorems usually provide *necessary and sufficient* conditions, due to the continuous invertibility of the Fourier–Laplace transformation. Yet, the spectral density one should consider here would be that of the generator of the desired Zeno dynamics itself, and that object is generically unknown *a priori*. In essence, the desired conditions would depend on detailed information about the energy distribution, with respect to the original Hamiltonian, of the states in the Zeno subspace, i.e., one of the (possibly many), invariant subspaces for the Zeno dynamics. Results in this promising direction are as of yet scarce, some remarks can be found in Section 2 of [52]. We will present an example in Section 4.3, where projections of rank one are considered, and conditions on the energy density distribution of the decaying initial state are given which are equivalent to the existence of the Zeno limit.

We conclude this section with some remarks of a more technical nature.

Remark (On Semigroups with Bounded Generators). We omitted from consideration the simple case of C_0-semigroups with bounded generators, for which a convergence result was noted in [49].

Theorem 2.10 ([49, Theorem 2.1]). *Let A be a bounded operator on a Banach space $\mathcal{B}$, generating a C_0-semigroup $(\mathrm{e}^{tA})_{t\geq 0}$, and E a bounded projection on $\mathcal{B}$. Then*

$$\lim_{n\to\infty}\left[\mathrm{e}^{\frac{t}{n}A}E\right]^n\phi = \lim_{n\to\infty}\left[E\mathrm{e}^{\frac{t}{n}A}\right]^n\phi = \lim_{n\to\infty}\left[E\mathrm{e}^{\frac{t}{n}A}E\right]^n\phi = \mathrm{e}^{tEAE}\phi,$$

for all $\phi\in\mathcal{B}$ and uniformly in $t\in[0,T]$ for all $T\geq 0$.

This is another case in which the Zeno paradox emerges unconditionally (in the sense of the following remark), but here it is also directly possible to identify the generator of the Zeno dynamics as EAE, a result for which additional conditions are needed in the general case of unbounded generators, see Sections 3.3 and 3.4. The direct proof in [49] is based on Chernoff's product formula [12]. It uses the decomposition $\phi = E\phi + (\mathbb{1}-E)\phi$ iteratively in its estimations, in a manner that is similar to the method that was used independently in [64], and will be used in Proposition 3.2, to obtain a sufficient condition for the existence of the Zeno limit.

Remark (On Two Complete Characterisations). It recently turned out that the two cases of sectorial semigroups, and such with bounded generators are in fact particularly benign [46]. They are in fact, in the Hilbert space case, completely characterised by the unconditional emergence of the Zeno paradox. Here unconditional means in any possible case, i.e., for every projection.

Theorem 2.11 ([46, Theorem 3]). *Let A be the generator of the C_0-semigroup $(\mathrm{e}^{tA})_{t\geq 0}$ on the Hilbert space $\mathcal{H}$. Then A is bounded ($-A$ is associated with a densely-defined, closed, sectorial form) if and only if, for all $\phi\in\mathcal{H}$, $t>0$,*

$$\lim_{n\to\infty}\left[\mathrm{e}^{\frac{t}{n}A}E\right]^n\phi$$

exists for all bounded projections (orthogonal projections) E.

What this complete characterisation also says is that there will be counterexamples in the quantum mechanical, and all the more in the von Neumann case where the generic generator is only semibounded or even unbounded, and the associated semigroup cannot be extended to a sector around the real axis. That is, there will be some orthogonal projections for which the Zeno paradox does not emerge. Very recently, Matolcsi also reported on the analogous results in the Banach space case [47].

Remark (On CPT symmetry). A slight weakening of the assumptions of Theorem 2.3 is possible if the system under consideration possesses a time reversal or, more specially, a CPT symmetry, i.e., an antiunitary operator θ such that

$$\begin{aligned}&\theta E\theta^{-1} = E,\\ &\theta U(t)\theta^{-1} = U(-t)\quad\text{for all } t.\end{aligned}$$

Then it suffices to assume existence of the limit $T(t)$ only for $t > 0$, since

$$\begin{aligned} T(-t) &= \operatorname*{s-lim}_{n\to\infty} \big[EU(-t/n)E\big]^n \\ &= \operatorname*{s-lim}_{n\to\infty} \theta\big[EU(-t/n)E\big]^n \theta^{-1} = \theta T(t)\theta^{-1}. \end{aligned}$$

Although this simplification can usually be applied in the case of quantum systems — where CPT symmetry is generic — it is not all-too helpful in concrete models, since the conditions used there imply convergence on the whole axis anyway, see Section 3.

Remark (On the Zeno Subspace). In Sections 2.1 and 2.2 we found that the Zeno dynamics is confined to the subspace $E\mathcal{H}$. Yet, this is only the ***maximal* Zeno subspace** that is invariant under the Zeno dynamics, if it exists at all. And in fact we saw in Theorem 2.4 that there the Zeno dynamics can be confined to the generally smaller subspace

$$\mathcal{H}_E = \overline{\mathcal{A}_E\Omega} = \overline{E\mathcal{A}E\Omega} \not\subseteq \overline{E\mathcal{A}\Omega} = E\mathcal{H}.$$

The main reason that made this identification possible was that E was assumed to be an element of the von Neumann algebra $\mathcal{A}$ in question, i.e., using the terminology of algebraic quantum theory, that it was an *observable*. This seems to be a natural choice in this context for physical reasons, as can be seen in the models considered in [64], and it is also most closely related to the colloquial description of the Zeno effect as 'evolution interrupted by frequent measurement'. But in the general case of arbitrary projections on a separable Hilbert space, larger Zeno subspaces can appear, and have been characterised in [14]. We will come back to that issue in Section 3.4.

Remark (On Separability). The conditions of weak continuity in the two Theorems 2.3 and 2.4, can be relaxed if the Hilbert space considered is separable. To make the argument clear, consider for instance Theorem 2.3, where the following reasoning can be used to conclude that $W(t+\mathrm{i}\eta)$ weakly approximates $T(t)$, if $\mathcal{H}$ is separable. We saw that

$$\lim_{\eta\to 0}\,(\phi\,, W(t+\mathrm{i}\eta)\psi) = (\phi\,, T(t)\psi)$$

t-almost everywhere , i.e., outside a set $\mathcal{N}_{\phi,\psi}$ of Lebesgue measure zero. For a countable, dense set $\mathcal{D} \subset \mathcal{H}$ (and here is the only place where separability of $\mathcal{H}$ is used), set $\mathcal{N} = \cup_{\phi,\psi\in\mathcal{D}}\mathcal{N}_{\phi,\psi}$, which is a null set as the countable union of null sets. Now, approximate two arbitrary vectors ϕ, $\psi \in \mathcal{H}$ by sequences $\{\phi_n\}_{n\in\mathbb{N}}$, $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$. With $A(s,\eta) = W(s+\mathrm{i}\eta) - T(s)$ we have

$$(\phi\,, A(s,\eta)\psi) = (\phi-\phi_n\,, A(s,\eta)\psi) + (\phi_n\,, A(s,\eta)(\psi-\psi_n)) + (\phi_n\,, A(s,\eta)\psi_n)$$

For s outside $\mathcal{N}$, the third term tends to zero as $\eta \to 0_+$, since ϕ_n, $\psi_n \in \mathcal{D}$, while the first and second converge to zero as $n \to \infty$. Together, this shows $\text{w-lim}_{\eta\to 0_+} W(t+\mathrm{i}\eta) = T(t)$, t-almost everywhere, from which point one can proceed as in [51]. Essentially the same argument applies to Theorem 2.4 when the GNS–Hilbert space is separable, and this is in fact the prevalent case for models in quantum statistical mechanics, in which the states inducing the representations are, for instance, constructed as thermodynamic limits of locally normal states, cf. [8]. In particular in the von Neumann case we assumed this state to be normal, which implies separability, and thus the assumptions of Theorem 2.4 are actually a bit too strong.

3. Operator Theoretical Conditions

3.1. **The Asymptotic Zeno Condition.** The first condition for the existence of the Zeno limit we want to present was found in [64], and applied to quantum statistical mechanics. It is closely related to the quadratic short-time behaviour of quantum evolutions, which is commonly associated with the Zeno effect [2, 52], see Section 4.1. We present it in a more neutral form which makes clear that its scope is somewhat broader.

Definition 3.1. Let E be a projection on a Hilbert space $\mathcal{H}$ and $(U(t))_{t\in\mathbb{R}}$ a unitary group on $\mathcal{H}$. We say that (U,E) satisfies the **uniform (strong) asymptotic Zeno condition**, in short, **uAZC (sAZC)** if the asymptotic relation

$$E^{\perp}U(\tau)E = O(\tau) \quad \text{uniformly (strongly) as } \tau \to 0,$$

holds. This relation means that there shall exist $\tau_0 > 0$ and $C \geq 0$ such that for all τ with $|\tau| < \tau_\psi$ holds the estimate $\|E^{\perp}U(\tau)E\| \leq C^{1/2}|\tau|$ (respectively, for any $\psi \in \mathcal{H}$ exist $\tau_0 > 0$ and $C_\psi \geq 0$ such that for all τ with $|\tau| < \tau_0$ holds $\|E^{\perp}U(\tau)E\psi\| \leq C_\psi^{1/2}|\tau|$).

The (uAZC) is in fact equivalent to saying that the function $E^{\perp}U(t)E$ is uniformly Lipschitz continuous at the point $t = 0$. Not surprisingly, Lipschitz continuity is well known as a salient condition for the existence of solutions to (nonlinear) evolution equations. For us, it ensures existence of the evolution in the Zeno limit.

Proposition 3.2. *If (U,E) satisfies uAZC (sAZC) then*

$$F_n(t) \stackrel{\text{def}}{=} \big[EU(t/n)E\big]^n, \quad \text{for } t \in \mathbb{R},\ n \in \mathbb{N},$$

converge uniformly (strongly) to operators $W(t)$. Furthermore $W(t)$ is uniformly (strongly) continuous in t and the uniform (strong) limit as $t \to 0$ of $W(t)$ is E.

Proof. We present the proof of the uniform case of which the strong one is a straightforward generalisation. We want to see whether the $F_n(t)$ form a Cauchy sequence in n for given t. For that, we have to estimate the quantities

$$\|(F_n(t) - F_m(t))\| \leq \|(F_n(t) - F_{nm}(t))\| + \|(F_m(t) - F_{nm}(t))\|.$$

A double telescopic estimation yields

$$\begin{aligned}
&\|(F_n(t) - F_{nm}(t))\| \leq \\
&\quad \sum_{k=1}^{n}\sum_{l=1}^{m-1} \Big\| \big[EU(t/n)E\big]^{n-k} \Big(EU(t(m-l)/(nm))E\big[EU(t/(nm))E\big]^{l} - \\
&\qquad EU(t(m-l+1)/(nm))E\big[EU(t/(nm))E\big]^{l-1}\Big) \big[EU(t/(nm))E\big]^{m(k-1)} \Big\|.
\end{aligned}$$

Now, since with $E^{\perp} \stackrel{\text{def}}{=} \mathbb{1} - E$ we have

$$EU(t(m-l+1)/(nm))E = EU(t(m-l)/nm)(E + E^{\perp})U(t/(nm))E,$$

we find that the (k,l)th term in the sum is equal to

$$\begin{aligned}
&\Big\| \big[EU(t/n)E\big]^{n-k} \cdot EU(t(m-l)/(nm))E^{\perp} \cdot \\
&\qquad \cdot E^{\perp}U(t/(nm))E \cdot \big[EU(t/(nm))E\big]^{l-1} \big[EU(t/(nm))E\big]^{m(k-1)} \Big\|.
\end{aligned}$$

Multiplying out and using repeatedly $\|AB\| \leq \|A\|\|B\|$, we estimate this expression from above by

$$\big\|\big[EU(t/n)E\big]\big\|^{n-k} \cdot \big\|EU(t(m-l)/(nm))E^{\perp}\big\| \\ \cdot \big\|E^{\perp}U(t/(nm))E\big\| \cdot \big\|\big[EU(t/(nm))E\big]\big\|^{l-1}\big\|\big[EU(t/(nm))E\big]\big\|^{m(k-1)}.$$

Observing that all terms containing only the projection E have operator norm ≤ 1 and can thus be omitted in the estimation of $\big\|\big(F_n(t) - F_{nm}(t)\big)\big\|$, we arrive at

$$\big\|\big(F_n(t) - F_{nm}(t)\big)\big\| \leq \sum_{k=1}^{n}\sum_{l=1}^{m-1} \big\|EU(t(m-l)/(nm))E^{\perp}\big\|\big\|E^{\perp}U(t/(nm))E\big\|.$$

Now, for $n > n_0 \geq 1/\tau_0$, and $m \geq 2$, the uAZC implies

$$\begin{aligned}\big\|\big(F_n(t) - F_{nm}(t)\big)\big\| &\leq Ct^2 \sum_{k=1}^{n}\sum_{l=1}^{m-1} \frac{m-l}{n^2m^2} \\ &= Ct^2 \sum_{k=1}^{n} \frac{(m-1)m}{2n^2m^2} \\ &= \frac{Ct^2}{2}\frac{(m-1)m}{nm^2} \leq \frac{Ct^2}{2n}.\end{aligned}$$

An analogous estimate holds for $\big\|\big(F_m(t) - F_{nm}(t)\big)\big\|$, which yields for $m - 2 \geq n > n_0 \geq 1/\tau_0$ the overall result

$$\big\|\big(F_n(t) - F_m(t)\big)\big\| \leq \frac{Ct^2}{n}. \tag{4}$$

The first statement of this proposition is now clear since the estimate above shows that the $F_n(t)$ form Cauchy sequences which are therefore *a fortiori* convergent. The other statements follow from $F_n(0) = E$ for all n, and the fact that the convergence of $F_n(t)$ is uniform for t on compact subsets of $\mathbb{R}$. This follows in turn from the t-dependence of the final estimate. □

Combining this result with Theorem 2.3, we immediately obtain an efficient condition for Zeno effect and dynamics in the quantum mechanical case.

Corollary 3.3. *Let $U(t) = \mathrm{e}^{\mathrm{i}tH}$ be a unitary group with nonnegative, self-adjoint generator H. If (U, E) satisfies uAZC or sAZC then $W(t) = \operatorname*{s-lim}_{n\to\infty} F_n(t)$ is a degenerate group of unitaries on $E\mathcal{H}$.*

The AZC can be related to one well known, basic condition for the Zeno paradox in quantum mechanics, namely finiteness of the first moment of the Hamiltonian in the initial state. Consider the rank one projection E_ψ onto a vector in the domain of H, and denote by $\mathcal{A}(t) \stackrel{\text{def}}{=} \big(\psi\,, \mathrm{e}^{\mathrm{i}tH}\psi\big)$ the **survival amplitude** in this state. Then holds the elementary asymptotic expansion

$$\mathcal{A}(\tau) \sim 1 + \mathrm{i}\tau(\psi\,, H\psi) - \frac{\tau^2}{2}\big(\psi\,, H^2\psi\big) \quad (\tau \to 0),$$

with

$$(\psi\,, H\psi) < \infty, \quad \big(\psi\,, H^2\psi\big) = \|H\psi\|^2 < \infty.$$

This is, as will be seen in Section 4.1, a rather direct expression of the quadratic short time behaviour of quantum dynamics, which is commonly identified as the main cause for Zeno effect and paradox [52, 65, 16]. Obviously, it implies uAZC and therefore ensures convergence to the Zeno limit.

The main strength of the AZCs lies in the fact that they can be efficiently tested in concrete models. In particular they enable the use of perturbation theory to obtain conditions for Zeno dynamics. This has been used in [64] to treat models of quantum statistical mechanics accordingly, and we will present two more generic of the pertinent results below in Section 3.2. The AZCs are quite weak and thus indicate how generic a quantum phenomenon the Zeno effect indeed is. For example it is always satisfied if the generator H of the group U is bounded, or, more generally, if E projects onto a closed subspace of entire analytic elements for H, e.g., if E is contained in a bounded spectral projection of H. In those cases a power series expansion of $U(t) = \mathrm{e}^{\mathrm{i}tH}$ implies uAZC. However, if neither is the case, then uAZC will generally fail to hold in that its defining estimate is not uniform in $\psi \in \mathcal{H}$, respectively, sAZC will not hold for all $\psi \in \mathcal{H}$ (not even on a dense subset). It is also noteworthy that in showing the convergence of F_n to W, we have not used the unitarity of U. Thus an analogue of the Zeno effect is also possible for non-unitary (non-Hamiltonian, non-Schrödinger) evolutions, as already noted above. On the other hand, the group property of U was essential for obtaining the quadratic term that forced the convergence of the sequence. Asymptotic bounds on $E^{\perp}U(t)E$ have already been considered by other authors [50, 54, 15], in the context of short-time regeneration of an undecayed state. In particular in [15], the deviation of the 'reduced evolution' $EU(t)E$ from being a semigroup has been expressed by such (polynomial) bounds. We will obtain a similar yet somewhat coarser result in Section 3.2, and present a much more advanced one in Section 3.4.

3.2. **Application of the AZC to Quantum Statistical Mechanics.** One of the most successful mathematical theories for physical phenomena is the algebraic formulation of quantum statistical mechanics and quantum field theory [8, 33]. Its basic tenet is the viewpoint that all relevant information about a system resides in its **observable algebra** $\mathcal{A}$, a topological algebra which captures all *finite* measurements that can be performed on the system, where finiteness is to be understood with respect to time, space, and energy resources. In this context it became clear that *weakly closed*, i.e., W^* or von Neumann algebras are the natural objects to consider, and this revealed another deep connection between 'pure' mathematics and physics. In particular the theoretical development of quantum statistical mechanics was spurred by the close relation between the modular dynamics of von Neumann algebras and the notion of thermal equilibrium states of C^*, respectively, W^*-dynamical systems, incorporated in the **KMS condition**. This connection, given by Takesaki's theorem [8, Theorem 5.3.10], is the fundament for the application of the mathematical result Theorem 2.4 to a general W^*-dynamical system $(\mathcal{A}, \tau)$ with faithful, normal KMS state ω at **inverse temperature** β (termed (τ, β)–KMS state). This was laid out in [64], and we report on the central results in the present section.

To fix the context, we denote by Ω the vector representative of ω in the associated representation π_ω on the (separable) GNS–Hilbert space $\mathcal{H}$. The automorphism group τ is assumed to be implemented covariantly, i.e., by a strongly continuous group of unitary operators $U(t)$ on $\mathcal{H}$. The representation π_ω will be omitted from the notation, when no confusion is possible. We are now ready to combine Theorem 2.4 and the AZC to obtain an effective condition for the emergence of Zeno dynamics in quantum statistical mechanics.

Theorem 3.4. *Under the conditions described above, let $\beta > 0$, assume $\mathcal{A}$ to be unital, let $E \in \mathcal{A}$ be a projection, and set $E^{\perp} \stackrel{\mathrm{def}}{=} \mathbb{1} - E$. Assume that the* ***asymptotic Zeno condition*** *holds: For $A \in \mathcal{A}$, the estimate*

$$\left\| E^{\perp} U(\zeta) E A \Omega \right\| \leq C \cdot \|A\Omega\| \cdot |\zeta|$$

is valid for $\zeta \in \mathbb{C}$ *with* $|\zeta| < r_0$ *for some fixed* $r_0 > 0$ *and* $\operatorname{Im}\zeta \geq 0$. *In short:* (U, E) *satisfies AZC for* $\mathcal{A}$. *Then the strong operator limits*

$$W(t) \stackrel{\text{def}}{=} \operatorname*{s-lim}_{n\to\infty} \big[EU(t/n)E\big]^n$$

exist, and form a strongly continuous group of unitary operators on the ***Zeno subspace*** $\mathcal{H}_E \stackrel{\text{def}}{=} \overline{\mathcal{A}_E\Omega} \subset E\mathcal{H}$*, where* $\mathcal{A}_E \stackrel{\text{def}}{=} E\mathcal{A}E$*. The group* $W(t)$ *induces an automorphism group* τ^E *of* $\mathcal{A}_E$*, such that* $(\mathcal{A}_E, \tau^E)$ *is a* W^**-dynamical system. The vectors* $W(z)A_E\Omega$*,* $A_E \in \mathcal{A}_E$*, extend analytically to the strip* $0 < \operatorname{Im} z < \beta/2$ *and are continuous on its boundary.*

The slight modification of the AZC is needed here, because to satisfy the conditions of Theorem 2.4, we need convergence on *both* boundaries of the strip $\{0 \leq \operatorname{Im} z \leq \beta/2\}$ in the complex plane, which would in general not hold if we assumed AZC only on the real axis. As was shown in [64], this version of AZC is satisfied in many model cases, of which we will present a generic one further down in this section.

Proof. We show that the assumptions of Theorem 2.4 are satisfied, from which we obtain the stated conclusions. First, for real τ, the AZC implies $E^\perp U(\tau)E = O(\tau)$ uniformly since the operators in question are bounded, $\mathcal{A}\Omega$ is dense in $\mathcal{H}$, and AZC holds uniformly in A on a fixed real neighbourhood of 0. Therefore Proposition 3.2 yields the existence of $W(t)$, $t \in \mathbb{R}$, its weak continuity in t and the initial condition $\operatorname{w-lim}_{t\to 0} W(t) = E$. These facts comprise condition i) of Theorem 2.4. For the second condition of the cited theorem, we need only to show that $W(t + \mathrm{i}\beta/2)$ exist as strong operator limits on the common, dense domain $\mathcal{A}\Omega$. For this notice that the calculations yielding Equation (4) are applicable to $(F_n(t + \mathrm{i}\beta/2) - F_m(t + \mathrm{i}\beta/2))A\Omega$, leading to the estimate

$$\big\|\big(F_n(t + \mathrm{i}\beta/2) - F_m(t + \mathrm{i}\beta/2)\big)A\Omega\big\| \leq \frac{C\|A\||t + \mathrm{i}\beta/2|^2}{n}$$

for $A \in \mathcal{A}\Omega$, and $m - 2 \geq n > n_0 \geq 1/r_0$. Thus, also condition ii) of Theorem 2.4 is satisfied and the stated conclusions follow from it. □

It should be noted that we restrict our discussion completely to a concrete realisation of a W^*-dynamical system given by the GNS representation π_ω of a fixed, *a priori* chosen KMS state ω. That is we consider the von Neumann algebra $\pi_\omega(\mathcal{A})$ on the GNS Hilbert space $\mathcal{H}$ and assume the dynamical automorphism group to be π_ω-covariant, i.e., to be realised by a strongly continuous, unitary group of operators. This notably simplifies our treatment, but also restricts it to a single superselection sector of the theory. Nevertheless, the results in Section 3.3 below are essentially independent of the chosen representation.

Theorem 3.4 leads directly to what may be seen as *the* proper manifestation of the Zeno paradox in quantum statistical mechanics, and what is the closest counterpart to the Zeno paradox in quantum mechanics, i.e., the prevention of a decay process [37, 17]. In the present context, the Zeno effect can prevent the *return to an equilibrium state*, as we will show now.

As said, The power of the AZC lies to a great extent in the fact that it yields perturbative conditions for the occurrence of the Zeno effect. For it is known that a perturbed semigroup U_t^P, resulting from adding a bounded perturbation P to a C_0-semigroup U_t, is close to U_t for small times in the sense that $\|U_t - U_t^P\| = O(t)$, as $t \to 0$, see [8, Theorem 3.1.33]. Now if E projects onto a subspace which is invariant under U_t, then this asymptotic behaviour implies that the Zeno dynamics of the pair (U_t^P, E) exists. We exemplify this basic mechanism in the following.

It is well known [60] that a quantum system will under general conditions, e.g., if $(\mathcal{A}, \tau)$ is asymptotically Abelian, return to equilibrium for large times. This means that if the system is prepared in an equilibrium state ω^P for the perturbed evolution τ^P, where $P = P^* \in \mathcal{A}_\tau$ is a bounded perturbation, which is in the set of entire analytic elements $\mathcal{A}_\tau$ for τ (termed local perturbation), and thereafter evolves under the unperturbed dynamics τ, one recovers a τ-equilibrium state $\omega_\pm$ for $t \to \pm\infty$, in the weak* topology. Assume that the perturbed and unperturbed dynamics are implemented by unitaries U^P and U, respectively, which is always possible if either τ or τ^P is covariant in the chosen representation [60, Theorem 1]. Then, the unperturbed dynamics can be written in terms of the perturbed one by the perturbation expansion [8, Theorem 3.1.33 and Proposition 5.4.1]

$$U(t) = U^P(t) + \sum_{n\geq 1} \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n U^P(t_1) P U^P(t_2 - t_1) P \cdots P U^P(t - t_n),$$

where the n-th term in the sum is bounded by $\|P\|^n t^n / n!$. Let the system be prepared in any τ^P-invariant state φ^P. In the representation π^P induced by the chosen τ^P-KMS state ω^P the corresponding vector states are denoted by Φ^P and Ω^P respectively. Let E be the projection onto the space spanned by the vector Φ^P and assume $E \in \mathcal{A}$. Then the above expansion readily yields $E^\perp U(t) E = O(t)$ uniformly, since the τ^P-invariance of φ^P implies $U^P(t)\Phi^P = \Phi^P$. In application to vectors in $\mathcal{A}\Omega$ this estimate extends to a fixed, small neighbourhood of 0 in the upper half-plane and is uniform in those vectors. Thus the AZC holds, the Zeno dynamics converges, and the system remains in the state φ^P. The same reasoning is applicable if E projects onto a τ^P-invariant subspace. Let us resume what we have proved.

Corollary 3.5. *Let $(\tau, \mathcal{A})$ be as above. Let $P \in \mathcal{A}$ be a local perturbation, and denote by τ^P perturbed dynamics as constructed in* [8, Proposition 5.4.1 and Corollary 5.4.2]. *Let $E \in \mathcal{A}$ be a τ^P-invariant projection, i.e., $\tau^P(E) = E$. Then the (τ, E)-Zeno dynamics τ^E is an automorphism group of $\mathcal{A}_E$, and $\mathcal{H}_E$ is τ^E-invariant.*

3.3. **Zeno Generator and Zeno Equilibria.** Let us remain for another while in the context of quantum statistical mechanics to emphasise how favourable its mathematical framework is for the study of the Zeno effect. In particular we want to see that the AZC can be used to identify the generator of the Zeno dynamics explicitly. Let H be the generator of $U(t) = \mathrm{e}^{\mathrm{i}tH}$. The unitary group $U_E(t) \stackrel{\mathrm{def}}{=} \mathrm{e}^{\mathrm{i}tEHE}$ is called the **reduced dynamics** associated with (U, E), whenever it is defined on the Zeno subspace $\mathcal{H}_E$. To be able to compare the reduced with the Zeno dynamics, we need a technical condition, which has been shown in [64] to be satisfied in many models where also the AZC holds: We call (U, E) **regular** if $\mathcal{A}_E$ contains a dense set of elements which are analytic for τ in an arbitrary neighbourhood of zero. The condition of regularity will be required to have enough analytic vectors in $\mathcal{H}_E$ at hand for the proof below to work. It excludes pathological cases, e.g., when E projects onto a subspace of states with properly infinite energy.

Proposition 3.6. *Let (U, E) be regular and satisfy AZC for $\mathcal{A}$. Then $U_E(t)$ equals $W(t)$, when restricted to $\mathcal{H}_E$.*

Throughout the proof below let $\psi_E \in \mathcal{A}_{E,\tau}\Omega \subset \mathcal{H}_E$, where $\mathcal{A}_{E,\tau}$ is a dense set of elements in $\mathcal{A}_E$, which are analytic for τ. Record that, by the discussion following [8, Definition 3.1.17], the τ-analyticity of ψ_E is equivalent to analyticity with respect to U and this is in turn equivalent to the convergence of power series of analytic functions in σH applied to ψ_E, for $\sigma \in \mathbb{C}$ small enough, as given in the cited definition.

Proof of Proposition 3.6. We first derive a useful asymptotic estimate: Setting $\psi_E(\sigma) \stackrel{\text{def}}{=} U_E(\sigma)\psi_E$ holds

$$\begin{aligned}\big\|\big(U_E(\tau) - EU(\tau)E\big)\psi_E(\sigma)\big\| &= \left\|\left\{\sum_{k=0}^{\infty}\frac{(\mathrm{i}\tau)^k(EHE)^k}{k!} - E\sum_{l=0}^{\infty}\frac{(\mathrm{i}\tau)^l H^l}{l!}E\right\}\psi_E(\sigma)\right\| \\ &= \left\|\sum_{k=2}^{\infty}\frac{(\mathrm{i}\tau)^k}{k!}\big[(EHE)^k - EH^kE\big]\psi_E(\sigma)\right\|,\end{aligned}$$

using $E\psi_E(\sigma) = \psi_E(\sigma)$, which is clear since U_E commutes with E. By using $\|E\| = 1$, this can be estimated further as

$$\leq 2\sum_{k=2}^{\infty}\frac{|\tau|^k}{k!}\big\|H^k\psi_E(\sigma)\big\|.$$

Since ψ_E is analytic for U in a neighbourhood of 0, also the translates $\psi_E(\sigma) = U_E(\sigma)\psi_E$, for σ small enough, will be analytic for U in a somewhat smaller neighbourhood of 0. This can be seen by noting that the power series of $U_E(\sigma)$ is term-wise bounded in norm by a convergent one, where EHE is replaced by H, using $\|E\| = 1$. The composition of power series in question then amounts to the composition of analytic functions of H for σ, τ, small enough. Therefore the power series on the right hand side of the last inequality is convergent for σ, τ small, and defines an analytic function in τ which is $O(|\tau|^2)$ as $|\tau| \to 0$. Thus, we finally obtain for small enough σ, τ the estimate

$$\big\|\big(U_E(\tau) - EU(\tau)E\big)\psi_E(\sigma)\big\| \leq \tau^2 \cdot C_{\psi_E,\sigma} < \infty. \tag{5}$$

Now, from $U_E(t)\psi_E = EU_E(t)E\psi_E$, follows the identity

$$U_E(t)\psi_E = \big[EU_E(t/n)E\big]^n\psi_E, \quad \text{for all } n, \tag{6}$$

by iteration. Exploiting this, we can rewrite $F_n(t) - U_E(t)$ to yield

$$\big\|F_n(t)\psi_E - U_E(t)\psi_E\big\| = \big\|\big[EU(t/n)E\big]^n\psi_E - \big[EU_E(t/n)E\big]^n\psi_E\big\|.$$

A telescopic estimate shows

$$\leq \sum_{i=1}^{n}\left\|\left\{\big[EU(t/n)E\big]^{n-i}\big(EU(t/n)E - EU_E(t/n)E\big)\big[EU_E(t/n)E\big]^{i-1}\right\}\psi_E\right\|.$$

The norm of the vector under the sum is, using (6),

$$\big\|[EU(t/n)E]^{n-i}(EU(t/n)E - EU_E(t/n)E)\Psi_E(t(i-1)/n)\big\|$$

Using commutativity of U_E with E, and the invariance of $\Psi_E(\sigma)$ under E, we have $EU_E(t/n)E\Psi_E(t(i-1)/n) = U_E(t/n)\Psi_E(t(i-1)/n)$, and use this to rewrite the above expression as

$$= \big\|[EU(t/n)E]^{n-i}(EU(t/n)E - U_E(t/n))\Psi_E(t(i-1)/n)\big\|$$

Now, with $\|[EU(t/n)E]^{n-i}\| \leq 1$ and $\|AB\Psi\| \leq \|A\|\|B\Psi\|$, this is bounded by

$$\leq \|(EU(t/n)E - U_E(t/n))\Psi_E(t(i-1)/n)\|.$$

Putting this together, we obtain the estimate

$$\big\|F_n(t)\Psi_E - U_E(t)\Psi_E\big\| \leq \sum_{i=1}^{n} \|(EU(t/n)E - U_E(t/n))\Psi_E(t(i-1)/n)\|.$$

We can now apply (5) to obtain, for $n > M$ large enough,

$$\|F_n(t)\psi_E - U_E(t)\psi_E\| \leq \sum_{i=1}^{n} \left(\frac{t}{n}\right)^2 \cdot \sup_{|\sigma|\leq|t|} C_{\psi_E,\sigma} = \frac{t^2 C'_{\psi_E,t}}{n},$$

for some finite $C'_{\psi_E,t}$. Since F_n converges strongly to W by the AZC, it follows $W(t)\psi_E = U_E(t)\psi_E$. The density of the elements $\mathcal{A}_{E,\tau}\Omega$ in $\mathcal{H}_E$ then shows the claim. □

The explicit form of the generator for the Zeno dynamics yields an heuristic argument for the equivalence of the Zeno effects produced by 'pulsed' and 'continuous' measurement, respectively. The latter commonly denotes the simple model for the coupling of the quantum system to a measurement apparatus that results from adding to the original Hamiltonian a measurement Hamiltonian multiplied by a coupling constant, and letting the coupling constant tend to infinity [22, 24, 25]. The essential point here is that the degrees of freedom in the Zeno subspace $\mathcal{H}_E$ become energetically infinitely separated from those in its orthogonal complement. For this it suffices to set

$$H_K \stackrel{\text{def}}{=} H + KE^{\perp}, \quad U_K(t) \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}tH_K},$$

and to consider the limit $K \to \infty$. This can be done by applying analytic perturbation theory to

$$H_\lambda \stackrel{\text{def}}{=} \lambda H + E^{\perp}, \quad \text{with } \lambda \stackrel{\text{def}}{=} K^{-1},$$

and

$$U_\lambda(\tau) \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}\tau H_\lambda} = U_K(t), \quad \text{with } \tau \stackrel{\text{def}}{=} Kt = t/\lambda,$$

around $\lambda = 0$. The final result is

$$\lim_{K\to\infty} U_K(t)\psi = \mathrm{e}^{\mathrm{i}tEHE}\psi,$$

for any vector $\psi \in \mathcal{H}_E$. Details are to be found in [25, Section 7]. Nevertheless, this treatment of 'continuous measurement' is certainly the coarsest possible. To examine more deeply the relationship between the two manifestations of the Zeno effect, one should consider more refined models for the interaction of a quantum with a classical system, e.g., as in [7].

The identification of the Zeno dynamics with the reduced one is in perfect accordance with the structure of the Zeno subspace $\mathcal{H}_E$, showing that the latter is in this case indeed the minimal subspace to which the Zeno dynamics is restricted (apart from further reducibility of the Zeno generator). The knowledge about the Zeno generator can be used to describe an important class of equilibrium states for the Zeno dynamics, namely **Gibbs equilibria**. For this note that U_E induces an automorphism group $\widehat{\tau}$ of $\mathcal{A}_E$, as follows from the reasoning of the proof of Lemma 2.9. Proposition 3.6 now amounts to the following.

Corollary 3.7. *If (U, E) is regular and satisfies AZC for $\mathcal{A}$ then, for every $\beta > 0$, the set of (τ^E, β)-KMS states of $\mathcal{A}_E$ equals the set of $(\widehat{\tau}^E, \beta)$-KMS states.*

This result is independent of the representation, since the reasoning of Proposition 3.6 can be repeated in any covariant representation. It applies, in particular, to the important case of Gibbs states for quantum **spin systems**. (for a detailed exposition of these, we refer the reader to [8, Section 6.2]). Consider a quantum spin system over the lattice $\mathbb{X} \stackrel{\text{def}}{=} \mathbb{Z}^d$ with interaction $\Phi\colon \mathbb{X} \supset X \longmapsto \mathcal{A}_X$. The local Hamiltonian of a bounded subset $\Lambda \subset \mathbb{X}$ is $H_\Phi(\Lambda) \stackrel{\text{def}}{=} \sum_{X\subset\Lambda} \Phi(X)$ and $U_\Lambda(t) \stackrel{\text{def}}{=} \mathrm{e}^{itH_\Phi(\Lambda)}$ is the associated group of unitaries on the finite dimensional, local Hilbert space $\mathcal{H}_\Lambda$. The ordinary local Gibbs states over bounded regions $\Lambda \subset \mathbb{Z}^d$ are

$$\omega_\Lambda(A) \stackrel{\text{def}}{=} \frac{\mathrm{Tr}_{\mathcal{H}_\Lambda}\big(\mathrm{e}^{-\beta H(\Lambda)}A\big)}{\mathrm{Tr}_{\mathcal{H}_\Lambda}\big(\mathrm{e}^{-\beta H(\Lambda)}\big)}, \quad \text{for } A \in \mathcal{A}(\Lambda),$$

and a candidate for a local Zeno equilibrium over Λ is thus

$$\omega_{E_\Lambda}(A_{E_\Lambda}) \stackrel{\text{def}}{=} \frac{\mathrm{Tr}_{\mathcal{H}_\Lambda}\big(\mathrm{e}^{-\beta E_\Lambda H(\Lambda)E_\Lambda}A_{E_\Lambda}\big)}{\mathrm{Tr}_{\mathcal{H}_\Lambda}\big(\mathrm{e}^{-\beta E_\Lambda H(\Lambda)E_\Lambda}\big)}, \quad \text{for } A_{E_\Lambda} \in \mathcal{A}(\Lambda)_{E_\Lambda},$$

if $E_\Lambda \in \mathcal{A}(\Lambda)$ is some collection of projections, and where as before, $\mathcal{A}(\Lambda)_{E_\Lambda} = E_\Lambda\mathcal{A}(\Lambda)E_\Lambda$. Here it is safe to take the trace over the full local space $\mathcal{H}_\Lambda$, since

$$\omega_{E_\Lambda}(AB_{E_\Lambda}C) = \omega_{E_\Lambda}(A_{E_\Lambda}B_{E_\Lambda}C_{E_\Lambda}),$$

for A, B, $C \in \mathcal{A}(\Lambda)$, as follows easily from $E\mathrm{e}^{-\beta E_\Lambda H(\Lambda)E_\Lambda} = \mathrm{e}^{-\beta E_\Lambda H(\Lambda)E_\Lambda}E = \mathrm{e}^{-\beta E_\Lambda H(\Lambda)E_\Lambda}$ and the invariance of the trace under cyclic permutations.

Assume that the local dynamics τ_t^Λ generated by $H(\Lambda)$ converges uniformly to an automorphism group τ of $\mathcal{A}$. Then we know [8, Proposition 6.2.15], that every thermodynamic limit point of the ordinary local Gibbs states, that is, a weak* limit of a net of extensions ω_Λ^G of ω_Λ to $\mathcal{A}$, is a (τ,β)-KMS state over $\mathcal{A}$. As a direct consequence of these considerations and Corollary 3.7, we obtain those equilibrium states for the Zeno dynamics which are limits of local Gibbs states.

Corollary 3.8. *Let $\beta > 0$. Let $\Lambda_\alpha \to \infty$ be such that the local dynamics converges uniformly to the global dynamics τ, and the net of local Gibbs states ω_{Λ_α} has a thermodynamic limit point ω. Let U be the unitary group representing τ in the GNS representation of ω. If a sequence of projections $E_{\Lambda_\alpha} \in \mathcal{A}(\Lambda_\alpha)$ converges in norm to a projection E in $\mathcal{A}$ such that (U,E) is regular and satisfies (AZC), then $\omega_E(A_E) \stackrel{\text{def}}{=} \lim_\alpha \omega^G_{E_{\Lambda_\alpha}}(A_E)$ defines a (τ^E,β)-KMS state on $\mathcal{A}_E$.*

Proof. The local Gibbs states $\omega_{E_{\Lambda_\alpha}}$ are the unique β-KMS states on the finite-dimensional algebras $\mathcal{A}_{\Lambda_\alpha}$ for the reduced dynamics $\widehat{\tau}^{E_{\Lambda_\alpha}}$. If $\{E_{\Lambda_\alpha}\}$ converges uniformly, these local Gibbs state possess ω_E as a weak-* limit, which is a KMS state on $\mathcal{A}_E$ for the reduced dynamics $\widehat{\tau}^E$ associated with τ. Then, by Corollary 3.7, ω_E is also a (τ^E,β)-KMS state. □

We finally note the *dynamical* manifestation of the Zeno paradox in quantum statistical mechanics. Assume that the difference between the Zeno generator EHE and the original one H is entire analytic for the Zeno dynamics τ^E. Then, the original dynamics is a local perturbation of the Zeno dynamics, and the general results about the return to equilibrium [60, Theorem 2], which have been described in Section 3.2, imply that the system starting in a global equilibrium state for the dynamics defined by H will spontaneously evolve toward a KMS state for the Zeno dynamics.

Corollary 3.9. *Let (U, E) be regular and satisfy AZC for $\mathcal{A}$. Let $\omega|_{\mathcal{A}_E}$ be the restriction of a (τ, β)-KMS state of $\mathcal{A}$ to $\mathcal{A}_E$. Assume that $(\mathcal{A}_E, \tau^E)$ is asymptotically Abelian, and that $H - EHE$ is entire analytic for τ^E. Then, every weak* limit point for $t \to \pm\infty$ of $\tau_t^E \omega|_{\mathcal{A}_E}$ is a (τ^E, β)-KMS state.*

3.4. **A Condition on the Zeno Generator.** We return from the operator algebraic framework of quantum statistical mechanics to general, unitary groups on a Hilbert space, to present what appears as the most advanced, functional analytical condition for the existence of Zeno dynamics so far. Although this very recent result [14] proposes, like all other ones in this Section, a *sufficient* condition, it seems to be the sharpest, general characterisation of Zeno dynamics presently available (the necessary and sufficient conditions below in Section 4.3 are less general and require detailed, additional information about Hamiltonian and projection). It also clarifies the exact form of the Zeno generator and determines the Zeno subspace completely.

Let H be a nonnegative self-adjoint operator on a separable Hilbert space $\mathcal{H}$, and E an orthogonal projection on $\mathcal{H}$. The quadratic form

$$\phi \longmapsto \|H^{1/2}E\phi\|, \quad \text{with form domain } D(H^{1/2}E),$$

has a self-adjoint operator

$$H_E \stackrel{\text{def}}{=} (H^{1/2}E)^*(H^{1/2}E)$$

associated with it. The necessary condition is now formulated in terms of H_E.

Theorem 3.10 ([14, Corollary 2.2])**.** *If H_E as defined above is densely defined on $\mathcal{H}$, then holds*

$$\operatorname*{s-lim}_{n\to\infty}\Big[E\mathrm{e}^{\mathrm{i}\frac{t}{n}H}\Big]^n = \operatorname*{s-lim}_{n\to\infty}\Big[\mathrm{e}^{\mathrm{i}\frac{t}{n}H}E\Big]^n = \operatorname*{s-lim}_{n\to\infty}\Big[E\mathrm{e}^{\mathrm{i}\frac{t}{n}H}E\Big]^n = \mathrm{e}^{\mathrm{i}tH_E}E,$$

uniformly in t on every compact interval in $\mathbb{R}$.

Note that the result in [14] concerns a yet more general case when the evolution is interrupted by *different* projections $E(\theta t/n)$ at the times t/n, which render a strongly continuous, non-increasing function $E(t)$, with the initial condition $\operatorname{s-lim}_{t\to 0} E(t) = E$. This generality is not needed for our present discussion. The proof of the main theorem in [14], which entails also the result above, is a clever combination of the known product formulae of Chernoff [11, 12] with the results of Kato [42] and Ichinose [38], and we will not go into details.

So, the Zeno dynamics exists, if only its generator H_E is densely defined — a very weak, yet nontrivial, condition, see the counterexample in [14]. It is also not completely dissimilar to the regularity condition that allowed us to identify the Zeno generator in the more benign case treated in Proposition 3.6. In general, H_E is not densely defined but is a self-adjoint operator on the closed subspace of $\mathcal{H}$, determined as the closure of the form domain $D(H^{1/2}E)$,

$$\widehat{\mathcal{H}}_E \stackrel{\text{def}}{=} D(H^{1/2}E) \subset E\mathcal{H},$$

which is now the relevant Zeno subspace, in general larger than the Zeno subspace $\mathcal{H}_E$ in the algebraic context above. The requirement that H_E be densely defined amounts to saying that the form domain $D(H^{1/2}E)$ is dense in $\mathcal{H}$. Furthermore, H_E differs from the reduced generator EHE, which is not necessarily closed, since EH does not need to be closed though HE is. In fact, H_E is generally a self-adjoint extension of EHE.

4. PHYSICAL CONSIDERATIONS AND THE ANTI–ZENO EFFECT

4.1. **Geometry of the Hilbert Space.** To show a further, and arguably more fundamental facet of the reasons leading to Zeno effect and paradox, we briefly describe the role that is played in that piece by the geometry of the Hilbert space. We follow mainly the clear account of [56], with some borrowings from [2] and [57]. We assume that some generalised quantum evolution — without any supposition about linearity, group structure, or unitarity — acts smoothly in the vicinity of a point $\psi(0)$ in a separable Hilbert space $\mathcal{H}$ (in the natural topology of $\mathcal{H}$). Let the system be prepared in the initial state $\psi(0)$, which shall be an eigenvector of the relevant observable $\mathcal{O}$. In turn, $\mathcal{O}$ is assumed to possess a complete set of eigenvalues $\{\mathcal{O}_n\}$ and eigenfunctions $\{\psi_n\}$. The **survival probability** of the initial state at time t is then

$$\mathcal{P}(t) \stackrel{\text{def}}{=} \left| \left(\frac{\psi(0)}{\|\psi(0)\|}, \frac{\psi(t)}{\|\psi(t)\|} \right) \right|^2,$$

since the evolving vector need not be normalised at later times, and can change its norm during the evolution. We introduce the vector $\chi(t) \stackrel{\text{def}}{=} \psi(t)/\|\psi(t)\|$, which is always normalised. Taylor expansion yields the asymptotics

$$\chi(\tau) = \chi(0) + \tau\dot{\chi}(0) + \frac{\tau^2}{2}\ddot{\chi}(0) + O(\tau^3) \quad (\tau \to 0),$$

and hence for the survival amplitude

$$\mathcal{A}(\tau) = (\chi(0), \chi(\tau)) = 1 + \tau(\chi(0), \dot{\chi}(0)) + \frac{\tau^2}{2}(\chi(0), \ddot{\chi}(0)) + O(\tau^3) \quad (\tau \to 0).$$

But since χ is kept normalised, the easy calculation

$$0 = \left. \frac{\mathrm{d}\|\chi(t)\|^2}{\mathrm{d}t} \right|_{t=0} = (\chi(0), \dot{\chi}(0)) + (\dot{\chi}(0), \chi(0)) = 2\,\mathrm{Re}\,(\chi(0), \dot{\chi}(0))$$

shows that $(\chi(0), \dot{\chi}(0))$ is purely imaginary. >From this, we obtain for the survival probability $\mathcal{P}(\tau) = |\mathcal{A}(\tau)|^2$ the expression

$$\begin{aligned}
\mathcal{P}(\tau) = 1&+\tau\Big[(\chi(0), \dot{\chi}(0)) + (\dot{\chi}(0), \chi(0))\Big] \\
&+\tau^2 \left[(\chi(0), \dot{\chi}(0))(\dot{\chi}(0), \chi(0)) + \frac{1}{2}\big\{(\chi(0), \ddot{\chi}(0)) + (\ddot{\chi}(0), \chi(0))\big\}\right] + O(\tau^3) \\
= 1&+2\tau\,\mathrm{Re}\,(\chi(0), \dot{\chi}(0)) + \tau^2 \left[|(\chi(0), \dot{\chi}(0))|^2 + \mathrm{Re}\,(\ddot{\chi}(0), \chi(0))\right] + O(\tau^3) \\
= 1&+\tau^2 \left[(\mathrm{Im}\,(\chi(0), \dot{\chi}(0)))^2 + \mathrm{Re}\,(\chi(0), \ddot{\chi}(0))\right] + O(\tau^3) \quad (\tau \to 0).
\end{aligned}$$

In a similar manner as above we see

$$0 = \left. \frac{\mathrm{d}^2\|\chi(t)\|^2}{\mathrm{d}t^2} \right|_{t=0} = 2\left[(\dot{\chi}(0), \dot{\chi}(0)) + \mathrm{Re}\,(\chi(0), \ddot{\chi}(0))\right],$$

that is, $\mathrm{Re}\,(\chi(0), \ddot{\chi}(0)) = -(\dot{\chi}(0), \dot{\chi}(0))$, allowing us to write

$$\mathcal{P}(\tau) = 1 - \tau^2 k + O(\tau^3)\ (\tau \to 0) \quad \text{with } k = (\dot{\chi}(0), \dot{\chi}(0)) - (\mathrm{i}(\chi(0), \dot{\chi}(0)))^2.$$

Let us assume for the moment that k is nonnegative. Then, if N measurements of $\mathcal{O}$ are performed at consecutive times $\tau_i = it/N$, $i = 1, \ldots, N$, the final survival probability of the initial state at $\tau_N = t$ is

$$\mathcal{P}_N(t) = [\mathcal{P}(t/N)]^N = \left[1 - \frac{t^2 k}{N^2}\right]^N + O(N^{-3N}) \sim \mathrm{e}^{kt^2/N^2} \quad \text{for } N \text{ large},$$

where, of course, the collapse *viz.* projection postulate has implicitly been applied by decomposing $\mathcal{O}$ into its eigenprojections. In the limit of infinitely frequent measurement, we then recover the Zeno paradox.

$$\lim_{N\to\infty} \mathcal{P}_N(t) = 1.$$

This means that, apart from the crucial assumption of non-negativity of the constant k to which we will come shortly, the Zeno effect is essentially a consequence of the *projective* nature of the quantum formalism. For the calculations above use at the decisive steps the postulate that probabilities are calculated from *normalised* vectors by taking absolute squares of their inner products, i.e., from unit rays in Hilbert space or *states*. The Zeno effect therefore does not hinge on the particularities of the evolution in question, but rather on the use of the Hilbert space formalism and the probability interpretation — indeed, two fundaments of quantum theory.

Let us come back to the constant k and the question of its non-negativity. In the case of unitary evolutions generated by a Hamiltonian H, it can be identified as the expected variance of H in the initial state

$$(\Delta H)^2 = \left(\psi\,, H^2\psi\right) - \left(\psi\,, H\psi\right)^2,$$

where $\psi = \chi(0)$ is the normalised, initial state. This holds if the first moment of H in the state ψ is finite, as follows from the asymptotic expansion of the survival amplitude at the end of Section 3.1. This physical quantity is always nonnegative and we obtain yet another proof that the Zeno paradox emerges in this case. We want to show that also in the general case k allows for a physical interpretation which makes its non-negativity very credible. In the general case at hand, we first have to take the correct perspective by looking at the projective state space $\mathbb{P} = \mathcal{H}^*/U(1)$, of the Hilbert space $\mathcal{H}$, where $\mathcal{H}^* = \{\psi \in \mathcal{H} \mid \|\psi\| = 1\}$ is the unit sphere of $\mathcal{H}$. The set $\mathbb{P}$ of unit rays of $\mathcal{H}$ can be equipped with a natural metric which arises from the inner product of vector representatives by

$$s^2 = 4\left(1 - \left|\left(\frac{\psi}{\|\psi\|}\,, \frac{\phi}{\|\phi\|}\right)\right|^2\right).$$

It is a measure of the distance between points in $\mathbb{P}$ and satisfies the usual metric axioms there. The metric is then given in infinitesimal form by its line element

$$\mathrm{d}s^2 = 4\left[(\dot\chi(t)\,, \dot\chi(t)) - (\mathrm{i}(\chi(t)\,, \ddot\chi(t)))^2\right]\mathrm{d}t^2,$$

where as before $\chi(t) \stackrel{\text{def}}{=} \psi(t)/\|\psi(t)\|$. This metric, constructed in [57], is a generalisation of the Fubini-Study metric [2] and reduces to it in the case of linear, unitary evolutions. Denoting by $v(t) = \dot s(t)$ the reparametrisation invariant speed at which a point in $\mathbb{P}$ travels under the evolution, we see immediately that

$$k = v(0)^2/4$$

is the square of the initial speed, a quantity whose non-negativity is guaranteed. Indeed at this point, the Zeno paradox appears as unavoidable, at least for rank-one projections onto initial states which evolve smoothly. The only conceivable possibility for not ending up in the Zeno regime remains when the asymptotics used in the above reasoning becomes unreliable hence, in the Hamiltonian context, for states with energetic singularities.

4.2. **The Zeno – Anti-Zeno Transition.** In contrast to the conclusions of the last section, we want to demonstrate that quantum evolution can not only be impeded by frequent measurement, but that it can also be *accelerated*, a surprising phenomenon which has aptly been termed **inverse Zeno**, **anti–Zeno**, or **Heraclitus** effect (due to Heraclitus' reply "everything flows" to Zeno's argument). We follow [20, 22, 23], and first rephrase the Zeno effect in terms of **decay rates**. It is well known that for sufficiently long times, an unstable quantum system shows exponential decay, i.e., the survival probability of the initial state approaches

$$\mathcal{P}(t) \sim \mathcal{Z}\mathrm{e}^{-\gamma_0 t} \quad \text{for large } t,$$

according to its natural decay rate γ_0, and where the positive constant $\mathcal{Z}$ can be identified in field theoretical models as the wave function renormalisation constant. On the other hand, we already learnt about the quadratic behaviour of $\mathcal{P}$ at short times (in the cases where the Zeno effect persists)

$$\mathcal{P}(\tau) \sim 1 - \tau^2/\tau_{\mathrm{Z}}^2 \quad (\tau \to 0),$$

where $\tau_{\mathrm{Z}}^{-2} = (\Delta H)^2$ is called the **Zeno time**. Thus, the survival probability will generally interpolate between these two regimes, which can be expressed, using an **effective decay rate** $\gamma_{\mathrm{eff}}(\tau)$, as

$$\mathcal{P}(\tau) = \mathrm{e}^{-\gamma_{\mathrm{eff}}\tau}, \quad \text{with } \gamma_{\mathrm{eff}}(\tau) = -\frac{1}{\tau}\ln \mathcal{P}(\tau).$$

The effective decay rate interpolates between the quadratic short time and exponential regimes as

$$\gamma_{\mathrm{eff}}(\tau) \sim \begin{cases} \tau^2/\tau_{\mathrm{Z}}^2 & (\tau \to 0), \\ \gamma_0 & (\tau \to \infty). \end{cases}$$

Now, if there exists a time τ^* with

$$\gamma_{\mathrm{eff}}(\tau^*) = \gamma_0$$

then measurements of the undecayed state performed at intervals τ^* let the system decay at its natural rate γ_0, that is, as if no measurements were performed. In turn, if such a (unique) τ^* exists it means that for shorter measurement intervals the decay will be inhibited, since γ_{eff} is smaller than γ_0 in that region — this is the regime of the quantum Zeno effect. Yet, for measurement intervals $\tau > \tau^*$ one then has generically $\gamma_{\mathrm{eff}} > \gamma_0$, corresponding to an accelerated decay — the anti–Zeno regime.

A sufficient, and physically meaningful, condition for the existence of at least one τ^* is $\mathcal{Z} < 1$. For then the graph of the survival probability starts out *above* the exponential $\mathrm{e}^{-\gamma_0\tau}$ (due to the quadratic short time behaviour), but ends up approximating the renormalised exponential $\mathcal{Z}\mathrm{e}^{-\gamma_0\tau} < \mathrm{e}^{-\gamma_0\tau}$, and thus must have an intersection with it. The appearance of the anti–Zeno effect has in fact been experimentally confirmed [30], see also [6, 44, 31] for further theoretical considerations.

4.3. **Asymptotics of State Energy Distribution.** We finally come to the very recent results of Atmanspacher, Ehm and Gneiting [5], who show that a sharp characterisation of the transition between Zeno and anti–Zeno effect can be given using the energy distribution of the decaying state. More precisely, the relevant information is encoded in the asymptotic decay of the cumulative energy density function of the initial state. The results are formulated in the framework of probability theory, and we briefly introduce the necessary notions. Consider independent random variables $X_1, X_2, \ldots$ distributed according to a common law $\Pr(X_k < x) = F(x)$, where F is some probability distribution on $\mathbb{R}$, i.e., an non-decreasing, left continuous function with $\lim_{x\to-\infty} = 0$, and $\lim_{x\to\infty} = 1$. Its **characteristic function** is given by the Fourier transform

$$\varphi(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx}\mathrm{d}F(x),$$

while its decay at infinity is captured in the quantity

$$\delta_F(x) = x\Pr\bigl(|X_k| > x\bigr) = x\bigl(F(-x) + 1 - F(x)\bigr),$$

where the last equality holds at all points of continuity of F.

Let us relate these notions to their physical counterparts. If we interpret $F(x)$ as the cumulative energy distribution of an initial, decaying quantum state ψ, i.e., as the probability to measure energies of absolute value larger than x in this state, then φ corresponds to the time evolution of this state, more precisely to the survival amplitude $\mathcal{A}(t)$. Namely, $\mathcal{A}(t)$ can be expressed as

$$\mathcal{A}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tE}|\lambda(E)|^2\mathrm{d}E,$$

(here we changed a sign in contrast to our previous notation) where $\mathrm{d}F$ is in fact identified as the absolute square of the energy density function $\lambda(E)$ of ψ, in its decomposition into energetic components. Note aside, that in F also negative energies are allowed, and that the results are insensitive to that, which is another example for the fact that semiboundedness is not required for the Zeno effect. Thus, the independent probability variables X_k are nothing but the outcomes of energy measurements of the system. We are now ready to state the first main result.

Theorem 4.1 ([5, Theorem 1]). *Equivalent are*
i) $\lim\limits_{n\to\infty} |\varphi(t/n)|^{2n} = 1$ *for all* $t \in \mathbb{R}$.
ii) $\lim\limits_{x\to\infty} \delta_F(x) = 0$.
iii) For all $\varepsilon > 0$ *holds*

$$\lim_{n\to\infty} \inf_{\mu\in\mathbb{R}} \Pr\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \varepsilon\right) = 0,$$

that is, there exists a sequence of real numbers μ_n *such that the distribution of the averages* $(X_1 + \cdots + X_n)/n - \mu_n$ *converges weakly to the Dirac measure concentrated at zero.*

Proof. The weak law of large numbers [40] ensures equivalence of ii) and iii), thus it remains to show the equivalence of i) and ii). The function $\gamma(t) = |\varphi(t)|^2$ is the characteristic function of the difference $Y = X' - X''$ of independent random variables X', X'', each with distribution F. If G denotes the distribution function of Y then $\gamma(t/n)^n$ is the characteristic function of the mean $(Y_1 + \cdots Y_N)/n$ of n independent random variables with distribution G. Then

$$\lim_{n\to\infty} |\varphi(t/n)|^{2n} = \lim_{n\to\infty} \gamma(t/n)^n = 1 \quad \text{for all } t \in \mathbb{R}$$

holds if and only if $(Y_1 + \cdots Y_N)/n$ converge to zero in distribution, which in turn is equivalent to

$$\lim_{x\to\infty} \delta_G(x) = 0,$$

since G is symmetric. But due to the symmetrisation inequalities

$$\exists a\colon \forall x > 0\colon \quad \frac{1}{2}\Pr\bigl(|X'| > x + a\bigr) \le \Pr\bigl(|X' - X''| > x\bigr) \le 2\Pr\bigl(|X'| > x/2\bigr),$$

for probability distributions [29, p. 149], the latter is equivalent to ii). $\square$

Very remarkably, a similar characterisation of the anti–Zeno effect was also achieved in [5]. Here, for the first time, the anti–Zeno effect is considered in the infinitely frequent measurement limit, i.e., the **anti–Zeno paradox** is treated, and shown to lead to a spontaneous decay, that is, to vanishing survival probability at arbitrary small times. This is the last result we reproduce, and we omit the proof, which is again based on the law of large numbers, but is a bit more involved.

We need a mild regularity condition on F, and say that F is **straight** if either $\sup_{x>0} \delta_F(x) < \infty$ or $\lim_{x\to\infty} \delta_F(x) = \infty$. This excludes cases where $\limsup_{x\to\infty} \delta_F(x) = \infty$ while $\lim_{x\to\infty} \delta_F(x)$ does not exist, roughly corresponding to energy spectra with a sequence of gaps of increasing size.

Theorem 4.2 ([5, Theorem 2]). *If F is straight then the following conditions are equivalent*
i) $\lim_{n\to\infty} |\varphi(t/n)|^{2n} = 0$ *in measure, i.e., for all $T > 0$ and $\varepsilon > 0$, the Lebesgue measure of the set of all $|t| < T$ with $|\varphi(t/n)|^{2n} > \varepsilon$ converges to zero.*
ii) $\lim_{x\to\infty} \delta_F(x) = \infty$.
iii) For all $c > 0$

$$\lim_{n\to\infty} \sup_{\mu\in\mathbb{R}} \Pr\left(\left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \le c \right) = 0,$$

that is, the distribution of $(X_1 + \cdots + X_n)/n$ spreads out over $\mathbb{R}$ as $n \to \infty$.

The crucial observation that enables the application of the weak law of large numbers in the proofs of both theorems, is the reinterpretation of the iterated survival amplitude $\mathcal{A}(t/n)^n$. It is now seen as the characteristic function of the mean value of n energy measurements, carried out on an ensemble of quantum systems prepared in the state ψ.

The physical interpretation of the results is that the critical transition between Zeno and anti–Zeno effect occurs roughly at state energy distributions for which the probability of measuring energies larger than E decays as $1/E$. More precisely, for instance condition ii) in Theorem 4.1 corresponds to $o(1/E)$. The characterisation of the anti–Zeno effect, or rather the anti–Zeno paradox of Theorem 4.2 is somewhat at variance with the heuristic presentation of the Zeno – anti–Zeno transition in the last section. The latter prevails only for *nonzero* measurement intervals, while here we are again concerned with the infinitely frequent measurement limit.

The distinction between Zeno and anti–Zeno regime shown by the above two theorems is much finer than that of Luo *et al.* [45] who showed that finiteness of the first absolute moment

$$\int_{-\infty}^{\infty} |E||\lambda(E)|^2 \mathrm{d}E,$$

of the energy density $\lambda(E)$ of the initial state, is sufficient for the Zeno paradox.

5. Concluding Remarks

The Zeno effect was once dismissed as a curious paradox that could only emerge in thought experiments based on wrong concepts of quantum theory and physical reality. Yet the phenomenon has lived through a renaissance. By now the effect appears as one of the most generic ones in quantum theory, and as extremely robust with respect to special formulations, ancillary conditions, and the wide range of specific models that has by now been considered. Even realistic possibilities for its application are under serious consideration [18].

The main line of thought of the present survey of the mathematics of the Zeno effect is, in retrospect, the challenge to delineate its domain of validity. The general picture that emerges is that the prevalence of the Zeno effect can only be broken in exceptional cases, for which also the nature of the mathematical counterexamples constructed so far [3, 48, 49] is an indication. In view of Section 4.1, what is required for leaving the Zeno regime, at arbitrary short times, is a certain amount of non-analyticity of the evolution starting in the initial state. Energetic singularities of this state, respectively, a state in the Zeno subspace seem to be a fundamental prerequisite for this, a view which is corroborated by the conditions for the anti–Zeno paradox of Theorem 4.2. On one hand such singularities *are* in fact present in field theoretical descriptions of decay processes, which is related to the discovery of the possibility of a regime governed by the anti–Zeno effect as described in Section 4.2, following [20, 22, 23], cf. also [26]. On the other hand, whether there exist physical systems which could exhibit the anti–Zeno paradox according to the conditions of Theorem 4.2, is an open question, and might even be considered doubtful.

>From a mathematical viewpoint, it would be most important to obtain sharp conditions for the Zeno paradox and the anti–Zeno paradox. For the former, the results of Section 3.4 are the best as yet, in a general operator theoretical framework, while for both Zeno and anti–Zeno paradox, the results of Section 4.3 are the most advanced for rank one projections. They are also closest in spirit to our heuristic reasoning of Section 2.4, which also led us to conjecture that the asymptotic growth of the energy density, set into proper relation to the projection, should be indicative for the Zeno effect. Yet, the probabilistic argument used in this section are of a quite different quality than the Payley–Wiener type, and complex analytic argument that was coarsely conceived in Section 2.4. The interesting question remains whether it is possible to find corresponding results in the latter framework, which would also generalise the results of Section 4.3 to projections of infinite rank.

Another subject worth further theoretical work is the identification of the Zeno dynamics and the Zeno subspace. It appears quite generally to be an ordinary quantum dynamics, which is confined, in the Zeno limit, to the Zeno subspace by additional boundary conditions. The prime example is the projection operator given by multiplication with the characteristic function of an interval of the real line, leading to Dirichlet boundary conditions [28, 14]. This has been followed by the identification of the Zeno generator in Section 3.3 as EHE, possible under certain technical conditions, and the finer characterisation of [14]. In [64] we have considered models of quantum spin systems, which exhibit the underlying mechanism, and point out possible manifestations of the Zeno effect in mesoscopic, or even macroscopic systems. Yet a general formulation and classification of the emerging boundary conditions would be desirable, and seems conceivable in the operator algebraic framework. Such general results would probably also be applicable in the context of algebraic quantum field theory [33], where the Zeno effect has, as of yet, not received much attention.

REFERENCES

[1] Jarkko Ahonen, Kari Enqvist, and Georg Raffelt, *The paradox of axions surviving primordial magnetic fields*, Phys. Lett. B **366** (1996), 224, hep-ph/9510211.

[2] J. Anandan and Y. Aharanov, *Geometry of Quantum Evolution*, Phys. Rev. Lett. **65** (1990), 1697–1700.

[3] W. Arendt and M. Ulm, *Trotter's Product Formula for Projections*, preprint, Universität Ulm, 1997. Problem Corner of the Bulletin Board for Evolution Equations and Semigroups. http://www.mathematik.uni-ulm.de/ees/.

[4] W. Arendt, S. Bu, and M. Haase, *Functional calculus, variational methods and Liapunov's theorem*, Arch. Math. **77** (2001), 65–75.

[5] H. Atmanspacher, W. Ehm, and T. Gneiting, *Necessary and sufficient conditions for the quantum Zeno and anti-Zeno effect*, J. Phys. A **36** (2003), 9899–9905.

[6] A. P. Balachandran and S. M. Roy, *Continuous Time-Dependent Measurements: Quantum Anti-Zeno Paradox with Applications*, Int. J. Mod. Phys. A **17** (2002), 4007–4024, quant-ph/0102019.

[7] Ph. Blanchard and A. Jadczyk, *Strongly coupled quantum and classical systems and Zeno's effect*, Phys. Lett. A **183** (1993), 272–276, hep-th/9309112. MR 95b:81017

[8] Ola Bratteli and Derek W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, vol. I & II, Springer-Verlag, Berlin, Heidelberg, New York, 1979 & 1981.

[9] Hans Bremermann, *Distributions, Complex Variables, and Fourier Transforms*, Addison–Wesley Publishing Company, Reading, Massachusetts, 1965.

[10] A. Camacho, *Quantum Zeno Effect and the Detection of Gravitomagnetism*, Internat. J. Modern Phys. D **10** (2001), 9–14.

[11] P. R. Chernoff, *Note on product formulas for operator semigroups*, J. Funct. Analysis **2** (1968), 238–242.

[12] ———, *Product Formulas, Nonlinear Semigroups, and Adition*, Mem. Am. Math. Soc., vol. 140, American Mathematical Society, Providence, RI, 1974.

[13] A. F. R. de Toledo Piza and M. C. Nemes, *Manipulating decay rates by entanglement and the Zeno effect*, Phys. Lett. A **290** (2001), 6–10. MR 1 877 695

[14] P. Exner and T. Ichinose, *Product formula related to quantum Zeno dynamics*, preprint, 2003, math-ph/0302060.

[15] Pavel Exner, *One more theorem on the short-time regeneration rate*, J. Math. Phys. **30** (1989), 2563–2564.

[16] P. Facchi, *Quantum Zeno effect, adiabaticity and dynamical superselection rules*, in: Fundamental Aspects of Quantum Physics, L. Accardi and S. Tasaki eds., Quantum Probability and White Noise Analysis, Vol. 17 (2003) p. 197, quant-ph/0202174.

[17] P. Facchi, V. Gorini, G. Marmo, S. Pascazio, and E. C. G. Sudarshan, *Quantum Zeno dynamics*, Phys. Lett. A **275** (2000), 12–19, quant-ph/0004040.

[18] P. Facchi, Z. Hradil, G. Krenn, S. Pascazio, and J. Řehaček, *Quantum Zeno tomography*, Phys. Rev. A **66** (2002), 012110, quant-ph/0104021.

[19] P. Facchi, A. G. Klein, S. Pascazio, and L. S. Schulman, *Berry phase from a quantum Zeno effect*, Phys. Lett. A **257** (2000), 232–240.

[20] P. Facchi, H. Nakazato, and S. Pascazio, *From the Quantum Zeno to the Inverse Quantum Zeno Effect*, Phys. Rev. Lett. **86** (2001), 2699–2703.

[21] P. Facchi and S. Pascazio, *Spontaneous emission and lifetime modification caused by an intense electromagnetic field*, Phys. Rev. A **62** (2000), 023804.

[22] ———, *Quantum Zeno effects with "pulsed" and "continuous" measurements*, preprint, 2001, Time's arrows, quantum measurements and superluminal behavior (D. Mugnai, R. Ranfagni and L. S. Schulman eds.) CNR, Rome, 2001, p. 139, quant-ph/0101044.

[23] ———, *Quantum Zeno phenomena: pulsed versus continuous measurement*, Fortschr. Phys. **49** (2001), 941–947, quant-ph/0106026.

[24] ———, *Quantum Zeno subspaces*, Phys. Rev. Lett. **89** (2002), 080401, quant-ph/0201115.

[25] ———, *Quantum Zeno subspaces and dynamical superselection rules*, in: Proceedings of the XXII Solvay Conference on Physics, I. Antoniou, V.A. Sadovnichy and H. Walther eds., World Scientific, Singapore, 2003, p. 251, quant-ph/0207030.

[26] ———, *Unstable systems and quantum Zeno phenomena in quantum field theory*, in: Fundamental Aspects of Quantum Physics, L. Accardi and S. Tasaki eds., Quantum Probability and White Noise Analysis, Vol. 17 (2003) p. 222, quant-ph/0202127.

[27] ———, *Three different manifestations of the quantum Zeno effect*, in: Irreversible Quantum Dynamics, F. Benatti and R. Floreanini eds., Lecture Notes in Physics, Vol. 622, Springer-Verlag Berlin, 2003, p. 141, quant-ph/0303161.

[28] P. Facchi, S. Pascazio, A. Scardicchio, and L. S. Schulman, *Zeno dynamics yields ordinary constraints*, Phys. Rev. A **65** (2001), 012108, `quant-ph/0101037`.

[29] W. Feller, *An Introduction to Probability Theory and its Applications*, 2nd ed., vol. 2, John Wiley & Sons, New York, 1971.

[30] M. C. Fischer, B. Gutiérrez-Medina, and M. G. Raizen, *Observation of the Quantum Zeno and Anti-Zeno Effects in an Unstable System*, Phys. Rev. Lett. **87** (2001), 040402, `quant-ph/0104035`.

[31] S. A. Gurvitz, *Quantum description of classical apparatus: Zeno effect and decoherence*, Quantum Information Processing **2** (2003), 15, `quant-ph/0212155`.

[32] Karl Gustafson, *A Zeno Story*, preprint, 2002, `quant-ph/0203032`.

[33] Rudolf Haag, *Local Quantum Physics*, Springer-Verlag, Berlin, Heidelberg, New York, 1992.

[34] Einar Hille and Ralph S. Phillips, *Functional Analysis and Semigroups*, revised ed., AMS Colloquium Publications, vol. XXXI, American Mathematical Society, Providence, RI, 1965.

[35] D. Home and M. A. B. Whitaker, *Reflections on the quantum Zeno paradox*, J. Phys. A **19** (1986), 1847–1854. MR 87m:81015

[36] ______, *Quantum Zeno Effect: Relevance for local realism, macroscopic realism, and non-invasive measurability at the macroscopic level*, Phys. Lett. A **239** (1998), 6–12.

[37] Zdenek Hradil, Hiromichi Nakazato, Mikio Namiki, Saverio Pascazio, and Helmut Rauch, *Infinitely frequent measurements and quantum Zeno effect*, Phys. Lett. A **239** (1998), 333–338.

[38] T. Ichinose, *A product formula and its application to the Schrödinger equation*, Publ. RIMS Kyoto Univ. **16** (1980), 585–600.

[39] Richard V. Kadison and John R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, vol. I & II, Academic Press, New York, London, Paris, San Diego, 1983 & 1986.

[40] Olav Kallenberg, *Foundations of modern probability*, 2nd ed., Springer-Verlag, New York, 2002.

[41] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg, 1976.

[42] ______, *Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups*, Topics in functional analysis (I. Gohberg and M. Kac, eds.), Academic Press, New York, 1978, pp. 185–195.

[43] Takahiro Kawai, *On the Theory of Fourier Hyperfunctions and its Applications to Linear Partial Differential Equations with Constant Coefficients*, Journ. Fac. Sci. Univ. Tokyo, Ser. IA **17** (1970), 467–515.

[44] Kazuki Koshino and Akira Shimizu, *Quantum Zeno and anti-Zeno effects by indirect measurement with finite errors*, Phys. Rev. A **67** (2003), 042101, `quant-ph/0210036`.

[45] Shunlong Luo, Zhen Wang, and Qiang Zhang, *An inequality for characteristic functions and its applications to uncertainty relations and the quantum Zeno effect*, J. Phys. A **35** (2002), 5935–5941.

[46] Máté Matolcsi, *On the relation of closed forms and Trotter's product formula*, J. Funct. Analysis, **205** (2003) 401–413.

[47] ______, *On quasi-contractivity of C_0-semigroups on Banach spaces*, preprint, 2003.

[48] Máté Matolcsi and Roman Shvidkoy, *Counterexample to the Trotter product formula for projections*, Ulmer Seminare 2001, pp. 259–262, `math.FA/0109049`.

[49] ______, *Trotter's product formula for projections*, to appear in Archiv der Mathematik.

[50] B. Misra and K. B. Sinha, *A remark on the rate of regeneration in decay processes*, Helv. Phys. Acta **50** (1977), 99–104.

[51] B. Misra and E. C. G. Sudarshan, *The Zeno's paradox in quantum theory*, J. Math. Phys. **18** (1977), 756–763.

[52] Hiromichi Nakazato, Mikio Namiki, and Saverio Pascazio, *Temporal behavior of quantum mechanical systems*, Internat. J. Modern Phys. B **10** (1996), 247, `quant-ph/9509016`.

[53] Hiromichi Nakazato, Tomoko Takazawa, and Kazuya Yuasa, *Purification through Zeno-Like Measurements*, Phys. Rev. Lett. **90** (2003), 060401, `quant-ph/0301026`.

[54] Michio Nishioka, *A theorem on the rate of regeneration in decay processes*, J. Math. Phys. **29** (1988), 1860–1861.

[55] Saverio Pascazio and Mikio Namiki, *Dynamical Quantum Zeno Effect*, Phys. Rev. A **50** (1994), 4582–4592.

[56] A. K. Pati and S. V. Lawande, *Geometry of the Hilbert space and the quantum Zeno effect*, Phys. Rev. A **58** (1998), 831–835.

[57] Arun Kumar Pati, *New derivation of the geometric phase*, Phys. Lett. A **202** (1995), 40–45.

[58] ______, *Limit on the frequency of measurements in the quantum Zeno effect*, Phys. Lett. A **215** (1996), 7–13.

[59] Michael Reed and Barry Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, London, San Diego, 1972.

[60] Derek W. Robinson, *Return to equilibrium*, Comm. Math. Phys. **31** (1973), 171–189.

[61] Walter Rudin, *Functional Analysis*, TMH ed., McGraw-Hill Book Company, New York, Singapore, 1974.

[62] Andreas U. Schmidt, *Asymptotic Hyperfunctions, Tempered Hyperfunctions, and Asymptotic Expansions*, preprint, 2002, `math.FA/0107058`.

[63] ______, *Zeno Dynamics of von Neumann Algebras*, J. Phys. A **35** (2002), 7817–7825, `math-ph/0203008`.

[64] ______, *Zeno Dynamics in Quantum Statistical Mechanics*, J. Phys. A **36** (2003), 1135–1148, `math-ph/0207013`.

[65] M. A. B. Whitaker, *Theory and experiment in the foundations of quantum theory*, Progress in Quantum Electronics **24** (2000), 1–106.

[66] David Wick, *The Infamous Boundary. Seven Decades of Heresy in Quantum Physics.*, Birkhäuser–Verlag, Boston, Basel, Stuttgart, 1995.

[67] Chr. Wunderlich, Chr. Balzer, and P. E. Toschek, *Evolution of an Atom Impeded by Measurement: The Quantum Zeno Effect*, Z. Naturforsch. **56a** (2001), 160–164, `quant-ph/0108040`.

Current address: Fraunhofer – Institute for Secure Information Technology, Dolivostraße 15, 64293 Darmstadt, Germany

FACHBEREICH MATHEMATIK, JOHANN WOLFGANG GOETHE-UNIVERSITÄT, 60054 FRANKFURT AM MAIN, GERMANY

E-mail address: `aschmidt@math.uni-frankfurt.de`

URL: `http://www.math.uni-frankfurt.de/~aschmidt`