ZENO DYNAMICS IN QUANTUM STATISTICAL MECHANICS

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ABSTRACT. We study the quantum Zeno effect in quantum statistical mechanics within the operator algebraic framework. We formulate a condition for the appearance of the effect in W^*-dynamical systems, in terms of the short-time behaviour of the dynamics. Examples of quantum spin systems show that this condition can be effectively applied to quantum statistical mechanical models. Further, we derive an explicit form of the Zeno generator, and use it to construct Gibbs equilibrium states for the Zeno dynamics. As a concrete example, we consider the X-Y model, for which we show that a frequent measurement at a microscopic level, e.g., a single lattice site, can produce a macroscopic effect in changing the global equilibrium.

1. INTRODUCTION

The Zeno effect consists of an impediment to the time evolution of a quantum system by frequent observation, for which it is nicknamed 'a watched pot never boils' or 'watchdog' effect. Research on the phenomenon has a long history dating back to the early days of quantum theory. It found its first explicit theoretical formulation in [13], and, after that, a vivid work in the field was initiated, stimulated also to a great extent by significant experimental advances. We will not review that development here, but see [14, 21] and references therein for further details.

In [20] we followed closely the reasoning of [13], and extended the theoretical treatment of the Zeno effect to modular flows of von Neumann algebras. Our results indicate that the effect can also appear in systems of quantum statistical mechanics at nonzero temperature. Furthermore to a given KMS, i.e., equilibrium state, one will, under favourable conditions, find an associated equilibrium state for the Zeno dynamics, i.e., the limit of unitary quantum evolution interrupted infinitely frequently by measurement events. This confounds the view that the induced Zeno dynamics consists mainly in an imposition of 'boundary conditions' on the original dynamics [9]. To show these things, theoretically and in concrete examples, is our objective here.

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To apply the abstract results of [20] we need a sufficient condition for the appearance of the Zeno effect, and in particular for the existence of the Zeno dynamics. Such a condition is derived in Section 2. It captures the essence of the quadratic short-time behaviour of quantum evolution [1, 14] which has since long been identified as an essential cause for the Zeno effect. As a direct consequence of our *asymptotic Zeno condition*, we find that the assumptions of the main Theorem 2.1 of [20] are satisfied. Thus the Zeno dynamics will exist and form a strongly continuous semigroup, whenever the condition holds. We derive this result in the context of W^*-dynamical systems, to open the way for its application in quantum statistical mechanics.

The asymptotic Zeno condition is formulated in terms of the short-time behaviour of the off-diagonal matrix elements of the original unitary evolution with respect to the decomposition $\mathbb{1} = E + E^{\perp}$, where E is the projection modelling the measurement. The most pleasing aspect of the condition with respect to applications is that it enables the use of perturbation theory for the examination of Zeno dynamics. We will show this in three examples in Section 3. The first one is a generic example for quantum evolution impeded by the Zeno effect in quantum statistical mechanics. We consider the return to equilibrium of a system which is subjected to a bounded, local perturbation. We show that this natural relaxation process will be inhibited by the Zeno effect, if the measurement controls the presence of a state which is invariant under the perturbed dynamics. The second example is in the general context of quantum spin systems, and shows that the Zeno effect can decouple a finite region of the system from its surrounding, if the interaction through the boundary stays finite in the thermodynamic limit. The third example presents, as a more concrete case of the phenomena observed in the previous two, a Zeno effect in the X-Y model of an infinite spin chain.

To consider the above-mentioned question of equilibria for the Zeno dynamics, it is necessary to find a more explicit form of it than that provided by the limit of infinitely frequent measurement, in which that problem would be hard to handle. This we have already noted in [20, Corollary 2.2], where we were only able to state a formal condition for a state to be a Zeno equilibrium. We will gain an instrument to improve on that in Section 4, where we rigorously identify the generator of the Zeno dynamics acting on the Zeno subspace to which the dynamics becomes confined as EHE, where H is the original Hamiltonian. This also provides a link to the Zeno effect induced by continuous observation in the limit of strong coupling between system and apparatus.

Having the Zeno generator at our disposal, it is easy to construct an important class of Zeno equilibria, namely Gibbs states, which we do in Section 5. We make this explicit for quantum spin systems and review the corresponding example 2 of Section 3 in that respect. The Zeno equilibrium on the bounded region in this case is the Gibbs state associated with a Hamiltonian which is averaged with respect to the state imposed on the region by the given rank one projection. In this way, the Zeno effects implements a boundary condition on the system, in accordance with results of Fannes and Werner [10]. In the X-Y model, we will be able to derive some physically remarkable results: First, a frequent measurement on the microscopic level, even a single lattice site, will significantly change the global equilibrium. In the concrete example considered, it will separate the left and right subchains. Secondly, this system will spontaneously evolve toward the Zeno equilibrium when the observation is turned on, rendering the effect macroscopically observable.

Finally, the last Section 6 contains some conclusions and an outlook to possible further applications in physical models.

It should be noted that we restrict our discussion completely to a concrete realisation of a W^*-dynamical system given by the GNS representation π_ω of a fixed, *a priori* chosen KMS state ω. That is we consider the von Neumann algebra $\pi_\omega(\mathcal{A})$ on the GNS Hilbert space $\mathcal{H}$ and assume the dynamical automorphism group to be π_ω-covariant, i.e., to be realised by a strongly continuous, unitary group of operators. This notably simplifies our treatment, but also restricts it to a single superselection sector of the theory. Nevertheless, the results in Sections 4 and 5 regarding Zeno equilibria are essentially independent of the chosen representation.

2. A Sufficient Asymptotic Condition for Zeno Dynamics

The Zeno effect is commonly attributed to the quadratic short-time behaviour of quantum evolution [16] which in turn is rooted deeply in the geometry of Hilbert space [1]. This quadratic behaviour seems so generic that one can hope to turn it into a sufficient condition for the effect to occur. This is what we will present in this section.

Let E be a projection, U a unitary group on a Hilbert space $\mathcal{H}$, and set

$$F_n(t) \stackrel{\text{def}}{=} \big[EU(t/n)E\big]^n, \quad \text{for } t \in \mathbb{R},\ n \in \mathbb{N}.$$

This is a time evolution of a system interrupted by frequent, instantaneous 'measurement' effects, coarsely modelled by projections (or, if one wishes, employing the projection postulate). The question whether the strong Zeno effect, or 'Zeno paradox' occurs is in essence equivalent to the question of strong convergence of the operator sequence F_n to a sensible, i.e., continuous, time evolution [13]. For then, the induced evolution will be confined to a 'Zeno subspace' within $E\mathcal{H}$ by 'infinitely frequent observation.' This limit is arguably unphysical [17], but still of conceptual interest, as will become evident below.

To exemplify the basic idea of our proceeding, we want to see whether the $F_n(t)$ form a Cauchy sequence in n for given t. For that, we have to estimate the quantities

$$\big\|\big(F_n(t)-F_m(t)\big)\big\| \le \big\|\big(F_n(t)-F_{nm}(t)\big)\big\| + \big\|\big(F_m(t)-F_{nm}(t)\big)\big\|.$$

A double telescopic estimation yields

$$\begin{aligned}
&\big\|\big(F_n(t)-F_{nm}(t)\big)\big\| \le \\
&\quad \sum_{k=1}^{n}\sum_{l=1}^{m-1} \Big\|\big[EU(t/n)E\big]^{n-k}\Big(EU(t(m-l)/(nm))E\big[EU(t/(nm))E\big]^{l} - \\
&\qquad EU(t(m-l+1)/(nm))E\big[EU(t/(nm))E\big]^{l-1}\Big)\big[EU(t/(nm))E\big]^{m(k-1)}\Big\|.
\end{aligned}$$

Now, since with $E^\perp \stackrel{\text{def}}{=} \mathbb{1}-E$ we have

$$EU(t(m-l+1)/(nm))E = EU(t(m-l)/nm)(E+E^\perp)U(t/(nm))E,$$

we find that the (k,l)th term in the sum is equal to

$$\begin{aligned}
&\Big\|\big[EU(t/n)E\big]^{n-k}\cdot EU(t(m-l)/(nm))E^\perp\cdot \\
&\qquad \cdot E^\perp U(t/(nm))E\cdot\big[EU(t/(nm))E\big]^{l-1}\big[EU(t/(nm))E\big]^{m(k-1)}\Big\|.
\end{aligned}$$

Multiplying out and using repeatedly $\|AB\| \leq \|A\|\|B\|$, we estimate this expression from above by

$$\big\| \big[EU(t/n)E\big] \big\|^{n-k} \cdot \big\| EU(t(m-l)/(nm))E^{\perp} \big\| \\ \cdot \big\| E^{\perp}U(t/(nm))E \big\| \cdot \big\| \big[EU(t/(nm))E\big] \big\|^{l-1} \big\| \big[EU(t/(nm))E\big] \big\|^{m(k-1)}.$$

Observing that all terms containing only the projection E have operator norm ≤ 1 and can thus be omitted in the estimation of $\big\|\big(F_n(t) - F_{nm}(t)\big)\big\|$, we arrive at

$$\big\|\big(F_n(t) - F_{nm}(t)\big)\big\| \leq \sum_{k=1}^{n} \sum_{l=1}^{m-1} \big\|EU(t(m-l)/(nm))E^{\perp}\big\| \big\|E^{\perp}U(t/(nm))E\big\|.$$

Now, we require $E^{\perp}U(\tau)E = O(\tau)$ uniformly as $\tau \to 0$. That is, there shall exist $\tau_0 > 0$ and $C \geq 0$ such that for all τ with $|\tau| < \tau_0$ holds the estimate $\big\|E^{\perp}U(\tau)E\big\| \leq C^{1/2}|\tau|$. Then, for $n > n_0 \geq 1/\tau_0$, and $m \geq 2$,

$$\begin{aligned} \big\|\big(F_n(t) - F_{nm}(t)\big)\big\| &\leq Ct^2 \sum_{k=1}^{n} \sum_{l=1}^{m-1} \frac{m-l}{n^2m^2} \\ &= Ct^2 \sum_{k=1}^{n} \frac{(m-1)m}{2n^2m^2} \\ &= \frac{Ct^2}{2} \frac{(m-1)m}{nm^2} \leq \frac{Ct^2}{2n}. \end{aligned}$$

An analogous estimate holds for $\big\|\big(F_m(t) - F_{nm}(t)\big)\big\|$, which yields for $m - 2 \geq n > n_0 \geq 1/\tau_0$ the overall result

$$\big\|\big(F_n(t) - F_m(t)\big)\big\| \leq \frac{Ct^2}{n}. \tag{$*$}$$

We have proved the essence of

Lemma 2.1. *If $E^{\perp}U(\tau)E = O(\tau)$ uniformly as $\tau \to 0$ then $F_n(t)$ converges uniformly as $n \to \infty$ for all $t \in \mathbb{R}$. Furthermore $W(t) \stackrel{\text{def}}{=} \text{s-lim}_{n\to\infty} F_n(t)$ is uniformly continuous in t and $\text{s-lim}_{t\to 0} W(t) = E$.*

Proof. The first statement is clear since $(*)$ shows that the $F_n(t)$ form Cauchy sequences which are therefore *a fortiori* convergent. The other statements follow from from $F_n(0) = E$ for all n, and the fact that the convergence of $F_n(t)$ is uniform for t on compact subsets of $\mathbb{R}$. This follows in turn from the t-dependence of the estimate $(*)$. □

We will now use the above result to reformulate the main Theorem 2.1 of [20] in a more effective way. The general setting is as follows: Let $(\mathcal{A}, \tau)$ be a W^*-dynamical system with faithful (τ, β)-KMS state ω which is assumed to be normal. Denote by Ω the vector representative of ω in the associated representation π_ω on the GNS–Hilbert space $\mathcal{H}$. The automorphism group τ is assumed to be implemented covariantly, i.e., by a strongly continuous group of unitary operators $U(t)$ on $\mathcal{H}$. The representation π_ω will be omitted from the notation, when no confusion is possible.

Proposition 2.2. *Under the conditions described above, let $\beta > 0$, assume $\mathcal{A}$ to be unital, let $E \in \mathcal{A}$ be a projection, and set $E^{\perp} \stackrel{\text{def}}{=} \mathbb{1} - E$. Assume that the* ***asymptotic Zeno condition*** *holds: For $A \in \mathcal{A}$, the estimate*

$$\big\|E^{\perp}U(\zeta)EA\Omega\big\| \leq C \cdot \|A\Omega\| \cdot |\zeta| \tag{AZC}$$

is valid for ζ with $|\zeta| < r_0$ for some fixed $r_0 > 0$ and $\operatorname{Im}\zeta \geq 0$. In short: (U, E) satisfies (AZC) *for $\mathcal{A}$. Then the strong operator limits*

$$W(t) \stackrel{\text{def}}{=} \underset{n\to\infty}{\text{s-lim}}\big[EU(t/n)E\big]^n$$

exist, and form a strongly continuous group of unitary operators on the ***Zeno subspace*** *$\mathcal{H}_E \stackrel{\text{def}}{=} \overline{\mathcal{A}_E\Omega} \subset E\mathcal{H}$, where $\mathcal{A}_E \stackrel{\text{def}}{=} E\mathcal{A}E$. The group $W(t)$ induces an automorphism group τ^E of $\mathcal{A}_E$, such that $(\mathcal{A}_E, \tau^E)$ is a W^*-dynamical system. The vectors $W(z)A_E\Omega$, $A_E \in \mathcal{A}_E$, extend analytically to the strip $0 < \operatorname{Im} z < \beta/2$ and are continuous on its boundary.*

Proof. We show that the assumptions of the main Theorem 2.1 of [20] are satisfied, from which we obtain the stated conclusions. First, for real τ, (AZC) implies $E^\perp U(\tau)E = O(\tau)$ uniformly since the operators in question are bounded, $\mathcal{A}\Omega$ is dense in $\mathcal{H}$, and (AZC) is uniform in A on a fixed real neighbourhood of 0. Therefore Lemma 2.1 yields the existence of $W(t)$, $t \in \mathbb{R}$, its weak continuity in t and the initial condition $\text{w-lim}_{t\to 0} W(t) = E$. These facts comprise condition i) of [20, Theorem 2.1] (keeping in mind, here and in the following, the connection between faithful states of von Neumann algebras and KMS states given by Takesaki's theorem [3, Theorem 5.3.10]). For the second condition of the cited theorem, we need only to show that $W(t + \mathrm{i}\beta/2)$ exist as strong operator limits on the common, dense domain $\mathcal{A}\Omega$. For this notice that the calculations leading to $(*)$ are applicable to $(F_n(t + \mathrm{i}\beta/2) - F_m(t + \mathrm{i}\beta/2))A\Omega$, leading to the estimate

$$\big\|\big(F_n(t + \mathrm{i}\beta/2) - F_m(t + \mathrm{i}\beta/2)\big)A\Omega\big\| \leq \frac{C\|A\||t + \mathrm{i}\beta/2|^2}{n}$$

for $A \in \mathcal{A}\Omega$, and $m - 2 \geq n > n_0 \geq 1/r_0$. Thus, also condition ii) of Theorem 2.1 of [20] is satisfied and the stated conclusions follow from it. □

Note that it would have been sufficient to test the asymptotic condition on any dense set of vectors which are analytic for $U(\zeta)$, in some strip $0 \leq \operatorname{Im}\zeta < \varepsilon$ for some $\varepsilon \geq \beta/2$. For simplicity, we restricted attention to $\mathcal{A}\Omega$. The AZC is strictly stronger than the assumptions of [20, Theorem 2.1], where no continuity at the boundary $\operatorname{Im} z = \beta/2$ was implied (as would also follow from the estimate in the proof above) and only weak continuity at the real axis needed to be assumed.

In the examples below weak closures will be understood for all observable algebras, i.e., we will always consider the von Neumann algebra $\pi_\omega(\mathcal{A})''$. That is, we will not exert the discrimination between C^*- and W^*-dynamical systems, which is not very important for our purpose here. The main merit of this simplification is that τ^E is guaranteed by [20, Lemma 2.6] to act by automorphisms on the von Neumann subalgebra $\mathcal{A}_E$ of $\mathcal{A}$. However it seems possible to obtain a result corresponding to Proposition 2.2 in the C^* case, repeating the arguments of [20] using the analogous analyticity properties of the vectors in $\mathcal{A}\Omega$ as detailed in the proof of [3, Theorem 5.4.4].

The condition (AZC) is quite weak and thus indicates how generic a quantum phenomenon the Zeno effect indeed is. For example it is always satisfied if the generator H of the group U is bounded, or, more generally, if E projects onto a closed subspace of entire analytic elements for H, e.g., if E is contained in a bounded spectral projection of H. In those cases a power series expansion of $U(t) = \mathrm{e}^{\mathrm{i}tH}$ implies (AZC). However, if neither is the case, then (AZC) will generally fail in that its defining estimate is not uniform in $A\Omega \in \mathcal{H}$.

It is also noteworthy that in showing the convergence of F_n to W, we have not used the unitarity of U. Thus an analogue of the Zeno effect is also possible for non-unitary (non-Hamiltonian, non-Schrödinger) evolutions, cf. [16]. On the other hand, the group property of U was essential for obtaining the quadratic term that forced the convergence of the sequence.

Condition (AZC) is comparable to other conditions for the appearance of the Zeno effect, which are commonly based on the finiteness of the moments of the Hamiltonian in the Zeno subspace [14]. The AZC, when restricted to the real axis, is equivalent to saying that the function $E^{\perp}U(t)E$ is uniformly Lipschitz continuous at the point $t = 0$. Not surprisingly, Lipschitz continuity is well known as a salient condition for the existence of solutions to (nonlinear) evolution equations.

Asymptotic bounds on $E^{\perp}U(t)E$ have already been considered by other authors [4, 15, 12], in the context of short-time regeneration of an undecayed state. In particular in [4], the deviation of the 'reduced evolution' $EU(t)E$ from being a semigroup has been expressed by such (polynomial) bounds. We will obtain a similar yet somewhat coarser result in Section 4.

3. Examples

The power of (AZC) lies to a great extent in that it yields perturbative conditions for the occurrence of the Zeno effect. For it is known that a perturbed semigroup U_t^P, resulting from adding a bounded perturbation P to a C_0-semigroup U_t, is close to U_t for small times in the sense that $\|U_t - U_t^P\| = O(t)$, as $t \to 0$, see [3, Theorem 3.1.33]. Now if E projects onto a subspace which is invariant under U_t, then this asymptotic behaviour implies that the Zeno dynamics of the pair (U_t^P, E) exists. We exemplify this basic mechanism in the following.

Example 1 (Non-Return to Equilibrium). It is well known [19] that a quantum system will under general conditions, e.g., if $(\mathcal{A}, \tau)$ is asymptotically Abelian, return to equilibrium for large times. This means the following: If the system is prepared in an equilibrium state ω^P for the perturbed evolution τ^P, where $P = P^* \in \mathcal{A}_\tau$ is a bounded perturbation, which is in the set of entire analytic elements $\mathcal{A}_\tau$ for τ (termed local perturbation), and thereafter evolves under the unperturbed dynamics τ, one recovers a τ-equilibrium state $\omega_\pm$ for $t \to \pm\infty$. Assume that the perturbed and unperturbed dynamics are implemented by unitaries U^P and U, respectively. This is always possible if either τ or τ^P is covariant in the chosen representation [19, Theorem 1]. Then, the unperturbed dynamics can be written in terms of the perturbed one by the perturbation expansion [3, Theorem 3.1.33 and Proposition 5.4.1]

$$U(t) = U^P(t) + \sum_{n\geq 1} \int_0^t \mathrm{d}t_1 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n U^P(t_1) P U^P(t_2 - t_1) P \cdots P U^P(t - t_n),$$

where the n-th term in the sum is bounded by $\|P\|^n t^n/n!$. Let the system be prepared in any τ^P-invariant state φ^P. In the representation π^P induced by the chosen τ^P-KMS state ω^P the corresponding vector states are denoted by Φ^P and Ω^P respectively. Let E be the projection onto the space spanned by the vector Φ^P and assume $E \in \mathcal{A}$. Then the above expansion readily yields $E^{\perp}U(t)E = O(t)$ uniformly, since the τ^P-invariance of φ^P implies $U^P(t)\Phi^P = \Phi^P$. In application to vectors in $\mathcal{A}\Omega$ this estimate extends to a fixed, small neighbourhood of 0 in the upper half-plane and is uniform in those vectors. Thus (AZC) holds, the Zeno dynamics converges, and the system remains in the state φ^P. The same reasoning is applicable if E projects onto a τ^P-invariant subspace.

The phenomenon described in this example is the direct counterpart, in the context of quantum statistical mechanics, of the most common example for the Zeno effect in quantum mechanics, i.e., the prevention of a decay process, see, e.g., [11, 5]. Its character is generic, and therefore we formulate it as a corollary.

Corollary 3.1. *Let $(\tau, \mathcal{A})$ be as above. Let $P \in \mathcal{A}$ be a local perturbation, and denote by τ^P perturbed dynamics as constructed in* [3, Proposition 5.4.1 and Corollary 5.4.2]. *Let $E \in \mathcal{A}$ be a τ^P-invariant projection, i.e., $\tau^P(E) = E$. Then the (τ, E)-Zeno dynamics τ^E is an automorphism group of $\mathcal{A}_E$, and $\mathcal{H}_E$ is τ^E-invariant.*

In view of the mechanism noted at the end of Example 2 below, this corollary could easily be reformulated in terms of the thermodynamic limit of local algebras over bounded regions. We omit the details.

Example 2 (Local Domains of Quantum Spin Systems*). For a detailed exposition of the notions and facts invoked below, we refer the reader to [3, Section 6.2]. Consider a quantum spin system over the lattice $\mathbb{X} \overset{\text{def}}{=} \mathbb{Z}^d$ with interaction $\Phi\colon \mathbb{X} \supset X \longmapsto \mathcal{A}_X$. The local Hamiltonian of a bounded subset $\Lambda \subset \mathbb{X}$ is $H_\Phi(\Lambda) \overset{\text{def}}{=} \sum_{X \subset \Lambda} \Phi(X)$ and $U_\Lambda(t) \overset{\text{def}}{=} \mathrm{e}^{itH_\Phi(\Lambda)}$ is the associated group of unitaries. Consider bounded subsets $\Lambda \subset \Lambda' \subset \mathbb{X}$. The surface interaction of Λ with Λ' is

$$W_\Phi(\Lambda; \Lambda') \overset{\text{def}}{=} \sum \bigl\{ \Phi(X) \bigm| X \subset \Lambda',\ X \cap \Lambda' \setminus \Lambda \neq \varnothing,\ X \cap \Lambda \neq \varnothing \bigr\}.$$

Then holds the decomposition

$$H_\Phi(\Lambda') = H_\Phi(\Lambda' \setminus \Lambda) + H_\Phi(\Lambda) + W_\Phi(\Lambda; \Lambda'),$$

and $\bigl[H_\Phi(\Lambda' \setminus \Lambda), H_\Phi(\Lambda)\bigr] = 0$. Let Φ_Λ be any vector in $\mathcal{H}_\Lambda$ and φ_Λ the associated local pure state of the closed subsystem localised in Λ. Define a projector on $\mathcal{H}_{\Lambda'} = \mathcal{H}_{\Lambda' \setminus \Lambda} \otimes \mathcal{H}_\Lambda$ by

$$E_{\varphi_\Lambda; \Lambda'} \overset{\text{def}}{=} \mathbb{1}_{\Lambda' \setminus \Lambda} \otimes P_{\Phi_\Lambda},$$

where P_{Φ_Λ} is the projector onto the one-dimensional subspace generated by Φ_Λ in $\mathcal{H}_\Lambda$ (note that this rank one projector is always in $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$ for bounded regions Λ). Now $P_\Phi(\Lambda; \Lambda') \overset{\text{def}}{=} H_\Phi(\Lambda) + W_\Phi(\Lambda; \Lambda')$ is the local perturbation which removes the effect of the region Λ on the local system in Λ'. The state φ_Λ is clearly invariant under the perturbed dynamics generated by $H_\Phi(\Lambda') - P_\Phi(\Lambda; \Lambda')$ and therefore Corollary 3.1 can be applied. Thus the local limit dynamics

$$W_{\varphi_\Lambda; \Lambda'}(t) = \lim_{n \to \infty} \bigl[E_{\varphi_\Lambda; \Lambda'} U_{\Lambda'}(t/n) E_{\varphi_\Lambda; \Lambda'}\bigr]^n$$

is well defined. This result persists in the thermodynamic limit if the global interaction energy

$$W_\Phi(\Lambda) \overset{\text{def}}{=} \sum \bigl\{ \Phi(X) \bigm| X \cap \Lambda \neq \varnothing,\ X \cap \Lambda^c \neq \varnothing \bigr\} = \lim_{\Lambda' \to \infty} W_\Phi(\Lambda; \Lambda')$$

is well defined, and then the local limits are uniform in Λ'. Choose a faithful normal state for the global algebra $\mathcal{A}$, and let U be the unitary generating the global dynamics τ in its GNS representation. Under the assumptions above we obtain that the Zeno limit

$$W_{\varphi_\Lambda}(t) = \lim_{n \to \infty} \bigl[E_{\varphi_\Lambda} U(t/n) E_{\varphi_\Lambda}\bigr]^n$$

*This example was suggested by G. Morchio.

exists, where $E_{\varphi_\Lambda} \stackrel{\text{def}}{=} \lim_{\Lambda'\to\infty} E_{\varphi_\Lambda} = \mathbb{1}_{\Lambda^c} \otimes P_{\Phi_\Lambda}$. It defines an automorphism group τ^E of $\mathcal{A}$. Furthermore, this automorphism group is the uniform limit of the local automorphism groups $\tau^{E_{\varphi_\Lambda;\Lambda'}}$ defined by $W_{\varphi_\Lambda;\Lambda'}$.

This simple method to obtain Zeno dynamics will in general only work for projections over bounded regions. The requirements one would have to fulfil over unbounded regions are severe: If we want to apply the perturbative method, the Hamiltonian over that region would have to be bounded. Or, if one looks for projectors onto states over the unbounded region which are invariant from the outset, those might be scarce if the system is sufficiently disordered, e.g., asymptotically Abelian. As a negative example, the projector onto a KMS equilibrium state over an unbounded region is generically not an observable.

Example 3 (The X-Y Model). We want to illustrate the two facets of the Zeno effect exhibited above in the more concrete model of the X-Y-spin chain. This model has been rigorously treated in [19], where all the facts used below are proved. It consists of a spin chain over $\mathbb{Z}$, where the state space over a point $x \in \mathbb{Z}$ is two-dimensional $\mathcal{H}_x \stackrel{\text{def}}{=} \mathbb{C}^2$. The local algebras over a bounded region $\mathcal{A}_{[n,m]}$, $n \le m \in \mathbb{Z}$, are generated by the fermionic generation and annihilation operators a_x, a_x^*, $n \le x \le m$, with commutation relations $[a_x, a_y] = 0 = [a_x, a_y^*]$, $x \ne y$, and $\{a_x, a_x^*\} = 1$, $\{a_x, a_x\} = 0$, where $\{\cdot,\cdot\}$ denotes the anti-commutator. The global algebra $\mathcal{A}$ is the weak closure of the union of the $\mathcal{A}_{[-n,n]}$. The local dynamics is given by the Hamiltonian

$$H_{[n,m]} \stackrel{\text{def}}{=} \frac{J}{2}\sum_{x=n}^{m-1}\left(a_x^* a_{x+1} + a_{x+1}^* a_x\right) + h\sum_{x=n}^{m} a_x^* a_x.$$

The global dynamics in the thermodynamic limit

$$\tau_t(A) \stackrel{\text{def}}{=} \lim_{n\to\infty} \mathrm{e}^{\mathrm{i}tH_{[-n,n]}} A_n \mathrm{e}^{-\mathrm{i}tH_{[-n,n]}},$$

for $A = \lim_{n\to\infty} A_n \in \mathcal{A}$, $A_n \in \mathcal{A}_{[-n,n]}$, $t \in \mathbb{R}$, exists, and renders $(\mathcal{A}, \tau)$ a W^*-dynamical system, whose unique (τ,β)-KMS state at given inverse temperature β is the unique weak-* limit ω_β of any increasing net of local Gibbs states over $[n,m]$. Now let P_0 be the perturbation which removes the particle at position 0:

$$P_0 \stackrel{\text{def}}{=} -\frac{J}{2}\left(a_{-1}^* a_0 + a_0^* a_1 + a_0^* a_{-1} + a_1^* a_0\right) - h a_0^* a_0,$$

such that the Hamiltonian over $[-n,n]$, $n \ge 1$, decomposes as

$$H_{[-n,n]} = H_{[-n,-1]} + P_0 + H_{[1,n]}.$$

Let $\omega_{L,\beta}$, $\omega_{R,\beta}$ be the Gibbs equilibrium states over the subchains $[-\infty,-1]$ and $[1,\infty]$, respectively, obtained as limits of local Gibbs states at inverse temperature β, and let ρ_0 be an arbitrary state over $\mathcal{A}_0$. Then the product state

$$\varphi_{\rho_0,\beta} \stackrel{\text{def}}{=} \omega_{L,\beta} \otimes \rho_0 \otimes \omega_{R,\beta}$$

is invariant under the perturbed dynamics τ^{P_0}. But as $(\mathcal{A},\tau)$ is asymptotically Abelian, it follows that return to equilibrium will occur, i.e.,

$$\lim_{|t|\to\infty} \varphi_{\rho_0,\beta}(\tau_t(A)) = \omega_\beta(A), \quad \text{for } A \in \mathcal{A}.$$

If we choose E to be the projection onto a vector representative of ρ_0 at site 0 and the identity on the rest of the chain, then we have a special instance of Corollary 3.1. Thus, the Zeno dynamics τ^E exists and prevents the return to the global equilibrium. The left and right subchains remain dynamically isolated, and the arbitrary state ρ_0 at the point 0 is preserved. The state $\varphi_{\rho_0,\beta}$

is preserved by the perturbed evolution and by E, however it is not a good candidate for a genuine equilibrium state for the Zeno dynamics τ^E. We will return to that matter at the end of Section 5

4. The Explicit Form of the Zeno Hamiltonian

We want to show that if the Zeno dynamics converges, it is possible to identify its generator explicitly. This will become useful in the sequel.

Let H be the generator of $U(t) = \mathrm{e}^{\mathrm{i}tH}$. The unitary group $U_E(t) \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}tEHE}$ is called the **reduced dynamics** associated to (U, E). Notice that U_E induces an automorphism group $\widehat{\tau}^E$ of $\mathcal{A}_E$ whenever the group τ is one for $\mathcal{A}$. To be able to compare the reduced with the Zeno dynamics, we need a technical condition: We call (U, E) **regular** if $\mathcal{A}_E$ contains a dense set of elements which are analytic for τ in an arbitrary neighbourhood of zero. The condition of regularity will be required to have enough analytic vectors in $\mathcal{H}_E$ at hand for the proof below to work. It excludes pathological cases, e.g., when E projects onto a subspace of states with properly infinite energy. It is automatically satisfied in all examples we consider, see the comment after the proof of the following proposition.

Proposition 4.1. *Let (U, E) be regular and satisfy* (AZC) *for $\mathcal{A}$. Then $U_E(t)$ equals $W(t)$, when restricted to $\mathcal{H}_E$.*

Throughout the proof below let $\Psi_E \in \mathcal{A}_{E,\tau}\Omega \subset \mathcal{H}_E$, where $\mathcal{A}_{E,\tau}$ is a dense set of elements in $\mathcal{A}_E$, which are analytic for τ. Record that, by the discussion following [3, Definition 3.1.17], the τ-analyticity of Ψ_E is equivalent to analyticity with respect to U and this is in turn equivalent to the convergence of power series of analytic functions in σH applied to Ψ_E, for $\sigma \in \mathbb{C}$ small enough, as given in the cited definition.

Proof of Proposition 4.1. We first derive a useful asymptotic estimate: Setting $\Psi_E(\sigma) \stackrel{\text{def}}{=} U_E(\sigma)\Psi_E$ holds

$$\begin{aligned}\big\|\big(U_E(\tau) - EU(\tau)E\big)\Psi_E(\sigma)\big\| &= \left\| \left\{ \sum_{k=0}^{\infty} \frac{(\mathrm{i}\tau)^k (EHE)^k}{k!} - E \sum_{l=0}^{\infty} \frac{(\mathrm{i}\tau)^l H^l}{l!} E \right\} \Psi_E(\sigma) \right\| \\ &= \left\| \sum_{k=2}^{\infty} \frac{(\mathrm{i}\tau)^k}{k!} \big[(EHE)^k - EH^kE\big] \Psi_E(\sigma) \right\|,\end{aligned}$$

using $E\Psi_E(\sigma) = \Psi_E(\sigma)$, which is clear since U_E commutes with E. By using $\|E\| = 1$, this can be estimated further as

$$\leq 2 \sum_{k=2}^{\infty} \frac{|\tau|^k}{k!} \big\|H^k \Psi_E(\sigma)\big\|.$$

Since Ψ_E is analytic for U in a neighbourhood of 0, also the translates $\Psi_E(\sigma) = U_E(\sigma)\Psi_E$, for σ small enough, will be analytic for U in a somewhat smaller neighbourhood of 0. This can be seen by noting that the power series of $U_E(\sigma)$ is term-wise bounded in norm by a convergent one, where EHE is replaced by H, using $\|E\| = 1$. The composition of power series in question then amounts to the composition of analytic functions of H for σ, τ, small enough. Therefore the power series on the right hand side of the last inequality is convergent for σ, τ small, and defines an analytic function in τ which is $O(|\tau|^2)$ as $|\tau| \to 0$. Thus, we finally obtain for small enough σ, τ the estimate

$$\big\|\big(U_E(\tau) - EU(\tau)E\big)\Psi_E(\sigma)\big\| \leq \tau^2 \cdot C_{\Psi_E,\sigma} < \infty. \qquad (\dagger)$$

Now, from $U_E(t)\Psi_E = EU_E(t)E\Psi_E$, follows the identity

$$U_E(t)\Psi_E = \big[EU_E(t/n)E\big]^n \Psi_E, \quad \text{for all } n, \tag{$**$}$$

by iteration. Exploiting this, we can rewrite $F_n(t) - U_E(t)$ to yield

$$\big\|F_n(t)\Psi_E - U_E(t)\Psi_E\big\| = \big\|\big[EU(t/n)E\big]^n\Psi_E - \big[EU_E(t/n)E\big]^n\Psi_E\big\|.$$

A telescopic estimate shows

$$\leq \sum_{i=1}^{n} \Big\|\Big\{\big[EU(t/n)E\big]^{n-i}\big(EU(t/n)E - EU_E(t/n)E\big)\big[EU_E(t/n)E\big]^{i-1}\Big\}\,\Psi_E\Big\|.$$

The norm of the vector under the sum is, using $(**)$,

$$\big\|[EU(t/n)E]^{n-i}(EU(t/n)E - EU_E(t/n)E)\Psi_E(t(i-1)/n)\big\|$$

Using commutativity of U_E with E, and the invariance of $\Psi_E(\sigma)$ under E, we have $EU_E(t/n)E\Psi_E(t(i-1)/n) = U_E(t/n)\Psi_E(t(i-1)/n)$, and use this to rewrite the above expression as

$$=\big\|[EU(t/n)E]^{n-i}(EU(t/n)E - U_E(t/n))\Psi_E(t(i-1)/n)\big\|$$

Now, with $\|[EU(t/n)E]^{n-i}\| \leq 1$ and $\|AB\Psi\| \leq \|A\|\|B\Psi\|$, this is bounded by

$$\leq \|(EU(t/n)E - U_E(t/n))\Psi_E(t(i-1)/n)\|.$$

Putting this together, we obtain the estimate

$$\big\|F_n(t)\Psi_E - U_E(t)\Psi_E\big\| \leq \sum_{i=1}^{n} \|(EU(t/n)E - U_E(t/n))\Psi_E(t(i-1)/n)\|.$$

We can now apply (†) to obtain, for $n > M$ large enough,

$$\|F_n(t)\Psi_E - U_E(t)\Psi_E\| \leq \sum_{i=1}^{n}\left(\frac{t}{n}\right)^2 \cdot \sup_{|\sigma|\leq|t|} C_{\Psi_E,\sigma} = \frac{t^2 C'_{\Psi_E,t}}{n},$$

for some finite $C'_{\Psi_E,t}$. Since F_n converges strongly to W by (AZC), it follows $W(t)\Psi_E = U_E(t)\Psi_E$. The density of the elements $\mathcal{A}_{E,\tau}\Omega$ in $\mathcal{H}_E$ then shows the claim. □

Let us return to Example 2 to see that the regularity condition is satisfied there. If Ψ is an entire analytic vector for U, then applying the projector E_{φ_Λ} changes its total energy only by a finite amount, since the perturbation $P_\Phi(\Lambda) \stackrel{\text{def}}{=} H_\Phi(\Lambda) + W_\Phi(\Lambda)$ is bounded. A finite change of energy does not affect the analyticity with respect to τ and thus the projected vector is τ-analytic. Since one always finds a dense set of analytic elements in $\mathcal{A}$ [3, Proposition 2.5.22], the regularity condition is satisfied in this case. This argument is also valid under the general assumptions of Corollary 3.1.

The explicit form of the generator for the Zeno dynamics also yields an heuristic argument for the equivalence of the Zeno effects produced by 'pulsed' and 'continuous' measurement, respectively. The latter commonly denotes the simple model for the coupling of the quantum system to a measurement apparatus by adding to the original Hamiltonian a measurement Hamiltonian multiplied by a coupling constant, and letting the coupling constant tend to infinity [6, 7, 8]. The essential point here is that the degrees of freedom in the Zeno subspace $\mathcal{H}_E$ become energetically infinitely separated from those in its orthogonal complement. For this it suffices to set

$$H_K \stackrel{\text{def}}{=} H + KE^\perp, \quad U_K(t) \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}tH_K},$$

and to consider the limit $K \to \infty$. This can be done by applying analytic perturbation theory to

$$H_\lambda \stackrel{\text{def}}{=} \lambda H + E^\perp, \quad \text{with } \lambda \stackrel{\text{def}}{=} K^{-1},$$

and

$$U_\lambda(\tau) \stackrel{\text{def}}{=} \mathrm{e}^{\mathrm{i}\tau H_\lambda} = U_K(t), \quad \text{with } \tau \stackrel{\text{def}}{=} Kt = t/\lambda,$$

around $\lambda = 0$. The final result is

$$\lim_{K\to\infty} U_K(t)\Phi = \mathrm{e}^{\mathrm{i}tEHE}\Phi,$$

for any vector $\Phi \in \mathcal{H}_E$. Details are to be found in [8, Section 7].

This treatment of 'continuous measurement' is certainly the coarsest possible. To examine more deeply the relationship between the two manifestations of the Zeno effect, one should consider more refined models for the interaction of a quantum with a classical system, e.g., as in [2].

5. Equilibrium States for Zeno Dynamics

The explicit form of the generator of the Zeno dynamics found in Proposition 4.1 readily provides us with examples for equilibrium states for the Zeno dynamics: Every equilibrium state for the reduced dynamics U_E will be one, since the original representation of $\mathcal{A}_E$ on $\mathcal{H}_E$ is faithful and thus the automorphism groups τ^E and $\widehat{\tau}^E$ of $\mathcal{A}_E$ are identical:

Corollary 5.1. *If (U, E) is regular and satisfies (AZC) for $\mathcal{A}$, then, for every $\beta > 0$, the set of (τ^E, β)-KMS states of $\mathcal{A}_E$ equals the set of $(\widehat{\tau}^E, \beta)$-KMS states.*

This result is independent of the representation, since the reasoning of Proposition 4.1 can be repeated in any covariant representation. It applies, in particular, to the important case of Gibbs states as we will now show for quantum spin systems. The ordinary local Gibbs states over bounded regions Λ are

$$\omega_\Lambda(A) \stackrel{\text{def}}{=} \frac{\mathrm{Tr}_{\mathcal{H}_\Lambda}\big(\mathrm{e}^{-\beta H(\Lambda)}A\big)}{\mathrm{Tr}_{\mathcal{H}_\Lambda}\big(\mathrm{e}^{-\beta H(\Lambda)}\big)}, \quad \text{for } A \in \mathcal{A}(\Lambda),$$

and a candidate for a local Zeno equilibrium over Λ is thus

$$\omega_{E_\Lambda}(A_{E_\Lambda}) \stackrel{\text{def}}{=} \frac{\mathrm{Tr}_{\mathcal{H}_\Lambda}\big(\mathrm{e}^{-\beta E_\Lambda H(\Lambda) E_\Lambda}A_{E_\Lambda}\big)}{\mathrm{Tr}_{\mathcal{H}_\Lambda}\big(\mathrm{e}^{-\beta E_\Lambda H(\Lambda) E_\Lambda}\big)}, \quad \text{for } A_{E_\Lambda} \in \mathcal{A}(\Lambda)_{E_\Lambda},$$

if $E_\Lambda \in \mathcal{A}(\Lambda)$ is some collection of projections, and where, as before, $\mathcal{A}(\Lambda)_{E_\Lambda} = E_\Lambda \mathcal{A}(\Lambda) E_\Lambda$. Here it is safe to take the trace over the full local space $\mathcal{H}_\Lambda$, since $\omega_{E_\Lambda}(AB_{E_\Lambda}C) = \omega_{E_\Lambda}(A_{E_\Lambda}B_{E_\Lambda}C_{E_\Lambda})$, for A, B, $C \in \mathcal{A}(\Lambda)$, as follows easily from $E\mathrm{e}^{-\beta E_\Lambda H(\Lambda)E_\Lambda} = \mathrm{e}^{-\beta E_\Lambda H(\Lambda)E_\Lambda}E = \mathrm{e}^{-\beta E_\Lambda H(\Lambda)E_\Lambda}$ and the invariance of the trace under cyclic permutations.

Assume that the local dynamics τ_t^Λ generated by $H(\Lambda)$ converges uniformly to an automorphism group τ of $\mathcal{A}$. Then we know [3, Proposition 6.2.15], that every thermodynamic limit point of the ordinary local Gibbs states, that is, a weak* limit of a net of extensions ω_Λ^G of ω_Λ to $\mathcal{A}$, is a (τ, β)-KMS state over $\mathcal{A}$. As a direct consequence of these considerations and Corollary 5.1, we obtain those equilibrium states for the Zeno dynamics which are limits of local Gibbs states.

Corollary 5.2. *Let $\beta > 0$. Let $\Lambda_\alpha \to \infty$ be such that the local dynamics converges uniformly to the global dynamics τ, and the net of local Gibbs states ω_{Λ_α} has a thermodynamic limit point ω. Let U be the unitary group representing τ in the GNS representation of ω. If a sequence of projections $E_{\Lambda_\alpha} \in \mathcal{A}(\Lambda_\alpha)$*

converges in norm to a projection E in $\mathcal{A}$ such that (U, E) is regular and satisfies (AZC), then $\omega_E(A_E) \stackrel{\text{def}}{=} \lim_\alpha \omega^G_{E_{\Lambda_\alpha}}(A_E)$ defines a (τ^E, β)-KMS state on $\mathcal{A}_E$.

Proof. The local Gibbs states $\omega_{E_{\Lambda_\alpha}}$ are the unique β-KMS states on the finite-dimensional algebras $\mathcal{A}_{\Lambda_\alpha}$ for the reduced dynamics $\widehat{\tau}^{E_{\Lambda_\alpha}}$. If $\{E_{\Lambda_\alpha}\}$ converges uniformly, these local Gibbs state possess ω_E as a weak-* limit, which is a KMS state on $\mathcal{A}_E$ for the reduced dynamics $\widehat{\tau}^E$ associated with τ. Then, by Corollary 5.1, ω_E is also a (τ^E, β)-KMS state. □

Example 4. Let us review Example 2, and assume again that the interaction Φ is such that the global surface energy $W_\Phi(\Lambda)$ is bounded. Then, given the family of projections $E_{\varphi_\Lambda;\Lambda'}$ of Example 2, and E_{φ_Λ} being its uniform limit point, (U, E_{φ_Λ}) satisfies AZC, and is regular as discussed in Section 4. Thus the conditions of Corollary 5.2 are satisfied, and we have to look at the state

$$\omega_{E_{\varphi_\Lambda}}(A_{E_{\varphi_\Lambda}}) \stackrel{\text{def}}{=} \lim_{\Lambda'\to\infty} \frac{\mathrm{Tr}_{\mathcal{H}_{\Lambda'}}\big(\exp(-\beta\, \mathbb{1}_{\Lambda'\setminus\Lambda}\otimes P_{\Phi_\Lambda}\, H(\Lambda')\, \mathbb{1}_{\Lambda'\setminus\Lambda}\otimes P_{\Phi_\Lambda}) A_{E_{\varphi_\Lambda;\Lambda'}}\big)}{\mathrm{Tr}_{\mathcal{H}_{\Lambda'}}\big(\exp(-\beta\, \mathbb{1}_{\Lambda'\setminus\Lambda}\otimes P_{\Phi_\Lambda}\, H(\Lambda')\, \mathbb{1}_{\Lambda'\setminus\Lambda}\otimes P_{\Phi_\Lambda})\big)}$$

where $A_{E_{\varphi_\Lambda;\Lambda'}} \in \mathcal{A}_{E_{\varphi_\Lambda;\Lambda'}}$ converges in $\mathcal{A}_{E_{\varphi_\Lambda}}$ to $A_{E_{\varphi_\Lambda}}$. This limit defines a $(\tau^{E_{\varphi_\Lambda}}, \beta)$-KMS state on $\mathcal{A}_{E_{\varphi_\Lambda}}$. If, in the decomposition $\mathcal{A} = \mathcal{A}_{\Lambda^c} \otimes \mathcal{A}_\Lambda$, the global Hamiltonian decomposes as

$$H = \sum_i H_{\Lambda^c,i} \otimes H_{\Lambda,i}$$

then $\omega_{E_{\varphi_\Lambda}}$ is exactly the Gibbs equilibrium for the **averaged Hamiltonian**

$$\mathbb{E}^{\varphi_\Lambda}(H) = \sum_i H_{\Lambda^c,i} \cdot \varphi_\Lambda(H_{\Lambda,i})$$

with respect to the local state φ_Λ over the interior region Λ. This state has been shown in [10, Section III] to be the strong-coupling limit of equilibrium states for the Hamiltonians $H_\lambda = H + \lambda \mathbb{1}_{\Lambda^c} \otimes P_{\Phi_\Lambda}$, in accordance with our results in the last section. Thus the Zeno dynamics effectively decouples the the interior Λ from the exterior part Λ^c of the system, while the influence of the interior is reduced to a 'mean field'-type interaction, where the inner part of the system is averaged out with respect to the chosen local state φ_Λ.

The above special result for Gibbs states has a counterpart for states which satisfy a *maximum entropy condition*: Let ω be a faithful, normal state on the von Neumann algebra $\mathcal{A}$. For the definition of the **relative entropy** $S(\omega, \varphi)$ of a state φ on $\mathcal{A}$ with respect to ω we refer the reader to [3, Definition 6.2.29]. Raggio and Werner have shown the following general result:

Theorem ([18], Theorem 7)**.** *Let $\widetilde{\omega}$ be a state on $\mathcal{A}$ with $\widetilde{\omega}(E) = 1$. Then holds the estimate $S(\omega, \widetilde{\omega}) \geq -\log\big(\omega(E)\big)$, with equality if and only if $\omega\big([E, A]\big) = 0$, and $\widetilde{\omega}(A) = \omega(EAE)/\omega(E)$, for all $A \in \mathcal{A}$.*

The state $\widetilde{\omega}$ with $\widetilde{\omega}(E) = 1$ is a natural candidate for a Zeno equilibrium state on $\mathcal{A}_E$. If the original state ω is a Gibbs state and the Hamiltonian commutes with E, then this conforms with our above result. However, these restrictions are too severe to identify general Zeno equilibria, which will therefore in general not maximise the relative entropy on the total algebra $\mathcal{A}$.

As a final application of our theoretical results, we reconsider the model of Example 3. As noted there, the τ^E-invariant state we chose was not an equilibrium state. We are now in a position to correct this.

Example 5 (Zeno Equilibria in the X-Y Model). We start by choosing again a fixed state $\rho_0 \in \mathcal{H}_0$ over the centre site of the chain. Again, we use

$$E_{\rho_0} \stackrel{\text{def}}{=} \mathbb{1}_{\mathcal{H}_{[-\infty,-1]}} \otimes P_{\rho_0} \otimes \mathbb{1}_{\mathcal{H}_{[1,\infty]}}$$

as the Zeno projection. Since in this model the interaction is has range 1, and the projection acts local, the Zeno dynamics $\tau^{E_{\rho_0}}$ will certainly exist, by the reasoning of Example 2. As in Example 4, the local Zeno Hamiltonians decompose into two commuting, nontrivial parts over the subchains $[-m,-1]$ and $[1,n]$ which are averaged with respect to ρ_0, and a scalar part:

$$E_{\rho_0} H E_{\rho_0} = H^{\rho_0}_{-m} + H^{\rho_0}_0 + H^{\rho_0}_{+n},$$

where E_{ρ_0} is restricted to $\mathcal{H}_{[-m,n]}$ in the natural way. Explicitly we obtain $H^{\rho_0}_0 = h\rho_0(a_0^* a_0)$, and

$$H^{\rho_0}_{+n} = \frac{J}{2}\big(\overline{\rho_0(a_0)}a_1 + \rho_0(a_0)a_1^*\big) + H_{[1,m]},$$

and likewise for $H^{\rho_0}_{-m}$. Straightforwardly, we obtain Gibbs states over the left and right subchains:

$$\omega^+_{\rho_0,\beta}(A_+) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{\mathrm{Tr}_{\mathcal{H}_{[1,n]}}\big(\mathrm{e}^{-\beta H^{\rho_0}_{+n}} A_{+n}\big)}{\mathrm{Tr}_{\mathcal{H}_{[1,n]}}\big(\mathrm{e}^{-\beta H^{\rho_0}_{+n}}\big)},$$

where $\{A_{+n} \in \mathcal{A}_{[1,n]}\}$ converges to A_+ in $\mathcal{A}_{[1,\infty]}$, which is the weak closure of the union of the local algebras $\mathcal{A}_{[1,n]}$. A similar result holds for the Gibbs state $\omega^-_{\rho_0,\beta}$ over the left subchain. Now the observables for the Zeno dynamics are in this model all of the form

$$A_{E_{\rho_0}} = \sum_i \rho_0(A_{0,i}) A_{-,i} \otimes P_{\rho_0} \otimes A_{+,i}$$

where $A_{\pm,i} \in \mathcal{A}_\pm$, $A_0 \in \mathcal{A}_0$, since the local observables are nothing but polynomials in the local generators a_x, a_x^*. Thus, since the scalar factor $\mathrm{e}^{-\beta H^{\rho_0}_0}$ cancels out in the definition of the Gibbs state, we finally obtain the global equilibrium state on $\mathcal{A}_{E_{\rho_0}}$:

$$\omega_{\rho_0,\beta}(A_{E_{\rho_0}}) \stackrel{\text{def}}{=} \sum_i \rho_0(A_{0,i})\omega^-_{\rho_0,\beta}(A_{-,i})\omega^+_{\rho_0,\beta}(A_{+,i}),$$

or

$$\omega_{\rho_0,\beta} = \omega^-_{\rho_0,\beta} \otimes \rho_0 \otimes \omega^+_{\rho_0,\beta}.$$

This is the desired equilibrium state for the Zeno dynamics $\tau^{E_{\rho_0}}$. Moreover, it is the unique $(\tau^{E_{\rho_0}},\beta)$-KMS state on $\mathcal{A}_{E_{\rho_0}}$, since the Gibbs states are the only KMS states in this class of models, as is shown in [19, Appendix] for the original spin chain by a calculation which depends only on the local CAR, and therefore also applies to the Zeno dynamics (this can also be seen by more general arguments [3, Theorem 6.2.47]). The state $\omega_{\rho_0,\beta}$ decomposes into a product state with respect to the decomposition $\{[-\infty,-1],0,[1,\infty]\}$ of the chain, which again shows that the Zeno dynamics decouples the left and right subchains. The equilibria of the lateral subchains are determined by ρ_0-averaged Hamiltonians, imposing boundary conditions as already exhibited in Example 4. For varying ρ_0, these equilibria are parametrised by $\rho_0(a_0)$. Yet there is somewhat more to say about this example: For here, the difference between the Zeno Hamiltonian and the original one is a finite combination of local generators a_x, a_x^*, $x = 0,\ \pm 1$, as can easily be seen from the explicit forms of H and $E_{\rho_0} H E_{\rho_0}$. This difference is therefore a bounded operator, and moreover it is entire analytic for $\tau^{E_{\rho_0}}$. Thus the original dynamics is a local perturbation of the Zeno dynamics. Under these conditions, the general results about the return to equilibrium [19, Theorem 2] imply that the system starting in a global equilibrium state for the dynamics defined by H will eventually evolve toward a KMS state for the Zeno dynamics. In fact it will approach the Zeno Gibbs state constructed above, since it is the unique KMS state as seen before.

The property of the Zeno dynamics to spontaneously approach a (τ^E, β)-KMS state does only depend on the properties of $EHE - H$. We conclude our discussion by noting this fact:

Corollary 5.3. *Let* (U, E) *be regular and satisfy* (AZC) *for* $\mathcal{A}$. *Let* $\omega|_{\mathcal{A}_E}$ *be the restriction of a* (τ, β)*-KMS state of* $\mathcal{A}$ *to* $\mathcal{A}_E$. *Assume that* $(\mathcal{A}_E, \tau^E)$ *is asymptotically Abelian, and that* $H - EHE$ *is entire analytic for* τ^E. *Then, every weak* limit point for* $t \to \pm\infty$ *of* $\tau_t^E \omega|_{\mathcal{A}_E}$ *is a* (τ^E, β)*-KMS state.*

6. CONCLUSIONS

The present and our previous work [20] have demonstrated that quantum statistical mechanics is another natural field for the exploration of the Zeno effect. In view of our general estimation of the status of the effect (see below), we think the examples shown to be the most important part of our work. Let us review the final Example 5: The decomposition of the global Gibbs equilibrium of the X-Y model into a product state under the special Zeno dynamics is hardly surprising. In fact, this behaviour is characteristic for Gibbs states, when the boundary interaction is removed [3, Definition 6.2.16]. But the new, and physically remarkable point in Example 5 is that a frequent observation at the microscopic level, even a single site, leads to a different equilibrium, namely that of two isolated subchains with a boundary condition. Moreover, we have seen that this behaviour is dynamically observable in the sense that the chain prepared in the Gibbs equilibrium will evolve to the lateral Zeno equilibria, under the Zeno dynamics. This shows that the context of quantum statistical mechanics can indeed exhibit new phenomenological aspects of the quantum Zeno effect. To give a tenable outlook toward further developments, it seems appropriate to give some epistemological rationale as to why the Zeno effect is worth any consideration at all. For its theoretical explanatory power is very limited, due to the very reason which lends it its heuristic appeal: It is an extremely generic phenomenon. But its ubiquity renders its value for basing theoretical explanations for physical phenomena on it small. The effect therefore seems more interesting if considered in special model cases, where it can yield real, and sometimes surprising, predictions of phenomena. Quantum statistical mechanics might provide a fruitful ground in that respect: Since characteristic lifetimes are generally longer for collective than for single- or few-particle phenomena, it is conceivable that the Zeno effect is easier to detect in this context than in many experiments devised so far in atomic and particle physics, see the reviews in [21, 14]. To give a theoretical treatment of further interesting phenomena, one would need a truly representation-independent formulation of the results shown here and in [20], as well as an independent treatment of the C^*-case. This is work in progress.

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