

# Lyapunov exponents on character varieties and Teichmüller curves

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# Introduction

Lyapunov exponents are characteristic numbers describing the behaviour of a cocycle over a dynamical system. If the cocycle satisfies an integrability property, Oseledets theorem states that there is a decomposition of the underlying vector bundle such that the norm of vectors in each component grows with different speed along the flow. The different possible growth rates are called Lyapunov exponents.

An interesting instance of a dynamical system is given by playing billiard on tables of polygonal shape. Lyapunov exponents describe the diffusion rate of the trajectories of the ball. Even in this special case, Lyapunov exponents are very hard to compute using standard ergodic theoretic tools. In a surprising way, algebraic geometry came into play and allowed to actually sometimes compute Lyapunov exponents associated to billiards. The first important brick of the bridge between Lyapunov exponent and algebraic geometry is given by flat surfaces. Trajectories in a polygonal billiard table can be identified with geodesics with respect to the flat metric canonically defined on the Riemann surface obtained by gluing the sides of the polygon. Such a flat metric on a Riemann surface can be in turn identified with an abelian differential, i.e. a global holomorphic 1-form. The set of flat surfaces of genus  $g$  can be made into an algebraic moduli space, known as Hodge bundle  $\Omega\mathcal{M}_g$ . On the strata of this moduli space given by fixing the partition defined by the zeroes of the differentials, there is a natural probability measure, called Masur-Veech volume. The natural  $\mathrm{SL}_2(\mathbb{R})$  action on this space given by shearing the polygon is ergodic and by the important result of Eskin and Mirzakhani orbit closures with respect to this action have an easy description in special coordinates. These orbit closures are called affine invariant submanifolds of the Hodge bundle. By a result of Filip, affine invariant submanifolds are even algebraic. Since affine invariant submanifolds describe families of Riemann surfaces, there is a canonical weight one variation of Hodge structures over them given by the flat vector bundle whose fiber over a Riemann surface  $X$  is the cohomology  $H^1(X, \mathbb{C})$ . It is a remarkable result that the Lyapunov exponents given as diffusion-rates of trajectories on a billiard given by a flat surface  $(X, \omega)$  are the same as the ones defined by the asymptotic growth rate of the Hodge norm of vectors in the variation of Hodge structures over the flow in the affine invariant manifold  $\overline{\mathrm{SL}_2(\mathbb{R})}(X, \omega)$ . It is at this point that algebraic geometry became handy. It is in [EKZ11] where Eskin, Kontsevich and Zorich proved that the sum of positive Lyapunov exponents of the Kontsevich-Zorich cocycle over an affine invariant manifold can be computed by computing the normalized degree of the Hodge bundle restricted to the affine invariant manifold. The surprising fact is that numbers defined only via a dynamical system approach that has no reason to be rational, turn out to be indeed the degree of a vector bundle. They moreover related Lyapunov exponents to other interesting invariants like the Masur-Veech volume of strata of abelian differentials and the Siegel-Veech constants.

## Introduction

Starting from billiards, algebraic geometry was used to investigate Lyapunov exponent in more general settings. For example, Kappes and Möller in [KM16] proved that a result analogous to the one of [EKZ11] was true for weight one variations of Hodge structures over ball quotients. This result allowed them to prove some results about commensurability questions for lattices. Later, Filip [Fil14] proved a similar result in the case of variations of Hodge structures given by one dimensional families of K3 surfaces. Variations of Hodge structures are a special case of flat vector bundles that are characterized by the existence of a special filtration and a compatible Hodge norm. However Lyapunov exponent can be defined for a general flat vector bundle. Indeed a flat vector defines a cocycle given by parallel transport over the geodesic flow. In [EKMZ18] Eskin, Kontsevich, Möller and Zorich proved that for a general flat vector bundle the sum of the first  $k$  Lyapunov exponents is bounded from below by the normalized degree of any rank  $k$  holomorphic subbundle of the flat bundle.

In the first part of this thesis we investigate Lyapunov exponents for general flat vector bundles over Riemann surfaces by refining the main result of [EKMZ18] and we describe properties of Lyapunov exponents on special loci of the moduli space of flat vector bundles. In the second part of the thesis we show how the knowledge of Lyapunov exponent over a sporadic Teichmüller curve can be used to compute the algebraic equation of the associated universal family of curves.

In Chapter 1 we first of all recall Riemann-Hilbert and Simpson correspondences which define an equivalence of categories between flat vector bundles over a Riemann surface  $C$ , reductive representations of the fundamental group of  $C$  and polystable Higgs bundles over  $C$  with vanishing Chern classes. These correspondences, that also induce an homeomorphism of the corresponding moduli spaces, will allow us to investigate Lyapunov exponents as functions on these moduli spaces. The moduli space of representations is called Betti moduli space or character variety, while the moduli space of flat bundles is called de Rham moduli space. In rank 2, the projective structures point of view will be useful to retrieve additional information. The main results of the chapter are a generalization of the result of [EKMZ18].

**Theorem** (Theorem 1.7.13). *Let  $\mathcal{V}$  be a holomorphic flat bundle over a hyperbolic Riemann surface  $C = \overline{C} \setminus \Delta$ . For any holomorphic subbundle  $\mathcal{E} \subset \mathcal{V}$  of rank  $k$ , then it holds*

$$\sum_{i=1}^k \lambda_i \geq \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_C^1(\log(\Delta)))} + \text{Err}^{\mathcal{E}}(u)$$

for almost any  $x \in C$  and Lebesgue almost any  $u \in \bigwedge^k \mathcal{V}_x^{\vee}$ . The error term is defined as

$$\text{Err}^{\mathcal{E}}(u) = 4\pi \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\#\{s_{\mathcal{E}}^{-1}(\ker u) \cap D_t\}}{\text{vol}(D_t)} dt.$$

Here we denoted by  $\deg_{\text{par}}(\Xi_h(\mathcal{E}))$  the parabolic degree of the metric extension of  $\mathcal{E}$ , by  $s_{\mathcal{E}} : \mathbb{H} \rightarrow \mathbb{P}(\bigwedge^k \mathcal{V}_x)$  the holomorphic classifying map defining  $\mathcal{E}$  and by  $D_t$  the hyperbolic ball of radius  $t$  in the hyperbolic plane whose center is a lift of  $x$ .

The proof of the above theorem generalizes to variation of Hodge structures of weight one over ball quotients or to the canonical variation of Hodge structures of weight one over affine invariant submanifolds of the Hodge bundle.

If the base curve is compact, we can even prove the following.

**Theorem** (Theorem 1.8.1). *If the Riemann surface  $C$  is compact, the above inequality is an equality.*

These results can be seen as a generalization of the main result of [DD15], where Deroin and Dujardin defined Lyapunov exponents of holonomies of projective structures in the context of the Brownian motion flow and related them to the covering degree of the associated developing map. Moreover, the above results can also be compared to the main result of [DD17a], in which Daniel and Deroin get a similar equality in the context of the Brownian motion flow over a Kähler manifold. They prove that the sum of the first Lyapunov exponents is the same as the sum of a normalized degree and a dynamical degree. As a corollary we get that the error term  $\text{Err}^{\mathcal{E}}(u)$  defined above, which has the same shape as the covering degree defined in [DD15], is the same as the dynamical degree defined in [DD17a].

Lyapunov exponents are not the only invariants on character varieties. We will recall the description of other invariants like the Toledo invariant, which is defined for representations into Lie groups of Hermitian type, or the critical exponent, which is defined for representation in  $\text{SL}_n(\mathbb{R})$ . Other invariants like the entropy or the minimal area are tightly related to the critical exponent, especially in the Hitchin components, where they satisfy similar bounds and the attainment of the bound characterizes symmetric powers of Fuchsian representations ([PS17]). With the support of computer experiment, we conjecture that the top Lyapunov exponent satisfy similar bounds as the critical exponent. The relation between Lyapunov exponents and the other cited invariants is still only speculation, but the geometric Oseledets theorem seems to help toward this direction. The main difference is given by the fact that Lyapunov exponents are naturally defined on the de Rham moduli space, since they depend on the complex structure of the base Riemann surface, while the other cited invariants are naturally defined on the character variety, since they depend only on the representation of the fundamental group.

We then focus on the properties of Lyapunov exponents on special loci of the de Rham moduli space. We recall the existence of a stratification given by Harder-Narasimhan type, called Shatz stratification. Via the identification of the maximal stratum with the oper locus, we get an explicit lower bound of the Lyapunov exponents on this stratum. Moreover, using a recent result [DF18], we can prove that the top Lyapunov exponent function is unbounded on the maximal Shatz stratum.

**Theorem** (Theorem 1.9.2). *The top Lyapunov exponent function is unbounded on the maximal Shatz stratum, the oper locus, with logarithmic growth near the boundary of the character variety.*

The next interesting locus to consider is the set of flat bundles underlying variations of Hodge structures. We prove a slight generalization of the results of [EKZ11] and [Fil14]

using a general condition for the vanishing of the error term. We also show that the triviality of the Lyapunov spectrum implies that the variation of Hodge structures is unitary and that for positive weight variations of Hodge structures the first term of the Hodge filtration induced a non-trivial lower bound for the sum of the first exponents.

We finally concentrate on the rank two de Rham moduli space and describe the Shatz stratification and the variations of Hodge structures loci more in detail. By identifying the oper locus with the set of holonomies of projective structures inducing the same complex structure, we give a generalization of the main result of [DD17b].

In Chapter 2, we compute the algebraic equation of the universal family over one of the two known sporadic Teichmüller curves, the Kenyon-Smillie  $(2, 3, 4)$ -Teichmüller curve. This is a joint work with André Kappes and appeared ad [CK17].

Teichmüller curves are projections to  $\mathcal{M}_g$  of closed  $\mathrm{SL}_2(\mathbb{R})$ -orbits in the Hodge bundle and so they are closed algebraic curves in the moduli space of curves. It was proven by Möller in [Möl06b] that Teichmüller curves, like Shimura curves, can be characterize by the special shape of their variation of Hodge structures. The Kenyon-Smillie Teichmüller curve parametrizes all affine deformations  $(S_t, \omega_t)$  of the translation surface  $(S, \omega) \in \Omega\mathcal{M}_3(3, 1)$  that is obtained from unfolding a Euclidean triangle with angles  $(\frac{2\pi}{9}, \frac{3\pi}{9}, \frac{4\pi}{9})$ . It was discovered by Kenyon and Smillie [KS00], who proved that  $(S, \omega)$  is a Veech surface with Veech group equal to the triangle group  $\Delta(9, \infty, \infty)$ . The translation surface  $(S, \omega)$  is the order 9 orbifold point of its associated Teichmüller curve, which is uniformized by  $\mathbb{H}/\Delta(9, \infty, \infty)$ .

**Theorem** (Theorem 2.1.1). *The universal family over the complement of the orbifold point of the Kenyon-Smillie  $(2, 3, 4)$ -Teichmüller curve is given by the family of plane quartics satisfying the equation*

$$X^4 + t(X^4 - 3X^3Y + 6X^3Z - 3X^2Y^2 - 6X^2YZ + 6X^2Z^2 + 4XY^3 - 6XY^2Z - 6XYZ^2 + XZ^3 + 3Y^4 + 3Y^3Z) = 0$$

where  $t$  varies in  $\mathbb{P}^1 - \{0, 1, \infty\}$ .

The triple zero of the differential  $\omega_t$  is the point  $p_t = (0 : 0 : 1) \in S_t$  and the simple zero is the point  $q_t = (0 : 1 : -1) \in S_t$ .

As above, the study of the variation of Hodge structures over the Teichmüller curve and of its Harder-Narasimhan filtration are central. Indeed, the characterization of Teichmüller curves via their associated variation of Hodge structure proved in [Möl06b] and the description of the associated Harder-Narasimhan type computed in [YZ13] are one of the main used tools.

Since in this case we are dealing with a family of hypersurfaces, we can use the Griffiths-Dwork method to compute the Picard-Fuchs equations associated to this family.

**Proposition** (Proposition 2.1.3). *The periods of  $\omega_t$  are solutions of the following differential equation:*

$$\frac{16}{81t(t-1)}y + \frac{17t-8}{9t(t-1)}y' + y'' = 0.$$



The local system defined by the solution of this differential equation is the one corresponding to an irreducible rank 2 subbundle of the variation of Hodge structure of the Teichmüller curve. By relating the local exponents of the above differential equation to the order of vanishing of the Higgs field associated to the variation of Hodge structures, we prove that the local system defined by the above equation is the one corresponding to the uniformizing representation of the Teichmüller curve. Hence, by the characterization of Teichmüller curves of [Möl06b], we independently reproved that the equation that we computed indeed is the one of the universal family over a Teichmüller curve.

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# 1 Lyapunov exponents on character varieties

Lyapunov exponents define an invariant of a flat vector bundle over a Riemann surface. They measure the asymptotic behavior of parallel transport over the geodesic flow. Starting from the work of [EKZ11], algebraic geometry started to come into play into the computation of Lyapunov exponents for flat bundles over affine invariant submanifolds of the Hodge bundle over the moduli space of curves. In [EKMZ18] it was investigated the relation between Lyapunov exponents and degrees of holomorphic subbundles of a general flat vector bundle. In this chapter we study Lyapunov exponents as invariants on the moduli space of flat vector bundles by refining the main result of [EKMZ18] and investigating the behavior of Lyapunov exponents on special subsets of the moduli space.

In Section 1.1 we state Oseledets multiplicative ergodic theorem and define Lyapunov exponents for flat vector bundles.

The set of flat vector bundles over  $C$  can be made into an algebraic variety called the de Rham moduli space. In Section 1.2 we recall Riemann-Hilbert and Simpson correspondences, which give an equivalence of categories and a homeomorphism between the de Rham moduli space, the Betti moduli space of reductive representations of the fundamental group of  $C$  and the Hitchin moduli space of polystable Higgs bundles with vanishing Chern classes.

In Section 1.3 we recall the definition of two stratifications on the de Rham moduli space. In particular we focus on the Shatz stratification, which is given by Harder-Narasimhan type, and describe the minimal and maximal strata.

In Section 1.4 we highlight the Betti point of view in rank two. We introduce projective structures and focus on representations given as holonomies of projective structures.

We then go more deeply into the description of special loci in rank two in Section 1.5. In particular in Proposition 1.5.1 we identify the oper locus, which is the same as the maximal Shatz stratum, with the set of holonomies of projective structures inducing the same complex structure.

In a more general situation, if a flat vector bundle corresponds to a representation into a reductive Lie group  $G$ , Riemann-Hilbert and Simpson correspondences still hold and if  $G$  has special properties, it is possible to define special invariants. We define them in Section 1.6 and recall some of their properties on special components. Using the geometric Oseledets theorem we define Lyapunov exponents for general  $G$ -flat bundles and we speculate a relation between them and the other described invariants.

In Section 1.7 we prove the main inequality of Theorem 1.7.13, which is a refinement of the result of [EKMZ18]. It describes the error term given by the difference between Lyapunov exponents and normalized degrees of holomorphic subbundles. The proof of the theorem

generalizes to variation of Hodge structures of weight one over ball quotients or to the canonical variation of Hodge structures of weight one over affine invariant submanifolds of the Hodge bundle over the moduli space of Riemann surfaces. In Proposition 1.7.18, we also derive a general condition to ensure that the error term is zero.

In Section 1.8 we present Theorem 1.8.1, which states that if the base curve is compact the main inequality is actually an equality. As a corollary we get that, in the case of compact base curve, the error term that we get is the same as the dynamical degree defined in [DD17a] (Corollary 1.8.9). Indeed in [DD17a], Daniel and Deroin provide a result analogous to ours in the context of Brownian motion, where the error term is given by a dynamical degree. It is defined as the intersection of the class induced by  $\mathcal{E}$  with a harmonic current. Notice that the error term can in principle be approximated with computer experiments, while the dynamical degree is harder to actually compute in experiments since it is defined in terms of a harmonic measure.

Using the above results we investigate how the Lyapunov exponent function behaves on special loci of the de Rham moduli space.

In Section 1.9 we describe in Proposition 1.9.1 a lower bound for the Lyapunov exponent functions on maximal Shatz strata, which we identify with the oper loci. We then use the main result of the recent work [DF18] to show in Theorem 1.9.2 that the top Lyapunov exponent function is unbounded on the oper loci with logarithmic growth near the boundary of the character variety.

In Section 1.10 we conjecture, with the support of computer experiments, that the top Lyapunov exponent function on the Hitchin components is bounded from below by the value of the top Lyapunov exponent of the appropriate symmetric power of the uniformizing representation of  $C$ . This would be a result analogous to the one of [PS17] about the critical exponent.

In Section 1.11 we then focus on other special flat bundles, namely the ones defining a variation of Hodge structures. Using the above cited general criterion for ensuring the vanishing of the error term, we reprove slightly more general versions of the original results of [EKZ11] and [Fil14] about weight one and real weight two variations of Hodge structures. For a general variation of Hodge structures  $\mathcal{V}$ , we prove in Proposition 1.11.6 that the Lyapunov spectrum is trivial if and only if  $\mathcal{V}$  is unitary and we show that if the weight is positive we get a non-trivial bound on the sum of the first  $\text{rk}(\mathcal{V}^{n,0})$ -exponents (Corollary 1.11.7).

We finally focus on the de Rham moduli space in rank two. In Section 1.12 we first of all prove the continuity of the top Lyapunov exponent function and we show that the locus of zero exponent is the locus of elementary representations. As in the proof of the same results in [DD15] in the context of Brownian motion, we use that the Lyapunov exponent with respect to the geodesic flow is the same as the one with respect to the random product on the image of the representation. We finally generalize for a compact base curve the main result of [DD17b], where it was described the top Lyapunov exponent function on the holonomies of projective structures giving the same complex structure. By identifying this set with the oper locus, we notice in Proposition 1.12.4 that the result of [DD17b] is a special case for the maximal Shatz strata.

## 1.1 Lyapunov exponents

We want to define the Lyapunov exponents associated to a flat vector bundle over a hyperbolic Riemann surface  $C$  of finite area.

We first of all recall Oseledets multiplicative ergodic theorem and the definition of Lyapunov exponents for a cocycle over an ergodic flow.

**Theorem/Definition 1.1.1.** *Let  $g_t : (M, \mu) \rightarrow (M, \mu)$  be an ergodic flow on a space  $M$  with finite measure  $\mu$ . Suppose that the action lifts equivariantly to a linear flow  $G_t$  on some measurable real bundle  $\mathcal{V}$  on  $M$ . Suppose there exists a (not equivariant) norm  $\|\cdot\|$  on  $\mathcal{V}$  such that the functions*

$$x \mapsto \sup_{t \in [0,1]} \log^+ \|G_t\|_x, \quad x \mapsto \sup_{t \in [0,1]} \log^+ \|G_{1-t}\|_{g_t(x)} \quad (1.1)$$

are in  $L^1(M, \mu)$  (we call such a norm integrable). Then there exist real constants  $\lambda_1 \geq \dots \geq \lambda_n$  and a decomposition

$$\mathcal{V} = \bigoplus_{i=1}^n \mathcal{V}_{\lambda_i}$$

by measurable real vector bundles such that for a.a.  $x \in M$  and all  $v \in (\mathcal{V}_{\lambda_i})_x - \{0\}$ , it holds

$$\lambda_i = \lim_{t \rightarrow \pm\infty} \frac{1}{t} \log \|G_t(v)\|.$$

The set of values  $\lambda_i$ , repeated with multiplicity  $\dim \mathcal{V}_{\lambda_i}$ , is called the set of Lyapunov exponents or Lyapunov spectrum of  $(M, \mu, g_t, \mathcal{V})$ .

The case we are interested in is when the space  $M$  is the unit tangent bundle  $T^1(C)$  of a hyperbolic curve  $C$  equipped with the measure induced by the hyperbolic metric with constant negative curvature  $-4$  (we keep the same convention as in [EKMZ18]). The geodesic flow  $g_t$  is then ergodic and we can lift it to a flat vector bundle  $(V, \nabla)$  using parallel transport. A flat vector bundle over  $C$  is a  $\mathcal{A}_C^0$ -vector bundle  $V$ , i.e. a smooth complex bundle, equipped with a connection  $\nabla : V \rightarrow V \otimes \mathcal{A}_C^1$  with vanishing curvature  $\nabla^2 = 0$ .

*Remark 1.1.2.* The Lyapunov spectrum is symmetric in this case. This follows since the negative time geodesic flow, which has negative Lyapunov spectrum, is conjugate to the positive time flow (because of the  $\mathrm{SL}_2(\mathbb{R})$ -action on  $\mathbb{H}$ ). Moreover, if the vector bundle  $\mathcal{V}$  is complex, it is possible to show that the Oseledets subvector bundles are complex subbundles of  $\mathcal{V}$ . In this case we then consider only half of the real Lyapunov spectrum, forgetting about the duplication phenomenon given by the complex structure.

We still need to define an integrable norm on the flat bundle  $(V, \nabla)$ . We defined the *constant norm* as the pullback to  $T^1(C)$  of the parallel transport of any norm over the fiber of some base point  $c \in C$  to a Dirichlet fundamental domain for  $\Gamma$  on  $\mathbb{H}$ . In particular this norm is not continuous across the boundary of the fundamental domain. We say

that  $(V, \nabla)$  has non-expanding cusp monodromies if the eigenvalues of the holonomy matrices  $\text{hol}_\nabla(\gamma)$  have absolute value one, for every simple loop  $\gamma$  around a cusp. We recall a result which ensures us that the constant norm is integrable.

**Theorem 1.1.3** ([EKMZ18]). *The constant norm over  $(V, \nabla)$  is integrable if and only if the associated local system has non-expanding cusp monodromies.*

Note that for any two integrable norms on  $(V, \nabla)$ , the Lyapunov filtrations and the Lyapunov spectra coincide (see [KM16, Lemma 2.6]).

*Remark 1.1.4.* The constant norm can be used to numerically compute Lyapunov exponents associated to  $(V, \nabla)$  when a coding for the geodesic flow on  $\mathbb{H}/\Gamma$  is available. If we denote by  $\gamma_n \in \Gamma$  the sequence of elements corresponding to the sequence of sides of the fundamental domain crossed by a generic geodesic, then by definition of constant norm the Lyapunov exponents are given as

$$\lambda_i = \lim_{n \rightarrow \infty} \frac{1}{n} \log(\mu_i(n)),$$

where  $\mu_i(n)$  are the eigenvalues of the matrix

$$\text{hol}_\nabla(\gamma_n) \cdots \text{hol}_\nabla(\gamma_1).$$

## 1.2 De Rham, Betti and Hitchin moduli spaces for compact curves

The set of flat vector bundles can be made into an algebraic variety called de Rham moduli space. Since we are interested in Lyapunov exponents as invariants on this moduli space, we recall its definition and its relation to other two moduli spaces, the Betti and the Hitchin moduli spaces. We will also recall the definition of special flat bundles, called variations of Hodge structures, and their Higgs bundles version called system of Hodge bundles. When we speak about moduli spaces, we are always assuming that the base Riemann surface is compact, since we do not want to deal here with representation varieties with fixed parabolic weights at the cusps. Let then  $C$  be a compact curve with negative Euler characteristic. The universal covering of  $C$  is the upper half-plane  $\mathbb{H}$ .

**1.2.1 De Rham moduli space** First of all we consider the de Rham moduli space  $\mathcal{M}_{\text{DR}}^{(n)}(C)$  defined as the moduli space of semisimple flat vector bundles  $(V, \nabla)$  of rank  $n$  over  $C$  with trivial determinant bundle modulo the action of the complex gauge group. A flat vector bundle can be identified with a holomorphic flat bundle, namely a holomorphic bundle  $\mathcal{V}$  together with a flat holomorphic connection  $\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_C^1$ . Here we denoted by  $\Omega_C^1 = \mathcal{K}_C$  the canonical bundle of holomorphic one forms over  $C$ . The identification is given by Koszul-Malgrange theorem (see [RR15, p. 2.2]), which asserts that the data of a  $\mathcal{C}_C^\infty$ -vector bundle  $V$  together with an operator  $\bar{\partial}_V : V \rightarrow V \otimes \mathcal{A}_C^{0,1}$  satisfying  $\bar{\partial}_V^2 = 0$  is the same data as the one of a holomorphic bundle. We can then associate to a flat

vector bundle  $(V, \nabla)$  the holomorphic structure  $\bar{\partial}_V := \nabla^{0,1} : V \rightarrow V \otimes \mathcal{A}_C^{0,1}$  on  $V$  and a holomorphic connection  $\nabla^{1,0}$  induced on the associate holomorphic bundle  $\mathcal{V} = \ker \bar{\partial}_V$ . Since we are working over a Riemann surface, the condition for a holomorphic connection to be flat is trivial.

**1.2.2 Betti moduli space** The second moduli space that we are interested in is the Betti moduli space

$$\mathcal{M}_B^{(n)} := \text{Hom}(\pi_1(C), \text{SL}_n(\mathbb{C})) // \text{SL}_n(\mathbb{C}),$$

defined as the GIT-quotient under the conjugation action of the affine variety given by the space of representations. The Betti moduli space is also called character variety, since the regular functions are given by the trace functions. This moduli space is an irreducible affine variety of complex dimension  $(n^2 - 1)(2g - 2)$ . It contains a Zariski open set of simple representations where the quotient is given by the geometric quotient. Points of  $\mathcal{M}_B^{(n)}$  are in one to one correspondence with conjugacy class of semisimple representations and every  $\text{SL}_n(\mathbb{C})$ -orbit contains a semisimple representation in its closure.

**1.2.3 Hitchin moduli space** The third moduli space we want to talk about is the Hitchin moduli space  $\mathcal{M}_H^{(n)}(C)$  defined as the moduli space of rank  $n$  polystable Higgs bundle over  $C$  with trivial determinat bundle and with vanishing Chern classes. Two Higgs bundles are identified in this moduli space if they induce the same graded object associated to the Seshadri filtration. A Higgs bundle over  $C$  is a pair  $(\mathcal{V}, \Phi)$  where  $\mathcal{V}$  is holomorphic vector bundle over  $C$  and  $\Phi$  is a global holomorphic section of  $\mathcal{K}_C \otimes \text{End}\mathcal{V}$ .

**1.2.4 Riemann-Hilbert and Simpson correspondences** All of the three described moduli spaces are homeomorphic (see [Sim94a], [Sim94b]). The Betti and the de Rham moduli space are even isomorphic as complex spaces but the isomorphism is not algebraic. We now want to describe the homeomorphisms between them.

The classical version of the Riemann-Hilbert correspondence defines the map between the Betti and the de Rham moduli spaces. First of all, recall that a representation  $\rho : \pi_1(C) \rightarrow \text{GL}_n(\mathbb{C})$  is the same as a local system  $\mathbb{V}_\rho$  over  $C$ , namely a locally constant sheaf defined as the sheaf of equivariant continuous functions

$$\mathbb{V}_\rho(U) := \{f : \pi^{-1}(U) \rightarrow \mathbb{C}^n : s(\gamma x) = \rho(\gamma)f(x), \forall \gamma \in \pi_1(C)\}$$

for any open  $U \subseteq C$ , where  $\pi : \mathbb{H} \rightarrow C$  is the universal covering map. The Riemann-Hilbert correspondence associates to the representation  $\rho$  the holomorphic flat bundle  $\mathcal{V}_\rho := \mathbb{V}_\rho \otimes \mathcal{O}_C$  equipped with the flat Gauss-Manin connection locally defined by

$$\nabla(s \otimes f) := s \otimes d(f)$$

where  $d(f)$  is the derivative of the local holomorphic function  $f$  (this is a well-defined connection since the cocycle of a local system is a constant function, so its derivative is zero). Equivalently, the total space of this holomorphic vector bundle is defined by

$$\mathcal{V}_\rho = \mathbb{H} \times \mathbb{C}^n / \sim, \quad (z, v) \sim (\gamma(z), \rho(\gamma)v), \quad \gamma \in \pi_1(C).$$

Vice versa, given a holomorphic flat vector bundle  $(\mathcal{V}, \nabla)$ , the associated local system is given by the sheaf of flat sections.

We want now to describe Simpson correspondence, which defines the map between the de Rham and Hitchin moduli spaces. The key idea is to factor through the space of harmonic bundles, which are  $C^\infty$ -vector bundles equipped with a harmonic metric. A harmonic metric on a rank  $n$  vector bundle  $\mathcal{V}$  is the same as a map to the space of metrics on  $\mathcal{V}$

$$C \rightarrow \mathrm{GL}(n)/\mathrm{U}(n)$$

that minimizes the energy functional. Corlette proved that each semisimple flat vector bundle and each polystable Higgs bundle with vanishing Chern classes admit a unique (up to scalar) harmonic metric (see [Cor88]). Using this fact, we recall the explicit map defined in [Sim92].

Given a flat vector bundle  $(V, \nabla)$  equipped with its harmonic metric, define the Higgs bundle  $(\mathcal{V}, \bar{\partial}_{\mathcal{V}}, \Phi)$  as the holomorphic vector bundle  $(\mathcal{V}, \bar{\partial}_{\mathcal{V}})$  together with the Higgs field  $\Phi$  defined as

$$\bar{\partial}_{\mathcal{V}} := \frac{\nabla^{0,1} + \delta''}{2}, \quad \Phi = \frac{\nabla^{1,0} - \delta'}{2}.$$

The connections  $\nabla^{1,0} + \delta''$  and  $\nabla^{0,1} + \delta'$  are the unique ones preserving the unique harmonic metric. Conversely, given a Higgs bundle  $(\mathcal{V}, \bar{\partial}_{\mathcal{V}}, \Phi)$ , the associated flat vector bundle  $(\mathcal{V}, \nabla)$  is given by

$$\nabla := \nabla^{1,0} + \nabla^{0,1}, \quad \nabla^{1,0} = \partial + \Phi, \quad \nabla^{0,1} = \bar{\partial} + \bar{\Phi}$$

where  $\partial + \bar{\partial}_{\mathcal{V}}$  is the harmonic metric connection and  $\bar{\Phi}$  is the metric adjoint of  $\Phi$ .

**1.2.5 Variation of Hodge structures and system of Hodge bundles** We recall the definition of variation of Hodge structures. They are special holomorphic flat bundles arising for example from the variation of the cohomology of families of algebraic varieties.

**Definition 1.2.1.** A complex variation of Hodge structures of weight  $k$  over  $C$  is a holomorphic flat vector bundle  $(\mathcal{V}, \nabla)$  over  $C$  together with a holomorphic filtration

$$F^{k+1} = 0 \subset \dots \subset F^0 = \mathcal{V}$$

which satisfies the Griffiths transversality condition

$$\nabla : F^p \rightarrow F^{p-1} \otimes \Omega_C^1$$

and such that furthermore there exists a  $\nabla$ -flat hermitian complex form  $H$  on  $\mathcal{V}$ , which is positive definite on  $F^i/F^{i+1}$  for  $i$  even and negative definite for  $i$  odd.

Via Simpson correspondence, variations of Hodge structures correspond to stable systems of Hodge bundles ([Sim88]).

**Definition 1.2.2.** A system of Hodge bundles is a Higgs bundle  $(\mathcal{V}, \Phi)$  together with a decomposition  $\mathcal{V} = \bigoplus \mathcal{V}^{p,q}$ , such that  $\Phi : \mathcal{V}^{p,q} \rightarrow \mathcal{V}^{p-1,q+1} \otimes \mathcal{K}_C$ .



Via Simpson correspondence a variation of Hodge structures is associated to the semistable system of Hodge bundles given by the graded object associated to the Hodge filtration equipped with the Higgs field defined by the graded pieces of the flat connection. In this case the harmonic metric is the Hodge metric coming from the hermitian form  $H$ . System of Hodge bundles can be also characterized ([Sim92]) as fixed point of the  $\mathbb{C}^*$ -action on  $\mathcal{M}_{\mathbb{H}}^{(n)}(C)$  given by  $t \cdot (\mathcal{V}, \Phi) = (\mathcal{V}, t \cdot \Phi)$  for  $t \in \mathbb{C}^*$ .

### 1.3 De Rham moduli space and Shatz stratification

Since we will relate Lyapunov exponents to degrees of holomorphic subbundles of holomorphic flat bundles, we recall here the existence of a stratification of the de Rham moduli space given by Harder-Narasimhan type, called Shatz stratification. We then describe more in detail the minimal stratum containing the unitary representations and the maximal stratum, which we identify with the open locus. Finally we recall the construction of another stratification defined by the  $\mathbb{C}^*$ -action on the Hitchin bundle moduli space. The minimal and maximal strata of these two stratifications coincide.

**1.3.1 Harder-Narasimhan filtration** We recall some basic definitions about Harder-Narasimhan filtrations and maximal degree subbundles following [HN74].

**Definition 1.3.1.** Let  $\mathcal{V}$  be a holomorphic vector bundle over  $C$  of rank  $n$ . The *degree*  $\deg(\mathcal{V})$  of  $\mathcal{V}$  is the first Chern class of  $\mathcal{V}$ , or equivalently the degree of the determinant bundle  $\det(\mathcal{V})$ . The *slope*  $\mu(\mathcal{V})$  of  $\mathcal{V}$  is defined as the quotient of the degree and the rank

$$\mu(\mathcal{V}) = \deg(\mathcal{V}) / \text{rk}(\mathcal{V}).$$

Notice that both degree and rank are additive functors.

**Definition 1.3.2.** A vector bundle  $\mathcal{V}$  is *(semi)stable* if, for every holomorphic subbundle  $\mathcal{E} \subset \mathcal{V}$ , it holds  $\mu(\mathcal{E}) < (\leq) \mu(\mathcal{V})$ , or equivalently  $\mu(\mathcal{V}) < (\leq) \mu(\mathcal{V}/\mathcal{E})$ .

*Remark 1.3.3.* The two following easy statements are useful for computations.

1. Let  $f : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  be a non-zero holomorphic map between two holomorphic vector bundles of the same rank. Then  $\mu(\mathcal{V}_2) \geq \mu(\mathcal{V}_1)$ . Indeed  $f$  induces a nonzero holomorphic section of the line bundle  $\det(\mathcal{V}_2) \otimes \det(\mathcal{V}_1)^\vee$ , which hence has positive degree since we are working over a curve.
2. Let  $f : \mathcal{V}_1 \rightarrow \mathcal{V}_2$  be a non-zero holomorphic map between two holomorphic semistable vector bundles. Then  $\mu(\mathcal{V}_2) \geq \mu(\mathcal{V}_1)$ . Indeed by semistability of  $\mathcal{V}_1$  first and of  $\mathcal{V}_2$  after we have  $\mu(\mathcal{V}_1) \leq \mu(\text{Im}(f)) \leq \mu(\mathcal{V}_2)$ .

We want now to define the Harder-Narasimhan filtration of a vector bundle.

If  $\mathcal{V}$  is not semistable, we say that  $\mathcal{E} \subset \mathcal{V}$  is maximal if  $\mathcal{E}$  is semistable and for every  $\mathcal{E}'$  such that  $\mathcal{E} \subsetneq \mathcal{E}' \subset \mathcal{V}$ , it holds  $\mu(\mathcal{E}) > \mu(\mathcal{E}')$ . In other words,  $\mathcal{E}$  is maximal if it is the

semistable subbundle with maximal slope. One can show that it exists and it is unique. One can moreover show that  $\mathcal{E}$  is maximal if and only if  $\mathcal{E}$  is semistable and for every  $\mathcal{Q} \subset \mathcal{V}/\mathcal{E}$ , it holds  $\mu(\mathcal{Q}) < \mu(\mathcal{E})$ .

**Definition 1.3.4.** The *Harder-Narashiman filtration* of a holomorphic vector bundle  $\mathcal{V}$  of rank  $n$  is a filtration by holomorphic subbundles

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_l = \mathcal{V}$$

such that  $\mathcal{V}_i/\mathcal{V}_{i-1}$  is semistable and  $\mu(\mathcal{V}_i/\mathcal{V}_{i-1}) > \mu(\mathcal{V}_{i+1}/\mathcal{V}_i)$ . This filtration always exists and the associated graded object  $\text{Gr}(\mathcal{V}) := \bigoplus_{i=1}^l \mathcal{V}_i/\mathcal{V}_{i-1}$  is uniquely determined by the isomorphism class of  $\mathcal{V}$ .

We call the collection

$$(\mu_1, \dots, \mu_n), \quad \mu_i = \mu(\mathcal{V}_i/\mathcal{V}_{i-1})$$

of slopes (possibly repeated depending on the rank of  $\mathcal{V}_i/\mathcal{V}_{i-1}$ ) the *Harder-Narashiman type* of  $\mathcal{V}$ .

For example, if  $\mathcal{V}$  is semistable, the Harder-Narashiman type of  $\mathcal{V}$  is simply given by  $(\mu_1 = \mu(\mathcal{V}), \dots, \mu_n = \mu(\mathcal{V}))$ .

*Remark 1.3.5.* The bundles  $\mathcal{V}_i$  appearing in the Harder-Narasimhan filtration satisfy  $\mu(\mathcal{V}_i) > \mu(\mathcal{V}_{i+1})$ . Moreover, each one of the following conditions can be substituted to the second condition in the definition of Harder-Narasimhan filtration:

1.  $\mu(\mathcal{V}_i/\mathcal{V}_{i-1}) > \mu(\mathcal{V}_{i+1}/\mathcal{V}_i)$ .
2.  $\mathcal{V}_i/\mathcal{V}_{i-1}$  is maximal in  $\mathcal{V}/\mathcal{V}_{i-1}$ .
3.  $\mu_i$  is the minimal slope among the slopes of quotients of  $\mathcal{V}_i$ .

In particular  $\mathcal{V}_1$  is the sub-line bundle with the maximal slope among all subbundles of  $\mathcal{V}$ . For the proof of the previous statements see [HN74].

**1.3.2 Shatz stratification of  $\mathcal{M}_{\text{DR}}^{(n)}(\mathbb{C})$**  We want to state a central theorem about the upper-semicontinuity of the Harder-Narashiman type. There is a partial ordering on vectors given by Harder-Narashiman types, namely

$$(\mu_1, \dots, \mu_n) \leq (\nu_1, \dots, \nu_n) \iff \sum_{i=1}^k \mu_i \leq \sum_{i=1}^k \nu_i \text{ for all } k = 1, \dots, n.$$

One can also visualize this partial ordering by drawing a semi-polygon in the plane that has vertices with coordinates  $(\text{rk}(\mathcal{V}_i), \text{deg}(\mathcal{V}_i))$ . The partial ordering is then given by checking if a semi-polygon is above another one.

Now we state the main theorem by Atiyah and Bott, which was proven in the general case of higher dimensional base space by Shatz.

**Theorem 1.3.6** ([AB83],[Sha77]). *The Harder-Narasimhan type defines an upper semi-continuous function, meaning that if  $\mathcal{C}_{\vec{\mu}}$  is the space of vector bundles with Harder-Narasimhan type  $\vec{\mu}$ , then*

$$\overline{\mathcal{C}_{\vec{\mu}}} \subseteq \bigcup_{\vec{\nu} \geq \vec{\mu}} \mathcal{C}_{\vec{\nu}}.$$

*In particular there is a stratification of  $\mathcal{M}_{\text{DR}}^{(n)}$  given by the Harder-Narasimhan type called Shatz stratification.*

We will now discuss more in detail the minimal and the maximal Shatz strata.

**1.3.3 Minimal Shatz stratum** The minimal stratum is easy to describe, since it is clearly given by the locus of semistable bundles. Note that the degree of every vector bundle in  $\mathcal{M}_{\text{DR}}^{(n)}(C)$  is zero, since this is the moduli space of flat vector bundles. Hence the Harder-Narasimhan type of a semistable bundle is  $(\mu_1 = 0, \dots, \mu_n = 0)$ . The minimal stratum is an open dense set. The subset of semistable but not stable bundles is the closed subset of the minimal stratum corresponding to the subset of reducible representations in  $\mathcal{M}_{\text{B}}^{(n)}$ .

*Remark 1.3.7.* The locus of stable bundles is isomorphic to the tangent bundle of the space of stable holomorphic vector bundles. Indeed the space of holomorphic connections on a fixed simple vector bundle  $\mathcal{V}$  is given by

$$H^0(\text{End}(\mathcal{V}) \otimes \mathcal{K}_C) \cong H^1(\text{End}(\mathcal{V})^\vee) \cong H^1(\text{End}(\mathcal{V}))$$

which is the tangent space to the space of stable holomorphic vector bundles (see [NS65]).

Using the last remark, it is easy to see that the open subset of the minimal stratum given by stable bundles is contained into the smooth locus of  $\mathcal{M}_{\text{DR}}^{(n)}(C)$ . Indeed if  $\mathcal{V}$  is stable, it is easy to see that it is also simple, i.e.  $H^0(\text{End}(\mathcal{V})) = 0$ . Note that here  $\text{End}(\mathcal{V})$  means the endomorphism of  $\mathcal{V}$  that are the identity on the determinant bundle, since we are only dealing with the category of vector bundles with trivial determinant. Using the Riemann-Roch theorem, we can then compute the dimension of the tangent space of  $\mathcal{M}_{\text{DR}}^{(n)}(C)$  at simple flat bundles

$$2h^1(\text{End}(\mathcal{V})) = 2h^0(\text{End}(\mathcal{V})) + 2 \text{rk}(\text{End}(\mathcal{V}))(g - 1) = 2(n^2 - 1)(g - 1)$$

which is equal to the dimension of  $\mathcal{M}_{\text{DR}}^{(n)}(C)$ .

Thanks to the theorem by Narasimhan and Seshadri we know what is the closed subset of the stable locus in  $\mathcal{M}_{\text{DR}}^{(n)}(C)$  corresponding to the unitary locus of  $\mathcal{M}_{\text{B}}^{(n)}$ .

**Theorem 1.3.8** ([NS65]). *The locus of unitary representations in  $\mathcal{M}_{\text{B}}^{(n)}$  corresponds to the locus of flat vector bundles  $(\mathcal{V}, \nabla) \in \mathcal{M}_{\text{DR}}^{(n)}(C)$  where  $\mathcal{V}$  is stable and  $\nabla$  is the harmonic metric connection. This subset is a  $(n^2 - 1)(g - 1)$ -dimensional closed subvariety of the stable locus of  $\mathcal{M}_{\text{DR}}^{(n)}(C)$ .*

In the Hitchin moduli space the unitary locus corresponds to the locus of Higgs bundles with zero Higgs field.

**1.3.4 Maximal Shatz stratum: oper locus** We want to describe the maximal locus of the Shatz stratification, namely the locus where the flat vector bundles have maximal Harder-Narashiman type. In order to do this, we have to introduce the notion of opers and state their main properties. We will follow the survey [Wen15].

**Definition 1.3.9.** A  $SL_n$ -oper is a rank  $n$  holomorphic vector bundle  $\mathcal{V}$  with trivial determinant bundle, equipped with a flat holomorphic connection  $\nabla$  and a filtration by holomorphic subbundles

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$$

such that

1.  $\nabla(\mathcal{V}_i) \subseteq \mathcal{V}_{i+1} \otimes \mathcal{K}_C$ ;
2.  $\nabla : \mathcal{V}_i/\mathcal{V}_{i-1} \rightarrow \mathcal{V}_{i+1}/\mathcal{V}_i \otimes \mathcal{K}_C$  is an isomorphism.

Let us now define the oper locus  $\text{Op}_n(C) \subset \mathcal{M}_{\text{DR}}^{(n)}(C)$  as the subset of the de Rham moduli space of flat vector bundles admitting an oper structure. This definition makes sense since the oper structure, i.e. the oper filtration, is unique for a fixed oper  $(\mathcal{V}, \nabla)$ . The uniqueness of the oper structure is a consequence of the following central proposition.

**Proposition 1.3.10** ([Wen15]). *Let  $(\mathcal{V}, \nabla)$  be a  $SL_n$ -oper. The oper structure on  $\mathcal{V}$  is uniquely determined by the line bundle*

$$\mathcal{V}/\mathcal{V}_{n-1} \cong \mathcal{V}_1 \otimes \mathcal{K}_C^{-(n-1)}.$$

Moreover

$$(\mathcal{V}/\mathcal{V}_{n-1})^n \cong \mathcal{K}_C^{-n(n-1)/2} \quad \text{and} \quad \det(\mathcal{V}_j) \cong \mathcal{V}/\mathcal{V}_{n-1} \otimes \mathcal{K}_C^{nj-(j(j+1)/2)}.$$

In particular the isomorphism class of  $\mathcal{V}$  is fixed on every connected component of  $\text{Op}_n(C)$  and each component parametrizes the space of holomorphic connections on a fixed holomorphic vector bundle. These components are classified by the choice of the line bundle  $\mathcal{V}/\mathcal{V}_{n-1}$  which is defined by the property  $(\mathcal{V}/\mathcal{V}_{n-1})^n \cong \mathcal{K}_X^{-n(n-1)/2}$ . Hence  $\text{Op}_n(C)$  has  $n^{2g}$  connected components, which also corresponds to the number of ways of lifting a monodromy representation in  $\text{PSL}_n(\mathbb{C})$  to  $\text{SL}_n(\mathbb{C})$  (see [Wen15, Remark 4.2]). Using Proposition 1.3.10 one can also prove that if a holomorphic bundle has the structure of an oper then it must be an irreducible flat vector bundle, or equivalently the representation that it defines is simple ([Wen15, Prop. 4.8]).

We will see in the next section that there is only one variation of Hodge structures on each connected component of the oper locus. It is given by the  $(n-1)$ -th symmetric power of the maximal Higgs one in rank 2 (see Proposition 1.3.15). The maximal Higgs variation of Hodge structures in rank 2 corresponds to the uniformizing representation of  $C$ .

We want now to parametrize the oper locus in such a way that it will be easy to see why in rank 2 the oper locus corresponds to the set of holonomies of projective structures inducing the same complex structure on  $C$  (see Proposition 1.5.1).

Consider the  $n$ -th order differential operator on  $\mathbb{H}$  locally of the form

$$Dy = y^{(n)} + Q_2 y^{(n-2)} + \dots + Q_n y \quad (1.2)$$

where  $Q_j$  are pull-backs of local sections of  $\mathcal{K}_C^j$ . This differential operator induces a short exact sequence of  $\mathbb{C}$ -modules

$$0 \rightarrow \mathbb{V} \xrightarrow{\varphi} \mathcal{K}_C^{1-(n+1/2)} \xrightarrow{D} \mathcal{K}_C^{n+1/2} \rightarrow 0 \quad (1.3)$$

and we say that the local system  $\mathbb{V}$  is realized in  $\mathcal{K}_C^{1-(n+1/2)}$ . Clearly the space of all such local systems is parametrized by the affine space modeled on  $\bigoplus_{j=2}^n H^0(C, \mathcal{K}_C^j)$ .

We can now state another characterization of opers.

**Proposition 1.3.11** ([Wen15]). *Let  $(\mathcal{V}, \nabla)$  be a flat vector bundle. Then  $(\mathcal{V}, \nabla)$  is an oper if and only if its associated local system is realized in  $\mathcal{K}_C^{1-(n+1/2)}$ .*

*This characterization defines an isomorphism between each connected component of  $\text{Op}_n(C)$  and the affine space modeled on the Hitchin base  $\bigoplus_{j=2}^n H^0(C, \mathcal{K}_C^j)$ . It follows also that the dimension of  $\text{Op}_n(C)$  is  $(n^2 - 1)(g - 1)$ .*

We give now the explicit map between local systems realized in  $\mathcal{K}_C^{1-(n+1/2)}$  and opers realizing the correspondence of the proposition above. Assume that we are given the exact sequence (1.3) and let  $\mathcal{V} := \mathcal{O}_C \otimes_{\mathbb{C}} \mathbb{V}$  be the associated flat vector bundle. Define the oper filtration by

$$\mathcal{V}_{n-k} := \left\{ \sum_{i=1}^n f_i \otimes v_i : \sum_{i=1}^n f_i^{(j)} \varphi(v_i) = 0, j = 0, \dots, k-1 \right\} \quad \text{for } k = 1, \dots, n-1, \quad (1.4)$$

where  $f_i^{(j)}$  is the  $j$ -th derivative of the local holomorphic function  $f_i$ . We will see that the construction (1.4) of the oper filtration is useful to relate opers and projective structures in rank 2.

Let us finally recall the theorem stating that the oper locus is the maximal Shatz stratum.

**Theorem 1.3.12** ([Wen15]). *The maximal stratum of  $\mathcal{M}_{DR}^{(n)}$  is the oper locus  $\text{Op}_n(C)$ . The Harder-Narashiman filtration of an oper is the oper filtration itself and the Harder-Narashiman type is given by*

$$\mu_i = \mu(\mathcal{K}_C^{(n+1)/2-i}) = (n+1-2i)(g-1).$$

*By upper-semicontinuity, the oper locus is a closed embedded subset of  $\mathcal{M}_{DR}^{(n)}$ .*

The previous theorem is a consequence of the following proposition, which one can directly use to compute the Harder-Narashiman type of an oper recursively.

**Proposition 1.3.13** ([Wen15]). *Let  $\mathcal{V}$  be an unstable vector bundle with an irreducible holomorphic connection and let  $(\mu_1 = \mu(\mathcal{V}_1), \dots, \mu_l = \mu(\mathcal{V}/\mathcal{V}_{l-1}))$  be its non repeated Harder-Narashiman type. Then for each  $i = 1, \dots, l-1$ ,*

$$\mu_i - \mu_{i+1} \leq 2g - 2.$$

*Moreover equality holds if and only if  $\mathcal{V}$  is an oper.*

**1.3.5 Bialynicki-Birula stratification** We describe another stratification of  $\mathcal{M}_{\text{DR}}^{(n)}(C)$  defined by using the  $\mathbb{C}^*$ -action on the Hitchin moduli space.

Recall that a variation of Hodge structures in  $\mathcal{M}_{\text{DR}}^{(n)}(C)$  corresponds to a system of Hodge bundles in  $\mathcal{M}_{\text{H}}^{(n)}(C)$ , which is equivalent to be a fixed point of the  $\mathbb{C}^*$ -action on  $\mathcal{M}_{\text{H}}^{(n)}(C)$ . Let us denote by

$$(\mathcal{M}_{\text{H}}^{(n)}(C))^{\mathbb{C}^*} = \bigsqcup_{\alpha} P_{\alpha}$$

the connected components decomposition of the fixed locus of the  $\mathbb{C}^*$ -action. We recall the construction of [Sim10], in which Simpson describes a limiting process that associates to any point  $(\mathcal{V}, \nabla) \in \mathcal{M}_{\text{DR}}^{(n)}(C)$  an element in  $(\mathcal{M}_{\text{H}}^{(n)}(C))^{\mathbb{C}^*}$ .

A *Griffiths transverse filtration* of  $(\mathcal{V}, \nabla)$  is a filtration

$$F^{k+1} = 0 \subset \dots \subset F^0 = \mathcal{V}$$

which satisfies the Griffiths transversality condition

$$\nabla : F^p \rightarrow F^{p-1} \otimes \Omega_C^1.$$

Note that a variation of Hodge structures is simply a flat vector bundle admitting a Griffiths transverse filtration together with the Hodge metric. Associate to such  $(\mathcal{V}, \nabla, F^{\bullet})$ , the graded Higgs bundle  $(\text{Gr}_F(\mathcal{V}) = \bigoplus_p F^p/F^{p-1}, \nabla^{gr})$ . In general this Higgs bundle will not be semistable, so it will not be a point of  $\mathcal{M}_{\text{H}}^{(n)}(C)$ . Clearly, if  $(\mathcal{V}, \nabla)$  is a variation of Hodge structures, this associated Higgs bundle is stable and it is the Higgs bundle in  $\mathcal{M}_{\text{H}}^{(n)}(C)$  corresponding to the point  $(\mathcal{V}, \nabla) \in \mathcal{M}_{\text{DR}}^{(n)}(C)$  under Simpson's correspondence.

**Theorem 1.3.14** ([Sim10]). *Let  $(\mathcal{V}, \nabla) \in \mathcal{M}_{\text{DR}}^{(n)}(C)$ . Then there exists a Griffiths transverse filtration such that its associated Higgs bundle is semistable, i.e. it is a system of Hodge bundles. This associated Higgs bundle depends only on  $(\mathcal{V}, \nabla)$  and not on the filtration. Moreover the Griffiths transverse filtration is unique if and only if the associated graded Higgs bundle is stable.*

Simpson proved that given  $(\mathcal{V}, \nabla)$ , the associated system of Hodge bundles is the limit point  $\lim_{t \rightarrow 0} t \cdot (\lambda, \mathcal{V}, \nabla)$  in the space

$$M_{\text{Hod}} := \{(\lambda, \mathcal{V}, \nabla), \nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_C^1, \nabla(ae) = a\nabla(e) + \lambda d(a)e\}$$

where  $t \cdot (\lambda, \mathcal{V}, \nabla) := (t\lambda, \mathcal{V}, t\nabla)$ .

Using Theorem 1.3.14 it is easy to show that there is only one variation of Hodge structures on an each connected component of the oper locus.

**Proposition 1.3.15.** *The only variation of Hodge structures on an each connected component of the oper locus is given by the  $(n-1)$ -th symmetric power of the maximal Higgs one in rank 2. The maximal Higgs variation of Hodge structures in rank 2 corresponds to the uniformizing representation of  $C$ .*

*Proof.* Let  $(\mathcal{V}, \nabla, \{\mathcal{V}_i\}_{i=0, \dots, n})$  be an oper. By Proposition 1.3.10 the holomorphic vector bundle  $\mathcal{V}$  is fixed on each connected component of the oper locus. Moreover the flat bundle  $(\mathcal{V}, \nabla)$  is irreducible. Let now

$$F^{k+1} = 0 \subset \dots \subset F^0 = \mathcal{V}$$

be the filtration associated to a VHS structure on  $\mathcal{V}$ . Since  $(\mathcal{V}, \nabla)$  is irreducible, the variation of Hodge structure  $(\mathcal{V}, \nabla, \{F^i\}_{i=0, \dots, k+1})$  is irreducible as VHS. Hence the corresponding system of Hodge bundles is stable. Since both the oper and the VHS filtrations are semistable, by Theorem 1.3.14 they are the same and the associated Higgs bundle has to be the maximal Higgs system of Hodge bundles. This system of Hodge bundles is clearly given by the symmetric power of the maximal Higgs in rank 2.

In order to check that the rank 2 maximal Higgs variation of Hodge structure defines the uniformizing representation of  $C$ , notice that the Higgs field can be identified with the derivative of the period map  $p : \mathbb{H} \rightarrow \mathbb{H}$ . Since the Riemann surface  $C$  is compact, the period map is proper. Hence if the Higgs field is an isomorphism, the period map is a covering map, since it is a proper local isomorphism. It then has to be an isomorphism since  $\mathbb{H}$  is simply connected. Hence the period map induces an isomorphism between  $C \cong \mathbb{H} / \pi_1(C)$  and  $\mathbb{H} / \rho(\pi_1(C))$ , where  $\rho$  is the corresponding representation. It follows that  $\rho$  is the uniformizing representation of  $C$ .  $\square$

We can now define another stratification, the Bialynicki-Birula stratification, of  $\mathcal{M}_{\text{DR}}^{(n)}(C)$ .

**Proposition 1.3.16** ([Sim10]). *For any  $\alpha$ , the subsets  $G_\alpha \subset \mathcal{M}_{\text{DR}}^{(n)}(C)$  consisting of all points  $(\lambda, \mathcal{V}, \nabla)$  such that  $\lim_{t \rightarrow 0} t \cdot (\lambda, \mathcal{V}, \nabla) \in P_\alpha$  is locally closed and it defines a stratification*

$$\mathcal{M}_{\text{DR}}^{(n)}(C) = \bigcup_{\alpha} G_\alpha$$

which is called *Bialynicki-Birula stratification*.

It is not hard to see ([Sim10]) that the minimal and maximal strata of the Bialynicki-Birula stratification coincide with the minimal and maximal strata of the Shatz stratification. More in detail, the unique open stratum  $G_0$  is the same as the minimal stratum of the Shatz stratification given by the locus of semistable vector bundles. The limiting locus  $P_0 \subset \mathcal{M}_{\mathbb{H}}^{(n)}(C)$  is simply the locus of Higgs bundles with zero Higgs field, corresponding to the locus of unitary representations in  $\mathcal{M}_{\mathbb{B}}^{(n)}$  by the Narashiman-Seshadri theorem. This is also the locus of weight zero variation of Hodge structures. The stratum with minimal dimension is the oper locus  $\text{Op}_n(C) \subset \mathcal{M}_{\text{DR}}^{(n)}(C)$ . The limiting locus  $P_n$  is given by the  $(n-1)$ -th symmetric power of the maximal Higgs variation of Hodge structures in rank 2 (see Proposition 1.3.15).

We want to add some remarks about the loci  $P_\alpha \subset \mathcal{M}_{\mathbb{H}}^{(n)}(C)$  corresponding to the loci of complex variation of Hodge structures in  $\mathcal{M}_{\text{DR}}^{(n)}(C)$ . Notice first of all even though these loci can have positive dimension, Deligne proved that there are only finitely many  $\mathbb{Q}$ -variation of Hodge structures.

**Theorem 1.3.17** ([Del87]). *Fix a smooth connected variety  $X$  and a number  $n$ . Then there are only finitely many  $\mathbb{Q}$ -variation of Hodge structures on  $X$  of rank  $n$ .*

In general, by the definition of the Hodge metric the monodromy group of a variation of Hodge structures is contained in some indefinite unitary group  $U(p, q)$ . We can say something more. Let  $\mathcal{M}_{U(p,q)}^{(n)} \subset \mathcal{M}_{\text{DR}}^{(n)}(\mathbb{C})$  be the real subspace corresponding to the set of representations with monodromy group contained into some possibly indefinite unitary group  $U(p, q)$ .

**Proposition 1.3.18** ([Sim10]). *Restricting to the subset of smooth points, the locus of variations of Hodge structures in  $G_\alpha$  is a connected component of  $\mathcal{M}_{U(p,q)}^{(n)} \cap G_\alpha$ .*

Finally we want to remark special properties of weight one and real weight two variations of Hodge structures what will be interesting to relate to Lyapunov exponents (see Section 1.11). A real weight two variation of Hodge structures is given by a real vector bundle  $\mathcal{V}_{\mathbb{R}}$  over  $C$  such that its base change to  $\mathbb{C}$  defines a complex weight 2 variation of Hodge structures  $\mathcal{V}$ .

**Proposition 1.3.19.** *1. Let  $\mathcal{V}$  be a weight one variation of Hodge structures. Then the first piece of the Hodge filtration  $F^1 = \mathcal{V}^{1,0}$  is the maximal degree subbundles among all subbundles of  $\mathcal{V}$ .*

*2. Let  $\mathcal{V}$  be a real variation of Hodge structures of weight two and let  $\mathcal{V}^{2,0}$  be the first piece of the Hodge filtration. Then for every subbundle  $\mathcal{E} \subseteq \mathcal{V}$  it holds  $\deg(\mathcal{E}) \leq 2 \deg(\mathcal{V}^{2,0})$ .*

*Proof.* If  $\mathcal{V}$  is a weight one variation of Hodge structures, the associated semistable system of Hodge bundles is given by  $\mathcal{V}^{1,0} \oplus \mathcal{V}^{0,1}$ , where  $\mathcal{V}^{0,1} = \mathcal{V}/\mathcal{V}^{1,0}$ . First of all note that if  $\mathcal{W} \subseteq \mathcal{V}^{1,0}$ , then  $\deg(\mathcal{W}) \leq \deg(\mathcal{V}^{1,0})$ . Indeed  $\mathcal{W} \oplus \mathcal{V}^{0,1}$  is sub-system of Hodge bundles. Hence by semistability  $\deg(\mathcal{W}) + \deg(\mathcal{V}^{0,1}) \leq 0$ . If  $\mathcal{E} \subset \mathcal{V}$  is any subbundle, then consider the short exact sequence

$$0 \rightarrow \mathcal{E} \cap \mathcal{V}^{1,0} \rightarrow \mathcal{E} \rightarrow \frac{\mathcal{E}}{\mathcal{E} \cap \mathcal{V}^{1,0}} \rightarrow 0.$$

Since the quotient  $\frac{\mathcal{E}}{\mathcal{E} \cap \mathcal{V}^{1,0}}$  is a subbundle of  $\mathcal{V}^{0,1}$ , it defines a sub-system of Hodge bundles and so it has negative degree. By additivity of the degree we finally get

$$\deg(\mathcal{E}) = \deg(\mathcal{E} \cap \mathcal{V}^{1,0}) + \deg\left(\frac{\mathcal{E}}{\mathcal{E} \cap \mathcal{V}^{1,0}}\right) \leq \deg(\mathcal{V}^{1,0}).$$

If  $\mathcal{V}$  is a real variation of Hodge structures of weight two, the associated semistable system of Hodge bundles is given by  $\mathcal{V}^{2,0} \oplus \mathcal{V}^{1,1} \oplus \mathcal{V}^{0,2}$  where by definition  $\mathcal{V}^{1,1} = F^1/\mathcal{V}^{2,0}$  and  $\mathcal{V}^{0,2} = \mathcal{V}/F^1$ . Let now  $\mathcal{E} \subseteq \mathcal{V}$  be a subbundle. First of all consider the bundle  $\frac{\mathcal{E}}{\mathcal{E} \cap F^1}$ : Since it is a subbundle of  $\mathcal{V}^{0,2}$ , it defines a sub-Higgs bundle of the associated system of Hodge bundles. By semistability it has then to have negative degree and so we get

$$\deg(\mathcal{E}) \leq \deg(\mathcal{E} \cap F^1).$$



## 1.4 Betti moduli space in rank two and projective structures

Moreover, since  $\mathcal{E} \cap \mathcal{V}^{0,2} \oplus \mathcal{V}^{1,1} \oplus \mathcal{V}^{0,2}$  also defines a sub-Higgs bundle and since  $\deg(\mathcal{V}^{0,2}) = -\deg(\mathcal{V}^{2,0})$  and  $\deg(\mathcal{V}^{1,1}) = 0$  (this is because  $\mathcal{V}$  is a real VHS), again by semistability we obtain

$$\deg(\mathcal{E} \cap \mathcal{V}^{2,0}) \leq \deg(\mathcal{V}^{2,0}).$$

To conclude consider the sub-Higgs bundle  $\frac{\mathcal{E} \cap F^1}{\mathcal{E} \cap \mathcal{V}^{0,2}} \oplus \mathcal{V}^{0,2}$ , which by semistability has negative degree. It follows that

$$\deg(\mathcal{E} \cap F^1) \leq \deg(\mathcal{E} \cap \mathcal{V}^{0,2}) + \deg(\mathcal{V}^{2,0}).$$

Using the three inequalities that we obtained we get

$$\deg(\mathcal{E}) \leq \deg(\mathcal{E} \cap F^1) \leq \deg(\mathcal{E} \cap \mathcal{V}^{0,2}) + \deg(\mathcal{V}^{2,0}) \leq 2 \deg(\mathcal{V}^{2,0}).$$

□

The last proposition will be related to the characterization of Lyapunov exponents from the main equality of Theorem 1.7.10 and Theorem 1.11.3.

## 1.4 Betti moduli space in rank two and projective structures

In this section we specialize to the rank 2 case and introduce the notion of projective structure. We will recall known results about the space of projective structures following [Dum09].

Recall that the rank 2 Betti moduli space

$$\mathcal{M}_B^{(2)} := \text{Hom}(\pi_1(C), \text{SL}_2(\mathbb{C})) // \text{SL}_2(\mathbb{C})$$

is given by the GIT quotient of the representation variety by the action of conjugation. Let  $S$  be the topological surface underlying  $C$ .

**Definition 1.4.1.** A *complex projective structure* on  $S$  is a maximal atlas of charts mapping open sets of  $S$  into  $\mathbb{P}_{\mathbb{C}}^1$  such that the transition functions are restriction of Möbius transformations. Equivalently, a projective structure on  $S$  is given by a pair  $(f, \rho)$  where  $\rho : \pi_1(C) \rightarrow \text{PSL}_2(\mathbb{C})$  is a representation called *holonomy representation* and  $f : \mathbb{H} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is a  $\rho$ -equivariant immersion called *developing map*.

The pair has to be considered modulo the natural equivalence relation given by precomposition of developing maps with orientation-preserving diffeomorphisms of  $S$  homotopic to the identity on one side and by conjugation of  $\text{PSL}_2(\mathbb{C})$  on the other.

Let  $\mathcal{P}(S)$  be space of projective structures on  $S$  and  $\mathcal{T}(S)$  be the Teichmüller space. Note that since Möbius transformations are holomorphic, a projective structure also determines a complex structure. We can then consider the forgetful map

$$p : \mathcal{P}(S) \rightarrow \mathcal{T}(S).$$

Denote by  $\mathcal{P}(C) := p^{-1}(C)$  the fiber over a Riemann surface  $C$ , which is the space of projective structures inducing the same holomorphic structure.

*Remark 1.4.2.* The space  $\mathcal{P}(C)$  can be identified with the space of quadratic differentials  $H^0(C, \mathcal{K}_C^2)$ , using the Schwarzian derivative. The identification is given by associating to  $(f, \rho) \in \mathcal{P}(C)$  the push-forward to  $C$  of the quadratic differential  $S(f)$  on  $\mathbb{H}$ , where  $S(f)$  is the Schwarzian derivative. In the other direction, one associates to  $\phi \in H^0(C, \mathcal{K}_C^2)$  the projective structure given by  $(u_1(z)/u_2(z), \rho)$  where  $u_1, u_2$  is a basis of solution of the differential equation

$$u''(z) + \frac{1}{2}\tilde{\phi}(z)u(z) = 0. \quad (1.5)$$

Here  $\tilde{\phi}(z)dz^2$  is the pull-back of  $\phi$  to the universal cover  $\mathbb{H}$  and  $\rho$  is the monodromy associated to the differential equation.

Recall that a representation is called *elementary* if its action on  $\mathbb{H}^3$  fixes a point or an ideal point, or if it preserves an unoriented geodesic. Otherwise it is called *non-elementary*. Equivalently a representation is elementary if and only if it is unitary or reducible or if the image is contained in the subgroup generated by  $\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$ . We denote by  $\mathcal{M}_B^{(2)'}$  the subspace of non-elementary representations. It is a Zariski-dense subset contained in the smooth locus of  $\mathcal{M}_B^{(2)}$ .

*Remark 1.4.3.* In general, there is another important dense subset of  $\mathcal{M}_B^{(n)}$  defined by representations with Zariski dense image. In the case of the  $\mathrm{SL}_2(\mathbb{C})$ -representation variety, this locus is the same as the one of simple representations (see [Mon17, Remark 2.13]).

We can relate the set of projective structures and the Betti moduli space via the holonomy map, which sends a projective structure to its associated holonomy representation.

**Theorem 1.4.4** ([GKM00],[Hej75],[Ear81],[Hub81]). *The holonomy map*

$$\mathrm{hol} : \mathcal{P}(S) \rightarrow \mathcal{M}_B^{(2)}$$

*has image in  $\mathcal{M}_B^{(2)'}$ , it is surjective on this set and it is a local biholomorphism.*

The holonomy map is not injective, in fact all the fibers are infinite, and it is not a covering map.

We will now restrict the holonomy map to the fibers  $\mathcal{P}(C)$  of the forgetful map  $p : \mathcal{P}(S) \rightarrow \mathcal{T}(S)$  and recall the following result.

**Theorem 1.4.5** ([Dum09]). *For every  $C \in \mathcal{T}(S)$ , the restriction  $\mathrm{hol}|_{\mathcal{P}(C)}$  is a proper holomorphic embedding. Consequently the image  $\mathrm{hol}(\mathcal{P}(C))$  is a complex-analytic subvariety of  $\mathcal{M}_B^{(2)'}$ .*

Notice that, since  $\mathcal{P}(C)$  is an affine space modeled on  $H^0(C, \mathcal{K}_C^2) \cong \mathbb{C}^{3g-3}$ , then also the image  $\mathrm{hol}(\mathcal{P}(C))$  is.

We want now to recall some definitions of special subgroups of  $\mathrm{SL}_2(\mathbb{C})$ . A discrete subgroup  $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$  is called *Fuchsian*. A discrete subgroup  $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$  is called *Kleinian*. A *quasi-Fuchsian group* is a Kleinian group  $\Gamma$  such that the accumulation

points of the  $\Gamma$ -action on  $\partial \mathbb{H}^3$  is a quasi-circle. They are quasi-conformal deformations of Fuchsian groups. These special subgroups of  $\mathrm{SL}_2(\mathbb{C})$  define special subsets of  $\mathcal{M}_B^{(2)}$ . We recall now their properties. Let  $\mathcal{QF}(S)$  be the subspace of quasi-Fuchsian representations and  $\mathcal{D}(S)$  be the subspace of discrete representations.

**Theorem 1.4.6** ([Dum09]). *The subset of discrete representations  $\mathcal{D}(S) \subset \mathcal{M}_B^{(2)}$  is closed in the analytic topology. Its interior is the locus  $\mathcal{QF}(S)$  of quasi-Fuchsian representations.*

Let the real Betti moduli space defined as

$$\mathcal{M}_{B,\mathbb{R}}^{(2)} := \mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{R})) // \mathrm{SL}_2(\mathbb{R}).$$

We want to recall some results about this space.

**Theorem 1.4.7** ([Gol88]). *The real representation variety  $\mathcal{M}_{B,\mathbb{R}}^{(2)}$  has one connected component for each even integer  $e$  with  $0 \leq |e| \leq 2g - 2$ . The integer  $e$  corresponds to the Toledo invariant or the Euler number associated to a representation.*

*In the case of the maximal integer  $e = 2g - 2$ , the connected component  $\mathcal{M}_{B,\mathbb{R},2g-2}^{(2)}$  is the same as the space of Fuchsian representations.*

*In the case of the minimal integer  $e = 0$ , the connected component  $\mathcal{M}_{B,\mathbb{R},0}^{(2)}$  is contained in the space of elementary representations.*

*Remark 1.4.8* ([Dum09]). The space  $\mathcal{QF}(S)$  can be identified, using the Bers simultaneous uniformization theorem, with the space  $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$  where  $\bar{S}$  represents the surface  $S$  with opposite orientation. This identification gives a holomorphic embedding

$$\mathcal{QF}(S) \cong \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \hookrightarrow \mathcal{M}_B^{(2)}.$$

The diagonal  $\{(C, \bar{C}) : C \in \mathcal{T}(S)\}$  corresponds to the space of Fuchsian representation  $\mathcal{M}_B^{\mathbb{R}}(S)_{2g-2} \cong \mathcal{T}(S)$  but the induced embedding is not holomorphic. The image is a totally real submanifold.

## 1.5 Special loci of Betti and De Rham moduli spaces in rank 2

We want to give a more detailed description of some special loci of the Betti and the de Rham moduli spaces in rank 2. In particular, we want to use the geometric tool given by projective structures to understand better some special loci. For example, we will notice that the oper locus of  $\mathcal{M}_{\mathrm{DR}}^{(2)}(C)$  corresponds to the locus  $\mathcal{P}(C)$  of projective structures on  $C$ . We will conclude this section with a summary schematic picture of these loci in the Betti and the de Rham moduli spaces.

**1.5.1 Shatz stratification in rank 2** In rank two, the Shatz stratification consists of  $g$  Shatz strata. The degree of the maximal destabilizing subbundle varies in the set  $\{0, 1, \dots, g - 1\}$ . Indeed flat bundles in the oper locus, which is the maximal stratum, have maximal destabilizing subbundle of degree  $g - 1$  (see Theorem 1.3.12).

**1.5.2 Locus of semistable bundles and unitary locus in rank 2** The minimal Shatz stratum is the same as the minimal Bialynicki-Birula stratum in rank 2 and it is given by semistable flat bundles. This is an open dense subset containing the space of unitary representations as a closed subspace. The space of unitary representations is the same as the space  $P_0$  defined in section 1.3.5 defined by variation of Hodge structures of weight zero. Moreover, the space of stable bundles can be identified with the tangent bundle to  $P_0$  (see Section 1.3.3).

The subspace of the semistable locus given by semistable but not stable bundle is a closed subset and corresponds to non simple representations. This subspace is contained in the space of elementary representations.

**1.5.3 Oper locus in rank 2** We want to describe more in detail the maximal Shatz stratum, which we identified with the oper locus  $\text{Op}_2(C) \subset \mathcal{M}_{\text{DR}}^{(2)}(C)$ . Recall that it is also the maximal Bialynicki-Birula stratum. It is a  $(3g-3)$ -dimensional closed subvariety given by the locus of flat bundles having a sub-line bundle with maximal possible degree, namely  $g-1$ . Indeed in this case the oper filtration is given by  $0 \subset \mathcal{V}_1 \subset \mathcal{V}$  with  $\mathcal{V}_1 \cong \mathcal{K}_C^{1/2}$  (see Proposition 1.3.10).

We want first of all describe the correspondence between the oper locus and set of holonomies of the projective structures  $\mathcal{P}(C)$  inducing the same complex structure  $C$ .

**Proposition 1.5.1.** *The subset  $\text{hol}(\mathcal{P}(C)) \subset \mathcal{M}_{\mathbb{B}}^{(n)}$  corresponds via the Riemann-Hilbert correspondence to the oper locus  $\text{Op}_2(C) \subset \mathcal{M}_{\text{DR}}^{(2)}(C)$ . Moreover, if  $\mathcal{V}$  is an oper, the meromorphic map*

$$s_{\mathcal{V}_1} : \mathbb{H} \rightarrow \mathbb{P}^1$$

*defined by the inclusion of the sub-line bundle  $\mathcal{V}_1 \subset \mathcal{V}$  given by the oper filtration is the developing map associated to the projective structure corresponding to the oper.*

*Proof.* For the proof of the first statement, it is enough to combine the description of opers given by Proposition 1.3.11 and Remark 1.4.2 about the relation of projective structures and differential equations.

In order to prove the second statement we need to use the construction of the oper filtration (1.4) starting from the local system defined by the differential equation. Using the notation of as in (1.4), the inclusion  $\mathcal{V}_1 \cong \mathcal{K}_C^{1/2} \subset \mathcal{V} = \mathcal{O}_C \otimes \mathbb{V}$  is given by

$$\mathcal{V}_1 = \{f_1 \otimes v_1 + f_2 \otimes v_2 : f_1\varphi(v_1) + f_2\varphi(v_2) = 0\} \subset \mathcal{V}$$

where  $v_1, v_2$  are a local basis of the solution of the local system. Recall that the developing map in Remark 1.4.2 was exactly defined as the meromorphic function  $\frac{\varphi(v_1)}{\varphi(v_2)}$ . It is immediate to see that the map  $s_{\mathcal{V}_1}$  defined by the inclusion of the pull-back of  $\mathcal{V}_1$  into the trivial vector bundle on  $\mathbb{H}$  is given by the same meromorphic map.  $\square$

Now that we know that the oper locus corresponds to the space  $\text{hol}(\mathcal{P}(C))$  in  $\mathcal{M}_{\mathbb{B}}^{(2)}$  defined by holonomies of projective structures inducing the same complex structure  $C$ , we can understand better the oper locus in rank 2 and its intersection with the locus of real representations.

**Theorem 1.5.2** ([Fal83],[Dum09]). *For each  $C \in \mathcal{T}(S)$ , the intersection of  $\text{hol}(\mathcal{P}(C))$  and  $\mathcal{M}_{B,\mathbb{R}}^{(2)}$  is transversal. Moreover, the intersection of  $\text{hol}(\mathcal{P}(C))$  and the Fuchsian locus  $\mathcal{M}_{B,\mathbb{R},2g-2}^{(2)}$  is countable. Each one of these Fuchsian points is contained in the open set of quasi-Fuchsian holonomy representations in  $\text{hol}(\mathcal{P}(C))$ . These open sets are connected, contractible and biholomorphic to  $\mathcal{T}(S)$ . The closure of these open sets gives the space of discrete representations in  $\text{hol}(\mathcal{P}(C))$ .*

The proof of the transversality of the intersection of  $\text{hol}(\mathcal{P}(C))$  and the real rank 2 character was proved by Faltings by showing that the tangent space  $H^0(C, \mathcal{K}_C^2)$  of the oper locus does not intersect the tangent space  $H^1(\pi_1(C), \mathfrak{sl}_2(\mathbb{R}))$  of the real character variety in the tangent space  $H^1(\pi_1(C), \mathfrak{sl}_2(\mathbb{C}))$  of the rank 2 character variety. Another interesting result proved in the same paper is that the situation described in the last theorem persists with the deformation of the complex structure of  $C$ . More specifically, Faltings proved the following.

**Theorem 1.5.3** ([Fal83]). *Let  $C$  be a hyperbolic Riemann surface with  $n$  cusps. The space  $\{\text{Op}_2(C)\}_{C \in \mathcal{T}_{g,n}}$  defines a vector bundle of rank  $3g - 3 + n$  over the Teichmüller space  $\mathcal{T}_{g,n}$ . Moreover the holonomy mapping is a local isomorphism of the total space of this vector bundle into the rank 2 character variety.*

*Remark 1.5.4.* There is a special Fuchsian representation point in  $\text{hol}(\mathcal{P}(C))$ , namely the point corresponding the uniformizing representation of  $C$ . The open connected set  $B(C)$  of quasi-Fuchsian representations containing this point is given by the image under the holonomy map of the Bers embedding

$$B(C) = \text{hol}(\text{Im}(\mathcal{T}(S) \hookrightarrow \mathcal{P}(C))) \subset \mathcal{M}_{\mathbb{B}}^{(2)}.$$

The Bers embedding is defined by

$$\mathcal{T}(S) \hookrightarrow \mathcal{P}(C), \quad Y \mapsto \Sigma_Y(C)$$

where  $\Sigma_Y(C)$  is the projective structure on  $C$  induced by the quasi-Fuchsian group  $Q(C, Y)$  given by the simultaneous uniformization theorem.

**1.5.4 Variation of Hodge structures locus in rank 2** Thanks to Hitchin in [Hit87], we know how the connected components  $P_e \subset \mathcal{M}_{\text{DR}}^{(2)}(C)$  of the variation of Hodge structures locus look like. More in detail, we know the corresponding system of Hodge bundles. These are indexed by an integer  $0 \leq e \leq g - 1$ . For  $e = 0$ , we already recalled that  $P_0$  is the space of variations of Hodge structures of weight 0, which corresponds to unitary representations. For  $e > 0$ , the space  $P_e$  parametrizes Higgs bundles of the form

$$\mathcal{V} = \mathcal{V}^0 \oplus \mathcal{V}^1, \quad \Phi : \mathcal{V}^1 \rightarrow \mathcal{V}^0 \otimes \mathcal{K}_C,$$

where  $\mathcal{V}^0$  and  $\mathcal{V}^1$  are line bundles of degrees  $-e$  and  $e$  respectively. These systems of Hodge bundles correspond to weight one complex variations of Hodge structures. By the trivial determinant condition  $\det(\mathcal{V}^0 \oplus \mathcal{V}^1) = \mathcal{V}^0 \otimes \mathcal{V}^1 \cong \mathcal{O}_C$ , and so  $\mathcal{V}^0 \cong (\mathcal{V}^1)^*$ . Note that the Higgs field  $\Phi$  is a section of the line bundle  $(\mathcal{V}^1)^* \otimes \mathcal{V}^0 \otimes \mathcal{K}_C$  of degree  $2g - 2 - 2e$ , hence  $e \leq g - 1$  and in case of equality,  $\Phi$  is an isomorphism. Let  $D \in \text{Sym}^{2g-2-2e}(C)$  be the divisor of  $\Phi$ . Then we have only finitely many possibilities for  $\mathcal{V}^1$ , determined by the isomorphism  $(\mathcal{V}^1)^2 \cong \mathcal{K}_C \otimes \mathcal{O}_C(-D)$ . We have then the parametrization

$$P_e \cong \text{Sym}^{2g-2-2e}(C) \times \text{finite\#}$$

which gives that the dimension of  $P_e$  is  $2g - 2 - 2e$ . For  $e = g - 1$ , the Higgs field is an isomorphism and we get the only variation of Hodge structure on the oper locus, which corresponds to the uniformizing representation of  $C$  (see Proposition 1.3.15).

Note finally that the locus  $P_e$  belongs to the  $e$ -th Shatz stratum, which is defined as the locus of flat bundles such that the degree of the maximal destabilizing subline bundle is  $e$ , for  $0 \leq e \leq g - 1$ . Indeed, by Proposition 1.3.19 the first piece of the Hodge filtration  $\mathcal{V}^1$  is the maximal destabilizing subsheaf of  $\mathcal{V}$ .

**1.5.5 Real representations locus** Finally, we would like to recall that the real representation locus  $\mathcal{M}_{B,\mathbb{R}}^{(2)} \subset \mathcal{M}_B^{(2)}$  given by representation with image in  $\text{SL}_2(\mathbb{R})$ , has one connected component for each integer  $0 \leq e \leq 2g - 2$ . The connected component  $\mathcal{M}_{B,\mathbb{R},2g-2}^{(2)}$  is the same as the space of Fuchsian representations and the connected component  $\mathcal{M}_{B,\mathbb{R},0}^{(2)}$  is contained in the space of elementary representations. It would be interesting to understand better the intersection of the other real components with the oper locus.

**1.5.6 Summarizing picture** We present a summarizing picture representing the special loci that we described. In each one of the  $g$  Shatz strata we find the complex variation of Hodge structures Loci  $P_e$ . The locus  $P_0$  of weight zero variation of Hodge structures corresponds to the locus of unitary representations and it is contained in the minimal open Shatz stratum of semistable flat bundles. The locus  $P_{g-1}$  of maximal Higgs variation of Hodge structures, corresponding to the  $s^{2g}$  lifts to  $SL_2(\mathbb{C})$  of the uniformizing representation of  $C$ , is contained in the oper locus, the maximal Shatz stratum. Using the interpretation available in rank 2 of representations as holonomies of projective structures, we can understand better the oper locus, since it is the same as the set  $\mathcal{P}(C)$  of projective structures on  $C$  inducing the same original complex structure. Finally the blue locus is the maximal real representation locus of Fuchsian representations.

## 1.6 Character varieties for general reductive groups, special components, dynamical invariants and geometric Oseledets

In this section we want to recall the general picture of the Riemann-Hilbert and Simpson correspondences in the case of representations in a reductive Lie group  $G$ . This description

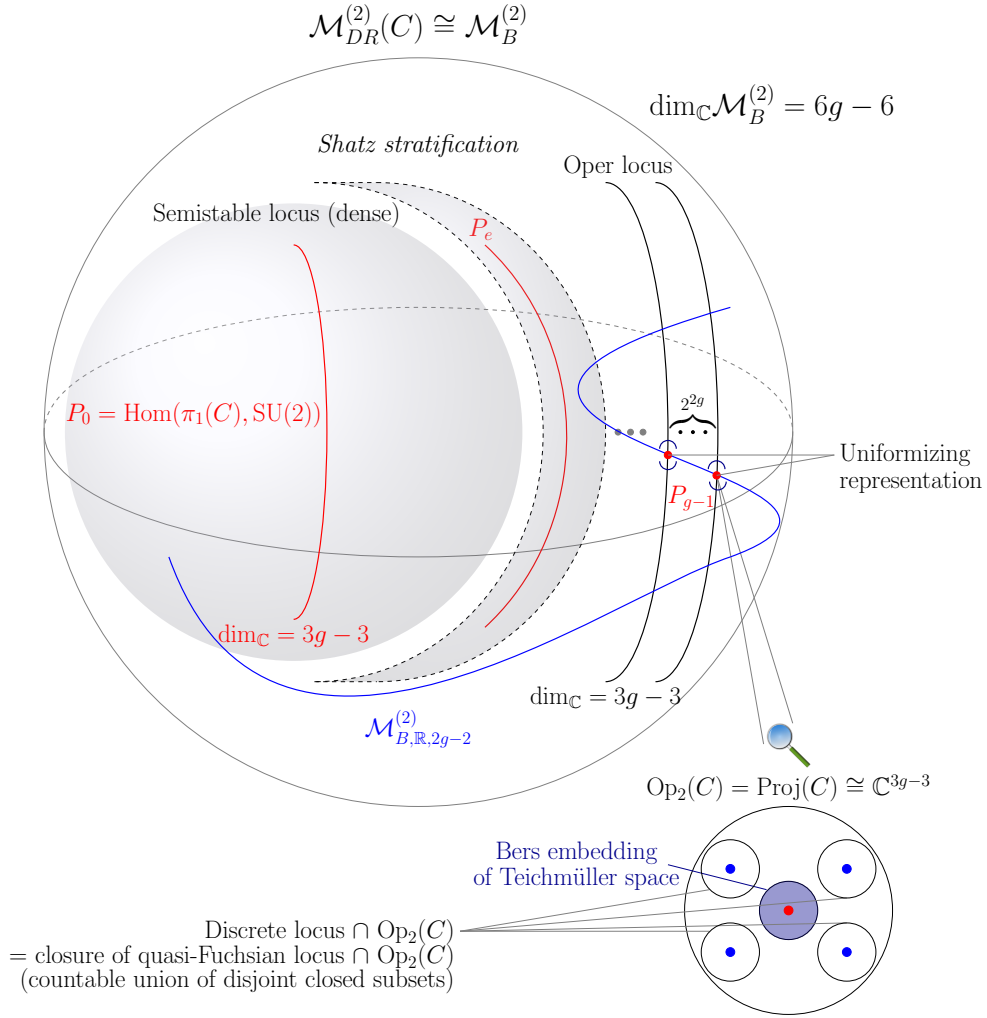


Figure 1.1: Betti and de Rham moduli spaces in rank 2 and special loci.

will allow us to define Hitchin and maximal representations. We will follow the description given in [Mau15]. We also recall the definition of important invariants like the Toledo invariant, the critical exponent, the entropy and the minimal area. The geometric Oseledets ergodic theorem will allow us to talk about Lyapunov exponents associated to a  $G$ -flat bundle. We finally speculate that this point of view can help to relate Lyapunov exponents to the other cited invariants.

**1.6.1 Flat  $G$ -bundles and  $G$ -Higgs bundles** Let  $M$  be a Kähler manifold,  $\tilde{M}$  be its universal cover and  $G$  be a reductive real linear algebraic group. Let  $\rho : \pi_1(M) \rightarrow G$  be a representation. Define the principal  $G$ -bundle

$$P_G := \tilde{M} \times_{\rho} G := \pi_1(M) \backslash (\tilde{M} \times G).$$

Since  $P_G$  is the quotient of the trivial  $G$ -bundle  $\tilde{M} \times G$ , it has a canonical flat Ehresman connection. Let  $K$  be a maximal compact subgroup of  $G$  and  $\mathcal{X} := G/K$  be the

associated symmetric space. A reduction to the structure group of  $P_G$  to  $K$  is a  $K$ -principal subbundle  $P_K \subset P_G$  together with an isomorphism  $P_K \times_K G \cong P_G$ . A reduction of the structure group of  $P_G$  to  $K$  is equivalent to a global section of the associated  $\mathcal{X}$ -fiber bundle  $P_G/K \cong \tilde{M} \times_\rho \mathcal{X}$ , or equivalently to a  $\rho$ -equivariant map  $f : \tilde{M} \rightarrow \mathcal{X}$ . The bundle  $P_K$  is retrieved as the pull-back

$$P_K = \pi_1(M) \backslash f^*(G)$$

of the principal  $K$ -bundle  $G \rightarrow \mathcal{X}$ .

Recall that the connection form associated to a Ehresman connection is the projection  $\eta : TP_G \rightarrow TP_G$  to the vertical bundle, i.e.  $\eta \in \mathcal{A}^1(P_G, TP_G)$ . A connection  $\eta$  is a principal connection if it behaves equivariantly with respect to the adjoint action of the group  $G$ . In this case we can view the principal connection  $\eta$  as an element of  $\mathcal{A}^1(P_G, \mathfrak{g})$ , where  $\mathfrak{g}$  is the Lie algebra of  $G$ . Consider the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , where  $\mathfrak{k}$  is the Lie algebra of  $K$ . Let  $\eta_G \in \mathcal{A}^1(P_G, \mathfrak{g})$  be the canonical flat connection on  $P_G$ . Let  $\eta_K \in \mathcal{A}^1(P_K, \mathfrak{k})$  be the connection induced on  $P_K$  by the pullback  $f^*\lambda$  of the canonical connection  $\lambda \in \mathcal{A}^1(G, T\mathcal{X})$  of the  $K$ -bundle  $G \rightarrow \mathcal{X}$ . Let finally  $\Theta \in \mathcal{A}^1(P_G, \mathfrak{p})$  be the connection induced by the differential of  $f$  seen as a section of  $\pi_1(M) \backslash f^*(T\mathcal{X}) \rightarrow M$ . We then get that on  $P_K$

$$\eta_G = \eta_K + \Theta.$$

The previous equality corresponds in the classical Simpson correspondence to the equality of the flat connection with the sum of the harmonic Chern connection and the Higgs field, which here are represented respectively by  $\eta_K$  and  $\Theta$ . In order to make the last sentence precise one has to pass to the complexified principal bundles. A Higgs principal  $G$ -bundle is defined as a holomorphic  $K_{\mathbb{C}}$ -bundle  $P_{K_{\mathbb{C}}}$  together with a holomorphic  $(1, 0)$ -form  $\theta \in \mathcal{A}^{1,0}(M, P_{K_{\mathbb{C}}} \times_{\text{Ad}} \mathfrak{p}_{\mathbb{C}})$  satisfying  $\theta \wedge \theta = 0$ . Here  $\theta$  is the Higgs field and it defined as the  $(1, 0)$ -part of the complexified version of  $\Theta = df$ . In this general setting Simpson correspondence still holds, meaning that there is a correspondence between Higgs polystable principal  $G$ -bundles with vanishing Chern classes and flat principal  $G$ -bundles.

For every finite dimensional representation  $\sigma : G \rightarrow \text{Aut}(V)$  we can construct the associated vector bundle  $P_G \times_\sigma V$ . Under this operation, applying the construction previously described we retrieve the usual correspondence between flat vector bundles and Higgs vector bundles.

Finally we recall the definition of Toledo invariant, which allows us to define maximal representations. Let  $C$  be a hyperbolic Riemann surface and  $\rho : \pi_1(C) \rightarrow G$  be a reductive representation into a group of Hermitian type. Let  $\mathcal{X}$  be the Hermitian symmetric space associated to  $G$  and  $T^1\mathcal{X}$  be its holomorphic tangent bundle. Let  $f : \mathbb{H} \rightarrow \mathcal{X}$  be the associated harmonic  $\rho$ -equivariant map. Let  $f^*(T^1\mathcal{X})$  be the bundle over  $C$  induced by the pull-back. Since  $\mathcal{X}$  is Kähler-Einstein, there is a constant  $\lambda_{\mathcal{X}}$  such that  $\lambda_{\mathcal{X}}c_1(T^1\mathcal{X}) = \omega_{\mathcal{X}}$ , where  $\omega_{\mathcal{X}}$  is the  $G$ -invariant Kähler form.

**Definition 1.6.1.** The *Toledo invariant* associated to  $\rho$  is defined as

$$\text{Tol}(\rho) = \int_C f^*(\omega_{\mathcal{X}}) = \lambda_{\mathcal{X}} \deg(f^*(T^1\mathcal{X})).$$



The Toledo invariant is constant on the connected components of the character variety  $\text{Hom}(\pi_1(S), G)//G$  and it satisfies a Milnor-Wood type inequality, namely:

$$|\text{To}(\rho)| \leq \text{rk}(\mathcal{X}) \text{vol}(C) = \frac{1}{2\pi} \text{rk}(\mathcal{X}) \deg(\Omega_C^1).$$

A representation  $\rho$  for which  $\text{To}(\rho) = \frac{1}{2\pi} \text{rk}(\mathcal{X}) \text{vol}(C)$  is called *maximal representation*.

We recall an important example from [Mau15] where  $G = \text{SU}(p, q)$ , which is a simple Lie group of Hermitian type. Let  $\rho : \pi_1(X) \rightarrow G$  be a reductive representation and let  $\sigma$  be a reductive representation of  $G$ . Then the Higgs vector bundle  $(E, \theta)$  associated to  $\rho$  and  $\sigma$  splits as a direct sum  $E = V \oplus W$ , where  $\text{rk}(V) = p$  and  $\text{rk}(W) = q$ , and the Higgs field has the form

$$\theta = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \quad \beta : W \rightarrow V \otimes \Omega_M^1, \quad \gamma : V \rightarrow W \otimes \Omega_M^1.$$

If  $f : \tilde{M} \rightarrow \mathcal{X} = \text{SU}(p, q)/S(\text{U}(p) \times \text{U}(q))$  is the corresponding harmonic map, since  $\theta = \text{d}f^{1,0}$  we get that  $W^* \otimes V \cong \pi_1(M) \backslash f^*(T^{1,0}\mathcal{X})$ . The Toledo invariant is then essentially given by the degree of  $W$ :

$$\deg(f^*(T^1\mathcal{X})) = p \deg(W^*) + q \deg(V) = -(p + q) \deg(W).$$

If  $p \geq q$ , the Milnor-Wood inequality can be rewritten in this case as

$$|\deg(W)| \leq \frac{q}{2} \deg(\Omega_C^1).$$

This inequality is not too hard to prove. We recall the proof from [Mau15]. If  $\beta = 0$  then  $W$  is a Higgs subsheaf of  $E$  and by the polystability of  $E$  the degree of  $W$  is less or equal than zero. If  $\beta \neq 0$ , then

$$\deg(W) + q \deg(T_C^1) = \deg(\ker(\beta)) + \deg(\text{Im}(\beta)).$$

Since  $W \oplus \text{Im}(\beta)$  is a Higgs subsheaf of  $E$ , by polystability of  $E$  we get  $\deg(\text{Im}(\beta)) \leq -\deg(W)$ . Also  $\ker(\beta) \otimes \Omega_C^1$  is a Higgs subsheaf of  $E$ , and so

$$\deg(\ker(\beta)) + (q - \text{rk}(\beta)) \deg(\Omega_C^1) \leq 0.$$

Summing up we get the desired bound. The other inequality  $\deg(W) \geq -\frac{q}{2} \deg(\Omega_C^1)$  can be obtained in the same way using  $\gamma^t : W^* \otimes T_C^1 \rightarrow V^*$ .

*Remark 1.6.2.* If the the Higgs vector bundle  $(E, \theta)$  is a system of Hodge bundles, then the Hodge metric defines a structure of  $\text{SU}(p, q)$ -Higgs bundle. The vectors bundles  $V$  and  $W$  defined above are given by

$$V = \bigoplus_{i \equiv 0(2)} E^{i,j}, \quad W = \bigoplus_{i \equiv 1(2)} E^{i,j}.$$

For example if the system of Hodge bundles corresponds to a variation of Hodge structures of weight one, then  $V = E^{0,1}$ ,  $W = E^{1,0}$  and by Griffiths transversality  $\gamma = 0$ . The Milnor-Wood inequality is the same as the Arakelov inequality and maximality implies that the Higgs field is an isomorphism and the base curve is a Shimura curve (see [VZ04]). A similar result holds true for variation of Hodge structures of real K3 type, namely such that  $\text{rk}(E^{0,2}) = \text{rk}(E^{2,0}) = 1$ . Also in this case the Milnor-Wood inequality is the same as the Arakelov inequality. Maximal K3 variation of Hodge structures were described in [STZ03].

**1.6.2 Maximal representations and Hitchin components** In this section we recall the definition of special components of the character variety  $\text{Hom}(\pi_1(C), G)//G$  where  $C$  is a hyperbolic Riemann surface. They are examples of higher Teichmüller spaces, which means that they are connected components consisting of discrete and faithful representations. Moreover we describe some known invariants and recall how these invariants behave on these higher Teichmüller spaces.

Recall that if  $G$  is a split real simple Lie group, then there is an embedding  $i : \text{SL}_2(\mathbb{R}) \hookrightarrow G$  which is unique up to conjugation.

**Definition 1.6.3.** Let  $G$  be a split real simple Lie group. The *Hitchin component* of the character variety  $\text{Hom}(\pi_1(C), G)//G$  is the connected component containing a principal Fuchsian representation, i.e. a representation given as the composition of  $i$  with a Fuchsian representation of  $\pi_1(C)$  in  $\text{SL}_2(\mathbb{R})$ .

Representations in the Hitchin component are discrete and faithful. For  $G = \text{SL}_2(\mathbb{R})$  and  $G = \text{SL}_3(\mathbb{R})$  the Hitchin component parametrizes hyperbolic structures, resp. convex real projective structures on  $S$ . In [Hit92], Hitchin showed that the Hitchin component is homeomorphic to a vector space of dimension  $-\dim(G)\chi(C)$ . He proved this result by giving a parametrization of the Higgs bundles corresponding to representations in the Hitchin component. He proved that the Hitchin component can be parametrized by  $\bigoplus_{j=2}^n \text{H}^0(C, \mathcal{K}_C^j)$ . The embedded copy of Teichmüller space is given by setting all but the quadratic differentials equal to zero. The Higgs bundle associated to  $(q_j) \in \bigoplus_{j=2}^n \text{H}^0(C, \mathcal{K}_C^j)$  is given by

$$E = \mathcal{K}_C^{\frac{n-1}{2}} \oplus \mathcal{K}_C^{\frac{n-3}{2}} \oplus \cdots \oplus \mathcal{K}_C^{-\frac{n-3}{2}} \oplus \mathcal{K}_C^{-\frac{n-1}{2}}$$

with Higgs field  $\phi$  depends explicitly on the  $(q_j) \in \bigoplus_{j=2}^n \text{H}^0(C, \mathcal{K}_C^j)$ .

If  $G$  is a simple Lie group of Hermitian type, we defined the Toledo invariant associated to a reductive representation  $\rho : \pi_1(S) \rightarrow G$ , see Definition 1.6.1.

**Definition 1.6.4.** Let  $G$  be a simple Lie group of Hermitian type. The *space of maximal representations* is the subset of the character variety  $\text{Hom}(\pi_1(S), G)//G$  consisting of maximal representations, i.e. the space of representations whose Toledo invariant attains the maximal possible value with respect to the Milnor-Wood inequality.

The space of maximal representations is a union of connected components and it consists of faithful and discrete representations.

For  $G = \text{PSL}_2(\mathbb{R})$  the Hitchin component is the same as the space of maximal representations. It is Teichmüller space. For groups  $G$  which are both split and of Hermitian type not locally isomorphic to  $\text{PSL}_2(\mathbb{R})$  like  $\text{Sp}_{2n}(\mathbb{R})$ , the Hitchin component is a proper subset of the space of maximal representations.

Both maximal representations and representations in the Hitchin components are examples of Anosov representations. A representation is called Anosov if it satisfies special dynamical properties. It would be very interesting to understand if the Anosov property can be related to Lyapunov exponent, but we will not investigate this further here.

**1.6.3 Critical exponents, entropy and minimal area of a representation** In this section we want to describe invariants of the character variety, in particular of Hitchin components, that were investigated in the last years. Let  $S$  be a closed surface of genus at least two.

**Definition 1.6.5.** The *length spectrum* of a representation  $\rho : \pi_1(S) \rightarrow \mathrm{SL}_n(\mathbb{C})$  is the function

$$L_\rho : \pi_1(S) \rightarrow \mathbb{R}_+, \quad L_\rho(\gamma) = \frac{1}{2} \log \left| \frac{\lambda_1(\rho(\gamma))}{\lambda_n(\rho(\gamma))} \right|$$

where  $\lambda_1$  and  $\lambda_n$  denote the eigenvalues with highest and lowest modulus respectively.

If  $\rho$  is a Fuchsian representation,  $L_\rho(\gamma)$  is the length of the closed geodesic in  $S$  corresponding to  $\gamma$  with respect to the hyperbolic metric defined by  $\rho$ .

**Definition 1.6.6.** Let  $\rho : \pi_1(S) \rightarrow \mathrm{SL}_n(\mathbb{R})$  be a representation. The *entropy* of  $\rho$  is the number

$$h_{\mathrm{top}}(\rho) = \limsup_{t \rightarrow \infty} \frac{1}{t} \log \#\{\gamma \in \pi_1(S) : L_\rho(\gamma) < t\}.$$

The definition of the entropy is justified by the construction of Labourie [Lab08] where he identifies  $L_\rho(\gamma)$  as the period of the closed orbit corresponding to  $\gamma$  with respect to a flow  $(\phi_\rho)_t$ . The entropy of  $\rho$  is then the same as the topological entropy of the flow  $(\phi_\rho)_t$ .

Let us define another related invariant. Let  $\mathcal{X}$  be the symmetric space of  $\mathrm{SL}_n(\mathbb{R})$  and normalize the invariant metric  $d_{\mathcal{X}}$  so that the totally geodesic embedding of  $\mathbb{H}$  defined by  $\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_n(\mathbb{R})$  has curvature  $-1$ .

**Definition 1.6.7.** The *critical exponent* of a representation  $\rho : \pi_1(S) \rightarrow \mathrm{SL}_n(\mathbb{R})$  is defined by

$$h_{\mathcal{X}}(\rho) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \#\{\gamma \in \pi_1(S) : d_{\mathcal{X}}(o, \rho(\gamma)(o)) < t\}$$

for some (and hence any)  $o \in \mathcal{X}$ .

As explained in [Zha15], the relation between these two invariants is given by the equality

$$d_{\mathcal{X}}(o, \rho(\gamma)(o)) = \|\mu(\rho(\gamma))\| = c_n \sqrt{\sum_{i=1}^n \mu_i(\rho(\gamma))^2} \quad (1.6)$$

where  $\mu : \mathrm{SL}_n(\mathbb{R}) \rightarrow \mathfrak{a}^+$  is the Cartan projection,  $\mu_i$  are the singular values and  $c_n$  is a positive constant depending on  $n$ . Using the inequality

$$\|\mu(\rho(\gamma))\| \geq \frac{c_n}{n} |\mu_1(\rho(\gamma)) - \mu_n(\rho(\gamma))|$$

one then gets that for any  $\rho$  in the rank  $n$  Hitchin component

$$h_{\mathrm{top}}(\rho) \geq \frac{c_n}{n} h_{\mathcal{X}}(\rho).$$

The last invariant that we want to define is the minimal area of a representation.

**Definition 1.6.8.** The *minimal area* of a representation  $\rho : \pi_1(S) \rightarrow G$  into a semi-simple Lie group  $G$  is defined by

$$\text{MinArea}(\rho) = \inf_{J \in \mathcal{T}(S)} e_\rho(J), \quad e_\rho(J) = \int_{(S,J)} \|\mathrm{d}(f)\|$$

where  $e_\rho(J)$  is the energy of the unique harmonic  $\rho$ -equivariant map  $f$  from the Riemann surface  $(S, J)$  to  $\rho(\pi_1(S)) \backslash \mathcal{X}$ .

If  $\rho$  is in the Hitchin component then Sanders [San14] proved that harmonic equivariant maps  $f : (S, J) \rightarrow \rho(\pi_1(S)) \backslash \mathcal{X}$  are immersions and hence  $\text{MinArea}(\rho)$  can be interpreted as the area of the minimal immersion of  $(S, J)$  in  $\rho(\pi_1(S)) \backslash \mathcal{X}$ . It can be showed that the metric induced on this immersed surface is negatively curved and so the topological entropy is bounded above by  $h_{\mathcal{X}}(\rho)$ . By using a theorem of Katok relating the topological entropy to the area of the surface, we get

$$\text{MinArea}(\rho) \geq \frac{-2\pi\chi(S)}{h_{\mathcal{X}}(\rho)}.$$

In the case of  $G = \text{Sp}_{2n}(\mathbb{R})$ , Labourie [Lab08] proved that the minimal area can be also related to the Toledo invariant.

**Theorem 1.6.9** ([Lab08]). *For every representation  $\rho : \pi_1(S) \rightarrow \text{Sp}_{2n}(\mathbb{R})$  we have*

$$\text{MinArea}(\rho) \geq |\text{Tol}(\rho)|.$$

*Moreover if  $\rho$  is maximal and the last inequality is an equality, then  $\rho$  comes from the diagonal representation  $\prod_{i=1}^n \text{SL}_2(\mathbb{R}) \subset \text{Sp}_{2n}(\mathbb{R})$ .*

Finally we want to recall the main result of Poitrie and Sambarino [PS17], which is the following bound on the Hitchin component together with a characterization of symmetric powers of Fuchsian representations.

**Theorem 1.6.10** ([PS17]). *If  $\rho : \pi_1(S) \rightarrow \text{SL}_n(\mathbb{R})$  is the Hitchin component, then*

$$h_{\text{top}}(\rho) \leq \frac{2}{n-1}, \quad h_{\mathcal{X}}(\rho) \leq 1, \quad \text{MinArea}(\rho) \geq -2\pi\chi(S)$$

*with equalities if and only if  $\rho$  is the symmetric power of a Fuchsian representation.*

There are analogous results for example for maximal representations into  $\text{SO}_0(2, n)$  ([CTT17]).

**1.6.4 Geometric Oseledets Theorem** In this section we describe a geometric version of the Oseledets theorem due to Kaimanovich, which allows us to naturally talk about Lyapunov exponent associated to  $G$ -principal flat bundles. We present the version of the geometric Oseledets theorem given in [Fil17b]. Given a finite dimensional representation of  $G$ , passing to the associated vector bundle we retrieve the usual Oseledets theorem. We will also recall the main idea of [Fil17b], namely that we can give a bound on the number of zero exponent depending on the real rank of  $G$ , and we apply it to the case of variations of Hodge structures. We finally speculate over a potential application of the geometric Oseledets point of view, which we believe could lead to a possible relation between critical exponent and the top Lyapunov exponent.

Let  $G$  be a real semisimple Lie group with Lie algebra  $\mathfrak{g}$ . Let  $K$  be a maximal compact with Lie algebra  $\mathfrak{k}$ . Then we have a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . Choose a maximal abelian subalgebra  $\mathfrak{a} \subset \mathfrak{p}$  and consider the root system given by the weights  $\Phi_{\mathfrak{a}} \subset \mathfrak{a}^{\vee}$ . Let  $\mathcal{X} = G/K$  be the associated symmetric space. Recall that a geodesic in  $\mathcal{X}$  is uniquely determined by a choice of a vector in  $\mathfrak{a}$  and an element of  $K$ . Indeed, using the Iwasawa decomposition  $G = KAK$  one can show that a geodesic starting at  $e \in \mathcal{X}$  is given as

$$\gamma(t) = k \exp(t \cdot \alpha) e$$

where  $k \in K$  is the direction and  $\alpha \in \mathfrak{a}^+$  is in the positive Weyl chamber and can be considered as the speed. A geodesic has unit speed if  $\|\alpha\| = 1$ . We can now state Kaimanovich version of Oseledets theorem for principal  $G$ -flat bundles over a Riemann surface.

**Theorem 1.6.11** (Geometric Oseledets [Fil17b]). *Let  $C$  be a hyperbolic Riemann surface. Let  $G$  be a semisimple group and let  $P_G \rightarrow C$  be a principal  $G$ -flat bundle. Let  $G_t$  be the lift of the geodesic to the pull-back of  $P_G$  to  $T^1C$ . Let  $f : \mathbb{H} \rightarrow \mathcal{X}$  be a reduction of the structure group to  $K$ , which is the same as a section  $s : C \rightarrow P_G/K$  of the  $\mathcal{X}$ -fiber bundle, such that the following integrability condition holds:*

$$\int_C \left( \sup_{t \in [-1,1]} d_{\mathcal{X}}(s(g_t(x)), G_t(s(x))) \right) dx < \infty.$$

*Then there is a vector  $\Lambda \in \mathfrak{a}^+$ , called the Lyapunov vector, satisfying the following property. For almost any  $x \in C$ , let  $\gamma_x(t) = k_1 A_t k_2$  be the geodesic in  $\mathcal{X}$  with  $\gamma(0) = s(g_t(x))$  and  $\gamma(1) = G_t(s(x))$ , which is the same as the geodesic from  $f(\tilde{x})$  and  $f(g_t(\tilde{x}))$ , where  $\tilde{x} \in \mathbb{H}$  is a lift of  $x$ . Then*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log(A_t) = \Lambda.$$

*Moreover there is a convergence of directions in  $K$  which corresponds to the Oseledets decomposition.*

The main example to keep in mind in order to see the relation to the standard Oseledets theorem is  $G = \mathrm{GL}_n(\mathbb{R})$ . Then  $K = \mathrm{O}_n(\mathbb{R})$  and the associated symmetric space  $\mathcal{X}$  is the space of metrics coming from inner products on  $\mathbb{R}^n$ , with distinguished point  $eK$  given by the euclidean metric  $\|\cdot\|_e$ . Then a point  $gK \in \mathcal{X}$  corresponds to the metric

## 1 Lyapunov exponents on character varieties

$\|g^{-1}(\cdot)\|_e$ . The standard Oseledets theorem for a vector bundle  $E \rightarrow C$  is equivalent to the geometric version for the associated principal  $G$ -bundle given by the frame bundle (see for example [Fil17a]).

Conversely, given a finite dimensional representation  $\sigma : G \rightarrow \text{Aut}(V)$  we can read the Lyapunov exponents of the vector bundle  $P_G \times_\sigma V$  associated to the principal bundle  $P_G$  from a Lyapunov vector  $\Lambda \in \mathfrak{a}^+$  in the following way. Let  $\Sigma_\sigma \subset \mathfrak{h}^\vee$  be the weights of  $\sigma$ , where  $\mathfrak{h}$  is a maximal Cartan subalgebra of  $\mathfrak{g}$ . The Lyapunov exponent of the vector bundle  $P_G \times_\sigma V$ , which are the same as the Lyapunov exponents of the linear representation of  $\pi_1(C)$  given as the composition of  $\sigma$  with the representation defining the flat  $G$ -bundle  $P_G$ , are given by evaluation:

$$\Sigma_\sigma \hookrightarrow \mathfrak{h}^\vee \longrightarrow \mathfrak{a}^\vee \xrightarrow{\text{ev}_\Lambda} \mathbb{R}.$$

For example, if we consider  $G = \text{SL}_n(\mathbb{R})$  with the standard representation, the top Lyapunov exponent of a  $G$ -flat bundle is given by

$$\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \mu_1(f(g_t(\tilde{x})))$$

where  $f : \mathbb{H} \rightarrow \mathcal{X}$  is the unique harmonic equivariant map and  $\mu_n : G \rightarrow \mathfrak{a}^+ \rightarrow \mathbb{R}$  is the top singular value of the matrix given by the the Cartan projection (here we have chosen  $\tilde{x} \in \mathbb{H}$  so that  $f(\tilde{x}) = 0 \in \mathcal{X}$  is the point stabilized by  $K = \text{SO}_n$ ). Notice that, by equality (1.6)

$$\lambda_1 \leq c_n \lim_{t \rightarrow \infty} \frac{1}{t} d_{\mathcal{X}}(f(\tilde{x}), f(g_t(\tilde{x})))$$

where  $c_n$  was a positive constant depending only on  $n$ . We speculate that the right hand side can be related to the critical exponent for Hitchin representations via a inequality of the kind  $\lambda_1(\rho) \leq \frac{2}{h_{\text{top}}(\rho)}$ , but we have not yet investigated it further. Such a bound would yield immediately a lower bound the we conjecture later for the top Lyapunov exponent on Hitchin components using [PS17].

*Remark 1.6.12.* Geometric Oseledets theorem sets a bound to the number of zero Lyapunov exponents of a flat  $G$ -bundle depending only on  $G$  ([Fil17b, Cor.4.4]). The number of zero exponents of a flat  $G$ -bundle is greater or equal than the number of weights mapped to zero via the map  $\Sigma_\sigma \hookrightarrow \mathfrak{h}^\vee \longrightarrow \mathfrak{a}^\vee$ . For example if  $G = \text{SU}(p, q)$  acts with the standard representation on  $\mathbb{C}^{p+q}$ , any vector bundle coming from a  $G$ -flat bundle has at least  $|p - q|$  zero exponents ([Fil17b, Ex.4.5]). This example can be applied to complex variations of Hodge structures, since by definition the image of the monodromy representation is in  $\text{SU}(p, q)$  where  $p = \sum_{i \equiv 0(2)} h^{i,j}$  is the dimension of the even Hodge bundles and  $q = \sum_{i \equiv 1(2)} h^{i,j}$  is the dimension of the odd ones.

## 1.7 Lyapunov exponents and holomorphic subbundles

In this section we prove one of the main results, namely a refinement of the main theorem of [EKMZ18]. We will work in the general setting of non-necessarily compact base curve, since the results that we will show can be proven in the general case. Let then

$C = \mathbb{H}/\Gamma$  be a Riemann surface of finite area. We denote by  $\Delta := \overline{C} \setminus C$  the boundary points with respect to the smooth compactification of  $C$ . Let  $\rho : \pi_1(C) \rightarrow \mathrm{GL}_n(\mathbb{C})$  be a representation of the fundamental group. We denote by  $\mathbb{V}_\rho$  the local system on  $C$  induced by  $\rho$  and by  $\mathcal{V}_C := \mathbb{V}_\rho \otimes_{\mathbb{C}} \mathcal{O}_C$  the associated holomorphic bundle equipped with the flat Gauss-Manin connection. Recall that we say that  $\mathbb{V}_\rho$  has non-expanding cusp monodromies if the eigenvalues of  $\rho(\gamma)$  have absolute value one, for every simple loop  $\gamma$  around a cusp  $c \in \Delta$ .

**1.7.1 Parabolic bundles and Metric extensions** This section summarizes results and notions of [Sim88], [Sim90] and [EKMZ18]. We will introduce parabolic bundles and metric extensions, which are needed to treat the case of non compact base curves.

**Definition 1.7.1.** A *parabolic bundle*  $\mathcal{V}$  over  $\overline{C}$  is a holomorphic vector bundle together with a  $[0, 1)$ -filtration  $F \cdot \mathcal{V}_c$  on the fiber  $\mathcal{V}_c$

$$\mathcal{V}_c = \mathcal{V}_c^{\geq \alpha_1} \supsetneq \mathcal{V}_c^{\geq \alpha_2} \supsetneq \dots \supsetneq \mathcal{V}_c^{\geq \alpha_{n+1}} = \mathcal{V}_c^{\geq 1} = 0$$

for every  $c \in \Delta$ .

If we denote by  $0 \leq \alpha_1 < \alpha_2 < \dots < \alpha_n < \alpha_{n+1} = 1$  the weights of the filtration of a fiber  $\mathcal{V}_c$ , the filtered dimension of  $(\mathcal{V}_c, F \cdot)$  is defined as

$$\dim_{F \cdot}(\mathcal{V}_c) = \sum_{i=1}^n \alpha_i \dim \mathrm{gr}_{\alpha_i}(\mathcal{V}_c)$$

where  $\mathrm{gr}_{\alpha_i}(\mathcal{V}_c)$  is the graded piece at weight  $\alpha_i$ .

**Definition 1.7.2.** The *parabolic degree* of a parabolic bundle  $(\mathcal{V}, F \cdot)$  is defined to be

$$\mathrm{deg}_{\mathrm{par}}(\mathcal{V}, F \cdot) = \mathrm{deg}(\mathcal{V}) + \sum_{c \in \Delta} \dim_{F \cdot} \mathcal{V}_c.$$

Following [EKMZ18], we define acceptable metrics. This notion is useful in order to compute parabolic degrees of parabolic bundles.

**Definition 1.7.3.** A smooth metric  $h$  on a holomorphic vector bundle  $\mathcal{V}_C$  over  $C$  is called *acceptable* if the curvature  $R_h$  of the metric admits locally near every cusp  $c \in \Delta$ , a bound

$$|R_h| \leq f + \frac{M}{|q|^2 |\log(q)|^2}$$

with  $f \in L^p(C)$  for some  $p > 1$  and some constant  $M$ .

When we consider a holomorphic vector bundle  $\mathcal{V}_C$  over  $C$  equipped with a smooth metric  $h$ , we can talk about a canonical metric extension of  $\mathcal{V}_C$  on  $\overline{C}$ , which in general is just a coherent sheaf.

Let  $j : C \hookrightarrow \overline{C}$  be the inclusion.

**Definition 1.7.4.** The *metric extension*  $\Xi_h(\mathcal{V}_C)$  of  $\mathcal{V}_C$  to  $\overline{C}$  with respect to the metric  $h$  is given by the subsheaf of  $j_*(\mathcal{V}_C)$  defined by following growth condition. If  $s(q)$  is a local section of  $j_*(\mathcal{V}_C)$  over  $U \subset \overline{C}$  around a cusp  $c \in U \cap \Delta$ , we set  $s(q) \in \Xi_h(\mathcal{V}_C)(U)$  if for all  $\epsilon > 0$  there exists a constant  $M(\epsilon)$  such that  $|s(q)|_h \leq M(\epsilon)|q|^{-\epsilon}$ .

The parabolic structure of  $\Xi_h(\mathcal{V}_C)$  over the cusps is given by the following filtration. If  $s(q)$  is a local section of  $\Xi_h(\mathcal{V}_C)$  around a cusp  $c \in \Delta$ , we set  $s \in (\Xi_h(\mathcal{V}_C))_c^{\geq \alpha}$  if for all  $\epsilon > 0$  there exists a constant  $M(\epsilon)$  such that

$$|s(q)|_h \leq M(\epsilon)|q|^{\alpha-\epsilon}.$$

We want now to generalize the definition of acceptable metric to parabolic bundles over  $\overline{C}$ .

**Definition 1.7.5.** A smooth metric  $h$  on a parabolic vector bundle  $(\mathcal{V}, F^\cdot)$  over  $\overline{C}$  is called *acceptable* if  $h$  is an acceptable metric for the holomorphic bundle  $\mathcal{V}_C$  and  $(\mathcal{V}, F^\cdot) = \Xi_h(\mathcal{V})$ .

We recall now a result which allows us to compute the parabolic degree of a parabolic vector bundle using any acceptable metric.

**Proposition 1.7.6** ([EKMZ18]). *If  $(\mathcal{V}, F^\cdot)$  is a parabolic vector bundle over  $\overline{C}$  of rank  $k$ , then*

$$\deg_{\text{par}}(\mathcal{V}) = \deg_{\text{par}}(\wedge^k \mathcal{V}).$$

Moreover if  $h$  is an acceptable metric, then

$$\deg_{\text{par}}(\mathcal{V}, F^\cdot) = \frac{1}{2\pi i} \int_C \partial \bar{\partial} \log(\det(h_{ij}))$$

where  $h_{ij} = h(e_i, e_j)$  is the Gram matrix of the metric.

Finally we can define the notion of admissible metric on a holomorphic flat bundle over  $C$ . Admissible metrics are the ones that can be used to compute Lyapunov exponents.

For any flat holomorphic vector bundle over  $C$  there is a canonical extension, which is called the Deligne extension. It is a holomorphic vector bundle  $\mathcal{V}$  on  $\overline{C}$  with a logarithmic connection  $\nabla : \mathcal{V} \rightarrow \mathcal{V}_\rho \otimes \Omega_{\overline{C}}^1(\log(\Delta))$ . Note that for holomorphic flat bundles  $(\mathcal{V}_C, \nabla)$  over  $C$  with non-expanding cusp monodromies the Deligne extension has a canonical parabolic structure (see [EKMZ18]).

**Definition 1.7.7.** A smooth metric  $h$  on the holomorphic flat bundle  $\mathcal{V}_C$  over  $C$  is called *admissible* if the following conditions hold:

1. The metric  $h$  is acceptable for the Deligne extension  $\mathcal{V}$  of  $\mathcal{V}_C$  with respect to its canonical parabolic structure.
2. For every cusp  $c \in \Delta$  with coordinate  $q$ , there is some  $n \in \mathbb{N}$  such that for any  $e \in \mathcal{V}_c^{\geq \alpha}$  and  $e' \in \mathcal{V}_c^{\geq \alpha'}$  it holds

$$h(e(q), e'(q)) \leq M_1 |q|^{\alpha+\alpha'} (\log |q|)^{2n}, \quad \text{for some } M_1 > 0.$$



3. For every cusp  $c \in \Delta$  with coordinate  $q$ , there is some  $n \in \mathbb{N}$  such that a local generating section  $e$  of  $\det(\mathcal{V})$  has the lower bound

$$|e(q)|_h \geq M_2 |q|^{2 \dim_F(\mathcal{V}_c)} (\log |q|)^{-2n}, \quad \text{for some } M_2 > 0.$$

We want to highlight that the first condition of the above definitions simply says that for an admissible metric the Deligne extension is the same as the metric extension and that the curvature does not grow too fast around the cusps.

We recall the existence lemma for such metrics based on the result by Simpson [Sim90, Theorem 4].

**Lemma 1.7.8** ([EKMZ18]). *If a holomorphic flat bundle  $(\mathcal{V}_C, \nabla)$  has non-expanding cusp monodromies, then it admits an admissible metric.*

We say that a norm on  $\mathcal{V}_C$  is admissible if it is induced by an admissible metric. Recall that if  $\mathcal{V}_C$  has non-expanding cusp monodromies, by Theorem 1.1.3 the constant norm is integrable in the sense of Oseledets theorem. In this case the same property holds for admissible norms.

**Theorem 1.7.9** ([EKMZ18]). *If a holomorphic flat bundle  $(\mathcal{V}_C, \nabla)$  has non-expanding cusp monodromies, then any admissible norm is integrable in the sense of Oseledets theorem.*

Admissible norms can then be used to compute Lyapunov exponents.

**1.7.2 Holomorphic subbundles, bad locus and main inequality** In this section we will refine the following main bound computed by Eskin-Kontsevich-Möller-Zorich in [EKMZ18].

**Theorem 1.7.10** ([EKMZ18]). *Let  $(\mathcal{V}_C, \nabla)$  be a holomorphic flat bundle with non-expanding cusp monodromies. If  $\mathcal{E} \subset \mathcal{V}$  is a holomorphic parabolic subbundle of rank  $k$  of the Deligne extension  $\mathcal{V}$  of  $\mathcal{V}_C$ , then*

$$\sum_{i=1}^k \lambda_i(\mathcal{V}_C) \geq \frac{2 \deg_{\text{par}}(\mathcal{E})}{\deg(\Omega_C^1(\log(\Delta)))}. \quad (1.7)$$

In order to state our refinement of inequality (1.7), we need to recall the definition of a seminorm on  $\bigwedge^k \mathcal{V}_C^\vee$  introduced in [EKMZ18] in the proof of Theorem 1.7.10.

Let  $h$  be an admissible metric for  $\mathcal{V}_C$ . By abuse of notation, we will denote also by  $h$  the induced admissible metric on dual exterior powers of  $\mathcal{V}_C$ . We denote by  $|\cdot|_h$  the induced norm. Following [EKMZ18], we define a seminorm on  $\bigwedge^k \mathcal{V}_C^\vee$ . For every point  $c \in C$ , consider a small open set  $U_c \subset C$  and fix a local basis  $\omega_1, \dots, \omega_k$  of  $H^0(U_c, \mathcal{E})$ . We define locally the seminorm on  $\bigwedge^k \mathcal{V}_C^\vee$  as

$$\|u\|_{\mathcal{E}} = \frac{|u(\omega_1(c) \wedge \dots \wedge \omega_k(c))|}{\|\omega_1(c) \wedge \dots \wedge \omega_k(c)\|_h}, \quad u \in \bigwedge^k \mathcal{V}_c^\vee \cong \left( \bigwedge^k \mathcal{V}_c \right)^\vee. \quad (1.8)$$

## 1 Lyapunov exponents on character varieties

The seminorm does not depend on the choice of the local frame  $(\omega_i)$  of  $\mathcal{E}$  since numerator and denominator are homogenous of the same degree in  $\omega_i$ 's.

The subvector bundle defined by the zero locus of the seminorm will appear in the refined inequality that we want to prove. In this regard, let us pull-back the vector bundles to the universal covering  $\pi : \mathbb{H} \rightarrow C$  of  $C$ . Since any local system on a simply connected space is trivial, the pull-back of the flat bundle  $\mathcal{V}_C$  is trivial. Let us fix  $c \in C$  and the isomorphism

$$\pi^*(\mathcal{V}_C) \cong \mathbb{H} \times \mathcal{V}_c.$$

The pull-back  $\pi^*(\mathcal{E}|_C) \subset \pi^*(\mathcal{V}_C)$  defines a  $\rho_{\mathcal{V}_C}$ -equivariant subbundle, where  $\rho_{\mathcal{V}_C}$  is the representation associated to the flat bundle  $\mathcal{V}_C$ . This means that

$$\pi^*(\mathcal{E}|_C)_{\gamma(c)} = \{\gamma(c)\} \times \rho_{\mathcal{V}_C}(\gamma)(\mathcal{E}_c) \subset \mathbb{H} \times \mathcal{V}_c \quad (1.9)$$

for every  $\gamma \in \pi_1(C, c)$ .

**Definition 1.7.11.** We define the 'trouble making set', or bad locus,

$$\mathrm{T}_{\mathrm{bad}}^{\mathcal{E}} := \ker \left( \bigwedge^k \mathcal{V}_C^{\vee} \rightarrow \bigwedge^k \mathcal{E}|_C^{\vee} \right) \subseteq \bigwedge^k \mathcal{V}_C^{\vee}$$

as the kernel of the map induced by the inclusion  $\mathcal{E} \subseteq \mathcal{V}$ .

Moreover, for any  $u \in \bigwedge^k \mathcal{V}_c^{\vee} = H^0(\mathbb{H}, \pi^*(\bigwedge^k \mathcal{V}_C^{\vee}))$ , the 'trouble making set' associated to  $u$ , or bad locus of  $u$ , is defined as

$$\mathrm{T}_{\mathrm{bad}}^{\mathcal{E}}(u) := \{z \in \mathbb{H} : \|u_z\|_{\pi^*(\mathcal{E})} = 0\} \subseteq \mathbb{H}.$$

Notice that  $\mathrm{T}_{\mathrm{bad}}^{\mathcal{E}} \subseteq \bigwedge^k \mathcal{V}_C^{\vee}$  is a holomorphic subbundle of corank 1 and  $\mathrm{T}_{\mathrm{bad}}^{\mathcal{E}}(u)$  can be identified with the intersection of the pull-back  $\pi^*(\mathrm{T}_{\mathrm{bad}}^{\mathcal{E}})$  to  $\mathbb{H}$  with the horizontal foliation induced by the flat bundle  $\bigwedge^k \mathcal{V}_C^{\vee}$ .

*Remark 1.7.12.* To give an inclusion of a rank  $k$  holomorphic sub-vector bundle  $\mathcal{E}|_C \subset \mathcal{V}_C$  over  $C$  is equivalent to give a section

$$s_{\mathcal{E}} : C \rightarrow \mathcal{G}_k(\mathcal{V}_C)$$

of the Grassmanian bundle  $\mathcal{G}_k(\mathcal{V}_C)$  of  $k$ -planes of  $\mathcal{V}_C$ . Let us fix as above  $c \in C$  and the isomorphism  $\pi^*(\mathcal{V}_C) \cong \mathbb{H} \times \mathcal{V}_c$ . Then the pull-back section  $\pi^*(s_{\mathcal{E}})$  defines a  $\rho_{\mathcal{V}_C}$ -equivariant holomorphic map

$$s_{\mathcal{E}} : \mathbb{H} \rightarrow \mathrm{Grass}(k, \mathcal{V}_c) \hookrightarrow \mathbb{P}(\bigwedge^k \mathcal{V}_c), \quad s_{\mathcal{E}}(z) = (\pi^*(\mathcal{E})_z \subset \mathcal{V}_c).$$

The equivariance property is defined by the equality

$$s_{\mathcal{E}}(\gamma \cdot z) = \rho_{\mathcal{V}_C}(\gamma) \cdot s_{\mathcal{E}}(z), \quad \forall \gamma \in \pi_1(C, c).$$

For any  $u \in \bigwedge^k \mathcal{V}_c^{\vee} \cong \left(\bigwedge^k \mathcal{V}_c\right)^{\vee}$ , we can now rewrite its bad locus as

$$\mathrm{T}_{\mathrm{bad}}^{\mathcal{E}}(u) = \{z \in \mathbb{H} : s_{\mathcal{E}}(z) \in \ker(u) \subset \mathbb{P}(\bigwedge^k \mathcal{V}_c)\}. \quad (1.10)$$

From this description it is clear that either  $T_{\text{bad}}(u) = \mathbb{H}$  if  $\ker(u) \supseteq \text{Im}(s_{\mathcal{E}})$  or it is a discrete subset given as the zero set of an holomorphic (non  $\rho_{\mathcal{V}_C}$ -equivariant) function on  $\mathbb{H}$ .

Notice that parallel transport on the trivial bundle given by the pull-back of  $\mathcal{V}^\vee$  to  $T^1\mathbb{H}$  is simply given, after the choice of a trivialization, by the constant transport

$$G_t : T^1\mathbb{H} \times \mathcal{V}_c^\vee \rightarrow T^1\mathbb{H} \times \mathcal{V}_c^\vee, \quad G_t(x, u) = (g_t(x), u).$$

From now on we fix a choice of a trivialization and we denote a point in the pull-back bundle by  $u_z := (z, u) \in T^1\mathbb{H} \times \mathcal{V}_c^\vee$  and a lift of  $c \in C$  by  $\tilde{c} \in \mathbb{H}$ .

We will state now a refinement of the main Theorem of [EKMZ18]. Let the notation be as above. In particular  $\mathcal{V}_C$  is a holomorphic flat bundle over  $C$  defined by a representation  $\rho_{\mathcal{V}_C}$ .

**Theorem 1.7.13.** *For any holomorphic subbundle  $\mathcal{E} \subset \mathcal{V}_C$  of rank  $k$  over  $C$ , then*

$$\sum_{i=1}^k \lambda_i(\mathcal{V}_C) \geq \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_C^1(\log(\Delta)))} + 4\pi \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\#\{T_{\text{bad}}(u) \cap D_t(\tilde{c})\}}{\text{vol}(D_t(\tilde{c}))} dt \quad (1.11)$$

for almost any  $c \in C$  and Lebesgue almost any  $u \in \bigwedge^k \mathcal{V}_c^\vee$ . Here  $D_t(\tilde{c})$  denotes the hyperbolic ball of radius  $t$  in  $\mathbb{H}$  with center  $\tilde{c}$ .

*Proof.* First of all note that it suffices to prove the theorem in the case where  $\mathcal{E}$  is a line bundle. Indeed if it is not the case, consider the line bundle  $\mathcal{L} := \bigwedge^k \mathcal{E} \subset \bigwedge^k \mathcal{V}_C$ . Then  $\deg_{\text{par}}(\mathcal{L}) = \deg_{\text{par}}(\mathcal{E})$  by Lemma 1.7.6 and the top Lyapunov exponent of  $\mathcal{L}$  is just the sum  $\sum_{i=1}^k \lambda_i(\mathcal{V}_C)$  of the first  $k$  exponents. Hence from now on  $\mathcal{E} = \mathcal{L}$  is a sub-line bundle of  $\mathcal{V}_C$ . Moreover, since the Lyapunov spectrum is symmetric (see Remark 1.1.2), the Lyapunov spectrum of the dual local system  $\mathbb{V}^\vee$  is the same as the one of  $\mathbb{V}$ . We will then focus on computing the top Lyapunov exponent  $\lambda_1(\mathcal{V}_C^\vee) = \lambda_1(\mathcal{V}_C)$ .

Note that the Cauchy-Schwartz inequality implies that the admissible norm  $\|\cdot\|_h$  induced by  $h$  is greater or equal than the  $\mathcal{L}$ -seminorm  $\|\cdot\|_{\mathcal{L}}$  defined in (1.8). Indeed, for any  $c \in C$  and any  $u \in \mathcal{V}_c^\vee$  it holds

$$\|u\|_{\mathcal{L}} = \frac{|u(\omega_c)|}{\|\omega_c\|_h} \leq \frac{\|u\|_h \|\omega_c\|_h}{\|\omega_c\|_h} = \|u\|_h \quad (1.12)$$

where  $\omega$  is a local non-zero section of  $\mathcal{L}$  near  $c \in C$ .

By Theorem 1.7.9, the norm induced by the admissible metric  $h$  is integrable, meaning that it computes the Lyapunov exponents. We can then write the top Lyapunov exponent as

$$\lambda_1(\mathcal{V}_C) = \lim_{t \rightarrow \infty} \frac{1}{t} (\log \|G_t(u)\|_h)$$

for almost any  $(c, v) \in T^1(C)$  and Lebesgue almost any  $u \in T^1\mathcal{V}_{(c,v)}^\vee$ . Here we denoted by  $T^1\mathcal{V}^\vee$  the pull-back of  $\mathcal{V}^\vee$  to  $T^1(C)$ .

We apply the usual chain of equalities as in [EKZ11] or [EKMZ18] to rewrite the expression

above. We first average over the circle and then use the above Cauchy–Schwarz inequality (1.12). After that we take the integral of the derivative and then we apply a version of Green’s formula ([EKZ11, Lemma 3.6]) for the hyperbolic disc  $D_t(\tilde{c})$  centered in  $\tilde{c} \in \mathbb{H}$  with hyperbolic radius  $t$  (here the term  $\log \|G_{Tr\theta}u\|_{\mathcal{L}}$  is considered in the distributional sense). Finally we split the integral using the definition of the  $\|\cdot\|_{\mathcal{L}}$ -seminorm and rewrite directly the second term of the expression in terms of the degree of  $\mathcal{L}$  as in [EKMZ18].

$$\begin{aligned}
\lambda_1(\mathcal{V}_C) &= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{1}{2\pi} \int_0^{2\pi} \log \|G_{Tr\theta}u\|_h d\theta \geq \lim_{T \rightarrow \infty} \frac{1}{T} \frac{1}{2\pi} \int_0^{2\pi} \log \|G_{Tr\theta}u\|_{\mathcal{L}} d\theta \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \frac{1}{2\pi} \int_0^T \frac{d}{dt} \int_0^{2\pi} \log \|G_{Tr\theta}u\|_{\mathcal{L}} d\theta \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{2 \operatorname{vol}(D_t(\tilde{c}))} \int_{D_t(\tilde{c})} \Delta_{\text{hyp}} \log \|u_z\|_{\mathcal{L}} dg_{\text{hyp}}(z) dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{2 \operatorname{vol}(D_t(\tilde{c}))} \int_{D_t(\tilde{c})} \Delta_{\text{hyp}} \log |u_z(\omega_z)| dg_{\text{hyp}}(z) dt + \\
&\quad - \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{2 \operatorname{vol}(D_t(\tilde{c}))} \int_{D_t(\tilde{c})} \Delta_{\text{hyp}} \log \|\omega_z\|_h dg_{\text{hyp}}(z) dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{2 \operatorname{vol}(D_t(\tilde{c}))} \int_{D_t(\tilde{c})} \Delta_{\text{hyp}} \log |u_z(\omega_z)| dg_{\text{hyp}}(z) dt + \\
&\quad + \frac{2 \operatorname{deg}_{\text{par}}(\Xi_h(\mathcal{L}))}{\operatorname{deg}(\Omega_C^1(\log(\Delta)))}
\end{aligned}$$

Note that we could split the log only in the fourth line, since the Laplacian makes the numerator and the denominator of the  $\mathcal{L}$ -norm become well-defined functions.

We need to treat the first summand. We write explicitly the hyperbolic Laplacian and the hyperbolic area form and simplify. We then use that the integral over the ball of the distribution  $\bar{\partial}\partial \log(|u_z(\omega_z)|)$  gives the number of zeros of the holomorphic function  $u_z(\omega_z)$  inside the ball times  $2\pi i$  (cf. [GH94, Poincaré-Lelong Equation]). The last equality follows since  $\tanh(t)$  is bounded and asymptotic to 1 for large  $t$ .

$$\begin{aligned}
&\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{2 \operatorname{vol}(D_t(\tilde{c}))} \int_{D_t(\tilde{c})} \Delta_{\text{hyp}} \log |u_z(\omega_z)| dg_{\text{hyp}}(z) dt \\
&= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{2 \operatorname{vol}(D_t(\tilde{c}))} \left( \int_{D_t(\tilde{c})} 4 \frac{\partial^2}{\partial z \partial \bar{z}} \log |u_z(\omega_z)| \frac{i}{2} |dz|^2 \right) dt \\
&= -2i \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{\operatorname{vol}(D_t(\tilde{c}))} \left( \int_{D_t(\tilde{c})} \bar{\partial}\partial \log |u_z(\omega_z)| \right) dt \\
&= 4\pi \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\tanh(t)}{\operatorname{vol}(D_t(\tilde{c}))} \#\{z \in D_t(\tilde{c}) : u_z(\omega_z) = 0\} dt \\
&= 4\pi \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\#\{z \in D_t(\tilde{c}) : u_z(\omega_z) = 0\}}{\operatorname{vol}(D_t(\tilde{c}))} dt
\end{aligned}$$

□

**Definition 1.7.14.** We define the second term in formula 1.11 as the *error term*

$$\text{Err}^{\mathcal{E}}(u) := 4\pi \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\#\{\mathbb{T}_{\text{bad}}^{\mathcal{E}}(u) \cap D_t(\tilde{c})\}}{\text{vol}(D_t(\tilde{c}))} dt$$

for  $u \in \bigwedge^k \mathcal{V}_c^{\vee}$ .

*Remark 1.7.15.* If the limit  $\lim_{t \rightarrow \infty} \frac{\#\{\mathbb{T}_{\text{bad}}^{\mathcal{E}}(u) \cap D_t(\tilde{c})\}}{\text{vol}(D_t(\tilde{c}))}$  exists, then the error term is equal to this limit. We conjecture that this is the case. Note moreover that  $\text{Err}^{\mathcal{E}}(u) = \text{Err}^{\mathcal{E}}(\lambda u)$ , for any  $\lambda \in \mathbb{C}^*$ . Hence the error term defines a function

$$\text{Err}^{\mathcal{E}} : \mathbb{P}\left(\bigwedge^k \mathcal{V}_C^{\vee}\right) \rightarrow \mathbb{R}^+.$$

**1.7.3 Higher dimensional analogues** Using the same argument as in the proof of Theorem 1.7.13, one can prove analogous statements in the case where the base manifold is a ball quotient or an affine invariant manifold of a stratum of abelian differentials. In the first case, since ball quotients are locally symmetric spaces of rank 1, the geodesic flow is ergodic and so the Oseledets multiplicative ergodic theorem can be applied. In the second case there is a natural  $\text{SL}_2(\mathbb{R})$  ergodic action on affine invariant manifolds. We omit the details of the proofs since the computations are analogous to the the ones of the last theorem.

The next Proposition is a generalization of the main result of [KM16].

**Proposition 1.7.16.** *Let  $\mathcal{V}$  be a weight one variation of Hodge structures over a ball quotient  $B = \mathbb{B}^n/\Gamma$  of constant curvature  $-4$ , where  $\Gamma$  is a torsionfree lattice in  $\text{PU}(1, n)$ . Let  $\bar{B}$  be a smooth compactification of  $B$  with normal crossing boundary divisor  $\Delta$ . Let  $\mathcal{E} \subset \mathcal{V}$  be a holomorphic sub-vector bundle of rank  $k$ . Then*

$$\sum_{i=1}^k \lambda_i \geq \frac{(n+1)c_1(\Xi_h(\mathcal{E})) \cdot c_1(\omega_{\bar{B}})^{n-1}}{c_1(\omega_{\bar{B}})^n} + \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_{\mathbb{B}_t^n(\tilde{c})} \partial\bar{\partial}(\log |u(s_z)|) \wedge \omega_{\text{hyp}}^{n-1} dg_{\text{hyp}}(z) dt$$

for almost any  $c \in B$  and Lebesgue almost any vector  $u \in \bigwedge^k \mathcal{V}_c^{\vee}$ . Here  $s$  is a local generator of  $\bigwedge^k \mathcal{E}$  and  $\mathbb{B}_t^n(\tilde{c})$  is the hyperbolic ball of radius  $t$  around the lift  $\tilde{c} \in \mathbb{B}^n$  of  $c \in B$ . Finally  $\omega_{\bar{B}} = \bigwedge^n \Omega_{\bar{B}}^1(\log(\Delta))$  is the log-canonical bundle and  $\Xi_h(\mathcal{E})$  is the metric extension of  $\mathcal{E}$  with respect to the Hodge metric  $h$ .

In the last proposition we only considered weight one variation of Hodge structures since the integrability of the Hodge norm was proven in this case in [KM16] using the geometry of the period domain and Royden's theorem. It is worth investigating if there is a notion of admissible norm for any flat vector bundle with non expanding monodromy around the boundary components and if this norm is integrable in the Oseledets sense.

The next proposition is about the case of affine invariant manifolds. It is a generalization of [EKZ11].

**Proposition 1.7.17.** *Let  $\mathcal{M}_1$  be an affine invariant manifold in some stratum of abelian differentials. Let  $\mathcal{H}$  be the Hodge bundle and  $\mathcal{E} \subset \mathcal{H}$  be a holomorphic sub-vector bundle of rank  $k$ . Then the sum of the top  $k$  Lyapunov exponents associated to the ergodic probability measure  $\nu_1$  corresponding to  $\mathcal{M}_1$  satisfy the bound:*

$$\sum_{i=1}^k \lambda_i \geq \int_{\mathcal{M}_1} \Delta(\log \|\omega\|_h) d\nu_1 + \text{Err}^{\mathcal{E}}(u)$$

where the Laplacian is the leafwise Laplacian along Teichmueller disks and  $\|\omega\|_h$  is the Hodge norm of a local section  $\omega$  of  $\bigwedge^k \mathcal{E}$ . Finally the error term is considered along the Teichmueller disk passing through the base point of  $u$ , for almost any  $c \in \mathcal{M}_1$  and all  $u \in \bigwedge^k \mathcal{H}_c^\vee$ .

Notice that the error term in the last proposition only depends on the Teichmueller disk passing through the base point of  $u$  and the restriction of  $\mathcal{E}$  to this Teichmueller disk. Since the error term  $\text{Err}^{\mathcal{E}}(u)$  is an almost everywhere constant function in  $u \in \bigwedge^k \mathcal{H}^\vee$ , the proposition implies that if the error term is zero for an affine invariant manifold  $\mathcal{M}_1$ , then it has to be zero for all generic affine invariant submanifolds of  $\mathcal{M}_1$ .

**1.7.4 Condition for rationality of Lyapunov exponents** We want now to state a sufficient condition for the sum of the top Lyapunov exponents being equal to the first term of inequality (1.11). This gives in particular a sufficient condition for the sum of the top Lyapunov exponents to be rational.

**Proposition 1.7.18.** *Let  $\mathcal{S} \subset \mathbb{P}(\bigwedge^k \mathcal{V}^\vee)$  be a  $G_t$ -invariant closed subset such that there is a vector  $u \in \mathcal{S}$  computing the top Lyapunov exponents, namely such that  $\sum_{i=1}^k \lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} (\log \|G_t(u)\|_h)$ . If there is a rank  $k$  holomorphic subbundle  $\mathcal{E} \subset \mathcal{V}$  such that*

$$\text{T}_{\text{bad}}^{\mathcal{E}}(u) = \emptyset, \quad \text{for all } u \in \mathcal{S}$$

then

$$\sum_{i=0}^k \lambda_i = \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_C^1(\log(\Delta)))}.$$

*Remark 1.7.19.* This last proposition is the analogous of [DD17a, Prop. 3.15] in which they require a strong irreducibility property of the flat bundle in order to have the right harmonic measure. We do not need any irreducibility property, but the drawback is that we need the existence of a vector computing the sum of the top exponents. In [DD17a] they do not need this assumption since for any closed  $\mathcal{S}$  there is always a harmonic measure with support in  $\mathcal{S}$ .

Note that if the main inequality (1.11) of Theorem 1.7.13 were an equality, we would not need the existence of the additional subbundle  $\mathcal{S}$  but only the existence of a vector  $u \in \bigwedge^k \mathcal{V}^\vee$  computing the top Lyapunov exponents with  $\text{T}_{\text{bad}}^{\mathcal{E}}(u) = \emptyset$ . Indeed in this case the error term  $\text{Err}^{\mathcal{E}}(u)$  would be zero and this would suffice. Since we will prove that over compact base curve (1.11) is an equality (Theorem 1.8.1), we can apply the previous argument to this situation (Corollary 1.8.2).

*Remark 1.7.20.* Consider the Grassmanian bundle  $\text{Gr}(n-k, \mathcal{V})$  of  $(n-k)$ -planes in  $\mathcal{V}$  as a subset of  $\mathbb{P}(\bigwedge^k \mathcal{V}^\vee)$  via the Plucker embedding  $\text{Gr}(n-k, \mathcal{V}) \subset \mathbb{P}(\bigwedge^{n-k} \mathcal{V})$  and the isomorphism  $\mathbb{P}(\bigwedge^k \mathcal{V}^\vee) \cong \mathbb{P}(\bigwedge^{n-k} \mathcal{V})$ . Then the condition that a  $(n-k)$ -plane  $u \in \text{Gr}(n-k, \mathcal{V})$  has empty bad locus, i.e.  $T_{\text{bad}}^\mathcal{E}(u) = \emptyset$ , is equivalent to the condition that the  $(n-k)$ -plane in  $\mathcal{V}$  represented by  $u$  intersects trivially the  $k$ -plane defined by the subbundle  $\mathcal{E}$ . We will indeed use this criterion to reprove rationality of Lyapunov exponents for weight 1 and K3 variation of Hodge structures in Section 1.11. The  $(n-k)$ -plane computing the top Lyapunov exponents will be constructed from the Oseledets subspaces  $\mathcal{V}_{\lambda_i}$ .

*Proof of Proposition 1.7.18.* We want to prove that if the bad locus is empty for all  $u \in \mathcal{S}$ , then we can use the  $\mathcal{E}$ -norm to compute Lyapunov exponents.

The argument is the standard one relying on the equivalence of any two norms on a finite dimensional vector space. If the bad locus  $T_{\text{bad}}^\mathcal{E}(u) = \emptyset$  is empty for all  $u \in \mathcal{S}$ , it means that the  $\mathcal{E}$ -norm  $\|\cdot\|_\mathcal{E}$  is a norm on the  $G_t$ -invariant closed subset  $\mathcal{S} \subset \mathbb{P}(\bigwedge^k \mathcal{V}^\vee)$ . Let  $K \subset T^1C$  be a compact positive measure set. Then  $\mathcal{S}|_K \subset \mathbb{P}(\bigwedge^k \mathcal{V}^\vee)$  is a compact subset and the quotient of the norms  $\|\cdot\|_\mathcal{E}$  and  $\|\cdot\|_h$  defines a bounded function on  $\mathcal{S}|_K$  with minimum greater than zero. This means that there exist two positive constants  $C_1$  and  $C_2$  such that

$$C_1 \|u\|_h \leq \|u\|_\mathcal{E} \leq C_2 \|u\|_h, \quad \forall u \in \mathcal{S}_K.$$

Now by Poincaré recurrence Theorem, the geodesic flow on  $T^1C$  comes back infinitely many times to  $K$  since it has positive measure. Moreover  $\mathcal{S}$  is  $G_t$ -invariant by assumption. Let  $t_j$  be a sequence of times tending to infinity for which  $g_{t_j}(c) \in K$ . Now let  $u \in \mathcal{S}$  be the vector computing the top Lyapunov exponents, which exists by assumption. We get then

$$\begin{aligned} \sum_{i=1}^k \lambda_i &= \lim_{t \rightarrow \infty} \frac{1}{t} (\log \|G_t(u)\|_h) = \lim_{t_j \rightarrow \infty} \frac{1}{t_j} (C_1 \log \|G_{t_j}(u)\|_h) \leq \\ &\leq \lim_{t_j \rightarrow \infty} \frac{1}{t_j} (\log \|G_{t_j}(u)\|_\mathcal{E}) \leq \lim_{t_j \rightarrow \infty} \frac{1}{t_j} (C_2 \log \|G_{t_j}(u)\|_h) = \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} (\log \|G_t(u)\|_h) = \sum_{i=1}^k \lambda_i. \end{aligned}$$

By following the proof of Theorem 1.7.13, we see that if the error term is computed with respect to the vector  $u$  used above the inequality (1.11) becomes an equality. The claim then follows directly from it.  $\square$

The condition to have empty bad locus  $T_{\text{bad}}^\mathcal{E}(u)$  for  $u \in \bigwedge^k \mathcal{V}^\vee$  can be rephrased using the equivalent definition of  $T_{\text{bad}}^\mathcal{E}(u)$  given by expression (1.10) via the equivariant map  $s_\mathcal{E} : \mathbb{H} \rightarrow \mathbb{P}(\bigwedge^k \mathcal{V}_c)$  defining  $\mathcal{E}$ :

$$T_{\text{bad}}^\mathcal{E}(u) = \emptyset \quad \text{if and only if} \quad \text{Im}(s_\mathcal{E}) \cap \ker(u) = \emptyset \subset \mathbb{P}(\bigwedge^k \mathcal{V}_c). \quad (1.13)$$

## 1 Lyapunov exponents on character varieties

*Remark 1.7.21.* If the vector bundle  $\mathcal{V}$  is of rank 2, then an element  $u \in \bigwedge^1 \mathcal{V}_c^\vee$  defines a point in  $\mathbb{P}(\mathcal{V}_c)$ . If  $\mathcal{E} \subset \mathcal{V}$  is a sub-line bundle, the developing map defining  $\mathcal{E}$

$$s_{\mathcal{E}} : \mathbb{H} \rightarrow \mathbb{P}_{\mathbb{C}}^1$$

is simply a meromorphic function on  $\mathbb{H}$  equivariant with respect to the action of the representation  $\rho_{\mathcal{V}}$  defined by  $\mathcal{V}$ . Then

$$T_{\text{bad}}^{\mathcal{E}}(u) = \{z \in \mathbb{H} : s_{\mathcal{E}}(z) = u \in \mathbb{P}_{\mathbb{C}}^1\}.$$

Moreover in this case there is only one line  $u \in \mathbb{P}(\mathcal{V}_c)$  not computing the top Lyapunov exponent, namely the line corresponding to the second Oseledets space  $\mathcal{V}_{\lambda_2}$ .

The previous remark together with Proposition 1.7.18 imply the following condition for equality in the rank 2 situation .

**Corollary 1.7.22.** *Let  $\mathcal{V}$  be a rank 2 flat bundle over a hyperbolic Riemann surface  $C$  and  $\mathcal{E} \subset \mathcal{V}$  a sub-line bundle. If there is a  $\rho_{\mathcal{V}}$ -invariant subset  $S \subset \mathbb{P}_{\mathbb{C}}^1$  containing more than one point and such that  $s_{\mathcal{E}}(\mathbb{H}) \cap S = \emptyset$ , then*

$$\lambda_1 = \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_{\mathbb{C}}^1(\log(\Delta)))}.$$

*Proof.* The pull-back of the restriction of the tautological bundle  $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(-1)|_S$  defines a  $G_t$ -invariant closed subset  $\mathcal{S} \subset \mathbb{P}(\mathcal{V})$ . Since  $S$  contains more than two points, by the previous remark there is at least one line  $u \in S$  computing the top Lyapunov exponent. By Proposition 1.7.18 we then have equality.  $\square$

An important example of an invariant closed subset  $S \subset \mathbb{P}_{\mathbb{C}}^1$  containing more than one point is the limit set of a discrete faithful representation. We can then specialize the last corollary in this setting.

**Corollary 1.7.23.** *Let  $\mathcal{V}$  be a rank 2 flat bundle over a hyperbolic Riemann surface  $C$  corresponding to a faithful discrete representation  $\rho_{\mathcal{V}}$ . If there is a projective structure  $(\rho_{\mathcal{V}}, \text{dev})$  such that the image of the developing map  $\text{dev} : \mathbb{H} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  is disjoint to the limit set of  $\rho_{\mathcal{V}}$ , then*

$$\lambda_1 = \frac{2 \deg_{\text{par}}(\Xi_h(\text{dev}^*(\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(-1))))}{\deg(\Omega_{\mathbb{C}}^1(\log(\Delta)))}.$$

Here  $\text{dev}^*(\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(-1))$  is an abuse of notation for the line bundle on  $C$  defined by the developing map  $\text{dev}$ .

## 1.8 Main equality in the compact case

In this section we show that inequality (1.11) is an equality if the base curve is compact.



**Theorem 1.8.1.** *Let  $\mathcal{V}$  be a flat bundle over a compact hyperbolic Riemann surface  $C$ . For every holomorphic subbundle  $\mathcal{E} \subset \mathcal{V}$ , it holds*

$$\sum_{i=0}^k \lambda_i(\mathcal{V}_C) = \frac{2 \deg(\mathcal{E})}{\deg(\mathcal{K}_C)} + \text{Err}^{\mathcal{E}}(u)$$

for almost any  $c \in C$  and Lebesgue almost any  $u \in \bigwedge^k \mathcal{V}_c^{\vee}$ .

We will first state some applications of the previous result and then go on with its proof. The proof of Theorem 1.8.1 is quite technical and is based on finer estimates on the bad locus. We finally recall the main result of [DD17a] and get as a corollary that the dynamical degree defined in [DD17a] is the same as our error term if the base curve is compact.

**1.8.1 Applications** Thanks to the equality proven in Theorem 1.8.1 we get a better condition in the case of compact base curves for checking rationality of Lyapunov exponents than the one given by Proposition 1.7.18.

**Corollary 1.8.2.** *Let  $\mathcal{V}$  be a flat bundle over a compact hyperbolic Riemann surface  $C$ . If there is a rank  $k$  holomorphic subbundle  $\mathcal{E} \subset \mathcal{V}$  such that*

$$\text{T}_{\text{bad}}^{\mathcal{E}}(u) = \emptyset$$

for a vector  $u \in \bigwedge^k \mathcal{V}^{\vee}$  that computes the sum of the top Lyapunov exponents, then

$$\sum_{i=0}^k \lambda_i = \frac{2 \deg(\mathcal{E})}{\deg(\mathcal{K}_C)}.$$

Note that the previous corollary can be used for example if one considers the vector  $u = \sum_{i=k}^n \mathcal{V}_{\lambda_i} \in \bigwedge^k \mathcal{V}^{\vee}$  given by the sum of the last Oseledets spaces. Then only the emptiness condition  $\text{T}_{\text{bad}}^{\mathcal{E}}(u) = \emptyset$  has to be checked.

In the rank 2 situation we get a better version of Corollary 1.7.22 in the case of compact base curve. In this case we know indeed that there is only one line  $\mathcal{V}_{\lambda_2}$  not computing the top Lyapunov exponent.

**Corollary 1.8.3.** *Let  $\mathcal{V}$  be a rank 2 flat bundle over a compact hyperbolic Riemann surface  $C$  and  $\mathcal{E} \subset \mathcal{V}$  a sub-line bundle. If the complement  $\mathbb{P}_{\mathbb{C}}^1 \setminus s_{\mathcal{E}}(\mathbb{H})$  of the image of the corresponding equivariant map  $s_{\mathcal{E}} : \mathbb{H} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  contains more than one point, then*

$$\lambda_1 = \frac{2 \deg(\mathcal{E})}{\deg(\mathcal{K}_C)}.$$

Since a rank  $k$  holomorphic subbundle of the flat bundle corresponding to a representation  $\rho$  is the same as a  $\wedge^k \rho$ -equivariant holomorphic map  $f : \mathbb{H} \rightarrow \mathbb{P}(\wedge^k \mathbb{C}^n)$ , we get also the following corollary.

**Corollary 1.8.4.** *Let  $C$  be a compact Riemann surface. For any representation of the fundamental group  $\rho : \pi_1(C) \rightarrow \mathrm{GL}_n(\mathbb{C})$  and any  $\wedge^k \rho$ -equivariant holomorphic map  $f : \mathbb{H} \rightarrow \mathbb{P}(\wedge^k \mathbb{C}^n)$  the error term function*

$$\mathbb{P}(\wedge^k \mathbb{C}^n)^\vee \rightarrow \mathbb{R}^+, \quad u \mapsto \mathrm{Err}^f(u) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\#\{f^{-1}(\ker(u)) \cap D_t(z)\}}{\mathrm{vol}(D_t(z))} dt$$

*is Lebesgue almost everywhere constant, for almost all  $z \in \mathbb{H}$ .*

**1.8.2 Proof of Theorem 1.8.1** We will prove Theorem 1.8.1 by proving that in the compact base curve case, the  $\mathcal{E}$ -seminorm can be used to compute Lyapunov exponents as any other integrable norm. We want to remark that in order to prove that the  $\mathcal{E}$ -seminorm is as good as any other integrable norm, one could try the naive approach via Poincaré recurrence theorem used for proving that any norm computes the same Lyapunov exponents (see for example [KM16, Lemma 2.6]). The key idea of that approach is that any two norms are uniformly bounded with respect to each other on a projective bundle over a compact subset. In this case the bad locus breaks the compactness of the projective bundle since the  $\mathcal{E}$ -seminorm is a norm on the complement of the bad locus, which is not compact.

In order to prove that the  $\mathcal{E}$ -seminorm can be used to compute Lyapunov exponents, it suffices as before to only consider the case where  $\mathcal{E} = \mathcal{L}$  is a line bundle. Theorem 1.8.1 is a direct consequence of the following proposition, whose proof will take up the rest of this section.

**Proposition 1.8.5.** *Let  $C$  be compact. Let  $\mathcal{L} \subset \mathcal{V}$  be a holomorphic subline bundle. For almost any  $c \in C$  and any vector  $u \in \mathcal{V}_c^\vee - \bigoplus_{i=2}^n \mathcal{V}_{\lambda_i, c}^\vee$  it holds*

$$\lambda_1(\mathcal{V}) = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{1}{2\pi} \int_0^{2\pi} \log \|G_t r_\theta u\|_{\mathcal{L}} d\theta$$

From now on we will denote the trouble making sets introduced in Definition 1.7.11 as

$$T := T_{\mathrm{bad}}^{\mathcal{L}} \subset \mathbb{P}(\mathcal{V}^\vee), \quad T(u) := T_{\mathrm{bad}}^{\mathcal{L}}(u) \subset \mathbb{H} \text{ for any } u \in \mathbb{P}(\mathcal{V}^\vee).$$

Let  $\pi : \mathbb{H} \rightarrow C$  be the universal covering map and  $\pi^*(\mathbb{P}(\mathcal{V}^\vee))$  be the pullback of the projective bundle associated to  $\mathcal{V}^\vee$ . Since  $\mathcal{V}^\vee$  is a flat bundle, the pull-back  $\pi^*(\mathbb{P}(\mathcal{V}^\vee))$  is isomorphic to the trivial projective bundle. let us fix an isomorphism

$$\psi : \mathbb{H} \times \mathbb{P}(\mathcal{V}_c^\vee) \xrightarrow{\sim} \pi^*(\mathbb{P}(\mathcal{V}^\vee)), \quad (z, u) \mapsto u_z := \psi(z, u)$$

for some  $c \in C$ .

Consider the function

$$\phi : \mathbb{H} \times \mathbb{P}(\mathcal{V}_c^\vee) \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$$

$$(z, u) \longmapsto \log \|u_z\|_h - \log \|u_z\|_{\mathcal{L}} = \log \left( \frac{\|u_z\|_h}{\|u_z\|_{\mathcal{L}}} \right) = \log \left( \frac{\|u_z\|_h \|\omega_z\|_h}{|u_z(\omega_z)|} \right)$$

where  $\omega$  is a local frame of  $\pi^*(\mathcal{L})$ . Notice that this function is positive by Cauchy-Schwartz (see inequality (1.12)).

We denote by  $\phi_u : \mathbb{H} \rightarrow \mathbb{R} \cup \{\infty\}$  the map  $\phi(-, u)$ , for  $u \in \mathbb{P}(\mathcal{V}_c^\vee)$ .

*Remark 1.8.6.* Recall that the function appearing in the denominator of  $\phi$  comes from the norm of the holomorphic function

$$\mathbb{H} \times \mathcal{V}_c^\vee \longrightarrow \mathbb{C}, \quad (z, u) \mapsto u_z(\omega_z).$$

The bad locus  $\psi^{-1}(T) \subset \mathbb{H} \times \mathbb{P}^\vee(\mathcal{V}_c)$  is its zero locus and the bad locus for the vector  $u \in \mathbb{P}(\mathcal{V}_c)$  is given by the slice  $T(u) = \psi^{-1}(T) \cap (\mathbb{H} \times \{u\})$ .

Let  $\epsilon > 0$  be a positive constant. We define a tubular neighborhood of the bad locus  $\psi^{-1}(T) \subset \mathbb{H} \times \mathbb{P}(\mathcal{V}_c^\vee)$  to be

$$B(T, \epsilon) := \{(z, u) \in \mathbb{H} \times \mathbb{P}(\mathcal{V}_c^\vee) : d_{\text{hyp}}(z, T(u)) < \epsilon\} \subset \mathbb{H} \times \mathbb{P}(\mathcal{V}_c^\vee)$$

and the slice

$$B(T(u), \epsilon) := B(T, \epsilon) \cap (\mathbb{H} \times \{u\}).$$

Let  $B(T, \epsilon)^\complement \subset \mathbb{H} \times \mathbb{P}^\vee(\mathcal{V}_c)$  be the complement of the tubular neighborhood  $B(T, \epsilon)$ . In the next lemma we obtain a bound on the behavior of the function  $\phi$  on  $B(T, \epsilon)$  and on  $B(T, \epsilon)^\complement$ . The main ingredients used in the proof of the next lemma are the compactness of the curve and the equivariance property of  $\phi$ .

**Lemma 1.8.7.** *There exist constants  $M, N > 0$  such that the function  $\phi$  outside the tubular neighborhood  $B(T, \epsilon)$  satisfies the following bound*

$$\|\phi|_{B(T, \epsilon)^\complement}\|_\infty \leq M + N |\log(\epsilon)|.$$

Moreover there is a constant  $M' > 0$  such that for every  $u \in \mathbb{P}(\mathcal{V}_c^\vee)$  and any  $w \in T(u)$  the function  $\phi_u$  restricted to the ball  $B_{\text{hyp}}(w, \epsilon)$  around  $w$  satisfies the following bound:

$$\phi_u(z)|_{B_{\text{hyp}}(w, \epsilon)} \leq M' + N \sum_{z' \in T(u) \cap B_{\text{hyp}}(w, \epsilon)} |\log(d_{\text{hyp}}(z, z'))|.$$

*Proof.* Let us choose a compact fundamental domain  $F \subset \mathbb{H}$  for  $C$  and consider the restrictions

$$T|_F := \psi^{-1}(T) \cap (F \times \mathbb{P}(\mathcal{V}_c^\vee)) \subset \mathbb{H} \times \mathbb{P}(\mathcal{V}_c^\vee)$$

and

$$B(T, \epsilon)|_F := B(T, \epsilon) \cap (F \times \mathbb{P}(\mathcal{V}_c^\vee)).$$

Note that if we prove the two claims of the proposition restricting ourselves to the subset  $F \times \mathbb{P}(\mathcal{V}_c^\vee)$ , then we can use the equivariance property

$$\phi(\gamma z, u) = \phi(z, \rho(\gamma^{-1})u), \quad \gamma \in \pi_1(C, c)$$

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to extend the results to all of the upper half plane since the constants involved in the expressions are independent of  $u \in \mathbb{P}(\mathcal{V}_c^\vee)$  and since  $\pi_1(C, c)$  acts via isometries on  $\mathbb{H}$ .

The main idea now is considering the holomorphic function of expression (1.8.6) locally as a power series. Then by a compactness argument we can control the coefficients of this power series. The main technical problem is that two zeros of this function, which are the bad points, can collide for some values of  $u \in \mathbb{P}(\mathcal{V}^\vee)$ . This has to be taken into account in order to correctly prove the second statement of the lemma.

Since  $F \times \mathbb{P}(\mathcal{V}_c^\vee)$  is compact, by the Weierstrass preparation theorem we can choose finitely many points  $(z_j, u_j) \in T|_F$  and compact neighborhoods of the points  $(z_j, u_j) \in U_j \subset F \times \mathbb{P}(\mathcal{V}_c^\vee)$  such that  $T|_F \subseteq \bigcup_{j=1}^m U_j$  and

$$\phi(z, u)|_{U_j} = \log \left( \frac{\|u_z\|_h \|\omega_z\|_h}{|h_j(z, u) P_j(z, u)|} \right)$$

where the holomorphic functions  $h_j$  are never zero and the polynomials  $P_j$  are given as

$$P_j(z, u) = \sum_{i=0}^{n_j-1} a_{i,j}(u) (z - z_j)^i + (z - z_j)^{n_j}.$$

The coefficients  $a_{i,j}(u)$  are holomorphic functions with  $a_{i,j}(u_j) = 0$  for all  $i$  and  $j$ . Notice that  $T|_F \cap U_j$  is the zero locus of  $P_j$ .

On each  $U_j$  we get then the following bound:

$$\phi(z, u)|_{U_j} = \log \left( \frac{\|u_z\|_h \|\omega_z\|_h}{|h_j(z, u)|} \right) - \log |P_j(z, u)| \leq M'' - \log |P_j(z, u)|$$

where the constant

$$M'' := \max_{j=1, \dots, m} \left( \max_{(z, u) \in U_j} \log \left( \frac{\|u_z\|_h \|\omega_z\|_h}{|h_j(z, u)|} \right) \right)$$

is well defined since the functions  $h_j$  are never zero on the compact subsets  $U_j$ .

We then rewrite for every  $(z, u) \in U_j$  the roots decomposition of the polynomial  $P_j(z, u)$  with respect to the variable  $z$  to get

$$\phi(z, u)|_{U_j} \leq M'' - \log |P_j(z, u)| = M'' - \sum_{i=1}^{n_j} \log |z - z_{i,j}(u)|$$

where  $z_{i,j}(u) \in \mathbb{C}$  are possibly equal to each other. Since  $F \subset \mathbb{H}$  is compact, the euclidean and the hyperbolic distances are comparable to each other. In particular there is a constant  $L > 0$  such that  $|x - y| \geq L \cdot d_{\text{hyp}}(x, y)$  for all  $x, y \in F$ . Hence we can rewrite the last inequality as

$$\phi(z, u)|_{U_j} \leq M'' - \sum_{i=1}^{n_j} \log |z - z_{i,j}(u)| \leq M'' - \sum_{i=1}^{n_j} \log |L \cdot d_{\text{hyp}}(z, z_{i,j}(u))|. \quad (1.14)$$

If  $\epsilon$  is chosen small enough, we can assume that  $B(T, \epsilon)|_F$  is contained in the union  $U := \bigcup_{j=1}^m U_j$ . Since  $(z, u) \in B(T, \epsilon)^c$  implies that  $d_{\text{hyp}}(z, z_{i,j}(u)) > \epsilon$ , we find the bound

$$\|\phi|_{B(T, \epsilon)^c}\|_\infty \leq M - N \log(\epsilon) = M + N |\log(\epsilon)|$$

where

$$N := \sum_{j=1, \dots, m} n_j, \quad M := \max\left\{ \max_{(z, u) \in \tilde{U}^c} \phi(z, u), M'' \right\} - N \log(L).$$

Hence the first statement of the proposition is proven.

In order to prove the second statement of the lemma we consider a bad point  $(w, u) \in T|_F$ . Since  $\epsilon$  is small, we can assume that the tubular neighborhood  $B(T, \epsilon)|_F$  of  $T|_F$  is contained in  $U = \bigcup_{j=1}^m U_j$ . Moreover, if  $\epsilon$  is small enough we can also assume that  $U_j$  is a product  $U_j = K_j \times V_j$  for  $K_j \subset F$  and  $V_j \subset \mathbb{P}(\mathcal{V}_c^\vee)$  compact subsets. From now on, for the sake of a simpler notation we set  $K := K_j$  and  $V := V_j$ . Moreover we set  $P := P_j$ , so that the bad locus  $T|_F \cap U_j = Z(P)$  is the zero set of  $P$ . We will prove the second statement of the lemma restricting to  $U_j = K \times V$ . This is sufficient since there are only finitely many  $U_j$ .

We need to prove that for any  $(w, u) \in Z(P) \subset K \times V$  the following bound holds:

$$\phi_u(z)|_{B_{\text{hyp}}(w, \epsilon)} \leq M' + N \sum_{z' \in Z(P_u) \cap B_{\text{hyp}}(w, \epsilon)} |\log(d_{\text{hyp}}(z, z'))|$$

for some constant  $M'$ . For any  $(z, u) \in K \times V$  we decompose the polynomial  $P(z, u)$  into roots

$$P(z, u) = \prod_{i=1}^n (z - z_i(u)).$$

Without loss of generality we can assume that the roots  $z_i : V \rightarrow \mathbb{C}$  are well-defined holomorphic functions. Indeed, even if in general they are only multi-valued functions, there is a finite covering  $\pi : V' \rightarrow V$  such that the pullbacks  $\pi^*(z_i) : V' \rightarrow \mathbb{C}$  are single valued functions. If we then can prove the desired bound for these pull-back roots, namely if for any root  $(w, v) \in Z(\pi^*(P)) \subset K \times V'$  it holds

$$\pi^*(\phi)(z, v)|_{B_{\text{hyp}}(w, \epsilon)} \leq M' + N \sum_{z' \in Z(\pi^*(P)_v) \cap B_{\text{hyp}}(w, \epsilon)} |\log(d_{\text{hyp}}(z, z'))|$$

then it is clear that we get the same bound for the original function. Indeed it is enough to choose  $v \in \pi^{-1}(u)$  for any  $(w, u) \in Z(P) \subset K \times V$  and use the definition of pull-back  $\pi^*(\phi)(z, v) = \phi(z, \pi(v))$  and  $\pi^*(P)_v = P_{\pi(v)}$  to get the original bound.

Since we reduced to the case where  $P(z, u) = \prod_{i=1}^n (z - z_i(u))$  and  $z_i : V \rightarrow \mathbb{C}$  are holomorphic functions, we can consider the irreducible components of the zero locus  $Z(P)$  which are now given as graphs

$$\Gamma_i := \{(z_i(u), u) \in K \times V : u \in V\}.$$

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Let us define the tubular neighborhood around  $\Gamma_i$  as

$$B(\Gamma_i) := \{(z, u) \in K \times V : d_{\text{hyp}}(z, z_i(u)) \leq \epsilon\}.$$

Notice that by inequality (1.14) we have

$$\phi(z, u)|_{K \times V} \leq M'' - \sum_{i=1}^n \log |L \cdot d_{\text{hyp}}(z, z_i(u))|.$$

In order to define a global constant  $M'$  independent of  $u \in V$  in the bound that we are trying to prove, we need to define the maximal compact subset of  $B(\Gamma_i)$  where the function  $\log |L \cdot d_{\text{hyp}}(z, z_k(u))|_{|B_{\text{hyp}}(z_i(u), \epsilon)}$  is well define and then take the maximum over this set for all  $k \neq i$ . This set can be defined as the  $k$ -th complement

$$\begin{aligned} B(\Gamma_i)(k) &:= \{(z, u) \in B(\Gamma_i) : d_{\text{hyp}}(z_i(u), z_k(u)) \geq \epsilon\} \\ &= \{(z, u) \in B(\Gamma_i) : (z_k(u), u) \notin B(\Gamma_i)\}. \end{aligned}$$

The number

$$M'_i := \max_{k \neq i} \left( \max_{(z, u) \in B(\Gamma_i)(k)} -\log(L \cdot d_{\text{hyp}}(z, z_k(u))) \right)$$

is well-defined since the sets we are taking the maximum on are compact and the functions  $\log(L \cdot d_{\text{hyp}}(z, z_k(u)))$  are well-defined in these sets. We finally define

$$M' := N \cdot \max_i M'_i + M'' - N \log(L).$$

Then by rewriting once again inequality (1.14) and using the definition of  $M'$  we find that for every  $i \in \{1, \dots, n\}$  it holds

$$\begin{aligned} \phi(z, u)|_{B(\Gamma_i)} &\leq M' - \sum_{z_k(u) \in B_{\text{hyp}}(z_i(u), \epsilon)} \log(d_{\text{hyp}}(z, z_k(u))) \\ &\leq M' + N \sum_{z_k(u) \in B_{\text{hyp}}(z_i(u), \epsilon)} |\log(d_{\text{hyp}}(z, z_k(u)))| \end{aligned}$$

where we added the constant  $N > 0$  in the second inequality because in the second claim of the proposition we are summing over the bad points not taking multiplicities into account. The second claim of the proposition is then proven.  $\square$

*Remark 1.8.8.* The positive constant  $N > 0$  of the previous lemma gives a uniform bound of the number of bad points in any compact fundamental domain for a compact curve  $C$ . In the case of a non compact curve, it is unclear if such a uniform bound exists. Notice that there is an alternative way of proving that the number of bad points is uniformly bounded, without using the Weierstrass preparation theorem. Indeed consider the map

$$\mathbb{P}(\mathcal{V}^\vee) \rightarrow \mathbb{R}, \quad u \mapsto \int_{\mathbb{H}} \tilde{\chi}_F(z) \chi_{T(u)}(z)$$

where  $\tilde{\chi}_F(z)$  is the smoothed characteristic function of a compact fundamental domain  $F \subset \mathbb{H}$  and  $\chi_{T(u)}(z)$  is the characteristic function of the set  $T(u)$ . This map is continuous and so it is bounded. The bound is independent of the choice of fundamental domain by the equivariance property

$$\chi_{T(u)}(\gamma(z)) = \chi_{T(\rho(\gamma^{-1})u)}(z), \quad \text{for any } \gamma \in \pi_1(C, c).$$

Using the bounds of the last lemma, we now prove Proposition 1.8.5 which, as we already noticed, implies Theorem 1.8.1. The main strategy is to separate the study of the integral of the seminorm near and far from the bad points.

*Proof of Proposition 1.8.5.* Let us choose  $u \in \mathcal{V}^\vee$  that computes the Lyapunov exponent, meaning that

$$\lambda_1(\mathcal{V}) = \lim_{t \rightarrow \infty} \frac{1}{t} \frac{1}{2\pi} \int_0^{2\pi} \log \|u_{g_t r_\theta(z_0)}\|_h d\theta$$

where  $z_0 \in T^1 \mathbb{H}$  is the base point of  $u$ . Let  $\epsilon' > 0$  and fix  $t \gg 0$ . Set  $\epsilon = \epsilon(t) := e^{-t\epsilon'}$ . By abuse of notation, we will not distinguish a point in  $T^1 \mathbb{H}$  from its base point in  $\mathbb{H}$ . We define now

$$S(t)_{\text{near}} := \{\theta \in [0, 2\pi] : g_t r_\theta(z_0) \in B(T(u), \epsilon)\}$$

and  $S(t)_{\text{far}} := [0, 2\pi] - S(t)_{\text{near}}$ .

We want to prove that the difference between the norm and the seminorm

$$\begin{aligned} \frac{1}{t} \int_0^{2\pi} \log \left( \frac{\|u_{g_t r_\theta(z_0)}\|_h}{\|u_{g_t r_\theta(z_0)}\|_{\mathcal{L}}} \right) d\theta &= \frac{1}{t} \int_0^{2\pi} \phi(g_t r_\theta(z_0), u) d\theta \\ &= \frac{1}{t} \left( \int_{S(t)_{\text{near}}} \phi(g_t r_\theta(z_0), u) d\theta + \int_{S(t)_{\text{far}}} \phi(g_t r_\theta(z_0), u) d\theta \right) \end{aligned}$$

tends to zero.

We treat first the integral near the bad locus. Define the hyperbolic annulus

$$A(t, \epsilon) := \{z \in \mathbb{H} : t - \epsilon \leq d_{\text{hyp}}(z, z_0) \leq t + \epsilon\}$$

and for any  $w \in \mathbb{H}$  define the arc portion

$$C_t(w, \epsilon) := \{\theta \in [0, 2\pi] : g_t r_\theta(z_0) \in B_{\text{hyp}}(w, \epsilon)\}.$$

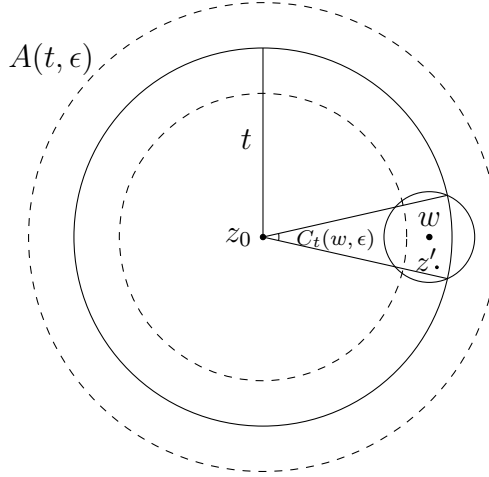
It follows from the above definitions and from the second statement of Lemma 1.8.7 that

$$\begin{aligned} \int_{S(t)_{\text{near}}} \phi(g_t r_\theta(z_0), u) d\theta &\leq \sum_{w \in T(u) \cap A(t, \epsilon)} \int_{C_t(w, \epsilon)} \phi(g_t r_\theta(z_0), u) d\theta \\ &\leq \sum_{w \in T(u) \cap A(t, \epsilon)} M' \int_{C_t(w, \epsilon)} d\theta + \\ &\quad \sum_{w \in T(u) \cap A(t, \epsilon)} N \sum_{z' \in T(u) \cap B_{\text{hyp}}(w, \epsilon)} \int_{C_t(w, \epsilon)} |\log(d_{\text{hyp}}(g_t r_\theta(z_0), z'))| d\theta \end{aligned}$$

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$$\leq \sum_{w \in T(u) \cap A(t, \epsilon)} M' \int_{C_t(w, \epsilon)} d\theta + \sum_{w \in T(u) \cap A(t, \epsilon)} N^2 \max_{z' \in B_{\text{hyp}}(w, \epsilon)} \left( \int_{C_t(w, \epsilon)} |\log(d_{\text{hyp}}(g_t r_\theta(z_0), z'))| d\theta \right).$$

where the last inequality follows since we can assume that  $B_{\text{hyp}}(w, \epsilon)$  is small enough to be contained in a fundamental domain and  $N$  is the the uniform bound for number of bad points in a fundamental domain.



First of all notice that

$$\int_{C_t(w, \epsilon)} d\theta \leq \frac{\sinh(\epsilon)}{\sinh(t)} =: \theta(\epsilon, t)$$

since the measure of the angle  $C_t(w, \epsilon)$  is the same as the quotient of the hyperbolic length of the arc that the angle defines and the hyperbolic length of the circumference  $S_t(z_0)$ . It is moreover clear that the hyperbolic length of the arc defined by  $C_t(w, \epsilon)$  is less than the length of the circumference  $\partial B_{\text{hyp}}(w, \epsilon)$ .

We hence get the following bound:

$$\int_{S(t)_{\text{near}}} \phi(g_t r_\theta(z_0), u) d\theta \leq M' \cdot \#\{T(u) \cap A(t, \epsilon)\} \cdot \theta(\epsilon, t) + N^2 \cdot \#\{T(u) \cap A(t, \epsilon)\} \cdot \max_{w \in A(t, \epsilon)} \left( \max_{z' \in B_{\text{hyp}}(w, \epsilon)} \int_{C_t(w, \epsilon)} |\log(d_{\text{hyp}}(g_t r_\theta(z_0), z'))| d\theta \right).$$

In order to bound the second summand, we notice that this term is invariant under isometries. Hence we can work in the Poincaré disk  $\mathcal{D}$  and we can assume that the starting point is  $z_0 = (0, (1, 0)) \in T^1\mathcal{D}$ , namely the center together with the horizontal direction. Moreover, since the term

$$\max_{z' \in B_{\text{hyp}}(w, \epsilon)} \left( \int_{C_t(w, \epsilon)} |\log(d_{\text{hyp}}(g_t r_\theta(z_0), z'))| d\theta \right)$$



is invariant under rotation, we can assume that  $w$  is on the horizontal ray

$$w \in R := A(t, \epsilon) \cap [0, 1] = [\tanh(t - \epsilon), \tanh(t + \epsilon)].$$

We define the tubular neighborhood of the ray as

$$U := \bigcup_{r \in [t - \epsilon, t + \epsilon]} B_{\text{hyp}}(\tanh(r/2), \epsilon).$$

We then get the following bound:

$$\begin{aligned} & \max_{w \in R} \left( \max_{z' \in B_{\text{hyp}}(w, \epsilon)} \left( \int_{C_t(w, \epsilon)} |\log(d_{\text{hyp}}(g_t r_\theta(z_0), z'))| d\theta \right) \right) \\ & \leq \max_{z' \in U} \left( \int_{-\frac{\theta(\epsilon, t)}{2}}^{\frac{\theta(\epsilon, t)}{2}} |\log d_{\text{hyp}}(\tanh(t/2)e^{i\theta}, z')| d\theta \right) \\ & \leq \max_{z' \in R} \left( \int_{-\theta(\epsilon, t)}^{\theta(\epsilon, t)} |\log d_{\text{hyp}}(\tanh(t/2)e^{i\theta}, z')| d\theta \right) \end{aligned}$$

where the last inequality follows again by rotational invariance (up to enlarging the angle of integration we can assume that  $z'$  is on the ray that cuts the angle in two equal parts and then we can rotate to have  $z' \in R$ ). Notice now that

$$\begin{aligned} & \max_{z' \in R} \left( \int_{-\theta(\epsilon, t)}^{\theta(\epsilon, t)} |\log d_{\text{hyp}}(\tanh(t/2)e^{i\theta}, z')| d\theta \right) \\ & \leq \int_{-\theta(\epsilon, t)}^{\theta(\epsilon, t)} |\log d_{\text{hyp}}(\tanh(t/2)e^{i\theta}, R)| d\theta \end{aligned}$$

where by definition

$$d_{\text{hyp}}(\tanh(t/2)e^{i\theta}, R) := \inf_{z' \in R} d_{\text{hyp}}(\tanh(t/2)e^{i\theta}, z').$$

Using the hyperbolic sine rule we get

$$d_{\text{hyp}}(\tanh(t/2)e^{i\theta}, R) = \sinh^{-1}(\sinh(\tanh(t/2)) \cdot \sin(\theta)) = \theta + o(\theta)$$

where the little- $o$  notation is with respect to  $t$  going to infinity (hence  $\theta = \theta(t)$  going to zero). We finally can then rewrite

$$\begin{aligned} & \int_{-\theta(\epsilon, t)}^{\theta(\epsilon, t)} |\log d_{\text{hyp}}(\tanh(t/2)e^{i\theta}, R)| d\theta = 2 \int_0^{\theta(\epsilon, t)} |\log \theta| d\theta + o(\theta(\epsilon, t)) \\ & = |2\theta(\epsilon, t) \log(\theta(\epsilon, t)) - 2\theta(\epsilon, t)| + o(\theta(\epsilon, t)) \sim \log(\theta(\epsilon, t))\theta(\epsilon, t) \sim \frac{t(\epsilon' + 1)}{e^{t(\epsilon' + 1)}} \end{aligned}$$

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where the last asymptotic follows from the definition

$$\theta(\epsilon, t) := \frac{\sinh(\epsilon)}{\sinh(t)} = \frac{\sinh(e^{-t\epsilon'})}{\sinh(t)}$$

and from the fact that for large  $t$  large the function  $\sinh(t)$  is asymptotic to  $e^t$  and for  $\epsilon$  small  $\sinh(\epsilon)$  is asymptotic  $\epsilon$ .

Putting together all the inequalities we showed that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_{S(t)_{\text{near}}} \phi(g_t r_\theta(z_0), u) d\theta &\leq \liminf_{t \rightarrow \infty} \frac{1}{t} \#\{T(u) \cap A(t, \epsilon)\} \cdot \frac{t(\epsilon' + 1)}{e^{t(\epsilon' + 1)}} \\ &\leq \liminf_{t \rightarrow \infty} \#\{T(u) \cap D_{t+\epsilon}(z_0)\} \cdot \frac{\epsilon' + 1}{e^{t(\epsilon' + 1)}}. \end{aligned}$$

By Theorem 1.7.13, we know that the limit defining the error term converge:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\#\{T(u) \cap D_{t+\epsilon}(z_0)\}}{\text{vol}(D_{t+\epsilon}(z_0))} dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\#\{T(u) \cap D_{t+\epsilon}(z_0)\}}{4\pi \sinh^2((t+\epsilon)/2)} dt < \infty.$$

This implies that

$$\liminf_{t \rightarrow \infty} \#\{T(u) \cap D_{t+\epsilon}(z_0)\} \cdot \frac{(\epsilon' + 1)}{e^{t(\epsilon' + 1)}} = 0.$$

Indeed if this is not the case then there is a constant  $c > 0$  and  $t'$  such that for all  $t > t'$  it holds  $\#\{T(u) \cap D_{t+\epsilon}(z_0)\} > c \cdot (e^{t(\epsilon' + 1)}/(\epsilon' + 1))$  which implies the contradiction

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t'}^T \frac{\#\{T(u) \cap D_{t+\epsilon}(z_0)\}}{4\pi \sinh^2((t+\epsilon)/2)} dt > \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t'}^T \frac{c \cdot (e^{t(\epsilon' + 1)}/(\epsilon' + 1))}{4\pi \sinh^2((t+\epsilon)/2)} dt = \infty.$$

Using the first statement of Lemma 1.8.7 we can compute a bound of the integral over the points which are not near  $T(u)$ :

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_{S(t)_{\text{far}}} \phi(g_t r_\theta(z_0), u) d\theta \leq \lim_{t \rightarrow \infty} \frac{2\pi}{t} (M - N(\log(\epsilon))) = 2\pi N\epsilon'.$$

By letting  $\epsilon'$  tend to zero, we finally get

$$\lambda_1(\mathcal{V}) = \lim_{T \rightarrow \infty} \frac{1}{T} \frac{1}{2\pi} \int_0^{2\pi} \log \|G_{T r_\theta} u\|_{\mathcal{L}} d\theta$$

and Proposition 1.8.5 is proven. □

**1.8.3 Harmonic measures and Brownian motion** In this section we compare Theorem 1.8.1 to the main results of [DD17b] and [DD17a]. We will describe how in the case of compact base curve Theorem 1.8.1 can be used to identify the dynamical degree defined in [DD17a] with our error term, which in turn can be viewed as a generalization of the asymptotic covering degree of developing maps defined in [DD17b].

In [DD15] and [DD17b], Deroin and Dujardin defined the Lyapunov exponents associated to holonomies of parabolic projective structure on hyperbolic surfaces in the context of Brownian motion. The definition of Lyapunov exponents in this context is essentially the same as our definition provided by Oseledets multiplicative ergodic theorem, but the cocycle is defined over the Brownian motion on the Riemann surface instead of on the geodesic flow. The two definitions of Lyapunov exponents provide the same numbers since the Brownian motion tracks the geodesic flow sublinearly on hyperbolic Riemann surfaces (see [Anc90]). In [DD15] it was proven the main equality of Theorem 1.8.1 in the specific case of rank 2 representations given as holonomies of projective structures inducing the same holomorphic structure of the base curve. In Proposition 1.5.1 we identified the locus of such representations with the oper locus if the base curve is compact. The error term was identified with the asymptotic covering degree of the developing map of the projective structure (the different constants appearing are due to a different normalization of the hyperbolic metric). Our error term is a generalization of this asymptotic degree (see Proposition 1.12.4 for a specific comparison in rank two). Notice however that the equality proven in [DD15] is more general than ours since it works for parabolic representations over non compact curves and since in the error term they do not need the integral defining the mean of the counting function since they can prove that the counting function converges.

In [DD17a], Daniel and Deroin generalized the definition of Lyapunov exponents in the context of Brownian motion on Kähler manifolds. The result they provide is analogous to the main equality of Theorem 1.8.1, where the error term is called dynamical degree. Recall that a measure  $\nu$  on the projective bundle associated to a flat bundle  $\mathcal{V}$  is called harmonic if it is invariant under the heat semigroup action. The dynamical degree associated to a sub-vector bundle  $\mathcal{E} \subset \mathcal{V}$  is defined as the intersection number

$$\delta_{\mathcal{E}} := T_{\nu} \cap [\mathbb{P}(\mathcal{E})]$$

where  $T_{\nu}$  is the harmonic current associated to  $\nu$ .

**Corollary 1.8.9.** *Over a compact Riemann surface, the error term and the dynamical degree coincide*

$$\delta_{\mathcal{E}} = \text{Err}^{\mathcal{E}}(u)$$

for Lebesgue almost all  $u \in \bigwedge^k \mathcal{V}^{\vee}$ .

*Proof.* As recalled above, the Lyapunov exponents defined in the context of Brownian motion and the one defined for the geodesic flow coincide on a hyperbolic curve. The result then follows by comparing the equality of Theorem 1.8.1 and the formula in [DD17a, Theorem 5].  $\square$

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Notice that the error term, contrary to the dynamical degree, can in principle be approximated with computer experiments.

Recall that  $G_t : \mathcal{V} \rightarrow \mathcal{V}$  is the lift via parallel transport of the geodesic flow over  $T^1C$ . The main theorem of [BEW17] in our setting implies that if the flat bundle  $\mathcal{V}_C$  over  $T^1C$  is irreducible, then there exists a measure  $\nu$  on the projective bundle  $\mathbb{P}(\mathcal{V}_C)$  that is  $G_t$ -invariant, projects to the hyperbolic measure on the base and it is fiberwise supported on the projectivization  $\mathbb{P}(\mathcal{V}_{\lambda_1})$  of the first Oseledets subspace. Using an abuse of notation we call  $\nu$  the corresponding measure on the wedge products  $\mathbb{P}(\bigwedge^k \mathcal{V}_C)$  if they are irreducible.

The property of our error term to be Lebesgue almost everywhere constant differentiate our result to the one in [DD17a]. If we allow ourselves to consider an error term which is almost everywhere constant with respect to a  $G_t$ -invariant measure  $\nu$ , we are able to more easily show a weaker version of Theorem 1.8.1 equivalent to the result of [DD17a].

**Proposition 1.8.10.** *Let  $\mathcal{V}_C$  be a flat bundle such that  $\bigwedge^k \mathcal{V}_C$  is irreducible. For any holomorphic subbundle  $\mathcal{E} \subset \mathcal{V}_C$  of rank  $k$ , if*

$$\int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|u\|_{\mathcal{L}}}{\|u\|_h} \right) d\nu(u) < \infty$$

then

$$\sum_{i=0}^k \lambda_i = \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_C^1(\log(\Delta)))} + \text{Err}^{\mathcal{E}}(u)$$

for  $\nu$ -almost any  $u \in \mathbb{P}(\bigwedge^k \mathcal{V}^\vee)$ .

If the base curve is compact, the integrability assumption holds.

*Proof of Proposition 1.8.10.* As in the proof of Theorem 1.7.13, it is enough to prove the result for the top Lyapunov exponent and consider the case where  $\mathcal{E} = \mathcal{L}$  is a line bundle. Since the measure  $\nu$  is fiberwise supported on the first Oseledets space, the top Lyapunov exponent of  $\mathcal{V}$  is given by

$$\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_h}{\|u\|_h} \right) d\nu(u)$$

where  $h$  is an integrable norm. Integrating inequality (1.11) over  $\mathbb{P}(\mathcal{V}^\vee)$  with respect to the measure  $\nu$  and rewriting backwards the equalities of the proof of Theorem 1.7.13, we find

$$\lambda_1 \geq \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_C^1(\log(\Delta)))} + \int_{\mathbb{P}(\mathcal{V}^\vee)} \text{Err}^{\mathcal{E}}(u) d\nu(u) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right) d\nu(u)$$

where  $\|\cdot\|_{\mathcal{L}}$  is the  $\mathcal{L}$ -seminorm defined in (1.8). We finally compute

$$\lambda_1 - \left( \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_C^1(\log(\Delta)))} + \int_{\mathbb{P}(\mathcal{V}^\vee)} \text{Err}^{\mathcal{E}}(u) d\nu(u) \right)$$

$$\begin{aligned}
 &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_h}{\|u\|_h} \right) d\nu(u) - \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right) d\nu(u) \\
 &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_h \cdot \|u\|_{\mathcal{L}}}{\|G_t u\|_{\mathcal{L}} \cdot \|u\|_h} \right) d\nu(u).
 \end{aligned}$$

Since by assumption  $\int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|u\|_{\mathcal{L}}}{\|u\|_h} \right) d\nu(u) < \infty$ , we can split the integral

$$\begin{aligned}
 &\int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_h \cdot \|u\|_{\mathcal{L}}}{\|G_t u\|_{\mathcal{L}} \cdot \|u\|_h} \right) d\nu(u) = \\
 &= \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_h}{\|G_t u\|_{\mathcal{L}}} \right) d\nu(u) - \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|u\|_h}{\|u\|_{\mathcal{L}}} \right) d\nu(u) = 0
 \end{aligned}$$

where the last equality follows from the  $G_t$ -invariance of the measure  $\nu$ . We have then proved that

$$\begin{aligned}
 \lambda_1 &= \left( \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_{\mathcal{C}}^1(\log(\Delta)))} + \int_{\mathbb{P}(\mathcal{V}^\vee)} \text{Err}^{\mathcal{E}}(u) d\nu(u) \right) = \\
 &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right) d\nu(u).
 \end{aligned}$$

In order to prove that the function  $\text{Err}^{\mathcal{E}}(u)$  is  $\nu$ -almost everywhere constant notice that the function

$$\mathbb{P}(\mathcal{V}^\vee) \longrightarrow \mathbb{R}, \quad u \mapsto \log \left( \frac{\|G_1 u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right)$$

is  $\nu$ -integrable since

$$\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_t u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right) d\nu(u) = \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_1 u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right) d\nu(u)$$

where the second equality comes from the  $G_t$ -invariance of  $\nu$ . Then applying the Birkhoff ergodic theorem to this function and the measure  $\nu$ , it follows that for  $\nu$ -almost any  $u \in \mathbb{P}(\mathcal{V}^\vee)$ :

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \left( \frac{\|G_t u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right) = \int_{\mathbb{P}(\mathcal{V}^\vee)} \log \left( \frac{\|G_1 u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right) d\nu(u) = \lambda_1.$$

Hence we finally get for  $\nu$ -almost any  $u \in \mathbb{P}(\mathcal{V}^\vee)$ :

$$\lambda_1 = \lim_{t \rightarrow \infty} \frac{1}{t} \log \left( \frac{\|G_t u\|_{\mathcal{L}}}{\|u\|_{\mathcal{L}}} \right) = \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_{\mathcal{C}}^1(\log(\Delta)))} + \text{Err}^{\mathcal{E}}(u).$$

The second claim of the proposition about the integrability condition in the compact case can be showed using Lemma 1.8.7 and arguing in an analogous way as in the proof of Proposition 1.8.5.  $\square$

## 1.9 Lyapunov exponents on Shatz strata

Let  $C$  be a compact Riemann surface. Recall that the Shatz stratification is the stratification of  $\mathcal{M}_{\text{DR}}^{(n)}(C)$  given by Harder-Narasimhan type, namely by slopes of maximal subbundles. The Harder-Narasimhan type provide then a natural bound for the sum of Lyapunov exponents.

The bound on the Lyapunov exponent functions on the minimal stratum defined by semistable bundles is trivial, since there the Harder-Narasimhan filtration is trivial. Notice that all Lyapunov exponent functions restricted to the closed subset of this stratum given by the unitary locus are zeros.

On the maximal stratum, which by Theorem 1.3.12 is the oper locus  $\text{Op}_n(C)$ , we have the maximal possible bound of the Lyapunov exponent functions. We can actually compute the bound, since we know the Harder-Narasimhan type of an oper.

**Proposition 1.9.1.** *If  $\mathcal{V} \in \text{Op}_n(C)$  is in the oper locus then*

$$\sum_{i=1}^k \lambda_i(\mathcal{V}) \geq k(n-k), \quad k = 1, \dots, n,$$

*with equality in the case of the only variation of Hodge structures point, given by the  $(n-1)$ -th symmetric power of the maximal Higgs one in rank 2.*

*Proof.* Using Theorem 1.7.10 we see that the Lyapunov spectrum dominates the Harder-Narasimhan type. By Theorem 1.3.12 we know that the Harder-Narasimhan filtration of an oper is the oper filtration itself. Using Proposition 1.3.13 and arguing inductively we can compute the slopes of the subbundles  $\mathcal{V}_k$  defining the oper filtration. We see then that

$$\deg(\mathcal{V}_k) = k(n-k).$$

To prove equality in the only variation of Hodge structures point (see Proposition 1.3.15) notice that in the rank 2 case the uniformizing variation of Hodge structure is of weight one. Hence by Theorem 1.11.2 we can compute the top Lyapunov exponent, which is 1. Now it is enough to compute how Lyapunov exponents change under symmetric power and check that we get indeed the equality.  $\square$

Now that we have proved a lower bound for the sum of Lyapunov exponents of Shatz strata, it is natural to ask if the Lyapunov exponent functions are unbounded on these strata. Using the recent result of Dujardin and Favre [DF18] about the growth of Lyapunov exponents for meromorphic families, we are able to show the following result for the maximal stratum.

**Theorem 1.9.2.** *The top Lyapunov exponent function is unbounded on the maximal Shatz stratum, the oper locus, with logarithmic growth near the boundary of the character variety.*

*Proof.* First of all note that the character variety and the de Rham moduli space are biholomorphic, hence by Theorem 1.3.12 the oper locus, being the maximal Shatz stratum, is a closed embedded subset of the character variety. Now recall that by Proposition 1.3.11 each connected component of the oper locus in rank  $n$  is biholomorphic to the Hitchin base  $\bigoplus_{j=2}^n H^0(C; \mathcal{K}_C^j)$ .

By the result of [DF18], a meromorphic family from the unit disk to the space of non-elementary representations in the character variety in rank 2 which is holomorphic outside of zero and cannot be holomorphically extended in 0 yields a logarithmic growth of the Lyapunov exponent near zero. A connected component of the oper locus in rank 2 is given by the properly embedded vector space  $H^0(C; \mathcal{K}_C^2)$  in the character variety and does not intersect the space of elementary representations. Since on a complex vector space there are a lot of meromorphic maps from the disk  $\mathbb{D} \rightarrow H^0(C; \mathcal{K}_C^2)$  holomorphic in  $\mathbb{D} \setminus \{0\}$  and not holomorphic in zero, we have shown that the Lyapunov exponent function is unbounded in rank 2. To generalize the result to any rank it is enough to recall that, by Proposition 1.3.10, each component of the oper locus parametrizes the space of holomorphic connections on a fixed holomorphic vector bundle. Since the rank  $n$  oper locus  $\text{Op}_n(C)$  contains the  $(n - 1)$ -symmetric power of the uniformizing representation, it is clear that the  $(n - 1)$ -symmetric power of any representation in  $\text{Op}_2(C)$  is contained in  $\text{Op}_n(C)$ . Indeed the symmetric power of a representation in  $\text{Op}_2(C)$  defines a holomorphic connection on the holomorphic vector bundle given as the symmetric power of the vector bundle underlying the uniformizing representation. Since the top Lyapunov exponent of the  $n$ -symmetric power of a representation  $\rho$  is  $n \times \lambda_1(\rho)$ , the result in rank 2 implies the general result for any rank.  $\square$

## 1.10 Top Lyapunov exponent on Hitchin components

Let  $C$  be a compact Riemann surface. Recall that Hitchin components are connected components of the real character variety  $\text{Hom}(\pi_1(C), \text{SL}_n(\mathbb{R})) // \text{SL}_n(\mathbb{R})$  containing symmetric powers of Fuchsian representations (see Section 1.6.2). In particular the Hitchin component in rank 2 is Teichmüller space  $\mathcal{T}(C)$ .

As recalled in Section 1.6.3 there have been lately a focus on the study of dynamical invariants on Hitchin components. Recall for example the main result of [PS17] that gives a bound of the critical exponent on Hitchin components that is attained if and only if the representation is the symmetric power of a Fuchsian representation. Unlike for critical exponent, we still cannot prove a lower bound for the top Lyapunov exponent function on Hitchin components. We performed computer experiments computing the top Lyapunov exponent on the Hitchin component of rank three character variety  $\text{Hom}(\Delta(3, 3, 4), \text{SL}_3(\mathbb{R})) // \text{SL}_3(\mathbb{R})$  associated to the triangle group  $\Delta(3, 3, 4)$ . We were able to perform experiments thanks to the explicit description of matrices in the Hitchin components given in [LRT11]. The experiments indicated that as shown above the top Lyapunov exponent function grows logarithmically near the boundary of the character variety and moreover this function is greater than 2 on this family, where 2 here represents the top Lyapunov exponent of the second symmetric power of the uniformizing representation. Relating critical exponent and top Lyapunov exponent as suggested in

Section 1.6.4, together with the main result of [PS17] could lead to prove a lower bound for the top Lyapunov exponent function on Hitchin components. Even though such a bound has still to be proven and investigated, we want to state the following conjecture, which should be analogous to the bound of the critical exponent recalled above.

*Conjecture 1.* The top Lyapunov exponent function on the  $n$ -th Hitchin component is greater or equal than  $n - 1$ , which is the top Lyapunov exponent of the  $(n - 1)$ -th symmetric power of the uniformizing representation.

It is not clear to expect if, as for the critical exponent, the attainment of the bound would imply that the representation is the symmetric power of the uniformizing one.

For representation which are maximal, namely that reach the Milnor-Wood inequality (see Section 1.6.2), there are not many results. The first and easiest instance to consider is a maximal  $SU(p, q)$ -representation which is also a variation of Hodge structure. For example, for weight 1 and real K3 representation the Milnor-Wood inequality is attained if and only if the Arakelov inequality is attained (see Remark 1.6.2). In the case of a maximal variation of Hodge structure of weight 1, by Theorem 1.11.2 the sum of the first  $q = \text{rk}(\mathcal{H}^{1,0})$ -Lyapunov exponent is equal to  $q$  and the top Lyapunov exponent is one (the last result follows from the characterization of maximal weight one VHS given in [VZ04]). In the case of a maximal variation of Hodge structure of real K3 type, by Theorem 1.11.3 the top Lyapunov exponent is  $p/2 = 1$ . It is still not known if for general maximal representations the Lyapunov exponent functions any bounds hold.

## 1.11 Lyapunov exponents for variations of Hodge structures

In this section we focus on variation of Hodge structures over a hyperbolic Riemann surface  $C$  not necessarily compact. We denote by  $\Delta$  the cusps of  $C$ . Recall that a variation of Hodge structure is a special flat bundle equipped with the Hodge filtration and an indefinite Hermitian form  $H$  (see definition 1.2.1). Recall also that the Hodge norm coming from the Hodge metric  $h$  defined by the hermitian form  $H$  on  $\mathcal{V}_C$  is admissible (see [EKMZ18, Prop. 3.1]).

First of all we will use the condition given by Proposition 1.7.18 to prove results about rationality of Lyapunov exponents for variations of Hodge structures in low weight and describe what is known in these situations. More specifically, we will prove a slightly generalized version of the results of [EKZ11] and [Fil14] about equality of Lyapunov exponents and degrees of Hodge bundles in the weight 1 and in the real weight 2 variation of Hodge structures case. The core of the arguments of the proofs that we present are similar as the one in the original proofs of the papers cited above or the ones of the most recent paper [DD17a]. The main idea is to relate the Lyapunov exponents to properties of the period maps. In particular, we will use that the image of the period map cannot contain the point corresponding to some Oseledets spaces in the two cases of weight 1 and real weight 2 variation of Hodge structures.

We then concentrate on the number of zero exponents of a general variation of Hodge structures over a compact base curve. Using Simpson correspondence, we will classify



unitary representations as VHS with trivial Lyapunov spectrum and deduce from the proof of this last statement a simplicity result for non-unitary VHS. We finally use geometric Oseledets theorem to get a lower bound on the number on zero exponent for a general VHS.

In the case of variation of Hodge structures we can use the Hodge norm to get results about the Lyapunov spectrum thanks to the next well-known proposition.

**Proposition 1.11.1.** *The Oseledets subspace  $\mathcal{V}_{\lambda_i}$  are totally isotropic with respect to the  $\nabla$ -flat indefinite hermitian form  $H$  unless  $\lambda_i = 0$  and they are pairwise orthogonal unless  $\lambda_i = -\lambda_j$ .*

We recall the proof of this statement, which can be found for example in [KM16].

*Proof.* Let  $K$  be a positive measure compact subset of  $T^1C$ , let  $t_k$  be a sequence going to infinity such that  $g_{t_k}(c) \in K$  for  $t_k \rightarrow \pm\infty$  and for almost all  $c \in T^1C$ . The existence of the sequence is justified by Poincaré recurrence theorem. Let  $u_i \in \mathcal{V}_{\lambda_i}$ . Then by  $G_t$ -invariance of  $H$  and by the Cauchy-Schwartz inequality we get

$$H(u_i, u_j) = H(g_{t_k} u_i, g_{t_k} u_j) \leq c(K) \|g_{t_k} u_i\|_H \|g_{t_k} u_j\|_H \sim e^{(\lambda_i + \lambda_j) \cdot t_k}$$

where  $c(K) > 0$  is a positive constant depending only on  $K$ . Since  $e^{(\lambda_i + \lambda_j) \cdot t_k} \rightarrow 0$  for  $\lambda_i \neq \lambda_j$  and for  $t_k \rightarrow \infty$  or  $t_k \rightarrow -\infty$ , we get the result.  $\square$

**1.11.1 Weight 1 variation of Hodge structures** Recall that a weight 1 complex variation of Hodge structure of rank  $n$  over  $C$  is given by a flat vector bundle  $\mathcal{H}$  of rank  $n$  together with a holomorphic subbundle  $\mathcal{H}^{1,0} \subset \mathcal{H}$  of rank  $k$  and a  $\nabla$ -flat hermitian complex form  $H$  on  $\mathcal{H}$  that is positive definite on  $\mathcal{H}^{1,0}$  and negative definite on  $\mathcal{H}/\mathcal{H}^{1,0}$ .

We can reprove the result of [EKZ11] in the case of complex weight 1 variation of Hodge structures over hyperbolic curves.

**Theorem 1.11.2.** *If  $\mathcal{H}$  is a weight 1 complex variation of Hodge structures of rank  $n$  with  $\text{rk}(\mathcal{H}^{1,0}) = k$  over  $C$ , then*

$$\sum_{i=1}^k \lambda_i = \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{H}^{1,0}))}{\deg(\Omega_C^1(\log(\Delta)))}$$

*Proof.* We want to use Proposition 1.7.18 and prove that for any vector  $u$  of the  $G_t$ -invariant closed subspace  $\mathcal{S} \subset \mathbb{P}(\bigwedge^k \mathcal{H}^\vee)$  given by totally isotropic  $(n-k)$ -planes the bad locus  $T_{\text{bad}}^{\mathcal{H}^{1,0}}(u)$  is empty. This locus is  $G_t$ -invariant because the indefinite metric  $H$  is  $G_t$ -invariant. Moreover, we need to prove that there is a totally isotropic  $(n-k)$ -plane computing the top  $k$ -Lyapunov exponents.

Assume that  $k \geq n-k$ . In order to prove that the bad locus  $T_{\text{bad}}^{\mathcal{H}^{1,0}}(u)$  is empty, notice that the image of the period map

$$s_{\mathcal{H}^{1,0}} : \mathbb{H} \rightarrow P \left( \bigwedge^k \mathcal{H}_c \right)$$

is contained in the space of positive definite  $k$ -planes since  $H$  is positive definite on  $\mathcal{H}^{1,0}$ . Hence every  $k$ -plane  $s_{\mathcal{H}^{1,0}}(z)$  has to intersect trivially any totally isotropic  $(n - k)$ -plane, which means by condition (1.13) that the bad locus is empty for any  $u \in \mathcal{S}$ . We now only need to find a totally isotropic  $(n - k)$ -plane computing the top Lyapunov exponents. Since  $k \geq n - k$  and the Hermitian form  $H$  has signature  $(k, n - k)$ , using Proposition 1.11.1 it is easy to see that there is a totally isotropic  $(n - k)$ -plane contained in the positive Oseledets space  $\mathcal{V}_{\geq 0}$ . By definition of the Oseledets space this plane computes the sum of the top  $k$  Lyapunov exponents.

If  $k \leq n - k$  we consider the complex conjugate variation of Hodge structure. It has the same Lyapunov exponent and now the complex conjugate of the bundle  $\mathcal{H}^{0,1} := \mathcal{H}/\mathcal{H}^{1,0}$  is a holomorphic subbundle with degree equal to  $\deg(\mathcal{H}^{0,1}) = -\deg(\mathcal{H}^{1,0})$ . Now we can use the first part of the proof

$$\sum_{i=1}^k \lambda_i = - \sum_{i=1}^{n-k} \lambda_i = - \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{H}^{0,1}))}{\deg(\Omega_C^1(\log(\Delta)))} = \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{H}^{1,0}))}{\deg(\Omega_C^1(\log(\Delta)))}.$$

□

Notice that as a corollary of the last theorem we get back the first part of the statement of Proposition 1.3.19, namely that  $\mathcal{H}^{1,0}$  is the maximal degree subbundle among all the subbundles of  $\mathcal{H}$ .

We want to conclude by recalling that in the case of a weight one variation of Hodge structures corresponding to a maximal representation, we can say something more on the Lyapunov spectrum. Indeed as explained in Remark 1.6.2, in this case the Milnor-Wood inequality is the same as the Arakelov inequality. In [VZ04], weight one VHS attaining the Arakelov inequality were classified. If the base curve is not compact, the VHS is given as the direct sum of a unitary part and the direct sum of the uniformizing representations. The Lyapunov spectrum is then degenerate, namely given only by  $\text{rk}(\mathcal{H}^{1,0})$ -ones and by zeroes. Moreover in this case the base curve is a Shimura curve in the moduli space of abelian varieties. It would be really interesting to understand if the other implication holds, namely to investigate if the degenerateness of the Lyapunov spectrum in the case of non-compact base curve implies that the curve is a Shimura curve. If the base curve is not compact, the characterization of [VZ04] does not help to understand better the Lyapunov spectrum. However, by maximality we at least know that the sum of the first  $\text{rk}(\mathcal{H}^{1,0})$ -exponents is equal to  $\text{rk}(\mathcal{H}^{1,0})$ .

**1.11.2 Real variations of Hodge structures of K3 type** A weight 2 real variation of Hodge structures over  $C$  is given by a real vector bundle  $\mathcal{H}_{\mathbb{R}}$  over  $C$  such that its base change to  $\mathbb{C}$  defines a complex variation of Hodge structures  $\mathcal{H}$ . Let  $F^2 = \mathcal{H}^{2,0} \subset F^1 \subset \mathcal{H}$  be the Hodge filtration and note that  $\overline{F^2} = \mathcal{H}^{0,2} := \mathcal{H}/F^1$ . Recall that by definition there is a  $\nabla$ -flat hermitian complex form  $H$  on  $\mathcal{H}$  which is positive definite on  $\mathcal{H}^{2,0}$  and  $\mathcal{H}^{0,2}$ , and negative definite on  $\mathcal{H}^{1,1} := F^1/\mathcal{H}^{2,0}$ . Let  $n := \text{rk}(\mathcal{H})$  and  $k := \text{rk}(\mathcal{H}^{2,0})$ .

### 1.11 Lyapunov exponents for variations of Hodge structures

Recall first of all that in the case of real variations of Hodge structures of weight 2, by the geometric Oseledets theorem, and more specifically by Remark 1.6.12, there are at least  $n - 4k$  zero Lyapunov exponents. Hence the Lyapunov spectrum takes the form

$$\lambda_1, \dots, \lambda_{2k}, 0, \dots, 0, -\lambda_{2k}, \dots, -\lambda_1.$$

In [Fil14], it is proven that for a real variation of Hodge structures of weight 2 of K3 type, i.e. for  $k = 1$ , the bound of Theorem 1.7.10 is an equality for the top Lyapunov exponent when we use the holomorphic sub-line bundle  $\mathcal{E} = \mathcal{H}^{2,0}$ . We can prove a generalization of this result for any weight 2 real variation of Hodge structures using Proposition 1.7.18. The idea of the proof was suggested by Simion Filip.

**Theorem 1.11.3.** *Let  $\mathcal{H}$  be a flat vector bundle of rank  $n$  corresponding to a real variation of Hodge structures of weight two. Let  $k := \text{rk}(\mathcal{H}^{2,0})$ . Then*

$$\sum_{i=1}^k \lambda_i = \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{H}^{2,0}))}{\deg(\Omega_{\mathbb{C}}^1(\log(\Delta)))}.$$

*Proof.* We will use a simple trick, namely we will compute the sum of the first  $(n - k)$ -exponents, which is equal to the sum of the first  $k$ -exponents, using the holomorphic subbundle  $F^1$ , which has the same degree as  $\mathcal{H}^{2,0}$  since the VHS is real. Summarizing, we will prove

$$\sum_{i=1}^k \lambda_i = \sum_{i=1}^{n-k} \lambda_i = \frac{2 \deg_{\text{par}}(\Xi_h(F^1))}{\deg(\Omega_{\mathbb{C}}^1(\log(\Delta)))} = \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{H}^{2,0}))}{\deg(\Omega_{\mathbb{C}}^1(\log(\Delta)))}.$$

As in the proof of weight one case we want to use Proposition 1.7.18. Hence we will prove that for any vector  $u$  in the  $G_t$ -invariant subset  $\mathcal{S} \subset \mathbb{P}(\bigwedge^{n-k} \mathcal{H}^\vee)$  given by real isotropic  $k$ -planes, the bad locus  $\text{T}_{\text{bad}}^{F^1}(u)$  is empty. Moreover we will prove that there is one hyperplane in  $\mathcal{S}$  computing the top Lyapunov exponent.

First of all we check that all  $(n - k)$ -planes in the image of the period map

$$s_{F^1} : \mathbb{H} \rightarrow \mathbb{P}(\bigwedge^{n-k} \mathcal{H}_c)$$

do not intersect any  $k$ -plane in  $\mathcal{S}$ . Assume by contradiction that for some  $z \in \mathbb{H}$ , the  $(n - k)$ -plane  $s_{F^1}(z)$  intersects non-trivially a real isotropic  $k$ -plane in a non-zero vector  $v$ . Since the plane is real, the vector  $v$  has to belong to the  $H^{(1,1)}$  part of  $F^1$ , which is negative definite. Hence we get a contradiction, since there cannot be isotropic vectors in a negative definite vector space. Finally, by Proposition 1.11.1, the Oseledets space  $u = \sum_{i=1}^k \mathcal{V}_{\lambda_i}$  is a totally isotropic real  $k$ -plane. By definition  $u$  computes the top Lyapunov exponent.  $\square$

Notice that as a corollary of the last theorem we get a similar statement as the one of Proposition 1.3.19.

**Corollary 1.11.4.** *Let  $\mathcal{H}$  be a flat vector bundle corresponding to a real variation of Hodge structures of weight two. Then for any subbundle  $\mathcal{E} \subset \mathcal{H}$  with  $\mathrm{rk}(\mathcal{E}) \leq \mathrm{rk}(\mathcal{H}^{2,0})$  it holds*

$$\mathrm{deg}(\mathcal{E}) \leq \mathrm{deg}(\mathcal{H}^{2,0}).$$

Recall finally that in the special case of a real K3 VHS corresponding to a maximal representation, the top Lyapunov exponent is 2, since the Milnor-Wood inequality is the same as the Arakelov inequality (see Remark 1.6.2). Moreover, by the results of [STZ03] we know that this is the case if and only if the variation of Hodge structures comes from the Kummer construction applied to the product of two isogenous modular elliptic curve. This in particular means that the base curve is a Shimura curve in the moduli space of K3 surfaces, which is the same as the period domain of variation of Hodge structures of real K3 type.

### 1.11.3 Zero exponents and simplicity results for variation of Hodge structures.

First of all, using the geometric Oseledets theorem (Theorem 1.6.11) we can give a bound on the number of zeroes of variation of Hodge structures.

**Proposition 1.11.5.** *Let  $(\mathcal{V}, \nabla)$  be a rank  $n$  irreducible complex variation of Hodge structures of weight  $k$  with Hodge filtration  $(F^i)$ . Let  $p = \sum_{i \equiv 0(2)} \mathrm{rk}(F^i/F^{i+1})$  be the rank of the even part of  $\mathcal{V}$ . Then there are at least  $|n - 2p|$  zero Lyapunov exponents.*

*Proof.* By definition of complex variation of Hodge structures, the monodromy representation has values in  $SU(p, n - p)$ . By the geometric Oseledets theorem and in particular by Remark 1.6.12 where the  $SU(p, q)$  case is discussed, there are at least  $|n - 2p|$  zero Lyapunov exponents.  $\square$

We now prove that for a variation of Hodge structure over a compact curve, triviality of the Lyapunov spectrum is equivalent to having weight 0, which means corresponding to a unitary representation.

**Proposition 1.11.6.** *Let  $(\mathcal{V}, \nabla)$  be a rank  $n$  irreducible variation of Hodge structure. Then the following are equivalent:*

1.  $\mathcal{V}$  is a variation of Hodge structures of weight zero.
2. the corresponding monodromy representation is unitary.
3.  $\mathcal{V}$  is stable.

Moreover the conditions of above are equivalent to have trivial Lyapunov spectrum.

*Proof.* Recall that by the result of Narasimhan-Seshadri (Theorem 1.3.8), the locus of unitary representations corresponds to the locus of stable vector bundles  $\mathcal{V}$  equipped with the harmonic metric connection or, by Simpson correspondence, to the locus of Higgs bundles with zero Higgs field. If a variation of Hodge structures is of weight zero, then the Griffiths filtration is trivial and so the associated system of Hodge bundles

has zero Higgs field. So the associated representation is unitary. Moreover, again by Narasimhan-Seshadri's result, if the associated representation is unitary, then  $\mathcal{V}$  is stable. We will prove now that stability implies weight zero. Assume by contradiction that the weight is bigger than zero, so that we have a non-trivial filtration

$$F^{k+1} = 0 \subset \dots \subset F^0 = \mathcal{V}$$

corresponding to the system of Hodge bundles

$$(\mathrm{Gr}_F(\mathcal{V}) = \bigoplus_p \mathcal{V}^{p,n-p}, \nabla^{gr}), \quad \mathcal{V}^{p,n-p} := F^p / F^{p+1}.$$

By Griffiths transversality condition,  $(\mathcal{V}^{0,n}, \nabla_{|\mathcal{V}^{0,n}}^{gr} = 0) \subset (\mathrm{Gr}_F(\mathcal{V}), \nabla^{gr})$  is a sub-Higgs bundle, and so by semistability of Higgs bundles in  $\mathcal{M}_{\mathbb{H}}^{(n)}(\mathbb{C})$ , we have that  $\mu(\mathcal{V}^{0,n}) \leq \mu(\mathrm{Gr}_F(\mathcal{V})) = 0$ . Since  $\deg(\mathrm{Gr}_F(\mathcal{V})) = \sum_p \deg(\mathcal{V}^{p,n-p})$ , it follows that

$$\deg(\mathcal{V}^{0,n}) = -\deg\left(\bigoplus_{p \neq 0} \mathcal{V}^{p,n-p}\right) = -\deg(F^1) \leq 0.$$

So we have found that  $\deg(F^1) \geq 0 = \mu(\mathcal{V})$  which is impossible by the stability of  $\mathcal{V}$ . We prove now the last statement of the proposition. We know that a unitary representation has trivial Lyapunov spectrum since in this case the norm used to compute the Lyapunov exponent is invariant under parallel transport. Conversely, if the Lyapunov spectrum is zero then, since  $F^1 \subset \mathcal{V}$  is a holomorphic subbundle, we have that  $0 = \sum_{i=1}^{\mathrm{rk}(F^1)} \lambda_i \geq \deg(F^1)$  by Theorem 1.7.10. But as we already saw, if the weight is positive then it holds also  $0 \leq \deg(F^1)$ . If  $F^1$  has zero degree then it defines a flat subbundle, contradicting the hypothesis of irreducibility of  $\mathcal{V}$ .  $\square$

From the last Theorem we get a direct corollary about the sum of the first  $\mathrm{rk}(F_1)$  Lyapunov exponents of a positive weight variation of Hodge structures.

**Corollary 1.11.7.** *Let  $(\mathcal{V}, \nabla)$  be a rank  $n$  irreducible variation of Hodge structures of positive weight. Then the the sum of the first  $\mathrm{rk}(F^1)$  Lyapunov exponents is positive and the following non-trivial bound holds:*

$$\sum_{i=1}^{\mathrm{rk}(F_1)} \lambda_i(\mathcal{V}) \geq \deg(F^1) = \deg(\mathcal{V}^{n,0}) > 0.$$

*Proof.* From the proof of the last Theorem we see that if the weight is positive and the variation of Hodge structures is irreducible then the degree of  $F^1$  is strictly positive. Hence the bound (1.7) gives the result.  $\square$

## 1.12 Lyapunov exponent function in rank two

From now on we will return to consider only the case of a compact Riemann surface  $C$ , since we want to deal with the moduli spaces described in the first sections. We saw that Lyapunov exponents are defined for every representation of the fundamental group  $\pi_1(C) \rightarrow \mathrm{SL}_n(\mathbb{C})$  with non-expanding cusp monodromies. Hence if  $C$  is compact, they are defined for every point of the character variety. Moreover Lyapunov exponents are obviously independent of the conjugacy class of the representation, so they give functions on the character variety. Notice that Lyapunov exponents are naturally defined on the de Rham moduli space since they depend both on a representation and on the complex structure of the base curve, which determines the hyperbolic metric and hence the geodesic flow. We can then define functions, which we will call the Lyapunov exponent functions, from the de Rham moduli space

$$\lambda_i : \mathcal{M}_{\mathrm{DR}}^{(n)}(C) \rightarrow \mathbb{R}, \quad \mathcal{V} \mapsto \lambda_i(\mathcal{V}), \quad i = 1, \dots, n$$

that send a flat holomorphic bundle over  $C$  to its  $i$ th Lyapunov exponent. Recall that we have to take care only of half of the Lyapunov spectrum since it is symmetric (see Remark 1.1.2).

It seems natural now to ask what are the properties of these functions. Note that for a rank 1 holomorphic flat bundle the Lyapunov spectrum has to be trivial since it is symmetric. Hence the first non trivial case to study is the one of rank 2 flat bundles. In this case we care only about the top Lyapunov exponent because of the symmetry of the Lyapunov spectrum. From now on we focus on this case. Using the relation of the geodesic flow and the random product on the fundamental group, we reprove continuity of the top Lyapunov exponent function and the characterization of the locus of zero exponent (see [DD17b] for the proof in the Brownian motion setup). We then use the Shatz stratification of  $\mathcal{M}_{\mathrm{DR}}^{(2)}(C)$  together with the bound of Proposition 1.9.1, which is an equality in the compact case thanks to Theorem 1.8.1, in order to describe the Lyapunov exponent function restricted to the Shatz strata. This provides a generalization of the main result of [DD17b] that was only about the maximal stratum, the oper locus. Using the theory of projective structures and the knowledge of the VHS loci in rank 2, we describe how the top Lyapunov exponent function behaves on the special loci described in Figure 1.1.

**1.12.1 Geodesic flow and random products** Before describing some properties of the top Lyapunov exponent function, we need a lemma that relates the cocycle  $G_t : \mathcal{V}_\rho \rightarrow \mathcal{V}_\rho$  defined by parallel transport over the geodesic flow to the random product of matrices in the monodromy group. We will argue in a similar way as in [EM15, Proof of Th. 1]. By a result of H. Furstenberg [Fur71], there is a probability measure  $\nu$  on the uniformizing group  $\Gamma < \mathrm{SL}_2(\mathbb{R})$  of  $C$  with support equal to  $\Gamma$  such that the Poisson boundary of  $(\Gamma, \nu)$  is  $(\mathrm{SO}(2, \mathbb{R}), \mathrm{Leb})$ . We will denote by  $\lambda_{\rho(\nu)}(u)$  the Lyapunov exponents of  $u \in \mathcal{V}_c$  with respect to the random walk of law  $\rho(\nu)$  on the monodromy group  $\rho(\pi_1(C, c)) = \rho(\Gamma)$ .

By definition, for  $\nu^{\mathbb{N}}$ -almost any  $(\gamma_1, \dots, \gamma_n, \dots) \in \Gamma^{\mathbb{N}}$  and any norm  $\|\cdot\|$  on the vector space  $\mathcal{V}_c$  it holds

$$\lambda_{\rho(\nu)}(u) = \lim_{n \rightarrow \infty} \frac{1}{n} \|\rho(\gamma_n) \cdots \rho(\gamma_1)u\|.$$

**Lemma 1.12.1.** *The Lyapunov exponents  $\lambda(\mathcal{V}_\rho)$  defined by the cocycle given by parallel transport over the geodesic flow of  $T^1C$  and the Lyapunov exponents  $\lambda_{\rho(\nu)}(u)$  given by the random walk on the monodromy group  $\rho(\pi_1(C, c))$  coincide.*

*Proof.* Recall that by definition the Lyapunov exponent  $\lambda(\mathcal{V}_\rho)$  is defined to be

$$\lambda(\mathcal{V}_\rho) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|G_t(u)\|$$

for almost any  $c \in T^1C$  and almost all  $u \in \mathcal{V}_c$ . Here  $\|\cdot\|$  is the constant norm which is integrable by Theorem 1.1.3. Since we are dealing with the case of a compact Riemann surface  $C$ , we can prove a stronger property than integrability of this norm. Consider the lift of the geodesic flow to  $T^1\mathbb{H}$ . Since  $C$  is compact, a geodesic segment of unit length can cross the boundary of a fundamental domain only a finite number of times. This number is uniformly bounded for all starting points. Hence, there is a constant  $M > 0$  such that

$$\|\log(G_t(u))\| \leq M \cdot t \tag{1.15}$$

for all  $u \in \mathcal{V}_\rho$  and all  $t \in \mathbb{R}$ .

Let us denote by  $(z, \theta) \in T^1\mathbb{H}$  a lift of a point  $c \in T^1C$ . It is well known that a typical trajectory of the random walk in  $\Gamma \cdot i \subset \mathbb{H}$  tracks a geodesic ray in  $\mathbb{H}$  up to sublinear error (see [CE15, Lemma 4.1]). This means that for almost all  $(\gamma_1, \dots, \gamma_n, \dots) \in \Gamma^{\mathbb{N}}$  there exists a geodesic ray  $\{g_t(z, \theta) : t \in \mathbb{R}\} \subset \mathbb{H}$  such that

$$\text{dist}_{\text{hyp}}(\gamma_n \cdots \gamma_1 \cdot i, g_n(z, \theta)) = o(n) \tag{1.16}$$

for almost any  $(z, \theta) \in T^1\mathbb{H}$ .

Putting together the bound (1.15) and the tracking property (1.16) we get

$$\log \left( \frac{\|G_n(u)\|}{\|\rho(\gamma_n) \cdots \rho(\gamma_1)u\|} \right) \leq M \cdot \text{dist}_{\text{hyp}}(\gamma_n \cdots \gamma_1 \cdot i, g_n(z, \theta)) = o(n)$$

for any  $u \in \mathcal{V}_\rho$  in the fiber over  $(z, \theta)$ . We conclude with the desired result

$$\lambda(\mathcal{V}_\rho) - \lambda_{\rho(\nu)}(u) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \frac{\|G_n(u)\|}{\|\rho(\gamma_n) \cdots \rho(\gamma_1)u\|} \right) \leq \lim_{n \rightarrow \infty} \frac{1}{n} o(n) = 0$$

□

Thanks to the last lemma we can prove some properties of the top Lyapunov exponent function using known results about random products of matrices. The next two propositions were already proven in [DD17b] in the context of Brownian motion using the same core arguments about random walks. The main difference is that we related the geodesic flow to the random walk while they related the Brownian motion to the random walk.

**1.12.2 Locus of zero top Lyapunov exponent** We can describe locus where the top Lyapunov exponent vanishes.

**Proposition 1.12.2.** *In rank 2, the Lyapunov exponent associated to a representation  $\rho$  is zero if and only if  $\rho$  is elementary.*

*Proof.* By Lemma 1.12.1, we can use classical results about random walks in the monodromy group to establish if the top Lyapunov exponent vanishes. By a Theorem of Furstenberg (see [Via14, Th. 6.11]), the Lyapunov exponents associated to a random walk in rank 2 are non zeroes if and only if the cocycle is pinching and twisting.

If the cocycle is non-pinching, then the monodromy group is contained in a compact subgroup of  $SL_2(\mathbb{C})$ , so it is the unitary case. If the cocycle is non-twisting then the monodromy group is a diagonal subgroup or a triangular subgroup or the image is contained in the subgroup generated by  $\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle$ . These cases are exactly the cases of non-elementary representations.  $\square$

**1.12.3 Continuity of the top Lyapunov exponent function** Using ones again Lemma 1.12.1, we can use known results about random walks to prove continuity of the top Lyapunov exponent function.

**Proposition 1.12.3.** *The top Lyapunov exponent function*

$$\lambda_1 : \mathcal{M}_B^{(2)} \rightarrow \mathbb{R}_{\geq 0}$$

*is a continuous function. Moreover it is locally Holder continuous on the set of non-elementary representations.*

*Proof.* By Lemma 1.12.1, we can use continuity results of Lyapunov exponents proved in the context of random walks. By [LP89] the top Lyapunov exponent function is locally Holder continuous on the set of non-elementary representations. Moreover, since the set of elementary representations coincides with the set of zero exponents, the top Lyapunov exponent function is continuous also at these points since all exponents are equal (see for example [Via14, Corollary 9.3]).  $\square$

We will now describe special loci where we can say something about the Lyapunov exponent. The main idea is to use Theorem 1.7.13 to get a lower bound for the top Lyapunov exponent on the Shatz strata. We can then describe the special loci where we have more information about the Lyapunov exponent. These special loci are the ones described in Picture 1.1.

**1.12.4 Lyapunov exponent function is unbounded** Recall that by Theorem 1.9.2 the top Lyapunov exponent function is unbounded on the oper locus with logarithmic growth near the boundary of the character variety. The Lyapunov exponent function should be unbounded also on the space of Fuchsian representation, which can be identified with the Teichmüller space, and hence on any Hitchin component, but it was still not proven at this moment.



**1.12.5 Lyapunov exponents and Shatz stratification in rank 2** Recall that the Shatz stratification is defined by Harder-Narasimhan type. In rank 2, there is a stratum for each integer  $0 \leq e \leq g - 1$  and for opers, which define the maximal stratum, the maximal destabilizing subbundle has degree  $g - 1$ . Recall moreover that the holonomy map  $\text{hol} : \mathcal{P}(S) \rightarrow \mathcal{M}_{\mathbb{B}}^{(2)}$  has image equal to the space of non-elementary representations (see Theorem 1.4.4).

**Proposition 1.12.4.** *Let  $\mathcal{V}$  be a flat vector bundle in the  $e$ -th Shatz stratum and let  $\mathcal{L} \subset \mathcal{V}$  be the maximal destabilizing sub-line bundle of degree  $e$ . Let  $\text{dev}_{\mathcal{L}} : \mathbb{H} \rightarrow \mathbb{P}_{\mathbb{C}}^1$  be the associated developing map. Then the first Lyapunov exponent associated to  $\mathcal{V}$  is given by*

$$\lambda_1(\mathcal{V}) = \frac{e}{g-1} + 4\pi \lim_{r \rightarrow \infty} \frac{1}{r} \int_0^r \frac{\#\{\text{dev}^{-1}(x) \cap D_r(z)\}}{\text{vol}(D_r(z))} dr$$

for almost any  $z \in \mathbb{H}$  and almost any  $x \in \mathbb{P}_{\mathbb{C}}^1$ .

*Proof.* The statement follows directly from Theorem 1.8.1 and the equivalent definition of bad locus given in Remark 1.7.12.  $\square$

*Remark 1.12.5.* The main result of [DD17b] relating Lyapunov exponents to the covering degree defined in term of the developing map of a projective structure is a special case of the last theorem for  $e = g - 1$ , meaning in the case of the oper locus or equivalently of  $\text{hol}(\mathcal{P}(C))$ . The difference in the above equality and the one in [DD17b] is given by the different normalization of the hyperbolic metric on  $C$ .

In the case of the maximal stratum, which is the oper locus  $\text{Op}_2(C)$ , we can say something more using the description via holonomy of projective structures given by Proposition 1.5.1. Recall that in remark 1.5.4 we defined the subset  $B(C) \subset \text{Op}_2(C)$  given by the Bers embedding.

**Proposition 1.12.6.** *The Lyapunov exponent function restricted to the oper locus is greater or equal than one and unbounded. Moreover it is one if the representation belongs to the closed subset  $\overline{B(C)} \subset \text{Op}_2(C)$  containing the uniformizing representation.*

*Proof.* On the oper locus the maximal degree sub-line bundle is isomorphic to a square root  $\mathcal{K}_C^{1/2}$  of the canonical bundle which has degree equal to  $g - 1$ . Hence the Proposition 1.7.10 gives that the top Lyapunov exponent has to be greater than  $\lambda_1 \geq 1$ . Moreover the top Lyapunov exponent function is unbounded by Theorem 1.9.2.

Since the closed locus  $\overline{B(C)} \subset \text{Op}_2(C)$  is defined by the Bers embedding, by the density theorem the standard developing map giving the inclusion of the maximal sub-line bundle  $\mathcal{V}_1 \subset \mathcal{V}$  (see Proposition 1.5.1) is a biholomorphism of  $\mathbb{H}$  onto one of the two domain of discontinuity of the associated representation. Hence the image of the developing map does not intersect the limit set of the representation. By the condition of Corollary 1.7.22 we get equality.  $\square$

## 1 Lyapunov exponents on character varieties

*Remark 1.12.7.* In [DD17b] they are able to prove the inverse of the second statement of the last proposition, namely that if the representation is in the complement  $\mathrm{Op}_2(C) \setminus \overline{B(C)}$  of the Bers embedding, then  $\lambda_1 > 1$ . The tool that they can use is that they know that the support of the harmonic measure is in the limit set and that the error term vanishes if and only if the image of the developing map intersect this support.

### Variations of Hodge structures locus

Recall that the locus in  $\mathcal{M}_{\mathrm{DR}}^{(2)}(C)$  corresponding to complex variation of Hodge structures is quite well understood. We described its connected components  $P_e$  in Section 1.5.4. Since for  $e = 0$  we get weight zero variations of Hodge structures and for  $0 < e \leq g - 1$  we get weight 1 variation of Hodge structures, we can compute exactly the associated top Lyapunov exponent in all of these loci.

**Proposition 1.12.8.** *The top Lyapunov exponent function restricted on the connected component  $P_e$  of the variation of Hodge structure locus is constant and given by*

$$\lambda_{1|P_e} = \frac{e}{g-1}, \quad e = 0, \dots, g-1.$$

*Proof.* We already saw in Proposition 1.11.6 that the Lyapunov spectrum of a weight zero variation of Hodge structure is trivial. By the description of  $P_e$  we know that if  $\mathcal{H} \in P_e$ , then the Hodge bundle  $\mathcal{H}^{1,0} \subset \mathcal{H}$  has degree  $e$  and the result follows from Theorem 1.11.2 since we are dealing with weight 1 variation of Hodge structures.  $\square$

### Real representations locus

By Proposition 1.12.2 the top Lyapunov exponent is the constant zero function on the connected component  $\mathcal{M}_{B,\mathbb{R},0}^{(2)}$  with zero Toledo invariant, since it is contained in the locus of elementary representations. Moreover there are points on the Fuchsian locus with Lyapunov exponent equal to one, namely the uniformizing representation associated to  $C$ . The Fuchsian locus intersects the oper locus countably many times and in these points the top Lyapunov exponent is strictly greater than one by Remark 1.12.7. Moreover, if Conjecture 1 is true the top Lyapunov exponent function is greater than one on the full Fuchsian locus.

## 2 The Equation of the Kenyon-Smillie (2, 3, 4)-Teichmüller curve

The relation of Lyapunov exponents to algebraic geometry originated from the study of the dynamics of Teichmüller curves. Almost all known examples of Teichmüller curves come in infinite families. In this chapter we compute the equations of the algebraic family of curves over one of the two sporadic Teichmüller curves, the Kenyon-Smillie (2, 3, 4)-Teichmüller curve. The characterization of Teichmüller curves via their associated variation of Hodge structure proved in [Möl06b] and the description of the associated Harder-Narasimhan type computed in [YZ13] are ones of the main used tools. Once we compute the equation of the family, we test that the family of plane quartics that we found is indeed a Teichmüller curve by computing the Picard-Fuchs differential equation associated to the corresponding variation of Hodge structures using the Griffiths-Dwork method. The content of this chapter is joint work with André Kappes and has appeared as [CK17].

### 2.1 Introduction

Almost all known primitive Teichmüller curves fall in very few series. Currently, an infinite series in genus 2 is known by independent work of Calta and McMullen ([McM03], [Cal04]), which generalizes to the construction of the infinite Prym families ([McM06]). In addition, there is the infinite series of Bouw-Möller curves ([BM10b]) generalizing [Vee89] and [War98]. Moreover, recently McMullen, Mukamel and Wright discovered a new series of Teichmüller curves in genus 4.

There are only two known primitive Teichmüller curves that do not belong to any of these families. One of them parametrizes all affine deformations  $\{(S_t, \omega_t)\}_t$  of the translation surface  $(S, \omega) \in \Omega\mathcal{M}_3(3, 1)$  that is obtained from unfolding a Euclidean triangle with angles  $(\frac{2\pi}{9}, \frac{3\pi}{9}, \frac{4\pi}{9})$ . It was discovered by Kenyon and Smillie [KS00], who proved that  $(S, \omega)$  is a lattice surface with Veech group equal to the triangle group  $\Delta(9, \infty, \infty)$ . The translation surface  $(S, \omega)$  is the order 9 orbifold point of its associated Teichmüller curve, which is uniformized by  $\mathbb{H}/\Delta(9, \infty, \infty)$ .

**The equation.** In this paper, we discuss the Kenyon-Smillie (2, 3, 4)-Teichmüller curve from an algebro-geometric perspective. First, we derive the equations of the algebraic curves  $\{S_t\}_t$  parametrized by this Teichmüller curve.

## 2 The Equation of the Kenyon-Smillie (2, 3, 4)-Teichmüller curve

**Theorem 2.1.1.** *The universal family over the complement of the orbifold point of the Kenyon-Smillie (2, 3, 4)-Teichmüller curve is given by the family of plane quartics satisfying the equation*

$$X^4 + t(X^4 - 3X^3Y + 6X^3Z - 3X^2Y^2 - 6X^2YZ + 6X^2Z^2 + 4XY^3 - 6XY^2Z - 6XYZ^2 + XZ^3 + 3Y^4 + 3Y^3Z) = 0 \quad (2.1)$$

where  $t$  varies in  $\mathbb{P}^1 - \{0, 1, \infty\}$ .

The triple zero of the differential  $\omega_t$  is the point  $p_t = (0 : 0 : 1) \in S_t$  and the simple zero is the point  $q_t = (0 : 1 : -1) \in S_t$ .

Since this Teichmüller curve has one orbifold point, which is  $t = 0$  in our presentation, the family we give is universal over the complement of this point. See formula (2.10) for the universal family over a cover of the full Teichmüller curve.

We prove Theorem 2.1.1 in two different ways. Firstly we prove it constructing the equation using considerations on the conditions imposed by the three special points of the Teichmüller curve, the orbifold point  $S_0$  and the two cusps  $S_1$  and  $S_\infty$ . Secondly, we show again that Equation (2.1) defines a Teichmüller curve by constructing the Picard-Fuchs differential equation associated to the variation of Hodge structure over the Teichmüller curve.

**The Torsion map.** By [Möl06a], the difference between two zeros of  $\omega_t$  is a torsion point of the Jacobian of  $S_t$ . Thus, there is a minimal  $n$  such that for all  $t$ , the divisor  $n(p_t - q_t)$  is the divisor of a meromorphic function, which we call the *torsion map*  $\text{Tor}$ . Using Theorem 2.1.1, we can explicitly determine this map for the Kenyon-Smillie Teichmüller curve and we notice that  $q_t$  is a hyperflex for every  $t$ .

**Proposition 2.1.2.** *The torsion map is given by the projection from  $q_t$ , which is the map induced by the linear system  $|K_{S_t}(-q_t)|$ . The point  $q_t$  is a hyperflex for every  $t$ , which means that the tangent to  $S_t$  in  $q_t$  is of order 4.*

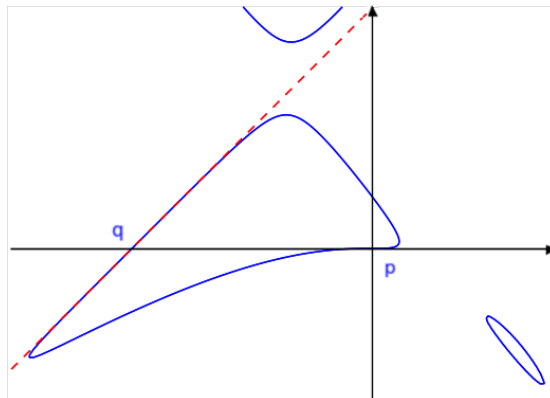


Figure 2.1: Real points of the curve  $S_t$  for  $t = 3$  near  $p = (0, 0)$  and the hyperflex  $q = (0, -1)$ .

In the above coordinates, the torsion map is the degree 3 map totally ramified over  $p_t$  and  $q_t$  given by

$$\text{Tor} : S_t \rightarrow \mathbb{P}^1, \quad \begin{cases} (X : Y : Z) \mapsto (X : Y + Z) & \text{for } (X : Y : Z) \neq (0 : 1 : -1) \\ q_t = (0 : 1 : -1) \mapsto (-1 : 1) \end{cases}$$

We will give also a proof of the above Proposition in which we do not use the explicit equation of the universal family. Indeed, it is enough to know that the Teichmüller curve parametrizes quartics in  $\mathbb{P}^2$  together with a differential in the stratum  $(3, 1)$ .

Notice that, since every  $S_t$  has a hyperflex in  $q_t$ , the Teichmüller curve is also in the image of the projection of the stratum  $\Omega\mathcal{M}_3(4)^{\text{odd}}$ , which is a divisor in  $\mathcal{M}_3$ .

The initial motivation of the paper was to investigate a question of Alex Wright about the relation between real multiplication and the torsion map. One can see such a relation in the Veech-Ward-Bouw-Möller Teichmüller curves, where real multiplication is induced by the correspondence given by the graph of an automorphism coming from the Galois normalization map (see Section 2.4.8 for details). As in the Veech-Ward-Bouw-Möller case, the Kenyon-Smillie Teichmüller curve is also uniformized by a triangle group. However, we will deduce from Proposition 2.1.2 that for the Kenyon-Smillie-Teichmüller curve, real multiplication does not come from the normalization of the torsion map. This gives some evidence in support of the Kenyon-Smillie example being sporadic.

**Picard-Fuchs equation.** Given only Equation (2.1), we can prove independently that it is indeed the equation of the universal family over a Teichmüller curve by computing the Picard-Fuchs equation satisfied by the periods of  $\omega_t$ .

**Proposition 2.1.3.** *The periods of  $\omega_t$  are solutions of the following differential equation:*

$$\frac{16}{81t(t-1)}y + \frac{17t-8}{9t(t-1)}y' + y'' = 0.$$

Note that this equation is a hypergeometric differential equation, meaning that it has three regular singular points.

The main ingredient to compute the above Picard-Fuchs equation is the well-known Griffiths-Dwork method, which is an algorithm to compute Picard-Fuchs equations of family of projective hypersurfaces.

At this point, it is easy to show that that the absolute cohomology bundle has a maximal Higgs rank 2 subbundle.

**Corollary 2.1.4.** *The differential  $\omega_t$  defines a maximal Higgs, irreducible, rank 2 subbundle of the absolute cohomology bundle.*

By the characterization of Teichmüller curves of [Möl06b, Theorem 5.3], Corollary 2.1.4 shows again that Equation (2.1) defines a Teichmüller curve.

**Structure of the chapter.** In Section 2.2, we set up the notation and we gather known facts about the Kenyon-Smillie Teichmüller curve and the splitting of  $H^0(S_t, \Omega^1)$  into one-dimensional eigenspaces given by real multiplication on the Jacobian of  $S_t$ . In Section 2.3, the three special points, the two cusps and the orbifold point, of the Kenyon-Smillie-curve are discussed. Section 2.4 contains the proof of Theorem 2.1.1 and Proposition 2.1.2. The starting point is that every non-hyperelliptic curve in genus 3 is a canonically embedded quartic in  $\mathbb{P}^2$ . The main ingredient in the proof of Theorem 2.1.1 is the use of a structural result for the relative canonical ring of a family of quartics of [CP06]. This, together with the computation of the degree of the eigenspace bundles, enables us to obtain an a priori bound on the degree of the coefficients of the family. It then suffices to consider the three special points on the Teichmüller curve to compute these coefficients. In Section 2.5, we derive the Picard-Fuchs equation proving Proposition 2.1.3 and we show again that Equation (2.1) defines a Teichmüller curve.

**Notes and references.** The strategy that we use to determine the equation of the universal family has already been successfully implemented before by Bouw and Möller [BM10a], who have worked out the equations of two of the Weierstraß curves in genus two. The main difference here is that we work with a family of smooth quartic hypersurfaces in  $\mathbb{P}^2$  instead that with a family of hyperelliptic curves.

For an algebraic description of the other Weierstraß curves in genus two with fundamental discriminant  $D < 100$ , see [KM14]. The methods that they use can be applied to other Weierstraß curves in genus two and possibly also to the Prym curves in genus three and four.

The second Teichmüller curve that is not known to belong to an infinite series has been found by Vorobets [Vor96] and is generated by the surface in  $\Omega\mathcal{M}_4(6)$  obtained from unfolding a billiard in a triangle with angles  $(\frac{\pi}{5}, \frac{\pi}{3}, \frac{7\pi}{15})$ . To the best of our knowledge, no algebro-geometric construction for its Teichmüller curve is known so far.

It is an open question, whether these two Teichmüller curves are truly sporadic. One argument in support of this hypothesis is given in [Lei04, Section 7], from which one can deduce that these Teichmüller curves do not fit into a family, where the Veech group is generated by two Dehn multi-twists. Since not all parabolics are multi-twists, this is however not enough to prove sporadicity among Veech groups generated by two parabolics. It is also worth noting that both the Kenyon-Smillie Teichmüller curve and the Vorobets Teichmüller curve correspond to two of the three exceptional billiard triangles that are naturally attached to the sporadic Coxeter groups  $E_6$ ,  $E_7$  and  $E_8$ .

Notice that, curiously, the Kenyon-Smillie Teichmüller curve is the only known primitive Teichmüller curve where the zeros of the differential have different orders and the only known primitive Teichmüller curve that does not possess an involution negating  $\omega$ .

It would be interesting to find the correspondence that gives real multiplication for the Kenyon-Smillie Teichmüller curve. A related open question is whether real multiplication for this curve is of Hecke type as in the Veech-Ward-Bouw-Möller case (cf. [Wri13]).

## 2.2 Preliminaries

In this section, we gather building blocks for the proof of the main theorem. We assume that the reader is familiar with the basic notions on Teichmüller curves. For background reading one may consult for example [Möl13].

**2.2.1 Orbifold uniformization and universal family** Let  $C$  denote the Kenyon-Smillie-Teichmüller curve associated to the flat surface  $(S, \omega)$  shown in Figure 2.2.

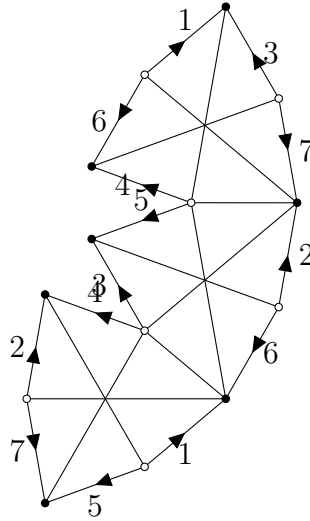


Figure 2.2: The Kenyon-Smillie  $(2, 3, 4)$ -lattice surface. Sides are labeled by powers of  $\zeta_9 = \exp(2\pi i/9)$ . The triple (simple) zero is marked by a white (black) dot.

In [KS00], Kenyon and Smillie proved that the Veech group of  $(S, \omega)$  is the triangle group  $\Delta(9, \infty, \infty)$ . Thus,  $C$  is uniformized by  $\mathbb{H}/\Delta(9, \infty, \infty)$  and the completion of  $C$  is isomorphic to  $\mathbb{P}^1$ . Let  $t \in \mathbb{P}^1$  be a parameter such that  $t = 0$  is the orbifold point of order 9,  $t = 1$  is the cusp corresponding to the limit deformation of the horizontal cylinder decomposition of  $(S, \omega)$  and  $t = \infty$  the cusp corresponding to the vertical one. Let  $S_t$  denote the (stable) Riemann surface parametrized by  $t$  and let  $\omega_t$  denote the (stable) holomorphic 1-form on  $S_t$  that is obtained by affinely deforming  $\omega$ . By our choice of  $t$ , we have  $(S, \omega) = (S_0, \omega_0)$ . Let further  $p_t, q_t \in S_t$  be such that

$$\operatorname{div}(\omega_t) = 3p_t + q_t.$$

We will abbreviate  $p = p_0$  and  $q = q_0$ .

To avoid working with orbifold line bundles, we pass to a finite étale covering  $\tilde{C}$  of  $C$ . In order to unfold the orbifold structure of  $C$ , we perform the 9-sheeted covering

$$\tilde{C} \rightarrow \mathbb{P}^1, \quad s \mapsto t = s^9$$

## 2 The Equation of the Kenyon-Smillie (2, 3, 4)-Teichmüller curve

which is totally ramified over the orbifold point and the cusp at  $\infty$ . The corresponding subgroup  $\Gamma'$  of the Veech group is free and the quotient by  $\Gamma'$  of the universal family over the Teichmüller disk gives the family of curves  $\tilde{\phi} : \tilde{\mathcal{X}} \rightarrow \tilde{C}$ . By the Riemann-Hurwitz theorem the genus of  $\tilde{C}$  is still 0 and by construction it has 10 cusps, which are given by the points  $\{\zeta_9^i, \infty\}_{i=1, \dots, 9}$  where  $\zeta_9 := \exp(2\pi i/9)$ .

Let  $\overline{C} \cong \mathbb{P}^1$  denote the completion of  $\tilde{C}$ . The family  $\tilde{\mathcal{X}}$  is extended to the family  $\overline{\phi} : \overline{\mathcal{X}} \rightarrow \overline{C}$ , by adding stable curves at the cusps of  $\tilde{C}$ .

The *absolute cohomology bundle*, which will come up in Section 2.5, is the flat vector bundle on  $\tilde{C}$  with fiber  $H^1(\tilde{\mathcal{X}}_s, \mathbb{C})$  over a point  $s \in \tilde{C}$ . Formally, it is defined as  $R^1\tilde{\phi}_* \mathbb{Q} \otimes_{\mathbb{C}} \mathcal{O}_{\tilde{C}}$ . It has a canonical extension, due to Deligne, to a vector bundle on all of  $\overline{C}$ , which we will still refer to as the absolute cohomology bundle.

**2.2.2 Real multiplication** By [Möl06b, Theorem 2.7], the Jacobian of  $\overline{\mathcal{X}}_s$  has real multiplication by the trace field  $\mathbb{K}(S, \omega)$ , which in our case is the cubic number field

$$\mathbb{K}(S, \omega) = \mathbb{Q}(v), \quad v = 2 \cos\left(\frac{2\pi}{9}\right).$$

In particular,  $C$  is an algebraically primitive Teichmüller curve. Note that  $v$  is a root of  $P(v) = v^3 - 3v + 1$ . The field  $\mathbb{K}(S, \omega)$  has three embeddings  $\sigma_i$ ,  $i = 1, 2, 3$ , into  $\mathbb{R}$ . We denote the image of an element  $\lambda \in \mathbb{K}(X, \omega)$  under the  $i$ -th embedding by  $\lambda^{(i)}$ . We order the three embeddings by requiring that

$$v^{(1)} = v, \quad v^{(2)} = 2 - v - v^2, \quad v^{(3)} = -2 + v^2.$$

This choice fits together with the ordering of the corresponding eigenspace bundles described later, as one can see by looking at the differentials over the cusps given in Section 2.3.

Having real multiplication implies that the direct image  $\mathcal{V} := \overline{\phi}_* \omega_{\overline{\mathcal{X}}/\overline{C}}$  of the relative dualizing sheaf  $\omega_{\overline{\mathcal{X}}/\overline{C}}$ , is a rank 3 vector bundle which splits as a direct sum of three line bundles

$$\mathcal{V} = \bigoplus_{i=1}^3 \mathcal{L}_i$$

where  $\mathcal{L}_i$  is the eigenspace bundle for real multiplication via the  $i$ -th embedding.

We order the line bundles  $\mathcal{L}_i$  in such a way that  $\mathcal{L}_1$  is the one that is generated by  $\omega_s$  (and is therefore maximal Higgs by [Möl06b]), and such that  $\mathcal{L}_1 \oplus \mathcal{L}_2$  is the next step in the Harder-Narasimhan filtration of  $\mathcal{V}$  (see [BHM14, Prop. 4.3]).

## 2.3 Special points of the Teichmüller curve

In this section, we study the three special points of the Kenyon-Smillie Teichmüller curve. We compute how real multiplication acts in the orbifold point and we compute the equations of the orbifold point and the two cusps as plane quartics in the distinguished coordinate system given by sections in the eigenform line bundles.



**2.3.1 Orbifold point** The orbifold point of the Teichmüller curve corresponds to the surface  $(X, \omega)$ , which admits an automorphism of order 9. We want to compute the action of this automorphism on the space of holomorphic differentials.

**Proposition 2.3.1.** *The translation surface  $(S, \omega)$  corresponding to the orbifold point on  $C$  is the complete non-singular curve  $\tilde{S}$  with singular model given by the affine equation*

$$y^9 = x^2(x - 1)^3.$$

*The curve  $\tilde{S}$  is a 9-sheeted cyclic covering of  $\mathbb{P}^1$  totally branched over 0 and  $\infty$  and branched of order 3 over 1. The deck transformation is given by*

$$g : (x, y) \mapsto (x, \zeta_9 \cdot y).$$

*Up to scalar multiples, the eigendifferentials for real multiplication  $\omega_0^{(i)}$  are given by*

$$\omega_0^{(1)} = \frac{y \, dx}{x(x - 1)}, \quad \omega_0^{(2)} = \frac{y^5 \, dx}{x^2(x - 1)^2}, \quad \omega_0^{(3)} = \frac{y^7 \, dx}{x^2(x - 1)^3}. \quad (2.2)$$

*Proof.* Up to isomorphism, there is only one curve in genus 3 admitting a  $\mathbb{Z}/9\mathbb{Z}$ -action (cf. [KK77]), so  $(S, \omega)$  and  $\tilde{S}$  are isomorphic. To see that the given differentials form a basis of holomorphic differentials see [Bou01]. The divisors of these differentials are given by

$$\operatorname{div}(\omega_0^{(1)}) = 3p + q, \quad \operatorname{div}(\omega_0^{(2)}) = p + \sum_{i=1}^3 l_i, \quad \operatorname{div}(\omega_0^{(3)}) = 4q$$

where  $p$  is the preimage of  $\infty \in \mathbb{P}^1$ ,  $q$  is the preimage of  $0 \in \mathbb{P}^1$  and  $l_i$  are the three preimages of  $1 \in \mathbb{P}^1$  under the cyclic covering map.

Note that the cyclic group generated by  $g$  must be the same as the cyclic group generated by the automorphism of the flat surface  $(S, \omega)$ . Hence  $\omega_0^{(1)}$  in (2.2) is proportional to  $\omega$ , since it is the only eigendifferential of  $g^*$  with the right divisor. By the construction of real multiplication on Teichmüller curves, we now know that its restriction to the fiber over 0 is given by the action of the totally real subfield  $\mathbb{Q}(\zeta_9 + \zeta_9^{-1}) = \mathbb{Q}(v) \leq \mathbb{Q}(\zeta_9)$ . Therefore, the  $\omega_0^{(i)}$  given in (2.2) are indeed eigendifferentials for the action of real multiplication, and from their divisors and expression (2.7) we see that  $\omega_0^{(i)} \in (\mathcal{L}_i)_0$ .  $\square$

**Corollary 2.3.2.** *The action of the order 9 automorphism  $g$  on  $H^0(S, \Omega^1)$  with respect to the basis  $\{\omega_0^{(i)}\}$  is given by the diagonal matrix*

$$A_g = \operatorname{diag}(\zeta, \zeta^5, \zeta^7) \in \operatorname{PGL}_3(\mathbb{C}).$$

Even though we will not need it in the next sections, it is nice to present the smooth quartic model of the curve  $S$  given via the canonical embedding.

**Corollary 2.3.3.** *Up to isomorphism, the curve  $S = S_0$  is given as the vanishing locus in  $\mathbb{P}^2$  of*

$$F_0(X, Y, Z) = X^4 + XZ^3 + 3Y^3Z.$$

*Proof.* One checks that putting  $X = -\zeta_3\omega_0^{(1)}$ ,  $Y = \sqrt[3]{\frac{\zeta_3}{3}}\omega_0^{(2)}$ ,  $Z = \omega_0^{(3)}$ , where  $\zeta_3$  is a third root of unity, we obtain the above relation in  $H^0(S, \omega_S^4)$ . The normalizing coefficients are put in order to have an equation consistent with the choices made later on.  $\square$

**2.3.2 Cusps** One of the two cusps of the Teichmüller curve is an irreducible stable curve, while the other one is reducible. We choose  $t$  such that the reducible stable curve lies over  $t = \infty$  and the irreducible stable curve over  $t = 1$ .



Figure 2.3: Dual graphs of the two cusps of the Teichmüller curve. The vertices represent the connected components and the edges correspond to the nodes of the stable curves associated to the cusps.

Recall that the topological type of a cusp is obtained by contracting the core curves of cylinders in the cylinder decomposition of  $(S, \omega)$  in the periodic direction corresponding to the cusp (see for example [Möl13, Prop. 5.9]).

**2.3.3 Reducible cusp** The reducible cusp corresponds to the vertical cylinder decomposition of  $(S, \omega)$  shown in Figure 2.4.

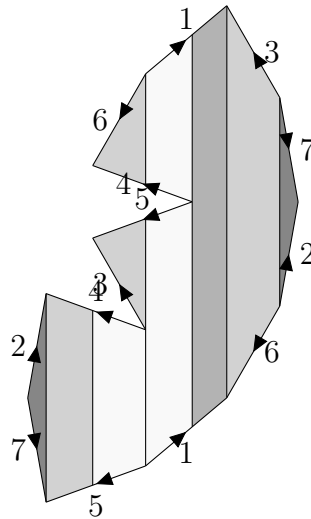


Figure 2.4: Vertical cylinder decomposition of  $S_\infty$  with cylinders  $A, B, C, D$  (from light to dark).

**Proposition 2.3.4.** 1. The stable curve  $(S_\infty, \omega_\infty)$  consists of two  $\mathbb{P}^1$

$$S_\infty = S_\infty^{(1)} \cup S_\infty^{(2)}$$

that meet at 3 points  $B, C, D$ . The only other singular point is a node  $A$ , where  $S_\infty^{(1)}$  is glued to itself.

2. The projective tuple of residues of the stable differential is given by

$$(r_A : r_B : r_C : r_D) = (-v^2 - v + 3 : 1 : v^2 - 3 : -v^2 + 2).$$

3. Up to scaling and isomorphism, the stable differential  $\omega_\infty$  on the normalization is given by

$$\omega_\infty|_{S_\infty^{(1)}} = \mu_\infty \left( \frac{r_A}{z-1} + \frac{-r_A}{z+1} + \frac{r_B}{z-B} + \frac{r_C}{z-C} + \frac{r_D}{z-D} \right) dz \quad (2.3)$$

$$\omega_\infty|_{S_\infty^{(2)}} = \mu_\infty \left( \frac{-r_B}{z} + \frac{-r_C}{z-1} + \frac{-r_D}{z+1} \right) dz \quad (2.4)$$

where

$$B = \frac{1}{17}(-2v^2 + 6v + 5), \quad C = \frac{1}{17}(-6v^2 - 8v + 13), \quad D = \frac{1}{17}(8v^2 + 2v - 15),$$

and  $\mu_\infty = v^2 + 2v - 2$ .

The factor  $\mu_\infty$  in front is chosen in such a way that the equation for the stable curve in Corollary 2.3.5 has rational coefficients.

*Proof.* The topological type of  $S_\infty$  and the projective tuple of residues of the stable differential can be read off directly from the flat picture. The cusp at  $\infty$  corresponds to replacing the cylinders of the cylinder decomposition of  $S$  in the vertical direction (see Figure 2.4) by infinitely long cylinders. The residues of the stable differential  $\omega_\infty$  at the nodes are given by the circumferences of the cylinders up to simultaneous multiplication by a non-zero scalar.

We identify the normalization of  $S_\infty^{(1)}$  with  $\mathbb{P}^1$  by sending the two preimages of  $A$  to 1 and  $-1$  and the triple zero of  $\omega_\infty$  to 0. One checks that there are precisely two differentials with simple poles at 1,  $-1$  and at three other points  $B, C, D$  with the residues given above and a threefold zero at  $P = 0$ . These two differentials corresponds to the triples

$$B_1 = \frac{1}{17}(-2v^2 + 6v + 5), \quad C_1 = \frac{1}{17}(-6v^2 - 8v + 13), \quad D_1 = \frac{1}{17}(8v^2 + 2v - 15)$$

and

$$B_2 = \frac{1}{19}(6v^2 + 18v - 21), \quad C_2 = \frac{1}{19}(-18v^2 - 12v + 27), \quad D_2 = \frac{1}{19}(12v^2 - 6v - 33).$$

After applying Galois conjugation to the residues, exchanging  $v = v^{(1)}$  with  $v^{(2)}$ , by Remark 2.4.5 the Galois conjugate  $\omega_\infty^{(2)}$  has to have a zero at  $z = 0$ . One checks that only the differential corresponding to the triple  $(B_1, C_1, D_1)$  has this property.

In the same way, we identify  $S_\infty^{(2)}$  with  $\mathbb{P}^1$  by sending the three nodes  $B, C, D$  to 0, 1 and  $-1$  and we obtain  $\omega_\infty|_{X_\infty^{(2)}}$ .  $\square$

**Corollary 2.3.5.** *Up to isomorphism, the stable curve  $S_\infty$  is given as the vanishing locus in  $\mathbb{P}^2$  of*

$$\begin{aligned} F_\infty(X, Y, Z) &= X^4 - 3X^3Y + 6X^3Z - 3X^2Y^2 - 6X^2YZ + 6X^2Z^2 \\ &\quad + 4XY^3 - 6XY^2Z - 6XYZ^2 + XZ^3 + 3Y^4 + 3Y^3Z \\ &= (X + Y + Z) \\ &\quad \cdot (X^3 - 4X^2Y + 5X^2Z + XY^2 - 7XYZ + XZ^2 + 3Y^3). \end{aligned} \tag{2.5}$$

*Proof.* One checks that putting  $X = \omega_\infty^{(1)}$ ,  $Y = \omega_\infty^{(2)}$ ,  $Z = \omega_\infty^{(3)}$ , where

$$\begin{aligned} \omega_\infty^{(i)}|_{X_\infty^{(1)}} &= \mu_\infty^{(i)} \left( \frac{r_A^{(i)}}{z-1} + \frac{-r_A^{(i)}}{z+1} + \frac{r_B^{(i)}}{z-B} + \frac{r_C^{(i)}}{z-C} + \frac{r_D^{(i)}}{z-D} \right) dz \\ \omega_\infty^{(i)}|_{X_\infty^{(2)}} &= \mu_\infty^{(i)} \left( \frac{-r_B^{(i)}}{z} + \frac{-r_C^{(i)}}{z-1} + \frac{-r_D^{(i)}}{z+1} \right) dz \quad (i = 1, 2, 3) \end{aligned}$$

are the three Galois conjugates of  $\omega_\infty$ , we obtain the above relation in  $H^0(S_\infty, \omega_{S_\infty}^4)$ .  $\square$

**2.3.4 Irreducible cusp** The irreducible cusp and its stable differential has been described in [BM12, Example 13.8]. We recall their description. It is obtained from the horizontal cylinder decomposition of  $S_0$  shown in Figure 2.5 by replacing each half of a cylinder by a half-infinite strip of the same width.

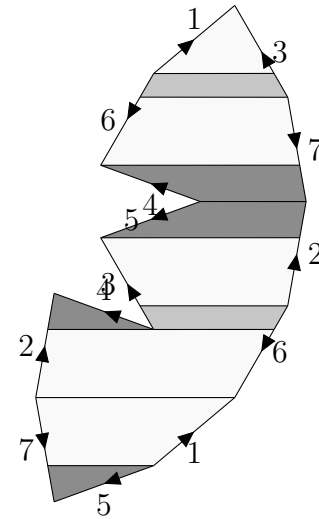


Figure 2.5: Horizontal cylinder decomposition of  $S_0$  with cylinders  $C_1, C_2, C_3$  (from light to dark).

**Proposition 2.3.6.** *1. The stable curve  $S_1$  is isomorphic to a  $\mathbb{P}^1$  with 3 pairs of points identified.*

## 2.4 Equation of the the universal family over the Teichmüller curve

2. The projective tuple of residues of the stable differential  $\omega_1$  is given by

$$(r_1 : r_2 : r_3) = (-v^2 - v : v + 1 : -2v^2 - 3v + 2).$$

3. Up to scaling and isomorphism, the stable differential  $\omega_1$  on the normalization is given by

$$\omega_1 = \mu_1 \cdot \sum_{i=1}^3 \left( \frac{r_i}{z - x_i} - \frac{r_i}{z - \zeta_3 x_i} \right),$$

where  $\zeta_3 = \exp(2\pi i/3)$ ,  $\mu_1 = -v^2 + 2$  and

$$x_1 = 1, \quad x_2 = 2 - v^2, \quad x_3 = v^2 - 3.$$

Again, the factor  $\mu_1$  is chosen in order that the coefficients of the equation below become rational.

**Corollary 2.3.7.** *Up to isomorphism, the stable curve  $S_1$  is given as the vanishing locus in  $\mathbb{P}^2$  of*

$$\begin{aligned} F_1(X, Y, Z) = & 2X^4 - 3X^3Y + 6X^3Z - 3X^2Y^2 - 6X^2YZ + 6X^2Z^2 \\ & + 4XY^3 - 6XY^2Z - 6XYZ^2 + XZ^3 + 3Y^4 + 3Y^3Z \end{aligned} \quad (2.6)$$

*Proof.* As in the proof of Corollary 2.3.5, one checks that putting  $X = -\omega_1^{(1)}$ ,  $Y = \omega_1^{(2)}$ ,  $Z = \omega_1^{(3)}$ , where the  $\omega_1^{(i)}$  are the three Galois conjugates of  $\omega_1$ , we obtain the quartic relation (2.6).  $\square$

Note that the choice of the coordinates in the previous corollary is made in such a way that the choice of coordinates of  $\mathbb{P}^2$  at  $s = 1$  is consistent with the choice of coordinates at  $s = \infty$ . This will be clear after Proposition 2.4.7.

## 2.4 Equation of the the universal family over the Teichmüller curve

In this section we will prove the main Theorem 2.1.1.

**2.4.1 The Teichmüller curve as a family of plane quartics** We want to realize the family  $\bar{\phi} : \bar{\mathcal{X}} \rightarrow \bar{\mathcal{C}}$  as a family of plane quartics. Since some of the fibers are stable curves, we need to employ the relative dualizing sheaf  $\omega_{\bar{\mathcal{X}}/\bar{\mathcal{C}}}$ . It coincides with the relative canonical sheaf off the singular fibers. On a singular stable curve, its sections can be understood as meromorphic 1-forms on the normalization, whose only poles are simple poles at the preimages of a node and such that the residues have opposite sign.

For every  $s \in \mathbb{P}^1$ , let

$$\varphi_s : \bar{\mathcal{X}}_s \rightarrow \mathbb{P}(\mathrm{H}^0(\bar{\mathcal{X}}_s, \omega_{\bar{\mathcal{X}}_s})^\vee), \quad x \mapsto (\omega \mapsto \omega(x))$$

be the canonical map associated to  $\bar{\mathcal{X}}_s$ . Note that if we fix a basis  $\{\omega_s^{(i)}\}_{i=1,2,3}$  of  $\mathrm{H}^0(\bar{\mathcal{X}}_s, \omega_{\bar{\mathcal{X}}_s})$ , the map above is simply given by

$$\varphi_s : \bar{\mathcal{X}}_s \rightarrow \mathbb{P}^2, \quad p \mapsto (\omega_s^{(1)}(p) : \omega_s^{(2)}(p) : \omega_s^{(3)}(p)).$$

**Proposition 2.4.1.** *The canonical map  $\varphi_s$  is an embedding for every  $s \in \mathbb{P}^1$ . As a consequence, each  $\varphi_s(\bar{\mathcal{X}}_s)$  is a plane quartic.*

In order to prove the above proposition, we only have to check that there are no hyperelliptic curves in the family.

**Lemma 2.4.2.** *None of the curves  $\bar{\mathcal{X}}_s$  is hyperelliptic.*

*Proof.* We first show that the surface  $(S, \omega)$  is not hyperelliptic. Let  $T$  denote the order 9 automorphism induced by rotation of one triangle by  $\frac{2\pi}{9}$ . If  $S$  were hyperelliptic, the hyperelliptic involution  $J$  would commute with  $T$ , and thus it would descend to an order 2 automorphism  $\bar{J}$  of  $S/\langle T \rangle \cong \mathbb{P}^1$ . This map  $\bar{J}$  would have to fix three points, namely the images of  $p$  and  $q$  and the image of a fixed point of  $T^3$ . Since  $\bar{J}$  would fix 3 points on  $\mathbb{P}^1$ , we would have  $\bar{J} = \mathrm{id}$ , which is impossible.

The claim now follows from [Möl13, Prop. 2.3] for the non-singular fibers and from [Möl13, Prop. 5.13] for the fibers over the cusps.  $\square$

The above Lemma is clear since we are in the (3, 1)-Stratum, however it is nice to have a different geometric proof.

*Proof of Proposition 2.4.1.* For every  $s$  outside the set of cusps, the canonical map is an embedding onto a non-singular plane quartic in  $\mathbb{P}^2$ , since by Lemma 2.4.2 every  $\bar{\mathcal{X}}_s$  is a non-hyperelliptic curve. Since the singular fibers  $\bar{\mathcal{X}}_{\zeta_9^i}$  and  $\bar{\mathcal{X}}_\infty$  are 3-connected (see Figure 2.3), by [CFHR99, Theorem 3.6] or [Art04, Theorem 1.2] their canonical map is still an embedding and the image is still a quartic.  $\square$

**2.4.2 Harder-Narasimhan filtration and degrees of eigenspace bundles** Recall that the direct image  $\mathcal{V}$  of the relative dualizing sheaf is a rank 3 vector bundle which splits as a direct sum of three line bundles

$$\mathcal{V} = \bigoplus_{i=1}^3 \mathcal{L}_i$$

where  $\mathcal{L}_i$  is the eigenspace bundle for real multiplication via the  $i$ -th embedding. They were ordered in such a way that  $\mathcal{L}_1$  is the one that is generated by  $\omega_s$  (and is therefore maximal Higgs by [Möl06b]), and such that  $\mathcal{L}_1 \oplus \mathcal{L}_2$  is the next step in the Harder-Narasimhan filtration of  $\mathcal{V}$ .

Since  $\bar{C} \cong \mathbb{P}^1$ , the line bundle  $\mathcal{L}_i$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(k_i)$ , where  $k_i = \deg(\mathcal{L}_i)$ , and these degrees are readily computed.

**Lemma 2.4.3.** *The degrees of the line bundles  $\mathcal{L}_i$  over  $\bar{C}$  are given by*

$$\deg(\mathcal{L}_1) = 4 \qquad \deg(\mathcal{L}_2) = 2 \qquad \deg(\mathcal{L}_3) = 1$$

*Proof.* First of all note that

$$\deg(\Omega_{\bar{C}}^1(\log(S))) = -\chi(\tilde{C}) = -(\chi(\bar{C}) - \#\text{cusps}) = 8.$$

Since we have a splitting of  $\mathcal{V}$  into line bundles, we can consider the numbers

$$\lambda_i = \frac{2 \deg(\mathcal{L}_i)}{\deg(\Omega_{\bar{C}}^1(\log(S)))} = \frac{1}{4} \deg(\mathcal{L}_i).$$

Since  $\mathcal{L}_1$  is maximal Higgs,  $\lambda_1 = 1$ .

From [BHM14, Prop. 4.3], we know that the Harder-Narasimhan filtration of  $\mathcal{V}$  is given by the filtration of eigenspace bundles. By our ordering choice, this is

$$\mathcal{L}_1 \subset \mathcal{L}_1 \oplus \mathcal{L}_2 \subset \mathcal{V}.$$

The quotient of the degrees of the steps of the Harder-Narasimhan filtration and the degree of  $\Omega_{\bar{C}}^1(\log(S))$  were computed in [YZ13, Table 1] for the (3, 1)-Stratum under the name  $w_i$  of Weierstraß exponents.

Hence, since we established that the  $\lambda_i$  are the same as  $w_i$ , we see from [YZ13, Table 1] that  $\lambda_2 = \frac{1}{2}$ ,  $\lambda_3 = \frac{1}{4}$  and we can conclude. □

*Remark 2.4.4.* By Kontsevich's formula (see e. g. [BM10b, Theorem 9.2]), the  $\lambda_i$  are the Lyapunov exponents of the Kontsevich-Zorich cocycle on  $C$ .

*Remark 2.4.5.* The ordering of the line bundles  $\mathcal{L}_i$  is also reflected in the divisors of its sections. If we let  $\omega^{(i)}$  denote a local section of  $\mathcal{L}_i$  ( $i = 1, 2, 3$ ), then by [BHM14, Prop. 4.1]

$$\begin{aligned} \operatorname{div}(\omega_s^{(1)}) &= 3p_s + q_s \in \operatorname{Pic}(\bar{\mathcal{X}}_s) \\ \operatorname{div}(\omega_s^{(2)}) &\geq p_s \in \operatorname{Pic}(\bar{\mathcal{X}}_s) \end{aligned} \tag{2.7}$$

This is used to label the embeddings of the trace field  $\mathbb{K}(X, \omega)$  consistently.

**2.4.3 Setup for the equation** The canonical embedding of the fibers of the family  $\bar{\phi} : \bar{\mathcal{X}} \rightarrow \bar{C}$  provides a rational map

$$\varphi : \bar{\mathcal{X}} \rightarrow \mathbb{P}^2 \times \bar{C}, \quad p_s \mapsto ((\omega_s^{(1)}(p_s) : \omega_s^{(2)}(p_s) : \omega_s^{(3)}(p_s)), s)$$

which is an isomorphism onto its image. By Lemma 2.4.3, the global sections  $\omega^{(i)}$  can be chosen such that

$$\operatorname{div}(\omega^{(i)}) = k_i \cdot \infty, \quad (i = 1, 2, 3), \quad (2.8)$$

where  $k_1 = 4$ ,  $k_2 = 2$  and  $k_3 = 1$ . In the sequel, it will be convenient to use projective coordinates  $(s_1 : s_2) \in \mathbb{P}^1$  for the base, so that  $s \in \mathbb{C}$  is identified with  $(1 : s) \in \mathbb{P}^1$ .

The image of the map  $\varphi$  is the zero locus of a degree 4 primitive homogeneous polynomial whose coefficients are homogeneous polynomials in  $(s_1 : s_2) \in \mathbb{P}^1$ . We denote this polynomial by

$$F = \sum_{i+j+k=4} a_{i,j,k}(s_1, s_2) X^i Y^j Z^k \in \mathbb{C}[s_1, s_2][X, Y, Z]_4.$$

The aim is to compute the coefficients  $a_{i,j,k}(s_1, s_2)$ .

**2.4.4 Degrees of the coefficients** In this section, we exhibit the main ingredient, which is central in the proof of Theorem 2.1.1. Using a structural result for the relative canonical ring of a family of quartics (cf. [CP06, Proposition 7.9, (2)]), we compute the degrees of the coefficients  $a_{i,j,k}(s_1, s_2)$ .

**Proposition 2.4.6.** *For every  $(i, j, k)$  with  $i + j + k = 4$ , the coefficient  $a_{i,j,k}$  of the primitive polynomial  $F$  is a homogeneous polynomial in  $(s_1 : s_2) \in \mathbb{P}^1$  of degree*

$$\deg(a_{i,j,k}(s_1, s_2)) = 4i + 2j + k - 7$$

if  $4i + 2j + k - 7 \geq 0$  or the zero polynomial otherwise.

$i$	4	3	3	2	2	2	1	1	1	1	0	0	0	0
$j$	0	1	0	2	1	0	3	2	1	0	4	3	2	1
$k$	0	0	1	0	1	2	0	1	2	3	0	1	2	3
$4i + 2j + k - 7$	9	7	6	5	4	3	3	2	1	0	1	0	-1	-2

Table 2.1: Degree of  $a_{i,j,k}(s_1, s_2)$ .

An important element in the proof of this proposition is the fact that there is an identification between maps  $\mathcal{O}_{\mathbb{P}^1}(l) \rightarrow \mathcal{O}_{\mathbb{P}^1}(m)$  and global sections of  $\mathcal{O}_{\mathbb{P}^1}(m - l)$ , which are identified with homogeneous polynomials if we consider the standard trivialization of  $\mathcal{O}_{\mathbb{P}^1}(m - l)$ . The choice of global sections (2.8) is made exactly in order to have this identification when we consider maps of line bundles  $\mathcal{L}_i \rightarrow \mathcal{L}_j$  and their canonical trivializations given by taking  $\omega^{(i)}$  as a basis over  $\mathbb{P}^1 - \{\infty\}$  and  $s_2^{k_i} \cdot \omega^{(i)}$  over  $\mathbb{P}^1 - \{0\}$ .



*Proof.* As in [CP06], we consider the relative canonical algebra associated with  $\bar{\phi} : \bar{\mathcal{X}} \rightarrow \bar{C}$  whose graded pieces are direct images of powers of the relative dualizing sheaf

$$V_n = \bar{\phi}_* \omega_{\bar{\mathcal{X}}/\bar{C}}^{\otimes n}$$

and the multiplication map

$$\sigma_n : \text{Sym}^n(V_1) \rightarrow V_n$$

given on a stalk over  $s \in \bar{C}$  by

$$(\sigma_n)_s : \text{Sym}^n(\mathbb{H}^0(\bar{\mathcal{X}}_s, \omega_{\bar{\mathcal{X}}_s})) \rightarrow \mathbb{H}^0(\bar{\mathcal{X}}_s, \omega_{\bar{\mathcal{X}}_s}^{\otimes n}), \quad \omega_1 \otimes \cdots \otimes \omega_n \mapsto \omega_1 \cdots \omega_n.$$

Since the canonical embedding of  $\bar{\mathcal{X}}_s$  is a smooth plane quartic defined the equation  $F_s$ , the standard exact sequence of sheaves induced by the embedding is given by

$$0 \rightarrow \mathcal{I}(F_s)(4) \rightarrow \mathcal{O}_{\mathbb{P}(\mathbb{H}^0(\bar{\mathcal{X}}_s, \omega_{\bar{\mathcal{X}}_s}^\vee))}(4) \rightarrow \omega_{\bar{\mathcal{X}}_s}^{\otimes 4} \rightarrow 0.$$

This shows that the kernel  $\mathcal{K}_4$  of  $\sigma_4$  is precisely the ideal sheaf generated by  $F$ .

Now we can use [CP06, Proposition 7.9, (2)], which states that in our case, namely in the case of a family of genus 3 non-hyperelliptic curves, the kernel  $\mathcal{K}_4$  of  $\sigma_4$  is isomorphic to the determinant bundle of the push forward of the relative dualizing sheaf

$$\mathcal{K}_4 \cong \det(V_1).$$

To extract the coefficients of  $F$ , we look at the projections onto the direct factors of  $\text{Sym}^4(V_1)$  given by the decomposition  $V_1 = \bigoplus_{i=1}^3 \mathcal{L}_i$  into eigenspace bundles for real multiplication. Let

$$\delta(i, j, k) : \mathcal{K}_4 \cong \det(V_1) \cong \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \rightarrow \text{Sym}^4(V_1) \rightarrow \mathcal{L}_1^{\otimes i} \otimes \mathcal{L}_2^{\otimes j} \otimes \mathcal{L}_3^{\otimes k}$$

denote the composition of the above maps. By Lemma 2.4.3,

$$\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \cong \mathcal{O}_{\mathbb{P}^1}(7)$$

and

$$\mathcal{L}_1^{\otimes i} \otimes \mathcal{L}_2^{\otimes j} \otimes \mathcal{L}_3^{\otimes k} \cong \mathcal{O}_{\mathbb{P}^1}(4i + 2j + k)$$

where the isomorphisms are determined by the choice of the basis (2.8). Thanks to our choice of isomorphisms, the map  $\delta(i, j, k)$  is given by a section of  $\mathcal{O}_{\mathbb{P}^1}(4i + 2j + k - 7)$ . Therefore,

$$\delta(i, j, k)(\omega^{(1)} \otimes \omega^{(2)} \otimes \omega^{(3)}) = a_{i,j,k}(s) \cdot \omega^{(1)\otimes i} \otimes \omega^{(2)\otimes j} \otimes \omega^{(3)\otimes k}$$

where  $a_{i,j,k}$  is a homogeneous polynomial in  $(s_1 : s_2) \in \mathbb{P}^1$  of degree  $4i + 2j + k - 7$  or 0. By our construction, they are the coefficient of  $F$  that we were searching for.  $\square$

Note that the equality

$$a_{0,0,4} = a_{0,1,3} = a_{0,2,2} = 0$$

can be easily deduced also from the form of the divisors of  $\omega_s^{(i)}$  given by Condition (2.7) with a local computation around the zeroes of the differentials.

**2.4.5 Conditions from the orbifold point** Since the family over  $\overline{C}$  comes from a degree 9 covering of the original Teichmüller curve, the coefficients of the polynomial describing this family must have an order 9 symmetry, which we want to understand. We use this symmetry in order to show that most of the coefficients of the polynomials  $a_{i,j,k}(s_1, s_2)$  vanish.

**Proposition 2.4.7.** *The coefficients of the dehomogenized polynomial  $a_{4,0,0}(1 : s)$  are all zero with the exception of the top and the constant one, while for all the other non-trivial triples  $(i, j, k)$ , all the coefficients of  $a_{i,j,k}(1 : s)$  are zero other than the top one, namely*

$$a_{i,j,k}(1 : s) = \begin{cases} \alpha_{4,0,0}s^9 + \beta, & (i, j, k) = (4, 0, 0) \\ \alpha_{i,j,k}s^{4i+2j+k-7}, & \text{otherwise} \end{cases}$$

for  $\alpha_{i,j,k}, \beta \in \mathbb{C}$ .

*Remark 2.4.8.* The lifting of the action of the Galois group of the 9-sheeted covering  $\tilde{C} \rightarrow C$  to an action on the family  $\overline{\phi} : \overline{\mathcal{X}} \rightarrow \overline{C}$  can be seen as follows.

A generator of the Galois group has a lift to an element of the Veech group, corresponding to an elliptic automorphism of  $\mathbb{H}$  of order 9. Since the Veech group embeds naturally into the mapping class group, we obtain in fact an automorphism of the Teichmüller space  $\mathcal{T}_3$ . We pass to an appropriate quotient  $\mathcal{M}'$  of  $\mathcal{T}_3$  that is a fine moduli space finitely covering  $\mathcal{M}_3$  and such that we have a factorization  $\tilde{C} \rightarrow \mathcal{M}'$  and obtain an automorphism of  $\mathcal{M}'$  fixing the image of  $\tilde{C}$ . By the universal property of  $\mathcal{M}'$ , being a fine moduli space, this automorphism lifts to an automorphism of the universal family over  $\mathcal{M}'$ , hence its restriction to  $\tilde{C}$  gives an automorphism  $\overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}$ .

We fix the Galois automorphism

$$h : \overline{C} \rightarrow \overline{C}, \quad s \mapsto \zeta_9^2 \cdot s$$

of the base. The lifted automorphism  $H : \overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}$  over  $h$  induces an automorphism of  $\mathcal{V}$ , which in the basis (2.8) is given as an element of  $A_H \in \text{PGL}_3(\mathbb{C}(s))$ . The proof of Proposition 2.4.7 essentially boils down to using the identity

$$\lambda \cdot F_s(X, Y, Z) = F_{h(s)} \circ A_H(X, Y, Z)$$

with  $\lambda \in \mathbb{C}^\times$ . We first need a lemma on the shape of the matrix  $A_H$ .

**Lemma 2.4.9.** *1.  $H$  preserves the eigenspace bundles  $\mathcal{L}_i$  and thus acts on the basis (2.8) by a diagonal matrix.*

*2. The matrix  $A_H$  is constant and given by*

$$A_H = \text{diag}(\zeta_9 : \zeta_9^5 : \zeta_9^7) \in \text{PGL}_3(\mathbb{C}).$$

*Proof.* That  $H$  preserves the eigenspace bundles  $\mathcal{L}_i$  follows from the  $\mathcal{L}_i$  being irreducible and from Schur's Lemma. The diagonal entries of  $A_H$  must be constant in  $s$  since they

are maps between line bundles on  $\mathbb{P}^1$  of the same degree. Hence, we just need to find the action of  $A_H$  on one fiber in order to determine it completely.

Consider now the fiber over 0, which corresponds to the curve with the order 9 automorphism. The lifting automorphism  $H : \overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}$  must specialize to a primitive element of the automorphism group of  $\overline{\mathcal{X}}_0$ . Thus by Corollary 2.3.2, the matrix  $A_H$  must specialize to a primitive element of the group generated by the matrix  $A_g$ . In order to determine this element, we compute how  $A_H$  acts on  $\mathcal{L}_1$ .

Since  $\mathcal{L}_1$  is maximal Higgs, it is a Theta characteristic, i. e.

$$\mathcal{L}_1^2 \cong \Omega_{\overline{C}}^1.$$

Since the action of  $H$  on  $(\Omega_{\overline{C}}^1)_0$  is given by multiplication by  $\zeta_9^2$ , the action on  $(\mathcal{L}_1)_0$  must be given by multiplication by  $\zeta_9$ . This fixes the element in the group generated by the matrix  $A_g$  of Corollary 2.3.2, and so  $A_H = A_g = \text{diag}(\zeta_9 : \zeta_9^5 : \zeta_9^7) \in \text{PGL}_3(\mathbb{C})$ .  $\square$

Now we can use the automorphism  $H$  in order to find symmetries of the polynomial  $F$ .

*Proof of Proposition 2.4.7.* The existence of  $H$  implies that the fiber  $\overline{\mathcal{X}}_s$  is isomorphic to the fiber  $\overline{\mathcal{X}}_{h(s)}$ . However, since in our case we are considering canonical embeddings, we know more, namely that the locus of the polynomial  $F_{h(s)}$  must be the same as the one of  $F_s$  up to a projective linear transformation of  $\mathbb{P}^2$ . Recall that with the fixed isomorphism  $\mathbb{P}(\mathbb{H}^0(X, \Omega^1)^\vee) \cong \mathbb{P}^2$  given by the choice of basis (2.8), the canonical embedding is given by the map

$$\varphi_s : \overline{\mathcal{X}}_s \rightarrow \mathbb{P}^2, \quad \varphi_s(x) = (\omega_s^{(1)}(x) : \omega_s^{(2)}(x) : \omega_s^{(3)}(x))$$

and so the projective linear transformation of  $\mathbb{P}^2$  that we are looking for is given by the matrix  $A_H$ .

Hence the condition imposed by the existence of  $H$  is

$$\lambda \cdot F_s(X, Y, Z) = F_{h(s)} \circ A_H(X, Y, Z) \tag{2.9}$$

for some  $\lambda \in \mathbb{C}^\times$ .

In order to compute  $\lambda$ , we look at the condition for the constant coefficient  $a_{0,3,1}$ . This coefficient cannot be zero, since it is not zero for the fiber over 1 or  $\infty$  by Corollary 2.3.5 or Corollary 2.3.7. Condition (2.9) yields

$$\lambda \cdot a_{0,3,1} = a_{0,3,1} \cdot \zeta_9^{3 \cdot 5 + 7} = \zeta_9^4 \cdot a_{0,3,1},$$

whence we have

$$\lambda = \zeta_9^4.$$

Therefore, the other coefficients must satisfy

$$\zeta_9^4 \cdot a_{i,j,k}(1, s) = \zeta_9^{i+5 \cdot j+7 \cdot k} a_{i,j,k}(1, \zeta_9^2 s).$$

Using this condition together with Proposition 2.4.6, it is immediate to conclude.  $\square$

**2.4.6 Computation of the family** Now we have gathered all the ingredients in order to prove the main Theorem and write down the algebraic equation of the universal family over the Kenyon-Smillie-Teichmüller curve.

*Proof of Theorem 2.1.1.* By Proposition 2.4.7, it is sufficient to know the values of the coefficients  $a_{i,j,k}(1 : s)$  in two special point in order to determine them completely. Using the computation of  $a_{i,j,k}(0 : 1)$  over the special point  $\infty = (0 : 1) \in \overline{C}$  given by Corollary 2.3.5, we can compute all coefficients other than  $a_{4,0,0}(1, s)$ , of which we have determined just the constant term. Now we can use Corollary 2.3.7, where we computed the coefficients  $a_{i,j,k}(1 : 1)$ . The coefficient of the monomial  $X^4$  of (2.6) tells us the sum of the top and the constant coefficient of  $a_{4,0,0}(1 : s)$ . The explicit form of the universal family  $\overline{\phi} : \overline{\mathcal{X}} \rightarrow \overline{C}$  is then given by

$$(s^9 + 1)X^4 - 3s^7X^3Y + 6s^6X^3Z - 3s^5X^2Y^2 - 6s^4X^2YZ + s^3(6X^2Z^2 + 4XY^3) - 6s^2XY^2Z + s(-6XYZ^2 + 3Y^4) + XZ^3 + 3Y^3Z = 0. \quad (2.10)$$

If we exclude  $s = 0$  and  $s = \infty$ , we can apply the projective linear transformation

$$(X : Y : Z) \mapsto (X : s^2Y : s^3Z)$$

of  $\mathbb{P}^2$  and see that the family  $\overline{\phi} : \overline{\mathcal{X}} \rightarrow \overline{C}$  indeed descends to the family  $\phi : \mathcal{X} \rightarrow C - \{0\}$  described by Equation (2.1).

We now identify, with a slight abuse of notation,  $p_t$  with  $\varphi_t(p_t)$  and  $q_t$  with  $\varphi_t(q_t)$ , where  $\text{div}(\omega_t^{(1)}) = 3p_t + q_t$ . Then, by Condition (2.7),

$$p_t = (0 : 0 : 1) \quad \text{and} \quad q_t = (0 : * : *).$$

Hence, the point  $p_t$  is given by the intersection of the curve  $\mathcal{X}_t$  with the line  $X = 0$ , thus  $p_t = (0 : 1 : -1)$ .  $\square$

**2.4.7 The torsion map and the hyperflex** The torsion map is the map (unique up to multiplication by an element of  $\mathbb{C}^\times$ ) exhibiting the torsion condition  $n(P_t - Q_t) = 0 \in \text{Pic}^0(\mathcal{X}_t)$ .

In order to compute it, we need to find a map  $\mathcal{X}_t \rightarrow \mathbb{P}^1$  totally ramified at  $p_t$  and  $q_t$ . In fact, [BM12, Section 13] suggests that the degree of this map is  $n = 3$ , since this is the case for the torsion map over the irreducible cusp.

*Proof of Proposition 2.1.2.* From Equation 2.1, it is easy to check that the point  $q_t$  is a hyperflex and the tangent at  $p_t$  is a triple tangent passing through  $q_t$ . It is trivial now that the projection from  $q_t$  onto a line is the desired torsion map. The explicit expression of the torsion map can be given by projecting from  $q_t$  onto the line  $\{Z = 0\} \subset \mathbb{P}^2$ .  $\square$

It is interesting to notice how Proposition 2.1.2 can be proven without using Equation 2.1, only using that we are in the (3, 1)-stratum and that the points of the Teichmüller curve are canonically embedded as quartics in  $\mathbb{P}^2$ .

*Alternative proof of Proposition 2.1.2.* The torsion map is a degree 3 map from  $\mathcal{X}_t \rightarrow \mathbb{P}^1$  totally ramified at  $p_t$  and  $q_t$ . It is not difficult to prove that every degree 3 map from a smooth quartic of  $\mathbb{P}^2$  to  $\mathbb{P}^1$  is a central projection from a point  $x$  on the quartic. Since  $\text{div}(\omega_t) = 3p_t + q_t$  and the curves  $S_t$  are canonically embedded, the line  $\overline{p_t q_t}$  is a triple tangent at  $p_t$ . This triple tangent is also the projection line  $\overline{x p_t}$  since the torsion map is totally ramified at  $p_t$ . Hence  $x = q_t$ , namely the torsion map is the central projection from  $q_t$ . Moreover, since  $q_t$  is a totally ramified point of this map, it must be a hyperflex of the quartic curve  $S_t$ .

□

**2.4.8 Torsion map and real multiplication** Now we want to briefly explain the relation between the torsion map and real multiplication in the case of Veech-Ward-Bouw-Möller Teichmüller curves and compare this situation with ours.

Let us recall the Bouw-Möller construction of the universal family  $\mathcal{H} \rightarrow \mathbb{P}^1$  over the completion of the Teichmüller curve uniformized by a triangle group. For simplicity, we restrict to the case  $\Delta(n, \infty, \infty)$  discussed in [BM10b, Section 5]. They construct a particular family of cyclic coverings  $\mathcal{Y}_t \rightarrow \mathbb{P}^1$ , parametrized by  $t \in \mathbb{P}^1 - \{0, 1, \infty\}$ , which is branched over 4 points and admits an involution  $\sigma$ . This family descends to the universal family  $\mathcal{H} := \mathcal{Y} / \langle \sigma \rangle \rightarrow \mathbb{P}^1$  over the Teichmüller curve. One can construct the following commutative diagram

$$\begin{array}{ccc} \mathcal{Y}_t & \xrightarrow{\mathbb{Z}/n\mathbb{Z}} & \mathbb{P}^1 \\ \mathbb{Z}/2\mathbb{Z} \downarrow & & \downarrow \\ \mathcal{H}_t & \xrightarrow{\text{Tor}} & \mathbb{P}^1 \end{array}$$

where Tor is the torsion map.

Note that, even though we constructed the above diagram starting from the upper  $n$ -cyclic covering, one can build it also only from the torsion map, since this is indeed its Galois normalization diagram. The relation between real multiplication and torsion map is now easy to explain. Let  $\tau : \mathcal{Y}_t \rightarrow \mathcal{Y}_t$  be the Galois automorphism of  $\mathcal{Y}_t \rightarrow \mathbb{P}^1$ . It induces complex multiplication by  $\mathbb{Q}(\zeta_n)$  on the Jacobian of  $\mathcal{Y}_t$ . One can check that real multiplication is given by the push-forward to  $\mathcal{H}_t$  of the correspondence induced by the graph of the endomorphism  $\tau + \tau^* \in \text{End}_{\mathbb{Q}} \text{Jac}(\mathcal{Y}_t)$ , which generates the totally real subfield of  $\mathbb{Q}(\zeta_n)$ .

Since by Proposition 2.1.2 we know the explicit form of the torsion map in the Kenyon-Smillie-Teichmüller curve case, we can compute its Galois normalization and check if real multiplication comes from a diagram analogous to the one constructed in the Veech-Ward-Bouw-Möller case.

Note that one can easily compute the ramification points of the torsion map. There are the two expected triple ramification points  $\text{Tor}(Q_t)$  and  $\text{Tor}(P_t)$  and other 6 double

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ramification points, which we denote by  $x_i(t) \in \mathbb{P}^1$  for  $i = 1, \dots, 6$ . Let  $\mathcal{Z}_t$  be the genus 2 curve defined by the affine equation

$$\mathcal{Z}_t : \quad v^2 = \prod_{i=1}^6 (x - x_i(t)).$$

By general considerations on how the Galois normalization has to behave on the ramification points of the map Tor, we find that the normalization diagram is given by the fibered product diagram

$$\begin{array}{ccc} \mathcal{Y}_t & \xrightarrow{\mathbb{Z}/3\mathbb{Z}} & \mathcal{Z}_t \\ \downarrow & & \downarrow p \\ \mathcal{X}_t & \xrightarrow{\text{Tor}} & \mathbb{P}^1 \end{array}$$

where  $\mathcal{Y}_t := \mathcal{X}_t \times_{\mathbb{P}^1} \mathcal{Z}_t$  is the fibered product. The map

$$p : \mathcal{Z}_t \rightarrow \mathbb{P}^1, \quad (x, v) \mapsto v$$

is the quotient by the hyperelliptic involution, thus it is ramified over the points  $x_i$ .

Now we want to show that real multiplication cannot be constructed from the Galois normalization diagram of the torsion map as in the Veech-Ward-Bouw-Möller case.

**Proposition 2.4.10.** *Real multiplication is not given by a correspondence induced by the graph of an automorphism of  $\mathcal{Y}_t$  that descends to the quotient  $\mathcal{Z}_t$ .*

*Proof.* Recall that the trace field of the Kenyon-Smillie-Teichmüller curve is cubic. Hence if real multiplication is given by a totally real subfield of the complex field induced by an automorphism of  $\mathcal{Y}_t$ , this automorphism has to be of order 9. Since the Galois automorphism of  $\mathcal{Y}_t \rightarrow \mathcal{Z}_t$  is of order 3, this cannot work. We can already see that this is not as in the Bouw-Möller case.

Consider now an order 9 covering automorphism of  $\mathcal{Y}_t$  descending to the quotient  $\mathcal{Z}_t$ . Since the order of the descending automorphism on  $\mathcal{Z}_t$  has to divide 9, then it has to be of order 3 because there are no order 9 automorphisms of a genus 2 curve. We show that there is no such automorphism of  $\mathcal{Z}_t$ .

Recall that  $p : \mathcal{Z}_t \rightarrow \mathbb{P}^1$  is the hyperelliptic family. Hence if there were an order 3 automorphism of  $\mathcal{Z}_t$ , it would commute with the hyperelliptic involution and thus would descend to multiplication by a 3rd root of unity on  $\mathbb{P}^1$  in the appropriate coordinates. Since the 6 ramification points  $x_i(t)$  must be preserved by the automorphism, they have to lie on two circles (any change of coordinates of  $\mathbb{P}^1$  preserves circles). Using Proposition 2.1.2, one can however compute explicitly  $x_i(t)$  and check that this is not the case.  $\square$

## 2.5 Picard-Fuchs equations

In this section, we forget how we found Equation (2.1) and we prove independently that this equation defines a Teichmüller curve. We will do this by showing that the absolute cohomology bundle splits as a direct sum of three rank two subbundles and that one of them is maximal Higgs. The main tool is the computation of the Picard-Fuchs equation associated to a local section of the  $(1, 0)$ -part of the absolute cohomology bundle of the family via the Griffiths-Dwork method.

**2.5.1 Griffiths-Dwork method** We quickly recall the Griffiths-Dwork algorithm for the computation of the Picard-Fuchs equation of a family of projective hypersurfaces. This method can more generally be used in the case of projective toric varieties. See [CK99] for more details.

Let  $V \subset \mathbb{P}^n$  be a hypersurface of degree  $d$  defined by a homogeneous equation  $f = 0$ . We want to identify elements of  $H^{n-1}(V)$  with elements in  $H^n(\mathbb{P}^n - V)$  via the residue map. Any element of  $H^n(\mathbb{P}^n - V)$  can be represented by a form

$$\frac{P\Omega_0}{f^k}, \quad \deg(P) = kd - (n + 1)$$

where  $\Omega_0$  is a section of the sheaf  $\Omega_{\mathbb{P}^n}^n(n + 1)$ , which in fact is trivial, and  $P$  is a homogeneous polynomial. The residue map

$$\text{Res} : H^n(\mathbb{P}^n - V) \rightarrow H^{n-1}(V)$$

is defined by the property that

$$\int_{\gamma} \text{Res} \left( \frac{P\Omega_0}{f^k} \right) = \int_{T(\gamma)} \frac{P\Omega_0}{f^k}$$

for any  $(n - 1)$ -cycle  $\gamma$  in  $V$  and its tubular neighborhood  $T(\gamma)$ .

Let  $J(f) := \langle \partial f / \partial x_0, \dots, \partial f / \partial x_n \rangle$  be the Jacobian ideal of  $V$ . The key ingredient of the Griffiths-Dwork method is the isomorphism

$$\left( \frac{\mathbb{C}[x_0, \dots, x_n]}{J(f)} \right)_{kd-(n+1)} \cong \text{P} H^{n-k, k-1}(V) \quad \text{for } k = 1, \dots, n$$

where the subscript denotes the  $kd - (n + 1)$ -graded piece and  $\text{P} H^{n-k, k-1}(V)$  is the primitive part of  $H^{n-k, k-1}(V)$ .

In other words, if a form  $\frac{P\Omega_0}{f^k}$  has a high order pole, namely if  $k > n$ , we can find a representative of its cohomology class with a pole of order less than  $n$ . The key equality that allows to compute representatives with lower order poles is given by

$$\left( \sum_i G_i \frac{\partial f}{\partial x_j} \right) \frac{\Omega_0}{f^k} = \frac{1}{k-1} \left( \sum_i \frac{\partial G_i}{\partial x_j} \right) \frac{\Omega_0}{f^{k-1}} \in H^n(\mathbb{P}^n - V). \quad (2.11)$$

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Now consider a family of hypersurfaces  $\{V_s\}$  defined by a varying polynomial  $f_s$ , and a form represented by  $\omega_s = \frac{P_s \Omega_0}{f_s^k}$ . The action of the Gauss-Manin connection is given by

$$\nabla\left(\frac{\partial}{\partial s}\right)(\omega_s) = \frac{(-kP_s f'_s + f_s P'_s)\Omega_0}{f_s^{k+1}}.$$

If we iterate the derivation many times, we find a form with poles of order greater than  $n$ . We can then find a representative of its cohomology class with a lower order pole by using Equation 2.11.

**2.5.2 Picard-Fuchs equations of the family** The Picard-Fuchs equation of a family of curves with respect to a local section of the  $(1, 0)$ -part of its cohomology bundle is the differential equation satisfied by the periods of the chosen local section. The set of solutions of the Picard-Fuchs equation forms a local system isomorphic to the dual of the local system associated with the irreducible part of absolute cohomology bundle containing the chosen local section. Therefore Picard-Fuchs equations characterize the local system underlying the absolute cohomology bundle in a unique way. A nice exposition of Picard-Fuchs equations can be found in [BM10b, Section 3]. We will now prove Proposition 2.1.3 using the Griffiths-Dwork method.

*Proof of Proposition 2.1.3.* Let  $\bar{\phi} : \bar{\mathcal{X}} \rightarrow \bar{C}$  be the family of curves described by Equation (2.10). We use this family since it has unipotent monodromy around the 10 cusps. This will be useful when we check that the absolute cohomology bundle has a maximal Higgs rank 2 subbundle.

Let  $F_s$  be the polynomial describing  $\bar{\mathcal{X}}_s \subset \mathbb{P}^2$ . Let  $D := \nabla\left(\frac{\partial}{\partial s}\right)$ , where  $\nabla$  is the Gauss-Manin connection.

Let  $\omega$  be a local section of the  $(1, 0)$ -part of the cohomology bundle at  $0 \in \mathbb{P}^1$ . Since the cohomology bundle is of rank 6, the local section  $\omega$  must a priori satisfy

$$a_0\omega + a_1D(\omega) + \cdots + D^6(\omega) = 0$$

where  $a_i$ ,  $i = 0, \dots, 5$ , is a rational function in  $s$  with poles of order at most  $6 - i$ . The associated differential equation

$$a_0y + a_1\frac{\partial y}{\partial s} + \cdots + \frac{\partial^6 y}{\partial s^6} = 0$$

is satisfied by the periods  $s \mapsto \int_\gamma \omega(s)$  for any locally constant 1-cycle  $\gamma$  of  $\bar{\mathcal{X}}_s$ .

We now want to use the identification of  $H^1(\bar{\mathcal{X}}_s)$  with  $H^2(\mathbb{P}^2 - \bar{\mathcal{X}}_s)$  via the residue map, and in particular the isomorphisms

$$\left(\frac{\mathbb{C}[X, Y, Z]}{J(F_s)}\right)_{4k-3} \cong H^{2-k, k-1}(\bar{\mathcal{X}}_s), \quad k = 1, 2.$$

Note that in this case the primitive part is the full cohomology space.



Using the above identifications, one can check that a basis for the cohomology groups is given by

$$H^{1,0}(\overline{\mathcal{X}}_s) = \left\langle X \frac{\Omega_0}{F_s}, Y \frac{\Omega_0}{F_s}, Z \frac{\Omega_0}{F_s} \right\rangle, \quad H^{0,1}(\overline{\mathcal{X}}_s) = \left\langle X^5 \frac{\Omega_0}{F_s^2}, Y^5 \frac{\Omega_0}{F_s^2}, Z^5 \frac{\Omega_0}{F_s^2} \right\rangle$$

where the coefficient of  $\frac{\Omega_0}{F_s^k}$  are considered modulo the Jacobian ideal.

We fix the local section  $\omega(s) = X \frac{\Omega_0}{F_s}$  and we want to find the differential equation satisfied by the periods of  $\omega$ . A priori, we are searching for an order 6 differential equation, but we can check that indeed  $\omega(s)$  satisfies an order 2 differential equation. We want to find rational functions  $a_0$  and  $a_1$  such that

$$a_0(s) \int X \frac{\Omega_0}{F_s} + a_1(s) \int \frac{\partial}{\partial s} \left( X \frac{\Omega_0}{F_s} \right) + \int \frac{\partial^2}{\partial s^2} \left( X \frac{\Omega_0}{F_s} \right) = 0.$$

We can get rid of the integral sign and rewrite the above equation in De Rham-cohomology as

$$a_0(s) X \frac{\Omega_0}{F_s} - (a_1(s) F'(s) + F''(s)) X \frac{\Omega_0}{F_s^2} + 2F'(s)^2 X \frac{\Omega_0}{F_s^3} = 0 \in H^1(\mathcal{X}_s).$$

Now we begin to apply Griffiths-Dwork method.

We can check that  $2F'(s)^2 X$  is in the Jacobian ideal  $J(F_s)$ . Hence we can apply Equality (2.11) and compute

$$2F'(s)^2 X \frac{\Omega_0}{F_s^3} = \left( G_1 \frac{\partial F_s}{\partial X} + G_2 \frac{\partial F_s}{\partial Y} + G_3 \frac{\partial F_s}{\partial Z} \right) \frac{\Omega_0}{F_s^3} = \frac{1}{2} G \frac{\Omega_0}{F_s^2} \in H^1(\mathcal{X}_s)$$

for  $G = \left( \frac{\partial G_1}{\partial X} + \frac{\partial G_2}{\partial Y} + \frac{\partial G_3}{\partial Z} \right)$ , where the  $G_i(s)$  are some homogeneous polynomials in function of the rational parameter  $s$ .

We can now compute that

$$-X F'_s = \frac{9}{s} X^5, \quad \frac{1}{2} G - F''_s X = -\frac{81s^7}{s^9 - 1} X^5 \quad \text{in } \frac{\mathbb{C}[X, Y, Z]}{J(F_s)}.$$

Hence, we must have that

$$a_1(s) = \frac{9s^8}{s^9 - 1}.$$

By our choice of  $a_1(s)$ , we have

$$-(a_1(s) F'(s) + F''(s)) X \frac{\Omega_0}{F_s^2} + 2F'(s)^2 X \frac{\Omega_0}{F_s^3} = H \frac{\Omega_0}{F_s^2} \in H^1(\mathcal{X}_s)$$

for some  $H = \left( H_1 \frac{\partial F_s}{\partial X} + H_2 \frac{\partial F_s}{\partial Y} + H_3 \frac{\partial F_s}{\partial Z} \right) \in J(F_s)$ .

Therefore, we can apply Equation 2.11 again finding that

$$H \frac{\Omega_0}{F_s^2} = \left( \frac{\partial H_1}{\partial X} + \frac{\partial H_2}{\partial Y} + \frac{\partial H_3}{\partial Z} \right) \frac{\Omega_0}{F_s} = -\frac{16s^7}{s^9 - 1} X \frac{\Omega_0}{F_s} \in H^1(\mathcal{X}_s).$$

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Hence, by setting

$$a_0(s) = \frac{16s^7}{s^9 - 1}$$

we are done.

Since we have found a differential equation of order 2, we have proved that the absolute cohomology bundle has a subbundle of rank 2, whose dual is isomorphic to the space of solution of the differential equation

$$\frac{16s^7}{s^9 - 1}y + \frac{9s^8}{s^9 - 1}y' + y'' = 0 \quad (2.12)$$

which has indeed regular singularities at the 9-th roots of unity and at infinity. It is easy to show that this differential equation is the pull-back of a unique hypergeometric differential equation

$$L_1(y) = \frac{16}{81t(t-1)}y + \frac{17t-8}{9t(t-1)}y' + y'' = 0$$

under the map  $s \mapsto t = s^9$ , which is the one in the statement of Proposition 2.1.3

Using the same algorithm, one can compute the differential equations satisfied by  $\omega(s)^{(2)} = Y \frac{\Omega_0}{F_s}$  and  $\omega(s)^{(3)} = Z \frac{\Omega_0}{F_s}$  and check that they are respectively the pull-back under the map  $s \mapsto t = s^9$  of the unique differential equations

$$L_2(y) = \frac{4}{81t(t-1)}y + \frac{13t-4}{9t(t-1)}y' + y'' = 0$$

$$L_3(y) = \frac{1}{81t(t-1)}y + \frac{11t-2}{9t(t-1)}y' + y'' = 0.$$

Hence we have proved that the cohomology bundle splits as a sum of three rank 2 subbundles.  $\square$

Now, we want to prove that the local system defined by the differential equation  $L_1(y) = 0$ , or equivalently by the differential equation (2.12), is maximal Higgs.

*Proof of Corollary 2.1.4.* We can use [BM10b, Proposition 3.2] which relates the order of vanishing of the Kodaira-Spencer map to the local exponent of the associated differential equation in the case of unipotent monodromy. One can easily compute the local exponent of the differential Equation (2.12) and see that the Riemann scheme is given by

$\zeta_9^1$	$\zeta_9^2$	$\zeta_9^3$	$\zeta_9^4$	$\zeta_9^5$	$\zeta_9^6$	$\zeta_9^7$	$\zeta_9^8$	$\zeta_9^9$	$\infty$
0	0	0	0	0	0	0	0	0	4
0	0	0	0	0	0	0	0	0	4

Since Equation (2.12) corresponds to a family with unipotent monodromies, by [BM10b, Proposition 3.2], the order of vanishing of the Kodaira-Spencer map at these cusps is 0. Hence by definition, the associated local system is maximal Higgs and it has to be irreducible.  $\square$

By [Möl06b, Theorem 5.3], if the absolute cohomology bundle associated to a family of curves over  $C$  has a maximal Higgs rank two subbundle, then the family is a finite unramified covering of a Teichmüller curve. By Proposition 2.1.3, this is the case for the family of curves described by Equation (2.1).



## Zusammenfassung

Lyapunov-Exponenten sind charakteristische Zahlen, die das Verhalten eines Kozykels über einem dynamischen System beschreiben. Wenn der Kozykel eine Integrierbarkeits-eigenschaft erfüllt, liefert der Satz von Oseledets eine Zerlegung des unterliegenden Vektorbündels, so dass die Norm der Vektoren in jeder Komponente mit unterschiedlicher Geschwindigkeit entlang des Flusses wächst. Die verschiedenen möglichen Wachstumsraten werden Lyapunov-Exponenten genannt.

Ein interessantes Beispiel für ein dynamisches System ist das Billardspielen auf polygonalen Tischen. Lyapunov-Exponenten beschreiben die Diffusionsrate der Bahnen des Balles. Selbst in diesem speziellen Fall sind Lyapunov-Exponenten sehr schwer zu berechnen, zumindest wenn man sich auf die Werkzeuge der klassischen Ergodentheorie beschränkt. Überraschenderweise kann man die algebraische Geometrie ins Spiel bringen und diese erlaubt es manchmal, die mit einem Billard verbundenen Lyapunov-Exponenten zu berechnen. Der erste wichtige Baustein der Brücke zwischen dem Lyapunov-Exponenten und der algebraischen Geometrie ist durch flache Flächen gegeben. Trajektorien auf einem polygonalen Billardtisch können mit Geodäten bezüglich der flachen Metrik identifiziert werden, die kanonisch auf der Riemannschen Fläche, die man durch Verkleben der Seiten des Polygons erhält, definiert ist. Eine solche flache Metrik auf einer Riemannschen Fläche kann wiederum mit einem abelschen Differential identifiziert werden, das heißt mit einer globalen holomorphen 1-Form. Die Menge flacher Flächen vom Geschlecht  $g$  kann zu einem algebraischen Modulraum gemacht werden, dem Hodge-Bündel  $\Omega\mathcal{M}_g$ . Dieser Modulraum hat eine natürliche Stratifizierung durch Vorgabe der Nullstellenordnungen der Differentiale. Auf jedem Stratum gibt es ein natürliches Wahrscheinlichkeitsmaß, das Masur–Veech-Volumen. Die natürliche  $SL_2(\mathbb{R})$ -Wirkung auf diesem Raum, die durch Scherung des Polygons gegeben ist, ist ergodisch und ein wichtiges Ergebnis von Eskin und Mirzakhani liefert eine einfache Beschreibung der Bahnabschlüsse in geeigneten Koordinaten. Diese Bahnabschlüsse werden affine invariante Untermannigfaltigkeiten des Hodge-Bündels genannt. Filip konnte sogar zeigen, dass affine invariante Untermannigfaltigkeiten algebraisch sind. Da eine affine invariante Untermannigfaltigkeit eine Familie Riemannscher Flächen beschreibt, gibt es darauf eine zugehörige kanonische Variation von Hodge-Strukturen vom Gewicht eins, die durch das flache Vektorbündel gegeben ist, dessen Faser über einer Riemannschen Fläche  $X$  die Kohomologie  $H^1(X, \mathbb{C})$  ist. Überraschenderweise sind die Lyapunov-Exponenten, die als Diffusionsraten von Billardbahnen auf  $(X, \omega)$  gegeben sind, dieselben wie die, die durch die asymptotische Wachstumsrate der Hodgenorm von Vektoren in der Variation von Hodge-Strukturen über dem Fluss auf der affinen invarianten Mannigfaltigkeit  $SL_2(\mathbb{R})(X, \omega)$  definiert sind. An diesem Punkt kommt die algebraische Geometrie ins Spiel: Eskin, Kontsevich und Zorich [EKZ11] zeigten, dass die Summe der positiven Lyapunov-Exponenten des Kontsevich-Zorich-Kozykels über einer affinen invarianten Mannigfaltigkeit  $\mathcal{M}$  als der normalisierte Grad

des auf  $\mathcal{M}$  eingeschränkten Hodge-Bündels berechnet werden kann. Die überraschende Konsequenz ist, dass Zahlen, die nur über ein dynamisches System definiert wurden, die also keinen Grund haben, rational zu sein, tatsächlich der Grad eines Vektorbündels sind. Außerdem verbindet dies die Lyapunov-Exponenten mit anderen interessanten Invarianten, wie beispielsweise dem Masur-Veech-Volumen der Strata von  $\Omega\mathcal{M}_g$  oder den Siegel-Veech-Konstanten.

Ausgehend von Billards wurden Methoden der algebraischen Geometrie verwendet, um allgemeinere Lyapunov-Exponenten zu studieren. Beispielsweise bewiesen Kappes und Möller in [KM16] ein analoges Ergebnis zu dem von [EKZ11] für die Gewicht-1-Variationen von Hodge-Strukturen über Ballquotienten. Dieses Ergebnis erlaubte ihnen unter anderem, Fragen der Kommensurabilität von Gittern zu beantworten. Später zeigte Filip [Fil14] ein ähnliches Ergebnis für Variationen von Hodge-Strukturen, die durch eindimensionale Familien von K3-Flächen gegeben sind. Variationen von Hodge-Strukturen sind ein Spezialfall flacher Vektorbündel, die durch eine spezielle Filtrierung und eine kompatible Hodgenorm ausgezeichnet sind. Lyapunov-Exponenten können jedoch für ein allgemeines flaches Vektorbündel definiert werden. In [EKMZ18] haben Eskin, Kontsevich, Möller und Zorich die Summe der ersten  $k$  Lyapunov-Exponenten eines allgemeinen flachen Vektorbündels von unten durch den normalisierten Grad eines holomorphen Unterbündels vom Rang  $k$  beschränkt.

Das Hauptziel dieser Arbeit ist die Untersuchung von Lyapunov-Exponenten für allgemeine flache Vektorbündel über Riemannschen Flächen durch eine Verallgemeinerung des Hauptergebnisses von [EKMZ18] und die Beschreibung von Eigenschaften von Lyapunov-Exponenten an speziellen Loci des Modulraumes flacher Vektorbündel. Des Weiteren zeigen wir, wie die Kenntnis der Lyapunov-Exponenten über einer sporadischen Teichmüller-Kurve verwendet werden kann, um die algebraische Gleichung der universellen Familie darüber zu berechnen.

Im Kapitel 1 erinnern wir zunächst an die Riemann-Hilbert- und Simpson-Korrespondenzen, die eine Kategorie-Äquivalenz zwischen flachen Vektorbündeln über einer Riemannschen Fläche  $C$ , reduktiven Darstellungen der Fundamentalgruppe von  $C$  und polystabilen Higgs-Bündeln auf  $C$  mit verschwindenden Chern-Klassen definieren. Diese Korrespondenzen, die auch einen Homöomorphismus der entsprechenden Modulräume induzieren, erlauben es, Lyapunov-Exponenten als Funktionen auf diesen Modulräumen zu untersuchen. Der Modulraum von Darstellungen wird als Betti-Modulraum oder Charaktervarietät, der von flachen Bündeln als de-Rham-Modulraum bezeichnet. In Rang 2 wird die Sichtweise projektiver Strukturen nützlich sein, um zusätzliche Informationen zu erhalten. Die Hauptergebnis des Kapitels ist eine Verallgemeinerung des Ergebnisses von [EKMZ18].

**Theorem** (Theorem 1.7.13). *Sei  $\mathcal{V}$  ein holomorphes flaches Bündel über einer hyperbolischen Riemannschen Fläche  $C = \overline{C} \setminus \Delta$ . Dann gilt für jedes holomorphe Unterbündel  $\mathcal{E} \subset \mathcal{V}$  vom Rang  $k$ ,*

$$\sum_{i=1}^k \lambda_i \geq \frac{2 \deg_{\text{par}}(\Xi_h(\mathcal{E}))}{\deg(\Omega_C^1(\log(\Delta)))} + \text{Err}^{\mathcal{E}}(u)$$

für fast alle  $x \in C$  und Lebesgue-fast alle  $u \in \bigwedge^k \mathcal{V}_x^\vee$ . Dabei ist der Fehler durch

$$\text{Err}^\mathcal{E}(u) = 4\pi \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\#\{s_\mathcal{E}^{-1}(\ker u) \cap D_t\}}{\text{vol}(D_t)} dt$$

gegeben. Hierbei bezeichnet  $\text{deg}_{\text{par}}(\Xi_h(\mathcal{E}))$  den parabolischen Grad der metrischen Fortsetzung von  $\mathcal{E}$ ,  $s_\mathcal{E} : \mathbb{H} \rightarrow \mathbb{P}(\bigwedge^k \mathcal{V}_x)$  die holomorphe klassifizierende Abbildung, die  $\mathcal{E}$  definiert und  $D_t$  einen hyperbolische Kreisscheibe vom Radius  $t$  um einen Lift von  $x$  in der hyperbolischen Ebene.

Der Beweis des obigen Satzes lässt sich auf Variationen von Hodge-Strukturen vom Gewicht eins über Ballquotienten oder affinen invarianten Untermannigfaltigkeiten des Hodge-Bündels verallgemeinern.

Im Falle einer kompakten Basiskurve können wir sogar folgendes zeigen.

**Theorem** (Theorem 1.8.1). *Wenn die Riemannsche Fläche  $C$  kompakt ist, ist die obige Ungleichung eine Gleichheit.*

Diese Ergebnisse können als eine Verallgemeinerung des Hauptergebnisses von [DD15] angesehen werden, wobei Deroin und Dujardin Lyapunov-Exponenten von Holonomien projektiver Strukturen im Zusammenhang mit dem Fluss, der aus der Brownschen Bewegung hervorgeht, definieren und diese mit dem Grad der zugehörigen Entwicklungsabbildung in Beziehung setzen. Darüber hinaus können die obigen Ergebnisse auch mit dem Hauptergebnis von [DD17a] verglichen werden, in dem Daniel und Deroin eine ähnliche Gleichheit im Kontext des Brownschen Bewegungsflusses über einer Kähler-Mannigfaltigkeit erhalten. Sie beweisen, dass die Summe der ersten Lyapunov-Exponenten gleich der Summe eines normalisierten Grades und eines dynamischen Grades ist; als Konsequenz erhält man, dass der oben definierte Fehlerterm  $\text{Err}^\mathcal{E}(u)$ , der die gleiche Form wie der Überlagerungsgrad aus [DD15] hat, derselbe ist, wie der in [DD17a] definierte dynamische Grad.

Lyapunov-Exponenten sind nicht die einzigen Invarianten von Darstellungen, die man auf Charaktervarietäten definieren kann. Wir erinnern hier an die Toledo-Invariante, die für Darstellungen in Lie-Gruppen vom hermiteschen Typ definiert ist, oder den kritischen Exponenten, der für Darstellungen nach  $\text{SL}_n(\mathbb{R})$  definiert ist. Weitere Invarianten wie die Entropie oder die Minimalfläche sind eng mit dem kritischen Exponenten verbunden, vor allem in den Hitchin-Komponenten, wo sie ähnlichen Schranken genügen. Das Erreichen dieser Schranken charakterisiert die symmetrischen Potenzen Fuchs'scher Darstellungen. Darüber hinaus vermuten wir auf Grund von Computer-Experimenten, dass der oberste Lyapunov-Exponent ähnliche Schranken wie der kritische Exponent genügt. Die genaue Beziehung zwischen Lyapunov-Exponenten und den anderen erwähnten Invarianten ist zwar immer noch spekulativ, aber das geometrische Oseledets-Theorem liefert eine hilfreiche Sichtweise. Der Hauptunterschied liegt in der Tatsache, dass Lyapunov-Exponenten auf dem de-Rham-Modulraum natürlich definiert sind, da sie von der komplexen Struktur der Basis-Riemannschen Fläche abhängen, während die anderen erwähnten Invarianten auf der Charaktervarietät natürlich definiert sind, da sie nur von der Darstellung der Fundamentalgruppe abhängen.

Wir konzentrieren uns nun auf die Eigenschaften von Lyapunov-Exponenten auf speziellen Loci des De-Rham-Modulraums. Wir erinnern an die Existenz einer Stratifizierung vom Harder-Narasimhan-Typ, der Shatz-Stratifizierung. Durch Identifikation des maximalen Stratum mit dem Oper-Locus erhalten wir eine explizite Untergrenze der Lyapunov-Exponenten auf diesem Stratum. Mit Hilfe eines neuen Resultats [DF18] bezüglich der Oper-Loci können wir beweisen, dass der oberste Lyapunov-Exponent unbeschränkt ist.

**Theorem** (Theorem 1.9.2). *Der oberste Lyapunov-Exponent ist auf jedem Oper-Locus unbeschränkt. Das Wachstum ist in der Nähe des Randes der Charaktervarietät logarithmisch.*

Der nächste interessante Ort, den es zu untersuchen gilt, ist die Menge der flachen Bündel, die den Variationen von Hodge-Strukturen zugrunde liegen. Wir zeigen eine leichte Verallgemeinerung der Ergebnisse von [EKZ11] und [Fil14] unter Verwendung einer allgemeinen Bedingung für das Verschwinden des Fehlerterms. Wir zeigen auch, dass die Trivialität des Lyapunov-Spektrums impliziert, dass die Variation von Hodge-Strukturen unitär ist und dass für Variationen von Hodge-Strukturen positiven Gewichts der erste Term der Hodge-Filtrierung eine nichttriviale untere Schranke für die Summe der ersten Exponenten induziert.

Schließlich konzentrieren wir uns auf den Rang-2-De-Rham-Modulraum und beschreiben die Shatz-Stratifizierung und die Loci der Variationen von Hodge-Strukturen genauer. Durch Identifikation des Oper-Locus mit der Menge der Holonomien projektiver Strukturen, die dieselbe komplexe Struktur induzieren, geben wir eine Verallgemeinerung des Hauptergebnisses von [DD17b].

Im Kapitel 2 berechnen wir die algebraische Gleichung der universellen Familie über einer der beiden sporadischen Teichmüller-Kurven, der Kenyon-Smillie-(2, 3, 4)-Teichmüller-Kurve. Dies ist eine gemeinsame Arbeit mit André Kappes und ist als [CK17] erschienen.

Teichmüller-Kurven sind Projektionen abgeschlossener  $SL_2(\mathbb{R})$ -Bahnen im Hodge-Bündel nach  $\mathcal{M}_g$  und sind somit abgeschlossene algebraische Kurven im Modulraum von Kurven. Möller hat in [Möl06b] gezeigt, dass Teichmüller-Kurven, wie Shimura-Kurven, durch die spezielle Form ihrer Variation von Hodge-Strukturen charakterisiert werden können. Die Kenyon-Smillie-Teichmüller-Kurve parametrisiert alle affinen Deformationen  $(S_t, \omega_t)$  der Translationsfläche  $(S, \omega) \in \Omega\mathcal{M}_3(3, 1)$ , die durch Entfalten eines euklidischen Dreiecks mit Winkeln  $(\frac{2\pi}{9}, \frac{3\pi}{9}, \frac{4\pi}{9})$  entsteht. Diese wurde von Kenyon und Smillie [KS00] entdeckt, die bewiesen, dass  $(S, \omega)$  eine Veech-Fläche mit Veech-Gruppe die Dreiecksgruppe  $\Delta(9, \infty, \infty)$  ist. Die Translationsfläche  $(S, \omega)$  ist ein Orbifoldpunkt der Ordnung 9 der zugehörigen Teichmüller-Kurve, die durch  $\mathbb{H}/\Delta(9, \infty, \infty)$  uniformisiert wird.

**Theorem** (Theorem 2.1.1). *Die universelle Familie über dem Komplement des Orbifold-Punkts der Kenyon-Smillie-(2, 3, 4)-Teichmüller-Kurve ist durch die Familie ebener Quartiken gegeben, die durch die Gleichung*

$$X^4 + t(X^4 - 3X^3Y + 6X^3Z - 3X^2Y^2 - 6X^2YZ + 6X^2Z^2 + 4XY^3 - 6XY^2Z - 6XYZ^2 + XZ^3 + 3Y^4 + 3Y^3Z) = 0$$



mit  $t$  in  $\mathbb{P}^1 - \{0, 1, \infty\}$  definiert ist.

Die dreifache Nullstelle des Differentials  $\omega_t$  ist der Punkt  $p_t = (0 : 0 : 1) \in S_t$  und die einfache Nullstelle ist der Punkt  $q_t = (0 : 1 : -1) \in S_t$ .

Wie oben ist das Studium der Variation von Hodge-Strukturen über der Teichmüller-Kurve und ihrer Harder-Narasimhan-Filtrierung von zentraler Bedeutung. Tatsächlich sind die Charakterisierung von Teichmüller-Kurven über ihre assoziierte Variation von Hodge-Strukturen aus [Möl06b] und die Beschreibung des zugehörigen Harder-Narasimhan-Typs, der in [YZ13] berechnet wird, die wichtigsten verwendeten Werkzeuge.

Da es sich in diesem Fall um eine Familie von Hyperflächen handelt, können wir die Griffiths-Dwork-Methode verwenden, um die Picard-Fuchs-Gleichungen zu berechnen, die zu dieser Familie gehören.

**Proposition** (Proposition 2.1.3). *Die Perioden von  $\omega_t$  sind Lösungen der folgenden Differentialgleichung:*

$$\frac{16}{81t(t-1)}y + \frac{17t-8}{9t(t-1)}y' + y'' = 0.$$

Das lokale System, das durch die Lösung dieser Differentialgleichung definiert ist, entspricht einem irreduziblen Rang 2 Unterbündel der Variation von Hodge-Strukturen der Teichmüller-Kurve. Indem wir die lokalen Exponenten der obigen Differentialgleichung mit der Verschwindungsordnung des Higgs-Felds, das der Variation von Hodge-Strukturen zugeordnet ist, in Beziehung setzen, beweisen wir, dass das lokale System, das durch die obige Gleichung definiert ist, der uniformisierenden Darstellung der Teichmüller-Kurve entspricht. Durch die Charakterisierung von Teichmüller-Kurven aus [Möl06b] haben wir einen unabhängigen Beweis, dass die Gleichung, die wir berechnet haben, tatsächlich die der universellen Familie über einer Teichmüller-Kurve ist.



# Lebenslauf

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