MATHEMATICAL PROBLEMS OF GAUGE QUANTUM FIELD THEORY: A SURVEY OF THE SCHWINGER MODEL

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Abstract. This extended write–up of my talk gives an introductory survey of mathematical problems of the quantization of gauge systems. Using the Schwinger model as an exactly tractable but nontrivial example which exhibits general features of gauge quantum field theory, I cover the following subjects: The axiomatics of quantum field theory, formulation of quantum field theory in terms of Wightman functions, reconstruction of the state space, the local formulation of gauge theories, indefiniteness of the Wightman functions in general and in the special case of the Schwinger model, the state space of the Schwinger model, special features of the model.

New results are contained in the Mathematical Appendix, where I consider in an abstract setting the Pontrjagin space structure of a special class of indefinite inner product spaces — the so called *quasi–positive* ones. This is motivated by the indefinite inner product space structure appearing in the above context and generalizes results of Morchio and Strocchi in [2], and Dubin and Tarski in [12].

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0. Introduction

The mathematical problems that gauge quantum field theory raises are so severe and manifold that not few mathematicians have turned their back on this part of physics. On the other hand the trials to catch the mathematical essence of quantum field theory in a small set of axioms has led to fundamental insights in the general structure of such theories. But it has also shown the difficulties that the quest for a rigoruous definition of *interacting* theories faces (one should better use the word *impossibility*, see the famous book [SW]).

In this record of my talk I review in a fairly mathematical language a model theory that gives us some hope — the Schwinger model, that is, quantum electrodynamics in two spacetime dimensions. It is exactly solvable in two senses: One can determine the correlation functions of the model, as has already been done by Schwinger in [4]. But from that solution one can also (re–) construct the space of physical states as a Hilbert space, and this is what will mainly concern us below. The mathematically interested reader could take a glance at the Mathematical Appendix, where I embed part of the problem into the framework of indefinite inner product spaces and gain some results about quasi–positive spaces which generalize those of [2, 12].

The promising features of the Schwinger model are that shared by other nontrivial gauge quantum field theories in four dimensions, like for instance, the confinement of charged particles, nontrivial vacuum structure and the U(1)–problem (see [1—7]) and one can gain appreciable insights in general quantum field theories by investigating the model.

The literature on the Schwinger model is vast, and we do not intend here to give a complete bibliography. Our exposition relies on [1—3], but the interested reader should also consult the earlier references [4, 5, 8, 11, 12] and the further results in [6, 7].

1. The Mathematical Setting of Quantum Field Theory

In this section I state the general mathematical stipulations for what follows. I briefly recall the axiomatic framework of quantum field theory, in its two formulations.

The Twofold Axiomatics of Quantum Field Theory. The first is the Hilbert space formulation — the *Wightman axioms* (see, e.g., [SW]):

0. Relativistic Quantum Theory. *States* of the theory are unit rays in a separable Hilbert space $\mathfrak{H}$. There exists a strongly continuous unitary representation U of the universal covering $\widetilde{\mathcal{P}}$ of the Poincaré–group on $\mathfrak{H}$:

$$\widetilde{\mathcal{P}} \ni \widetilde{\{a, \Lambda\}} \longmapsto U(a, \Lambda) \in \mathcal{U}(\mathfrak{H}).$$

There exists further a *unique* translationally invariant state called the *vacuum* represented by an unit vector $\Omega \in \mathfrak{H}$, $U(a,1)\Omega = \Omega$, for all a in Minkowski spacetime $\mathcal{M} \equiv \mathbf{R}^4$. The generators P^μ, $\mu = 0, \dots, 3$, of translations (which exist due to Stones Theorem) shall have joint spectrum contained in the closed forward lightcone: $\mathrm{spec}\{P^\mu\} \subset \overline{V}^+$ (*spectrum condition*).

I. Fields are a set $\{\varphi_1, \dots, \varphi_k\}$ of operator valued tempered distributions. That is, for all test functions $f \in \mathcal{S}(\mathcal{M})$ are the *smeared fields*

$$\varphi_i(f) \equiv \int_{\mathcal{M}} d^4x \, \varphi_i(x) f(x) : \ \mathcal{D} \longrightarrow \mathcal{D}$$

operators on a common domain $\mathcal{D}$ of definition. The vacuum is *cyclic* for the polynomial algebra $\mathcal{F} \equiv \mathcal{P}\left(\{\varphi_i(f), \varphi_i(f)^* | f \in \mathcal{S}(\mathcal{M})\}\right)$ of the smeared fields:

$$\mathfrak{H} = \overline{\mathcal{D}_0} \equiv \overline{\mathcal{F}\Omega}.$$

Comment: The above integral notation for the smeared fields is merely to be understood symbolically. For one can easily show (see, e.g., [1], sect. 1.2) that the only way to define fields as *functions*, which is compatible with axioms 0 — III, is to take trivial fields $\varphi_i(f) = \lambda\mathbf{1}$.

II. Poincaré–Covariance. The fields transform *covariantly* under the adjoint action of U:

$$U(a,\Lambda)\varphi_i(x)U(a,\Lambda)^{-1} = S^k{}_j(\widetilde{\Lambda}^{-1})\varphi_k(\Lambda x + a).$$

Here S is a finite–dimensional representation of $\mathrm{SL}(2,\mathbf{C})$, the universal covering of the proper orthochronuous Lorentz group $\mathcal{L}_+^\uparrow$. The above equation shall hold in the strong sense on $\mathcal{D}$, when both sides are smeared with $f \in \mathcal{S}(\mathcal{M})$.

III. Locality/Causality. If $\mathrm{supp}f$ is spacelike to $\mathrm{supp}g$, $f, g \in \mathcal{S}(\mathcal{M})$ (i.e., $(x-y)^2 < 0$, $\forall x \in \mathrm{supp}f$, $y \in \mathrm{supp}g$), then the smeared fields $\varphi_i(f)$ and $\varphi_j(g)$ either commute or anticommute: $[\varphi_i(f), \varphi_j(g)]_\pm = 0$, representing the *Bose–Fermi–alternative*.

The second formulation is that in terms of vacuum expectaion values or *Wightman functions*. First, one point about notation. We introduce the space $\underline{\mathcal{S}}$ of *test function sequences*

$$\underline{\mathcal{S}} \equiv \left\{\underline{f} = (f_0, f_1, \dots) | f_0 \in \mathbf{C}, \, f_i \in \mathcal{S}(\mathcal{M}^i), \, \#\{i, f_i \neq 0\} < \infty\right\},$$

which will play an important role in our discussion of the reconstruction theorem below. Now to the second set of axioms for relativistic quantum field theory.

A. Relativistic n–Point–Functions. There exist tempered distributions, also called *correlation functions*, $\mathfrak{W}_n \equiv \mathfrak{W}(x_1, \ldots, x_n) \in \mathcal{S}(\mathcal{M}^n)'$ for all $n \in \mathbf{N}$. The $\mathfrak{W}_n$ are *translationally invariant*, i.e., there exist distributions $\mathbf{W}_n$ such that

$$\mathfrak{W}(x_1, \ldots, x_n) = \mathbf{W}(\xi_1, \ldots, \xi_{n-1}), \quad \xi_i = x_{i+1} - x_i.$$

Further, the n–point-functions shall be *Lorentz–invariant*: $\mathfrak{W}(\Lambda x_1, \ldots, \Lambda x_n) = \mathfrak{W}(x_1, \ldots, x_n), \forall \Lambda \in \mathcal{L}_+^\uparrow$. A *spectrum condition* is demanded, i.e., the Fourier–transforms $\widetilde{\mathbf{W}}_n$ shall have their support in the $n-1$–fold product of the forward lightcone: $\mathrm{supp}\,\widetilde{\mathbf{W}}_n \subset \overline{V}^{+^{n-1}}$. Another important condition imposed is that of *hermiticity*. For $f_n \in \mathcal{S}(\mathcal{M}^n)$ define $f_n^*(x_1, \ldots, x_n) \equiv \overline{f_n}(x_n, \ldots, x_1)$. Then

$$\mathfrak{W}_n(f_n) = \overline{\mathfrak{W}_n(f_n^*)}.$$

B. Positity means that, for $\underline{f}, \underline{g} \in \underline{\mathcal{S}}$, the *sesquilinear form* given by

$$<\underline{f}|\underline{g}> \equiv \sum_{m,n} \mathfrak{W}_{m+n}(f_n^* \otimes g_m) \tag{1.1}$$

shall be positive, i.e., $<\underline{f}|\underline{f}> \geq 0$. (We have set $f_n \otimes g_m(x_1, \ldots, x_{m+n}) \equiv f_n(x_1, \ldots, x_n) g_m(x_{n+1}, \ldots, x_{m+n})$).

C. Cluster Property. For every spacelike vector $a \in \mathcal{M}$ and $\mathbf{R} \ni \lambda \to \infty$ the condition

$$\lim_{\lambda \to \infty} \mathfrak{W}(x_1, \ldots, x_j, x_{j+1} + \lambda a, \ldots, x_n + \lambda a) - \mathfrak{W}(x_1, \ldots, x_j)\mathfrak{W}(x_{j+1}, \ldots, x_n) = 0$$

shall hold in the sense of distributions.

Comment: This is a less intuitive axiom from the mathematical standpoint. It means roughly that the correlation of clusters of fields, which become infinitely spacelike separated, factorizes (i.e., a decorrelation takes place). This is tied to the independence of events in the two clusters in that limit and motivated by results from (Haag–Ruelle) scattering theory (see, e.g., [BLT]). It is remarkable that the cluster property serves also to ensure the uniqueness of the vacuum state in the reconstruction theorem discussed below.

D. Local Commutativity. Whenever $(x_j - x_{j+1})^2 < 0$,

$$\mathfrak{W}(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = \mathfrak{W}(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n)$$

follows.

Equivalence of the Two Formulations. It is relatively clear that we can gain n–point functions by the simple definition of vacuum expectation values

$$\mathfrak{W}_n(x_1, \ldots, x_n) \equiv <\Omega | \phi(x_1) \ldots \phi(x_n) \Omega>.$$

(For simplicity we consider only one single hermitean scalar field in all the following. Otherwise, one would have to take more than one sort of Wightman functions into account, i.e., one would have to go over to tensor products of distribution spaces). With the above definition, the properties A — D follow rather directly from 0 — III (see, e.g., [SW]). The idea how to reconstruct the theory, in terms of a Hilbert space and fields, from the Wightman functions, is also simple but mathematically a bit more involved. Roughly, one uses the space $\underline{\mathcal{S}}$ as the raw material for building the Hilbert space $\mathfrak{H}$.

Sketch of the Reconstruction Theorem. On $\underline{\mathcal{S}}$ one immediately has a linear structure given by $(\alpha \underline{f} + \beta \underline{g})_i \equiv \alpha(\underline{f})_i + \beta(\underline{g})_i$, for $\alpha, \beta \in \mathbf{C}$. A linear representation of the Poincaré–group is induced by setting

$$(U(a, \Lambda)\underline{f})_i \equiv (\underline{f})_i(\Lambda^{-1}(x - a)). \tag{1.2}$$

The positivity condition B gives the obvious idea, to take the form $<.|.>$ as a candidate for a scalar product on $\underline{\mathcal{S}}$. What we shall call fields will then act simply as a sort of "creation operators" on $\underline{\mathcal{S}}$: That is, for all $h \in \mathcal{S}(\mathcal{M})$ the action of the map $\varphi(h) : \underline{\mathcal{S}} \to \underline{\mathcal{S}}$ is defined by

$$\varphi(h)\underline{f} \equiv (0, hf_0, h \otimes f_1, \ldots). \tag{1.3}$$

The condition of translational invariance formulated in 0 gives a simple choice for a vacuum state, namely $\Omega \equiv (c, 0, 0, \ldots)$, $c \in \mathbf{C}$. The main technical hurdle to take is the fibering of $\underline{\mathcal{S}}$ into equivalence classes and subsequent Hilbert space completion. One has to quotient out the isotropic part $\underline{\mathcal{S}}^{\perp}$ (see the appendix for its definition) to render the scalar product positive definite, and finally arrives in Hilbert space:

$$\mathfrak{H} \equiv \overline{\underline{\mathcal{S}} \Big/ \underline{\mathcal{S}}^{\perp}}.$$

Some comments about the interdependence of the axioms: First, the definition (1.1) together with the invariance properties of the $\mathfrak{W}_n$ makes the representation (1.2) unitary and the fields defined by (1.3) will transform under the adjoint action of U (for the case of a single scalar field), thus they fulfill II. Locality follows directly from D. The only remarkable thing to note is, that the cluster property C is necessarily needed to show that the vacuum is the only translationally invariant state. To see a simple physical argument for that the reader might take a look at [1], pp. 15. But see also the complete proof of the reconstruction theorem in [SW], chapter III.

2. The Schwinger Model

Definition of the Model. Now we will introduce the model we are about to consider, in the symbolic notation common to physicists.

The *Schwinger model*, also called QED$_2$, is a Quantum Field Theory on two–dimensional Minkowski–spacetime, i.e., $\mathbf{R}^2$, $x = (x^0, x^1)$, with diagonal metric $g^{00} = -g^{11} = 1$. Let ε be the antisymmetric rank–2–tensor, $\varepsilon^{10} = -\varepsilon^{01} = 1$, and define the two–dimensional gamma matrices by

$$\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \gamma^5 \equiv \gamma^0\gamma^1.$$

The ingredients of the theory are two fields, as usual in quantum electrodynamics: First, a *fermion field*

$$\psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \overline{\psi} \equiv \psi^*\gamma^0.$$

Second, a *gauge potential* A_μ — a two–dimensional vector field with affiliated *field strength*

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.$$

The *equations of motion* are the *Dirac equation*

$$i\gamma^\mu\partial_\mu\psi + g\gamma^\mu[A_\mu\psi]_{\text{ren}} = 0, \tag{2.1}$$

where g is the *coupling constant* (replacing the electric charge unit in usual QED), and the *local Gauß law*

$$\partial^\nu F_{\nu\mu} = -gj_\mu, \tag{2.2}$$

where j_μ is the *"electric" current* to be defined below.

The product of fields $A_\mu\psi$ at a point in spacetime is, at first, an ill–defined object as well as the current operator j_μ. Namely, when one imposes the canonical commutation relations on the fields A_μ, ψ and their conjugated momenta, one immediately encounters singularities if one tries to define the operator products in a naïve way. To give sense to them as operator–valued distributions one has to adopt a regularization procedure. It reads

$$j_\mu(x) \equiv \lim_{\substack{\varepsilon\to 0 \\ \varepsilon^2 < 0}} \left\{\overline{\psi}(x+\varepsilon)\gamma_\mu\psi(x) - <\Omega|\overline{\psi}(x+\varepsilon)\gamma_\mu\psi(x)\Omega>\right\}[1 - ig\varepsilon^\nu A_\nu(x)] \tag{2.3}$$

for the current. The last term makes the procedure of subtracting the singularity, which is given by the vacuum expectation value, gauge invariant. And for the renormalized operator product one sets

$$[A_\mu(x)\psi(x)]_{\mathrm{ren}} \equiv \lim_{\substack{\varepsilon\to 0 \\ \varepsilon^2<0}} \frac{1}{2}\big\{A_\mu(x+\varepsilon)\psi(x) + \psi(x)A_\mu(x-\varepsilon)\big\}. \qquad (2.4)$$

At this point a comment about the mathematical status of the above notions is at hand. They are "hypothetical," in the sense that there is a priori no Hilbert space of states in which the above expressions could be defined as weak operator limits. Nevertheless, by the canonical quantization procedure and by the Lagrange–equation of the model, one has enough data to determine the singularities exact enough to subtract (as in (2.3)) or regularize them away (as in (2.4)).

To complete this "bootstrap approach" the central task is the construction of the state space. Before addressing this issue in the context of the Schwinger model, I will first mention a general problem, which every gauge quantum field theory raises. In this I will assume that the entities "vacuum" and "Hilbert space," in the sense of the Wightman axioms, exist.

Local Formulation of Gauge Theories. The main question in gauge quantum field theory is: Can one construct *charged states*, and is it possible to give sense to *charge operators* as observables? Charged states are classically characterized by the Gauß law, i.e., the conservation of the Noether current corresponding to the gauge symmetry. The first best guess for a quantum charge operator is to smear out the zero–component of the current (in the meantime I am talking about general QFT in four dimensions):

$$Q_R \equiv \int_{\mathcal{M}} d^4x\, j_0(\boldsymbol{x},t) f_r(\boldsymbol{x})\alpha(t), \qquad (2.5)$$

where $f_R(\boldsymbol{x}) \equiv f(|\boldsymbol{x}|/R)$, with f a test function obeying $f(\boldsymbol{x}) = 1$ for $|\boldsymbol{x}| < 1$, $f(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| > 1 + \varepsilon$ and $\alpha \in \mathcal{D}(\mathbf{R})$, $\int dt\,\alpha(t) = 1$. In a quantum theory this charge should fulfill the commutation relations of a generator of gauge transformations

$$\lim_{R\to\infty} [Q_R, A] = qA$$

with every charged field A carrying total charge q, which is a scalar in the abelian case, as we assume here.

The problem one has to face now is that a theory in which the local Gauß law holds will not contain any *local* charged fields. Namely one has

$$[Q_R, A] = \int_{\mathcal{M}} d^4x\, f_R(\boldsymbol{x})\alpha(t)[\partial^i F_{i0}, A] = \int_{\mathcal{M}} d^4x\, \partial^i f_R(\boldsymbol{x})\alpha(t)[F_{i0}, A]$$

by antisymmetry of F and (2.2). Now the function $\partial^i f_R \alpha$ has support in the shell $R < |\boldsymbol{x}| < R(1+\varepsilon)$ and compact support in time direction. If A is a local observable in the sense of axiom III, then its own support will be bounded and become spacelike to the support of $\partial^i f_R \alpha$ for large enough R. It follows that $\lim_{R\to\infty}[Q_R, A] = 0$ and A has zero charge.

There are two ways to overcome this: Either one could use non–local fields or one could retain locality and modify Gauß' law. The second approach is what we call the *local formulation of gauge QFT's*: We insist on locality of all fields on the expense of introducing an unphysical *longitudinal field* j_μ^{L} (a term coming from QED), which adds to the current in (2.2). The result is the *modified Gauß law*

$$\partial^\nu F_{\nu\mu} = -gj_\mu + gj_\mu^{\mathrm{L}}. \tag{2.6}$$

Clearly, the effect of the additional term is to give a nontrivial contribution for the quantum charge. But, as j_μ^{L} is an artifact of quantization, it should have no observable content by itself. That is, one has to impose the condition

$$<\Phi|j_\mu^{\mathrm{L}}\Psi> = 0, \quad \forall \Phi, \Psi \in \mathfrak{H}_{\mathrm{phys}}, \tag{2.7}$$

on the vacuum expectation values of states in the *physical* Hilbert space $\mathfrak{H}_{\mathrm{phys}}$, which has yet to be defined. Equation (2.6) together with condition (2.7) is called *weak Gauß law*. The subspace $\mathfrak{H}_{\mathrm{phys}}$ is singled out from the total space $\mathfrak{H}$ (which is created by acting on the vacuum by *all* fields, even the unphysical) by the so–called *subsidiary* or *BRST–condition*

$$Q_{\mathrm{BRST}} \equiv (j_\mu^{\mathrm{L}})^- \Psi = 0, \quad \forall \Psi \in \mathfrak{H}_{\mathrm{phys}}. \tag{2.8}$$

Here $(.)^-$ means taking the negative–energy–, i.e., annihilator–part of a given field. This suffices to ensure (2.7), as one easily sees using hermiticity of j_μ^{L}.

There is, however, a price to be paid for the local formulation: The Wightman functions will, under very general conditions, no longer satisfy the positivity axiom B. For it holds

PROPOSITION. *A local formulation of a gauge theory satisfying all Wightman axioms 0 — III contains no charged states in* $\mathfrak{H}_{\mathrm{phys}}$.

Sketch of Proof.   From $\Omega \in \mathfrak{H}_{\mathrm{phys}}$ and the fact that j_μ and $F_{\mu\nu}$ are observable fields, it follows that

$$j_\mu^{\mathrm{L}}(f)\Omega = \left(j_\mu + \frac{1}{g}\partial^\nu F_{\nu\mu}\right)(f)\Omega \in \mathfrak{H}_{\mathrm{phys}}.$$

Then the weak Gauß law implies $<j_\mu^{\mathrm{L}}(f)\Omega|j_\mu^{\mathrm{L}}(f)\Omega> = 0$, $\forall f \in \mathcal{S}(\mathcal{M})$. Since the vacuum is not only cyclic but also a separating vector for the field algebra (an important consequence of the Wightman axioms called the *Reeh–Schlieder–Theorem*, cf. [SW]), it follows that $j_\mu^{\mathrm{L}}(f)\Omega = 0$, *provided positivity holds*. If now $j_\mu^{\mathrm{L}}(f)$ is localized in a bounded region
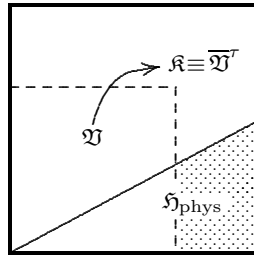
$\mathcal{O} \subset \mathcal{M}$ and P_1, P_2 are polynomials in fields localized in regions $\mathcal{O}_1$, $\mathcal{O}_2$ respectively, both spacelike separated from $\mathcal{O}$, then $<P_1\Omega|j_\mu^L(f)P_2\Omega> = <P_1\Omega|P_2 j_\mu^L(f)\Omega> = 0$. Since the states generated by polynomials in the local fields form a dense subset in $\mathfrak{H}$ (Axiom 0), we conclude that $j_\mu^L(f) = 0$ as an operator in $\mathfrak{H}$. This means nothing but that the Gauß law holds in its original form, which, as we have seen above, implies that $\mathfrak{H}_{phys}$ contains no charged states.

It should be clear from the above proof that indeed the properties in conflict are locality and positivity. This means that the space generated by the polynomial algebra $\mathcal{P}$ of the local fields from the vacuum must be an indefinite inner product space $\mathfrak{V} \equiv \mathcal{P}\Omega$, and the physical Hilbert space should be the positive part of this space determined by condition (2.8).

Now that we have saved locality of our fields and possibly have gained by the subsidiary condition a physical space $\mathfrak{H}_{phys}$, the next question to ask (which is mostly neglected in literature, see the discussion in [1], pp. 91) runs: Is $\mathfrak{V}$ modulo (2.8) large enough to contain charged states? Unfortunately the answer is negative, at least for abelian gauge theories (and there seems to be no reason why this should become better in non–abelian theories), as the following result shows.

PROPOSITION [1, Prop. 6.4]. *In a local formulation of QED all the physical local states have zero charge.*

The situation so far is sketched below:

We have an indefinite inner product space $\mathfrak{V}$ (we assume $\mathfrak{V}$ to be nondegenerate, otherwise one would have to quotient away its isotropic part) containing physical as well as unphysical states, but no charged states since $\mathfrak{V}$ is generated from the vacuum by polynomials in *local* fields. To find charged states as suitable limits of states in $\mathfrak{V}$, one has to look for a completion of $\mathfrak{V}$ in an appropriate topology τ. After that, one should impose the subsidiary condition to get the physical Hilbert space $\mathfrak{H}_{phys}$. The charged and presumably nonlocal states are then contained in the shaded area of the above figure.

In our quest of a topology the framework of majorant Hilbert topologies and Krein spaces, as developed in the first part of the appendix, comes in

place (consult this section for definitions). Let us first replace the by now unsufficient positivity condition B by the weaker assumption

B′. Hilbert space structure condition. Assume that there exist Hilbert seminorms p_n on $\mathcal{S}(\mathcal{M}^n)$ such that

$$\left|\mathfrak{W}_{n+m}(f_n^* \otimes g_m)\right| \leq p_n(f_n)p_m(g_m) \tag{2.9}$$

for all $f_n \in \mathcal{S}(\mathcal{M}^n)$, $g_m \in \mathcal{S}(\mathcal{M}^m)$.

Taking a look at lemma A.7 in the appendix we see that, in a theory where B′ holds, $\mathfrak{V}$ will possess a majorant Hilbert topology τ induced by a single majorant Hilbert norm p. Then A.9 — A.12 tell us that there exists a completion of $\mathfrak{V}$ w.r.t. a minimal majorant Hilbert topology τ_* to a Krein space $\mathfrak{K}$, i.e., a maximal Hilbert space structure $(\mathfrak{K}, G)$, with metric operator G, for $\mathfrak{V}$. This is the best we can do with indefinite inner product spaces. Let us now see how B′ works in the concrete example.

The State Space of the Schwinger Model. Lowenstein and Swieca have shown in [5] that the solution of QED_2 in terms of Wightman functions in the original paper [4] can be rewritten formally in terms of the so–called *building block fields*. These are three free fields η, Σ and ψ_0. $\psi_0(x)$ is a massless free Dirac fermion field. $\Sigma(x)$ is a massive scalar field of mass $M = g/\sqrt{\pi}$ obeying the equation of motion

$$\Box\Sigma(x) = M^2\Sigma(x).$$

Finally η is a massless scalar field, on which we will concentrate our attention below. First, let us write down the formal solution to (2.1) and (2.6) in terms of these fields. It reads

$$\psi(x) = :e^{i\sqrt{\pi}\gamma^5(\eta+\Sigma)}:(x)\psi_0(x),$$
$$A_\mu(x) = -\frac{\sqrt{\pi}}{g}\varepsilon_{\mu\nu}\{\partial^\nu\eta(x) + \partial^\nu\Sigma(x)\}. \tag{2.10}$$

Here the Wick–ordered exponential in the expression for $\psi(x)$ is defined as usual by its formal power series (see, e.g., [BLT]). Using the equations of motion for η and Σ, and inserting (2.10) into the regularization prescription (2.3) for the current, one gets explicit expressions for the physical and the longitudinal current in (2.6) (see [5] for details):

$$j_\mu(x) = j_\mu^{\text{free}} + \frac{g}{\pi}A_\mu(x), \quad \text{where } j_\mu^{\text{free}}(x) \equiv :\psi_0\gamma_\mu\psi_0:(x), \quad \text{and}$$
$$j_\mu^{\text{L}}(x) = j_\mu^{\text{free}}(x) - \frac{1}{\sqrt{\pi}}\varepsilon_{\mu\nu}\partial^\nu\eta(x).$$

Let us reconsider the nonpositivity problem above in the example. As the two–point functions of the free fields in question are well known, one can, at first, conclude that nonpositivity can only come from the two–point function of the massless scalar field η. For it has the special form in two dimensions (see [1] and [8] for definitions and notations):

$$(\eta(x)\Omega|\eta(y)\Omega) = -D^+(x-y), \text{ where } D^+(x) \equiv \lim_{\varepsilon \searrow 0} \frac{1}{4\pi} \ln\left(x^2 + i\varepsilon x^0\right). \quad (2.11)$$

The limit has been taken to define the expression as a distribution on $\mathcal{S}(\mathbf{R}^2)$ and we have used the notation $(.|.)$ for the indefinite form. The indefiniteness can be made explicit using the momentum–space representation of $D^+(x)$. Going over to *lightcone coordinates* $p_\pm \equiv p_0 \pm p_1$, one finds that the Fourier–transform $\widetilde{D^+}(p)$ is given by

$$\widetilde{D^+}(p) = \left[\left(\frac{1}{p_+}\right)_+ \delta(p_-) + \left(\frac{1}{p_-}\right)_+ \delta(p_+)\right] \Theta(p_0) \quad (2.12)$$

(see [8]). (2.12) is — up to a constant which we suppressed — the most general Lorentz invariant Distribution satisfying $p^2 \widetilde{D^+}(p) = 0$, and concentrated on the plus lightcone C^+, the surface on which $p^2 = 0$, $p_0 \geq 0$. In this expression we used the canonically regularized distribution (cf. [GS], §I.3)

$$\left(\frac{1}{p}\right)_+ (\tilde{f}) \equiv \int_0^\infty dp \, \frac{\tilde{f}(p) - \tilde{f}(0)}{p}.$$

The necessity of a regularization at $p = 0$ makes it clear that the two–point function will violate the positivity postulate B for general $f \in \mathcal{S}(\mathbf{R}^2)$, with $\tilde{f}(0) \neq 0$.

Now we can again apply the results of the appendix — especially of the last section — to reconstruct the state space of the theory from its Wightman functions, as sketched in section 1. This will now actually be a Krein space. We begin with the subspace $\mathcal{S}(\mathbf{R}^2)$ of $\underline{\mathcal{S}}(\mathbf{R}^2)$ corresponding to the part of the inner product coming from the two–point function. On this, the indefinite inner product (A.5) has the special form

$$(f|g)_{(1)} \equiv -D^+\left(\overline{f}(x)g(y)\right) = -\widetilde{D^+}\left(\overline{\tilde{f}}(-p)\tilde{g}(p)\right).$$

By (2.12), the singular integral kernel K_λ of (A.4) is $1/p_\pm$ acting on two copies of the real half–axis $[0, \infty)$. One finds a suitable test function $\chi_0 \in \mathcal{S}(\mathbf{R}^2)$ with

Fourier–transform that is unity at the origin, $\widetilde{\chi_0}(0) = 1$, and $(\chi_0|\chi_0)_{(1)} = 0$. Then the linear decomposition of $\mathcal{S}(\mathbf{R}^2)$ as in (A.6), (A.7) takes place. The Fourier–transform of every $f \in \mathcal{S}(\mathbf{R}^2)$ decomposes into

$$\widetilde{f} = \widetilde{f^+} + \chi_0 \widetilde{f}(0),$$

with a function $f^+ \in \mathcal{S}(\mathbf{R}^2)$, satisfying $\widetilde{f^+}(0) = 0$. We find a majorant Hilbert seminorm p_η and a positive inner product $<.|.>_{(1)}$ on $\mathcal{S}(\mathbf{R}^2)$ as in (A.8). It is explicitly given by

$$<f|g>_{(1)} \equiv -(f^+|g^+)_{(1)} + (f|\chi_0)_{(1)}(\chi_0|g)_{(1)} + \overline{\widetilde{f}(0)}\widetilde{g}(0).$$

The minus sign in the first term is compensating the one in the expression (2.11) for the two–point function (we are actually dealing here with a quasi–negative instead of a quasi–positive space, but this poses no problem for the general procedure). Finally, the completion of $\mathcal{S}(\mathbf{R}^2)$ w.r.t. the topology $\tau(\eta)$ induced by p_η will be

$$\mathfrak{K}_\eta^{(1)} \equiv \overline{\mathcal{S}(\mathbf{R}^2) \Big/ \mathcal{S}(\mathbf{R}^2)^\perp}^{\widehat{\tau(\eta)}} = \mathrm{L}^2\left(C^+, \frac{dp}{|p|}\right) \oplus \langle v_0 \rangle \oplus \langle \chi_0 \rangle,$$

like in (A.9), where $\widehat{\tau(\eta)}$ is the induced topology on the quotient. This is what we could call the "one–particle space" of η. It is a Pontrjagin space with rank of positivity equal to one. Since the n–point functions of a free scalar field decompose tensorially into sums of products of the two–point function, one can render the total *Fock space* of this field as an orthogonal sum of tensor products

$$\mathfrak{K}_\eta = \bigoplus_n \left(\mathfrak{K}_\eta^{(1)}\right)^{\otimes n}.$$

Further analysis of the state–space structure yields: p_η, together with the positive seminorms given by the two–point functions of Σ and ψ_0, induces a minimal majorant Hilbert topology τ on the space $\mathcal{F}\Omega$, generated from the Fock vacuum by the action of the polynomial algebra $\mathcal{F} = \mathcal{P}(\{A_\mu, \psi\})$ of the physical fields. Setting $\mathfrak{K} \equiv \overline{\mathcal{F}\Omega}^\tau$, one finds (see [1, 3])

$$\mathfrak{K} = \mathfrak{H}_\Sigma \otimes \mathfrak{K}_{\eta,\psi_0}.$$

Here $\mathfrak{H}_\Sigma$ is the Fock space of Σ, and $\mathfrak{K}_{\eta,\psi_0}$ is $\overline{\mathcal{F}_{\eta,\psi_0}\Omega}^\tau$, where

$$\mathcal{F}_{\eta,\psi_0} \equiv \mathcal{P}\left(\left\{j_\mu^{\text{free}}, j_\mu^{\text{L}}, :e^{i\sqrt{\pi}\gamma^5\eta}:\psi_0\right\}\right)$$

is the field algebra generated by all physical fields not containing Σ. The last step is to impose the subsidiary condition to get the physical Hilbert space. Denote by $\widehat{\mathfrak{K}} \equiv \left\{ \Psi \in \mathfrak{K} \mid (j_\mu^{\mathrm{L}})^- \Psi = 0 \right\}$ the subspace of solutions to this condition. Then finally,

$$\mathfrak{H}_{\mathrm{phys}} = \overline{\widehat{\mathfrak{K}} \Big/ \widehat{\mathfrak{K}}^\perp}^{\,\widehat{\tau}}.$$

Physical Features of the Model. Coming to the end, let me say a few words about the interesting features of the Schwinger model, which can be rigoruously derived, now that we have identified its physical state space. These features are interesting because they are shared by nontrivial gauge quantum field theories in four dimensions like QCD.

The first to mention is the *confinement of charged particles*. That is, if we define the "electric charge" as in (2.5) by

$$Q_R^{\mathrm{el}} \equiv \int_{\mathbf{R}^2} d^2x \, j_0(x^0, x^1) f_R(x^1) \alpha(x^0),$$

then

$$\mathrm{s-}\lim_{R \to \infty} \exp\left(i\lambda Q_R^{\mathrm{el}} \right) = \mathbf{1}_{\mathfrak{H}_{\mathrm{phys}}}, \quad \forall \lambda \in \mathbf{C}$$

(s$-$lim denotes the limit in the strong operator sense). That means, the electric charge vanishes on the physical space — charges are "invisible."

Much more would be left to tell about, e.g., the *nontrivial vacuum structure*, the *energy gap* and the U(1)–*problem*, but lack of space urges me to stop here. The physically interested reader might consider to take a look at [1, 3, 6 and 7] for discussions of these issues.

A. Mathematical Appendix

Generalities About Indefinite Inner Product Spaces. In this subsection we recall some facts about indefinite inner product, Krein and Pontrjagin spaces needed in the subsequent sections and the main text. This is just to be self–contained. For an extensive discussion of the subject see [BOG].

First some notations: Let $\mathfrak{V}$ be a vector space equipped with an indefinite inner product $(.|.)$ (antilinear in the first, linear in the second argument). The linear span of a subset $\mathfrak{A}$ of vectors in $\mathfrak{V}$ is denoted by $\langle \mathfrak{A} \rangle$. The *linear sum* of subspaces $\mathfrak{V}_1, \dots, \mathfrak{V}_n$ of $\mathfrak{V}$ is given by $\langle \mathfrak{V}_1 \cup \cdots \cup \mathfrak{V}_n \rangle$ and denoted by $\mathfrak{V}_1 + \cdots + \mathfrak{V}_n$. If the spaces $\mathfrak{V}_1, \dots, \mathfrak{V}_n$ are linearly independent, their linear sum is termed *direct sum* and denoted by $\mathfrak{V}_1 \dotplus \cdots \dotplus \mathfrak{V}_n$. *Orthogonality* w.r.t. $(.|.)$ is defined, and denoted by the binary relation $\perp$ as usual (but clearly does not have the same strong consequences as in definite inner product spaces). If the $\mathfrak{V}_1, \dots, \mathfrak{V}_n$ are mutually orthogonal, their *orthogonal direct*

sum is denoted by $\mathfrak{V}_1 \,(\dotplus) \cdots (\dotplus)\, \mathfrak{V}_n$, whereas the symbol $\oplus$ is reserved for orthogonal sums w.r.t. a positive definite inner product, which we will casually denote with $<.|.>$. By *positive definite* we mean as usual $<x|x> \geq 0$, $\forall x \neq 0$, and $<x|x> = 0 \Rightarrow x = 0$. A subspace $\mathfrak{A}$ of $\mathfrak{V}$ is called *positve*, *negative*, or *neutral*, respectively, if one of the possibilities $(x|x) > 0$, $(x|x) < 0$ or $(x|x) = 0$ holds for all $x \in \mathfrak{A}$, $x \neq 0$. One sets

$$\mathfrak{V}^{++} \equiv \{x \in \mathfrak{V} |\, (x|x) > 0 \text{ or } x = 0\},$$

and calls this subset the *positive part* of $\mathfrak{V}$. The *negative* and *neutral* parts $\mathfrak{V}^{--}$ and $\mathfrak{V}^0$ are defined alike. A subspace $\mathfrak{A}$ of $\mathfrak{V}$ is called *degenerate*, if its *isotropic part* $\mathfrak{A} \cap \mathfrak{A}^\perp$ does not only consist of the zero vector. In the following we will deal merely with *non–degenerate* spaces, i.e., spaces with $\mathfrak{V}^\perp = \{0\}$.

LEMMA A.1 [BOG, Lemma I.2.1]. *Every indefinite inner product space contains at least one nonzero neutral vector.*

LEMMA A.2 [BOG, Cor. I.4.6]. *If two vectors in an inner product space satisfy $(x|y) \neq 0$, $(x|x) = 0$, then the subspace $\langle x, y \rangle$ is indefinite.*

A non–degenerate inner product space $\mathfrak{V}$ is said to be *decomposable* if it admits a *fundamental decomposition*

$$\mathfrak{V} = \mathfrak{V}^\perp \,(\dotplus)\, \mathfrak{V}^+ \,(\dotplus)\, \mathfrak{V}^-, \quad \text{with } \mathfrak{V}^+ \subset \mathfrak{V}^{++},\ \mathfrak{V}^- \subset \mathfrak{V}^{--}.$$

For non–degenerate spaces the isotropic part of the decomposition vanishes. The special species of decomposable inner product spaces that we will consider in the next section is defined as follows:

DEFINITION A.3. $\mathfrak{V}$ is called a *quasi–positive* (*quasi–negative*) inner product space, if it does not contain any negative definite (positive definite) subspace of infinite dimension.

LEMMA A.4 [BOG, Thm. I.11.7]. *Every quasi–positive (quasi–negative) inner product space is decomposable.*

The dimension of a maximal negative definite subspace $\mathfrak{V}^- \subset \mathfrak{V}^{--}$ of a non–degenerate, quasi–positive space is called the *rank of negativity* of $\mathfrak{V}$. It is an unique positive cardinal ([BOG, Cor. II.10.4]) denoted by $\varkappa^-(\mathfrak{V})$. The *rank of positivity* $\varkappa^+(\mathfrak{V})$ is defined in analogy to that. We set $\varkappa \equiv \min\{\varkappa^-(\mathfrak{V}), \varkappa^+(\mathfrak{V})\}$ and call this number the *rank of indefiniteness* of $\mathfrak{V}$.

Now some less trivial things about the topology of indefinite inner product spaces: A locally convex topology τ on $\mathfrak{V}$ defined by a single seminorm p, which is then actually a norm, is called *normed*. If $\mathfrak{V}$ is τ–complete, we say that τ is

a *Banach topology*. If τ can be defined by a *quadratic norm* $p(x) = <x|x>^{1/2}$, where $<.|.>$ is a positive inner product on $\mathfrak{V}$, then τ is called a *quadratic normed topology*. Again, if $\mathfrak{V}$ is τ–complete, then τ is termed *Hilbert topology*. A normed topology τ_1 is *stronger* than another τ_2, written $\tau_1 \geq \tau_2$, iff every τ_2–open set is also a τ_1–open set. This is the case, iff the relation $p_1(x) \geq \alpha p_2(x)$ holds, with $\alpha > 0$ for all $x \in \mathfrak{V}$. Two norms that define the same topology are called *equivalent*.

A locally convex topology τ on $\mathfrak{V}$ is called a *partial majorant* of the inner product, iff $(.|.)$ is separately τ–continuous. The *weak topology* on $\mathfrak{V}$ is the topology defined by the family of seminorms

$$p_y(x) \equiv |(y|x)|, \quad \forall x \in \mathfrak{V}.$$

LEMMA A.5 [BOG, Thm. II.2.1]. *The weak topology is the weakest partial majorant on $\mathfrak{V}$. If a locally convex topology on $\mathfrak{V}$ is stronger than the weak topology, then it is a partial majorant.*

Below we will encounter the stronger concept of a *majorant topology*:

DEFINITION A.6. A locally convex topology τ on $\mathfrak{V}$ is called *majorant topology*, if the inner product $(.|..)$ is jointly τ–continuous.

The following properties of majorant topologies were used in section 2, equation (2.9), to modify the Wightman axioms for local gauge theories.

LEMMA A.7 [BOG, Lemma IV.1.1 & 1.2].
  i) *To every majorant there exists a weaker majorant defined by a single seminorm.*
  ii) *For a locally convex topology defined by a single seminorm p to be a majorant it is sufficient that p dominates the inner square:*

$$|(x|x)| \leq \alpha p(x)^2, \quad \alpha > 0, \ \forall x \in \mathfrak{V}.$$

Majorant topologies — especially majorant Hilbert topologies — have many advantages over partial majorants. Before we describe them, let us see why one would not like to use the weak topology on general indefinite inner product spaces:

LEMMA A.8 [BOG, Thm. IV.1.4]. *The weak topology on the non–degenerate indefinite inner product space $\mathfrak{V}$ is a majorant, iff $\dim \mathfrak{V} < \infty$.*

The indefinite inner product on a space equipped with a majorant Hilbert topology admits a simple description by the so–called *metric operator*.

PROPOSITION A.9 [BOG, Thm. IV.5.2]. *Let $\mathfrak{V}$ be an indefinite inner product space with a majorant Hilbert topology τ defined by a norm $\|.\|$. Then there exists a Hermitean linear operator, called* metric *(or* Gram*) operator, G on $\mathfrak{V}$ such that*

$$(x|y) = <x|Gy>, \quad \forall x, y \in \mathfrak{V},$$

where $<.|.>$ is the positive inner product on $\mathfrak{V}$ that defines $\|.\|$. Moreover, in this case $\mathfrak{V}$ is decomposable and the fundamental decomposition can be chosen so that each of the three components is τ–closed.

The spaces we want to construct in the next section should be — in a sense — complete.

DEFINITION A.10. If a non–degenerate indefinite inner product space $\mathfrak{K}$ admits a decomposition

$$\mathfrak{K} = \mathfrak{K}^{+} \, (\dot{+}) \, \mathfrak{K}^{-}, \quad \mathfrak{K}^{+} \subset \mathfrak{K}^{++}, \ \mathfrak{K}^{-} \subset \mathfrak{K}^{--},$$

where $\mathfrak{K}^{+}$, $\mathfrak{K}^{-}$ are complete respectively w.r.t. the restriction of the weak topology to them (termed *intrinsically complete*), then $\mathfrak{K}$ is called a *Krein space*. If moreover the rank of indefiniteness of $\mathfrak{K}$ is finite, $\varkappa(\mathfrak{K}) < \infty$, then $\mathfrak{K}$ is called a *Pontrjagin space*.

Krein spaces can easily be characterized:

PROPOSITION A.11 [BOG, Thm. V.1.3]. *An indefinite inner product space $\mathfrak{V}$ is a Krein space iff*

    i) *There exists a majorant Hilbert topology τ on $\mathfrak{V}$ and*
    ii) *The metric operator G is completely invertible.*

The Hilbert–space–completion $\mathfrak{H}$ of an indefinite inner product space $\mathfrak{V}$, if it exists together with its metric operator G, is called the *Hilbert space structure* $(\mathfrak{H}, G)$ associated to $\mathfrak{V}$. In applications one would like to find the largest Hilbert space associated to an indefinite inner product space. For that, one considers *minimal* majorant topologies, i.e., topologies τ_{*} such that no majorant τ is weaker than τ_{*}. Hilbert space structures given by the completion of $\mathfrak{V}$ w.r.t. a minimal majorant are correspondingly called *maximal*. We find that the Hilbert space structure is maximal, iff it leads actually to a Krein space:

LEMMA A.12 [1, App. A.1]. *A majorant Hilbert topology leads to a maximal Hilbert space structure $(\mathfrak{K}, G)$, iff G has a bounded inverse. Given a Hilbert space structure one can always construct a maximal one.*

The last statement means that every space admitting some majorant Hilbert topology can be completed to a Krein space.

The Geometry of Quasi–Positive Spaces. From now on we assume $\mathfrak{V}$ to be an infinite–dimensional, non–degenerate, quasi–positive inner product space with rank of negativity $\varkappa^-(\mathfrak{V}) = N$, $N \in \mathbf{N}$. The following remark sets up our framework.

REMARK A.13. Let $\mathfrak{V}$ be as stated above. Equivalent are:

i) $\varkappa^-(\mathfrak{V}) = N$, $N \in \mathbf{N}$.

ii) There exists a maximal neutral subspace $\mathfrak{V}_N \subset \mathfrak{V}^0$ with $\dim \mathfrak{V}_N = N$ such that $\mathfrak{V} = \mathfrak{V}^+ \dotplus \mathfrak{V}_N$, $\mathfrak{V}^+ \subset \mathfrak{V}^{++}$. Furthermore exists a maximal orthogonal system of N neutral vectors that spans $\mathfrak{V}_N$: $\mathfrak{V}_N = \langle \chi_1, \dots, \chi_N \rangle$, $(\chi_i | \chi_j) = 0$, $\forall i, j \in \{1, \dots, N\}$.

Proof. Assume that i) holds. Then Lemma A.4 shows that $\mathfrak{V}$ admits a fundamental decomposition $\mathfrak{V} = \mathfrak{V}^+ (\dotplus) \mathfrak{V}^-$, $\mathfrak{V}^+ \subset \mathfrak{V}^{++}$, $\mathfrak{V}^- \subset \mathfrak{V}^{--}$, and $\mathfrak{V}^- = \langle v_1^-, \dots, v_N^- \rangle$, where the $v_1^-, \dots, v_N^-$ are assumed to form an orthonormal Basis of $\mathfrak{V}^-$. Moreover, since $\mathfrak{V}$ is infinite–dimensional, we find a orthonormal system $\{v_1^+, \dots, v_N^+\} \subset \mathfrak{V}^+$ of positive vectors. Then the subspaces $\mathfrak{V}_i \equiv \langle v_i^+, v_i^- \rangle$ are by construction linearly independent and mutually orthogonal, and each $\mathfrak{V}_i$ contains a nonzero neutral vector χ_i, due to Lemma A.1. These vectors are an orthogonal system spanning a subspace $\mathfrak{V}_N$. $\mathfrak{V}_N$ is neutral, since the χ_i are mutually orthogonal[1]. We show that $\mathfrak{V}_N$ is maximal: Choose $x_i \in \mathfrak{V}$, $(x_i | \chi_i) \neq 0$, which is possible since $\mathfrak{V}$ is non–degenerate. Then (Lemma A.2) $\langle x_i, \chi_i \rangle$ contains a negative vector w_i^- and these vectors are again linearly independent. Since by assumption the maximal number of linearly independent negative vectors in $\mathfrak{V}$ is N, this shows that $\mathfrak{V}_N$ is maximal. The other direction follows easily from Lemma A.2. $\qquad\square$

$\mathfrak{V}$ admits a majorant Hilbert topology, as we will now show. First note that every $x \in \mathfrak{V}$ admits an unique decomposition

$$x = x^+ + \sum_{i=1}^N x^i \chi_i, \quad x^+ \in \mathfrak{V}^+, \ x^i \in \mathbf{C}. \tag{A.1}$$

PROPOSITION A.14. *The seminorm*

$$p(x)^2 \equiv (x^+ | x^+) + \sum_{i=1}^N \left\{ |(x|\chi_i)|^2 + |x^i|^2 \right\}, \quad \forall x \in \mathfrak{V}, \tag{A.2}$$

defines a majorant topology τ on $\mathfrak{V}$.

Proof. Using the decomposition (A.1) and $(x^+ | \chi) = (x | \chi)$, $\forall x \in \mathfrak{V}$, $\chi \in \mathfrak{V}_N$, the inner product on $\mathfrak{V}$ takes the form

$$(x|y) = (x^+ | y^+) + \sum_{i=1}^N \left\{ \overline{x^i}(\chi_i | y) + (x|\chi_i) y^i \right\}, \quad \forall x, y \in \mathfrak{V}.$$

---

[1] Note that this construction would also apply under the weaker assumption $\varkappa^+(\mathfrak{V}) \geq N$ instead of $\dim \mathfrak{V} = \infty$, but not for general indefinite inner product spaces. A counterexample would be $2 + 1$–dimensional spacetime, which does not contain a neutral subspace of dimension 2. I am indebted to G. Hofmann for bringing this point to my attention.

Now the seminorm p majorizes the inner square $(x|x)$, i.e., $|(x|x)| \leq p(x)^2$, $\forall x \in \mathfrak{V}$, since in every term of the sum the estimate $|\overline{x^i}(\chi_i|x) + (x|\chi_i)x^i| \leq |(\chi_i|x)|^2 + |x^i|^2$ holds. Then Lemma A.7ii) shows that $(.|.)$ is jointly τ–continuous. Therefore τ is a majorant topology on $\mathfrak{V}$. $\qquad\square$

COROLLARY A.15. *On the closure* $\mathfrak{K} \equiv \overline{\mathfrak{V}}^\tau$ *of* $\mathfrak{V}$ *w.r.t.* τ *we can define a Hilbert scalar product by*

$$<x|y> \equiv (x^+|y^+) + \sum_{i=1}^{N} \left\{ (x|\chi_i)(\chi_i|y) + \overline{x^i}y^i \right\}, \ \ \forall x, y \in \mathfrak{K}. \qquad (A.3)$$

We denote the Hilbert norm on $\mathfrak{K}$ *by* $||.|| \equiv p(.)$.

Proof. $<.|.>$ is well defined on whole $\mathfrak{K}$, since $(.|.)$ has an unique extension to $\mathfrak{K}$ (denoted by the same symbol) by its joint continuouity w.r.t. τ. $\qquad\square$

The following lemma sheds a first light on the structure of $\mathfrak{K}$.

LEMMA A.16. *The subspace* $\overline{\mathfrak{V}^+}^\tau$ *of* $\mathfrak{K}$ *is orthogonal to* $\mathfrak{V}_N$ *w.r.t.* $<.|.>$. *That is,* $\mathfrak{K} = \overline{\mathfrak{V}^+}^\tau \oplus \mathfrak{V}_N$.

Proof. For $x^+ \in \mathfrak{V}^+$ and $\chi \in \mathfrak{V}_N$ (A.3) reduces to $<x^+|\chi> = \sum_{i=1}^{N}(x^+|\chi_i)(\chi_i|\chi) = 0$, because x^+ and χ have decompositions with $(x^+)^i = 0$ and $\chi^+ = 0$ respectively, and since all inner products vanish in the neutral subspace $\mathfrak{V}_N$. Continuouity of $(.|.)$ then implies the statement. $\qquad\square$

To take a closer look on $\mathfrak{K}$ we consider the linear functionals

$$F_i(x) \equiv (\chi_i|x)$$

on $\mathfrak{V}$. These functionals are nonzero since $\mathfrak{V}$ is non–degenerate, vanish on $\mathfrak{V}_N$, and are clearly bounded w.r.t. the seminorm p. Then, by the Hahn–Banach Theorem, F_i has an extension to $\mathfrak{K}$ (denoted by the same symbol), which is also bounded, and by (A.2) we have $0 < ||F_i|| \leq 1$. From now on we assume the F_i to be normalized, $||F_i|| = 1$, e.g., by choosing a new Basis $\tilde{\chi}_i \equiv \chi_i/||F_i||$ in $\mathfrak{V}_N$. We have

LEMMA A.17. *The vectors* $v_i \in \mathfrak{K}$ *representing* F_i *by* $F_i(x) = <v_i|x>$, $\forall x \in \mathfrak{K}$ *are actually in the closure of* $\mathfrak{V}^+$: $v_i \in \overline{\mathfrak{V}^+}^\tau$.

Proof. v_i exist in $\mathfrak{K}$ due to the Riesz Representation Theorem. Assume that $(v_{in})_{n\in\mathbf{N}} \subset \mathfrak{K}$ is a sequence converging to v_i, i.e., $||v_i - v_{in}||^2 \longrightarrow 0$. Then, using decomposition (A.1) for v_{in} (and adopting Einstein's summation convention), we find

$$||v_i - v_{in}||^2 = <v_i - (v_{in})^+ - (v_{in})^j\chi_j|v_i - (v_{in})^+ - (v_{in})^k\chi_k>$$
$$= ||v_i - (v_{in})^+||^2 - <v_i - (v_{in})^+ - (v_{in})^j\chi_j|(v_{in})^k\chi_k> - <(v_{in})^j\chi_j|v_i - (v_{in})^+>$$
$$= ||v_i - (v_{in})^+||^2 - <v_i - v_{in}|(v_{in})^j\chi_j> - <(v_{in})^j\chi_j|v_i> + <(v_{in})^j\chi_j|(v_{in})^+>.$$

The last term on the right hand side vanishes identically, due to Lemma A.16. The third term is zero, since $<v_i|\chi_j> = F_i(\chi_j) = 0$. The second term is bounded by $|<v_i - v_{in}|(v_{in})^j\chi_j>|$ $\leq ||v_i - v_{in}|| \sum_{j=1}^{N} |(v_{in})^j|$. In this expression the sum stays bounded, whereas the first factor tends to zero for $n \to \infty$. Hence necessarily $||v_i - (v_{in})^+|| \longrightarrow 0$, which proves the statement. $\qquad\square$

Using this result we can collect some properties of the v_i:

LEMMA A.18. *It holds*

   i) $<v_i|v_j> = (\chi_i|v_j) = 0$, *for* $i \neq j$, *and* $||v_i||^2 = <v_i|v_i> = (\chi_i|v_i) = 1$.
   ii) $(v_i|v_i) = 0$.
   iii) $(v_i|x) = x^i, \ \forall x \in \mathfrak{V}$.

Proof. The second part of i) is clear. Assume v_{in} to be a sequence in $\mathfrak{V}^+$ converging to v_i in $\mathfrak{K}$. Then with (A.3)

$$\left(\frac{|(\chi_i|v_{in})|^2}{||v_{in}||^2}\right)^{-1} = \frac{(v_{in}|v_{in})}{|(\chi_i|v_{in})|^2} + 1 + \frac{\sum\limits_{j \neq i} |(\chi_j|v_{in})|^2}{|(\chi_i|v_{in})|^2}.$$

Now $|(\chi_i|v_{in})|^2 = |<v_i|v_{in}>|^2 \longrightarrow 1$, since $||v_i||^2 = 1$, so that the left hand side tends to one for $n \to \infty$. By the same argument the denominators on the right hand side stay bounded. Thus necessarily $|(\chi_j|v_{in})|^2 \longrightarrow 0$, showing the first part of i). Furthermore $(v_{in}|v_{in}) \longrightarrow 0$, which shows ii), since v_{in} converges w.r.t. $||.|| = p(.)$ and $p(x)^2$ majorizes $|(x|x)|$. To show iii), we consider again the decomposition (A.1) for a vector $x \in \mathfrak{V}$ which yields

$$(v_{in}|x) = (v_{in}|x^+) + x^i(v_{in}|\chi_i) + \sum_{j \neq i} x^j(v_{in}|\chi_j).$$

In this expression $(v_{in}|x^+) \longrightarrow 0$, since by ii) v_{in} converges strongly to zero in $\overline{\mathfrak{V}^+}^\tau$. The sum tends to zero due to i), leaving us with the second term, which tends to x^i for $n \to \infty$ by i), as proposed. $\qquad\square$

So the v_i form an orthonormal system in $\mathfrak{K}$. It is clear by now that the space $\langle v_i|i = 1, \dots, N\rangle \subset \mathfrak{K}$ is isomorphic to the dual space $\mathfrak{V}_N^*$ of $\mathfrak{V}_N$ w.r.t. $(.|.)$. We are now ready to state our main result, which says roughly that every quasi–positive space admits a maximal Hilbert space completion to a Pontrjagin space. Note that this result has already been found by G. Hofmann, cf. [9], in a different context, but without explicit construction of the minimal majorant topology.

THEOREM A.19. *The space $\mathfrak{K}$ is a Pontrjagin space with $\varkappa^-(\mathfrak{K}) = N$. Its Hilbert space structure is maximal and given as follows:*

   a) *If $\mathfrak{H}$ is the Hilbert space closure of $\mathfrak{V}^+$ w.r.t. the topology τ_+ induced by the Norm $p_+(x^+)^2 \equiv (x^+|x^+), \ \forall x^+ \in \mathfrak{V}^+$, then $\mathfrak{K}$ admits the orthogonal decomposition*

$$\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{V}_N^* \oplus \mathfrak{V}_N.$$

b) *The metric operator G on $\mathfrak{K}$, $(.|.) = <.|G.>$ has the form*

$$G = \mathbf{1}_{\mathfrak{H}} \oplus \underbrace{J \oplus \ldots \oplus J}_{N \text{ times}}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \widehat{\mathfrak{V}}_i \to \widehat{\mathfrak{V}}_i,$$

where $\widehat{\mathfrak{V}}_i = \langle v_i \rangle \oplus \langle \chi_i \rangle$, $i = 1, \ldots, N$.

Proof. To prove a), we have to show that $\overline{\mathfrak{V}^{+}}^{\tau} = \mathfrak{H} \oplus \langle v_i | i = 1, \ldots, N \rangle$, taking Lemma A.16 into account. First, note that according to Lemma A.18 the v_i form an orthonormal basis for $\mathfrak{V}_N^*$. For $x^+, y^+ \in \mathfrak{V}^+$ we have by definition of the v_i and (A.3)

$$<x^+|y^+> = (x^+|y^+) + \sum_{i=1}^{N} <x^+|v_i><v_i|y^+>.$$

This shows that a sequence $(x_n^+)_{n \in \mathbf{N}} \subset \mathfrak{V}^+$ converges to $x \in \mathfrak{K}$, iff $p_+(x_n^+ - x) \longrightarrow 0$ and independently the orthogonal projections of $x_n^+ - x$ onto $\mathfrak{V}_N^*$ converge to zero. This means that $\mathfrak{V}^+ \cap (\mathfrak{V}_N^*)^\perp$ is dense in $\mathfrak{H}$ w.r.t. τ_+, which shows that the decomposition of $\overline{\mathfrak{V}^{+}}^{\tau}$ is indeed orthogonal.

If we can render the metric operator as stated in b), it follows that the negative subspace of $\mathfrak{K}$ has dimension N, since every $\widehat{\mathfrak{V}}_i$ contains exactly one one–dimensional negative subspace, which is clear from the action of J on these spaces. Then, since this metric operator is clearly invertible on whole $\mathfrak{K}$, Proposition A.11 shows that the components of $\mathfrak{K}$ are intrinsically complete, and therefore $\mathfrak{K}$ is a Krein space and actually a Pontrjagin space, after Definition A.10. Lemma A.12 then tells us that $(\mathfrak{K}, G)$ is maximal.

First for $x^\perp, y^\perp \in \mathfrak{V}^+ \cap (\mathfrak{V}_N^*)^\perp$ we have $<x^\perp|y^\perp> = (x^\perp|y^\perp)$. Thus G acts as the identity on this dense set in $\mathfrak{H}$, and therefore also on whole $\mathfrak{H}$. Further by definition $(x|\chi_i) = <x|G\chi_i> = <x|v_i>$, $\forall x \in \mathfrak{K}$, showing that $G\chi_i = v_i$. On the other hand $(x|v_i) = \overline{x^i} = <x|\chi_i>$, $\forall x \in \mathfrak{K}$ by Lemma A.18iii) and (A.3). This shows $Gv_i = \chi_i$, which yields the desired result. $\qquad \square$

Application: Integral Kernels with Algebraic Singularities. We consider the space $\mathcal{S}(\mathbf{R})$ of smooth, complex valued test functions, decreasing faster (together with all derivatives) than $1/|x|^n$ for all $n \in \mathbf{N}$. We assume K_λ to be a *positive integral Kernel with algebraic singularity* of strength $\lambda \in \mathbf{C}$ at 0. By this we mean, that K_λ defines a linear functional K_λ on $\mathcal{S}(\mathbf{R})$ by the *canonical regularization* (cf. [GS], §I.3), which is given for all $f \in \mathcal{S}(\mathbf{R})$ by

$$K_\lambda(f) \equiv \int_{-\infty}^{\infty} dx\, K_\lambda(x) \left\{ f(x) - \sum_{i=0}^{N-1} \frac{x^i}{i!} f^{(i)}(0) \right\}, \tag{A.4}$$

where $-N - 1 < \text{Re}\lambda < N$, $N \in \mathbf{N}$, and $f^{(i)}$ denotes the i–th derivative of f, $f^{(0)} \equiv f$. We make $\mathcal{S}(\mathbf{R})$ an inner product space by introducing the sesquilinear form

$$(f|g) \equiv K_\lambda(\overline{f}g), \quad \forall f, g \in \mathcal{S}(\mathbf{R}). \tag{A.5}$$

This is clearly an indefinite inner product on $\mathcal{S}(\mathbf{R})$, the indefiniteness coming from the N regularization terms at the origin. For simplicity we assume this inner product to be non–degenerate (otherwise one would just have to go over to equivalence classes in $\mathcal{S}(\mathbf{R})$). Furthermore, by the term *positive* kernel we mean that $(.|.)$ shall be a positive form, or equivalently that K_λ is a positive measure on the subspace $\mathcal{S}^+(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$, which will be defined below.

For the sake of concreteness, consider the real functions $\tilde{\chi}_i(x) \equiv x^{i/2}\vartheta_i(x) \in \mathcal{S}(\mathbf{R})$ for $i = 0,\dots,N-1$, where $\vartheta_i \in \mathcal{S}(\mathbf{R})$ is a function which is unity in a neighbourhood of the origin. The ϑ_i shall now be chosen such that the condition $(\tilde{\chi}_i|\tilde{\chi}_i) = 0$ holds, i.e., such that the $\tilde{\chi}_i$ are neutral vectors. Also, ϑ_i must be chosen such that we can obtain an orthogonal system of neutral vectors χ_i, which span the same subspace as the $\tilde{\chi}_i$:

$$\mathfrak{V}_N \equiv \langle \chi_i | i = 0,\dots,N-1 \rangle \equiv \langle \tilde{\chi}_i | i = 0,\dots,N-1 \rangle \subset \mathcal{S}(\mathbf{R}),$$
$$(\chi_i|\chi_j) = 0, \quad \forall i,j \in \{0,\dots,N-1\}.$$

A concrete construction of such a function system can be found in [10], Appendix A. Then every $f \in \mathcal{S}(\mathbf{R})$ admits a linear decomposition

$$f = f^+ + \sum_{i=0}^{N-1} f^{(i)}(0)\chi_i, \quad f^+ \in \mathcal{S}^+(\mathbf{R}). \tag{A.6}$$

The remaining part f^+ of f is then in the subspace $\mathcal{S}^+(\mathbf{R}) \subset \mathcal{S}(\mathbf{R})$ of functions for which the unregularized Integral

$$(f^+|f^+) \equiv K_\lambda(|f^+|^2) \equiv \int_{-\infty}^{\infty} dx\, K_\lambda(x)\,|f^+(x)|^2 \tag{A.7}$$

is positive and finite. The positive definite scalar product (A.3) then takes the special form

$$<f|g> \equiv (f^+|g^+) + \sum_{i=0}^{N-1} \left\{ (f|\chi_i)(\chi_i|g) + \overline{f^{(i)}(0)}g^{(i)}(0) \right\} \tag{A.8}$$

for all $f,g \in \mathcal{S}(\mathbf{R})$, inducing a seminorm $p(f)^2 \equiv <f|f>$, and by that a majorant Hilbert topology τ on $\mathcal{S}(\mathbf{R})$.

We are now in a position to apply the results of the last section to $\mathcal{S}(\mathbf{R})$. The main Theorem A.19 shows that $\overline{\mathcal{S}(\mathbf{R})}^\tau$ has the structure $\mathfrak{H} \oplus \mathfrak{V}_N^* \oplus \mathfrak{V}_N$. Now the restriction of the norm $||f||^2 \equiv <f|f>$ to $\mathcal{S}^+(\mathbf{R})$ is nothing but the L^2–norm for functions on $\mathbf{R}$, measurable w.r.t. $dx\, K_\lambda$, showing that the Pontrjagin space with rank of negativity $\varkappa^-\left(\overline{\mathcal{S}(\mathbf{R})}^\tau\right) = N$ in this case is

$$\overline{\mathcal{S}(\mathbf{R})}^\tau \equiv L^2(\mathbf{R}, dx\, K_\lambda) \oplus \mathfrak{V}_N^* \oplus \mathfrak{V}_N. \tag{A.9}$$

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