Smooth valuations on convex functions

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Contents

1 Introduction

Valuation theory has its origin at the beginning of the twentieth century in Dehn's solution of Hilbert's third problem [24], which showed that a three dimensional cube cannot be dissected into a finite number of convex polytopes such that these polytopes can be rearranged to form a tetrahedron of the same volume. His proof relied on the construction of a finitely additive measure, i.e. a valuation, on the set of convex polytopes that is invariant with respect to Euclidean motions and associates different values to the cube and tetrahedron of equal volume. While this sparked the development of a rather rich combinatorial theory, see for example [45, 54], the theory of *continuous* valuations on convex bodies has also seen remarkable developments in recent years with many applications to geometric inequalities and integral geometry. As the notion of valuations on functions is heavily influenced by this part of valuation theory, we will start our discussion with these functionals.

Let V be a real vector space of dimension $\dim V = n$ and let $\mathcal{K}(V)$ denote the space of all convex bodies in V, i.e. the set of all compact convex subsets of V. Equipped with the Hausdorff metric, $\mathcal{K}(V)$ is a locally compact, σ-compact metric space. Let F denote a real topological vector space. A functional $\mu : \mathcal{K}(V) \to F$ is called a valuation if it satisfies

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

for all $K, L \in \mathcal{K}(V)$ such that $K \cup L \in \mathcal{K}(V)$. Note that the intersection $K \cap L$ is non-empty in this case and thus belongs to $\mathcal{K}(V)$. Let us denote the space of all continuous, translation invariant valuations on $\mathcal{K}(V)$ with values in F by $\mathrm{Val}(V, F)$. For $F = \mathbb{R}$, we will also set $\mathrm{Val}(V) := \mathrm{Val}(V, \mathbb{R})$. We equip these spaces with the compact-open topology, which coincides with the topology of locally uniform convergence, as $\mathcal{K}(V)$ is a locally compact metric space.

A valuation $\mu \in \mathrm{Val}(V, F)$ is called *k-homogeneous* or homogeneous of degree $k \in \mathbb{R}$ if $\mu(tK) = t^k \mu(K)$ for all $K \in \mathcal{K}(V)$ and $t \geq 0$. We will denote the subspace of k-homogeneous elements in $\mathrm{Val}(V, F)$ by $\mathrm{Val}_k(V, F)$, and we set $\mathrm{Val}_k(V) := \mathrm{Val}_k(V, \mathbb{R})$.

One of the most striking properties of continuous, translation invariant valuations is the following homogeneous decomposition, also called *McMullen decomposition*.

Theorem 1.0.1 (McMullen [43]). *Let F be a Hausdorff real topological vector space. Then*

$$\mathrm{Val}(V, F) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(V, F).$$

If F is Banach space with norm $|\cdot|$, this theorem implies that the topological vector space $\mathrm{Val}(V,F)$ is also a Banach space with respect to the norm

$$\|\mu\| := \sup_{K\subset B} |\mu(K)|,$$

where $B \in \mathcal{K}(V)$ is any convex body with non-empty interior. Similarly, $\mathrm{Val}(V,F)$ becomes a Fréchet space if F carries a Fréchet topology.
Let us restate the homogeneous decomposition: For any valuation $\mu \in \mathrm{Val}(V,F)$, the map $t \mapsto \mu(tK)$ is a polynomial in $t \geq 0$, whose degree is bounded by the dimension of V. Starting with a homogeneous element $\mu \in \mathrm{Val}_k(V,F)$, this theorem allows us to define the *polarization* of μ, given by

$$\bar{\mu}(K_1,\dots,K_k) := \frac{1}{k!}\frac{\partial}{\partial\lambda_1}\Big|_0 \cdots \frac{\partial}{\partial\lambda_k}\Big|_0 \mu\left(\sum_{i=1}^k \lambda_i K_i\right)$$

for $K_1,\dots,K_k \in \mathcal{K}(V)$. We obtain a symmetric functional $\bar{\mu} : \mathcal{K}(V)^k \to F$ with the following properties (see [54] Theorem 6.3.6):

1. $\bar{\mu}$ is a continuous, translation invariant valuation in each argument.
2. $\bar{\mu}$ is additive in each argument: For $K, L, K_2, \dots, K_k \in \mathcal{K}(V)$:
$$\bar{\mu}(K+L,K_2,\dots,K_k) = \bar{\mu}(K,K_2,\dots,K_k) + \bar{\mu}(L,K_2,\dots,K_k).$$
3. $\bar{\mu}(K,\dots,K) = \mu(K)$ for all $K \in \mathcal{K}(V)$.

Starting with a Lebesgue measure vol_V, which can be considered as an element of $\mathrm{Val}_n(V)$, we recover the well known *mixed volumes* $V(K_1,\dots,K_n)$. These functionals can be used to construct a large class of translation invariant valuations: Let $L_1,\dots,L_{n-k} \in \mathcal{K}(V)$. Then the functional $K \mapsto V(K[k],L_1,\dots,L_{n-k})$ belongs to $\mathrm{Val}_k(V)$, where K is taken with multiplicity k in this expression. In fact, valuations of this type are dense in $\mathrm{Val}_k(V)$ with respect to the topology of uniform convergence on compact subsets. This result, known as McMullen's conjecture, was proved by Alesker in [2]. In fact, he obtained a much more general result:

Theorem 1.0.2 (Alesker's irreducibility theorem [2])**.** *The natural representation of* $\mathrm{GL}(V)$ *on* $\mathrm{Val}_k^{\pm}(V) := \{\mu \in \mathrm{Val}_k(V) : \mu(-K) = \pm\mu(K)\ \forall K \in \mathcal{K}(V)\}$ *is topologically irreducible.*

Here $\mathrm{GL}(V)$ acts on $\mu \in \mathrm{Val}(V)$ by $\pi(g)\mu(K) := \mu(g^{-1}K)$, and a representation is called topologically irreducible if there exists no proper, non-trivial, closed invariant subspace, i.e. if every invariant subspace is either 0 or dense. McMullen's conjecture follows directly from this theorem, since one can construct linear combinations of mixed volumes that intersect the spaces $\mathrm{Val}_k^{\pm}(V)$ non-trivially, see [2]. This theorem relies on a number of embedding theorems for translation invariant valuations. For our purposes, the most important construction is the *Goodey-Weil embedding.* Recall that a convex body $K \in \mathcal{K}(V)$ is uniquely determined by its *support function* $h_K : V^* \to \mathbb{R}$, given by

$$h_K(y) = \sup_{x \in K} \langle y, x \rangle \quad \text{for } y \in V^*,$$

where the brackets denote the natural pairing between V^* and V. Note that h_K is 1-homogeneous by definition. Identifying a convex body with its support function, the polarization of a homogeneous valuation can be considered as a multilinear functional on the cone of support functions. By considering differences of support functions, Goodey and Weil [30] obtained the following result:

Theorem 1.0.3. *Let F be a locally convex vector space. For every $\mu \in \mathrm{Val}_k(V, F)$ there exists a unique distribution $\mathrm{GW}(\mu) \in \mathcal{D}'(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}; \bar{F})$, called the Goodey-Weil distribution of μ, such that*

$$\mathrm{GW}(\mu)\,[h_{K_1} \otimes \cdots \otimes h_{K_k}] = \bar{\mu}(K_1, \ldots, K_k)$$

for all $K_1, \ldots, K_k \in \mathcal{K}(V)$ smooth and strictly convex.
In addition, the order of $\mathrm{GW}(\mu)$ is uniformly bounded for all $\mu \in \mathrm{Val}_k(V, F)$, and the map $\mathrm{GW} : \mathrm{Val}_k(V, F) \to \mathcal{D}'(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}; \bar{F})$ is continuous and injective.
Furthermore, the support of $\mathrm{GW}(\mu)$ is contained in the diagonal of $\mathbb{P}_+(V^*)^k$.

Here, $L \to \mathbb{P}_+(V^*)$ is a certain $\mathrm{GL}(V)$-equivariant line bundle over the space of oriented lines in V^*, and $\mathcal{D}'(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}; \bar{F})$ denotes the space of all $\bar{F}$-valued distributions, i.e. all continuous linear maps $C^\infty(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}) \to \bar{F}$, which is equipped with the strong topology. $\bar{F}$ denotes the completion of F. The space of continuous sections of the line bundle L can be canonically identified with the space of continuous, 1-homogeneous functions on V^*, so in particular, any support function induces a section of L. The Goodey-Weil embedding was originally introduced in [30] to prove McMullen's conjecture for 1-homogeneous valuations. Alesker observed that the support of these distributions is contained in the diagonal (see [1]), which was a major step in the proof of the irreducibility theorem.

Let us remark that the Goodey-Weil embedding was originally only defined for real valued valuations, but the construction can easily be extended to the more general case. We will omit the precise details, as we will encounter identical estimates in the construction of the Goodey-Weil embedding for valuations on convex functions in Section 5.4.1.

In recent years, many results from the theory of convex bodies were extended to functional versions. This includes extensions of mixed volumes [46, 47] as well as functional versions of various inequalities, see for example [41, 58]. Many of the functionals considered in these works turn out to be valuations on functions in the following sense: Let X denote some class of real-valued functions. A map $\mu : X \to (G, +)$ into some Abelian semi-group G is called a valuation if

$$\mu(f) + \mu(h) = \mu(f \vee h) + \mu(f \wedge h)$$

for all $f, h \in X$ such that the pointwise maximum $f \vee h$ and minimum $f \wedge h$ also belong to X. Take for example the set of *convex indicator functions* of convex bodies in $\mathcal{K}(V)$. The convex indicator function I_A^∞ of a set $A \subset V$ is defined by

$$I_A^\infty(x) := \begin{cases} \infty & x \notin A \\ 0 & x \in A \end{cases},$$

so any valuation on these indicator functions recovers the notion of a valuation on $\mathcal{K}(V)$ as

$$I_K^\infty \vee I_L^\infty = I_{K\cap L}^\infty, \qquad\qquad I_K^\infty \wedge I_L^\infty = I_{K\cup L}^\infty$$

for all $K, L \in \mathcal{K}(V)$. The same reasoning applies to indicator functions in $\mathrm{L}^1(\mathbb{R}^n)$. In this sense, the Lebesgue integral defines a valuation $I : \mathrm{L}^1(\mathbb{R}^n) \to \mathbb{R}$, which satisfies $I(\alpha 1_K) = \alpha \operatorname{vol}_n(K)$ for all $K \in \mathcal{K}(\mathbb{R}^n)$. Of course, this is not the only possibility to extend the volume. Take for example $F \in C(\mathbb{R})$ with $|F(t)| \le C|t|$ for some $C > 0$ and all $t \in \mathbb{R}$. Then

$$\tilde{I}(f) := \int_{\mathbb{R}^n} F(f(x))dx$$

defines a continuous valuation on $\mathrm{L}^1(\mathbb{R}^n)$, which satisfies $\tilde{I}(\alpha 1_K) = F(\alpha)\operatorname{vol}_n(K)$ for all $K \in \mathcal{K}(\mathbb{R}^n)$. The problem of extending classical functionals to valuations on functions thus usually involves dealing with a large number of degrees of freedoms, although there are exceptions, see for example [22].

Nevertheless, the problem of finding geometrically or analytically meaningful valuations remains. It is thus not surprising that a large number of results is focused on the classification of certain valuations in terms of their invariance properties. For example, the functionals considered above give a complete characterization of all continuous, translation invariant valuations on $\mathrm{L}^1(\mathbb{R}^n)$, see [56]. Similar results exist for Sobolev-spaces [38, 39, 42], L^p-spaces [11, 37, 40, 50, 56, 57], quasi-concave functions [10, 15, 16], Orlicz-spaces [36], Lipschitz functions [22, 23], definable functions [10], functions of bounded variation [59], and convex functions [12, 17, 18, 48, 49]. There also exist some general results on analytic properties of valuations on Banach lattices [55]. Let us also remark that valuations on convex functions were also used to give a first example of a Spin(9)-invariant valuation on convex bodies on the octonionic plane [6] as well as examples of invariant valuations on quaternionic spaces [4].

In this thesis, we will also try to characterize a certain space of invariant valuations on convex functions, however, we will focus on the construction of some dense subspaces. A general classification of all valuations in this class seems to be out of reach, at least currently. These valuations can be considered, in a way made precise below, as translation invariant valuations on convex bodies, where a full classification is unknown.

Let us introduce the general setting. For the most part, we will be interested in valuations on subspaces of

$$\operatorname{Conv}(V) := \{f : V \to \mathbb{R} \cup \{+\infty\} \ : \ f \text{ convex, lower semi-continuous}, f \not\equiv +\infty\}.$$

This space carries a standard metrizable topology, induced by the notion of epi-convergence. We will recall the necessary definitions in Chapter 4, for the purpose of this introduction, it is sufficient to note that this topology coincides with the topology of locally uniform convergence on the space $\mathrm{Conv}(V,\mathbb{R}) := \{f \in \mathrm{Conv}(V) : f < \infty\}$ of finite-valued convex functions.

Let us consider some examples of valuations on convex functions. We start with the following observation: If a continuous valuation $\mu : \mathrm{Conv}(V,\mathbb{R}) \to \mathbb{R}$ is invariant with respect to the addition of linear functionals, then

$$\mu(h_{K+y}) = \mu(h_K + \langle y, \cdot\rangle) = \mu(h_K)$$

for all $K \in \mathcal{K}(V^*)$, $y \in V^*$, i.e. the map $K \mapsto \tilde{\mu}(K) := \mu(h_K)$ defines an element of $\mathrm{Val}(V^*)$. By the McMullen decomposition, $\tilde{\mu}$ decomposes into its homogeneous components. These valuations were considered by Alesker [7], who showed that the construction above gives raise to a dense subspace of $\mathrm{Val}(V^*)$, although the kernel is infinite dimensional. This directly leads to the question, whether this space of invariant valuations decomposes into homogeneous components. Unsurprisingly, this is not the case.

Example 1.0.4. *Take $p \geq 0$ and define*

$$\mu(f) := |f(0)|^p \quad \textit{for } f \in \mathrm{Conv}(V,\mathbb{R}).$$

It is easy to see that μ is a valuation, which is in addition p-homogeneous.

Obviously, these valuations are all invariant under the addition of linear functionals, so there does not exist a homogeneous decomposition for this space of invariant valuations. However, all of the examples constructed in [7] are homogeneous of degree $0 \leq k \leq n$. They are all of the following form:

Example 1.0.5. *Let V be a Euclidean vector space. For $0 \leq k \leq n$ let $B \in C_c(V)$, $A_1, \dots A_{n-k} \in C_c(V, \mathcal{H}(V))$, where $\mathcal{H}(V)$ denotes the space of symmetric endomorphisms of V. Alesker showed [7] that the functional*

$$\begin{aligned} \mathrm{Conv}(V,\mathbb{R}) \cap C^2(V) &\to \mathbb{R} \\ f &\mapsto \int_V B(x) \det\left(H_f(x)[k], A_1(x), \dots, A_{n-k}(x)\right) dx \end{aligned}$$

extends uniquely to a k-homogeneous, continuous valuation on $\mathrm{Conv}(V,\mathbb{R})$, which is in addition invariant with respect to the addition of linear functionals. Here det *denotes the mixed determinant of n symmetric endomorphisms and H_f is the Hessian of $f \in C^2(V)$. We will call all valuations of this type* Alesker valuations*.*

Note that Alesker valuations satisfy an additional invariance property: They are invariant under the addition of constants.
These functionals are in some sense all derived from the following construction.

Example 1.0.6. *Let V be a Euclidean vector space. Consider the functional*

$$\begin{aligned}\mathrm{Conv}(V,\mathbb{R})\cap C^2(V)&\to\mathcal{M}(V)\\ f&\mapsto\left[U\mapsto\int_U\det(H_f(x))dx\right].\end{aligned}$$

As shown in [20] (or [7]), this functional extends to a continuous valuation Hess_n *on* $\mathrm{Conv}(V,\mathbb{R})$ *with values in the space $\mathcal{M}(V)$ of signed Radon measures on V equipped with the vague topology, i.e. the topology induced by the family of semi-norms*

$$\|\nu\|_\phi:=\left|\int_V\phi(x)d\nu(x)\right|\quad \text{for } \nu\in\mathcal{M}(V),$$

for $\phi\in C_c(V)$. We will call this valuation the n-th Hessian measure*, or simply the Hessian measure (see [20] for more general versions of this valuation).*

Although we have chosen a scalar product in the previous example, Hess_n only depends on a choice of a density on V^*: Take $f\in\mathrm{Conv}(V,\mathbb{R})\cap C^2(V)$ strictly convex. Then $df:V\to V^*$ is a homeomorphism, so given a Lebesgue measure vol_{V^*} on V^* the pushforward $(df^{-1})_*\mathrm{vol}_{V^*}$ defines a measure on V. If we choose a scalar product on V such that the vol_{V^*} coincides with the induced Lebesgue measure under the isomorphism $V\cong V^*$, then

$$(df^{-1})_*\mathrm{vol}_{V^*}(U)=\int_U\det(H_f(x))d\,\mathrm{vol}_V(x).$$

This also allows for an interpretation in terms of differential forms: Assume that we have chosen an orientation on V with induced orientation on V^*. Then vol_{V^*} can be considered as an n-form on V^*, which we can pull back to $V\times V^*=T^*V$ using the natural projection onto V^*. The graph of the differential of $f\in\mathrm{Conv}(V,\mathbb{R})\cap C^2(V)$ defines a C^1-submanifold of T^*V, and the pullback of the restriction of vol_{V^*} by the natural map $x\mapsto(x,df(x))$ from V to the graph of df corresponds exactly to the Hessian measure. In Chapter 7 we will give a more general construction of valuations defined by integrating certain differential forms over the differential of a convex function.

Example 1.0.7. *Let ν denote a compactly supported, signed Radon measure on V. Then*

$$\mu(f):=\int_V f(x)d\nu(x)$$

defines a continuous valuation on $\mathrm{Conv}(V,\mathbb{R})$*. The class of valuations of this type includes examples such as*

$$\begin{aligned}\mu'(f)&:=f(x)+f(-x)-2f(0) &&\text{for } f\in\mathrm{Conv}(V,\mathbb{R}),\ \text{for some fixed } x\in V,\\ \mu''(f)&:=\frac{1}{\mathcal{H}^{n-1}(S^{n-1})}\int_{S^{n-1}}fd\mathcal{H}^{n-1}-f(0) &&\text{for } f\in\mathrm{Conv}(\mathbb{R}^n,\mathbb{R}).\end{aligned}$$

Note that both μ' and μ'' are invariant under the addition of linear and constant functionals.

The additional property that has to be imposed on the valuations to obtain a homogeneous decomposition is *vertical invariance*, i.e. the invariance with respect to the addition of constants.

Definition 1.0.8. *Let $C\subset \mathrm{Conv}(V)$ be a non-empty subset. A valuation μ on C with values in some real topological vector space F is called* dually epi-translation invariant *if*

$$\mu(f+\lambda+c)=\mu(f)\quad \textit{for all } f\in C,\lambda\in V^*,c\in\mathbb{R}$$

such that $f+\lambda+c\in C$. The vector space of all continuous, dually epi-translation invariant valuations will be denoted by $\mathrm{VConv}(C;V,F)$.

For $C=\mathrm{Conv}(V,\mathbb{R})$ we will denote these spaces by $\mathrm{VConv}(V,F)$, and we will drop the dependency on F for $F=\mathbb{R}$. Let us remark that the name $\mathrm{VConv}(V)$ was used by Alesker in [7] to denote the space of all continuous valuations on $\mathrm{Conv}(V,\mathbb{R})$ that are in addition invariant under the addition of linear functionals. As his examples are all dually epi-translation invariant, we will borrow the name from his work. This also implies that the main result from [7] applies to our space of valuations, i.e. the map

$$\begin{aligned}\mathrm{VConv}(V)&\to \mathrm{Val}(V^*)\\ \mu&\mapsto [K\mapsto \mu(h_K)]\end{aligned}$$

has dense image and infinite dimensional kernel. We will, however, not need this result.

For the geometric significance of this invariance property, let us return to the definition of a valuation on a space of functions. Assume that the functions under consideration are defined on some set A. For a given function f, the *epi-graph* $\mathrm{epi}(f):=\{(x,t)\in A\times\mathbb{R}: f(x)\le t\}$ satisfies

$$\mathrm{epi}(f)\cap\mathrm{epi}(h)=\mathrm{epi}(f\vee h),\qquad\qquad \mathrm{epi}(f)\cup\mathrm{epi}(h)=\mathrm{epi}(f\wedge h),$$

so a valuation on functions may also be interpreted as a set theoretic valuation on epigraphs. For every $f\in\mathrm{Conv}(V)$, its Legendre transform $f^*\in\mathrm{Conv}(V^*)$ is given by

$$f^*(y)=\sup_{x\in V}\langle y,x\rangle-f(x)=h_{\mathrm{epi}\,f}(y,-1)\quad\text{for } y\in V^*,$$

where the brackets denote the natural pairing between V and its dual space V^*. As $f^{**}=f$, we may consider $f=h_{\mathrm{epi}\,f^*}(\cdot,-1)$ as the support functional of a non-compact convex set in $V^*\times\mathbb{R}$. The invariance property thus asserts that the valuation is invariant with respect to translations of this set in $V^*\times\mathbb{R}$. In particular, we can restrict these valuations to support functions of convex bodies in $V^*\times\mathbb{R}$.

Theorem 1.0.9 (Theorem 5.2.5). *Let $C\subset\mathrm{Conv}(V)$ be a subset with $\mathrm{Conv}(V,\mathbb{R})\subset C$, F a Hausdorff real topological vector space. The map*

$$\begin{aligned}T:\mathrm{VConv}(C;V,F)&\to \mathrm{Val}(V^*\times\mathbb{R},F)\\ \mu&\mapsto [K\mapsto \mu(h_K(\cdot,-1))]\end{aligned}$$

is well defined, continuous, and injective. Here, $\mathrm{Val}(V, F)$ *denotes the space of all continuous, translation invariant valuations that take values in* F*, and both spaces are equipped with the compact-open topology.*

We can thus interpret elements of $\mathrm{VConv}(C; V, F)$ as valuations on higher dimensional convex bodies. The McMullen decomposition for $\mathrm{Val}(V^* \times \mathbb{R}, F)$ thus implies the following homogeneous decomposition.

Theorem 1.0.10. *Let* $C \subset \mathrm{Conv}(V)$ *be invariant under scaling, i.e.* $tf \in C$ *for all* $f \in C$*,* $t > 0$*, and such that* $\mathrm{Conv}(V, \mathbb{R}) \subset C$*. Assume that* F *is a Hausdorff real topological vector space. Then*

$$\mathrm{VConv}(C; V, F) = \bigoplus_{k=0}^{n} \mathrm{VConv}_k(C; V, F).$$

Here $\mathrm{VConv}_k(C; V, F)$ *denotes the space of all* k*-homogeneous valuations, i.e. all valuations* $\mu \in \mathrm{VConv}(C; V, F)$ *that satisfy* $\mu(tf) = t^k \mu(f)$ *for all* $f \in C$*,* $t > 0$*.*

For $C = \mathrm{Conv}(V, \mathbb{R})$ and $F = \mathbb{R}$, this was already proved by Colesanti, Ludwig and Mussnig [21] using translates of support functions. Their proof easily generalizes to the more general setting. As the map T will play a crucial part in many of our constructions, we will include an alternative proof of this result, see Theorem 5.3.4.
Similar to the McMullen decomposition of $\mathrm{Val}(V)$, the homogeneous decomposition for $\mathrm{VConv}(C; V, F)$ allows us to define the polarization of a homogeneous valuation if $C \subset \mathrm{Conv}(V)$ is a *cone*, i.e. if $tf + g \in C$ for all $f, g \in C$, $t > 0$. However, this requires the following regularity assumption: We will call a cone C *regular* if the domain $\mathrm{dom}(f) := \{x \in V : f(x) < \infty\}$ has non-empty interior for all $f \in C$. This assumption seems to be necessary, as the definition of the polarization relies on the continuity of the addition map on $C \subset \mathrm{Conv}(V)$, which is not continuous on arbitrary cones. From now on, we will assume that C is a regular cone containing $\mathrm{Conv}(V, \mathbb{R})$. For $\mu \in \mathrm{VConv}_k(C; V, F)$ we obtain a symmetric functional $\bar{\mu} : C^k \to F$ with the following properties:

1. $\bar{\mu}$ is a continuous, dually epi-translation invariant valuation in each argument.

2. $\bar{\mu}$ is additive: For $f, h, f_2, \dots, f_k \in C$

$$\bar{\mu}(f + h, f_2, \dots, f_k) = \bar{\mu}(f, f_2, \dots, f_k) + \bar{\mu}(h, f_2, \dots, f_k).$$

3. $\bar{\mu}(f, \dots, f) = \mu(f)$ for all $f \in C$.

Thus, $\bar{\mu}$ is a multilinear functional on the cone C, and by representing a smooth function $\phi \in C_c^\infty(V)$ as a difference of convex functions we can use $\bar{\mu}$ to define a *continuous* multilinear functional on $C_c^\infty(V)$. If F is a locally convex vector space, the L. Schwartz kernel theorem implies that this functional lifts to a unique distribution on V^k.

Theorem 1.0.11 (Theorem 5.4.9 and 5.4.7)**.** *Let $\mu \in \mathrm{VConv}_k(C;V,F)$ and assume that F admits a continuous norm. Then there exists a unique distribution $\overline{\mathrm{GW}}(\mu) \in \mathcal{D}'(V^k, \bar{F})$ with compact support such that*

$$\overline{\mathrm{GW}}(f_1 \otimes \cdots \otimes f_k) = \bar{\mu}(f_1, \dots, f_k) \quad \text{for all } f_1, \dots, f_k \in \mathrm{Conv}(V,\mathbb{R}) \cap C^\infty(V),$$

which will be called the Goodey-Weil distribution of μ. Furthermore, the support of this distribution is contained in the diagonal in V^k.

Note that this implies that the map $\overline{\mathrm{GW}} : \mathrm{VConv}(C;V,F) \to \mathcal{D}'(V^k, \bar{F})$ is injective, and we will call this map the Goodey-Weil embedding for $\mathrm{VConv}(C;V,F)$. We also define a version of the Goodey-Weil distributions for valuations with values in an arbitrary locally convex vector space, that may not admit a continuous norm. In this case, the support of the Goodey-Weil distribution is still contained in the diagonal but in general not compact. Nevertheless, the valuation is uniquely determined by its associated distribution.
As the support of the Goodey-Weil distribution $\overline{\mathrm{GW}}(\mu)$ is contained in the diagonal, we may think of this set as the image of the *support of μ* under the diagonal embedding $\Delta : V \to V^k$. While this definition relies on the Goodey-Weil distributions, it is straightforward to check that $\mu(f) = \mu(h)$ for all functions $f, h \in \mathrm{Conv}(V,\mathbb{R})$ with $f = h$ on a neighborhood of the support of μ. In fact, this can be used to characterize the support without reference to the Goodey-Weil embedding, see Proposition 6.1.3.

The support imposes a number of restrictions on the valuations, in particular, real valued valuations that are invariant with respect to non-compact subgroups of the general linear group $\mathrm{GL}(V)$. For example, Corollary 6.3.6 implies that there exist no non-constant valuations in $\mathrm{VConv}(V)$ that are invariant with respect to translations or $\mathrm{SL}(V)$ (for $\dim V \geq 2$). In addition, there is a very interesting connection between the cone C and the supports of the valuations.

Theorem 1.0.12 (Theorem 6.3.5)**.** *Let $C \subset \mathrm{Conv}(V)$ be a regular cone containing $\mathrm{Conv}(V,\mathbb{R})$. Consider the set $\mathrm{dom}(C) := \bigcap_{f \in C} \overline{\mathrm{dom}\, f}$. Then the following holds:*

1. *The support of any valuation in $\mathrm{VConv}(C;V,F)$ is contained in $\mathrm{dom}(C)$.*
2. *If F admits a continuous norm, then every valuation in $\mathrm{VConv}(V,F)$ with support contained in the interior of $\mathrm{dom}(C)$ extends uniquely to a continuous valuation in $\mathrm{VConv}(C;V,F)$.*

If F admits a continuous norm, we thus have inclusions

$$\mathrm{VConv}_{\mathrm{int}\,\mathrm{dom}(C)}(V,F) \hookrightarrow \mathrm{VConv}(C;V,F) \hookrightarrow \mathrm{VConv}_{\mathrm{dom}(C)}(V,F).$$

While both of these inclusions are strict in general, there are certain cones where the first inclusion becomes a bijection. More precisely, this applies to the regular cone $C_U := \{f \in \mathrm{Conv}(V) : f|_U < \infty\}$ for an open, convex subset $U \subset V$. We also set $\mathrm{Conv}(U,\mathbb{R}) :=$

$\{f : U \to \mathbb{R} : f \text{ convex}\}$, which is a metrizable topological space with respect to the topology of uniform convergence on compact subsets. Let $\mathrm{VConv}(U,F)$ denote the space of all continuous valuations on $\mathrm{Conv}(U,\mathbb{R})$ that are dually epi-translation invariant. As usual, we equip this space with the compact-open topology.

Theorem 1.0.13 (Theorem 6.3.12). *If $U \subset V$ is an open, convex subset and F is a locally convex vector space admitting a continuous norm, then the map*

$$\begin{aligned} \mathrm{res}^* : \mathrm{VConv}(U,F) &\to \mathrm{VConv}(C_U;V,F) \\ \mu &\mapsto [f \mapsto \mu(f|_U)] \end{aligned}$$

is well defined and a topological isomorphism.

We also consider subspaces of $\mathrm{VConv}(V,F)$ of compactly supported valuations. If we fix a compact subset $A \subset V$, the value of a valuation whose support is contained in A only depends on the restriction of its argument to an arbitrary neighborhood of A. We will use this fact, to construct a semi-norm on the space $\mathrm{VConv}_A(V,F)$ of valuations with support in A for every continuous semi-norm on F. These semi-norms turn out to generate the subspace topology in $\mathrm{VConv}_A(V,F)$. In particular, $\mathrm{VConv}_A(V)$ is a Banach space for every compact subset $A \subset V$.

Let us return to the inclusion $T : \mathrm{VConv}(V,F) \to \mathrm{Val}(V^* \times \mathbb{R}, F)$. To describe its image, we have to return to the Goodey-Weil embedding for $\mathrm{Val}(V,F)$. Similar to the support of a dually epi-translation invariant valuation, we can use the support of the Goodey-Weil distributions to define the *vertical support* of $\mu \in \mathrm{Val}(V,F)$, which is a subset of $\mathbb{P}_+(V^*)$. This leads to the following description of the image of T:

Theorem 1.0.14 (Theorem 6.3.2). *Let F be a locally convex vector space that admits a continuous norm. The image of $T : \mathrm{VConv}_k(V,F) \to \mathrm{Val}_k(V^* \times \mathbb{R}, F)$ consists precisely of all valuations in $\mathrm{Val}_k(V^* \times \mathbb{R}, F)$ whose vertical support is contained in the negative half sphere $\mathbb{P}_+(V \times \mathbb{R})_- := \{[(y,s)] \in \mathbb{P}_+(V \times \mathbb{R}) : s < 0\}$. If F is a Fréchet space, $T : \mathrm{VConv}_A(V) \to \mathrm{Val}_{P(A)}(V^* \times \mathbb{R}, F)$ is a topological isomorphism for any compact subset $A \subset V$, where*

$$\begin{aligned} P : V &\to \mathbb{P}_+(V \times \mathbb{R})_- \\ x &\mapsto [(x,-1)], \end{aligned}$$

which is a diffeomorphism onto its image.

The proof uses the characterizing property of the (vertical) support to replace a given convex function by a suitable convex body such that the support function of this body coincides with the original function on a neighborhood of the (vertical) support. The second part is just an application of Banach's inversion theorem. Note that this theorem gives us a very powerful tool to obtain approximation results: If we are interested in a special class of valuations on convex functions, we can try to identify these valuations with some corresponding class in $\mathrm{Val}(V^* \times \mathbb{R})$ using T. If this class of valuations behaves well with respect to the vertical support, i.e. if we can approximate any given valuation in $\mathrm{Val}(V^*\times\mathbb{R})$ by a sequence of valuations in the given class such that the vertical supports of these valuations are eventually contained in an arbitrary neighborhood of the vertical support of the limit, then the theorem gives us a corresponding density result for $\mathrm{VConv}(V)$.

We will apply this method to *smooth valuations*. Recall that a valuation $\mu\in\mathrm{Val}(V)$ is called smooth if the map

$$\begin{aligned}\mathrm{GL}(V)&\to\mathrm{Val}(V)\\ g&\mapsto\pi(g)\mu\end{aligned}$$

is smooth. It is a standard result from representation theory that the space $\mathrm{Val}(V)^{sm}$ of smooth valuations is dense in $\mathrm{Val}(V)$. This is usually proved by considering the convolution of a given valuation with a smooth approximation of the δ-distribution at the identity in $\mathrm{GL}(V)$, and this type of approximations turns out to be compatible with the vertical support.
To describe the corresponding space of valuations in $\mathrm{VConv}(V)$, we will use the differential cycle introduced by Fu in [26]. This cycle associates to any convex function $f\in\mathrm{Conv}(V,\mathbb{R})$ an integral n-current $D(f)$ on the cotangent bundle T^*V, which coincides with the graph of df if the function f is twice differentiable. In addition, it satisfies the valuation property, so by integrating suitable differential forms, we obtain real-valued valuations. To be more precise, let $\Omega^n_{hc}(T^*V)$ denote the space of smooth n-forms on T^*V with *horizontally compact support*, i.e. all forms $\tau\in\Omega^n(T^*V)$ that satisfy $\operatorname{supp}\tau\subset\pi^{-1}(K)$ for some compact subset $K\subset V$. Here $\pi:T^*V\to V$ denotes the natural projection. We will call a valuation $\mu:\mathrm{Conv}(V,\mathbb{R})\to\mathbb{R}$ *smooth* if it is given by

$$\mu(f)=D(f)[\tau]\quad\text{for all } f\in\mathrm{Conv}(V,\mathbb{R})$$

for some $\tau\in\Omega^n_{hc}(T^*V)$. This representing form τ is highly non-unique. To describe the kernel of this procedure, we use a certain second order differential operator $\bar{\mathrm{D}}$, called *symplectic Rumin differential.*

Theorem 1.0.15 (Theorem 7.2.5)**.** *$\tau\in\Omega^n_{hc}(T^*V)$ satisfies $D(f)[\tau]=0$ for all $f\in\mathrm{Conv}(V,\mathbb{R})$ if and only if*

1. *$\bar{\mathrm{D}}\,\tau=0$,*
2. *$\int_V\tau=0$, where we consider the zero section $V\hookrightarrow T^*V$ as a submanifold.*

We will denote the subspace of smooth valuations in $\mathrm{VConv}(V)$ and $\mathrm{VConv}_k(V)$ by $\mathrm{VConv}(V)^{sm}$ and $\mathrm{VConv}_k(V)^{sm}$ respectively. In Section 7.3, we show that this space can indeed be identified with $\mathrm{Val}(V^*\times\mathbb{R})^{sm}\cap\mathrm{Im}T$. Applying the necessary approximation result for smooth valuations in $\mathrm{Val}(V^*\times\mathbb{R})$, this implies

Theorem 1.0.16 (Theorem 7.3.5)**.** $\mathrm{VConv}(V)^{sm}$ *is dense in* $\mathrm{VConv}(V)$*.*

By averaging the differential forms in an approximating sequence with respect to the Haar measure, we obtain the following Corollary.

Corollary 1.0.17 (Corollary 7.3.6). *Let $G \subset \mathrm{GL}(V)$ be a compact subgroup. Then the space of smooth G-invariant valuations is dense in the space $\mathrm{VConv}(V)^G$ of continuous G-invariant valuations in $\mathrm{VConv}(V)$.*

We will apply this result to $\mathrm{SO}(n)$-invariant valuations. For $\mathrm{Val}(\mathbb{R}^n)$, a famous theorem due to Hadwiger asserts that the space of $\mathrm{SO}(n)$-invariant, continuous, translation invariant valuations is spanned by the intrinsic volumes. By considering the relevant invariant differential forms, we obtain the following classification of smooth, rotation invariant valuations in $\mathrm{VConv}(V)$:

Theorem 1.0.18 (Theorem 9.4.4). *For every $\mu \in \mathrm{VConv}_k(\mathbb{R}^n)^{sm} \cap \mathrm{VConv}(\mathbb{R}^n)^{\mathrm{SO}(n)}$ of degree $k > 0$ there exists a unique function $\phi \in C_c^\infty([0,\infty))$ such that*

$$\mu(f) = \int_{\mathbb{R}^n} \phi(|x|^2)[D^2 f(x)]_k dx \quad \text{for } f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n),$$

where $[D^2 f(x)]_k$ denotes the k-th elementary symmetric polynomial in the eigenvalues of the Hessian of f.

Informally speaking, every intrinsic volume gives raise to a family of rotation invariant valuations on convex functions, which can be parametrized by some suitable class of functions. As a Corollary we also obtain that every $\mathrm{SO}(n)$-invariant valuation in $\mathrm{VConv}(\mathbb{R}^n)$ is actually $\mathrm{O}(n)$-invariant.

Remark 1.0.19. *After this thesis was handed in for review, a full classification of all $\mathrm{SO}(n)$-invariant valuations in $\mathrm{VConv}(\mathbb{R}^n)$ was obtained by Colesanti, Ludwig and Mussnig in [19].*
To state their result we need to introduce some notation. Let $C_b((0,\infty))$ denote the space of all continuous functions on $(0,\infty)$ with bounded support. For $1 \le k \le n-1$ set

$$D_k^n := \left\{ \zeta \in C_b((0,\infty)) : \lim_{s\to 0^+} s^{n-k}\zeta(s) = 0, \lim_{s\to 0^+} \int_s^\infty t^{n-k-1}\zeta(t)dt \text{ exists and is finite} \right\}.$$

In addition, let D_n^n denote the space of all $\zeta \in C_b((0,\infty))$ such that $\lim_{s\to 0^+}\zeta(s)$ exists and is finite.

Theorem 1.0.20 (Colesanti-Ludwig-Mussnig [19] Theorems 1.4 and 1.5). *Let $1 \le k \le n$. For every $\mu \in \mathrm{VConv}_k(\mathbb{R}^n)^{\mathrm{SO}(n)}$ there exists a unique function $\zeta \in D_k^n$ such that*

$$\mu(f) = \int_{\mathbb{R}^n} \zeta(|x|)[D^2 f(x)]_k dx \tag{1.1}$$

for all $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n)$ that are strictly convex.
Conversely, for any $\zeta \in D_k^n$ there exists a unique valuation $\mu \in \mathrm{VConv}_k(\mathbb{R}^n)^{\mathrm{SO}(n)}$ that extends the right hand side of Equation (1.1).

At its core, Theorem 1.0.16 relies on the representation of a smooth valuation in $\mathrm{Val}(V)$ by integrating a differential form over the conormal cycle of a convex body and the relation of this cycle to the differential cycle of the corresponding support function. In [3] Alesker gave a second construction of smooth valuations: He constructed a continuous multilinear functional $\mathrm{Dens}(V)\times C^\infty(\mathbb{P}_+(V^*),L)^k\to\mathrm{Val}_{n-k}(V)$, which maps a product of support functions to the corresponding mixed volume. By the L. Schwartz kernel theorem this functional induces a continuous map from $\mathrm{Dens}(V)\otimes C^\infty(\mathbb{P}_+(V^*)^k,L^{\boxtimes k})$ to $\mathrm{Val}_{n-k}(V)$, which, in fact, maps onto the space of smooth valuations. This expresses any smooth valuation as a converging sum of mixed volumes.
To imitate his construction, let us take a slightly more general approach to the valuations considered by Alesker in [7]. Let $\mathcal{M}(V)$ denote the space of signed Radon measures on V equipped with the vague topology, i.e. the topology induced by the semi-norms

$$|\nu|_\phi:=\left|\int_V\phi(x)d\nu(x)\right|\quad\text{for }\nu\in\mathcal{M}(V),$$

where $\phi\in C_c(V)$. If $f\in\mathrm{Conv}(V,\mathbb{R})\cap C^2(V)$ is a strictly convex function, the differential $df:V\to V^*$ is a homeomorphism, so given a density $\mathrm{vol}_{V^*}\in\mathrm{Dens}(V^*)$ on V^* we can consider the pushforward $(df^{-1})_*\mathrm{vol}_{V^*}\in\mathcal{M}(V)$. If we choose a scalar product on V such that vol_{V^*} coincides with the induced Lebesgue measure under the induced isomorphism $V\cong V^*$, this pushforward is given by

$$(df^{-1})_*\mathrm{vol}_{V^*}(U)=\int_U\det(H_f(x))dx\quad\text{for all Borel sets }U\subset V,$$

which extends to a unique valuation $\mathrm{Hess}_n\in\mathrm{VConv}_n(V,\mathcal{M}(V))$ (see [7] or [20] for details). Note that this valuation only depends on the choice of $\mathrm{vol}_{V^*}\in\mathrm{Dens}(V^*)$ and not the specific scalar product. We can consider the polarization of this functional and define the measure valued valuations

$$f\mapsto\mathrm{Hess}_n(f[n-k],f_1,\dots,f_k)$$

for $f_1,\dots,f_k\in\mathrm{Conv}(V,\mathbb{R})$, where Hess_n denotes the polarization, abusing notation. By considering differences of these valuations, we extend this definition to differences of convex functions using the multilinearity of the polarization. We thus obtain the multilinear map

$$\begin{aligned}&C_c(V)\times C^2_c(V)^k\to\mathrm{VConv}_{n-k}(V)\\&(\phi_0,\phi_1,\dots,\phi_k)\mapsto\left[f\mapsto\int_V\phi_0d\,\mathrm{Hess}_n(f[n-k],\phi_1,\dots,\phi_k)\right],\end{aligned}$$

which turns out to be continuous. These valuations are all examples of *mixed Hessian valuations* and are also special cases of the valuations considered by Alesker in [7]. Using the L. Schwartz kernel theorem once again, this functional induces a continuous linear map

$$C^\infty_c(V^{k+1})\to\mathrm{VConv}_{n-k}(V).$$

We compare this construction with the one used by Alesker to show that the image of this map coincides with $\mathrm{VConv}_{n-k}(V)^{sm}$. This uses the key observation that the surface area measure of a convex body $K \in \mathcal{K}(V^* \times \mathbb{R})$ is related to the Hessian measure of $h_K(\cdot, -1)$ by a very simple formula. We thus obtain the following version of McMullen's conjecture:

Theorem 1.0.21 (Theorem 8.3.5)**.** *For every $\mu \in \mathrm{VConv}_{n-k}(V)^{sm}$ and every open neighborhood U of $\operatorname{supp}\mu$, there exist functions $\phi_i^j \in C_c^\infty(U)$ for $0 \le i \le k$, $j \in \mathbb{N}$ such that*

$$\mu(f) = \sum_{j=1}^{\infty} \int_V \phi_0^j d\,\mathrm{Hess}_n(f[n-k], \phi_1^j, \dots, \phi_k^j).$$

In particular, the space generated by smooth mixed Hessian valuations is a dense subspace of $\mathrm{VConv}(V)^{sm}$ and $\mathrm{VConv}(V)$.

1.1 Plan of this thesis

Chapter 2 introduces some basic notation and presents some well known results concerning convex bodies. It also discusses a version of the L. Schwartz kernel theorem.
In Chapter 3, we will present the relevant results from the theory of translation invariant valuations on convex bodies and introduce the notion of vertical support using the Goodey-Weil embedding. We then apply this concept to smooth valuations and the two constructions of these valuations mentioned in the introduction.
Chapter 4 collects some results on convex functions and the topology on some spaces of convex functions. We also use these facts to obtain density results for certain classes of convex functions as well as a characterization of relatively compact subsets of the space of finite-valued convex functions.

Valuations on convex functions are introduced in Chapter 5. We present two embeddings of the space $\mathrm{VConv}(C; V, F)$ into spaces related to valuations on convex bodies and use these embeddings to give an alternative proof of the homogeneous decomposition from [21]. The second part of this chapter is used to define the Goodey-Weil embedding for dually epi-translation invariant valuations and establish its main properties, the diagonality of the support as well as its compactness, depending on the topology of the target space.
Chapter 6 introduces the notion of support for dually epi-translation invariant valuations. We use this concept to construct suitable semi-norms on spaces of compactly supported valuations and discuss various applications. These include the characterization of the image of one of the embeddings from Chapter 5, triviality results for certain spaces of valuations that are invariant with respect to non-compact subgroups of the general linear group, as well as restrictions on the support imposed by the domain of the valuation. These restrictions allow us to reinterpret valuations on certain cones of

convex functions as valuations defined on finite-valued convex functions over open, convex subsets.

Smooth valuations are introduced in Chapter 7. We recall the definition of the differential cycle as well as some of its properties, and we show that it is continuous on the class of finite-valued convex functions with respect to the local flat topology on the space of integral currents. The second part of this chapter establishes a kernel theorem for the differential cycle and characterizes the subspace of smooth dually epi-translation invariant valuations in terms of differential forms. In the last section, we interpret dually epi-translation invariant valuations as certain valuations on convex bodies to show that the space of smooth dually epi-translation invariant valuations on convex functions coincides with a certain space of smooth valuations on convex bodies, where the subspace is characterized by a geometric restriction on the vertical support. This will also imply that smooth valuations on convex functions are dense in $\mathrm{VConv}(V)$.
Chapter 8 examines smooth mixed Hessian valuations. We first show that the surface area measure of a convex body is related to the Hessian measure of the restriction of its support function and discuss how mixed Hessian valuations are related to the valuations considered by Alesker in [7]. We then use the results obtained in previous chapters to relate a construction of smooth valuations on convex functions to a similar construction used in [3] to construct smooth valuations on convex bodies, which gives us a representation of any smooth dually epi-translation invariant valuation as a converging sum of smooth mixed Hessian valuations.
In Chapter 9 we consider smooth dually epi-translation invariant valuations on $\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$ that are invariant under $\mathrm{SO}(n)$ and give a classification of these functionals.

2 Preliminaries

In this chapter we are going to introduce some basic notation and discuss a version of the well known Schwartz kernel theorem, which we will use in Chapter 5 and 8. We also collect some results on convex bodies, in particular the relation to their support functions and the approximation of convex bodies by smooth, strictly convex bodies.

2.1 Basic notation

Let V denote a finite dimensional real vector space of dimension $\dim V = n$. Given a scalar product $\langle \cdot, \cdot \rangle$ on V, we will denote the induced Euclidean norm on V by $|\cdot|$. We equip the algebraic dual V^* of V with the unique scalar product such that the natural isomorphism $V \cong V^*$ induced by the scalar product becomes an isometry. Abusing notation, we will also denote the natural pairing $V^* \times V \to \mathbb{R}$ by $(y, x) \mapsto \langle y, x \rangle$ for $y \in V^*$, $x \in V$. Note that this notation coincides with the notation used for the scalar product if we identify $V^* \cong V$ (and we are actually given a scalar product on V). We will also write $S(V)$ for the unit sphere in V.

Given a subsets A of some topological space X, we will denote the interior of A by $\operatorname{int} A$ and the topological closure by $\bar{A}$. If X is a normed vector space, we will denote the closed ball with radius $R > 0$ centered at x by $B_R(x)$ and the open ball by $U_R(x)$. For $x = 0$ we will also write $B_R := B_R(0)$.

If f, h are two functions with values in the extended real line $(-\infty, +\infty]$ that are defined on the same domain, we denote their pointwise maximum and minimum by either $f \vee h$ and $f \wedge h$, or $\max(f, h)$ and $\min(f, h)$ respectively. The first convention is usually used whenever we want to emphasize the connection of these operations with the intersection and union of sets. The second convention is usually used if we are dealing with explicit expressions.

2.2 Distributions and the L. Schwartz kernel theorem

Recall that a Hausdorff real topological vector space F is called *locally convex* if its topology is generated by the family of its open, convex subsets. Equivalently, the topology is generated by a separating family of semi-norms, where we call a family of semi-norms on F separating if $|v|_F = 0$ for all semi-norms $|\cdot|_F$ of the family implies $v = 0$. Any such space is naturally a uniform space and we will denote its completion by $\bar{F}$, which is again a locally convex vectors space. Its topological dual space will be denoted by F'.

For $U \subset V$ the space $C_c^\infty(U)$ of test functions is equipped with the inductive topology with respect to the inclusions of the Fréchet spaces

$$C_K^\infty(U) := \{\phi \in C_c^\infty(U) : \operatorname{supp} \phi \subset K\},$$

where $K \subset U$ is compact and these spaces are equipped with the topology of uniform convergence of all derivatives. Equivalently, the locally convex topology on $C_c^\infty(U)$ is characterized by the following property: A linear map $T : C_c^\infty(U) \to F$ into a locally convex vector space F is continuous if and only if for every compact subset $K \subset V$ and every continuous semi-norm $|\cdot|_F$ on F, there exists $k \in \mathbb{N}$ and a constant $C = C(k, K)$ such that

$$|T(\phi)|_F \leq C\|\phi\|_{C^k(U)} \quad \text{for all } \phi \in C_K^\infty(U). \tag{2.1}$$

The *local order of* T *on* K (with respect to the semi-norm $|\cdot|_F$) is defined as the smallest number $k \in \mathbb{N}$ such that an estimate of the form above holds for all $\phi \in C_K^\infty(U)$.
Any continuous functional $T : C_c^\infty(U) \to F$ is called a *distribution* and the space of distributions on U with values in F will be denoted by $\mathcal{D}'(U, F)$.

To any distribution $T \in \mathcal{D}'(U, F)$ one associates a closed subset in U, called the support of T. It is defined as the complement of the set of all points $x \in U$ such that there exists a neighborhood $O \subset U$ of x with $T(\phi) = 0$ for all $\phi \in C_c^\infty(O)$. Using a partition of unity, it is easy to see that $T(\phi) = 0$ for all $\phi \in C_c^\infty(U)$ with $\operatorname{supp} \phi \cap \operatorname{supp} T = \emptyset$. If the support of a distribution is compact, then the estimate in Equation 2.1 holds for some constant $C > 0$ independent of the compact set $K \subset U$. The converse is also true, but we will not need this fact. However, any distribution $T \in \mathcal{D}'(U, F)$ with compact support induces a continuous functional on $C^\infty(U)$ equipped with the topology of uniform convergence of all derivatives on compact subsets: Take a smooth cut-off function $\phi \in C_c^\infty(U)$ with $\phi = 1$ on a neighborhood of $\operatorname{supp} T$ and define $T(f) := T(\phi \cdot f)$ for $f \in C^\infty(U)$. Using the Leibniz rule, it is easy to see that this functional is continuous on $C^\infty(U)$. Moreover, the properties of the support of T guaranty that this definition does not depend on the choice of $\phi \in C_c^\infty(U)$.

One of our main constructions uses the well known Schwartz kernel theorem. We will only state the following basic version for the trivial line bundle over a finite dimensional vector space.

Theorem 2.2.1 (L. Schwartz kernel theorem, [29])**.** *Let F be a complete locally convex vector space and let V, W be finite dimensional real vector spaces. For every continuous bilinear map*

$$b : C_c^\infty(V) \times C_c^\infty(W) \to F$$

there exists a unique continuous linear map

$$B : C_c^\infty(V \times W) \to F$$

such that $B(f \otimes h) = b(f, h)$ for all $f \in C_c^\infty(V)$, $h \in C_c^\infty(W)$. Moreover, the local order of B is bounded by a polynomial expression in the local order of b: If $A \subset V \times W$ is compact and $U_1 \subset V$, $U_2 \subset W$ are two relatively compact, open subsets with $A \subset U_1 \times U_2$ such that

$$|b(g, h)|_F \leq C\|g\|_{C^k(U_1)}\|h\|_{C^l(U_2)}$$

for all $g \in C_c^\infty(U_1)$, $h \in C^\infty(U_2)$ for some $C > 0$, $k, l \in \mathbb{N}$, then there exists $\tilde{C} > 0$ such that

$$|B(f)|_F \leq \tilde{C}\|f\|_{C^M(A)}$$

for all $f \in C_c^\infty(V \times W)$ with $\operatorname{supp} f \subset A$ and some $M \leq k + l + \dim V + \dim W + 2$.

At its heart, the Schwartz kernel theorem relies on a decomposition of a given function into a converging sum of products of simpler functions. We will state a version of this decomposition for smooth sections of vector bundles over compact manifolds, as well as a version for compactly supported functions on a finite dimensional real vector space. The first proposition follows from the second using a partition of unity.

Proposition 2.2.2. *Let X_1, X_2 be two compact manifolds, $\mathcal{E}_1$ and $\mathcal{E}_2$ two finite dimensional vector bundles over X_1 and X_2 respectively. For every $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ and $C > 0$ such that the following holds: For every $f \in C^\infty(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2)$ there exist sections $g_j \in C^\infty(X_1, \mathcal{E}_1)$, $h_j \in C^\infty(X_2, \mathcal{E}_2)$ with*

$$\sum_{j=1}^\infty ||g_j||_{C^N(X_1)} \cdot ||h_j||_{C^N(X_2)} \leq C\|f\|_{C^M(X_1 \times X_2)}$$

such that

$$f = \sum_{j=1}^\infty g_j \otimes h_j.$$

Proposition 2.2.3 ([29] Lemma 1)**.** *Let V_1, V_2 be finite dimensional real vector spaces. For every $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that the following holds: For every compact subset $K \subset V_1 \times V_2$ and for all open sets $U_1 \subset V_1$ and $U_2 \subset V_2$ with $K \subset U_1 \times U_2$, there is a constant $C = C(U_1, U_2, K, M)$ such that for all $f \in C_c^\infty(V_1 \times V_2)$ with $\operatorname{supp} f \subset K$ there exist functions $g_j \in C_c^\infty(U_1)$, $h_j \in C_c^\infty(U_2)$ with*

$$\sum_{j=1}^\infty \|g_j\|_{C^N(V_1)} \cdot \|h_j\|_{C^N(V_2)} \leq C\|f\|_{C^M(V_1 \times V_2)}$$

and

$$f = \sum_{j=1}^\infty g_j \otimes h_j,$$

i.e. the sum converges absolutely to f in the C^N-topology.

Note that these propositions imply that a continuous linear map $\Phi : C_c^\infty(V_1 \times V_2) \to F$ into some locally convex vector space F is uniquely determined by its values on functions of the form $g \otimes h \in C_c^\infty(V_1 \times V_2)$ for $g \in C_c^\infty(V_1)$, $h \in C_c^\infty(V_2)$. A similar statement holds in the compact case.

Theorem 2.2.4 ([29] Theorem 2)**.** *Let* $\mathrm{Bil}(C_c^\infty(V), C_c^\infty(W); F)$ *denote the space of all continuous bilinear maps*

$$b : C_c^\infty(V) \times C_c^\infty(W) \to F$$

into some complete locally convex vector space. Then the natural map

$$\mathcal{D}'(V \times W, F) \to \mathrm{Bil}(C_c^\infty(V), C_c^\infty(W); F)$$

is a topological isomorphism if the spaces are equipped with strong topology, i.e. the topology induced the semi-norms

$$\|b\|_{F;K,L} := \sup_{g \in K, h \in L} |b(g,h)|_F \quad \text{on } \mathrm{Bil}(C_c^\infty(V), C_c^\infty(W); F)$$

for $K \subset C_c^\infty(V)$, $L \subset C_c^\infty(W)$ *relatively compact (or equivalently, bounded), and*

$$\|B\|_{F;K} := \sup_{f \in K} |B(f)|_F \quad \text{on } \mathcal{D}'(V \times W, F)$$

for $K \subset C_c^\infty(V \times W)$ *relatively compact.*

2.3 The compact-open topology

Let (X, d) be a metric space. Given a topological space Y, we equip the space $C(X, Y)$ of all continuous functions from X to Y with the *compact-open topology*. A basis for this topology is given by the open sets

$$\mathcal{M}(K, O) := \{f \in C(X, Y) : f(K) \subset O\}$$

for all compact sets $K \subset X$ and open subsets $O \subset Y$. If F is a locally convex vector space, then $C(X, F)$ is a locally convex vector space and the compact-open topology is induced by the family of semi-norms

$$\|f\|_{F;K} := \sup_{x \in K} |f(x)|_F \quad \text{for } f \in C(X, F),$$

where $|\cdot|_F$ is a continuous semi-norm on F and $K \subset X$ is compact. Note that $C(X, F)$ is complete with respect to the compact-open topology if F is complete. Furthermore, the evaluation map

$$\begin{aligned} \mathrm{ev} : X \times C(X, F) &\to F \\ (x, f) &\mapsto f(x) \end{aligned}$$

is continuous.

2.4 Background on convex bodies

Given a finite dimensional real vector space V, we consider the set $\mathcal{K}(V)$ of *convex bodies*, i.e. all convex and compact subsets in V. The space $\mathcal{K}(V)$ carries a standard topology, induced by the *Hausdorff metric*. Given a scalar product on V, this metric is given by

$$d_H(K,L) = \inf\{\epsilon > 0 : K \subset L + \epsilon B_1, L \subset K + \epsilon B_1\}.$$

Here $A + B := \{a + b : a \in A, b \in B\}$ denotes the *Minkowski sum* of two subsets $A, B \subset V$. While the definition of the Hausdorff metric depends on the choice of a scalar product on V, different choices lead to equivalent metrics, so the topology is independent of this choice.
Equipped with this metric, $\mathcal{K}(V)$ becomes a complete, σ-compact, and locally compact metric space. More precisely, we have the following description of its relatively compact subsets.

Theorem 2.4.1 (Blaschke selection theorem,[54] Theorem 1.8.7)**.** *$A \subset \mathcal{K}(V)$ is relatively compact if and only if it is bounded.*

Also note that the general linear group $\mathrm{GL}(V)$ and the affine group $\mathrm{Aff}(V) = \mathrm{GL}(V) \ltimes V$ act continuously on $\mathcal{K}(V)$ by

$$\begin{aligned} &(g,K) \mapsto gK := \{gx : x \in K\} && \text{for } g \in \mathrm{GL}(V), K \in \mathcal{K}(V),\\ &(z,K) \mapsto z + K := \{z + x : x \in K\} && \text{for } z \in V, K \in \mathcal{K}(V). \end{aligned}$$

For our purposes, the most important object associated to a convex body $K \in \mathcal{K}(V)$ is its *support function* $h_K : V^* \to \mathbb{R}$, defined by

$$h_K(y) := \sup_{x\in K}\langle y, x\rangle \quad \text{for } y \in V^*.$$

K is uniquely determined by its support function, and one easily deduces that h_K is a 1-homogeneous and convex function. Conversely, any convex and 1-homogeneous function $h : V^* \to \mathbb{R}$ is the support function of a unique convex body.
Note that the basic properties of the support function imply that it is continuous and uniquely determined by its restriction to the unit sphere in V^*, assuming that we have fixed a scalar product. This allows for the following alternative characterization of the Hausdorff metric:

Lemma 2.4.2 ([54] Lemma 1.8.14)**.** *For $K, L \in \mathcal{K}(V)$*

$$d_H(K,L) = \sup_{y\in S(V^*)} |h_K(y) - h_L(y)|.$$

In other words, the map $K \mapsto h_K$ establishes an isometric embedding $\mathcal{K}(V) \to C(S(V^*))$.
Let $\mathcal{K}(V)^{sm}$ denote the subspace of all strictly convex bodies with smooth boundary. The following results are well known, see for example [54] Section 2.5.

Proposition 2.4.3. *$K \in \mathcal{K}(V)^{sm}$ if and only if $h_K : V^* \setminus \{0\} \to \mathbb{R}$ is smooth and its Hessian is of constant rank $n-1$.*

Note that the last condition implies that the restriction of dh_K to the unit sphere, which we will denote by $d'h_K$, establishes a diffeomorphism from $S(V^*)$ onto its image. To describe its inverse, recall that for such a body K and any point $x \in \partial K$ there exists a unique outer unit normal $\nu_K(x) \in S(V^*)$. The map $\nu_K : \partial K \to S(V^*)$ is called the *Gauss map.*

Lemma 2.4.4. *For $K \in \mathcal{K}(V)^{sm}$, $\nu_K = d'h_K^{-1}$.*

Smooth, strictly convex bodies are an essential tool in the study of convex bodies due to the following approximation result.

Proposition 2.4.5. *$\mathcal{K}(V)^{sm}$ is dense in $\mathcal{K}(V)$.*

Proof. This is well known, but we are going to need an explicit approximation in Chapter 3, so we will give a sketch of proof. Let us identify $V \cong \mathbb{R}^n$. Take a sequence U_j of open neighborhoods of the identity $e \in \mathrm{O}(n)$ such that the diameter of U_j converges to 0 for $j \to \infty$ with respect to some Riemannian metric on $\mathrm{O}(n)$. Now take non-negative functions $\phi_j \in C_c^\infty(U_j, [0,\infty))$ such that $\int_{\mathrm{O}(n)} \phi_j(k)dk = 1$ for all $j \in \mathbb{N}$, where dk denotes a Haar measure on $\mathrm{O}(n)$. Then the sequence $(\phi_j)_j$ is a smooth approximation of the δ-distribution in $e \in \mathrm{O}(n)$. Given a 1-homogeneous continuous function $f : V^* \to \mathbb{R}$, we consider the sequence $(f_j)_j$ of functions defined by

$$f_j(x) := \int_{\mathrm{O}(n)} \phi_j(k) f(k^{-1}x)dk \quad \text{for } x \in V^*.$$

Then $(f_j)_j$ converges uniformly on compact subsets to f. Furthermore, f_j is 1-homogeneous, and if f is convex, then so is f_j (this requires the non-negativity of the functions ϕ_j). We can consider the restriction of these functions to the unit sphere, which is preserved by the operation of $\mathrm{O}(n)$, i.e. we can use the same formula to define this convolution integral on continuous functions on the unit sphere. As $\mathrm{O}(n)$ operates transitively on the unit sphere, the restriction of f_j is smooth if and only if the map $g \mapsto f_j(g^{-1}x)$ is a smooth map on $\mathrm{O}(n)$ for some unit vector x. Using the invariance of the Haar measure with respect to translations, we need to consider the function

$$g \mapsto f_j(g^{-1}x) = \int_{\mathrm{O}(n)} \phi_j(k) f(k^{-1}g^{-1}x)dk = \int_{\mathrm{O}(n)} \phi_j(g^{-1}k) f(k^{-1}x)dk.$$

As ϕ_j is a smooth function, this function is smooth using standard results on integrals depending on a parameter. Thus $f_j : V^* \to \mathbb{R}$ is smooth outside of 0.
Let us take $K \in \mathcal{K}(V)$ and consider $f = h_K$. Then $f_j : V^* \to \mathbb{R}$ is smooth outside of 0, 1-homogeneous, and convex, so $\tilde{f}_j := f_j + \frac{1}{j} h_{B_1(0)}$ has the same properties and in addition the Hessian of $\tilde{f}_j$ has constant rank $n-1$. It is therefore the support function of a convex body K_j, which is smooth and strictly convex by Proposition 2.4.3. Lemma 2.4.2 implies that the sequence $(K_j)_j$ converges to K in the Hausdorff metric. □

Let us conclude this section with an invariant construction of the support function considered as a function on the unit sphere. Let $\mathbb{P}_+(V^*)$ denote the space of oriented lines in V^* through the origin. For a line $l \in \mathbb{P}_+(V^*)$, we denote its positive part with respect to its orientation by $l^+ \subset l \setminus \{0\}$. Consider the line bundle L over $\mathbb{P}_+(V^*)$ with fiber over $l \in \mathbb{P}_+(V^*)$ given by

$$L_l := \{h : l^+ \to \mathbb{R} : h \text{ 1-homogeneous}\}.$$

For any convex body $K \in \mathcal{K}(V)$, we consider its support function as a continuous section of L by defining $h_K(l)$ for $l \in \mathbb{P}_+(V^*)$ by

$$h_K(l)[y] := \sup_{x \in K} \langle y, x \rangle \quad \forall y \in l^+.$$

This construction has the advantage that it is equivariant with respect to the natural operation of $\mathrm{GL}(V)$ on $\mathcal{K}(V)$ and $C(\mathbb{P}_+(V^*), L)$: For $K \in \mathcal{K}(V)$, $l \in \mathbb{P}_+(V^*)$, $g \in \mathrm{GL}(V)$ and $y \in l^+$:

$$h_{gK}(l)[y] = \sup_{x \in gK} \langle y, x \rangle = \sup_{x \in K} \langle y, gx \rangle = \sup_{x \in K} \langle g^{-1}y, x \rangle = h_K(g^{-1}l)[g^{-1}y] =: [(gh_K)(l)]\,[y].$$

Note that any choice of a scalar product induces a trivialization $L \cong S(V^*) \times \mathbb{R}$ by mapping $(l, h) \in L$ to $(y, h(y))$, where $y \in l^+$ is the unique vector of unit length.
In the following chapters, we will consider the support function either as a section of the line bundle L over $\mathbb{P}_+(V^*)$ or as a 1-homogeneous, convex function on V^*, however, we will not distinguish notationally between these two perspectives. We will also need some additional properties of the support function (where we consider it as a function on V^*).

Proposition 2.4.6. *For $K, L \in \mathcal{K}(V)$ and $y \in V^*$ the following holds:*

1. $h_{K+x}(y) = h_K(y) + \langle y, x \rangle$ *for all* $x \in V$.
2. *If* $K \cup L$ *is convex, then* $h_K \vee h_L = h_{K \cup L}$ *and* $h_K \wedge h_L = h_{K \cap L}$.
3. $h_{K+L} = h_K + h_L$.
4. $h_{tK} = th_K$ *for all* $t \geq 0$.

3 Valuations on convex bodies and the vertical support

In this chapter we use the Goodey-Weil embedding to define the *vertical support* of a valuation in Section 3.1, and we discuss how this notion interacts with two different constructions of smooth valuations in Sections 3.2 and 3.3.

Section 3.1 will be published in [35] (except for the approximation result on smooth valuations, which can be found in [34]), while the results from Sections 3.2 and 3.3 can be found in [34].

3.1 Vertical support

As we have seen in the introduction, the Goodey-Weil distributions are uniquely determined by the underlying valuations. In particular, we can directly reconstruct the original valuation on smooth, strictly convex bodies by plugging in the corresponding smooth support functions. This leads to the question which properties of distributions translate in some meaningful way to properties of valuations. For our purposes, we will focus on the support of the Goodey-Weil distributions, which induces a corresponding notion of support for elements of $\mathrm{Val}(V,F)$. We start with the following result due to Alesker.

Proposition 3.1.1 (Alesker [1] Proposition 3.3). *For $\mu \in \mathrm{Val}_k(V,F)$ the support of $\mathrm{GW}(\mu)$ is contained in the diagonal in $\mathbb{P}_+(V^*)^k$.*

Proof. This was originally only proved for real valued valuations. For the general case, take $\lambda \in \bar{F}' \cong F'$. From the defining property of the Goodey-Weil distribution, one easily deduces that $\lambda \circ \mathrm{GW}(\mu) = \mathrm{GW}(\lambda \circ \mu)$ for all $\mu \in \mathrm{Val}_k(V,F)$. In particular, we can apply Alesker's result to the real valued valuation $\lambda \circ \mu$. Thus the restriction of $\lambda \circ \mathrm{GW}(\mu)$ to the complement of the diagonal vanishes. As this is true for all $\lambda \in \bar{F}'$, the same holds for $\mathrm{GW}(\mu)$, as $\bar{F}$ is locally convex. $\square$

Definition 3.1.2. *For $1 \leq k \leq n$ we define the vertical support of $\mu \in \mathrm{Val}_k(V,F)$ to be the set*

$$\text{v-supp}\,\mu := \bigcap_{\substack{\operatorname{supp}\mathrm{GW}(\mu) \subset \Delta A,\\ A \subset \mathbb{P}_+(V^*) \text{ compact}}} A,$$

where $\Delta : \mathbb{P}_+(V^*) \to \mathbb{P}_+(V^*)^k$ *is the diagonal embedding.*
For $k = 0$ *we set* $\text{v-supp}\,\mu = \emptyset$. *If* $\mu = \sum_{i=0}^n \mu_i$ *is the decomposition of* μ *into its homogeneous components, we set* $\text{v-supp}\,\mu := \bigcup_{i=0}^n \text{v-supp}\,\mu_i$.

The vertical support can also be characterized without reference to the Goodey-Weil embedding.

Proposition 3.1.3. *Let* $\mu \in \mathrm{Val}(V,F)$. *The vertical support is minimal (with respect to inclusion) amongst all closed sets* $A \subset \mathbb{P}_+(V^*)$ *with the following property: If* $K, L \in \mathcal{K}(V)$ *are two convex bodies with* $h_K = h_L$ *on a neighborhood of* A, *then* $\mu(K) = \mu(L)$.

Proof. Let us first show that $\text{v-supp}\,\mu$ satisfies this property. Using the homogeneous decomposition, we can assume that μ is k-homogeneous.
Let us identify $V \cong \mathbb{R}^n$ and take a sequence of smooth functions $\phi_j \in C^\infty(O(n))$ as in the proof of Proposition 2.4.5. Mirroring the argument of this proof, we see that the functions

$$(h_K)_j(x) := \int_{\mathrm{O}(n)} \phi_j(g) h_K(g^{-1}x)dg + \frac{1}{j} h_{B_1(0)}(x),$$
$$(h_L)_j(x) := \int_{\mathrm{O}(n)} \phi_j(g) h_L(g^{-1}x)dg + \frac{1}{j} h_{B_1(0)}(x)$$

are support functions of smooth and strictly convex bodies K_j, L_j for all $j \in \mathbb{N}$. Furthermore, $(K_j)_j$ and $(L_j)_j$ converge in the Hausdorff metric to K and L respectively.
As the support of ϕ_j is contained in a neighborhood of the identity of $\mathrm{O}(n)$, where the diameter of these neighborhoods converges to zero, we see that the equation $h_K = h_L$ on a neighborhood of $\text{v-supp}\,\mu$ implies that h_{K_j} and h_{L_j} coincide on a (smaller) neighborhood of $\text{v-supp}\,\mu$ for all $j \in \mathbb{N}$ large enough. Thus $h_{K_j}^{\otimes k} = h_{L_j}^{\otimes k}$ on a neighborhood of the support of $\mathrm{GW}(\mu)$, and we obtain

$$\mu(K) = \lim_{j\to\infty} \mu(K_j) = \lim_{j\to\infty} GW(\mu)\left(h_{K_j}^{\otimes k}\right) = \lim_{j\to\infty} GW(\mu)\left(h_{L_j}^{\otimes k}\right) = \lim_{j\to\infty} \mu(L_j) = \mu(L).$$

For the converse statement, we can again assume that $\mu \in \mathrm{Val}(V)$ is k-homogeneous. Let $A \subset \mathbb{P}_+(V^*)$ be a closed subset with the property stated above. Assume that the claim was false. Then we could find functions $\phi_1, \dots, \phi_k \in C^\infty(\mathbb{P}_+(V^*), L)$ with support contained in $\mathbb{P}_+(V^*) \setminus A$ such that $\mathrm{GW}(\mu)(\phi_1 \otimes \dots \otimes \phi_k) = 1$. Consider the function $h_{B_1} + \sum_{i=1}^k \delta_i \phi_i$ on $\mathbb{P}_+(V^*)$. As B_1 is strictly convex, this is the support function of a convex body K_δ for all $\delta_i > 0$ small enough, and $h_{K_\delta} = h_{B_1}$ on a neighborhood of A, so $\mu(K_\delta) = \mu(B)$ by assumption. Note that $\mu(K_\delta)$ is a polynomial in δ_i for δ_i small enough due to Theorem 1.0.3. The coefficient in front of $\delta_1 \dots \delta_k$ is exactly $k! GW(\mu)(\phi_1 \otimes \dots \otimes \phi_k) = k!$, while the right hand side does not depend on $\delta_i > 0$. Thus the coefficient has to vanish and we obtain a contradiction. ☐

For $A \subset \mathbb{P}_+(V^*)$ let $\mathrm{Val}_A(V,F)$ denote the subspace of all valuations with vertical support contained in A.

Corollary 3.1.4. *Let $A \subset \mathbb{P}_+(V^*)$ be closed. Then* $\mathrm{Val}_A(V,F)$ *is closed in* $\mathrm{Val}(V,F)$.

Proof. Let $(\mu_\alpha)_\alpha$ be a net in $\mathrm{Val}_A(V,F)$ converging to $\mu \in \mathrm{Val}(V,F)$. Given $K, L \in \mathcal{K}(V)$ with $h_K = h_L$ on a neighborhood of A, Proposition 3.1.3 implies $\mu_\alpha(K) = \mu_\alpha(L)$ for all α. Thus

$$\mu(K) = \lim_\alpha \mu_\alpha(K) = \lim_\alpha \mu_\alpha(L) = \mu(L).$$

As this holds for all $K, L \in \mathcal{K}(V)$ with $h_K = h_L$ on a neighborhood of A, we can apply Proposition 3.1.3 again to obtain $\text{v-supp}\,\mu \subset A$, i.e. $\mu \in \mathrm{Val}_A(V,F)$. □

We conclude this section with an approximation result on smooth real-valued valuations. The proof relies on the following lemma on the compatibility of the vertical support and the natural operation of $\mathrm{GL}(V)$ on both $\mathbb{P}_+(V^*)$ and $\mathrm{Val}(V)$. Recall that the operation of $g \in \mathrm{GL}(V)$ on a section f of L is given by $[(gf)(l)][y] = [f(g^{-1}l)][y \circ g]$ for $y \in l^+$.

Lemma 3.1.5. *For $g \in \mathrm{GL}(V)$, $\mu \in \mathrm{Val}(V)$:* $\text{v-supp}(\pi(g)\mu) = g(\text{v-supp}\,\mu)$.

Proof. Let $K, L \in \mathcal{K}(V)$ be two convex bodies with $h_K = h_L$ on a neighborhood U of $g(\text{v-supp}\,\mu)$. For $l \in g^{-1}(U)$ and $y \in l^+$

$$\begin{aligned}[h_{g^{-1}K}(l)]\,[y] &= \sup_{x \in g^{-1}K} \langle y, x\rangle = \sup_{x\in K}\langle y \circ g^{-1}, x\rangle = [h_K(gl)]\left[y \circ g^{-1}\right] \\ &= [h_L(gl)]\,[y \circ g^{-1}] = [h_{g^{-1}L}(l)][y],\end{aligned}$$

i.e. $h_{g^{-1}K} = h_{g^{-1}L}$ on the neighborhood $g^{-1}(U)$ of $\text{v-supp}\,\mu$. By Proposition 3.1.3,

$$[\pi(g)\mu]\,(K) = \mu(g^{-1}K) = \mu\left(g^{-1}L\right) = [\pi(g)\mu]\,(L).$$

As this is true for all $K, L \in \mathcal{K}(V)$ with $h_K = h_L$ on a neighborhood of $g(\text{v-supp}\,\mu)$, Proposition 3.1.3 implies $\text{v-supp}(\pi(g)\mu) \subset g(\text{v-supp}\,\mu)$ for $g \in \mathrm{GL}(V)$. Thus the converse inclusion follows from

$$g\,(\text{v-supp}\,\mu) = g\left(\text{v-supp}\left[\pi\left(g^{-1}\right)(\pi(g)\mu)\right]\right) \subset g\left(g^{-1}\left(\text{v-supp}\,(\pi(g)\mu)\right)\right) = \text{v-supp}(\pi(g)\mu).$$

□

Proposition 3.1.6. *Let $A \subset \mathbb{P}_+(V^*)$ be compact, $B \subset \mathbb{P}_+(V^*)$ a compact neighborhood of A. Then the following holds: For every $\mu \in \mathrm{Val}_A(V)$ there exists a sequence in* $\mathrm{Val}_B(V) \cap \mathrm{Val}(V)^{sm}$ *converging to μ.*

Proof. Take a sequence of relatively compact, open neighborhoods $(U_j)_j$ of the identity in $\mathrm{GL}(V)$ such that their diameter converges to zero with respect to some Riemannian metric on $\mathrm{GL}(V)$. Given $\mu \in \mathrm{Val}_A(V)$ and $g \in U_j$, Lemma 3.1.5 shows that $\text{v-supp}(\pi(g)\mu) \subset U_j \cdot A$. As B is a neighborhood of A, the fact that the diameter of the neighborhoods $(U_j)_j$ converges to zero implies that there exists $N \in \mathbb{N}$ such that $U_j \cdot A \subset B$ for all $j \geq N$. In particular, $\pi(g)\mu \in \mathrm{Val}_B(V)$ for $g \in U_j$ and $j \geq N$ by

Lemma 3.1.5.
Now take $\phi_j \in C_c^\infty(U_j)$ with $\int_{\mathrm{GL}(V)} \phi_j(g)dg = 1$, where have have equipped $\mathrm{GL}(V)$ with some left invariant Haar measure, and consider the valuations

$$\mu_j := \int_{\mathrm{GL}(V)} \phi_j(g) \cdot \pi(g)\mu \, dg.$$

For $j \geq N$, $\phi_j(g) \cdot \pi(g)\mu \in \mathrm{Val}_B(V)$ for all $g \in \mathrm{GL}(V)$ by construction. As this is a closed subspace, we deduce $\mu_j \in \mathrm{Val}_B(V)$ for all $j \geq N$.
For $h \in \mathrm{GL}(V)$ and $j \geq N$,

$$\pi(h)\mu_j = \int_{\mathrm{GL}(V)} \phi_j(g) \cdot \pi(hg)\mu \, dg = \int_{\mathrm{GL}(V)} \phi_j(h^{-1}g) \cdot \pi(g)\mu \, dg$$

is just the convolution of the $\mathrm{Val}(V)$-valued continuous function $g \mapsto \pi(g)\mu$ on $\mathrm{GL}(V)$ and the smooth function $\phi_j \in C_c^\infty(\mathrm{GL}(V))$. In particular, $h \mapsto \pi(h)\mu_j$ depends smoothly on h and thus μ_j is a smooth valuation, i.e. $\mu_j \in \mathrm{Val}_B(V) \cap \mathrm{Val}(V)^{sm}$ for $j \geq N$. Obviously, $(\mu_j)_j$ converges to μ in $\mathrm{Val}(V)$. The claim follows. □

3.2 Construction of smooth valuations using mixed volumes

In [3], Alesker considered a version of the following functional on convex bodies.

$$\begin{aligned} \mathrm{Dens}(V) \times \mathcal{K}(V)^k &\to \mathrm{Val}_{n-k}(V) \\ (\mathrm{vol}, L_1, \dots, L_k) &\mapsto \left[K \mapsto \frac{1}{k!} \frac{\partial}{\partial \lambda_1}\Big|_0 \dots \frac{\partial}{\partial \lambda_k}\Big|_0 \mathrm{vol}\left(K + \sum_{i=1}^k \lambda_i L_i \right) \right]. \end{aligned}$$

It is easy to see that this functional is additive in each component. By representing a section $\phi \in C^\infty(\mathbb{P}_+(V^*), L)$ as a difference of support functions as in the construction of the Goodey-Weil embedding, this functional can be extended to a continuous multilinear functional

$$\tilde{\Theta}_k : \mathrm{Dens}(V) \times C^\infty(\mathbb{P}_+(V^*), L)^k \to \mathrm{Val}_{n-k}(V).$$

By the Schwartz kernel theorem 2.2.1, this induces a continuous linear functional

$$\Theta_k : \mathrm{Dens}(V) \otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}) \to \mathrm{Val}_{n-k}(V).$$

In fact even more is true:

Theorem 3.2.1 (Alesker [3] Corollary 1.9)**.** $\Theta_k : \mathrm{Dens}(V) \otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}) \to \mathrm{Val}_{n-k}(V)^{sm}$ *is an epimorphism of Fréchet spaces.*

Note that the continuity of $\Theta_k : \mathrm{Dens}(V) \otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}) \to \mathrm{Val}_{n-k}(V)$ implies that there exists $N \in \mathbb{N}$ and $C > 0$ such that

$$\|\Theta_k(f)\| = \sup_{K \subset B_1} |\Theta_k(f)(K)| \leq C \|f\|_{\mathrm{Dens}(V)\otimes C^N(\mathbb{P}_+(V^*)^k, L^{\boxtimes k})}.$$

In particular, Θ_k extends to a continuous functional $\mathrm{Dens}(V) \otimes C^N(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}) \to \mathrm{Val}_{n-k}(V)$.

Proposition 3.2.2. *For $f \in \mathrm{Dens}(V) \otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\boxtimes k})$, $h \in C^\infty(\mathbb{P}_+(V^*)^{n-k}, L^{\boxtimes n-k})$:*

$$\binom{n}{k} \Theta_n(f \otimes h) = \mathrm{GW}(\Theta_k(f))[h] \cdot \chi,$$

where $\chi \in \mathrm{Val}_0(V)$ is the Euler characteristic.

Proof. As the Goodey-Weil embedding is continuous with respect to the strong topology, both sides define jointly continuous, multilinear maps

$$\mathrm{Dens}(V) \otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}) \times C^\infty(\mathbb{P}_+(V^*)^{n-k}, L^{\boxtimes n-k}) \to \mathrm{Val}_0(V) \cong \mathbb{R},$$

where the last isomorphism is given by evaluating a valuation in $\{0\} \in \mathcal{K}(V)$. We thus only need to consider the case

$$f = \mathrm{vol} \otimes h_{K_1} \otimes \cdots \otimes h_{K_k}, \qquad h = h_{K_{k+1}} \otimes \cdots \otimes h_{K_n}$$

for $K_1, \ldots, K_n \in \mathcal{K}(V)$ smooth and strictly convex, $\mathrm{vol} \in \mathrm{Dens}(V)$. Evaluating both sides in $\{0\}$, we obtain

$$\begin{aligned}
&\mathrm{GW}\left(\Theta_k\left(\mathrm{vol} \otimes h_{K_1} \otimes \cdots \otimes h_{K_k}\right)\right)\left[h_{K_{k+1}} \otimes \cdots \otimes h_{K_n}\right] \\
=& \frac{1}{(n-k)!} \frac{\partial}{\partial \lambda_{k+1}}\Big|_0 \cdots \frac{\partial}{\partial \lambda_n}\Big|_0 \Theta_k\left(\mathrm{vol} \otimes h_{K_1} \otimes \cdots \otimes h_{K_k}\right)\left(\{0\} + \sum_{i=k+1}^n \lambda_i K_i\right) \\
=& \frac{1}{(n-k)!} \frac{\partial}{\partial \lambda_{k+1}}\Big|_0 \cdots \frac{\partial}{\partial \lambda_n}\Big|_0 \frac{1}{k!} \frac{\partial}{\partial \lambda_1}\Big|_0 \cdots \frac{\partial}{\partial \lambda_k}\Big|_0 \mathrm{vol}\left(\{0\} + \sum_{i=1}^n \lambda_i K_i\right) \\
=& \frac{n!}{k!(n-k)!} \frac{1}{n!} \frac{\partial}{\partial \lambda_1}\Big|_0 \cdots \frac{\partial}{\partial \lambda_n}\Big|_0 \mathrm{vol}\left(\{0\} + \sum_{i=1}^n \lambda_i K_i\right) \\
=& \binom{n}{k} \Theta_n\left(\mathrm{vol} \otimes h_{K_1} \otimes \cdots \otimes h_{K_n}\right)(\{0\}).
\end{aligned}$$

□

For a closed set $A \subset \mathbb{P}_+(V^*)$, let $C^\infty_A(\mathbb{P}_+(V^*)^k, L^{\boxtimes k})$ denote the space of all smooth sections of L with support contained in A^k. We will prove the following refinement of Theorem 3.2.1:

Proposition 3.2.3. *Let $A \subset \mathbb{P}_+(V^*)$ be a closed subset, $B \subset \mathbb{P}_+(V^*)$ a compact neighborhood of A. Then the following holds: For every $\mu \in \mathrm{Val}_A(V) \cap \mathrm{Val}_{n-k}(V)^{sm}$ there exists a function $f \in \mathrm{Dens}(V) \otimes C^\infty_B(\mathbb{P}_+(V^*)^k, L^{\boxtimes k})$ such that $\Theta_k(f) = \mu$.*

Proof. Let $\phi \in C^\infty(\mathbb{P}_+(V^*))$ be a function with $\phi = 1$ on a neighborhood of A and $\operatorname{supp}\phi \subset B$. By Theorem 3.2.1, we can find $f \in \mathrm{Dens}(V) \otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\boxtimes k})$ such that $\Theta_k(f) = \mu$. We claim that $\tilde{f} := \phi^{\otimes k} \cdot f$ satisfies $\Theta_k(\tilde{f}) = \mu$.
By Theorem 1.0.3, the order of $\mathrm{GW}(\mu)$ is uniformly bounded by some $M \in \mathbb{N}$. We can thus extend $\mathrm{GW}(\mu)$ to a continuous linear functional on $C^M(\mathbb{P}_+(V^*)^l, L^{\boxtimes l})$ for all $\mu \in \mathrm{Val}_l(V)$ and all $0 \le l \le n$. Following the remark to Theorem 3.2.1, we extend Θ_l to a continuous linear functional $\Theta_l : \mathrm{Dens}(V) \otimes C^M(\mathbb{P}_+(V^*)^l, L^{\boxtimes l}) \to \mathrm{Val}_{n-l}(V)$ for all $0 \le l \le n$ (increasing M if necessary). Using Proposition 2.2.2, we can write f as a converging sum (with respect to the C^M-topology)

$$f = \mathrm{vol} \otimes \sum_{j=1}^\infty \phi_1^j \otimes \cdots \otimes \phi_k^j,$$

with $\mathrm{vol} \in \mathrm{Dens}(V)$, $\phi_i^j \in C^\infty(\mathbb{P}_+(V^*), L)$. Evaluating the Goodey-Weil distribution of $\Theta_k(f)$ in $\psi_{k+1}, \dots, \psi_n \in C^\infty(\mathbb{P}_+(V^*), L)$ and using Proposition 3.2.2, as well as the continuity of $\mathrm{GW}(\mu)$ and Θ_k with respect to the C^M-topology, we see that

$$\begin{aligned}
&\mathrm{GW}\left(\Theta_k(f)\right)\left[\psi_{k+1} \otimes \cdots \otimes \psi_n\right] \\
=&\binom{n}{k} \Theta_n\left(f \otimes \psi_{k+1} \otimes \cdots \otimes \psi_n\right) \cdot \chi(\{0\}) \\
=&\binom{n}{k} \sum_{j=1}^\infty \Theta_n\left(\mathrm{vol} \otimes \phi_1^j \otimes \cdots \otimes \phi_k^j \otimes \psi_{k+1} \otimes \cdots \otimes \psi_n\right)(\{0\}) \\
=&\binom{n}{k} \sum_{j=1}^\infty \mathrm{GW}(\mathrm{vol})\left[\phi_1^j \otimes \cdots \otimes \phi_k^j \otimes \psi_{k+1} \otimes \cdots \otimes \psi_n\right].
\end{aligned}$$

By definition, the support of $\mathrm{GW}(\Theta_k(f))$ is equal to $\Delta(\text{v-supp}\,\mu)$, so

$$\psi_{k+1} \otimes \cdots \otimes \psi_n = (\phi \cdot \psi_{k+1}) \otimes \cdots \otimes (\phi \cdot \psi_n)$$

on a neighborhood of the support of $\mathrm{GW}(\Theta_k(f))$. Using the same argument as before, we obtain

$$\begin{aligned}
&\mathrm{GW}\left(\Theta_k(f)\right)\left[\psi_{k+1} \otimes \cdots \otimes \psi_n\right] = \mathrm{GW}\left(\Theta_k(f)\right)\left[(\phi \cdot \psi_{k+1}) \otimes \cdots \otimes (\phi \cdot \psi_n)\right] \\
=&\binom{n}{k} \sum_{j=1}^\infty \mathrm{GW}(\mathrm{vol})\left[\phi_1^j \otimes \cdots \otimes \phi_k^j \otimes (\phi \cdot \psi_{k+1}) \otimes \cdots \otimes (\phi \cdot \psi_n)\right].
\end{aligned}$$

Now note that

$$\phi_1^j \otimes \cdots \otimes \phi_k^j \otimes (\phi \cdot \psi_{k+1}) \otimes \cdots \otimes (\phi \cdot \psi_n) = \left(\phi \cdot \phi_1^j\right) \otimes \cdots \otimes \left(\phi \cdot \phi_k^j\right) \otimes \psi_{k+1} \otimes \cdots \otimes \psi_n$$

on a neighborhood of the diagonal in $\mathbb{P}_+(V^*)^n$. Using again that the support of GW(vol) is contained in the diagonal, as well as the continuity of GW(vol) with respect to the C^M-topology, we arrive at

$$\begin{aligned}
&\mathrm{GW}(\Theta_k(f))[\psi_{k+1}\otimes\cdots\otimes\psi_n] \\
=&\binom{n}{k}\sum_{j=1}^{\infty}\mathrm{GW}(\mathrm{vol})[(\phi\cdot\phi_1^j)\otimes\cdots\otimes(\phi\cdot\phi_k^j)\otimes\psi_{k+1}\otimes\cdots\otimes\psi_n] \\
=&\binom{n}{k}\mathrm{GW}(\mathrm{vol})\left[\sum_{j=1}^{\infty}\left(\phi\cdot\phi_1^j\right)\otimes\cdots\otimes\left(\phi\cdot\phi_k^j\right)\otimes\psi_{k+1}\otimes\cdots\otimes\psi_n\right] \\
=&\binom{n}{k}\Theta_n\left[\tilde{f}\otimes\psi_{k+1}\otimes\cdots\otimes\psi_n\right]\{0\} \\
=&\mathrm{GW}\left(\Theta_k(\tilde{f})\right)[\psi_{k+1}\otimes\cdots\otimes\psi_n],
\end{aligned}$$

where we have applied Proposition 3.2.2 in the last two steps. Thus $\mathrm{GW}(\Theta_k(f)) = \mathrm{GW}(\Theta_k(\tilde{f}))$, so the injectivity of the Goodey-Weil embedding implies $\Theta_k(\tilde{f}) = \Theta_k(f) = \mu$. Obviously, $\tilde{f}$ has the desired property. □

3.3 Construction of smooth valuations using the conormal cycle

Let V be an oriented vector space. For a convex body $K \in \mathcal{K}(V)$, the set

$$\mathrm{N}^*(K) := \{(x,[v]) \in V \times \mathbb{P}_+(V^*) : v \text{ outer normal to } K \text{ in } x \in \partial K\}$$

is a Lipschitz submanifold of the co-sphere bundle $V \times \mathbb{P}_+(V^*)$ of dimension $n-1$, which carries a natural orientation induced by the orientation of V, and can thus be considered as an integral current, called the *conormal cycle* of K. We refer to [8] for further details (see also [28] for results on the conormal cycle for arbitrary compact, as well as subanalytic sets).
For smooth and strictly convex bodies the conormal cycle admits a very simple description. Recall that the support function $h_K : V^* \setminus \{0\} \to \mathbb{R}$ of any such convex body $K \in \mathcal{K}(V)$ is a smooth function. As it is 1-homogeneous, its differential $dh_K : V^* \setminus \{0\} \to (V^*)^* \cong V$ is 0-homogeneous and can thus be considered as a map $d'h_K : \mathbb{P}_+(V^*) \to V$.

Lemma 3.3.1. *If $K \in \mathcal{K}(V)$ is smooth and strictly convex, then*

$$\mathrm{N}^*(K) = (d'h_K \times Id)_* [\mathbb{P}_+(V^*)].$$

Proof. This follows directly from Lemma 2.4.4, as the conormal cycle is given by integration over the (projectivized) conormal bundle of K, which can be parametrized by the Gauss map. □

Let $I_{n-1}(V \times \mathbb{P}_+(V^*))$ denote the space of integral currents of dimension $n-1$ in $V \times \mathbb{P}_+(V^*)$.

Proposition 3.3.2 (Alesker-Fu [8] Proposition 2.1.12)**.** *The map* $\mathrm{N}^* : \mathcal{K}(V) \to I_{n-1}(V \times \mathbb{P}_+(V^*))$ *is continuous, where* $I_{n-1}(V \times \mathbb{P}_+(V^*))$ *is equipped with the local flat metric topology. Furthermore,* N^* *is a valuation: If* $K, L \in \mathcal{K}(V)$ *are convex bodies such that* $K \cup L \in \mathcal{K}(V)$*, then*

$$\mathrm{N}^*(K) + \mathrm{N}^*(L) = \mathrm{N}^*(K \cup L) + \mathrm{N}^*(K \cap L).$$

Let $\Omega^{n-1}(V \times \mathbb{P}_+(V^*))^{tr}$ denote the space of all translation invariant differential $(n-1)$-forms on $V \times \mathbb{P}_+(V^*)$. From the previous proposition, one easily deduces that we obtain a continuous, translation invariant valuation for every translation invariant differential form $\tau \in \Omega^{n-1}(V \times \mathbb{P}_+(V^*))$ by considering the map

$$\begin{aligned} \mathcal{K}(V) &\to \mathbb{R} \\ K &\mapsto \mathrm{N}^*(K)[\tau]. \end{aligned}$$

Set $\Omega^{k,n-k-1}(V \times \mathbb{P}_+(V^*))^{tr} := \Lambda^k V \otimes \Omega^{n-k-1}(\mathbb{P}_+(V^*))$. Then the space of translation invariant differential $n-1$-forms on $V \times \mathbb{P}_+(V^*)$ decomposes as

$$\Omega^{n-1}(V \times \mathbb{P}_+(V^*))^{tr} = \bigoplus_{k=0}^{n-1} \Omega^{k,n-k-1}(V \times \mathbb{P}_+(V^*))^{tr}.$$

Theorem 3.3.3 (Alesker [5] Theorem 5.2.1)**.** *The map*

$$\begin{aligned} \Omega^{k,n-k-1}(V \times \mathbb{P}_+(V^*))^{tr} &\to \mathrm{Val}_k(V)^{sm} \\ \tau &\mapsto \big(K \mapsto \mathrm{N}^*(K)[\tau]\big) \end{aligned}$$

is surjective for $0 \leq k \leq n-1$*.*

The kernel of this map was described in [9]. It uses a certain second order differential operator $D : \Omega^{n-1}(V \times \mathbb{P}_+(V^*)) \to \Omega^n(V \times \mathbb{P}_+(V^*))$ defined on the contact manifold $V \times \mathbb{P}_+(V^*)$, which is called the *Rumin differential* (see [53]).

Theorem 3.3.4 (Bernig-Bröcker [9] Theorem 2.2)**.** $\tau \in \Omega^{k,n-k-1}(V \times \mathbb{P}_+(V^*))^{tr}$ *induces the trivial valuation if and only if*

1. $D\tau = 0$,
2. $\pi_*\tau = 0$.

Here $\pi : V \times \mathbb{P}_+(V^*) \to V$ *is the natural projection and* $\pi_* : \Omega^{n-1}(V \times \mathbb{P}_+(V^*))^{tr} \to C^\infty(V)$ *denotes the fiber integration.*

Note that the second condition is always satisfied for $1 \leq k \leq n-1$, while the first is always true for $k = 0$.
As we will need it in the following chapters, let us discuss the Rumin differential in more detail. On $V \times \mathbb{P}_+(V^*)$, there exists a canonical distribution of hyperplanes $H \subset T(V \times \mathbb{P}_+(V^*))$, given by

$$H_{(x,[v])} = \ker(v \circ d\pi|_{(x,[v])}) \quad \text{for } (x,[v]) \in V \times \mathbb{P}_+(V^*).$$

Given a scalar product on V, we can identify $\mathbb{P}_+(V^*) \cong S(V^*)$. Then the hyperplane distribution is given by the kernel of the nowhere vanishing 1-form $\alpha \in \Omega^1(V \times S(V^*))$,

$$\alpha|_{(x,v)} := \langle v, d\pi \cdot \rangle \quad \text{for } (x,v) \in V \times S(V^*).$$

The restriction of $d\alpha$ to each hyperplane is non-degenerate, so the distribution H is a *contact distribution.* We will identify α with the corresponding 1-form on $V \times \mathbb{P}_+(V^*)$. Due to the non-degeneracy of $d\alpha$ on the contact hyperplanes, one can introduce a unique vector field R on $V \times \mathbb{P}_+(V^*)$ such that $i_R\alpha = 1$, $i_R d\alpha = 0$, which is called the *Reeb vector field.* Let us call a form $\tau \in \Omega^k(V \times \mathbb{P}_+(V^*))$ *vertical* if its restriction to the contact distribution vanishes. It is easy to see that this is equivalent to $\alpha \wedge \tau = 0$. Plugging in R, this yields $\tau = \alpha \wedge i_R\tau$ for vertical differential forms.
One can show that for any $\tau \in \Omega^{n-1}(V \times \mathbb{P}_+(V^*))$ there exists a unique vertical form ξ such that $d(\tau + \xi)$ is vertical. In this case, the Rumin differential of τ is defined as $D\tau := d(\tau + \xi)$.

For the next proposition, we need the following notion: A *smooth area measure* is any map of the form

$$\begin{aligned} \mathcal{K}(V) \times \mathcal{B}(\mathbb{P}_+(V^*)) &\to \mathbb{R} \\ (K,U) &\mapsto \left(N(K) \llcorner \pi_2^{-1}(U)\right)[\tau], \end{aligned}$$

where $\mathcal{B}(\mathbb{P}_+(V^*))$ denotes the family of Borel sets in $\mathbb{P}_+(V^*)$, $\tau \in \Omega^{k,n-k-1}(V \times \mathbb{P}_+(V^*))^{tr}$ is a translation invariant differential form, and $\pi_2 : V \times \mathbb{P}_+(V^*) \to \mathbb{P}_+(V^*)$ is the projection on the second factor. In other words, a smooth area measure associates to every convex body a signed measure on the unit sphere. It is known that a differential form τ induces the trivial area measure if and only if it is contained in the ideal generated by α and $d\alpha$. We refer to [60] for further details on smooth area measures.

Proposition 3.3.5. *For $1 \leq k \leq n-1$ let $\tau \in \Omega^{k,n-k-1}(V \times \mathbb{P}_+(V^*))^{tr}$ represent a smooth valuation $\mu \in \mathrm{Val}_k(V)^{sm}$. Then* $\text{v-supp}\,\mu = \pi_2(\operatorname{supp} D\tau)$.

Proof. We will assume that we are given a scalar product on V, so we can work with the induced structures. This also identifies $C^\infty(\mathbb{P}_+(V^*), L) \cong C^\infty(\mathbb{P}_+(V^*))$.
Let us start by showing

$$\begin{aligned} \mathrm{GW}(\mu)(\phi_1 \otimes \cdots \otimes \phi_k) &= \frac{1}{k!} \mathrm{N}^*(K)\left[(\phi_1 i_R D) \dots (\phi_k i_R D)\tau\right] \\ &= \frac{1}{k!} \int_{\{0\} \times \mathbb{P}_+(V^*)} (\phi_1 i_R D) \dots (\phi_k i_R D)\tau \end{aligned}$$

for any $K \in \mathcal{K}(V)$, $\phi_1, \dots, \phi_k \in C^\infty(\mathbb{P}_+(V^*))$. Wannerer observed in [60] Proposition 2.2 that the following holds for every smooth strictly convex body L:

$$\frac{d}{dt}\Big|_0 \mu(K + tL) = \mathrm{N}^*(K)[h_L i_R D\tau].$$

Iterating this formula for smooth convex bodies $L_1, \dots, L_k$, we obtain

$$\begin{aligned}\mathrm{GW}(\mu)(h_{L_1} \otimes \dots \otimes h_{L_k}) =& \frac{1}{k!}\frac{\partial}{\partial\lambda_1}\Big|_0 \dots \frac{\partial}{\partial\lambda_k}\Big|_0 \mathrm{N}^*\left(K + \sum_{i=1}^k \lambda_i L_i\right)[\tau]\\ =& \frac{1}{k!}\,\mathrm{N}^*(K)\left[(h_{L_1} i_R D)\dots(h_{L_k} i_R D\tau)\right].\end{aligned}$$

As $\mathrm{GW}(\mu)$ is uniquely determined by its values on functions of the form $h_{L_1} \otimes \dots \otimes h_{L_k}$, we see that

$$\mathrm{GW}(\mu)(\phi_1 \otimes \dots \otimes \phi_k) = \frac{1}{k!}\,\mathrm{N}^*(K)\left[(\phi_1 i_R D)\dots(\phi_k i_R D)\tau\right]$$

for any $K \in \mathcal{K}(V)$. Choosing $K = \{0\}$, we obtain the desired formula.

Let us show that $\text{v-supp}\,\mu \subset \pi_2(D\tau)$: Assume that one of the functions ϕ_i in the equation above satisfies $\operatorname{supp}\phi_i \cap \pi_2(D\tau) = \emptyset$. As the Goodey-Weil distribution of a valuation is symmetric, we can assume $i = k$, so $\phi_k i_R D\tau = 0$. Thus the formula above implies $\mathrm{GW}(\mu)(\phi_1 \otimes \dots \otimes \phi_k) = 0$, and we see that $\operatorname{supp}\mathrm{GW}(\mu) \subset \Delta(\pi_2(\operatorname{supp} D\tau))$, i.e. $\text{v-supp}\,\mu \subset \pi_2(\operatorname{supp} D\tau)$.
For the converse inclusion, let $v \in \pi_2(\operatorname{supp} D\tau)$ be an arbitrary point, U an arbitrary neighborhood of $v \in \mathbb{P}_+(V^*)$. Then $D\tau$ does not vanish identically on $V \times U$. We will construct functions $\phi_1, \dots, \phi_k$ with support in U such that $\mathrm{GW}(\mu)(\phi_1 \otimes \dots \otimes \phi_k) \neq 0$.
Let us start with ϕ_1: Consider the area measure induced by $i_R D\tau$. Let us show that the restriction of $i_R D\tau$ to the contact distribution H is not contained in the ideal generated by α and $d\alpha$ on $V \times U$. Using the Lefschetz decomposition (see Proposition 7.2.1), this is equivalent to $i_R D\tau$ being primitive, i.e. to $d\alpha \wedge i_R D\tau|_H = 0$ on the contact plane. However, $D\tau = \alpha \wedge i_R D\tau$ is closed, so

$$0 = d(\alpha \wedge i_R D\tau) = d\alpha \wedge i_R D\tau - \alpha \wedge d i_R D\tau.$$

Restricting the equation to the contact distribution, i.e. the kernel of α, we obtain the desired result. We can thus find $\phi_1 \in C_c^\infty(U)$ such that $\mathrm{N}^*(K)[\phi_1 i_R D\tau] \neq 0$ for some $K \in \mathcal{K}(V)$. In particular, the valuation induced by the differential form $\tau_1 := \phi_1 \wedge i_R D\tau$ is non-trivial. By construction, this differential form is of bidegree $(k-1, n-k)$, i.e. it defines a $(k-1)$-homogeneous valuation. Thus $D\tau_1 \neq 0$ if $k \neq 1$ by Theorem 3.3.4. Obviously, $\pi_2(\operatorname{supp} D\tau_1) \subset U$.
Repeating this construction, we obtain functions $\phi_1, \dots, \phi_k$ with the properties

1. $\operatorname{supp}\phi_i \subset U$,

2. $\tau_{i+1} := \phi_{i+1} i_R D \tau_i$ defines a non-trivial valuation of degree $k-i-1$.

In particular, the map

$$K \mapsto \mathrm{N}^*(K)[\tau_k] = \mathrm{N}^*(K)\left[(\phi_k i_R D) \dots (\phi_1 i_R D)\tau\right]$$

defines a 0-homogeneous, non-trivial valuation, i.e. it is a constant multiple of the Euler characteristic. Using the expression of the Goodey-Weil distribution derived above, we obtain

$$\mathrm{GW}(\mu)(\phi_1 \otimes \dots \otimes \phi_k) = \frac{1}{k!}\mathrm{N}^*(K)\left[(\phi_k i_R D) \dots (\phi_1 i_R D)\tau\right] \neq 0$$

for any $K \in \mathcal{K}(V)$. As this is true for any neighborhood U of v, $\Delta(v) \in \operatorname{supp}\mathrm{GW}(\mu)$, i.e. $v \in \text{v-supp}(\mu)$. $\square$

4 Convex functions

In this chapter we collect some facts about convex functions and their topology. It is also a part of [35].

For simplicity we will assume that V is a Euclidean vector space.

4.1 Topology on spaces of convex functions

Let $U \subset V$ be a convex subset. A function $f : U \to \mathbb{R} \cup \{+\infty\}$ is called convex if the following inequality holds for any $x_0, x_1 \in U$ and every $t \in [0,1]$:

$$f(tx_1 + (1-t)x_0) \leq tf(x_1) + (1-t)f(x_0).$$

Equivalently, $f : U \to \mathbb{R} \cup \{+\infty\}$ is convex if and only if its *epi-graph* $\mathrm{epi}(f) := \{(x,t) \in U \times \mathbb{R} \ : \ f(x) \leq t\}$ is a convex subset of $U \times \mathbb{R}$. Note that f is lower semi-continuous if and only if $\mathrm{epi}(f)$ is closed in $U \times \mathbb{R}$. In this thesis we are mostly interested in subsets of

$$\mathrm{Conv}(V) := \{f : V \to \mathbb{R} \cup \{+\infty\} \ : \ f \text{ convex, lower semi-continuous}, f \not\equiv +\infty\}.$$

For any $f \in \mathrm{Conv}(V)$, we define the *domain* of f

$$\mathrm{dom}(f) := \{x \in V \ : \ f(x) < +\infty\}.$$

By definition, $\mathrm{dom}(f)$ is a non-empty convex subset of V. f is always continuous on the interior of $\mathrm{dom}(f)$. In particular, the space of finite-valued convex functions

$$\mathrm{Conv}(V, \mathbb{R}) := \{f : V \to \mathbb{R} \ : \ f \text{ convex}\} \subset \mathrm{Conv}(V)$$

contains only continuous functions. Note that $\mathrm{Conv}(V, \mathbb{R})$ is closed with respect to the formation of the pointwise maximum, while the maximum of two elements of $\mathrm{Conv}(V)$ may be identical to $+\infty$.

We adopt the following notion of convergence for sequences in $\mathrm{Conv}(V)$. It is closely related to convergence with respect to the Hausdorff metric.

Definition 4.1.1. *A sequence $(f_j)_j$ in* $\mathrm{Conv}(V)$ epi-converges *to $f \in \mathrm{Conv}(V)$ if and only if for every $x \in V$ the following conditions hold:*

1. *For every sequence $(x_j)_j$ in V converging to x: $f(x) \leq \liminf_{j \to \infty} f_j(x_j)$.*
2. *There exists a sequence $(x_j)_j$ converging to x such that $f(x) = \lim_{j \to \infty} f_j(x_j)$.*

It is known that this notion of convergence is induced by a metrizable topology on $\mathrm{Conv}(V)$ (see for example [52] Theorem 7.58).
In the constructions used in the later chapters, the limit function $f \in \mathrm{Conv}(V)$ will usually be finite on some open subset of V. In this case epi-convergence, pointwise convergence, and locally uniform convergence are compatible in the following sense:

Proposition 4.1.2 ([52] Theorem 7.17)**.** *For a function $f \in \mathrm{Conv}(V)$ such that $\mathrm{dom}\, f$ has non-empty interior and a sequence $(f_j)_j$ in $\mathrm{Conv}(V)$ the following are equivalent:*

1. *$(f_j)_j$ epi-converges to f.*
2. *$(f_j)_j$ converges pointwise to f on a dense subset.*
3. *$(f_j)_j$ converges uniformly to f on all compact subsets that do not contain a boundary point of $\mathrm{dom}\, f$.*

In particular, a sequence $(f_j)_j$ in $\mathrm{Conv}(V,\mathbb{R})$ epi-converges to $f \in \mathrm{Conv}(V,\mathbb{R})$ if and only if it converges uniformly on compact subsets.

4.2 Compact subsets of the space of finite-valued convex functions

As we will equip our spaces of valuations with the compact-open topology, we need some useful characterization of (relatively) compact subsets. For our purposes, it will only be necessary to have some controle over compact subsets of the subspace $\mathrm{Conv}(V,\mathbb{R})$ of finite-valued convex functions. Due to Proposition 4.1.2, we can consider $\mathrm{Conv}(V,\mathbb{R})$ as a subspace of the space of all continuous functions on V. We will thus invoke the theorem of Arzelà-Ascoli, so we need to show that any compact subset of $\mathrm{Conv}(V,\mathbb{R})$ is locally uniformly equicontinuous. Sufficient for the uniform equicontinuity is a common bound on the Lipschitz constants. This common bound is established by the following proposition.

Proposition 4.2.1. *Let $U \subset V$ be a convex, open subset and $f : U \to \mathbb{R}$ a convex function. If $X \subset U$ is a set with $X + \epsilon B_1 \subset U$ such that f is bounded on $X + \epsilon B_1$, then f is Lipschitz continuous on X with Lipschitz constant $\frac{2}{\epsilon}\|f|_{X+\epsilon B_1}\|_\infty$.*

Proof. This is a special case of [52] 9.14. □

Proposition 4.2.2. *A subset $U \subset \mathrm{Conv}(V,\mathbb{R})$ is relatively compact if and only if it is bounded on compact subsets.*

Proof. As $U \subset C(V)$ is compact, the restriction to any compact subset $K \subset V$ leads to a compact subset of $C(K)$, which therefore has to be bounded. Thus the elements of U are locally uniformly bounded.
For the converse statement, observe that the topology on $\mathrm{Conv}(V,\mathbb{R})$ is metrizable, so we only have to show that the closure of any such subset is sequentially compact. Let

$(f_k)_k$ be a sequence in $\mathrm{Conv}(V,\mathbb{R})$ that is bounded on compact subsets of V. Then the Lipschitz constants of these functions are also uniformly bounded on B_j for all $j \in \mathbb{N}$ by Proposition 4.2.1. In particular, the set $\{f_k|_{B_j} : k \in \mathbb{N}\} \subset C(B_j)$ is equicontinuous. By the theorem of Arzelà-Ascoli, we can choose a subsequence $f_{j,k}$ that converges uniformly on B_j to some function $f_{j,\infty} \in C(B_j)$ for $k \to \infty$. Iterating this argument for all $j \in \mathbb{N}$ and taking an appropriate diagonal series, we find a subsequence that converges uniformly on B_j for all $j \in \mathbb{N}$ to some function $f \in C(V)$. It is easy to see that f is convex. Now the claim follows from Proposition 4.1.2. $\square$

4.3 Some dense families of convex functions

The *Legendre transform* or *convex dual* of a function $f \in \mathrm{Conv}(V)$ is the function $f^* : V^* \to (-\infty, \infty]$ given by

$$f^*(y) = \sup_{x\in V} \langle y, x\rangle - f(x) \quad \text{for } y \in V^*,$$

where $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ denotes the canonical pairing. As a consequence of [51] Theorem 12.2 and Corollary 12.2.1, we have

Proposition 4.3.1. *For $f \in \mathrm{Conv}(V)$, $f^* \in \mathrm{Conv}(V^*)$ and $f^{**} := (f^*)^* = f$.*

Proposition 4.3.2. *The map $\mathrm{Conv}(V) \to \mathrm{Conv}(V)$, $f \mapsto f^*$ has the following properties:*

1. *It is continuous with respect to the topology induced by epi-convergence.*
2. *If $f \vee h, f \wedge h \in \mathrm{Conv}(V)$, then*
$$(f \vee h)^* = f^* \wedge h^*, \qquad\qquad (f \wedge h)^* = f^* \vee h^*.$$
3. *If $x \in V$, $c \in \mathbb{R}$ and $f \in \mathrm{Conv}(V)$, then*
$$(f(\cdot - x) + c)^*(y) = f^*(y) + \langle y, x\rangle - c \quad \textit{for all } y \in V^*.$$

Proof. See [52] Theorem 11.34 for 1. and [20] Proposition 3.4. for 2. The last property follows directly from the definition. $\square$

Let $f \in \mathrm{Conv}(V)$. An element $y \in V^*$ is called a *subgradient* of f in $x_0 \in \mathrm{dom}\, f$ if

$$f(x_0) + \langle y, x - x_0\rangle \leq f(x) \quad \text{for all } x \in V.$$

The set of all subgradients of f in a point $x_0 \in \mathrm{dom}\, f$ is called the subdifferential of f in x_0 and will be denoted by $\partial f(x_0)$. Note that $\partial f(x) = \{df(x)\}$ if f is differentiable in $x \in V$.
We recall the following basic properties of the subdifferential:

Lemma 4.3.3 ([51] Theorem 23.5)**.** *For $f \in \mathrm{Conv}(V)$, $x \in V$, $y \in V^*$ the following are equivalent:*

1. $y \in \partial f(x)$,
2. $\langle y, x\rangle = f(x) + f^*(y)$,
3. $y \in \operatorname{argmax}_{x \in V} \langle y, x\rangle - f(x)$.

Lemma 4.3.4. *Let $f \in \mathrm{Conv}(V, \mathbb{R})$. Then $\partial f(x) \in \mathcal{K}(V^*)$ for all $x \in V$, and if f is Lipschitz continuous on a neighborhood of $x \in V$ with Lipschitz constant $L > 0$, then $\partial f(x) \subset B_L(0)$.*

Proof. This is a special case of [13] 2.1.2. □

Also note that $\partial f(x) \neq \emptyset$ for any $x \in V$ if $f \in \mathrm{Conv}(V, \mathbb{R})$.
In Section 5.2.2 we will relate valuations on convex functions to valuations on higher dimensional convex bodies. This construction relies on a density result contained in Corollary 4.3.6 below, which we will deduce from the following proposition.

Proposition 4.3.5. *Let $f \in \mathrm{Conv}(V, \mathbb{R})$ be a finite-valued convex function, f^* its convex dual, $R > 0$. If $\|f\|_{C(B_{R+2})} \leq c$, then the set*

$$K_{f,R} := \mathrm{epi}(f^*) \cap \{(y,t) \in V^* \times \mathbb{R} \ : \ |y| \leq 2c, |t| \leq (2R+3)c\}$$

is a convex body in $V^* \times \mathbb{R}$ and satisfies

$$f(x) = h_{K_{f,R}}(x, -1) \quad \text{for all } x \in B_{R+1}.$$

Proof. Consider the set $C := \{y \in V^* \ : \ y \in \partial f(x) \text{ for some } x \in B_{R+1}\}$. As $f|_{B_{R+1}}$ is Lipschitz continuous with Lipschitz constant $L = 2\|f\|_{C(B_{R+2})}$ by Proposition 4.2.1, Lemma 4.3.4 implies that C is contained in a ball of radius L centered at the origin.
Any $y \in C$ satisfies $f^*(y) = \langle x, y\rangle - f(x)$ for some $x \in B_{R+1}$ due to Lemma 4.3.3. Thus

$$|f^*(y)| \leq |\langle y, x\rangle| + |f(x)| \leq L(R+1) + \|f\|_{C(B_{R+2})} \leq (2R+3)\|f\|_{C(B_{R+2})} \leq (2R+3)c.$$

Let us show that $f(x) = h_{K_{f,R}}(x, -1)$ for all $x \in B_{R+1}$. Obviously, the left hand side is equal to or larger than the right hand side. By Lemma 4.3.4, we know that for any $x \in B_{R+1}$ there exists $y \in V^*$ such that $f(x) = \langle y, x\rangle - f^*(y)$. In particular $y \in C$. Then

$$(y, f^*(y)) \in \mathrm{epi}(f^*) \cap \{(y,t) \in V^* \times \mathbb{R} \ : \ |y| \leq 2c, |t| \leq (2R+3)c\} = K_{f,R}$$

by the previous discussion, so

$$f(x) = \langle y, x\rangle - f^*(y) \leq \sup_{(\tilde{y},t) \in K_{f,R}} \langle \tilde{y}, x\rangle - t = h_{K_{f,R}}(x, -1).$$

As f^* is lower semi-continuous, the set $\mathrm{epi}(f^*) \cap \{(y,t) \in V^* \times \mathbb{R} \ : \ |y| \leq 2(R+2)c, |t| \leq 3(R+2)c\}$ is closed. As it is also convex and bounded, it belongs to $\mathcal{K}(V^* \times \mathbb{R})$. □

We thus obtain the following density results.

Corollary 4.3.6. *The following families of functions are dense in* $\mathrm{Conv}(V,\mathbb{R})$*:*

1. $\{h_K(\cdot,-1)\ :\ K \in \mathcal{K}(V^* \times \mathbb{R})\}$,
2. $\{h_P(\cdot,-1)\ :\ P \in \mathcal{K}(V^* \times \mathbb{R})$ *polytope*$\}$,
3. $\{h_K(\cdot,-1)\ :\ K \in \mathcal{K}(V^* \times \mathbb{R})^{sm}\}$,
4. $\mathrm{Conv}(V,\mathbb{R}) \cap C^\infty(V)$.

Proof. For the first set this follows directly from Proposition 4.3.5 and the continuity of the map $\mathcal{K}(V^* \times \mathbb{R}) \to \mathrm{Conv}(V,\mathbb{R})$, $K \mapsto h_K(\cdot,-1)$, see Lemma 2.4.2. As $\{P \in \mathcal{K}(V^* \times \mathbb{R})\ :\ P \text{ polytope}\}$ and $\mathcal{K}(V^* \times \mathbb{R})^{sm}$ are dense subsets of $\mathcal{K}(V^* \times \mathbb{R})$ by [54] Theorem 1.8.16 and Proposition 2.4.5 respectively, this implies the density of the second and third set. For the last set, observe that the support function of any smooth and strictly convex body is smooth, so the last set contains a dense subset and is thus dense itself. □

4.4 Lipschitz regularization

Most of our results are actually results on valuations on $\mathrm{Conv}(V,\mathbb{R})$ which generalize to more general subspaces of $\mathrm{Conv}(V)$ by approximation.
For $r > 0$ the *Lipschitz regularization* or *Pasch-Hausdorff envelope* of a convex function $f \in \mathrm{Conv}(V)$ is defined as

$$\mathrm{reg}_r(f) := \left(f^* + 1^\infty_{B_{1/r}}\right)^*.$$

We will need the following properties:

Proposition 4.4.1 ([20] Propositions 4.1, 4.2, 4.3)**.** *For* $f, h \in \mathrm{Conv}(V)$ *and* $r > 0$*, the Lipschitz regularization has the following properties:*

i. *There exists* $r_0 > 0$ *such that* $\mathrm{reg}_r f \in \mathrm{Conv}(V,\mathbb{R})$ *for all* $0 < r \le r_0$.
ii. $\mathrm{reg}_r f$ *epi-converges to* f *for* $r \to 0$.
iii. *If* $x \in \mathrm{dom}(f)$ *and* $\partial f(x) \cap B_{1/r} \neq \emptyset$*, then* $\mathrm{reg}_r f(x) = f(x)$ *and* $\partial \mathrm{reg}_r f(x) = \partial f(x) \cap B_{1/r}$.
iv. *If* $(f_j)_j$ *is a sequence in* $\mathrm{Conv}(V)$ *that epi-converges to* f*, then there exists* $r_0 > 0$ *such that* $(\mathrm{reg}_r f_j)_j$ *epi-converges to* $\mathrm{reg}_r f$ *for all* $0 < r \le r_0$.
v. *If* $f \vee h, f \wedge h \in \mathrm{Conv}(V)$*, then there exists* $r_0 > 0$ *such that*
$$\mathrm{reg}_r(f \vee h)) = \mathrm{reg}_r f \vee \mathrm{reg}_r h, \qquad \mathrm{reg}_r(f \wedge h) = \mathrm{reg}_r f \wedge \mathrm{reg}_r h$$
for all $0 < r \le r_0$.

Note that ii. implies that $\mathrm{Conv}(V,\mathbb{R}) \subset \mathrm{Conv}(V)$ is dense. Thus the sets considered in Corollary 4.3.6 are also dense in $\mathrm{Conv}(V)$.

5 Dually epi-translation invariant valuations on convex functions

In this Chapter we examine dually epi-translation invariant valuations on certain cones of convex functions, restating the basic definitions from the introduction in Section 5.1. Their invariance properties will allow us to relate this type of valuations to translation invariant valuations on convex bodies in Section 5.2, where we also discuss a classification of n-homogeneous valuations from [21]. This relation is then used to establish the existence of a homogeneous decomposition in Section 5.3, which was also obtained in [21]. Section 5.4.1 establishes a version of the Goodey-Weil distributions derived from this decomposition and Sections 5.4.2 and 5.4.3 examine the diagonality and compactness of the support of these distributions.

The results of this chapter are to be published in [35].

5.1 Basic definitions

Let $C \subset \mathrm{Conv}(V)$ be a non-empty subset and let $(F, +)$ be an abelian semigroup.

Definition 5.1.1. *A map $\mu : C \to F$ is called a valuation if*

$$\mu(f) + \mu(h) = \mu(f \vee h) + \mu(f \wedge h)$$

for all $f, h \in C$ such that the pointwise maximum $f \vee h$ and minimum $f \wedge h$ belong to C.

We will only be interested in the case where F is a real topological vector space and $\mu : C \to F$ is continuous with respect to the metrizable topology induced by epi-convergence.

Definition 5.1.2. *Let $C \subset \mathrm{Conv}(V)$. A valuation $\mu : C \to F$ is called* dually epi-translation invariant *if*

$$\mu(f + \lambda + c) = \mu(f) \quad \textit{for all } f \in C, \lambda \in V^*, c \in \mathbb{R}$$

such that $f + \lambda + c \in C$.

From now on let $C \subset \mathrm{Conv}(V)$ be a subset with $\mathrm{Conv}(V, \mathbb{R}) \subset C$.

Definition 5.1.3. *If F is a topological vector space, let $\mathrm{VConv}(C; V, F)$ denote the space of all valuations $\mu : C \to F$ that are*

1. *continuous with respect to epi-convergence,*

2. *dually epi-translation invariant.*

If $C = \mathrm{Conv}(V,\mathbb{R})$, we will use the notation $\mathrm{VConv}(V,F)$ instead. For $F = \mathbb{R}$, we will write $\mathrm{VConv}(C;V) := \mathrm{VConv}(C;V,\mathbb{R})$ and $\mathrm{VConv}(V) := \mathrm{VConv}(V,\mathbb{R})$ for brevity. We also equip $\mathrm{VConv}(C;V,F)$ with the compact-open topology. Note that Proposition 4.2.2 provides a characterization of all compact subsets for the case $C = \mathrm{Conv}(V,\mathbb{R})$.

5.2 Relation to valuations on convex bodies

In this section we will construct injective maps from $\mathrm{VConv}(C;V,F)$ into spaces of translation invariant valuations on convex bodies. We start with a generalization of the map

$$\begin{aligned}\mathrm{VConv}(V) &\to \mathrm{Val}(V^*)\\ \mu &\mapsto [K \mapsto \mu(h_K)]\end{aligned}$$

considered by Alesker in [7].

5.2.1 Characteristic function

We first consider translates of support functions.

Lemma 5.2.1. *Let $(G,+)$ be an Abelian semi-group with cancellation law that carries a Hausdorff topology, and $\mu_1,\mu_2 : \mathrm{Conv}(V,\mathbb{R}) \to G$ two continuous valuations. If $\mu_1(h_P(\cdot - y)+c) = \mu_2(h_P(\cdot - y)+c)$ for all polytopes $P \in \mathcal{K}(V^*)$ with $0 \in \operatorname{int} P$, $y \in V$ and $c \in \mathbb{R}$, then $\mu_1 \equiv \mu_2$ on $\mathrm{Conv}(V,\mathbb{R})$.*

Proof. This is [49] Lemma 5.1. To be precise, the version in [49] considers translation invariant valuations, however, the proof only uses the weaker property stated above. □

Proposition 5.2.2. *Let $\mu \in \mathrm{VConv}(C;V,F)$, where F is Hausdorff and $\mathrm{Conv}(V,\mathbb{R}) \subset C$. For $x \in V$ define $S(\mu)[x] \in \mathrm{Val}(V,F)$ by $[S(\mu)[x]](K) := \mu(h_K(\cdot - x))$. Then the map*

$$S : \mathrm{VConv}(C;V,F) \to C(V,\mathrm{Val}(V^*,F))$$

is well defined and injective. We will call $S(\mu)$ the characteristic function *of $\mu \in \mathrm{VConv}(C;V,F)$.*

Proof. The properties of the support function imply that $S(\mu)[x] \in \mathrm{Val}(V^*,F)$ for all $x \in V$, see Proposition 2.4.6. To see that S is injective, assume that $S(\mu) = 0$. Then $\mu(h_P(\cdot - x)+c) = 0$ for all polytopes $P \in \mathcal{K}(V)$ with $0 \in \operatorname{int} P$ and all $x \in V$, $c \in \mathbb{R}$, so Lemma 5.2.1 implies $\mu \equiv 0$ on $\mathrm{Conv}(V,\mathbb{R})$, which is dense in C by Proposition 4.4.1, so $\mu = 0$ by continuity.

Now let $B \subset V^*$ be a convex body with non-empty interior and set $\tilde{B} := \{K \in \mathcal{K}(V^*) : K \subset B\}$. Recall that a basis for the topology of $\mathrm{Val}(V^*, F)$ is given by translates of the open subsets

$$\mathcal{M}(\tilde{B}, O) = \{\mu \in \mathrm{Val}(V^*, F) : \mu(K) \in O \ \forall K \in \tilde{B}\},$$

where $O \subset F$ is an open neighborhood of the origin. Let us show that the map

$$\begin{aligned} R : \mathcal{K}(V^*) \times V &\to \mathrm{Conv}(V, \mathbb{R}) \\ (K, x) &\mapsto h_K(\cdot - x) \end{aligned}$$

is continuous. Assume that (K_j, x_j) is a convergent sequence in $\mathcal{K}(V^*) \times V$ with limit (K, x). Then

$$\left|h_{K_j}(\cdot - x_j) - h_K(\cdot - x)\right| \leq \left|h_{K_j}(\cdot - x_j) - h_K(\cdot - x_j)\right| + \left|h_K(\cdot - x_j) - h_K(\cdot - x)\right|.$$

As the sequence of functions $(h_{K_j})_j$ converges uniformly on the compact subset $\{x_j : j \in \mathbb{N}\} \cup \{x\}$, the right hand side converges to 0.
We deduce that $(K, x) \mapsto S(\mu)[x](K) = [\mu \circ R](K, x)$ is uniformly continuous on compact subsets. In particular, we can find $\delta > 0$ such that

$$\mu(h_K(\cdot - x)) - \mu(h_K(\cdot - x')) \in O \quad \forall K \subset B, \forall x, x' \in V \text{ with } |x - x'| < \delta,$$

i.e.

$$S(\mu)[x] - S(\mu)[x'] \in \mathcal{M}(\tilde{B}, O) \quad \text{for all } x, x' \in V \text{ with } |x - x'| < \delta.$$

Thus $S(\mu)$ is continuous. $\square$

If $\mu \in \mathrm{VConv}(V)$ is n-homogeneous, then $S(\mu) \in C(V, \mathrm{Val}_n(V^*))$. By a classical result due to Hadwiger [31], $\mathrm{Val}_n(V^*)$ is 1-dimensional and spanned by a Lebesgue measure. In other words, we can interpret $S(\mu)$ as an element of $C(V) \otimes \mathrm{Dens}(V^*)$. Let us return to the n-homogeneous valuations considered in Example 1.0.5. All of these functionals are given by

$$f \mapsto \int_V \phi d \,\mathrm{Hess}_n(f)$$

for some $\phi \in C_c(V)$. This is in fact a complete characterization of $\mathrm{VConv}_n(V)$, as we will discuss below. To present a version of this result in terms suitable for our application in later chapters, we will need some additional properties of the Hessian measure. These results are well known, but for the convenience of the reader we will show how these basic properties follow from the characterization of Hess_n on smooth functions and the continuity with respect to the vague topology.

Lemma 5.2.3. *The Hessian measure has the following properties:*

1. *If $f=h$ on an open subset $U\subset V$, then*
$$\int_V \phi d\operatorname{Hess}_n(f)=\int_V \phi d\operatorname{Hess}_n(h)\quad \text{for all } \phi\in C_c(U).$$

2. $\operatorname{Hess}_n(f)$ *is a non-negative measure for all* $f\in\operatorname{Conv}(V,\mathbb{R})$.

3. $\operatorname{Hess}_n(h_K(\cdot-y))=\operatorname{vol}_{V^*}(K)\delta_y$ *for all* $y\in V$*, where* vol_{V^*} *is the unique volume form on* V^* *inducing* Hess_n*, see Chapter 1.*

Proof. Choose a scalar product on V and identify $V\cong\mathbb{R}^n$ using an orthonormal basis. For the first property, observe that the mollified functions f_ϵ and h_ϵ coincide on an open neighborhood of $\operatorname{supp}\phi$ for all $\epsilon>0$ small enough. Thus the defining property and the continuity of the Hessian measure imply

$$\begin{aligned}\int_V \phi d\operatorname{Hess}_n(f)=&\lim_{\epsilon\to0}\int_V \phi d\operatorname{Hess}_n(f_\epsilon)=\lim_{\epsilon\to0}\int_V\phi(x)\det(H_{f_\epsilon}(x))dx\\ =&\lim_{\epsilon\to0}\int_V\phi(x)\det(H_{h_\epsilon}(x))dx=\lim_{\epsilon\to0}\int_V\phi d\operatorname{Hess}_n(h_\epsilon)=\int_V\phi d\operatorname{Hess}_n(h).\end{aligned}$$

This also directly implies the second property.
For the last property, assume first that $K\in\mathcal{K}(V^*)$ is smooth and strictly convex. Then $h_K(\cdot-y)$ is smooth on $V\setminus\{y\}$. In addition, h_K is 1-homogeneous, so $\det H_{h_K(\cdot-y)}=0$ on $V\setminus\{y\}$. Mollifying h_K, the first property implies

$$\begin{aligned}&\int_V\phi d\operatorname{Hess}_n(h_K(\cdot-y))=\lim_{\epsilon\to0}\int_V\phi d\operatorname{Hess}_n((h_K(\cdot-y))_\epsilon)\\ =&\lim_{\epsilon\to0}\int_V\phi(x)\det H_{(h_K(\cdot-y))_\epsilon}(x)dx=\int_V\phi(x)\det H_{h_K(\cdot-y)}(x)dx=0\end{aligned}$$

for all $\phi\in C_c(V)$ with $\operatorname{supp}\phi\subset V\setminus\{y\}$. By continuity, this holds for all $K\in\mathcal{K}(V^*)$, so $\operatorname{Hess}_n(h_K(\cdot-y))=c(K)\delta_y$ for some $c(K)\in\mathbb{R}$. Now take a function $\phi\in C_c(V)$ with $\phi=1$ on a neighborhood of $y\in V$. Then

$$K\mapsto c(K)=\int_V\phi d\operatorname{Hess}_n(h_K(\cdot-y))$$

defines a continuous, n-homogeneous valuation on $\mathcal{K}(V^*)$, which is also translation invariant. By Hadwiger's characterization, $c(K)=C\cdot\operatorname{vol}_{V^*}(K)$ for some $C\in\mathbb{R}$. To determine C, take the function $f_\epsilon\in\operatorname{Conv}(V,\mathbb{R})$ given for $\epsilon>0$ by

$$f_\epsilon(x)=\sqrt{\epsilon^2+|x|^2}.$$

For $\epsilon\to0$, this function converges to the support function of the unit ball. Thus

$$C\cdot\operatorname{vol}_{V^*}(B_1)=c(B_1)=\lim_{\epsilon\to0}\int_V\phi d\operatorname{Hess}_n(f_\epsilon(\cdot-y)).$$

The Hessian of f_ϵ is given by

$$H_{f_\epsilon}(x) = \frac{1}{\sqrt{\epsilon^2 + |x|^2}^3}((\epsilon^2 + |x|^2)\mathrm{Id}_n - x \cdot x^T),$$

which has two distinct eigenvalues for $x \neq 0$: On the orthogonal complement of x, $H_{f_\epsilon}(x)$ operates by multiplication with $(\sqrt{\epsilon^2 + |x|^2})^{-1}$, while it is given by the multiplication with $\epsilon^2/\sqrt{\epsilon^2 + |x|^2}^3$ on the 1-dimensional space spanned by x. Thus

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_V \phi d\,\mathrm{Hess}_n(f_\epsilon(\cdot - y)) = \lim_{\epsilon\to 0}\int_V \phi(x)\det H_{f_\epsilon}(x-y)dx\\
=&\lim_{\epsilon\to 0}\int_V \phi(x+y)\det H_{f_\epsilon}(x)dx = \lim_{\epsilon\to 0}\int_V \phi(x+y)\frac{\epsilon^2}{\sqrt{\epsilon^2+|x|^2}^{n+2}}dx\\
=&\lim_{\epsilon\to 0}\int_V \phi(x+y)\frac{\epsilon^2}{\epsilon^{n+2}\sqrt{1+\frac{|x|^2}{\epsilon^2}}^{n+2}}dx = \lim_{\epsilon\to 0}\int_V \phi(\epsilon x+y)\frac{1}{\sqrt{1+|x|^2}^{n+2}}dx\\
=&\int_V \phi(y)\frac{1}{\sqrt{1+|x|^2}^{n+2}}dx = \phi(y)\int_V \frac{1}{\sqrt{1+|x|^2}^{n+2}}dx
\end{aligned}$$

by the theorem of dominated convergence. The last integral can be computed using polar coordinates and reduces to the volume of the unit ball. Thus $C = \phi(y) = 1$. $\square$

Now assume that $\mu \in \mathrm{VConv}(V)$ is n-homogeneous and choose $\mathrm{vol}_{V^*} \in \mathrm{Dens}(V^*)$. If $S(\mu) \in C(V, \mathrm{Val}_n(V^*)) \cong C(V)\otimes\mathrm{Dens}(V^*)$ has compact support, write $S(\mu) = \phi\otimes\mathrm{vol}_{V^*}$ for $\phi \in C_c(V)$. If Hess_n denotes the Hessian measure induced by vol_{V^*}, consider the continuous valuation

$$\tilde{\mu}(f) := \int_V \phi d\,\mathrm{Hess}_n(f).$$

By Lemma 5.2.3, $S(\tilde{\mu}) = \phi \otimes \mathrm{vol}_{V^*} = S(\mu)$, so Proposition 5.2.2 implies $\mu = \tilde{\mu}$. For a complete characterization of all n-homogeneous elements of $\mathrm{VConv}(V)$, we only need to show that the characteristic function of any n-homogeneous valuation has compact support. This was done by Colesanti, Ludwig and Mussnig in [21]. Let us state their result using our terminology:

Theorem 5.2.4 ([21] Theorem 5). *$\mu : \mathrm{Conv}(V, \mathbb{R}) \to \mathbb{R}$ is a continuous, n-homogeneous, dually epi-translation invariant valuation if and only if there exists $\phi \in C_c(V)$ such that*

$$\mu(f) = \int_V \phi d\,\mathrm{Hess}_n(f) \quad \textit{for all } f \in \mathrm{Conv}(V, \mathbb{R})$$

for some Hessian measure Hess_n. More precisely, $\phi\otimes\mathrm{vol}_{V^*} \in C_c(V, \mathrm{Val}_n(V^*))$ coincides with the characteristic function $S(\mu)$ of μ, where vol_{V^*} is the unique density inducing Hess_n.

5.2.2 Embedding into $\mathrm{Val}(V^* \times \mathbb{R}, F)$

By Proposition 2.4.6, the map

$$\begin{aligned} P : \mathcal{K}(V^* \times \mathbb{R}) &\to \mathrm{Conv}(V, \mathbb{R}) \\ K &\mapsto h_K(\cdot, -1) \end{aligned}$$

is continuous, where we have used the canonical isomorphism $(V \times \mathbb{R})^* \cong V^* \times \mathbb{R}$. For $\mu \in \mathrm{VConv}(C; V, F)$ define $T(\mu) \in \mathrm{Val}(V^*, F)$ by $T(\mu)[K] := \mu(h_K(\cdot, -1))$ for $K \in \mathcal{K}(V^* \times \mathbb{R})$.

Theorem 5.2.5. *Let F be a Hausdorff real topological vector space and $C \subset \mathrm{Conv}(V)$ a subset with $\mathrm{Conv}(V, \mathbb{R}) \subset C$. Then*

$$T : \mathrm{VConv}(C; V, F) \to \mathrm{Val}(V^* \times \mathbb{R}, F)$$

is well defined, continuous, and injective.

Proof. It is clear that $T(\mu) = \mu \circ P \in \mathrm{Val}(V^* \times \mathbb{R}, F)$.
Let us show that T is injective: If $T(\mu) = 0$, then $\mu(h_K(\cdot, -1)) = 0$ for all $K \in \mathcal{K}(V^* \times \mathbb{R})$. By Corollary 4.3.6, these functions form a dense subspace of $\mathrm{Conv}(V, \mathbb{R})$, which is dense in C, so the continuity of μ implies $\mu = 0$, as F is Hausdorff. Thus T is injective.
A basis for the topology of $\mathrm{Val}(V^* \times \mathbb{R}, F)$ is given by the open sets

$$\mathcal{M}(B, O) = \{\mu \in \mathrm{Val}(V^* \times \mathbb{R}, F) \ : \ \mu(K) \in O \quad \forall K \in B\},$$

where $O \subset F$ is open and $B \subset \mathcal{K}(V^* \times \mathbb{R})$ is compact. Then

$$\begin{aligned} T^{-1}(\mathcal{M}(B, O)) &= \{\mu \in \mathrm{VConv}(C; V, F) \ : \ \mu(h_K(\cdot, -1)) \in O \quad \forall K \in B\} \\ &= \{\mu \in \mathrm{VConv}(C; V, F) \ : \ \mu(f) \in O \quad \forall f \in P(B)\} \\ &= \mathcal{M}(P(B), O). \end{aligned}$$

As P is continuous, $P(B)$ is compact in C, so $T^{-1}(\mathcal{M}(B, O)) = \mathcal{M}(P(B), O)$ is open in $\mathrm{VConv}(C; V, F)$. □

5.3 Homogeneous decomposition

In this section we are going to prove that $\mathrm{VConv}(C; V, F)$ decomposes into a direct sum of spaces of homogeneous valuations. Colesanti, Ludwig and Mussnig [21] showed a version of this homogeneous decomposition for $C = \mathrm{Conv}(V, \mathbb{R})$, $F = \mathbb{R}$ using the characteristic function. Their approach easily generalizes to the more general case. We will thus present a different proof, where we use the embedding of $\mathrm{VConv}(C; V, F)$ into $\mathrm{Val}(V^* \times \mathbb{R}, F)$.
We will call a subset $C \subset \mathrm{Conv}(V)$ invariant under scaling if $tf \in C$ for all $f \in C$, $t > 0$. This notion will be replaced by a more restrictive property in the next section.

Definition 5.3.1. *Let $C \subset \mathrm{Conv}(V)$ be invariant under scaling. A continuous valuation $\mu : C \to F$ is called k-homogeneous if $\mu(tf) = t^k \mu(f)$ for all $f \in C$ and all $t > 0$. We will denote the space of k-homogeneous valuations in $\mathrm{VConv}(C;V,F)$ by $\mathrm{VConv}_k(C;V,F)$.*

Proposition 5.3.2. *Let F be a Hausdorff real topological vector space, $C \subset \mathrm{Conv}(V)$ a subset invariant under scaling that contains $\mathrm{Conv}(V,\mathbb{R})$, and $\mu \in \mathrm{VConv}(C;V,F)$. Then there exist valuations $\mu_i \in \mathrm{VConv}_i(C;V,F)$ for $i = 0,\dots,n+1$ such that*

$$\mu = \sum_{i=0}^{n+1} \mu_i.$$

Proof. Consider the injective map $T : \mathrm{VConv}(C;V,F) \to \mathrm{Val}(V^* \times \mathbb{R}, F)$ from Theorem 5.2.5, given by $T(\mu)[K] = \mu(h_K(\cdot,-1))$ for $K \in \mathcal{K}(V^* \times \mathbb{R})$.
For $t > 0$ define $\mu^t \in \mathrm{VConv}(C;V,F)$ by $\mu^t(f) := \mu(tf)$ for $f \in C$. Then $T(\mu^t)[K] = T(\mu)[tK]$, as $h_{tK} = t h_K$ for $t > 0$.
Using the McMullen decomposition (Theorem 1.0.1) for $\mathrm{Val}(V^* \times \mathbb{R}, F)$, we see that $T(\mu^t) = \sum_{i=0}^{n+1} t^i \tilde{\mu}_i$ for homogeneous elements $\tilde{\mu}_i \in \mathrm{Val}_i(V^* \times \mathbb{R}, F)$. Plugging in $0 < t_0 < \dots < t_{n+1}$ and using the inverse of the Vandermonde matrix, we obtain constants $c_{ij} \in \mathbb{R}$ such that $\tilde{\mu}_i = \sum_{j=0}^{n+1} c_{ij} T(\mu^{t_j})$.
Now define $\mu_i \in \mathrm{VConv}(C;V,F)$ by $\mu_i := \sum_{j=0}^{n+1} c_{ij} \mu^{t_j}$. Then, obviously, $T(\mu_i) = \tilde{\mu}_i$ and for any $K \in \mathcal{K}(V^* \times \mathbb{R})$, $t > 0$:

$$T\left(\mu_i^t\right)(K) = T(\mu_i)(tK) = \tilde{\mu}_i(tK) = t^i \tilde{\mu}_i(K) = t^i T(\mu_i)(K) = T\left(t^i \mu_i\right)(K).$$

The injectivity of T implies $t^i \mu_i = \mu_i^t$, i.e. μ_i is i-homogeneous. In addition,

$$T(\mu) = \sum_{i=0}^{n+1} \tilde{\mu}_i = \sum_{i=0}^{n+1} T(\mu_i) = T\left(\sum_{i=0}^{n+1} \mu_i\right).$$

Thus the injectivity of T implies $\mu = \sum_{i=0}^{n+1} \mu_i$. □

It remains to see that the top component vanishes.

Proposition 5.3.3. $\mathrm{VConv}_{n+1}(C;V,F) = 0$

Proof. Let $\mu \in \mathrm{VConv}_{n+1}(C;V,F)$. As $\mathrm{Conv}(V,\mathbb{R})$ is dense in C, we only need to show that μ vanishes on finite-valued convex functions. Using Proposition 5.2.2, it is sufficient to show $\mu(h_K(\cdot - x)) = 0$ for all $K \in \mathcal{K}(V^*)$ and $x \in V$. However, $K \mapsto \mu(h_K(\cdot - x))$ defines an element of $\mathrm{Val}_{n+1}(V^*, F) = 0$. The claim follows. □

We thus arrive at the main result of this section.

Theorem 5.3.4. *Let $C \subset \mathrm{Conv}(V)$ be subset invariant under scaling that contains $\mathrm{Conv}(V,\mathbb{R})$ and F a Hausdorff real topological vector space. Then*

$$\mathrm{VConv}(C;V,F) = \bigoplus_{k=0}^{n} \mathrm{VConv}_k(C;V,F).$$

Proof. This follows directly from Proposition 5.3.2 and Proposition 5.3.3. □

5.3.1 Polynomiality and polarization

Let us call a subset $C \subset \mathrm{Conv}(V)$ a cone if $tf + h \in C$ for $f, h \in C$ and $t > 0$. To define the polarization of a homogeneous valuation in $\mathrm{VConv}_k(C; V, F)$, we need the following regularity assumption on the cone C:

Definition 5.3.5. *A cone $C \subset \mathrm{Conv}(V)$ will be called* regular *if* $\mathrm{dom}\, f$ *has non-empty interior for all $f \in C$.*

Lemma 5.3.6. *If $C \subset \mathrm{Conv}(V)$ is a regular cone, then*

$$\begin{aligned} + : C \times C &\to C \\ (f, h) &\mapsto f + h \end{aligned}$$

is continuous.

Proof. This follows directly from Proposition 4.1.2. □

Note that this map is in general not continuous if C is not a regular cone: Take $h := I^{\infty}_{(-\infty,0]}$ and $f_j \in \mathrm{Conv}(V)$ given by

$$f_j(x) = I^{\infty}_{[0,\infty)} + j^2 \max(\frac{1}{j} - x, 0) = \begin{cases} \infty & x < 0, \\ j - j^2 x & 0 \le x \le \frac{1}{j}, \\ 0 & \frac{1}{j} < x. \end{cases}$$

Using Proposition 4.1.2, we see that $(f_j)_j$ epi-converges to $f = I^{\infty}_{[0,\infty)}$, but $h + f_j = j + I^{\infty}_{\{0\}}$ does not epi-converge to $h + f = I^{\infty}_{\{0\}}$.
From now on we will assume that F is a Hausdorff real topological vector space and that $C \subset \mathrm{Conv}(V)$ is a regular cone containing $\mathrm{Conv}(V, \mathbb{R})$. Then we can consider the question of polynomiality for elements of $\mathrm{VConv}(C; V, F)$. From Theorem 5.3.4 we deduce

Corollary 5.3.7. *Let $C \subset \mathrm{Conv}(V)$ be a regular cone containing $\mathrm{Conv}(V, \mathbb{R})$. For $\mu \in \mathrm{VConv}(C; V, F)$ and $f_1, \dots, f_m \in C$, the map $(\lambda_1, \dots, \lambda_m) \mapsto \mu(\sum_{j=1}^m \lambda_j f_j)$ is a polynomial in $\lambda_j > 0$.*

Proof. We will use induction on $m \in \mathbb{N}$. For $m = 1$, this is just Theorem 5.3.4. Assume we have shown the statement for $m \in \mathbb{N}$. The map

$$f \mapsto \mu(h + f)$$

belongs to $\mathrm{VConv}(C; V, F)$ for all $h \in C$ by Lemma 5.3.6. Using Theorem 5.3.4, we obtain

$$\mu(h + tf) = \sum_{i=0}^{n} t^i \mu_i(h, f),$$

where $\mu_i : C^2 \to F$ is an i-homogeneous, continuous, and dually epi-translation invariant valuation in the second argument and a dually epi-translation invariant valuation in the first. To see that μ_i is continuous in the first argument, apply the inverse of the Vandermonde matrix to the formula above to write $\mu_i(h, f)$ as a linear combination of elements of the form $\mu(h + tf)$ for a finite number of fixed values of $t > 0$.
The induction assumption implies that $(\lambda_1, \ldots, \lambda_m) \mapsto \mu_i(\sum_{j=1}^m \lambda_j f_j, f)$ is a polynomial in $\lambda_j > 0$, $1 \leq j \leq m$. The claim follows. $\square$

Definition 5.3.8. *A valuation $\mu \in \mathrm{VConv}(C; V, F)$ is called additive if $\mu(f + g) = \mu(f) + \mu(g)$ for all $f, g \in C$.*

By continuity any additive valuation is 1-homogeneous.

Theorem 5.3.9. *Let $C \subset \mathrm{Conv}(V)$ be a regular cone containing $\mathrm{Conv}(V, \mathbb{R})$. For every $\mu \in \mathrm{VConv}_k(C; V, F)$ there exists a unique map $\bar{\mu} : C^k \to F$, called the* polarization *of μ, with the following properties:*

1. *$\bar{\mu}$ is additive and 1-homogeneous in each argument,*
2. *$\bar{\mu}$ is symmetric,*
3. *$\mu(f) = \bar{\mu}(f, \ldots, f)$ for all $f \in C$.*

Proof. We start by showing uniqueness: Using 1. and 3., we obtain

$$\mu\left(\sum_{j=1}^k \lambda_j f_j\right) = \bar{\mu}\left(\sum_{j=1}^k \lambda_j f_j, \ldots, \sum_{j=1}^k \lambda_j f_j\right) = \sum_{j_1, \ldots, j_k = 1}^k \lambda_{j_1} \ldots \lambda_{j_k} \bar{\mu}\left(f_{j_1}, \ldots, f_{j_k}\right).$$

Differentiating and using 2., we obtain the formula

$$\bar{\mu}(f_1, \ldots, f_k) = \frac{1}{k!} \frac{\partial}{\partial \lambda_1}\Big|_0 \cdots \frac{\partial}{\partial \lambda_k}\Big|_0 \mu\left(\sum_{j=1}^k \lambda_j f_j\right). \tag{5.1}$$

Thus $\bar{\mu}$ is uniquely determined by μ. To see that such a functional exists, observe that 5.3.7 implies that the right-hand side of Equation (5.1) is well defined, so we can use this equation to define $\bar{\mu}$.
Obviously, the definition is symmetric in $f_1, \ldots, f_n$. To see that $\bar{\mu}$ is additive in each argument, we thus only need to consider one argument. Setting

$$\begin{aligned} F(t, s) :=& \frac{1}{k!} \frac{\partial}{\partial \lambda_1}\Big|_0 \cdots \frac{\partial}{\partial \lambda_{k-1}}\Big|_0 \mu\left(\sum_{j=1}^{k-1} \lambda_j f_j + tf + sg\right), \\ G(t) :=& F(t, t), \end{aligned}$$

we obtain

$$\begin{aligned} \bar{\mu}(f_1, \ldots, f_{k-1}, f + g) =& G'(0) = \frac{\partial F}{\partial t}\Big|_{(0,0)} + \frac{\partial F}{\partial s}\Big|_{(0,0)} \\ =& \bar{\mu}(f_1, \ldots, f_{k-1}, f) + \bar{\mu}(f_1, \ldots, f_{k-1}, g). \end{aligned}$$

For the last property, we calculate

$$\bar{\mu}(f,\dots,f)=\frac{1}{k!}\frac{\partial}{\partial\lambda_1}\Big|_0\cdots\frac{\partial}{\partial\lambda_k}\Big|_0\mu\left(\sum_{j=1}^k\lambda_j f\right)=\frac{1}{k!}\frac{\partial}{\partial\lambda_1}\Big|_0\cdots\frac{\partial}{\partial\lambda_k}\Big|_0\left(\sum_{j=1}^k\lambda_j\right)^k\cdot\mu(f),$$

where we used that μ is k-homogeneous in the last step. Thus $\bar{\mu}(f,\dots,f)=\mu(f)$. □

The construction shows that $\bar{\mu}$ is a dually epi-translation invariant valuation in each argument. We will now show that $\bar{\mu}$ is jointly continuous. From the defining properties of $\bar{\mu}$ we deduce the following corollary.

Corollary 5.3.10. *Let $C\subset\mathrm{Conv}(V)$ be a regular cone containing $\mathrm{Conv}(V,\mathbb{R})$. For $\mu\in\mathrm{VConv}_k(C;V,F)$, $m\in\mathbb{N}$, and $f_1,\dots,f_m\in C$, $\mu(\sum_{j=1}^m\lambda_jf_j)$ is a polynomial of degree at most k in $\lambda_j\geq 0$.*

Corollary 5.3.11. *If $C\subset\mathrm{Conv}(V)$ is a regular cone containing $\mathrm{Conv}(V,\mathbb{R})$, then $\bar{\mu}:C^k\to\mathbb{R}$ is continuous for $\mu\in\mathrm{VConv}_k(C;V,F)$.*

Proof. Assume that we are given sequences $(f_{i,j})_j$ in C, $1\leq i\leq k$, such that each sequence $(f_{i,j})_j$ converges to some $f_i\in C$. Then the polynomials $P_j(\lambda_1,\dots,\lambda_k):=\mu(\sum_{i=1}^k\lambda_if_{i,j})$ converge pointwise to $P(\lambda_1,\dots,\lambda_k):=\mu(\sum_{i=1}^k\lambda_if_i)$ for $\lambda_i\geq 0$. As the degree of P_j is bounded by k due to Corollary 5.3.10, this implies that the coefficient in front of $\lambda_1\dots\lambda_k$ converges. Now the claim follows from the definition of $\bar{\mu}$ in the proof of Theorem 5.3.9. □

We close this section with an inequality, which will be used in the construction of the Goodey-Weil embedding. It also shows that the map which associates the polarization to a given valuation is continuous with respect to the natural topologies.

Lemma 5.3.12. *There exists a constant $C_k>0$ depending on $0\leq k\leq n$ only such that the following holds: If $C\subset\mathrm{Conv}(V)$ is a regular cone containing $\mathrm{Conv}(V,\mathbb{R})$ and if $K\subset C$ is compact, then*

$$\|\bar{\mu}\|_{F;K}:=\sup_{f_1,\dots,f_k\in K}|\bar{\mu}(f_1,\dots,f_k)|_F\leq C_k\|\mu\|_{F;K'}$$

for every semi-norm $|\cdot|_F$ on F and all $\mu\in\mathrm{VConv}_k(V,F)$, where

$$K':=\sum_{j=1}^k\bigcup_{i=1}^{j+1}iK=\left\{\sum_{j=1}^k f_j\ :\ f_j\in\bigcup_{i=1}^{j+1}iK\right\}\subset C$$

is compact. If K is convex with $0\in K$, there exists a constant $C_k'>0$ independent of K such that

$$\|\bar{\mu}\|_{F;K}\leq C_k'\|\mu\|_{F;K}.$$

Proof. For $f, g \in C$ and $\lambda \geq 0$

$$\mu(f+\lambda g) = \sum_{i=0}^{k} \lambda^i i!(k-i)!\bar{\mu}(f[k-i], g[i]).$$

We are only interested in the linear term. Plugging in $\lambda = 1, \dots, k+1$ and setting $\mu_i'(f) := \mu(f + ig)$, we obtain a valuation $P_g(\mu) = (\mu_1', \dots, \mu_{k+1}') \in \mathrm{VConv}(C; V, F^{k+1})$. Let $\pi_i : F^{k+1} \to F$ denote the i-th projection and S_k the Vandermonde matrix with entries corresponding to $(1, \dots, k+1)$. Then

$$\bar{\mu}(f[k-1], g) = \frac{1}{(k-1)!}\pi_1[S_k^{-1}P_g(\mu)(f)].$$

If we equip F^{k+1} with the family of semi-norms $|(v_1, \dots, v_{k+1})|_F := \max_{i=1,\dots,k+1} |v_i|_F$, and denote by $\|S_k^{-1}\|_\infty$ the operator norm of $S_k^{-1} : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ with respect to the maximum norm on $\mathbb{R}^{k+1}$, we obtain

$$|\bar{\mu}(f[k-1], g)|_F \leq \frac{1}{(k-1)!}\|S_k^{-1}\|_\infty \; |P_g(\mu)(f)|_F.$$

For $g \in K$, $P_g : \mathrm{VConv}(C; V, F) \to \mathrm{VConv}(C; V, F^{k+1})$ satisfies

$$|P_g(\mu)(f)|_F = \max_{i=1,\dots,k+1} |\mu(f+ig)|_F \leq \sup\left\{ |\mu(f+\tilde{g})|_F : \tilde{g} \in \bigcup_{i=1}^{k+1} iK \right\}$$

independent of $g \in K$. Iterating this construction starting with the $(k-1)$-homogeneous valuation $\nu(f) := \bar{\mu}(f[k-1], g)$, we see that there exists $C_k > 0$ depending on k only such that for $f_1, \dots, f_k \in K$

$$|\bar{\mu}(f_1, \dots, f_k)|_F \leq C_k \sup\left\{ |\mu(\tilde{g})|_F : \tilde{g} \in \sum_{j=1}^{k} \bigcup_{i=1}^{j+1} iK \right\} = C_k \|\mu\|_{F,K'}$$

for every semi-norm $|\cdot|_F$ on F. Also note that K' is compact, as it is the image of a compact subset under the addition map, which is continuous on C by Lemma 5.3.6.
If K is convex with $0 \in K$, then $K' \subset (k+1)^2 K$, so

$$\|\mu\|_{F;K'} = \sup_{g\in K'} |\mu(g)|_F \leq \sup_{g \in (k+1)^2 K} |\mu(g)|_F \leq ((k+1)^2)^k \sup_{g\in K} |\mu(g)| = (k+1)^{2k}\|\mu\|_{F;K},$$

i.e. we can choose $C_k' := (k+1)^{2k} C_k$. $\square$

5.4 Goodey-Weil embedding

5.4.1 Construction and basic properties

In this section, we will assume that V carries a Euclidean structure. Let $C_b^2(V)$ denote the Banach space of twice differentiable functions with bounded C^2-norm

$$\|\phi\|_{C_b^2(V)} := \|\phi\|_\infty + \|\nabla\phi\|_\infty + \|H_\phi\|_\infty = \sup_{x\in V} |\phi(x)| + \sup_{x \in V} |\nabla\phi(x)| + \sup_{x\in V, v\in S(V)} |\langle H_\phi(x)v, v\rangle|.$$

Let us also set $c(A) := \sup_{x\in A} \frac{|x|^2}{2} + 1$ for any compact subset $A \subset V$.

Lemma 5.4.1. *For every $\phi \in C^2_b(V)$ there exist two convex functions $f, h \in \mathrm{Conv}(V,\mathbb{R})$ such that $f - h = \phi$ and such that $\|f|_A\|_\infty, \|h|_A\|_\infty \leq c(A)\|\phi\|_{C^2_b(V)}$ for all compact subsets $A \subset V$. These functions can be chosen in $C^\infty(V)$.*

Proof. Take $f(x) := c\frac{|x|^2}{2} + \phi(x)$, $h(x) = c\frac{|x|^2}{2}$, where $c := \|\phi\|_{C^2_b(V)}$. Then f and h are convex, as their Hessians are positive semi-definite. In addition

$$\|h|_A\|_\infty, \|f|_A\|_\infty \leq c \cdot \sup_{x\in A} \frac{|x|^2}{2} + \|\phi\|_\infty \leq \left(\sup_{x\in A}\frac{|x|^2}{2} + 1\right)\|\phi\|_{C^2_b(V)} = c(A)\|\phi\|_{C^2_b(V)}.$$

□

To every $\mu \in \mathrm{VConv}_k(C;V,F)$ we can associate a k-multilinear functional $\tilde{\mu}$ on formal differences of convex functions: Assume that $h_1 + \phi_1 = f_1, \dots, h_1 + \phi_k = f_k$ for convex functions $f_1, \dots, f_k, h_1, \dots, h_k \in C$. Using the polarization $\bar{\mu}$ from Theorem 5.3.9, we define the following functionals inductively for arbitrary convex functions $g_1, \dots, g_k \in \mathrm{Conv}(V,\mathbb{R})$ and $1 \leq i \leq k-1$:

$$\begin{aligned}
\mu^{(1)}(\phi_1, g_2, \dots, g_k) :=&\bar{\mu}(f_1, g_2, \dots, g_k) - \bar{\mu}(h_1, g_2, \dots, g_k),\\
\mu^{(i+1)}(\phi_1, \dots, \phi_{i+1}, g_{i+2}, \dots, g_k) :=&\mu^{(i)}(\phi_1, \dots, \phi_i, f_{i+1}, g_{i+1}, \dots, g_k)\\
&- \mu^{(i)}(\phi_1, \dots, \phi_i, h_{i+1}, g_{i+1}, \dots, g_k).
\end{aligned}$$

Then we set $\tilde{\mu}(\phi_1, \dots, \phi_k) := \mu^{(k)}(\phi_1, \dots, \phi_k)$. It is easy to see that this is equivalent to

$$\tilde{\mu}(\phi_1, .., \phi_k) = \sum_{l=0}^{k} (-1)^{k-l} \frac{1}{(k-l)!l!} \sum_{\sigma \in S_k} \bar{\mu}(f_{\sigma(1)}, \dots, f_{\sigma(l)}, h_{\sigma(l+1)}, \dots, h_{\sigma(k)}), \tag{5.2}$$

where $\bar{\mu}$ denotes the polarization of μ from Theorem 5.3.9. Thus $\tilde{\mu}$ is also symmetric. Using the additivity of $\bar{\mu}$ in each argument, one readily verifies that this definition only depends on the functions $\phi_1, \dots, \phi_k$ (and not the special choices of f_i and h_i) and that this functional is multilinear.

By Lemma 5.4.1, $C^2_c(V)$ is contained in the space generated by differences of elements of $\mathrm{Conv}(V,\mathbb{R})$. For the construction of the Goodey-Weil embedding, we will thus consider the restricted map

$$\tilde{\mu} : C^2_b(V)^k \to F.$$

Given functions $\phi_1, \dots, \phi_k \in C^2_b(V)$, we take the special convex functions $f_1, \dots, f_k$, $h_1, \dots, h_k$ in $\mathrm{Conv}(V,\mathbb{R})$ with $\phi_i = f_i - h_i$ satisfying the inequality in Lemma 5.4.1. The set K of all convex functions that are bounded by $c(A)$ on every compact set A (as defined above) is compact in $\mathrm{Conv}(V,\mathbb{R})$ by Proposition 4.2.2, so it is also compact in C. Note that the functions

$$\tilde{f}_i := \frac{f_i}{\|\phi_i\|_{C^2_b(V)}}, \qquad \tilde{h}_i := \frac{h_i}{\|\phi_i\|_{C^2_b(V)}}$$

belong to K by construction. As K is also convex with $0 \in K$, Lemma 5.3.12 and Equation (5.2) imply

$$\begin{aligned}
|\tilde{\mu}(\phi_1,\dots,\phi_k)|_F &= |\sum_{l=0}^{k}(-1)^{k-l}\frac{1}{(k-l)!l!}\sum_{\sigma\in S_k}\bar{\mu}(f_{\sigma(1)},\dots,f_{\sigma(l)},h_{\sigma(l+1)},\dots,h_{\sigma(k)})|_F\\
&\leq \sum_{l=0}^{k}\frac{1}{(k-l)!l!}\sum_{\sigma\in S_k}|\bar{\mu}(f_{\sigma(1)},\dots,f_{\sigma(l)},h_{\sigma(l+1)},\dots,h_{\sigma(k)})|_F\\
&= \sum_{l=0}^{k}\frac{1}{(k-l)!l!}\sum_{\sigma\in S_k}|\bar{\mu}(\tilde{f}_{\sigma(1)},\dots,\tilde{f}_{\sigma(l)},\tilde{h}_{\sigma(l+1)},\dots,\tilde{h}_{\sigma(k)})|_F\cdot\prod_{i=1}^{k}\|\phi_i\|_{C^2_b(V)}\\
&\leq c_k\|\mu\|_{F;K}\cdot\prod_{i=1}^{k}\|\phi_i\|_{C^2_b(V)} \qquad (5.3)
\end{aligned}$$

for any continuous semi-norm $|\cdot|_F$ on F for some constant $c_k > 0$ depending on k only (we can choose 2^k times the constant C'_k from Lemma 5.3.12). Here we have used that $\bar{\mu}$ is 1-homogeneous in each argument. As $\tilde{\mu}$ is multilinear, this inequality implies that $\tilde{\mu}$ is continuous. In particular, we can apply the L. Schwartz kernel theorem 2.2.1.

Definition 5.4.2. *Let F be a locally convex vector space, $1 \leq k \leq n$. To every $\mu \in \mathrm{VConv}_k(C;V,F)$ we associate the distribution $\overline{\mathrm{GW}}(\mu) \in \mathcal{D}'(V^k,\bar{F})$ determined by*

$$\overline{\mathrm{GW}}(\mu)(\phi_1\otimes\dots\otimes\phi_k) = \tilde{\mu}(\phi_1,\dots,\phi_k)$$

for $\phi_1,\dots,\phi_k \in C^\infty_c(V)$. This distribution will be called the Goodey-Weil distribution *of μ.*

Corollary 5.4.3. *The map $\overline{\mathrm{GW}} : \mathrm{VConv}_k(C;V,F) \to \mathcal{D}'(V^k,\bar{F})$, that maps each valuation to its Goodey-Weil distribution, is continuous.*

Proof. From Inequality (5.3) we deduce that the map that associates to any valuation $\mu \in \mathrm{VConv}_k(C;V,F)$ its multilinear extension $\tilde{\mu}$ is a continuous linear map from $\mathrm{VConv}_k(C;V,F)$ into the space of continuous, $\bar{F}$-valued, k-multilinear functionals on $C^\infty_c(V)$. By Theorem 2.2.4, the L. Schwartz kernel theorem establishes a topological isomorphism between this space and $\mathcal{D}'(V^k,\bar{F})$. Thus $\overline{\mathrm{GW}}$ is continuous. □

Before we discuss the Goodey-Weil distributions of some of the examples from Chapter 1, let us show a simple way to calculate these distributions.

Lemma 5.4.4. *Let $\mu \in \mathrm{VConv}_k(C;V,F)$ and let $f \in \mathrm{Conv}(V,\mathbb{R})$ be a strictly convex function. If $\phi_1,\dots,\phi_k \in C^\infty_c(V)$, then $\mu(f+\sum_{i=1}^k\delta_i\phi_i)$ is a polynomial in δ_i for all δ_i small enough and*

$$\overline{\mathrm{GW}}(\mu)[\phi_1\otimes\dots\otimes\phi_k] = \frac{1}{k!}\frac{\partial}{\partial\delta_1}\Big|_0\dots\frac{\partial}{\partial\delta_k}\Big|_0\mu\left(f+\sum_{i=1}^k\delta_i\phi_i\right).$$

Proof. Note that the strict convexity implies that $f+\sum_{i=1}^k \delta_i\phi_i$ is a convex function for all δ_i sufficiently small. Let us consider the multilinear functional $\tilde{\mu}$ from section 5.4.1. The construction of $\tilde{\mu}$ implies $\tilde{\mu}(h,\dots,h)=\mu(h)$ for $h\in\mathrm{Conv}(V,\mathbb{R})$, so

$$\mu\left(f+\sum_{i=1}^k \delta_i\phi_i\right)=\tilde{\mu}\left(f+\sum_{i=1}^k \delta_i\phi_i,\dots,f+\sum_{i=1}^k \delta_i\phi_i\right).$$

In particular, the left hand side is a polynomial in $\delta_i>0$ for all δ_i small enough and we calculate

$$\begin{aligned}\frac{1}{k!}\frac{\partial}{\partial\delta_1}\Big|_0\cdots\frac{\partial}{\partial\delta_k}\Big|_0\mu\left(f+\sum_{i=1}^k \delta_i\phi_i\right)=&\frac{1}{k!}\frac{\partial}{\partial\delta_1}\Big|_0\cdots\frac{\partial}{\partial\delta_k}\Big|_0\tilde{\mu}\left(f+\sum_{i=1}^k \delta_i\phi_i,\dots,f+\sum_{i=1}^k \delta_i\phi_i\right)\\ =&\tilde{\mu}(\phi_1,\dots,\phi_k)\\ =&\overline{\mathrm{GW}}(\mu)[\phi_1\otimes\dots\otimes\phi_k],\end{aligned}$$

where we have used that $\tilde{\mu}$ is multilinear and symmetric. □

Example 5.4.5. *Consider the Hessian measure* Hess_n *from Example 1.0.6, which belongs to* $\mathrm{VConv}_n(V,\mathcal{M}(V))$. *Given* $\phi_1,\dots,\phi_n\in C_c^\infty(V)$, *its Goodey-Weil distribution is given by*

$$\overline{\mathrm{GW}}(\mathrm{Hess}_n)[\phi_1\otimes\dots\otimes\phi_n](U)=\int_U \det(H_{\phi_1}(x),\dots,H_{\phi_n}(x))dx$$

for all Borel sets $U\subset V$. *In particular, the support of* $\overline{\mathrm{GW}}(\mathrm{Hess}_n)$ *is non-compact and coincides with the diagonal in* V^n.

Example 5.4.6. *Let* $A_1,\dots,A_{n-k}\in C_c(V,\mathcal{H}(V))$ *be functions with values in the space* $\mathcal{H}(V)$ *of symmetric endomorphisms with respect to a fixed Euclidean structure,* $B\in C_c(V)$. *The Alesker valuation* μ *extending*

$$\begin{aligned}\mathrm{Conv}(V,\mathbb{R})\cap C^2(V)&\to\mathbb{R}\\ f&\mapsto\int_V B(x)\det(H_f(x)[k],A_1(x),\dots,A_{n-k}(x))dx\end{aligned}$$

belongs to $\mathrm{VConv}_k(V)$. *Its Goodey-Weil distribution is given by*

$$\overline{\mathrm{GW}}(\mu)[\phi_1\otimes\dots,\otimes\phi_k]=\int_V B(x)\det(H_{\phi_1}(x),\dots,H_{\phi_k}(x),A_1(x),\dots,A_{n-k}(x))dx.$$

The support of this distribution is contained in the diagonal in V^k, *but this time the support is compact - it is contained in the intersection of the supports of the functions* $B,A_1,\dots,A_k$.

The diagonality of the support of the Goodey-Weil distributions is indeed a general property of these distributions and we will discuss this result in the next section. It turns out that the compactness of the support only depends on the topology of F: If F admits a continuous norm, the support of a Goodey-Weil distribution is always compact. Note that this is consistent with the first example. We will address this question in Section 5.4.3

5.4.2 Diagonality of the support of the Goodey-Weil distributions

The following theorem is a version of the proof of the corresponding statement for the Goodey-Weil embedding for translation invariant valuations on convex bodies, compare [1].

Theorem 5.4.7. *Let F be a locally convex vector space. For $\mu \in \mathrm{VConv}_k(C;V,F)$ the support of $\overline{\mathrm{GW}}(\mu)$ is contained in the diagonal in V^k.*

Proof. Let us assume that V carries a Euclidean structure. Using a partition of unity, it is sufficient to show that $\overline{\mathrm{GW}}(\mu)(h_1 \otimes \dots \otimes h_k) = 0$ if $h_1, \dots, h_k \in C_c^\infty(V)$ are smooth functions satisfying $\operatorname{supp} h_i \subset U_\epsilon(a_i)$, where $a_i \in V$, $1 \le i \le k$, are points with $U_\epsilon(a_i) \cap U_\epsilon(a_j) = \emptyset$ for $i \neq j$ and some fixed $\epsilon > 0$.
First, there exists $\delta > 0$ such that the function $x \mapsto 1 + |x|^2 + \sum_{i=1}^k \delta_i h_i$ is convex and non-negative for all $|\delta_i| \le \delta$. Set

$$H(x) := 1 + |x|^2 + \sum_{i=3}^k \delta_i h_i$$

and choose an affine hyperplane that separates $U_\epsilon(a_1)$ and $U_\epsilon(a_2)$. This plane is given by the equation $\langle y, x - x_0 \rangle = 0$ for some $y, x_0 \in V$. We can choose y such that $U_\epsilon(a_1)$ is contained in the positive half space with respect to the normal y. Define the convex functions

$$G_\pm(x) := \max(0, \pm\langle y, x - x_0\rangle) = \begin{cases} 0 & \pm\langle y, x - x_0\rangle \le 0, \\ \pm\langle y, x - x_0\rangle & \pm\langle y, x - x_0\rangle > 0. \end{cases}$$

As $G_\pm$ is positive on the supports of h_1 and h_2 respectively, we can rescale y such that G_+ is larger than $H + \delta_1 h_1$ on the support of h_1 and G_- is larger than $H + \delta_2 h_2$ on the support of h_2 for all $|\delta_i| \le \delta$. Now set $\tilde{H}_+ := \max(H + \delta_1 h_1, G_+)$ and $\tilde{H}_- := \max(H + \delta_1 h_1, G_-)$. Then $\tilde{H}_+$ and $\tilde{H}_-$ are convex functions with

$$\tilde{H}_+(x) = \begin{cases} H(x) + \delta_1 h_1(x) & \langle y, x - x_0\rangle \le 0, \\ \max(H(x) + \delta_1 h_1(x), \langle y, x - x_0\rangle) & \langle y, x - x_0\rangle > 0, \end{cases}$$

$$\tilde{H}_-(x) = \begin{cases} H(x) + \delta_1 h_1(x) & \langle y, x - x_0\rangle \ge 0, \\ \max(H(x) + \delta_1 h_1(x), -\langle y, x - x_0\rangle) & \langle y, x - x_0\rangle < 0. \end{cases}$$

In particular $\min(\tilde{H}_+, \tilde{H}_-) = \tilde{H} := H + \delta_1 h_1$ is convex. As the support of h_1 is contained in the positive half space with respect to y, we see that in fact

$$\tilde{H}_+(x) = \begin{cases} H(x) & \langle y, x - x_0\rangle \le 0, \\ \max(H(x), \langle y, x - x_0\rangle) & \langle y, x - x_0\rangle > 0, \end{cases}$$

$$\tilde{H}_-(x) = \begin{cases} H(x) + \delta_1 h_1(x) & \langle y, x - x_0\rangle \ge 0, \\ \max(H(x), -\langle y, x - x_0\rangle) & \langle y, x - x_0\rangle < 0. \end{cases}$$

Thus $\max(\tilde{H}_+, \tilde{H}_-) = \max(H, |\langle y, \cdot - x_0\rangle|)$.

Let us also define $H_\pm := \max(H, G_\pm)$. Then $H_+ = \tilde{H}_+$, and, using the non-negativity of H, it is easy to see that $\min(H_+, H_-) = H$ and $\max(H_+, H_-) = \max(H, |\langle y, \cdot - x_0\rangle|) = \max(\tilde{H}_+, \tilde{H}_-)$.
The valuations property implies

$$\begin{aligned}\mu(\tilde{H}_+) + \mu(\tilde{H}_-) &= \mu(\max(\tilde{H}_+, \tilde{H}_-)) + \mu(\min(\tilde{H}_+, \tilde{H}_-)),\\ \mu(H_+) + \mu(H_-) &= \mu(\max(H_+, H_-)) + \mu(\min(H_+, H_-)).\end{aligned}$$

Thus using $\max(H_+, H_-) = \max(\tilde{H}_+, \tilde{H}_-)$, and $\tilde{H}_+ = H_+$, we obtain

$$\mu(\tilde{H}_-) - \mu(H_-) = \mu(\min(\tilde{H}_+, \tilde{H}_-)) - \mu(\min(H_+, H_-))$$

by subtracting the two equations. Plugging in the relations for the minima, we arrive at

$$\mu(\tilde{H}_-) - \mu(H_-) = \mu(\tilde{H}) - \mu(H).$$

Set

$$\begin{aligned}\Delta(x) :=& H_-(x) = \begin{cases} H(x) & \langle y, x - x_0\rangle \geq 0,\\ \max(H(x), -\langle y, x - x_0\rangle) & \langle y, x - x_0\rangle < 0,\end{cases}\\ \tilde{\Delta}(x) :=& \tilde{H}_-(x) = \begin{cases} H(x) + \delta_1 h_1(x) & \langle y, x - x_0\rangle \geq 0,\\ \max(H(x), -\langle y, x - x_0\rangle) & \langle y, x - x_0\rangle < 0,\end{cases}\end{aligned}$$

to rewrite the previous equation as

$$\mu(\tilde{\Delta}) - \mu(\Delta) = \mu(\tilde{H}) - \mu(H) = \mu(H + \delta_1 h_1) - \mu(H).$$

Now, if we replace H by $H + \delta_2 h_2$ and repeat the argument, the convex functions Δ' and $\tilde{\Delta}'$ defined by

$$\begin{aligned}\Delta'(x) &= \begin{cases} H(x) + \delta_2 h_2(x) & \langle y, x - x_0\rangle \geq 0,\\ \max(H(x) + \delta_2 h(x), -\langle y, x - x_0\rangle) & \langle y, x - x_0\rangle < 0,\end{cases}\\ \tilde{\Delta}'(x) &= \begin{cases} H(x) + \delta_1 h_1(x) + \delta_2 h_2(x) & \langle y, x - x_0\rangle \geq 0,\\ \max(H(x) + \delta_2 h_2(x), -\langle y, x - x_0\rangle) & \langle y, x - x_0\rangle < 0,\end{cases}\end{aligned}$$

satisfy

$$\mu(\tilde{\Delta}') - \mu(\Delta') = \mu(H + \delta_1 h_1 + \delta_2 h_2) - \mu(H + \delta_2 h_2).$$

However, the support of h_2 is contained in the negative half space and $H + \delta_2 h_2$ is smaller than $-\langle y, \cdot - x_0\rangle$ on the support of h_2. Thus $\Delta' = \Delta$ and $\tilde{\Delta}' = \tilde{\Delta}$, and we obtain the equation

$$\mu(H + \delta_1 h_1) - \mu(H) = \mu(\tilde{\Delta}) - \mu(\Delta) = \mu(H + \delta_1 h_1 + \delta_2 h_2) - \mu(H + \delta_2 h_2)$$

for all δ_i with $|\delta_i| < \delta$. Both the left and the right hand side are polynomials in δ_i for all δ_i small enough by Lemma 5.4.4, but the left hand side is independent of δ_2. Thus Lemma 5.4.4 implies

$$\begin{aligned}
0 =& \frac{1}{k!}\frac{\partial}{\partial \delta_1}\Big|_0 \cdots \frac{\partial}{\partial \delta_k}\Big|_0 [\mu(H + \delta_1 h_1 + \delta_2 h_2) - \mu(H + \delta_2 h_2)] \\
=& \frac{1}{k!}\frac{\partial}{\partial \delta_1}\Big|_0 \cdots \frac{\partial}{\partial \delta_k}\Big|_0 \mu(H + \delta_1 h_1 + \delta_2 h_2) \\
=& \overline{\mathrm{GW}}(\mu)[h_1 \otimes \cdots \otimes h_k].
\end{aligned}$$

□

5.4.3 Compactness of the support

Let us start by considering the case $F = \mathbb{R}$. We have constructed the Goodey-Weil distribution of $\mu \in \mathrm{VConv}_k(C;V)$ as the unique continuous lift of the functional $\tilde{\mu} : C_c^\infty(V)^k \to \mathbb{R}$ to a linear functional on $C_c^\infty(V^k)$. However, $\tilde{\mu}$ is actually defined on arbitrary differences of convex functions, which includes all smooth functions, as shown below. As distributions with compact support are in natural duality with smooth functions, this suggests that the Goodey-Weil distributions should have compact support. However, in contrast to the restriction of $\tilde{\mu}$ to compactly supported smooth functions, we do not have estimates controlling the continuity of $\tilde{\mu}$ with respect to the natural topology on the space of all smooth functions, i.e. the topology of locally uniform convergence of all derivatives. On the other hand, $\overline{\mathrm{GW}}(\mu)$ is given by a very simple combinatorial formula involving only the polarization of μ on products of smooth functions. This polarization is continuous with respect to locally uniform convergence of convex functions, which is much more flexible than locally uniform convergence of all derivatives.
To show that the support is compact, we will thus argue by contradiction, using the diagonality of the support as well as the special formula for products of smooth functions. For this, we need the following lemma.

Lemma 5.4.8. *For every $\phi \in C^2(V)$ there exists a convex function $h \in \mathrm{Conv}(V, \mathbb{R})$ with the following property: If $\psi \in C^\infty(V)$ is a function with $\|\psi\|_{C^2(B_j)} \leq \|\phi\|_{C^2(B_j)}$ for all $j \in \mathbb{N}$, then $h + \psi$ is convex.*

Proof. Assume that we are given ϕ and let ψ be an arbitrary function with the property stated above. Let us inductively define a sequence of convex functions $h_j \in \mathrm{Conv}(V, \mathbb{R})$. Set $c_j := \|\phi\|_{C^2(B_{j+1})}$. Then $c_j \geq \sup_{x \in B_{j+1}, v \in S(V)} \langle H_\psi(x)v, v\rangle$ for all $j \in \mathbb{N}$.
For $j = 1$ define h_1 by $h_1(x) := c_1 \frac{|x|^2}{2}$. As its Hessian is positive semi-definite, $h_1 + \psi$ is convex on B_2.
Assume that we have already constructed h_j. Then the Hessian of $c_{j+1}\frac{|x|^2}{2} + \psi$ is positive semi-definite on B_{j+2}, so this function is convex on B_{j+2}. We set

$$h_{j+1}(x) := \max\left(c_{j+1}\frac{|x|^2 - j^2}{2}, 0\right) + h_j(x).$$

Thus h_{j+1} is a finite-valued convex function for all $j \in \mathbb{N}$, that coincides with h_j on B_j. We deduce that for each point $x \in V$ the sequence $(h_j(x))_j$ becomes constant for large j and thus $(h_j)_j$ converges pointwise to a function $h \in \mathrm{Conv}(V,\mathbb{R})$. By Proposition 4.1.2, this implies that this sequence epi-converges to h.
It remains to see that $h+\psi$ is convex. Observe that for every point $x \in V$ there exists an open, convex neighborhood such that the restriction of $h+\psi$ to this neighborhood is convex, i.e. $h+\psi$ is a locally convex function: On $U_{j+1} \setminus B_{j-1}$

$$\begin{aligned} h(x)+\psi(x) =& \max\left(c_{j+1}\frac{|x|^2-j^2}{2},0\right)+h_j(x)+\psi(x)\\ =& \max\left(c_{j+1}\frac{|x|^2-j^2}{2},0\right)+h_{j-1}(x)+c_j\frac{|x|^2-(j-1)^2}{2}+\psi(x), \end{aligned}$$

where $c_j\frac{|x|^2-(j-1)^2}{2}+\psi(x)$ is locally convex on this set, as its Hessian is positive semi-definite. Obviously, the other two functions are locally convex on this set as well, so the same applies to $h+\psi$.
As any locally convex function defined on V is convex, $h+\psi \in \mathrm{Conv}(V,\mathbb{R})$. □

Theorem 5.4.9. *Let F be a locally convex vector space admitting a continuous norm. For every $\mu \in \mathrm{VConv}_k(C;V,F)$ the distribution $\overline{\mathrm{GW}}(\mu) \in \mathcal{D}'(V^k,\bar{F})$ has compact support and is uniquely determined by the following property: If $f_1,\dots,f_k \in \mathrm{Conv}(V,\mathbb{R}) \cap C^\infty(V)$, then*

$$\overline{\mathrm{GW}}(\mu)(f_1\otimes\dots\otimes f_k)=\bar{\mu}(f_1,\dots,f_k), \tag{5.4}$$

where $\bar{\mu}$ denotes the polarization of μ.

Proof. Uniqueness follows directly from Equation (5.4), as every function $\phi \in C^\infty_c(V)$ can be written as a difference of two smooth convex functions due to Lemma 5.4.1 and a distribution on V^k is uniquely determined by its values on functions of the form $\phi_1\otimes\dots\otimes\phi_k$ for $\phi_1,\dots,\phi_k \in C^\infty_c(V)$ by the L. Schwartz kernel theorem.
Let us assume that $\overline{\mathrm{GW}}(\mu)$ does not have compact support and let $\|\cdot\|$ denote a continuous norm on F. Then we can inductively define a sequence of functions $(\phi_i^j)_j$ in $C^\infty_c(V)$ for each $1 \le i \le k$ and a strictly increasing sequence $(r_j)_j$ of positive real numbers with $\lim\limits_{j\to\infty} r_j=\infty$ with the following properties:

1. For each $1 \le i \le k$ the functions $(\phi_i^j)_j$ have pairwise disjoint support.
2. The support of ϕ_i^j is contained in $V \setminus B_{r_j}(0)$ for all $j \in \mathbb{N}$, $1 \le i \le k$.
3. $\|\overline{\mathrm{GW}}(\mu)(\phi_1^j\otimes\dots\otimes\phi_k^j)\| = \|\tilde{\mu}(\phi_1^j,\dots,\phi_k^j)\| = 1$.

To see this, assume that we have constructed $\phi_1^j,\dots,\phi_k^j$ as well as $r_j>0$. First choose $r_{j+1}>r_j+1$ such that the restriction of $\overline{\mathrm{GW}}(\mu)$ to $[U_{r_{j+1}}\setminus B_{r_j}]^k \subset V^k$ does not vanish. This is possible, as the support of $\overline{\mathrm{GW}}(\mu)$ is contained in the diagonal due to Theorem

5.4.7. Then take $\phi_1^{j+1},\dots,\phi_k^{j+1}\in C^\infty(U_{r_{j+1}}\setminus B_{r_j})$ with $\overline{\mathrm{GW}}(\mu)[\phi_1^{j+1}\otimes\dots\otimes\phi_k^{j+1}]\neq 0$ and rescale one function by an appropriate constant.
Note that $\phi_i:=\sum_{j=1}^\infty \phi_i^j\in C^\infty(V)$ is well defined as this sum is locally finite. More precisely, the supports of the functions ϕ_i^j are disjoint for each $1\le i\le k$, so $\|\phi_i^j\|_{C^2(B_N)}\le \|\phi_i\|_{C^2(B_N)}$ for all $N\in\mathbb{N}$ and $j\in\mathbb{N}$. Applying Lemma 5.4.8 to the functions ϕ_i, we find convex functions $h_1,\dots,h_k\in\mathrm{Conv}(V,\mathbb{R})$ such that for $1\le i\le k$ the function $f_i^j:=h_i+\phi_i^j$ is convex for all $j\in\mathbb{N}$. By Equation (5.2)

$$\tilde{\mu}(\phi_1^j,\dots,\phi_k^j)=\sum_{l=0}^k(-1)^{k-l}\frac{1}{(k-l)!l!}\sum_{\sigma\in S_k}\bar{\mu}\left(f^j_{\sigma(1)},\dots,f^j_{\sigma(l)},h_{\sigma(l+1)},\dots,h_{\sigma(k)}\right).$$

As $f_i^j\to h_i$ uniformly on compact subsets for all $1\le i\le k$, the joint continuity of the polarization $\bar{\mu}$ from Corollary 5.3.11 and the continuity of the norm $\|\cdot\|$ imply

$$\begin{aligned}\lim_{j\to\infty}\|\tilde{\mu}(\phi_1^j,\dots,\phi_k^j)\|=&\|\sum_{l=0}^k(-1)^{k-l}\frac{1}{(k-l)!l!}\sum_{\sigma\in S_k}\bar{\mu}\left(h_{\sigma(1)},\dots,h_{\sigma(l)},h_{\sigma(l+1)},\dots,h_{\sigma(k)}\right)\|\\ =&\|(-1)^k\sum_{l=0}^k(-1)^l\frac{k!}{(k-l)!l!}\bar{\mu}(h_1,\dots,h_k)\|\\ =&\|(-1)^k\bar{\mu}(h_1,\dots,h_k)\sum_{l=0}^k(-1)^l\binom{k}{l}\|\\ =&\|\bar{\mu}(h_1,\dots,h_k)\|\cdot 0=0.\end{aligned}$$

We arrive at a contradiction to $\|\tilde{\mu}(\phi_1^j,\dots,\phi_k^j)\|=1$ for all $j\in\mathbb{N}$, so the distribution $\overline{\mathrm{GW}}(\mu)$ must have compact support.
It remains to see that $\overline{\mathrm{GW}}(\mu)(f_1\otimes\dots\otimes f_k)=\bar{\mu}(f_1,\dots,f_k)$ for all convex functions $f_i\in\mathrm{Conv}(V,\mathbb{R})\cap C^\infty(V)$. Take a sequence of functions $\phi_j\in C_c^\infty(V)$ with $\phi_j\equiv 1$ on $B_j(0)$ and such that $\|\phi_j\|_{C^2(V)}\le C$ for all $j\in\mathbb{N}$ and some $C>0$. Such a sequence can be constructed by setting $\phi_j(x):=\psi(\frac{x}{j})$ for $\psi\in C_c^\infty(V)$ with $\psi\equiv 1$ on $B_1(0)$. As the support of $\overline{\mathrm{GW}}(\mu)$ is compact, we obtain $N\in\mathbb{N}$ such that

$$\overline{\mathrm{GW}}(\mu)(f_1\otimes\dots\otimes f_k)=\overline{\mathrm{GW}}(\mu)(\phi_jf_1\otimes\dots\otimes\phi_jf_k)\quad\forall j\ge N.$$

Using the Leibniz-rule, we see that there exists $C'>0$ such that for any compact set $A\subset V$ the inequality $\|\phi_jf_i\|_{C^2(A)}\le C'\|f_i\|_{C^2(A)}$ holds for all $j\in\mathbb{N}$. Now take the function h_i from Lemma 5.4.8 for the function $\phi=C'f_i$. Then $h_i+\phi_jf_i$ is convex for all $j\in\mathbb{N}$ and $h_i+\phi_jf_i$ converges to h_i+f_i uniformly on compact subsets, i.e. in $\mathrm{Conv}(V,\mathbb{R})$. Plugging in the definition of $\tilde{\mu}$ and using the joint continuity of the polarization $\bar{\mu}$, we obtain

$$\begin{aligned}
&\overline{\mathrm{GW}}(\mu)(f_1\otimes\cdots\otimes f_k) = \lim_{j\to\infty}\overline{\mathrm{GW}}(\mu)(\phi_j f_1\otimes\cdots\otimes\phi_j f_k)\\
=&\lim_{j\to\infty}\sum_{l=0}^k(-1)^{k-l}\frac{1}{(k-l)!l!}\sum_{\sigma\in S_k}\bar{\mu}(h_{\sigma(1)}+\phi_j f_{\sigma(1)},\dots,h_{\sigma(l)}+\phi_j f_{\sigma(l)},h_{\sigma(l+1)},\dots,h_{\sigma(k)})\\
=&\sum_{l=0}^k(-1)^{k-l}\frac{1}{(k-l)!l!}\sum_{\sigma\in S_k}\bar{\mu}(h_{\sigma(1)}+f_{\sigma(1)},\dots,h_{\sigma(l)}+f_{\sigma(l)},h_{\sigma(l+1)},\dots,h_{\sigma(k)})\\
=&\bar{\mu}(f_1,\dots,f_k),
\end{aligned}$$

where we have used the additivity of $\bar{\mu}$ in the last step. □

Corollary 5.4.10. $\overline{\mathrm{GW}}:\mathrm{VConv}_k(C;V,F)\to\mathcal{D}'(V^k,\bar{F})$ *is injective.*

Proof. Assume first that F admits a continuous norm. As $\mathrm{Conv}(V,\mathbb{R})\cap C^\infty(V)$ is dense in C due to Proposition 4.4.1, the claim follows from Theorem 5.4.9 and the continuity of μ.
If F is an arbitrary locally convex vector space, then the definition of $\overline{\mathrm{GW}}$ implies

$$\lambda\circ\overline{\mathrm{GW}}(\mu)=\overline{\mathrm{GW}}(\lambda\circ\mu)\quad\forall\lambda\in\bar{F}'\cong F',$$

where F' denotes the topological dual of F. In particular, $\overline{\mathrm{GW}}(\mu)=0$ if and only if $\overline{\mathrm{GW}}(\lambda\circ\mu)=0$ for all $\lambda\in F'$. By the previous discussion, $\overline{\mathrm{GW}}(\lambda\circ\mu)=0$ implies $\lambda\circ\mu=0$. If this holds for all $\lambda\in F'$, we obtain $\mu=0$, as F is locally convex. □

6 The support of a dually epi-translation invariant valuation

Similar to Section 3.1, the support of the Goodey-Weil distribution of a dually epi-translation invariant valuation can be used to define a notion of support for the underlying valuation. After establishing an intrinsic characterization of the support in Section 6.1, we consider spaces of valuations with compact support in Section 6.2 and define families of continuous semi-norms, which induce the subspace topology on these spaces. In Section 6.3 we will see some applications of this concept: We start by characterizing the image of the embedding from Section 5.2, and show some triviality results for valuations that are invariant under non-compact subgroups of the affine group. Lastly, we discuss how the cone on which the valuation is defined affects its support. This gives a partial answer to the question which valuations on finite-valued convex functions can be extended to larger cones of convex functions.

The results of this chapter are to be published in [35].

6.1 Definition and characterization of the support

Throughout this section, let F be a locally convex vector space. As in Section 3.1, the properties of the Goodey-Weil distributions suggest the following notion of support for dually epi-translation invariant valuations on convex functions.

Definition 6.1.1. *For $1 \leq k \leq n$ and $\mu \in \mathrm{VConv}_k(C;V,F)$, the support $\operatorname{supp}\mu \subset V$ of μ is the set*

$$\operatorname{supp}\mu := \bigcap_{\substack{A \subset V \text{ closed,} \\ \operatorname{supp}\overline{\mathrm{GW}}(\mu) \subset \Delta A}} A.$$

Here, $\Delta : V \to V^k$ is the diagonal embedding. For $\mu \in \mathrm{VConv}_0(C;V,F)$ we set $\operatorname{supp}\mu = \emptyset$. If $\mu = \sum_{i=0}^n \mu_i$ is the homogeneous decomposition of $\mu \in \mathrm{VConv}(C;V,F)$, we set $\operatorname{supp}\mu := \bigcup_{i=0}^n \operatorname{supp}\mu_i$.

Theorem 5.4.9 implies

Corollary 6.1.2. *If F admits a continuous norm, then any $\mu \in \mathrm{VConv}(C;V,F)$ has compact support.*

Similar to the vertical support of a translation invariant valuation on convex bodies, the support of a dually epi-translation invariant valuation can be characterized without reference to its Goodey-Weil distribution.

Proposition 6.1.3. *The support of* $\mu \in \mathrm{VConv}(C; V, F)$ *is minimal (with respect to inclusion) amongst the closed sets* $A \subset V$ *with the following property: If* $f, g \in \mathrm{Conv}(V, \mathbb{R})$ *satisfy* $f = g$ *on an open neighborhood of* A*, then* $\mu(f) = \mu(g)$*.*

Proof. Let us first show that any closed set A satisfying the property contains the support of μ. Using the homogeneous decomposition, we can assume that μ is k-homogeneous. We will argue by contradiction. Assume that the support of μ is not contained in A. Then $\operatorname{supp} \overline{\mathrm{GW}}(\mu) \setminus \Delta A \neq \emptyset$. In particular, we find functions $\phi_1, ..., \phi_k$ with support in $V \setminus A$ such that

$$\overline{\mathrm{GW}}(\mu)(\phi_1 \otimes ... \otimes \phi_k) \neq 0.$$

Using the Hahn-Banach theorem, we can choose $\lambda \in F'$ with

$$\lambda(\overline{\mathrm{GW}}(\mu)(\phi_1 \otimes \cdots \otimes \phi_k)) \neq 0.$$

Choose some Euclidean structure on V and let $f \in \mathrm{Conv}(V, \mathbb{R})$ be given by $f(x) := |x|^2$. Then $f + \sum_{i=1}^k \delta_i \phi_i$ is convex for all δ_i small enough. Let us compare $\lambda(\mu(f))$ and $\lambda(\mu(f + \sum_{i=1}^k \delta_i \phi_i))$. By construction, the two functions coincide on an open neighborhood of A and thus the assumption implies $\lambda(\mu(f)) = \lambda(\mu(f + \sum_{i=1}^k \delta_i \phi_i))$ for all δ_i small enough. Applying Theorem 5.4.9, we see that the right hand side is a polynomial in δ_i for all δ_i small enough and that the coefficient in front of $\delta_1 \dots \delta_k$ is exactly $k! \, \overline{\mathrm{GW}}(\lambda \circ \mu)(\phi_1 \otimes \cdots \otimes \phi_k) = k! \lambda(\overline{\mathrm{GW}}(\mu)(\phi_1 \otimes ... \otimes \phi_k))$. As the left hand side is independent of δ_i, this coefficient has to vanish, so we obtain a contradiction.

It remains to see that $\operatorname{supp} \mu$ actually satisfies the property. We can again assume that μ is k-homogeneous. As F is locally convex, it is sufficient to show the claim for all valuations $\lambda \circ \mu \in \mathrm{VConv}(C; V)$ for $\lambda \in F'$. This is a real-valued valuation and thus in particular compactly supported, so under the assumptions above the mollified functions $f_\epsilon, g_\epsilon \in \mathrm{Conv}(V, \mathbb{R}) \cap C^\infty(V)$ satisfy $f_\epsilon = g_\epsilon$ on an open neighborhood of the support of $\lambda \circ \mu$ for all $\epsilon > 0$ small enough. In particular, $f_\epsilon^{\otimes k} = g_\epsilon^{\otimes k}$ on a neighborhood of $\operatorname{supp} \overline{\mathrm{GW}}(\lambda \circ \mu)$, and using Theorem 5.4.9, we obtain

$$\begin{aligned}\lambda(\mu(f)) &= \lim_{\epsilon \to 0} \lambda(\mu(f_\epsilon)) = \lim_{\epsilon \to 0} \overline{\mathrm{GW}}(\lambda \circ \mu) \left(f_\epsilon^{\otimes k}\right) = \lim_{\epsilon \to 0} \overline{\mathrm{GW}}(\lambda \circ \mu) \left(g_\epsilon^{\otimes k}\right) \\ &= \lim_{\epsilon \to 0} \lambda(\mu(g_\epsilon)) = \lambda(\mu(g)).\end{aligned}$$

□

6.2 Subspaces of valuations with compact support

The goal of this section is to establish some useful results on the topology of spaces of valuations with support contained in a fixed (compact) set. Most notably, these spaces

turn out to be Banach spaces for real-valued valuations, which will be crucial for the density results in Chapter 7 and 8. We will start with some general considerations and then construct suitable semi-norms on spaces of valuations with compact support.
Throughout this section, let F be a locally convex vector space and let us assume for simplicity that V carries some Euclidean structure.

Definition 6.2.1. *For $A \subset V$ let $\mathrm{VConv}_A(V, F)$ denote the subspace of all valuations in $\mathrm{VConv}(V, F)$ with support contained in A.*

Lemma 6.2.2. *If $A \subset V$ is a closed subset, then $\mathrm{VConv}_A(V, F)$ is a closed subspace of $\mathrm{VConv}(V, F)$.*

Proof. If $(\mu_\alpha)_\alpha$ is a net in $\mathrm{VConv}_A(V, F)$ converging to μ in $\mathrm{VConv}(V, F)$ and $f, h \in \mathrm{Conv}(V, \mathbb{R})$ are two functions with $f = h$ on a neighborhood of A, we deduce $\mu_\alpha(f) = \mu_\alpha(h)$ for all α using Proposition 6.1.3. Taking the limit, we obtain $\mu(f) = \mu(h)$. As this is true for any $f, h \in \mathrm{Conv}(V, \mathbb{R})$ with $f = h$ on a neighborhood of A, the support of μ has to be contained in A by Proposition 6.1.3. Thus $\mathrm{VConv}_A(V, F)$ is closed in $\mathrm{VConv}(V, F)$. □

If F carries a continuous norm, all valuations in $\mathrm{VConv}(V, F)$ have compact support by Corollary 6.1.2, and, obviously, the inclusion $\mathrm{VConv}_A(V, F) \to \mathrm{VConv}_B(V, F)$ is continuous for $A \subset B$. We thus obtain the directed family of locally convex vector spaces $(\mathrm{VConv}_A(V, F))_{A \subset V}$, where $A \subset V$ is compact. The direct limit $\varinjlim \mathrm{VConv}_A(V)$ in the category of locally convex vector spaces can be identified with $\mathrm{VConv}(V, F)$ equipped with the final topology with respect to the inclusions $\mathrm{VConv}_A(V, F) \to \mathrm{VConv}(V, F)$. In other words, the natural map

$$\varinjlim \mathrm{VConv}_A(V) \to \mathrm{VConv}(V, F)$$

is continuous and bijective, but the topology on the left hand side might be finer than the original topology, i.e. the inverse might not be continuous. However, the following proposition implies that the inverse is at least sequentially continuous. While we do not have a direct application of this result, it highlights the fact that the support of valuations in $\mathrm{VConv}(V, F)$ and the compact-open topology on this space are heavily intertwined.

Proposition 6.2.3. *Let F be a locally convex vector space admitting a continuous norm. If a sequence $(\mu_j)_j$ converges to μ in $\mathrm{VConv}(V, F)$, then there exists a compact set $A \subset V$ such that the supports of μ and μ_j are contained in A for all $j \in \mathbb{N}$. In particular, $(\mu_j)_j$ converges to μ in $\mathrm{VConv}_A(V, F)$.*

Proof. Let us denote the continuous norm by $\|\cdot\|$ and let $U_R := U_R(0)$ denote the open ball in V with radius $R > 0$. Using the homogeneous decomposition, we can assume that all valuations are k-homogeneous.
Assume that the supports of the valuations μ_j are not bounded. Choosing a subsequence if necessary, we can assume that the following holds: There exists a strictly increasing sequence $(r_j)_j$ of positive real numbers converging to $+\infty$ such that

1. $\operatorname{supp}\mu \subset U_{r_0}$,
2. $\operatorname{supp}\mu_j \subset U_{r_j}$ for all $j \geq 1$,
3. $\operatorname{supp}\mu_{j+1} \setminus B_{r_j} \neq \emptyset$ for all $j \geq 1$.

In particular, for every $j \in \mathbb{N}$ we can inductively define functions $\phi_1^j, ..., \phi_k^j \in C_c^\infty(V)$ with the properties

1. $\operatorname{supp}\phi_i^j \subset U_{r_j} \setminus B_{r_{j-1}}$ for all $j \geq 1$,
2. $\| \sum\limits_{l=1}^{j} \overline{\mathrm{GW}}(\mu_j)(\phi_1^l \otimes ... \otimes \phi_k^l)\| \geq 1$ for all $j \geq 1$,

as follows: Assume that we have constructed the functions ϕ_i^l for all $1 \leq i \leq k$ and $l \leq j-1$. If $\| \sum_{l=1}^{j-1} \overline{\mathrm{GW}}(\mu_j)(\phi_1^l \otimes ... \otimes \phi_k^l)\| \geq 1$, choose $\phi_1^j = ... = \phi_k^j = 0$.
If $\| \sum_{l=1}^{j-1} \overline{\mathrm{GW}}(\mu_j)(\phi_1^l \otimes ... \otimes \phi_k^l)\| < 1$, choose $\phi_i^j \in C_c^\infty(U_{r_j} \setminus B_{j-1})$ such that $\overline{\mathrm{GW}}(\mu_j)(\phi_1^j \otimes \cdots \otimes \phi_k^j) \neq 0$. Then

$$\begin{aligned}\| \sum_{l=1}^{j} \overline{\mathrm{GW}}(\mu_j)(\phi_1^l \otimes ... \otimes \phi_k^l)\| \geq & \| \overline{\mathrm{GW}}(\mu_j)(\phi_1^j \otimes ... \otimes \phi_k^j)\| - \| \sum_{l=1}^{j-1} \overline{\mathrm{GW}}(\mu_j)(\phi_1^l \otimes ... \otimes \phi_k^l)\| \\ > & \| \overline{\mathrm{GW}}(\mu_j)(\phi_1^j \otimes ... \otimes \phi_k^j)\| - 1.\end{aligned}$$

Scaling one of the functions ϕ_i^j appropriately for $1 \leq i \leq k$, we can make the right hand side equal to 1.
In any case, we obtain functions satisfying $\| \sum_{l=1}^{j} \overline{\mathrm{GW}}(\mu_j)(\phi_1^l \otimes ... \otimes \phi_k^l)\| \geq 1$ for all $j \geq 1$.

For $1 \leq i \leq k$ define $\phi_i := \sum_{j=1}^\infty \phi_i^j$. By construction, this is a locally finite sum, so we obtain an element in $C^\infty(V)$. As the supports of the functions $(\phi_i^j)_j$ are pairwise disjoint for each $1 \leq i \leq k$, we can apply Lemma 5.4.8 to find functions $f_i \in \mathrm{Conv}(V, \mathbb{R})$, $1 \leq i \leq k$, such that $f_i^j := f_i + \sum_{l=1}^{j} \phi_i^l$ is convex for all $1 \leq i \leq k$, $j \in \mathbb{N}$. Then $(f_i^j)_j$ converges to $f_i + \phi_i$ uniformly on compact subsets, i.e. in $\mathrm{Conv}(V, \mathbb{R})$. Furthermore, $f_i^j = f_i$ on an open neighborhood of the support of μ, so $\mu(f_i) = \mu(f_i^j)$ for all j. As the polarization $\bar{\mu}$ is a linear combination of μ evaluated in positive linear combinations of the arguments, exchanging f_i and f_i^j does not change the value of $\bar{\mu}$. For any $j \in \mathbb{N}$ we thus obtain

$$\begin{aligned}0 =& \| \overline{\mathrm{GW}}(\mu)(0 \otimes ... \otimes 0)\| \\ =& \| \sum_{i=0}^{k} (-1)^{k-i} \frac{1}{(k-i)!i!} \sum_{\sigma \in S_k} \bar{\mu}\left(f_{\sigma(1)}, ..., f_{\sigma(i)}, f_{\sigma(i+1)}, ..., f_{\sigma(k)}\right) \| \\ =& \| \sum_{i=0}^{k} (-1)^{k-i} \frac{1}{(k-i)!i!} \sum_{\sigma \in S_k} \bar{\mu}\left(f_{\sigma(1)}^j, ..., f_{\sigma(i)}^j, f_{\sigma(i+1)}, ..., f_{\sigma(k)}\right) \|,\end{aligned}$$

i.e. $\sum_{i=0}^k(-1)^{k-i}\frac{1}{(k-i)!i!}\sum_{\sigma\in S_k}\bar{\mu}(f^j_{\sigma(1)},...,f^j_{\sigma(i)},f_{\sigma(i+1)},...,f_{\sigma(k)})=0$.
Set $K:=\{f_i^j\ :\ j\in\mathbb{N},1\le i\le k\}\cup\{f_1+\phi_1,...,f_k+\phi_k,f_1,...,f_k\}$. Then $K\subset\mathrm{Conv}(V,\mathbb{R})$ is compact, so $(\mu_j)_j$ converges to μ uniformly on K. By Lemma 5.3.12 the same holds for the polarizations $(\bar{\mu}_j)_j$. In particular, there exists $N\in\mathbb{N}$ such that

$$\begin{aligned}&\|\sum_{i=0}^k(-1)^{k-i}\frac{1}{(k-i)!i!}\sum_{\sigma\in S_k}\bar{\mu}_j(f^j_{\sigma(1)},...,f^j_{\sigma(i)},f_{\sigma(i+1)},...,f_{\sigma(k)})\|\\=&\|\sum_{i=0}^k(-1)^{k-i}\frac{1}{(k-i)!i!}\sum_{\sigma\in S_k}\bar{\mu}(f^j_{\sigma(1)},...,f^j_{\sigma(i)},f_{\sigma(i+1)},...,f_{\sigma(k)})\\&-\sum_{i=0}^k(-1)^{k-i}\frac{1}{(k-i)!i!}\sum_{\sigma\in S_k}\bar{\mu}_j(f^j_{\sigma(1)},...,f^j_{\sigma(i)},f_{\sigma(i+1)},...,f_{\sigma(k)})\|<\frac{1}{2}\end{aligned}$$

for all $j\ge N$. By definition

$$\begin{aligned}&\sum_{i=0}^k(-1)^{k-i}\frac{1}{(k-i)!i!}\sum_{\sigma\in S_k}\bar{\mu}_j(f^j_{\sigma(1)},...,f^j_{\sigma(i)},f_{\sigma(i+1)},...,f_{\sigma(k)})\\=&\overline{\mathrm{GW}}(\mu_j)\left(\sum_{l=1}^j\phi_1^l\otimes...\otimes\sum_{l=1}^j\phi_k^l\right).\end{aligned}$$

As the support of $\overline{\mathrm{GW}}(\mu_j)$ is contained in the diagonal and the functions belonging to different superscripts i have disjoint support, we obtain

$$\sum_{i=0}^k(-1)^{k-i}\frac{1}{(k-i)!i!}\sum_{\sigma\in S_k}\bar{\mu}_j(f^j_{\sigma(1)},...,f^j_{\sigma(i)},f_{\sigma(i+1)},...,f_{\sigma(k)})=\sum_{l=1}^j\overline{\mathrm{GW}}(\mu_j)(\phi_1^l\otimes...\otimes\phi_k^l).$$

Thus we arrive at

$$\|\sum_{l=1}^j\overline{\mathrm{GW}}(\mu_j)(\phi_1^l\otimes...\otimes\phi_k^l)\|<\frac{1}{2}$$

for all $j\ge N$, which is a contradiction. □

We will now focus on valuations with support contained in a fixed compact subset. In the remaining part of this section, we will construct special continuous semi-norms on these subspaces, and we will show that these semi-norms generate the subspace topology. This will also imply that the topology on any such subspace is much simpler than the topology on $\mathrm{VConv}(V,F)$: Instead of uniform convergence on all compact subsets in $\mathrm{Conv}(V,\mathbb{R})$, we only have to check the convergence on one simple subset.

Proposition 6.2.4. *Let $A\subset V$ be compact and convex. Let $|\cdot|_F$ denote a continuous semi-norm on F and choose $s>0$. For $\mu\in\mathrm{VConv}_A(V,F)$ define*

$$\|\mu\|_{F;A,s}:=\sup\{|\mu(f)|_F\ :\ f\in\mathrm{Conv}(V,\mathbb{R}),\|f\|_{C(A+2sB_1)}\le1\}.$$

This defines a continuous semi-norm on $\mathrm{VConv}_A(V)$*. If* $|\cdot|_F$ *is a norm, so is* $\|\cdot\|_{F;A,s}$*. In addition, the topology induced by the family* $\|\cdot\|_{F;A,s}$ *(for all continuous semi-norms* $|\cdot|_F$ *on* F*) on* $\mathrm{VConv}_A(V,F)$ *coincides with the relative topology.*

Proof. It is clear that $\|\cdot\|_{F;A,s}$ defines a semi-norm if it is finite. Let $f\in\mathrm{Conv}(V,\mathbb{R})$ with $\|f\|_{C(A+2sB_1)}\leq 1$ be given. By Proposition 4.2.1, f is Lipschitz continuous on B_{A+sB_1} with Lipschitz constant $L=\frac{2}{s}\|f|_{A+2sB_1}\|_\infty\leq\frac{2}{s}$. Consider the function

$$\tilde{f}(x):=\begin{cases}\sup\limits_{x=\lambda y+(1-\lambda)z,\lambda\geq 1}\lambda f(y)+(1-\lambda)f(z) & x\in V\setminus(A+sB_1),\\ f(x) & x\in A+sB_1.\end{cases}$$

By the proof of Theorem 4.1 in [61], $\tilde{f}$ is a finite-valued convex extension of the Lipschitz continuous function $f|_{A+sB_1}$. For any $\lambda\geq 1$, $y,z\in A+sB_1$ with $x=\lambda y+(1-\lambda)z$:

$$\begin{aligned}\lambda f(y)+(1-\lambda)f(z)\leq&|\lambda[f(y)-f(z)]|+|f(z)|\leq\frac{2}{s}\lambda|y-z|+\|f\|_{C(A+sB_1)}\\ \leq&\frac{2}{s}|\lambda y-\lambda z|+1=\frac{2}{s}|x-z|+1.\end{aligned}$$

For $x\in V\setminus(A+sB_1)$ we thus obtain

$$\tilde{f}(x)\leq\frac{2}{s}\sup_{z\in A+sB_1}|x-z|+1\leq\frac{2}{s}(\mathrm{dist}(x,A+sB_1)+\mathrm{diam}(A+sB_1))+1.$$

Choosing $\lambda=\frac{|z-x|}{s}$, $y=z+s\frac{x-z}{|z-x|}$ and $z\in A$, we also obtain the inequality

$$\frac{|z-x|}{s}f(z+s\frac{x-z}{|z-x|})+(1-\frac{|z-x|}{s})f(z)\leq\tilde{f}(x).$$

As

$$\begin{aligned}&|\frac{|z-x|}{s}f(z+s\frac{x-z}{|z-x|})+(1-\frac{|z-x|}{s})f(z)|\leq\frac{|z-x|}{s}+|(1-\frac{|z-x|}{s})|\\ \leq&2\frac{|z-x|}{s}+1\leq\frac{2}{s}(\mathrm{dist}(x,A+sB_1)+\mathrm{diam}(A+sB_1))+1,\end{aligned}$$

$|\tilde{f}(x)|\leq\frac{2}{s}(\mathrm{dist}(x,A+sB_1)+\mathrm{diam}(A+sB_1))+1$ for all $x\in V$, so the set

$$K:=\{f\in\mathrm{Conv}(V,\mathbb{R})\ :\ f=\tilde{h}\text{ for some }h\in\mathrm{Conv}(V,\mathbb{R})\text{ with }\ \|h\|_{C(A+2sB_1)}\leq 1\}$$

is uniformly bounded on compact subsets and therefore relatively compact in $\mathrm{Conv}(V,\mathbb{R})$ due to Proposition 4.2.2. In particular, μ is bounded on K, as it is continuous.
Any function $f\in\mathrm{Conv}(V,\mathbb{R})$ satisfies $\tilde{f}=f$ on $A+sB_1$, i.e. these functions coincide on an open neighborhood of the support of μ. Proposition 6.1.3 implies $\mu(f)=\mu(\tilde{f})$, and therefore

$$\|\mu\|_{F;A,s}=\sup\{|\mu(f)|_F\ :\ f\in\mathrm{Conv}(V,\mathbb{R}),\|f\|_{A+sB_1}\leq 1\}=\sup_{\tilde{f}\in K}|\mu(\tilde{f})|_F<\infty.$$

In addition, we see that the compact subset $\bar{K} \subset \mathrm{Conv}(V, \mathbb{R})$ satisfies

$$\|\mu\|_{F;A,s} \leq \|\mu\|_{F;\bar{K}} \quad \text{for all } \mu \in \mathrm{VConv}_A(V, F).$$

On the other hand, any $f \in \bar{K}$ satisfies $\|f\|_{C(A+2sB_1)} \leq \sup_{x\in A+2sB_1} \frac{2}{s}(\mathrm{dist}(x, A+sB_1) + \mathrm{diam}(A+sB_1)) + 1 \leq c_{A,s} := \frac{2}{s}(\mathrm{diam}(A) + 3s) + 1 = \frac{2}{s}\,\mathrm{diam}(A) + 7$. By considering the k-homogeneous component μ_k of μ, we obtain

$$\|\mu_k\|_{F;\bar{K}} = \sup_{f\in\bar{K}} |\mu_k(f)| = c_{A,s}^k \sup_{f\in\bar{K}} \left|\mu_k\left(\frac{f}{c_{A,s}}\right)\right| \leq c_{A,s}^k \|\mu_k\|_{F;A,s}.$$

Thus $\|\cdot\|_{F;A,s}$ and $\|\cdot\|_{F;\bar{K}}$ are equivalent, so the semi-norm $\|\cdot\|_{F;A,s}$ is in particular continuous on $\mathrm{VConv}_A(C; V, F)$.
More generally, any compact set $D \subset \mathrm{Conv}(V, \mathbb{R})$ satisfies $t := \sup_{f\in D, x\in A+2sB_1} |f(x)| < \infty$. Assuming $t > 0$, this implies

$$\|\mu_k\|_{F;D} = \sup_{f\in D} |\mu_k(f)|_F = t^k \sup_{f\in D} \left|\mu_k\left(\frac{f}{t}\right)\right|_F \leq t^k \|\mu_k\|_{F;A,s}.$$

If $t = 0$, then any $f \in D$ coincides with the zero function on a neighborhood of the support of μ, so $\mu_k(f) = \mu_k(0)$ for all $f \in D$ due to Proposition 6.1.3, i.e. $\|\mu_k\|_{F;D} \leq \|\mu_k\|_{F;A,s}$.
In any case, we see that $\|\cdot\|_{F;A,s}$ defines a continuous semi-norm on $\mathrm{VConv}_A(V, F)$ and that the family of these semi-norms generates the subspace topology.
Let us now assume that $|\cdot|_F$ is a norm. If $\mu \neq 0$, we can find $f \in \mathrm{Conv}(V, \mathbb{R})$ with $\mu(f) \neq 0$. Repeating the argument above for $D = \{f\}$, we see that $\|\mu\|_{F;D} > 0$ for $\mu \in \mathrm{VConv}_A(V, F)$ implies $\|\mu\|_{F;A,s} > 0$. Thus $\|\cdot\|_{F;A,s}$ is indeed a norm. □

For completeness, let us relate these semi-norms for different parameters $s > 0$:

Corollary 6.2.5. *Let $A \subset V$ be a compact convex subset. For $0 < s < t$*

$$\|\mu\|_{F;A,t} \leq \|\mu\|_{F;A,s} \leq \left(\frac{2}{s}(2t + \mathrm{diam}\, A) + 1\right)^k \|\mu\|_{F;A,t}$$

for all k-homogeneous $\mu \in \mathrm{VConv}_A(V, F)$.

Proof. The first inequality is obvious. For the second inequality, let $f \in \mathrm{Conv}(V, \mathbb{R})$ be a function with $\|f\|_{C(A+2sB_1)} \leq 1$. Considering the function $\tilde{f} \in \mathrm{Conv}(V, \mathbb{R})$ given by

$$\tilde{f}(x) := \begin{cases} \sup\limits_{x=\lambda y+(1-\lambda)z, \lambda\geq 1} \lambda f(y) + (1-\lambda) f(z) & x \in V \setminus (A + sB_1), \\ f(x) & x \in A + sB_1 \end{cases}$$

from the previous proof, we see that $|\tilde{f}(x)| \leq \frac{2}{s}(\mathrm{dist}(x, A+sB_1) + \mathrm{diam}(A+sB_1)) + 1$, so $\|\tilde{f}\|_{C(A+2tB_1)} \leq \frac{2}{s}(2t - s + \mathrm{diam}\, A + s) + 1 = \frac{2}{s}(2t + \mathrm{diam}\, A) + 1$. As $f = \tilde{f}$ on a neighborhood of the support of μ, we obtain

$$
\begin{aligned}
|\mu(f)|_F &= \left(\frac{2}{s}(2t+\operatorname{diam} A)+1\right)^k \left|\mu\left(\frac{f}{\frac{2}{s}(2t+\operatorname{diam} A)+1}\right)\right|_F \\
&\leq \left(\frac{2}{s}(2t+\operatorname{diam} A)+1\right)^k \|\mu\|_{F;A,s}.
\end{aligned}
$$

□

Corollary 6.2.6. *If A is compact and F is a Banach or Fréchet space, then* $\mathrm{VConv}_A(V,F)$ *is also a Banach or Fréchet space respectively.*

Proof. By Lemma 6.2.2, $\mathrm{VConv}_A(V,F)$ is a closed subspace of the complete locally convex space $\mathrm{VConv}(V,F)$ and so it is also complete.
If A is compact and convex, we can take one of the families of semi-norms from Proposition 6.2.4, which generates the subspace topology, so the space $\mathrm{VConv}_A(V,F)$ is complete with respect to these semi-norms. If F is a Banach space, we only obtain one norm, while we get a sequence of norms if F is a Fréchet space. In both cases, the claim follows
If A is not convex, choose $R>0$ such that $A\subset B_R(0)$. Using the same argument as in the proof of Lemma 6.2.2, we see that $\mathrm{VConv}_A(V,F)\subset \mathrm{VConv}_{B_R(0)}(V,F)$ is a closed subspace of a Banach or Fréchet space. The claim follows. □

6.3 Applications

6.3.1 The image of the embedding of $\mathrm{VConv}(V,F)$ into $\mathrm{Val}(V^*\times\mathbb{R},F)$

We are now able to describe the image of $T:\mathrm{VConv}(V,F)\to\mathrm{Val}(V^*\times\mathbb{R},F)$ in the case that F admits a continuous norm. Note that by Corollary 6.1.2, all valuations $\mu\in\mathrm{VConv}(V,F)$ have compact support in this case. We start with the following observation:

Proposition 6.3.1. *For $\mu\in\mathrm{VConv}(V)$,* $\text{v-supp}(T(\mu))\subset P(\operatorname{supp}\mu)$*, where*

$$
\begin{aligned}
P: V &\to \mathbb{P}_+(V\times\mathbb{R})\\
v &\mapsto [(v,-1)].
\end{aligned}
$$

Proof. By Proposition 3.1.3, we only need to show that $T(\mu)[K]=T(\mu)[L]$ whenever h_K and h_L coincide on an open neighborhood U of $P(\operatorname{supp}\mu)$. Considering h_K and h_L as 1-homogeneous functions on $V\times\mathbb{R}$, this implies that they coincide on the open set $\pi^{-1}(U)\subset V\times\mathbb{R}$, where $\pi:(V\times\mathbb{R})\setminus\{0\}\to\mathbb{P}_+(V\times\mathbb{R})$ is the natural projection. Obviously, this is an open neighborhood of $\operatorname{supp}\mu\times\{-1\}$, so we can apply Proposition 6.1.3 to obtain $\mu(h_K(\cdot,-1))=\mu(h_L(\cdot,-1))$, i.e. $T(\mu)(K)=T(\mu)(L)$. Thus $\text{v-supp}\,T(\mu)\subset P(\operatorname{supp}\mu)$. □

Theorem 6.3.2. *Let F be a locally convex vector space that admits a continuous norm. The image of $T : \mathrm{VConv}_k(V, F) \to \mathrm{Val}_k(V^* \times \mathbb{R}, F)$ consists precisely of the valuations $\mu \in \mathrm{Val}_k(V^* \times \mathbb{R}, F)$ whose vertical support is contained in the negative half sphere $\mathbb{P}_+(V \times \mathbb{R})_- := \{[(y, s)] \in \mathbb{P}_+(V \times \mathbb{R}) : s < 0\}$. If F is a Fréchet space, $T : \mathrm{VConv}_A(V, F) \to \mathrm{Val}_{P(A)}(V^* \times \mathbb{R}, F)$ is a topological isomorphism for any compact subset $A \subset V$.*

Proof. Starting with $\mu \in \mathrm{VConv}_k(V, F)$, Proposition 6.3.1 shows that $T(\mu)$ has vertical support contained in $\mathbb{P}_+(V \times \mathbb{R})_-$.
Conversely, let $\nu \in \mathrm{Val}_k(V^* \times \mathbb{R}, F)$ be a valuation with vertical support contained in $\mathbb{P}_+(V \times \mathbb{R})_-$. As $P : V \to \mathbb{P}_+(V \times \mathbb{R})_-$ is a diffeomorphism, $P^{-1}(\text{v-supp}\,\nu)$ is compact. Let us construct a functional μ on $\mathrm{Conv}(V, \mathbb{R})$ as follows: Given $f \in \mathrm{Conv}(V, \mathbb{R})$, let $K_f \in \mathcal{K}(V^* \times \mathbb{R})$ be a convex body with $h_{K_f}(\cdot, -1) = f$ on some neighborhood of $P^{-1}(\text{v-supp}\,\nu)$, which exists by Proposition 4.3.5. Now set

$$\mu(f) := \nu(K_f).$$

Note that this does not depend on the special choice of K_f: If K is another convex body with $h_K(\cdot, -1) = f$ on some neighborhood of $P^{-1}(\text{v-supp}\,\nu)$, then $h_K(\cdot, -1) = f = h_{K_f}(\cdot, -1)$ on a neighborhood of $P^{-1}(\text{v-supp}\,\nu)$, i.e. $h_K = h_{K_f}$ on a neighborhood of $\text{v-supp}\,\nu$, so Proposition 3.1.3 implies $\nu(K) = \nu(K_f)$.
The functional constructed this way is also a valuation: Choose a scalar product on V and let $R > 0$ be such that $P^{-1}(\text{v-supp}\,\nu)$ is contained in B_R. If $\min(f, h)$ is convex, then

$$\begin{aligned}
\operatorname{epi}\max(f, h)^* &= \operatorname{epi}\min(f^*, h^*) = \operatorname{epi}(f^*) \cup \operatorname{epi}(h^*),\\
\operatorname{epi}\min(f, h)^* &= \operatorname{epi}\max(f^*, h^*) = \operatorname{epi}(f^*) \cap \operatorname{epi}(h^*).
\end{aligned}$$

For $c = \max\{||f||_{C(B_{R+2})}, ||h||_{C(B_{R+2})}, ||\min\{f, h\}||_{C(B_{R+2})}, ||\max\{f, h\}||_{C(B_{R+2})}\}$ choose

$$\begin{aligned}
K_f &= \operatorname{epi}(f^*) \cap \{|y| \le 2(n+2)c, |t| \le 3(n+2)c\},\\
K_h &= \operatorname{epi}(h^*) \cap \{|y| \le 2(n+2)c, |t| \le 3(n+2)c\}.
\end{aligned}$$

Proposition 4.3.5 shows that

$$\max(f, h) = h_{K_f \cup K_h}(\cdot, -1), \qquad \min(f, h) = h_{K_f \cap K_h}(\cdot, -1) \quad \text{on } B_{R+1},$$

so the definition of μ implies

$$\begin{aligned}
\mu(\max(f, h)) + \mu(\min(f, h)) &= \nu(K_f \cup K_h) + \nu(K_f \cap K_h)\\
&= \nu(K_f) + \nu(K_h) = \mu(f) + \mu(h).
\end{aligned}$$

Furthermore, μ is invariant under the addition of linear or constant functions, as ν is translation invariant. It remains to show that μ is continuous. We will argue by contradiction.
Let $(f_j)_j$ be a sequence in $\mathrm{Conv}(V, \mathbb{R})$ converging to $f \in \mathrm{Conv}(V, \mathbb{R})$ uniformly on

compact subsets and assume that there exists $\epsilon > 0$ such that $|\mu(f_j) - \mu(f)| > \epsilon$ for all $j \in \mathbb{N}$ for some continuous semi-norm $|\cdot|$ on F. Recall that we have chosen $R > 0$ such that $P^{-1}(\text{v-supp}\,\nu) \subset B_R$. As the set $\{f_j | j \in \mathbb{N}\} \cup \{f\}$ is compact, these functions are bounded on B_{R+2} by some constant $c > 0$. Using Proposition 4.3.5, we see that the convex bodies

$$K_{f_j} = \text{epi}(f_j^*) \cap \{|y| \le 2(R+2)c, |t| \le 3(R+2)c\},$$
$$K_f = \text{epi}(f^*) \cap \{|y| \le 2(R+2)c, |t| \le 3(R+2)c\},$$

satisfy $h_{K_{f_j}} = f_j$ and $h_{K_f} = f$ on B_{R+1}. By construction, the sequence $(K_{f_j})_j$ of convex bodies is bounded, so by the Blaschke selection theorem 2.4.1 we find a subsequence $K_{f_{j_k}}$ converging to some convex body $K \in \mathcal{K}(V^* \times \mathbb{R}, F)$. Then $h_K(\cdot, -1) = h_{K_f}(\cdot, -1)$ on B_{R+1}, as $h_{K_{f_{j_k}}}(\cdot, -1) = f_{j_k}$ on B_{R+1} and $f_j \to f$. As $\mu(f)$ does not depend on the special choice of the convex body, we deduce that

$$\lim_{k\to\infty} \mu(f_{j_k}) = \lim_{k\to\infty} \nu(K_{f_{j_k}}) = \nu(K) = \nu(K_f) = \mu(f).$$

This is a contradiction to $|\mu(f_j) - \mu(f)| > \epsilon$ for all $j \in \mathbb{N}$. Thus μ has to be continuous. We have constructed $\mu \in \text{VConv}(V, F)$ with $T(\mu) = \nu$ and the support of μ is obviously contained in $P^{-1}(\text{v-supp}\,\nu)$.

Now let $A \subset V$ be compact, F a Fréchet space. Observe that the restriction $T : \text{VConv}_A(V, F) \to \text{Val}_{P(A)}(V^* \times \mathbb{R})$ is a well defined, injective, and continuous map between Fréchet spaces by Corollary 6.2.6 and Theorem 5.2.5. By the preceding discussion it is also surjective, so Banach's inversion theorem implies that $T^{-1} : \text{Val}_{P(A)}(V^* \times \mathbb{R}, F) \to \text{VConv}_A(V, F)$ is continuous, i.e. $T : \text{VConv}_A(V, F) \to \text{Val}_{P(A)}(V^* \times \mathbb{R}, F)$ is a topological isomorphism. □

Note that Proposition 6.2.3 shows that the inverse $T^{-1} : \text{Im}T \to \text{VConv}(V; F)$ is not continuous if F admits a continuous norm: If $(\mu_j)_j$ is a sequence in $\text{Val}(V^* \times \mathbb{R}, F)$ that converges to zero such that $\mu_j \in \text{Im}T$ and such that the distance of the vertical supports of these valuations to the set $\{[(v, s)] \in \mathbb{P}_+(V \times \mathbb{R}) : s = 0\}$ converges to zero, then $T^{-1}(\mu_j)$ defines a sequence of valuations in $\text{VConv}(V, F)$ with unbounded supports. Thus the sequence cannot converge in $\text{VConv}(V, F)$.

6.3.2 Triviality of certain invariant valuations

A classical problem in the theory of valuation is the classification of valuations in terms of their invariance properties. It is easy to see that the Goodey-Weil embedding for $\text{VConv}(C; V, F)$ is equivariant with respect to the natural operation of the affine group, so any invariance property of a valuation is reflected in the properties of its Goodey-Weil distribution. The compactness of the support thus imposes certain restrictions on the existence of invariant valuations. Let us start with the following observation.

Proposition 6.3.3. *If the support of $\mu \in \text{VConv}(C; V, F)$ is contained in a one-point set, then it is empty and μ is constant.*

Proof. By considering $\lambda \circ \mu$ for $\lambda \in F'$ again, it is enough to consider the case $F = \mathbb{R}$. Let us also assume $V = \mathbb{R}^n$ and, without loss of generality, let the support of μ be contained in $\{0\}$. By taking the homogeneous decomposition of μ, we can assume that μ is homogeneous of degree k. We thus only need to show that the assumptions imply $\mu = 0$ for $k > 0$.
If μ is 1-homogeneous, $\overline{\mathrm{GW}}(\mu)$ is a distribution with compact support of order at most 2 due to Inequality (5.3), so there exist constants $c_\alpha \in \mathbb{R}$ such that

$$\overline{\mathrm{GW}}(\mu) = \sum_{|\alpha| \leq 2} c_\alpha \partial^\alpha \delta_0.$$

Plugging in linear and constant functions, we see that $c_\alpha = 0$ for $|\alpha| < 2$. Thus for any $f \in C^\infty(V)$:

$$\overline{\mathrm{GW}}(\mu)(f) = \sum_{|\alpha|=2} c_\alpha \partial^\alpha f(0).$$

Fix $1 \leq i \leq n$ and consider the functions $f_\epsilon(x) = \sqrt{\epsilon^2 + x_i^2}$ for $\epsilon > 0$. Then

$$\partial^\alpha f_\epsilon(x) = \begin{cases} \frac{\epsilon^2}{\sqrt{\epsilon^2 + x_i^2}^3} & \alpha = (i,i), \\ 0 & \text{else.} \end{cases}$$

Moreover, $f_\epsilon(x) \to f(x) = |x_i|$ for $\epsilon \to 0$, so the continuity of μ implies

$$\mu(f) = \lim_{\epsilon \to 0} \mu(f_\epsilon) = \lim_{\epsilon \to 0} \overline{\mathrm{GW}}(\mu)(f_\epsilon) = \lim_{\epsilon \to 0} c_{(i,i)} \frac{1}{\epsilon}.$$

Thus we must have $c_{(i,i)} = 0$. In total, we are left with an expression of the form

$$\overline{\mathrm{GW}}(\mu) = \sum_{i<j} c_{ij} \partial_i \partial_j \delta_0.$$

Now consider $f_\epsilon(x) = \sqrt{\epsilon^2 + (x_i + x_j)^2}$ for $i \neq j$, which converges to $f(x) = |x_i + x_j|$ for $\epsilon \to 0$. Then $\partial_i \partial_j f_\epsilon(x) = \frac{\epsilon^2}{\sqrt{\epsilon^2 + (x_i + x_j)^2}^3}$ and all other mixed derivatives vanish, so the same argument as before shows that

$$\mu(f) = \lim_{\epsilon \to 0} \mu(f_\epsilon) = \lim_{\epsilon \to 0} \overline{\mathrm{GW}}(\mu)(f_\epsilon) = \lim_{\epsilon \to 0} c_{ij} \frac{1}{\epsilon}.$$

Thus $c_{ij} = 0$ for all $1 \leq i, j \leq n$, i.e. $\overline{\mathrm{GW}}(\mu) = 0$. The injectivity of $\overline{\mathrm{GW}}$ from Corollary 5.4.10 implies $\mu = 0$.

If μ is k-homogeneous, we consider the valuation

$$\mu_f := \bar{\mu}(\cdot, f[k-1])$$

for $f \in C$ obtained from $\bar{\mu}$ by setting the last $k-1$ arguments equal to f. Then μ_f is a 1-homogeneous valuation. Using Proposition 6.1.3, it is easy to see that the support of μ_f is a subset of the support of μ, so we deduce $\mu_f = 0$ from the case $k = 1$. In particular, $\mu(f) = \bar{\mu}(f, f[k-1]) = \mu_f(f) = 0$. $\square$

Corollary 6.3.4. *Let $G \subset Aff(V)$ be a subgroup such that either*

1. *there exists no compact orbit in V, or*
2. *the only compact orbit in V consists of a single point.*

Then any G-invariant valuation in $\mathrm{VConv}(V)$ *is constant. In particular, any translation or* $\mathrm{SL}(V)$*-invariant valuation (for* $\dim V \geq 2$*) is constant.*

Proof. Without loss of generality, we can assume that μ is homogeneous of degree k and G-invariant. We will show that μ has to vanish identically if $k > 0$.
Suppose $k > 0$. It is easy to see that $\overline{\mathrm{GW}} : \mathrm{VConv}_k(V) \to \mathcal{D}'(V^k)$ is equivariant with respect to the operation of the affine group. In particular, any G-invariant valuation induces a G-invariant distribution. As the support of any such distribution must be invariant with respect to the group, the same holds true for the support of μ. However, the support of μ is compact, so we directly see that the support of μ is either empty or consists of a single point. Due to Proposition 6.3.3 the second case cannot occur, so the support of μ is empty, i.e. $\mu = 0$. □

6.3.3 Restrictions on the support and extension of valuations to larger cones of convex functions

Up to this point, the cone $C \subset \mathrm{Conv}(V)$ did not play any role in our constructions. In particular, we could have defined the Goodey-Weil embedding by first restricting any valuation in $\mathrm{VConv}_k(C; V, F)$ to $\mathrm{Conv}(V, \mathbb{R})$ and then using the Goodey-Weil embedding for $\mathrm{VConv}(V, F)$. This naturally leads to the question if the cone C actually imposes any restrictions on the support. We will use this approach to give a partial answer to the question which valuations in $\mathrm{VConv}(V, F)$ can be extended to some larger cone. As usual, let F denote a locally convex vector space.

Theorem 6.3.5. *Let $C \subset \mathrm{Conv}(V)$ be a regular cone containing $\mathrm{Conv}(V, \mathbb{R})$. Consider the set $\mathrm{dom}(C) := \bigcap_{f \in C} \overline{\mathrm{dom}\, f}$. Then the following holds:*

1. *The support of any valuation in $\mathrm{VConv}(C; V, F)$ is contained in $\mathrm{dom}(C)$.*
2. *If F admits a continuous norm, then every valuation in $\mathrm{VConv}(V, F)$ with support contained in the interior of $\mathrm{dom}(C)$ extends uniquely to a continuous valuation in $\mathrm{VConv}(C; V, F)$.*

If F admits a continuous norm, we thus have inclusions

$$\mathrm{VConv}_{\mathrm{int}\,\mathrm{dom}(C)}(V, F) \hookrightarrow \mathrm{VConv}(C; V, F) \hookrightarrow \mathrm{VConv}_{\mathrm{dom}(C)}(V, F).$$

Proof. For the first statement, consider the Goodey-Weil distribution of a k-homogeneous valuations μ and let $\phi_1, \dots, \phi_k \in C_c^\infty(V \setminus \mathrm{dom}(C))$. We have to show that $\overline{\mathrm{GW}}(\mu)(\phi_1 \otimes \dots \otimes \phi_k) = 0$. Using a partition of unity, we can assume that $\mathrm{supp}\, \phi_i \subset U_\epsilon(x_i)$ for some

$x_i \in V \setminus \mathrm{dom}(C)$ and that $B_\epsilon(x_i) \subset V \setminus \mathrm{dom}(C)$. We claim that every point $y \in B_\epsilon(x_i)$ has a neighborhood where some $f_{x_i} \in C$ is identical to $+\infty$. Indeed, if $y \in B_\epsilon(x_i)$ is a point where the assertion is violated, then $y \in \overline{\mathrm{dom}\, f}$ for all $f \in C$. Thus $y \in \mathrm{dom}(C)$, which is a contradiction to $y \in B_\epsilon(x_i) \subset V \setminus \mathrm{dom}(C)$. As $B_\epsilon(x_i)$ is compact, we can thus find a finite number of functions $f_{1,i}, ..., f_{j,i} \in C$ such that $f_i := \sum_{l=1}^{j} f_{l,i}$ is identical to $+\infty$ on $B_\epsilon(x_i)$.
The Lipschitz regularization $f_{i,r} := \mathrm{reg}_r\, f_i$ belongs to $\mathrm{Conv}(V, \mathbb{R})$ for all $r > 0$ small enough. Let $h_i \in \mathrm{Conv}(V, \mathbb{R})$ be a convex function such that $h_i + \phi_i \in \mathrm{Conv}(V, \mathbb{R})$. Then $\tilde{h}_{i,r} := f_{i,r} + h_i \in C$ satisfies $\tilde{h}_{i,r} + \phi_i \in C$ as well, so

$$\begin{aligned}&\overline{\mathrm{GW}}(\mu)(\phi_1 \otimes \cdots \otimes \phi_k)\\ &= \sum_{l=0}^{k} (-1)^{k-l} \frac{1}{(k-l)!l!} \sum_{\sigma \in S_k} \bar{\mu}\left(\tilde{h}_{\sigma(1),r} + \phi_{\sigma(1)}, ..., \tilde{h}_{\sigma(l),r} + \phi_{\sigma(l)}, \tilde{h}_{\sigma(l+1),r}, ..., \tilde{h}_{\sigma(k),r}\right)\end{aligned}$$

for all $r > 0$ sufficiently small. Of course, $\tilde{h}_{i,r}$ epi-converges to $f_i + h_i$ for $r \to 0$ and $\tilde{h}_{i,r} + \phi_i$ epi-converges to $f_i + h_i + \phi_i = f_i + h_i$, as $f_i = \infty$ on the support of ϕ_i. The joint continuity of $\bar{\mu}$ implies

$$\begin{aligned}\overline{\mathrm{GW}}(\mu)(\phi_1 \otimes \cdots \otimes \phi_k) =& \sum_{l=0}^{k} (-1)^{k-l} \frac{1}{(k-l)!l!} \sum_{\sigma \in S_k} \bar{\mu}(f_{\sigma(1)} + h_{\sigma(1)}, ..., f_{\sigma(k)} + h_{\sigma(k)})\\ =& \bar{\mu}(f_1 + h_1, ..., f_k + h_k) \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} = 0.\end{aligned}$$

For the second statement, let $\mu \in \mathrm{VConv}_k(V, F)$ be a valuation with support in $\mathrm{int}\,\mathrm{dom}(C)$. If $f \in C$ is any function, it is finite and thus continuous on the interior of $\mathrm{dom}(C)$. In particular, it is bounded on a compact neighborhood A of the support of μ that is contained in $\mathrm{int}\,\mathrm{dom}(C)$. Taking a smaller open neighborhood U of the support of μ such that $\bar{U} \subset \mathrm{int}\, A$, Proposition 4.2.1 implies that f is Lipschitz continuous on U. In particular, the norm of any subgradient of f on U is bounded by the Lipschitz constant. Proposition 4.4.1 iv. implies that there exists $r_0 > 0$ such that $\mathrm{reg}_r\, f = f$ on U for all $0 < r \le r_0$. Thus Proposition 6.1.3 shows that $\mu(\mathrm{reg}_r\, f)$ does not depend on $0 < r \le r_0$, so

$$\mu'(f) := \lim_{r \to 0} \mu(\mathrm{reg}_r\, f)$$

defines an extension of μ to C. Due to Proposition 4.4.1 v., it is a valuation. We need to show that this extension is continuous. As the topology on C is metrizable, we only need to show that μ' is sequentially continuous. Let $(f_j)_j$ be a sequence in C epi-converging to $f \in C$. Then all functions are finite on the interior of $\mathrm{dom}(C)$ and thus they converge uniformly on the compact set A by Proposition 4.1.2. The estimate in Proposition 4.2.1 shows that $\{f_j \;:\; j \in \mathbb{N}\} \cup \{f\}$ is uniformly Lipschitz continuous on U, so Proposition 4.4.1 iii. implies that there exists $r_0 > 0$ such that $\mathrm{reg}_r\, f_j = f_j$ and $\mathrm{reg}_r\, f = f$ on U for all $0 < r \le r_0$ independent of $j \in \mathbb{N}$. In particular, using Proposition 6.1.3, we see that there exists $r_0 > 0$ such that $\mu(\operatorname{reg}_r f)$ and $\mu(\operatorname{reg}_r f_j)$ do not depend on $0 < r \leq r_0$. As $\operatorname{reg}_r f_j \to \operatorname{reg}_r f$ for $j \to \infty$ and all r sufficiently small, we obtain

$$\mu(\operatorname{reg}_r f) = \lim_{j\to\infty} \mu(\operatorname{reg}_r f_j) \quad \text{for all } r \text{ sufficiently small.}$$

However, $\mu(\operatorname{reg}_r f_j)$ and $\mu(\operatorname{reg}_r f)$ do not depend on r for $0 < r \leq r_0$ independent of $j \in \mathbb{N}$, so we conclude $\mu'(f) = \lim_{j\to\infty} \mu'(f_j)$.
Obviously, the inclusion constructed this way is injective. □

Let us show that both inclusions in Theorem 6.3.5 are strict in general:
Define $\mu(f) := f(0) + f(2) - 2f(1)$ for $f \in \operatorname{Conv}(\mathbb{R}, \mathbb{R})$. It is easy to see that μ is a dually epi-translation invariant valuation with support contained in $\{0, 1, 2\}$.
For the first inclusion, let C be the cone generated by $\operatorname{Conv}(\mathbb{R}, \mathbb{R})$ and the convex indicator functions $I^\infty_{[-\frac{1}{n},\infty)}$ for all $n \in \mathbb{N}$. Then $\operatorname{dom}(C) = [0, \infty)$, but any $f \in C$ contains $\operatorname{supp}\mu$ in the interior of its domain. Now let $(f_j)_j$ be a sequence in C that epi-converges to $f \in C$. Due to Proposition 4.1.2, the sequence converges locally uniformly on the interior of $\operatorname{dom} f$, so in particular on $\{0, 1, 2\}$, i.e. $\mu(f_j) = f_j(0) + f_j(2) - 2f_j(1) \to f(0) + f(2) - 2f(1)$. We can thus extend μ continuously to C by setting $\mu(f) := f(0) + f(2) - 2f(1)$ for $f \in C$.

For the second inclusion, let $C \subset \operatorname{Conv}(\mathbb{R})$ be the cone generated by $\operatorname{Conv}(\mathbb{R}, \mathbb{R})$ and the convex indicator $I^\infty_{[0,\infty)}$. Consider the sequence $(f_j)_j$ in C given by

$$f_j(x) = j^2 \max(\frac{1}{j} - x, 0) = \begin{cases} j - j^2 x & x \leq \frac{1}{j}, \\ 0 & x > \frac{1}{j}. \end{cases}$$

Using Proposition 4.1.2 again, we see that $(f_j)_j$ epi-converges to $I^\infty_{[0,\infty)}$, but $\mu(f_j) = j$ for all $j \in \mathbb{N}$, so μ does not extend to C by continuity.

The restrictions on the support apply in particular to cones that are invariant under large subgroups of the affine group.

Corollary 6.3.6. *Let $C \subset \operatorname{Conv}(V)$ be a regular cone containing $\operatorname{Conv}(V, \mathbb{R})$ that is invariant with respect to either translations or $\operatorname{SL}(V)$ (if $\dim V \geq 2$). If C contains a non-finite convex function, then the only dually epi-translation invariant valuations are the constant valuations.*

Proof. If C contains a non-finite convex function, $\operatorname{dom}(C)$ is either empty or contains only the origin due to the invariance of C. Due to Proposition 6.3.3, there are no non-trivial valuations with this support for any $1 \leq k \leq n$. Thus the only valuations are the constant valuations. □

Let us see that first inclusion in Theorem 6.3.5 is bijective for certain cones.

Proposition 6.3.7. *Let $U \subset V$ be an open, convex set, $C_U := \{f \in \mathrm{Conv}(V) : f|_U < \infty\}$, and F a locally convex vector space. Then the support of any valuation $\mu \in \mathrm{VConv}(C_U; V, F)$ is contained in U.*

Proof. This is trivial for $U = V$, thus let us assume $U \neq V$. Due to Theorem 6.3.5, it is enough to show that the support of any valuation $\mu \in \mathrm{VConv}(C_U; V, F)$ does not contain any point $x_0 \in \partial U$. By considering $\lambda \circ \mu$ for all $\lambda \in F'$, it is also sufficient to consider real-valued valuations. For simplicity, we will identify $V \cong \mathbb{R}^n$. Let us assume that $\mu \in \mathrm{VConv}(C_U; V, F)$ is k-homogeneous and that $x_0 \in \operatorname{supp}\mu \cap \partial U$. By taking a supporting hyperplane through x_0 and using translations as well as rotations, we can assume that $x_0 = 0$ and that $\bar{U} \subset [0, \infty) \times \mathbb{R}^{n-1}$.
As $0 \in \operatorname{supp}\mu$, we can choose functions $\phi_1^j, ..., \phi_k^j \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\phi_i^j \subset U_{\frac{1}{j}}(0)$ such that

$$\overline{\mathrm{GW}}(\mu)(\phi_1^j \otimes ... \otimes \phi_k^j) = 1 \quad \forall j \in \mathbb{N}.$$

Consider the function $h_j \in C_U$ given by

$$h_j(x) = \begin{cases} \infty & x \in (-\infty, 0) \times \mathbb{R}^{n-1}, \\ \max\left(\frac{(x_1 - (j + \frac{1}{j}))^2}{2} + \sum\limits_{i=2}^n \frac{x_i^2}{2}, \frac{j^2}{2}\right) - \frac{j^2}{2} & x \in [0, \infty) \times \mathbb{R}^{n-1}. \end{cases}$$

Then $h_j \equiv 0$ on $B_j(j + \frac{1}{j}, 0, \dots, 0)$. Setting $x_j := (j + \frac{1}{j}, 0, \dots, 0)$, we see that $x \in B_j(x_j)$ implies

$$|x - x_{j+1}| \leq |x - x_j| + |x_j - x_{j+1}| \leq j + \frac{1}{j} - \frac{1}{j+1} \leq j + 1,$$

so $B_j(x_j) \subset B_{j+1}(x_{j+1})$. If $y = (y_1, ..., y_n) \in (0, \infty) \times \mathbb{R}^{n-1}$ is given, then

$$|y - x_j|^2 - j^2 = -2\left(j + \frac{1}{j}\right) y_1 + \frac{1}{j^2} + 2 + \sum_{i=1}^n y_i^2 \to -\infty \quad \text{for } j \to \infty,$$

so $\bigcup_{j \in \mathbb{N}} B_j(x_j) = (0, \infty) \times \mathbb{R}^{n-1}$. In particular, the sequence $(h_j)_j$ converges pointwise to $h := I^\infty_{[0,\infty) \times \mathbb{R}^{n-1}}$ for all $x \notin \{0\} \times \mathbb{R}^{n-1}$. Proposition 4.1.2 implies that $(h_j)_j$ epi-converges to h.
Now set $c_j := \max_{i=1,\dots,k} \|\phi_i^j\|_{C^2(V)}$ and define $f_i^j := c_j h_j + \phi_i^j$. Then $f_i^j \in C_U$ for all $1 \leq i \leq k$, $j \in \mathbb{N}$. By construction, $\lim_{j \to \infty} f_i^j(x) = I^\infty_{[0,\infty) \times \mathbb{R}^{n-1}}(x) = h(x)$ for $x \notin \{0\} \times \mathbb{R}^{n-1}$, so Proposition 4.1.2 shows that $(f_i^j)_j$ epi-converges to h for $j \to \infty$. Using the definition of the Goodey-Weil embedding and the joint continuity of the polarization $\bar{\mu}$, we obtain

the contradiction

$$\begin{aligned}1 &= \lim_{j\to\infty} \overline{\mathrm{GW}}(\mu)(\phi_1^j \otimes ... \otimes \phi_k^j)\\ &= \lim_{j\to\infty} \sum_{l=0}^{k} (-1)^{k-l} \frac{1}{(k-l)!l!} \sum_{\sigma\in S_k} \bar{\mu}\left(f^j_{\sigma(1)}, ..., f^j_{\sigma(l)}, h_{\sigma(l+1)}, \dots, h_{\sigma(k)}\right)\\ &= \sum_{l=0}^{k} (-1)^{k-l} \frac{1}{(k-l)!l!} \sum_{\sigma\in S_k} \bar{\mu}(h[l], h[k-l])\\ &= (-1)^k \mu(h) \sum_{l=0}^{k} (-1)^l \frac{k!}{(k-l)!l!} = 0.\end{aligned}$$

Thus $0 \notin \operatorname{supp}\mu$. □

6.3.4 Valuations on convex functions defined on open subsets

For an open and convex subset $U \subset V$, let us denote the space of all convex functions $f : U \to \mathbb{R}$ by $\mathrm{Conv}(U, \mathbb{R})$, which is a subspace of $C(U)$. Equipped with the topology of uniform convergence on compact subset of U, $\mathrm{Conv}(U, \mathbb{R})$ becomes a metrizable topological space.

Lemma 6.3.8. *For $f \in \mathrm{Conv}(U, \mathbb{R})$ define $\tilde{f}$ by*

$$\tilde{f}(x_0) = \begin{cases} f(x_0) & x_0 \in U,\\ \liminf_{x\to x_0, x\in U} f(x) & x_0 \in \partial U,\\ \infty & x_0 \in V \setminus \bar{U}.\end{cases}$$

Then $\tilde{f} \in \mathrm{Conv}(V)$.

Proof. Observe that $\tilde{f}(x_0) = \liminf_{x\to x_0, x\in U} f(x)$ for all $x \in \bar{U}$, as f is continuous on U. Obviously, $\tilde{f}$ is lower semi-continuous. We need to show that $\tilde{f} > -\infty$ and that $\tilde{f}$ is convex.
Let $x \in \partial U$ be any point, $(x_j)_j$ a sequence in U converging to x such that $\lim_{j\to\infty} f(x_j) = \tilde{f}(x)$. For $y \in U$ and $\lambda \in (0, 1)$, the convexity of f implies

$$f(\lambda y + (1-\lambda)x_j) \le \lambda f(y) + (1-\lambda) f(x_j).$$

As U is open, $\lambda y + (1-\lambda)x_j \to \lambda y + (1-\lambda)x$ in U for all $\lambda \in (0, 1)$, so the continuity of f on U implies

$$f(\lambda y + (1-\lambda)x) \le \lambda f(y) + (1-\lambda)\tilde{f}(x).$$

In particular, $\tilde{f}(x) > -\infty$. In addition, we see that $\tilde{f}$ is convex along line segments $[x, y]$, where $x \in \partial U$ and $y \in U$. To see that $\tilde{f}$ is convex, the only non-trivial case remaining is a line segment $[x, y]$ where $x, y \in \partial U$. Take a sequence $(y_j)_j$ in U converging to y such that $\lim_{j\to\infty} f(y_j) = \tilde{f}(y)$. Using the inequality above, we see that for $\lambda \in (0,1)$

$$f(\lambda y_j + (1-\lambda)x) \leq \lambda f(y_j) + (1-\lambda)\tilde{f}(x).$$

Now, $\lambda y_j + (1-\lambda)x \in U$ defines a sequence converging to $\lambda y + (1-\lambda)x \in \bar{U}$. Thus taking limits and using the remark, we obtain

$$\tilde{f}(\lambda y + (1-\lambda)x) \leq \liminf_{j\to\infty} f(\lambda y_j + (1-\lambda)x) \leq \lambda \tilde{f}(y) + (1-\lambda)\tilde{f}(x).$$

□

Proposition 6.3.9. *The extension $f \mapsto \tilde{f}$ defines a continuous, injective map i_U : $\mathrm{Conv}(U,\mathbb{R}) \to C_U = \{f \in \mathrm{Conv}(V) : f|_U < \infty\}$. The inverse map is given by restricting the map*

$$\begin{aligned} \mathrm{res} : C_U &\to \mathrm{Conv}(U,\mathbb{R}) \\ f &\mapsto f|_U \end{aligned}$$

to the image of $\mathrm{Conv}(U,\mathbb{R})$ *in C_U and is also continuous. In addition, i_U and* res *are compatible with the formation of the pointwise maximum and minimum of two convex functions.*

Proof. It is clear that i_U is injective. To see that it is continuous, it is enough to show that it is sequentially continuous, as both spaces are metrizable.
Let $(f_j)_j \subset \mathrm{Conv}(U,\mathbb{R})$ be a sequence converging to $f \in \mathrm{Conv}(U,\mathbb{R})$. Then $(\tilde{f}_j)_j$ converges pointwise on the dense subset $V \setminus \partial U$ to $\tilde{f}$, so the claim follows from Proposition 4.1.2. Of course, the restriction map defines the inverse to this extension procedure. The continuity follows again from Proposition 4.1.2.
Obviously, the restriction map is compatible with the formation of the pointwise maximum and minimum. If $f, h \in \mathrm{Conv}(U,\mathbb{R})$, then $i_U(\max(f,h)) = \max(i_U(f), i_U(h))$ on $V \setminus \partial U$. Thus $i_U(\max(f,h))(x_0) \leq \max(i_U(f), i_U(h))(x_0)$ for $x_0 \in \partial U$ by definition of i_U. For the converse inequality, take a sequence $(x_j)_j$ in U converging to x_0 such that

$$\lim_{j\to\infty} \max(f(x_j), h(x_j)) = i_U(\max(f,h))(x_0).$$

As $i_U(f)$ and $i_U(h)$ are lower semi-continuous, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $i_U(f)(x_0) \leq f(x_j) + \epsilon$ and $i_U(h)(x_0) \leq h(x_j) + \epsilon$ for all $j \geq N$, and therefore

$$\max(i_U(f), i_U(h))(x_0) \leq \max(f(x_j), h(x_j)) + \epsilon \quad \forall j \geq N.$$

Thus $\max(i_U(f), i_U(h))(x_0) \leq i_U(\max(f,h))(x_0)$. The same argument can be applied to the minimum. □

Definition 6.3.10. *Let U be an open, convex subset, F a locally convex vector space. We will denote the space of all continuous valuations $\mu : \mathrm{Conv}(U,\mathbb{R}) \to F$ that are dually epi-translation invariant by* $\mathrm{VConv}(U,F)$.

As usual, we equip $\mathrm{VConv}(U,F)$ with the topology of uniform convergence on compact subsets, which is generated by the semi-norms $\|\mu\|_{F;K} := \sup_{f\in K}|\mu(f)|_F$ for all continuous semi-norms $|\cdot|_F$ of F and compact subsets $K\subset \mathrm{Conv}(U,\mathbb{R})$.
Let $C_U := \{f\in \mathrm{Conv}(V) \ : \ f|_U<\infty\}$ be the regular cone of convex functions that are finite on U. Using Proposition 6.3.9, we can consider the map

$$\begin{aligned}\mathrm{res}^*: \mathrm{VConv}(U,F)&\rightarrow \mathrm{VConv}(C_U;V,F)\\ \mu&\mapsto [f\mapsto \mu(f|_U)].\end{aligned}$$

Lemma 6.3.11. $\mathrm{res}^*: \mathrm{VConv}(U,F)\rightarrow \mathrm{VConv}(C_U;V,F)$ *is injective and continuous.*

Proof. Assume that $\mathrm{res}^*(\mu)=0$ and let $f\in\mathrm{Conv}(U,\mathbb{R})$. The Lipschitz regularization $\mathrm{reg}_r\tilde{f}$ belongs to $\mathrm{Conv}(V,\mathbb{R})$ for $r>0$ small enough, so Proposition 4.4.1 and Proposition 6.3.9 imply that $\mu([\mathrm{reg}_r\tilde{f}]|_U)$ converges to $\mu(f)$. However, $\mu([\mathrm{reg}_r\tilde{f}]|_U)=0$, so $\mu(f)=0$. As this holds for arbitrary $f\in\mathrm{Conv}(U,\mathbb{R})$, $\mu=0$.
To see that the map is continuous, let $K\subset C_U$ be a compact subset. As the restriction $\mathrm{res}: C_U\rightarrow\mathrm{Conv}(U,\mathbb{R})$ is continuous due to Proposition 6.3.9, $\mathrm{res}(K)\subset\mathrm{Conv}(U,\mathbb{R})$ is compact, and

$$\|\mathrm{res}^*\mu\|_{F;K}=\|\mu\|_{F;\mathrm{res}(K)}.$$

Thus res^* is continuous. □

In addition to res^*, we can also consider

$$\begin{aligned}i_U^*: \mathrm{VConv}(C_U;V,F)&\rightarrow \mathrm{VConv}(U,F)\\ \mu&\mapsto [f\mapsto \mu(i_U(f))].\end{aligned}$$

Using the same argument as in Lemma 6.3.11, we see that this is well defined and continuous.
We obtain the following identification between valuations on $\mathrm{Conv}(U,\mathbb{R})$ and valuations on C_U:

Theorem 6.3.12. *If $U\subset V$ is an open, convex subset and F is a locally convex vector space admitting a continuous norm, then the map*

$$\begin{aligned}\mathrm{res}^*: \mathrm{VConv}(U,F)&\rightarrow \mathrm{VConv}(C_U;V,F)\\ \mu&\mapsto [f\mapsto \mu(f|_U)]\end{aligned}$$

is a topological isomorphism with inverse i_U^*.

Proof. It is easy to see that $i_U^*\circ\mathrm{res}^*=Id_{\mathrm{VConv}(U,F)}$, so i_U^* is surjective. Let us show that i_U^* is injective. Assume that $\mu\in\mathrm{VConv}(C_U;V,F)$ satisfies $\mu(i_U(f))=0$ for all $f\in\mathrm{Conv}(U,\mathbb{R})$. Due to Proposition 6.3.7, the support of μ is compactly contained in U. Given $h\in\mathrm{Conv}(V,\mathbb{R})$, the function $i_U(h|_U)$ coincides with h on U, i.e. they coincide on a neighborhood of the support of μ. Proposition 6.1.3 implies $\mu(h)=\mu(i_U(h|_U))=0$. Thus μ vanishes on the dense subset $\mathrm{Conv}(V,\mathbb{R})\subset C_U$, i.e. $\mu=0$.
We obtain $(i_U^*)^{-1}=\mathrm{res}^*$, which is continuous. The same applies to $(\mathrm{res}^*)^{-1}=i_U$. □

7 Smooth valuations on convex functions

In this chapter we are going to construct a large class of valuations on finite-valued convex functions. For smooth convex functions, these functionals are obtained by integrating certain differential forms of appropriate degree over the graph of their differential, which can be considered as a smooth n-dimensional submanifold in the cotangent bundle. For non-smooth functions we replace the graph of the differential by the *differential cycle*, which was introduced by Fu in [26]. The valuations obtained this way will be called *smooth valuations*. Let us also remark that this cycle exists for a much larger class of functions, called *Monge-Ampère functions*, and smooth valuations extend naturally to this larger class.
In the first section, we are going to discuss some properties of the differential cycle and establish the continuity of this current on the class of convex functions.
The second section examines which differential forms induce the trivial valuation. This description involves a certain second order differential operator on the cotangent bundle, which is closely related to the Rumin operator. As we are mostly interested in dually epi-translation invariant valuations, we will also show that smooth dually epi-translation invariant valuations can be represented by suitable invariant differential forms.
In the third section, we will show that smooth dually epi-translation invariant valuations are dense in $\mathrm{VConv}(V)$. This result is based on the observation that the differential cycle of a support function is closely related to the conormal cycle of the corresponding convex body. We then use the description of the image of the embedding of $\mathrm{VConv}(V)$ into $\mathrm{Val}(V^*\times\mathbb{R})$ to show that the approximation of continuous valuations in $\mathrm{Val}(V^*\times\mathbb{R})$ by smooth valuations can be used to establish the corresponding statement for $\mathrm{VConv}(V)$.

The results of this chapter are to be published in [34].

7.1 Properties of the differential cycle

In this section we summarize the basic facts concerning Monge-Ampère functions established by Fu in [26]. For a generalization of this notion we also refer to [33]. Let V be an oriented vector space, $\mathrm{vol}\in\Lambda^nV^*$ a positive volume form, and let ω_s denote the natural symplectic form on T^*V. Recall that a current S on T^*V is called *locally vertically bounded* if $\operatorname{supp} S\cap\pi^{-1}(K)$ is compact for all compact subsets $K\subset V$. Here, $\pi: T^*V\to V$ denotes the natural projection. For a background on currents we refer to [25].

Theorem 7.1.1 (Fu [26] Theorem 2.0)**.** *Let $f : V \to \mathbb{R}$ be a locally Lipschitzian function. There exists at most one integral current $S \in I_n(T^*V)$ such that*

1. *S is closed, i.e. $\partial S = 0$,*
2. *S is Lagrangian, i.e. $S \llcorner \omega_s = 0$,*
3. *S is locally vertically bounded,*
4. *$S(\psi(x,y) \wedge \pi^* \operatorname{vol}) = \int_V \psi(x, df(x)) d\operatorname{vol}(x)$ for all $\psi \in C_c^\infty(T^*V)$.*

Note that the right hand side of the last equation is well defined due to Rademacher's theorem.

If such a current exists, the function f is called *Monge-Ampère*. The corresponding current is denoted by $D(f)$ and is called the *differential cycle of f*. Moreover, we have the following description of the support of $D(f)$: Let $\partial^* f : V \to \mathcal{K}(V^*)$ denote the unique upper semi-continuous multifunction with values in $\mathcal{K}(V^*)$ such that $df(x) \in \partial^* f(x)$ whenever f is differentiable at $x \in V$ (also called the generalized differential by Clarke [13]).

Theorem 7.1.2 (Fu [26] Theorem 2.2.)**.** *If $f : V \to \mathbb{R}$ is Monge-Ampère, then*

$$\operatorname{supp} D(f) \subset \textit{graph } \partial^* f := \{(x,y) \in T^*V : y \in \partial^* f(x)\}.$$

In particular, given an relatively compact, open set $U \subset V$,

$$\operatorname{supp} D(f) \cap \pi^{-1}(U) \subset U \times B_{\operatorname{lip}(f|_U)}(0),$$

where $\operatorname{lip}(f|_U)$ denotes the Lipschitz constant of $f|_U$ (with respect to some scalar product on V).

Let us summarize some additional properties.

Proposition 7.1.3 (Fu [26] Proposition 2.4)**.** *Let f be a Monge-Ampère function and $\phi \in C^{1,1}(V)$. Then $f + \phi$ is Monge-Ampère and*

$$F(f + \phi) = G_{\phi *} D(f),$$

where $G_\phi : T^*V \to T^*V$ is given by $(x, y) \mapsto (x, y + d\phi(x))$.

Proposition 7.1.4. *Let $\phi : V \to V$ be a diffeomorphism of class $C^{1,1}$. Then $f \circ \phi$ is Monge-Ampère and*

$$D(f \circ \phi) = \left(\phi^\#\right)_* D(f)$$

if ϕ is orientation preserving, and

$$D(f \circ \phi) = -\left(\phi^\#\right)_* D(f)$$

if ϕ is orientation reversing. Here $\phi^\# : T^*V \to T^*V$ is given by $(x, y) \mapsto (\phi^{-1}(x), \phi^* y)$.

Proof. For orientation preserving diffeomorphisms this was shown in [26] Proposition 2.5. The second case follows with the same argument: First note that the integral current $-\left(\phi^{\#}\right)_* D(f)$ is locally vertically bounded and satisfies $\partial\left[\left(\phi^{\#}\right)_* D(f)\right] = \left(\phi^{\#}\right)_* \partial D(f) = 0$. As $\phi^{\#}$ is a symplectomorphism, so is $(\phi^{\#})^{-1}$, and in particular

$$-\left[\left(\phi^{\#}\right)_* D(f)\right] \llcorner \omega_s = -\left(\phi^{\#}\right)_* [D(f)\llcorner((\phi^{\#})^{-1})^*\omega_s] = -\left(\phi^{\#}\right)_* [D(f)\llcorner\omega_s] = 0.$$

In addition,

$$\begin{aligned}
-\left[\left(\phi^{\#}\right)_* D(f)\right][\psi(x,y) \wedge \pi^* \mathrm{vol}] &= -D(f)\left[(\phi^{\#})^*[\psi(x,y) \wedge \pi^* \mathrm{vol}]\right] \\
&= -D(f)\left[\psi \circ \phi^{\#}(x,y) \wedge \pi^*(\phi^{-1})^* \mathrm{vol}\right] \\
&= -D(f)\left[\psi \circ \phi^{\#}(x,y) \wedge \pi^*(\det D\phi^{-1} \wedge \mathrm{vol})\right] \\
&= D(f)\left[\psi \circ \phi^{\#}(x,y) \wedge \pi^*|\det D\phi^{-1}| \wedge \pi^* \mathrm{vol}\right] \\
&= D(f)\left[\psi\left(\phi^{-1}(x), \phi^* y\right) \wedge \pi^*|\det D\phi^{-1}| \wedge \pi^* \mathrm{vol}\right] \\
&= \int_V \psi\left(\phi^{-1}(x), \phi^* df\left(\phi^{-1}(x)\right)\right) |\det D\phi^{-1}(x)| d\,\mathrm{vol}(x)
\end{aligned}$$

for all $\psi \in C_c^\infty(T^*V)$, where we have used the defining property of $D(f)$ in the last step. By a change of variables, we obtain

$$\begin{aligned}
-\left[\left(\phi^{\#}\right)_* D(f)\right](\psi(x,y) \wedge \pi^* \mathrm{vol}) &= \int_V \psi\left(\phi^{-1}(x), \phi^* df\left(\phi^{-1}(x)\right)\right) |\det D\phi^{-1}(x)| d\,\mathrm{vol}(x) \\
&= \int_V \psi\left(x, \phi^* df(x)\right) d\,\mathrm{vol}(x) \\
&= \int_V \psi(x, d(\phi^* f)(x)) d\,\mathrm{vol}(x).
\end{aligned}$$

Thus $-\left[\left(\phi^{\#}\right)_* D(f)\right]$ satisfies the defining properties of the differential cycle for $\phi^* f$, which is thus a Monge-Ampère function with $D(\phi^* f) = -\left[\left(\phi^{\#}\right)_* D(f)\right]$ by Theorem 7.1.1. $\square$

Also note that [26] Remark 2.1 shows that cf is Monge-Ampère for any $c \in \mathbb{R} \setminus \{0\}$ and any Monge-Ampère function f, with

$$D(cf) = C_* D(f), \tag{7.1}$$

where $C : T^*V \to T^*V$ is given by $(x,y) \mapsto (x, cy)$.
The differential cycle satisfies the following valuation property:

Proposition 7.1.5 (Fu [26] Proposition 2.9)**.** *Let $f, g : V \to \mathbb{R}$ be locally Lipschitzian. If any three of f, g, $f \vee g$ and $f \wedge g$ are Monge-Ampère, then so is the fourth, and*

$$D(f) + D(g) = D(f \wedge g) + D(f \vee g).$$

By [26] Proposition 3.1, all convex functions are Monge-Ampère. We will now show that $D : \mathrm{Conv}(V,\mathbb{R}) \to I_n(T^*V)$ is continuous with respect to the local flat topology on $I_n(T^*V)$, i.e. the topology induced by the family of semi-norms

$$\begin{aligned} \|T\|_{A,\flat} &:= \sup\left\{|T(\omega)| : \operatorname{supp}\omega \subset A, \|\omega\|^\flat \leq 1\right\} && \text{for } T \in I_n(T^*V), \text{ where} \\ \|\omega\|^\flat &:= \max(\|\omega\|_\infty, \|d\omega\|_\infty) && \text{for } \omega \in \Omega^n(T^*V), \end{aligned}$$

for $A \subset T^*V$ compact. The proof is based on the following approximation result.

Proposition 7.1.6 (Fu [26] Proposition 2.7.)**.** *Let $f_1, f_2, \ldots : V \to \mathbb{R}$ be a sequence of Monge-Ampère functions, and suppose that for each bounded open subset $U \subset V$ there exists a constant C such that*

1. $\mathrm{lip}(f_j|_U) \leq C$
2. $M_{\pi^{-1}(U)}(D(f_j)) \leq C$

for all $j \in \mathbb{N}$. If $f = \lim_{j\to\infty} f_j$ in the C^0-topology, then f is Monge-Ampère, with

$$D(f) = \lim_{j\to\infty} D(f_j)$$

in the local flat topology.

Here $M_{\pi^{-1}(U)}(T) := \sup\{|T(\phi)| : \operatorname{supp}\phi \subset \pi^{-1}(U), \|\phi\|_\infty \leq 1\}$ denotes the mass of a current T on $\pi^{-1}(U)$.
For convex functions the first bound follows directly from Proposition 4.2.1, while the second is established by the following lemma.

Lemma 7.1.7. *For $f \in \mathrm{Conv}(V,\mathbb{R})$, $M_{\pi^{-1}(U_R)}(D(f)) \leq 2^N\omega_N\|f\|^N_{C^0(U_{R+1})}$.*

Proof. We will prove the following estimate:

$$M_{\pi^{-1}(U_R)}(D(f+\epsilon|\cdot|^2)) \leq 2^N\omega_N\|f+\epsilon|\cdot|^2\|^N_{C^0(U_{R+1})}(1+\epsilon)^{\frac{N}{2}}.$$

As $D(f+\epsilon|\cdot|^2) = G_{\epsilon*}D(f_j)$ for $G_\epsilon(x,y) = (x, y+2\epsilon x)$ due to Proposition 7.1.3, we see that $D(f+\epsilon|\cdot|^2)$ converges to $D(f)$ weakly for $\epsilon \to 0$, so the claim follows from this inequality using the lower semi-continuity of the mass norm.

Considering the mollifications $(f+\epsilon|\cdot|^2)_h$ for $h > 0$, Fu observed in the proof of [26] Proposition 3.1. that

$$M_{\pi^{-1}(U)}(D((f+\epsilon|\cdot|^2)_h)) \leq \omega_N r^N(1+\epsilon)^{\frac{N}{2}}$$

for any bounded, open subset $U \subset V$, where $r > 0$ can be chosen to be the Lipschitz constant of $f+\epsilon|\cdot|^2$ on $\{x \in V | d(x,U) < h\}$. For $U = U_R$, we may thus choose $r = 2\|f+\epsilon|\cdot|^2\|_{C^0(U_{R+1+h})}$ by Proposition 4.2.1. Now, the proof of [26] Proposition 3.1.

shows that $D((f+\epsilon|\cdot|^2)_h) \to D(f)$ in the local flat topology for $h \to 0$, so in particular, $D((f+\epsilon|\cdot|^2)_h)$ converges weakly to $D(f+\epsilon|\cdot|^2)$. The lower semi-continuity of the mass norm thus implies

$$M_{\pi^{-1}(U_R)}(D(f+\epsilon|\cdot|^2)) \le \omega_N \left(2\|f+\epsilon|\cdot|^2\|_{C^0(U_{R+1})}\right)^N (1+\epsilon)^{\frac{N}{2}}.$$

□

Theorem 7.1.8. $D : \mathrm{Conv}(V,\mathbb{R}) \to I^n(T^*V)$ *is continuous with respect to the local flat topology on* $I^n(T^*V)$*.*

Proof. If $f_j \to f$ in $\mathrm{Conv}(V,\mathbb{R})$, their Lipschitz constants are locally uniformly bounded by Proposition 4.2.1. The mass estimate from Lemma 7.1.7 shows that the mass of $D(f_j)$ is locally uniformly bounded as well. Thus $D(f_j) \to D(f)$ in the local flat topology by Proposition 7.1.6. □

Let $\Omega^k_{hc}(T^*M)$ denote the space of all smooth k-forms τ on T^*V with horizontally compact support, i.e. all k-forms $\tau \in \Omega^k_{hc}(T^*V)$ with $\operatorname{supp}\tau \subset \pi^{-1}(K)$ for some compact set $K \subset V$.

Corollary 7.1.9. *For each* $\tau \in \Omega^n_{hc}(T^*V)$*,* $f \mapsto D(f)[\tau]$ *defines a continuous valuation on* $\mathrm{Conv}(V,\mathbb{R})$*.*

Proof. Let $K \subset V$ be a compact subset with $\operatorname{supp}\tau \subset \pi^{-1}(K)$. As the support of the differential cycle is vertically bounded, $\operatorname{supp} D(f) \cap \operatorname{supp}\tau$ is compact for every $f \in \mathrm{Conv}(V,\mathbb{R})$, so $D(f)[\tau]$ is well defined for all $f \in \mathrm{Conv}(V,\mathbb{R})$. Furthermore, Proposition 7.1.5 shows that this functional satisfies the valuations property.
To see that it is continuous, let $(f_j)_j$ be a sequence in $\mathrm{Conv}(V,\mathbb{R})$ converging to $f \in \mathrm{Conv}(V,\mathbb{R})$ uniformly on compact subsets. Choose $R > 0$ such that U_R contains K. As $f_j \to f$ uniformly on U_R, Proposition 4.2.1 implies that the Lipschitz constants of these functions are bounded on U_R by some $L > 0$. Now Theorem 7.1.2 shows that $\operatorname{supp} D(f_j) \cap \pi^{-1}(U_R) \subset U_R \times B_L(0)$. Let A be a compact neighborhood of $U_R \times B_L(0)$ and $\phi \in C^\infty_c(T^*V)$ a function with $\phi = 1$ on a neighborhood of $U_R \times B_L(0)$ and $\operatorname{supp}\phi \subset A$. Then

$$|D(f_j)[\tau] - D(f)[\tau]| = |(D(f_j) - D(f))[\phi\cdot\tau]| \le \|D(f_j) - D(f)\|_{A,\flat} \cdot \|\phi\cdot\tau\|^\flat.$$

Now the claim follows from Theorem 7.1.8, as $||\phi\cdot\tau||^\flat < \infty$. □

7.2 Kernel theorem for the differential cycle

By the previous section, any $\tau \in \Omega^n_{hc}(T^*V)$ defines a continuous valuation $\mathrm{Conv}(V,\mathbb{R}) \to \mathbb{R}$, $f \mapsto D(f)[\tau]$. To decide which differential forms induce the trivial valuation, we will need a symplectic version of the Rumin differential. The starting point is the Lefschetz decomposition for the space of smooth k-forms on a symplectic manifold (see [32] Proposition 1.2.30).

Proposition 7.2.1. *Let (M, ω_s) be a symplectic manifold of dimension $2n$ and let $L : \Omega^*(M) \to \Omega^*(M)$, $\tau \mapsto \omega_s \wedge \tau$ be the Lefschetz operator.*
For $0 \le k \le n$ let $P^k(M) := \{\tau \in \Omega^k(M) : L^{n-k+1}\tau = 0\}$ denote the space of primitive k-forms on M. Then the following holds:

1. *There exists a direct sum decomposition $\Omega^k(M) = \bigoplus_{i\ge 0} L^i P^{k-2i}(M)$.*

2. *$L^{n-k} : \Omega^k(M) \to \Omega^{2n-k}(M)$ is an isomorphism.*

In particular, $L : \Omega^{n-1}(M) \to \Omega^{n+1}(M)$ is an isomorphism.

Definition 7.2.2. *We define*

$$\begin{aligned} &\bar{d} : \Omega^n(M) \to \Omega^{n-1}(M), && \bar{d}\tau := L^{-1} d\tau \\ &\bar{\mathrm{D}} : \Omega^n(M) \to \Omega^n(M), && \bar{\mathrm{D}}\,\tau := d\bar{d}\tau = dL^{-1}d\tau \end{aligned}$$

and call $\bar{\mathrm{D}}$ the symplectic Rumin operator.

Note that $\bar{d}$ is a first order differential operator, while $\bar{\mathrm{D}}$ is of second order.

Proposition 7.2.3. *$\bar{\mathrm{D}}$ and $\bar{d}$ have the following properties:*

1. *$\bar{\mathrm{D}}\,\tau$ is primitive for all $\tau \in \Omega^n(M)$.*

2. *$\bar{\mathrm{D}}$ vanishes on multiples of ω_s.*

3. *$\bar{d}$ and $\bar{\mathrm{D}}$ vanish on closed forms.*

4. *If $\phi : M \to M$ is a symplectomorphism, then $\bar{d}$ and ϕ^* commute. The same holds for $\bar{\mathrm{D}}$.*

Proof.
1. As the degree of $\bar{\mathrm{D}}\,\tau$ is n, we only need to show that $\omega_s \wedge \bar{\mathrm{D}}\,\tau = 0$. Let ξ be the unique element such that $d\tau = \omega_s \wedge \xi$, i.e. $\xi = \bar{d}\tau = L^{-1}d\tau$. Then $\omega_s \wedge \bar{\mathrm{D}}\,\tau = \omega_s \wedge dL^{-1}d\tau = d(\omega_s \wedge L^{-1}d\tau) = d^2\tau = 0$, as ω_s is closed.

2. If $\tau = \omega_s \wedge \xi$, then $\bar{\mathrm{D}}\,\tau = dL^{-1}d(\omega_s \wedge \xi) = dL^{-1}(\omega_s \wedge d\xi) = d(d\xi) = 0$.

3. Trivial.

4. Set $\xi = \bar{d}\tau$, i.e. $\omega_s \wedge \xi = d\tau$. Then

$$d(\phi^*\tau) = \phi^* d\tau = \phi^*(\omega_s \wedge \xi) = \omega_s \wedge \phi^*\xi = \omega_s \wedge \phi^*\bar{d}\tau.$$

By dividing by ω_s we obtain $\bar{d}(\phi^*\tau) = \phi^*\bar{d}\tau$. $\bar{\mathrm{D}}(\phi^*\tau) = \phi^*\,\bar{\mathrm{D}}\,\tau$ follows by applying d to both sides.

□

For a Euclidean vector space V, consider the symplectic vector space $V \times V$ with symplectic form $\omega_s((v_1, v_2), (w_1, w_2)) := \langle v_1, w_2 \rangle - \langle v_2, w_1 \rangle$. An isotropic subspace $W \subset V \times V$ is called strictly positive if there exist k orthogonal vectors $u_1, \ldots, u_k \in V$ such that W is spanned by the vectors $w_i := (u_i, \lambda_i u_i)$ where $\lambda_i > 0$ for all $1 \le i \le k$. We will need the following lemma, which is due to Bernig. It is an easy generalization of [9] Lemma 1.4.

Lemma 7.2.4. *If $\tau \in \Omega^k(V \times V)$ vanishes on all strictly positive isotropic subspaces, then τ is a multiple of the symplectic form.*

Theorem 7.2.5. *$\tau \in \Omega^n_{hc}(T^*V)$ satisfies $D(f)[\tau] = 0$ for all $f \in \mathrm{Conv}(V, \mathbb{R})$ if and only if*

1. *$\bar{\mathrm{D}}\,\tau = 0$,*
2. *$\int_V \tau = 0$, where we consider the zero section $V \hookrightarrow T^*V$ as a submanifold.*

Proof. To fix some notation, let $\mu := D(\cdot)[\tau]$ be the valuation induced by τ and let $\alpha \in \Omega^1(T^*V)$ denote the canonical 1-form. Let us start by showing that the conditions above imply $\mu = 0$. Set $\xi := \bar{d}\tau$, i.e. $\omega_s \wedge \xi = d\tau$. Then $d(\alpha \wedge \xi) = -\omega_s \wedge \xi - \alpha \wedge d\xi = -d\tau - \alpha \wedge \bar{\mathrm{D}}\,\tau$. Thus, if $\bar{\mathrm{D}}\,\tau = 0$, we have $d(\tau + \alpha \wedge \xi) = 0$.
Consider the valuation $f \mapsto D(f)[\alpha \wedge \xi]$. By the main result of [27], $D(f)\llcorner\alpha = D(f)\llcorner\pi^* df$, so

$$\begin{aligned} D(f)[\alpha \wedge \xi] =& D(f)[d(\pi^* f) \wedge \xi] = D(f)[d(\pi^* f \wedge \xi) - \pi^* f \wedge d\xi] \\ =& - D(f)[\pi^* f \wedge \bar{\mathrm{D}}\,\tau] = 0. \end{aligned}$$

Here we have used that $D(f)$ is closed. Now observe that $D(f)$ and $[V \times \{0\}] = D(0)$ belong to the same homology class. Thus $d(\tau + \alpha \wedge \xi) = 0$ and $D(f)[\alpha \wedge \xi] = 0$ imply

$$D(f)[\tau] = D(f)[\tau + \alpha \wedge \xi] = D(0)[\tau + \alpha \wedge \xi] = \int_V (\tau + \alpha \wedge \xi) = \int_V \tau = 0,$$

as $\alpha|_V = 0$. Thus $\mu(f) = 0$.

Now let us assume that $\mu = 0$. Because $\int_V \tau = D(0)[\tau] = \mu(0) = 0$, the second condition follows directly.
Let f be a smooth, strictly convex function. Then $D(f)$ is given by integration over the graph of df. For $g \in C_c^\infty(V)$, Proposition 7.1.3 shows that

$$0 = D(f + tg)[\tau] = \Phi_{tg*} D(f)[\tau] = D(f)[\Phi_{tg}^* \tau],$$

where $\Phi_{tg} : T^*V \to T^*V$, $\Phi_{tg}(x, y) = (x, y + t dg(x))$. Differentiating, we obtain

$$0 = D(f)[\mathcal{L}_{X_g} \tau] = D(f)[(d \circ i_{X_g} + i_{X_g} \circ d)\tau] = D(f)[i_{X_g} d\tau],$$

as $D(f)$ is closed. Here $X_g := \frac{d}{dt}|_0 \Phi_{tg}$. Using $d\tau = \omega_s \wedge \bar{d}\tau$,

$$0 = D(f)[i_{X_g}(\omega_s \wedge \bar{d}\tau)] = D(f)[\omega_s \wedge i_{X_g} \bar{d}\tau + i_{X_g} \omega_s \wedge \bar{d}\tau].$$

As $D(f)$ is Lagrangian, the first term vanishes, so we are left with

$$0 = D(f)[i_{X_g}\omega_s \wedge \bar{d}\tau].$$

The map Φ_{tg} is a symplectomorphism and it is easy to see that $\Phi_{tg}^*\alpha = \alpha + t\pi^* dg$. Differentiating, we obtain $\pi^* dg = \mathcal{L}_{X_g}\alpha = i_{X_g}d\alpha + di_{X_g}\alpha = -i_{X_g}\omega_s$. Here we have used that $d\pi(X_g) = 0$, i.e. $i_{X_g}\alpha = 0$, as Φ_{tg} maps each fiber to itself. In particular,

$$0 = D(f)[d(\pi^* g) \wedge \bar{d}\tau].$$

Using once again that $D(f)$ is closed, we arrive at

$$0 = D(f)(\pi^* g \wedge d\bar{d}\tau) = D(f)(\pi^* g \wedge \bar{\mathrm{D}}\,\tau).$$

As this is true for all $g \in C_c^\infty(V)$, $\bar{\mathrm{D}}\,\tau$ vanishes on all spaces tangent to the graph of df. We are now going to apply Lemma 7.2.4: Fix a Euclidean structure on V and use the induced isomorphism $V \times V^* \cong V \times V$. It can be checked that this is a symplectomorphism. Fix a point $(x, y) \in T^*V$. We claim that the pullback of $\bar{\mathrm{D}}\,\tau$ vanishes on all strictly positive isotropic subspaces at the corresponding point in $TV = V \times V$. Given a strictly positive subspace W, we thus need to find a strictly convex function $f \in \mathrm{Conv}(V, \mathbb{R}) \cap C^\infty(V)$ such that $df(x) = y$ and such that the tangent space to the graph of ∇f is exactly W. By definition, there exist orthonormal vectors $u_1, \dots, u_n \in V$ and positive numbers $\lambda_1, \dots, \lambda_n$ such that W is spanned by $w_i = (u_i, \lambda_i u_i)$. With respect to the basis $u_1, \dots, u_n$, we obtain linear coordinates $z_1, \dots, z_n$ on V, and we define $f \in \mathrm{Conv}(V, \mathbb{R}) \cap C^\infty(V)$ by

$$f(z) := \sum_{i=1}^{n} \frac{1}{2}\lambda_i z_i^2 + (y_i - \lambda_i x_i)z_i,$$

where $(y_1, \dots, y_n)$ are the coordinates with respect to the basis $u_1, \dots, u_n$ of the image of $y \in V^*$ in V under the isomorphism above. Then f has the desired properties. Using that $\bar{\mathrm{D}}\,\tau$ vanishes on all spaces tangent to the graph of df, we see that the pullback of $\bar{\mathrm{D}}\,\tau$ to TV vanishes on W. As this is true for all strictly positive isotropic subspaces and all $(x, y) \in T^*V \cong V \times V$, this pullback must be a multiple of the symplectic form due to Lemma 7.2.4. As $T^*V \cong V \times V$ are symplectomorphic, $\bar{\mathrm{D}}\,\tau$ must be a multiple of the symplectic form on T^*V as well. However, $\bar{\mathrm{D}}\,\tau$ is primitive due to Proposition 7.2.3, thus the Lefschetz decomposition in Proposition 7.2.1 implies $\bar{\mathrm{D}}\,\tau = 0$. $\square$

Corollary 7.2.6. *If $\tau \in \Omega_{hc}^n(T^*V)$ satisfies $\bar{\mathrm{D}}\,\tau = 0$, then $D(f)[\tau] = \int_V \tau$ for all f.*

Proof. Choose $\phi \in C_c^\infty(V)$ such that $\int_V \tau = \int_V \phi(x) d\,\mathrm{vol}(x)$. Then $\bar{\mathrm{D}}(\pi^*(\phi \wedge \mathrm{vol})) = 0$, as $d\pi^*(\phi \wedge \mathrm{vol}) = 0$. By definition $\int_V (\tau - \pi^*(\phi \wedge \mathrm{vol})) = 0$, so the valuations induced by τ and $\pi^*(\phi \wedge \mathrm{vol})$ have to coincide by Theorem 7.2.5. But $D(f)[\pi^*(\phi \wedge \mathrm{vol})] = \int_V \phi(x) d\,\mathrm{vol}(x) = \int_V \tau$ by the defining property of the differential cycle. $\square$

Proposition 7.2.7. *Let $G \subset \mathrm{Aff}(V)$ be a subgroup and let $\mu = D(\cdot)[\tau]$ be a G-invariant valuation on $\mathrm{Conv}(V, \mathbb{R})$. Then $g^*\,\bar{\mathrm{D}}\,\tau = \mathrm{sign}(\det g) D\tau$.*

Proof. By Proposition 7.1.4 we have $D(f \circ g) = \operatorname{sign}(\det g)(g^{-1})_* D(f)$, where $g \in G$ operates on T^*V by $g(x,y) = (gx, y \circ g^{-1})$, which is a symplectomorphism. Thus the valuation $f \mapsto \mu(f \circ g)$ is represented by the differential form $\operatorname{sign}(\det g)(g^{-1})^*\tau$. Proposition 7.2.3 implies $\bar{\mathrm{D}}((g^{-1})^*\tau) = (g^{-1})^* \bar{\mathrm{D}}\,\tau$ for all $g \in G$.
As τ induces a G-invariant valuation, $\tau - \operatorname{sign}(\det g)(g^{-1})^*\tau$ induces the trivial valuation for all $g \in G$. Thus

$$\begin{aligned} \bar{\mathrm{D}}\,\tau - \operatorname{sign}(\det g)(g^{-1})^* \bar{\mathrm{D}}\,\tau &= \bar{\mathrm{D}}\,\tau - \bar{\mathrm{D}}(\operatorname{sign}(\det g)(g^{-1})^*\tau) \\ &= \bar{\mathrm{D}}(\tau - \operatorname{sign}(\det g)(g^{-1})^*\tau) = 0 \end{aligned}$$

for all $g \in G$ by Theorem 7.2.5. □

We will call a differential form on T^*V vertically translation invariant if it is invariant with respect to translations in the second component of $T^*V \cong V \times V^*$.

Corollary 7.2.8. *A differential form τ represents a dually epi-translation invariant valuation μ if and only if $\bar{\mathrm{D}}\,\tau$ is vertically translation invariant and $\int_V \phi_\lambda^*\tau = \int_V \tau$ for all $\lambda \in V^*$, where $\phi_\lambda : T^*V \to T^*V$, $\phi(x,y) = (x, y+\lambda)$.*

Proof. By Proposition 7.1.3, $D(f+\lambda) = \phi_{\lambda *}D(f)$. If μ is dually epi-translation invariant, this implies that τ and $\phi_\lambda^*\tau$ induce the same valuation μ. Theorem 7.2.5 shows $\bar{\mathrm{D}}(\tau - \phi_\lambda^*\tau) = 0$ and $\int_V \phi_\lambda^*\tau = \int_V \tau$. But it is easy to see that ϕ_λ is a symplectomorphism, so $\bar{\mathrm{D}}\,\tau = \phi_\lambda^* \bar{\mathrm{D}}\,\tau$ by Proposition 7.2.3, i.e. $\bar{\mathrm{D}}\,\tau$ is vertically translation invariant.

For the converse direction, note that this argument also shows that $\bar{\mathrm{D}}(\tau) = \bar{\mathrm{D}}(\phi_\lambda\tau)$ for all $\lambda \in V^*$ if $\bar{\mathrm{D}}\,\tau$ is vertically translation invariant. Together with the second property, Theorem 7.2.5 implies that τ and $\phi_\lambda^*\tau$ induce the same valuation. Of course, any valuation obtained from the differential cycle is invariant under the addition of constants, so τ induces a dually epi-translation invariant valuation. □

Consider the map $m_t : T^*V \to T^*V$, $(x,y) \mapsto (x, ty)$ for $t > 0$. We will call a differential form τ on T^*V homogeneous of degree $k \in \mathbb{R}$ if $m_t^*\tau = t^k\tau$ for all $t > 0$.

Corollary 7.2.9. *τ represents a smooth valuation μ homogeneous of degree $k \geq 0$ if and only if*

1. *$\bar{\mathrm{D}}\,\tau$ is $(k-1)$-homogeneous and $\int_V m_t^*\tau = 0$ for all $t > 0$ if $k \neq 0$,*
2. *$\bar{\mathrm{D}}\,\tau = 0$ if $k = 0$.*

In particular, τ induces a constant valuation if and only if $\bar{\mathrm{D}}\,\tau = 0$.

Proof. Using Equation (7.1), $D(tf) = m_{t*}D(f)$. m_t is not a symplectomorphism but $m_t^*\omega_s = t\omega_s$, i.e. we obtain a constant multiple of ω_s. Set $\xi = \bar{d}\tau$, i.e. $d\tau = \omega_s \wedge \xi$. Then

$$d(m_t^*\tau) = m_t^* d\tau = m_t^*(\omega_s \wedge \xi) = t\omega_s \wedge m_t^*\xi,$$

and thus $\bar{d}(m_t^*\tau) = tm_t^*\xi = tm_t^*\bar{d}\tau$ and $\bar{\mathrm{D}}(m_t^*\tau) = d(tm_t^*\bar{d}\tau) = tm_t^*\,\bar{\mathrm{D}}\,\tau$.
Let μ be k-homogeneous. Then

$$D(f)[m_t^*\tau] = \mu(tf) = t^k\mu(f) = D(f)[t^k\tau] \quad \text{for all } f \in \mathrm{Conv}(V,\mathbb{R}).$$

Theorem 7.2.5 implies $\bar{\mathrm{D}}(m_t^*\tau - t^k\tau) = 0$ and using the computation above, we obtain $tm_t^*\,\bar{\mathrm{D}}\,\tau = t^k\,\bar{\mathrm{D}}\,\tau$ for all $t > 0$.
If $k = 0$, we obtain $\bar{\mathrm{D}}\,\tau = 0$ by considering the limit $t \to 0$ and thus μ is constant by Corollary 7.2.6. If $k \neq 0$, we can divide by t to obtain $m_t^*\,\bar{\mathrm{D}}\,\tau = t^{k-1}\,\bar{\mathrm{D}}\,\tau$ for all $t > 0$, i.e. $\bar{\mathrm{D}}\,\tau$ is $(k-1)$-homogeneous. Obviously, $\int_V m_t^*\tau = D(0)[m_t^*\tau] = m_{t*}D(0)[\tau] = D(t\cdot 0)[\tau] = \mu(0) = 0^k \cdot \mu(0) = 0$ if $k > 0$.

Now assume that $\bar{\mathrm{D}}\,\tau$ is $k-1$ homogeneous, $k \neq 0$, and $\int_V m_t^*\tau = 0$ for all $t > 0$. With the same computation as before, we conclude that $\bar{\mathrm{D}}(m_t^*\tau - t^k\tau) = 0$ for all $t > 0$. As $\int_V m_t^*\tau = 0$ by assumption, $m_t^*\tau$ and $t^k\tau$ induce the same valuation by Theorem 7.2.5, i.e. $\mu(tf) = t^k\mu(f)$ for all $t > 0$.
If $\bar{\mathrm{D}}\,\tau = 0$, then μ is constant by Corollary 7.2.6 and in particular 0-homogeneous. □

Let us also make the following observation:

Lemma 7.2.10. *If $\tau \in \Omega_{hc}^n(T^*V)$ induces a k-homogeneous valuation for $k \neq 0$, then*

$$D(f)[\tau] = \frac{1}{k}D(f)\left[f\,\bar{\mathrm{D}}\,\tau\right]$$

for all $f \in \mathrm{Conv}(V,\mathbb{R}) \cap C^\infty(V)$.

Proof. Let $G_f : T^*V \to T^*V$, $(x,y) \mapsto (x, y + df(x))$. Using Proposition 7.1.3 we see that

$$(1+t)^k D(f)[\tau] = D(f+tf)[\tau] = D(f)\left[G_{tf}^*\tau\right].$$

Differentiating at $t = 0$ and setting $X_f = \frac{d}{dt}|_0 G_{tf}$, we obtain

$$kD(f)[\tau] = D(f)\left[\mathcal{L}_{X_f}\tau\right] = D(f)\left[i_{X_f}d\tau\right] = D(f)\left[i_{X_f}(\omega_s \wedge \bar{d}\tau)\right] = D(f)\left[i_{X_f}\omega_s \wedge \bar{d}\tau\right].$$

As in the proof of Theorem 7.2.5, $i_{X_f}\omega_s = -\pi^*df$, so we obtain

$$kD(f)[\tau] = -D(f)\left[df \wedge \bar{d}\tau\right] = D(f)\left[f \wedge \bar{\mathrm{D}}\,\tau\right].$$

□

In the rest of this section, we will show that any smooth dually epi-translation invariant valuation, which can be represented by some (not necessarily invariant) differential form, can actually be obtained by only considering vertically translation invariant differential forms.
Let $\Omega^{k,l} = \Omega_c^k(V) \otimes \Lambda^l V^* \subset \Omega_{hc}^{k+l}(T^*V)$ denote the space of differential forms of bidegree (k,l) with horizontally compact support that are in addition translation invariant in

the second component of $T^*V \cong V \times V^*$. The next lemma describes the image of $\bar{\mathrm{D}} : \Omega^{n-k,k} \to \Omega^{n-(k-1),k-1}$. Recall that the de Rham cohomology with compact support is given by

$$H_c^k(\mathbb{R}^n) \cong \begin{cases} 0 & k \neq n, \\ \mathbb{R} & k = n. \end{cases}$$

This isomorphism is realized by the map $[\tau] \mapsto \int_{\mathbb{R}^n} \tau$.

Lemma 7.2.11. *For $2 \leq k \leq n$:*

$$\operatorname{Im}(\bar{\mathrm{D}} : \Omega^{n-k,k} \to \Omega^{n-(k-1),k-1}) = \ker d \cap \ker L \cap \Omega^{n-(k-1),k-1}.$$

For $k = 1$:

$$\begin{aligned} &\operatorname{Im}(\bar{\mathrm{D}} : \Omega^{n-1,1} \to \Omega^{n,0}) \\ &= \left\{ \pi^*(\phi \wedge \mathrm{vol}) : \phi \in C_c^\infty(V), \int_V \phi(x) d\,\mathrm{vol}(x) = \int_V \lambda(x)\phi(x) d\,\mathrm{vol}(x) = 0 \quad \forall \lambda \in V^* \right\}. \end{aligned}$$

Proof. Let us start with the case $2 \leq k \leq n$. Examining the degrees and using Proposition 7.2.3, the image of $\bar{\mathrm{D}}$ is contained in the space on the right. For the converse, let $\tau \in \ker d \cap \ker L \cap \Omega^{n-(k-1),k-1}$ be given. Choosing a basis ξ_i, $1 \leq i \leq \binom{n}{k-1}$, of $\Lambda^{k-1}V^*$, we find differential forms $\phi_i \in \Omega_c^{n-(k-1)}(V)$ such that

$$\tau = \sum_i \xi_i \wedge \phi_i.$$

As τ is closed, $0 = (-1)^{k-1} \sum_i \xi_i \wedge d\phi_i$, and thus $d\phi_i = 0$ for all i. Using $H_c^{n-(k-1)}(V) = 0$ for $2 \leq k \leq n$, we see that there exists $\psi_i \in \Omega_c^{n-k}(V)$ such that $\phi_i = (-1)^{k-1} d\psi_i$. Set

$$\omega := \sum_i \xi_i \wedge \psi_i,$$

i.e. $\tau = d\omega$. Then $d\alpha \wedge \omega$ is closed, as $d(d\alpha \wedge \omega) = d\alpha \wedge d\omega = d\alpha \wedge \tau = 0$, because τ belongs to the kernel of L. We will need to find a vertically translation invariant n-form $\tilde{\tau}$ such that $d\tilde{\tau} = -d\alpha \wedge \omega$.

Note that $-d\alpha \wedge \omega \in \Omega^{n+1}(T^*V)$ is again a vertically translation invariant differential form, now of bidegree $(n-k+1, k)$. If $\tilde{\xi}_i$, $i = 1, \ldots, \binom{n}{k}$, denotes a basis of $\Lambda^k V^*$, there exist unique differential forms $\tilde{\phi}_i \in \Omega_c^{n-k+1}(V)$ such that

$$-d\alpha \wedge \omega = \sum_i \tilde{\xi}_i \wedge \tilde{\phi}_i.$$

As $d\alpha \wedge \omega$ is closed, we obtain $0 = (-1)^k \sum_i \tilde{\xi}_i \wedge d\tilde{\phi}_i$, i.e. $d\tilde{\phi}_i = 0$ for all i. Using $H_c^{n-k+1}(V) = 0$ for $k \neq 1$, we obtain $\tilde{\psi}_i \in \Omega_c^{n-k}(V)$ such that $\tilde{\phi}_i = (-1)^k d\tilde{\psi}_i$. Then $-d\alpha \wedge \omega = d\tilde{\tau}$, where

$$\tilde{\tau} = \sum_i \tilde{\xi}_i \wedge \tilde{\psi}_i \in \Omega^{n-k,k}.$$

Thus $\bar{\mathrm{D}}\,\tilde{\tau} = d\omega = \tau$.

For $k = 1$, choose oriented linear coordinates $(x_1, \dots, x_n)$ on V with induced coordinates $(y_1, \dots, y_n)$ on V^* and volume form $\mathrm{vol} = dx_1 \wedge \dots \wedge dx_n$. Let $\phi \in C_c^\infty(V)$ be a function with $\int_V \phi(x) d\,\mathrm{vol}(x) = 0$. Then $\phi \wedge \mathrm{vol}$ belongs to the trivial cohomology class of $H_c^n(V) \cong \mathbb{R}$, so there exists $\omega \in \Omega_c^{n-1}(V)$ such that $\phi \wedge \mathrm{vol} = d\omega$. Now $d\alpha \wedge \omega = d(\alpha \wedge \omega) - \alpha \wedge d\omega$. As $d\omega$ is a multiple of $\pi^* \mathrm{vol}$, the second term vanishes, and we are left with $d\alpha \wedge \omega = d(\alpha \wedge \omega)$. Thus $\bar{\mathrm{D}}(-\alpha \wedge \omega) = d\omega = \pi^*(\phi \wedge \mathrm{vol})$. Moreover, $\alpha \wedge \omega = \sum_{i=1}^n y_i \pi^*(\phi_i \,\mathrm{vol})$ for some $\phi_i \in C_c^\infty(V)$. If we consider the valuation induced by $-\alpha \wedge \omega$, the defining property of the differential cycle implies

$$D(f)[-\alpha \wedge \omega] = -D(f)\left[\sum_{i=1}^n y_i \pi^*(\phi_i \,\mathrm{vol})\right] = -\sum_{i=1}^n \int_V \partial_i f(x) \phi_i(x) d\,\mathrm{vol}(x). \tag{7.2}$$

On the other hand, Lemma 7.2.10 implies

$$D(f)[-\alpha \wedge \omega] = D(f)\left[f\,\bar{\mathrm{D}}(-\alpha \wedge \omega)\right] = D(f)\left[\pi^*(f\phi \wedge \mathrm{vol})\right] = \int_V f(x)\phi(x) d\,\mathrm{vol}(x)$$

for all $f \in \mathrm{Conv}(V, \mathbb{R}) \cap C^\infty(V)$. As $\int_V \phi(x) d\,\mathrm{vol}(x) = \int_V x_i \phi(x) d\,\mathrm{vol}(x) = 0$ for all $1 \le i \le n$, $D(\cdot)[-\alpha \wedge \omega]$ is invariant under the addition of constant and linear functions. From Equation (7.2) we deduce $0 = \sum_{i=1}^n l_i \int_V \phi_i(x) d\,\mathrm{vol}(x)$ for all $l = (l_1, \dots, l_n) \in V^*$, which implies $\int_V \phi_i(x) d\,\mathrm{vol}(x) = 0$ for all $i = 1, \dots, n$, i.e. $\phi_i \wedge \mathrm{vol}$ is trivial in cohomology. Thus we can find $\psi_i \in \Omega_c^{n-1}(V)$ such that $d\psi_i = \phi_i \wedge \mathrm{vol}$. In total, $\alpha \wedge \omega = \sum_{i=1}^n y_i d\psi_i = d(\sum_{i=1}^n y_i \psi_i) - \sum_{i=1}^n dy_i \wedge \psi_i$. Then $\tau := \sum_{i=1}^n dy_i \wedge \psi_i \in \Omega^{n-1,1}$ satisfies $\bar{\mathrm{D}}\,\tau = \bar{\mathrm{D}}(-\alpha \wedge \omega + d(\sum_{i=1}^n y_i \psi_i)) = -\bar{\mathrm{D}}(\alpha \wedge \omega) = \pi^*(\phi \wedge \mathrm{vol})$. □

Theorem 7.2.12. *Let* $\mathrm{VConv}(V)^{sm} \subset \mathrm{VConv}(V)$ *denote the space of all dually epi-translation invariant valuations of the form* $f \mapsto D(f)[\tau]$ *for some* $\tau \in \Omega_{hc}^n(T^*V)$. *Then the following holds:*

1. *The map* $\Omega^{n-k,k} \to \mathrm{VConv}_k(V)^{sm}$, $\tau \mapsto D(\cdot)[\tau]$ *is surjective for all* $0 \le k \le n$.
2. *For* $2 \le k \le n$ $\bar{\mathrm{D}}$ *induces an isomorphism*

$$\begin{aligned}\mathrm{VConv}_k(V)^{sm} \cong &\mathrm{Im}(\bar{\mathrm{D}} : \Omega^{n-k,k} \to \Omega^{n-(k-1),k-1}) \\ &= \ker d \cap \ker L \cap \Omega^{n-(k-1),k-1}.\end{aligned}$$

3. *For* $k = 1$ $\bar{\mathrm{D}}$ *induces an isomorphism*

$$\begin{aligned}&\mathrm{VConv}_1(V)^{sm} \cong \mathrm{Im}(\bar{\mathrm{D}} : \Omega^{n-1,1} \to \Omega^{n,0}) \\ &= \left\{\pi^*(\phi \wedge \mathrm{vol}) : \phi \in C_c^\infty(V), \int_V \phi(x) d\,\mathrm{vol}(x) = \int_V \lambda(x)\phi(x) d\,\mathrm{vol}(x) = 0 \ \forall \lambda \in V^*\right\}.\end{aligned}$$

Proof. As any k-homogeneous valuation of degree $k>0$ vanishes in $0\in \mathrm{Conv}(V,\mathbb{R})$, a valuation $\mu\in \mathrm{VConv}_k(V)^{sm}$ is uniquely determined by $\bar{\mathrm{D}}\,\tau$, where τ is any smooth differential form representing μ, due to Theorem 7.2.5. Thus 2. and 3. follow from 1. using Lemma 7.2.11. For $k=0$, the map in 1. is obviously surjective. Thus let $k>0$. As remarked before, any k-homogeneous valuation represented by $\tau\in\Omega^n_{hc}(T^*V)$ vanishes in 0 and $\bar{\mathrm{D}}\,\tau$ is vertically translation invariant and $k-1$-homogeneous. For $k\geq 2$, this implies that $\bar{\mathrm{D}}\,\tau$ belongs to the image of $\bar{\mathrm{D}}:\Omega^{n-k,k}\rightarrow\Omega^{n-(k-1),k-1}$, so we find some $\tilde{\tau}\in\Omega^{n-k,k}$ with $\bar{\mathrm{D}}(\tau-\tilde{\tau})=0$. Of course, any such differential form satisfies $\int_V\tilde{\tau}=0$, so Theorem 7.2.5 implies that $\tilde{\tau}$ and τ induce the same valuation.
For $k=1$, we need to show that $\bar{\mathrm{D}}\,\tau=\pi^*(\phi\wedge\mathrm{vol})$ is in the image of $\bar{\mathrm{D}}:\Omega^{n-1,1}\rightarrow\Omega^{n,0}$. However, this follows from the fact that $D(\cdot)[\tau]$ is dually epi-translation invariant together with Lemma 7.2.10. With the same argument as before we find $\tilde{\tau}\in\Omega^{n-1,1}$ with $\bar{\mathrm{D}}(\tau-\tilde{\tau})=0$ and $\int_V\tilde{\tau}=0=\int_V\tau$. Applying Theorem 7.2.5 again, we obtain the desired result. □

Let us also add the following observation:

Lemma 7.2.13. *Let $\mu\in\mathrm{VConv}(V)$ be a smooth valuation. For every function $\phi\in C^\infty(V)$ (not necessarily with compact support) there exists a unique smooth valuation μ_ϕ such that*

$$\mu_\phi(f)=\frac{d}{dt}\Big|_0\mu(f+t\phi)$$

for all $f\in\mathrm{Conv}(V,\mathbb{R})$.

Proof. First note that every smooth valuation naturally extends to a functional on all Monge-Ampère functions, so the right hand side is well defined for all $f\in\mathrm{Conv}(V,\mathbb{R})$ by Proposition 7.1.3. If μ is represented by some differential form τ, then

$$\frac{d}{dt}\Big|_0\mu(f+t\phi)=\frac{d}{dt}\Big|_0 D(f+t\phi)[\tau]=\frac{d}{dt}\Big|_0 D(f)[G^*_{t\phi}\tau]$$

for $G_{t\phi}(x,y):=(x,y+td\phi(x))$ by Proposition 7.1.3. Setting $X_\phi:=\frac{d}{dt}|_0G_{t\phi}$, we see that

$$\begin{aligned}\frac{d}{dt}\Big|_0\mu(f+t\phi)=&D(f)\left[\mathcal{L}_{X_\phi}\tau\right]=D(f)\left[i_{X_\phi}d\tau\right]=D(f)\left[i_{X_\phi}(\omega_s\wedge\bar{d}\tau)\right]\\ =&D(f)\left[i_{X_\phi}\omega_s\wedge\bar{d}\tau\right]=-D(f)\left[d\phi\wedge\bar{d}\tau\right]=D(f)\left[\phi\,\bar{\mathrm{D}}\,\tau\right].\end{aligned}$$

Thus $f\mapsto\mu_\phi(f):=D(f)\left[\phi\,\bar{\mathrm{D}}\,\tau\right]$ is the desired smooth valuation. □

7.3 Characterization of smooth valuations

Let us choose a scalar product on V with induced scalar products on V^* and $V\times\mathbb{R}$. In addition, let us fix an orientation on V (and thus V^*). If $\mathrm{vol}\in\Lambda^nV$ induces the orientation of V^*, we will equip $V^* \times \mathbb{R}$ with the orientation induced by $-dt \wedge \mathrm{vol}$, where dt is the standard coordinate form on $\mathbb{R}$. Consider the map

$$\begin{aligned} Q : (V^* \times \mathbb{R}) \times \mathbb{P}_+(V \times \mathbb{R})_- &\to V \times V^* = T^*V \\ (y, s, [(x,t)]) &\mapsto \left(-\frac{x}{t}, y\right). \end{aligned}$$

To simplify the notation, let $E := V^* \times \mathbb{R}$, such that $Q : \mathbb{P}E_- := E \times \mathbb{P}_+(E^*)_- \to T^*V$.

Proposition 7.3.1. *Let $K \in \mathcal{K}(V^* \times \mathbb{R})$. Then*

$$Q_* \left[\mathrm{N}^*(K)|_{\mathbb{P}E_-}\right] = D\left(h_K(\cdot, -1)\right).$$

Proof. As $\operatorname{supp} \mathrm{N}^*(K) \subset K \times \mathbb{P}_+(E)$, Q is proper on the support of $\mathrm{N}^*(K)\big|_{\mathbb{P}E_-}$. Now observe that both sides depend continuously on K in the local flat topology by Proposition 3.3.2 and Theorem 7.1.8. It is thus enough to prove the equation for $K \in \mathcal{K}(E)$ smooth and strictly convex. In this case, the support function of K is smooth outside of 0 and

$$\mathrm{N}^*(K) = (d'h_K \times Id)_* \left[\mathbb{P}_+(E^*)\right]$$

by Lemma 3.3.1. We therefore need to consider the map

$$\begin{aligned} Q \circ (d'h_K \times Id) : \mathbb{P}_+(E^*)_- &\to V \times V^* \\ [(x,t)] &\mapsto \left(-\frac{x}{t}, \partial_1 h_K(x,t)\right), \end{aligned}$$

where $\partial_1 h_K = (\partial_{x_1} h_K, \dots, \partial_{x_N} h_K)$. h_K is 1-homogeneous, so $\partial_1 h_K(x,t) = \partial_1 h_K(-\frac{x}{t}, -1) = df_K(-\frac{x}{t})$ for $t < 0$, where $f_K := h_K(\cdot, -1)$. Thus

$$Q \circ (d'h_K \times Id)\left([(x,t)]\right) = \left(-\frac{x}{t}, df_K\left(-\frac{x}{t}\right)\right)$$

for all $[(x,t)] \in \mathbb{P}_+(E^*)_-$. The map $\mathbb{P}_+(E^*)_- \to V$, $[(x,t)] \mapsto -\frac{x}{t}$ is a diffeomorphism and it is easy to see that it is orientation preserving for our choice of orientation. As $D(h_K(\cdot, -1))$ is given by integration over the graph of df_K, we see that both currents coincide. □

Let us choose orthonormal linear coordinates $x_1, \dots, x_n$ on V with induced coordinates $(y_1, \dots, y_n)$ on V^*. We will denote the induced contact form on $E \times \mathbb{P}_+(E^*)$ by α_E, $\omega_E := -d\alpha_E$. Then $\alpha_E = sdt + \sum_{i=1}^n x_i dy_i$ with respect to the coordinates (y, s, x, t) on $V^* \times \mathbb{R} \times S(V \times \mathbb{R}) \cong V^* \times \mathbb{R} \times \mathbb{P}_+(V \times \mathbb{R})$.

Lemma 7.3.2. *Let $\omega \in \Omega^k(E \times \mathbb{P}_+(E^*))$ be a translation invariant differential form. Then there exists a differential form $\omega' \in \Omega^k(T^*V)$ such that $\omega - Q^*\omega'$ is vertical on $E \times \mathbb{P}_+(E^*)_-$, i.e. a multiple of the contact form α_E.*

Proof. Any translation invariant differential form ω on $E \times \mathbb{P}_+(E^*)$ can be written as a sum of terms of the form $ds \wedge dy^I \wedge \tau$ or $dy^I \wedge \tau$, where τ is a form on $\mathbb{P}_+(V \times \mathbb{R})$ of degree $k - |I| - 1$ or $k - |I|$ respectively. As $\alpha_E = tds + \sum_{j=1}^n x_j dy_j$, we can replace ds by $-\frac{1}{t}\sum_{j=1}^n x_j dy_j$ while picking up a multiple of α_E. We can thus assume that ω only consists of terms of the form $dy^I \wedge \tau$ with $\tau \in \Omega^*(\mathbb{P}_+(V \times \mathbb{R}))$, i.e. ω is the pullback of a form $\tilde{\omega}$ on $V^* \times \mathbb{P}_+(V \times \mathbb{R})$. Obviously, $\tilde{Q} : V^* \times \mathbb{P}_+(V \times \mathbb{R})_- \to T^*V$, $(y, [(x,t)]) \mapsto (-\frac{x}{t}, y)$ is a diffeomorphism, and if we let $\tilde{\pi} : V^* \times \mathbb{R} \times \mathbb{P}_+(V \times \mathbb{R})_- \to V^* \times \mathbb{P}_+(V \times \mathbb{R})_-$ denote the obvious projection, we obtain $\tilde{Q} \circ \tilde{\pi} = Q$. The claim follows by setting $\omega' := (\tilde{Q}^{-1})^* \tilde{\omega}$. $\square$

Due to the kernel theorems 3.3.4 and 7.2.5, a smooth valuation is (up to its 0-homogeneous component) uniquely determined by the (symplectic) Rumin differential of a representing form. We will thus need the following compatibility between the two versions of the differential.

Corollary 7.3.3. *For any smooth differential form $\tau \in \Omega^n(T^*V)$, $DQ^*\tau = -\frac{1}{t}\alpha_E \wedge Q^* \bar{\mathrm{D}}\,\tau$.*

Proof. Let ω_V denote the symplectic form on T^*V. A short calculation shows $Q^*\omega_V = \frac{1}{t}\omega_E + \frac{1}{t^2} dt \wedge \alpha_E$. Let $\xi \in \Omega^{n-1}(T^*V)$ be the unique form with $\omega_V \wedge \xi = d\tau$. Pulling back this equation, we see that

$$dQ^*\tau = Q^*\omega_V \wedge Q^*\xi = \frac{1}{t}\omega_E \wedge Q^*\xi + \frac{1}{t^2} dt \wedge \alpha_E \wedge Q^*\xi.$$

Restricting this equation to the contact distribution H in $E \times P_+(E^*)$, we obtain

$$dQ^*\tau|_H = \frac{1}{t}\omega_E \wedge Q^*\xi|_H = \omega_E|_H \wedge \frac{1}{t} Q^*\xi|_H.$$

This implies

$$\begin{aligned} d(Q^*\tau + \alpha_E \wedge \frac{1}{t}Q^*\xi) =& dQ^*\tau - \omega_E \wedge \frac{1}{t}Q^*\xi - \alpha_E \wedge d(\frac{1}{t}Q^*\xi) \\ =& -\frac{1}{t^2}\alpha_E \wedge dt \wedge Q^*\xi + \frac{1}{t^2}\alpha_E \wedge dt \wedge Q^*\xi - \alpha_E \wedge \frac{1}{t} dQ^*\xi \\ =& -\frac{1}{t}\alpha_E \wedge Q^* d\xi = -\frac{1}{t}\alpha_E \wedge Q^* \bar{\mathrm{D}}\,\tau, \end{aligned}$$

which is vertical. Thus $D(Q^*\tau) = d(Q^*\tau + \alpha_E \wedge \frac{1}{t}Q^*\xi) = -\frac{1}{t}\alpha_E \wedge Q^* \bar{\mathrm{D}}\,\tau$. $\square$

Proposition 7.3.4. *Let $\mu \in \mathrm{VConv}(V)$ be a valuation such that $T(\mu) \in \mathrm{Val}(V^* \times \mathbb{R})^{sm}$. Then there exists a differential form $\tau \in \Omega^n_{hc}(T^*V)$ such that*

$$\mu(f) = D(f)[\tau] \quad \forall f \in \mathrm{Conv}(V, \mathbb{R}).$$

In particular $\mu \in \mathrm{VConv}(V)^{sm}$ if and only if $T(\mu) \in \mathrm{Val}(V^* \times \mathbb{R})^{sm}$.

Proof. Using the homogeneous decomposition, we can assume that μ is homogeneous of degree $1 \leq k \leq n$. As $T(\mu)$ is a smooth valuation, it can be represented by a smooth differential form $\omega \in \Omega^{k,n-k}(E \times \mathbb{P}_+(E^*))$. Using Lemma 7.3.2, we can find a differential form $\omega' \in \Omega^n(T^*V)$ such that $\omega - Q^*\omega'$ differ by a multiple of α on $E \times \mathbb{P}_+(E^*)_-$. Applying the Rumin differential and using Corollary 7.3.3, we obtain

$$D\omega = DQ^*\omega' = -\frac{1}{t}\alpha \wedge Q^* \bar{\mathrm{D}}\,\omega' \quad \text{on } E \times \mathbb{P}_+(E^*)_-.$$

By Theorem 6.3.2, the support of $T(\mu)$ is compactly contained in $\mathbb{P}_+(E^*)_-$. From Proposition 3.3.5 we deduce that $D\omega$ has support compactly contained in $E \times \mathbb{P}_+(E^*)_-$, so the same applies to $Q^* \bar{\mathrm{D}}\,\omega'$. Thus the support of $\bar{\mathrm{D}}\,\omega'$ is horizontally compact. By construction, this is a vertically translation invariant form of bidegree $(n+1-k, k-1)$. Using Lemma 7.2.11, we find a vertically translation invariant form $\tau \in \Omega^{n-k,k} \subset \Omega^n_{hc}(T^*V)$ such that $\bar{\mathrm{D}}\,\tau = \bar{\mathrm{D}}\,\omega$. It remains to see that μ is represented by the differential form τ. Observe that

$$D\omega = -\frac{1}{t}\alpha \wedge Q^* \bar{\mathrm{D}}\,\omega' = -\frac{1}{t}\alpha \wedge Q^* \bar{\mathrm{D}}\,\tau = D(Q^*\tau) \quad \text{on } E \times \mathbb{P}_+(E^*)_-.$$

By extending $Q^*\tau$ trivially to $E \times \mathbb{P}_+(E^*)$, we see that this equation holds on the whole space, so Theorem 3.3.4 implies that ω and $Q^*\tau$ induces the same valuation (note that the second property in Theorem 3.3.4 is satisfied as the degree of our valuation is positive). In particular

$$\begin{aligned} T(\mu)(K) =& \mathrm{N}^*(K)[\omega] = \mathrm{N}^*(K)\,[Q^*\tau] = \left[\mathrm{N}^*(K)\big|_{E\times\mathbb{P}_+(E^*)_-}\right][Q^*\tau] \\ =& D\left(h_K(\cdot,-1)\right)[\tau] = T\left(D(\cdot)[\tau]\right)(K) \end{aligned}$$

for any $K \in \mathcal{K}(V^* \times \mathbb{R})$, where we have used Proposition 7.3.1. The injectivity of T implies $\mu = D(\cdot)[\tau]$.
It remains to see that any valuation $\mu \in \mathrm{VConv}(V)^{sm}$ satisfies $T(\mu) \in \mathrm{Val}(V^* \times \mathbb{R})^{sm}$. This follows directly from Proposition 7.3.1 and the characterization of $\mathrm{Val}(V^* \times \mathbb{R})^{sm}$ in Theorem 3.3.3. □

Theorem 7.3.5. $\mathrm{VConv}(V)^{sm}$ *is dense in* $\mathrm{VConv}(V)$. *More precisely, the following holds: For every compact set $A \subset V$ and every compact neighborhood $B \subset V$ of A, there exists a sequence $(\mu_j)_j$ in $\mathrm{VConv}_B(V)^{sm}$ such that $(\mu_j)_j$ converges to μ.*

Proof. Let $\mu \in \mathrm{VConv}_A(V)$ be given and consider the following commutative diagram with the diffeomorphism

$$\begin{aligned} P : V &\to \mathbb{P}_+(V \times \mathbb{R})_- \\ v &\mapsto [(v,-1)] \end{aligned}$$

from Section 6.3.1:

$$
\begin{tikzcd}
\mathrm{VConv}_A(V) \arrow[r, "T"] \arrow[d] & \mathrm{Val}_{P(A)}(V^* \times \mathbb{R}) \arrow[d] \\
\mathrm{VConv}_B(V) \arrow[r, "T"] & \mathrm{Val}_{P(B)}(V^* \times \mathbb{R})
\end{tikzcd}
$$

The vertical maps are the natural inclusions, while the horizontal maps are topological isomorphisms due to Theorem 6.3.2. As P is a diffeomorphism, $P(B)$ is a compact neighborhood of $P(A)$, so using Proposition 3.1.6, we can find a sequence $(\nu_j)_j$ in $\mathrm{Val}_{P(B)}(V^* \times \mathbb{R}) \cap \mathrm{Val}(V^* \times \mathbb{R})^{sm}$ such that (ν_j) converges to $T(\mu)$. Then $(T^{-1}(\nu_j))_j$ is a sequence in $\mathrm{VConv}_B(V)$ that converges to μ in $\mathrm{VConv}_B(V)$ and, by Proposition 7.3.4, $T^{-1}(\nu_j) \in \mathrm{VConv}(V)^{sm}$. The claim follows. □

Corollary 7.3.6. *Let $G \subset \mathrm{GL}(V)$ be a compact subgroup. Then the space of smooth G-invariant valuations is dense in the space $\mathrm{VConv}(V)^G$ of all continuous G-invariant elements of $\mathrm{VConv}(V)$.*

Proof. Without loss of generality, we can assume that G is a subgroup of $O(n)$, $V = \mathbb{R}^n$. Let μ be a G-invariant valuation and let $R > 0$ be such that B_R is a neighborhood of $\operatorname{supp}\mu$. Applying Proposition 6.1.3, it is easy to deduce that G maps an element of $\mathrm{VConv}_{B_R}(\mathbb{R}^n)$ to an element of the same space. Using Theorem 7.3.5, choose a sequence $(\mu_j)_j$ of smooth valuations converging to μ such that the supports of the valuations μ_j are all contained in B_R. By the previous theorem, each μ_j can be represented by a vertically translation invariant differential form ω_j. By averaging μ_j with respect to the Haar measure, we obtain a G-invariant valuation $\tilde{\mu}_j \in \mathrm{VConv}_{B_R}(\mathbb{R}^n)$. We claim that this valuation is induced by the differential form $\tilde{\omega}_j$ obtained by averaging $g \mapsto \operatorname{sign}(\det g)(g^{-1})^*\omega_j$ with respect to the Haar measure. Using the relation $D(f \circ g)[\tau] = \operatorname{sign}(\det g)D(f)[(g^{-1})^*\tau]$ from Proposition 7.1.4, this is easily verified. Thus $\tilde{\mu}_j$ is a smooth G-invariant valuation.
It is easy to see that G acts continuously on $\mathrm{VConv}_{B_R}(\mathbb{R}^n)$, i.e. the map

$$
\begin{aligned}
G \times \mathrm{VConv}_{B_R}(\mathbb{R}^n) &\to \mathrm{VConv}_{B_R}(\mathbb{R}^n) \\
(g, \mu) &\mapsto [f \mapsto (g \cdot \mu)(f) := \mu(f \circ g)]
\end{aligned}
$$

is continuous. By Corollary 6.2.6, this is a Banach space with norm $\|\cdot\| := \|\cdot\|_{B_R,1}$, and the definition of the norm implies that G acts by isometries. We obtain

$$\|\mu - \tilde{\mu}_j\| = \left\| \int_G g \cdot (\mu - \mu_j) dg \right\| \le \int_G \|g \cdot (\mu - \mu_j)\| \, dg = \int_G \|\mu - \mu_j\| \, dg = \|\mu - \mu_j\| .$$

Thus $(\tilde{\mu}_j)_j$ is a sequence of smooth G-invariant valuations converging to μ. □

8 McMullen's conjecture for dually epi-translation invariant valuations

In this Chapter we will show that smooth mixed Hessian valuations are dense in $\mathrm{VConv}(V)$. We first establish a relation between the surface area measure of a convex body in $V^*\times\mathbb{R}$ and the Hessian measure of the restriction of its support function in Section 8.1. Section 8.2 introduces mixed Hessian valuations and examines their connection to Alesker valuations. Finally, McMullen's conjecture for $\mathrm{VConv}(V)$ is proved in Section 8.3 using a construction from Alesker [3] that represents a smooth valuation on convex bodies as a converging sum of mixed volumes.

The results of this chapter are to be published in [34].

8.1 The Relation between the n-th Hessian measure and the surface area measure

Throughout this section let us assume that V carries some Euclidean structure, which also induces a normalization of the Hessian measure Hess_n.
Let us recall the following well known theorem.

Theorem 8.1.1. *For every $K\in\mathcal{K}(V)$ there exists a measure $S_{n-1}(K)$ on $S(V^*)$, called the surface area measure, which as a measure-valued map on $\mathcal{K}(V)$ is uniquely determined by the following properties:*

1. *If K is a polytope with non-empty interior and unit normals $\{u_1,\dots,u_s\}\subset S(V^*)$, then*
$$S_{n-1}(K)=\sum_{i=1}^{s}\mathrm{vol}_{n-1}(F(K,u_i))\delta_{u_i}.$$
Here $F(K,u_i)$ is the face in direction u_i.
2. *If a sequence $(K_j)_j$ converges to K with respect to the Hausdorff metric, then $S_{n-1}(K_j)$ converges weakly to $S_{n-1}(K)$.*

Usually the surface area measure is interpreted as a measure on the unit sphere in V. We will work in the dual space due to the following characterization in terms of support functions.

Lemma 8.1.2. *For $K, L \in \mathcal{K}(V)$:*

$$\frac{d}{dt} \operatorname{vol}_n(K + tL) = \frac{1}{n} \int_{S(V^*)} h_L dS_{n-1}(K).$$

Proof. See [54] 5.1.7. $\square$

These measures also play an important role in the characterization of $\operatorname{Val}_{n-1}(V)$.

Theorem 8.1.3 (McMullen [44])**.** *For every $\mu \in \operatorname{Val}_{n-1}(V)$ there exists a function $f \in C(S(V^*))$ such that*

$$\mu(K) = \int_{S(V^*)} f dS_{n-1}(K) \quad \forall K \in \mathcal{K}(V).$$

Furthermore, this function is unique up to the restriction of linear functions.

Let $\mu \in \operatorname{VConv}_n(V)$. Then $T(\mu) \in \operatorname{Val}_n(V^* \times \mathbb{R})$ and can thus be written as

$$T(\mu) = \int_{S(V \times \mathbb{R})} f dS_n(\cdot)$$

for some function $f \in C(S(V \times \mathbb{R}))$ by McMullen's characterization. On the other hand, $\mu = \int_V \phi d \operatorname{Hess}_n$, for some $\phi \in C_c(V)$ due to Theorem 5.2.4. The next lemma shows how these two functions are related.

Lemma 8.1.4. *If $\mu = \int_V \phi d \operatorname{Hess}_n$ for $\phi \in C_c(V)$ and $T(\mu) = \int_{S(V \times \mathbb{R})} f dS_n$ for a function $f \in C(S(V \times \mathbb{R}))$, then*

$$\phi(y) = \sqrt{1 + |y|^2} \cdot \left[f\left(\frac{y}{\sqrt{1+|y|^2}}, -\frac{1}{\sqrt{1+|y|^2}} \right) + f\left(-\frac{y}{\sqrt{1+|y|^2}}, \frac{1}{\sqrt{1+|y|^2}} \right) \right].$$

Proof. Let us identify $V \cong V^*$. First note that Theorem 8.1.1 implies that

$$T(\mu)[P] = [f(u_0) + f(-u_0)] \operatorname{vol}_n(P) \quad \text{for all } P \in \mathcal{K}(\ker u_0),$$

where $u_0 \in S(V \times \mathbb{R})$ is an arbitrary unit direction. Choose an orthonormal basis $e_1, \dots, e_n$ of V and let P be the parallelotope spanned by these basis vectors. By Proposition 5.2.2,

$$\mu(h_P(\cdot - y)) = \phi(y) \cdot \operatorname{vol}_n(P) = \phi(y) \quad \forall y \in V.$$

On the other hand, considering $P \subset V \times \mathbb{R}$,

$$h_P(\cdot - y, -1) = [h_P \circ g_y](\cdot, -1)$$

for $g_y \in \operatorname{GL}(V \times \mathbb{R})$ given by

$$g_y := \begin{pmatrix} Id_n & y \\ 0 & 1 \end{pmatrix}.$$

Identifying $V \cong V^*$, we thus obtain

$$\begin{aligned}\mu\left(h_P(\cdot - y, -1)\right) =&\mu\left([h_P \circ g_y](\cdot, -1)\right) = \mu\left(h_{g_y^T P}(\cdot, -1)\right) = T(\mu)\left[g_y^T P\right]\\ =&[f(u_0) + f(-u_0)]\operatorname{vol}_n\left(g_y^T P\right),\end{aligned}$$

where $u_0 := \frac{1}{\sqrt{1+|y|^2}}(y, -1)$ is orthogonal to $g_y^T P$.
We thus only need to calculate $\operatorname{vol}_n(g_y^T P)$. This is equal to the $n+1$-dimensional volume of the parallelotope spanned by $g_y^T e_1, \dots, g_y^T e_n, u_0$, i.e. it is given by the absolute value of the determinant of the matrix

$$\begin{pmatrix} g_y^T e_1 & \dots & g_y^T e_n & u_0 \end{pmatrix} = \begin{pmatrix} e_1 & \dots & e_n & \frac{y}{\sqrt{1+|y|^2}} \\ y_1 & \dots & y_n & -\frac{1}{\sqrt{1+|y|^2}} \end{pmatrix},$$

which is $\sqrt{1+|y|^2}$. $\square$

Also note that, if we consider $f \in C(S(V \times \mathbb{R}))$ as the restriction of a 1-homogeneous function on $V \times \mathbb{R}$ to $S(V \times \mathbb{R})$, the formula in Lemma 8.1.4 simplifies to

$$\phi(y) = f(y, -1) + f(-y, 1) \quad \text{for } y \in V.$$

Assuming that the support of f is contained in the negative half sphere $S(V \times \mathbb{R})_- := \{(v, t) \in S(V \times \mathbb{R}) \ : \ t < 0\}$, this implies that f is the 1-homogeneous extension of ϕ to $V \times (-\infty, 0)$, extended by 0 on $V \times [0, \infty)$.

8.2 Alesker valuations and mixed Hessian valuations

Abusing notation, let us denote the polarization of the Hessian measure by Hess_n again. As in Section 5.4.1, we can extend this functional to a multilinear functional on differences of convex functions. We can thus consider the valuation $f \mapsto \operatorname{Hess}_n(f[n-k], \psi_1, \dots, \psi_k)$ for $\psi_1, \dots, \psi_k \in C_c^2(V)$, which belongs to $\operatorname{VConv}_{n-k}(V, \mathcal{M}(V))$. If $f \in \operatorname{Conv}(V, \mathbb{R}) \cap C^2(V)$, then this valuation is given by

$$\int_V \phi d \operatorname{Hess}_n\left(f[n-k], \psi_1, \dots, \psi_k\right) = \int_V \phi(x) \det\left(H_f(x)[n-k], H_{\psi_1}(x), \dots, H_{\psi_k}(x)\right) dx,$$

where det denotes the mixed determinant, and is thus an Alesker valuation. Let us also set

$$\mu_\phi(f) := \int_V \phi d \operatorname{Hess}_n(f)$$

for $\phi \in C_c(V)$.

Definition 8.2.1. *For $f_1, .., f_k \in \operatorname{Conv}(V, \mathbb{R})$, $\bar{\mu}_\phi(\cdot[n-k], f_1, \dots, f_k) \in \operatorname{VConv}_{n-k}(V)$ is called a mixed Hessian valuation.*

Proposition 8.2.2. *The space spanned by mixed Hessian valuations contains all Alesker valuations. Every mixed Hessian valuation $\bar{\mu}_\phi(\cdot[n-k], f_1, \dots, f_k)$ with $f_1, \dots, f_k \in \mathrm{Conv}(V) \cap C^2(V)$ is an Alesker valuation.*
Furthermore, if the coefficients of an Alesker valuation are smooth, so is the valuation.

Proof. For simplicity let us assume $V = \mathbb{R}^n$. Let E_{ij} denote the symmetric matrix that has 1 as its (i,j)-th and (j,i)-th entry and 0 everywhere else. Using the multilinearity of the mixed determinant, we see that any Alesker valuation can be written as a sum of terms of the form

$$\tilde{\mu}(f) := \int_V \phi(x) \det\left(H_f(x)[n-k], E_{i_1 j_1}, \dots, E_{i_k j_k}\right) \quad \forall f \in \mathrm{Conv}(V, \mathbb{R}) \cap C^2(V),$$

where $\phi \in C_c(V)$. Consider the functions $f_{ij}(x) := x_i x_j - \frac{1}{2}\delta_{ij} x_i x_j$. Then $H_{f_{ij}} = E_{ij}$ and the definition of the mixed determinant implies for $f \in \mathrm{Conv}(V, \mathbb{R}) \cap C^2(V)$

$$\begin{aligned}
\tilde{\mu}(f) =& \frac{(n-k)!}{n!} \frac{\partial}{\partial \lambda_1}\Big|_0 \cdots \frac{\partial}{\partial \lambda_k}\Big|_0 \int_V \phi(x) \det\left(H_f(x) + \sum_{i=1}^k \lambda_i E_{ij}\right) dx \\
=& \frac{(n-k)!}{n!} \frac{\partial}{\partial \lambda_1}\Big|_0 \cdots \frac{\partial}{\partial \lambda_k}\Big|_0 \int_V \phi d\,\mathrm{Hess}\left(f + \sum_{i=1}^k \lambda_i f_{ij}\right) \\
=& \frac{(n-k)!}{n!} \frac{\partial}{\partial \lambda_1}\Big|_0 \cdots \frac{\partial}{\partial \lambda_k}\Big|_0 \mu_\phi\left(f + \sum_{i=1}^k \lambda_i f_{ij}\right).
\end{aligned}$$

Thus we see that $\tilde{\mu}$ is a mixed Hessian valuation. Furthermore, the density ϕ in the equation above is smooth if all coefficients of the Alesker valuation are smooth. As f_{ij} is smooth as well, the last statement follows from Lemma 7.2.13.
Now let $f_1, \dots, f_k \in \mathrm{Conv}(V) \cap C^2(V)$ be given. For $f \in \mathrm{Conv}(V, \mathbb{R}) \cap C^2(V)$:

$$\begin{aligned}
&\bar{\mu}_\phi(f[n-k], f_1, \dots, f_k) \\
=& \frac{(n-k)!}{n!} \frac{\partial}{\partial \lambda_1}\Big|_0 \cdots \frac{\partial}{\partial \lambda_k}\Big|_0 \int_V \phi(x) \det\left(H_f(x) + \sum_{i=1}^k \lambda_i H_{f_i}(x)\right) dx \\
=& \int_V \phi(x) \det\left(H_f(x)[n-k], H_{f_1}(x), \dots, H_{f_k}(x)\right) dx \\
=& \int_V \phi(x) \det\left(H_f(x)[n-k], A_1(x), \dots, A_k(x)\right) dx,
\end{aligned}$$

where we have set $A_i(x) = \psi(x) H_{f_i}(x)$ for some function $\psi \in C_c(V)$ with $\psi \equiv 1$ on $\mathrm{supp}\,\mu$. Thus $f \mapsto \bar{\mu}_\phi(f[n-k], f_1, \dots, f_k)$ is indeed an Alesker valuation. □

8.3 Density result for mixed Hessian valuations

Before we prove the main result concerning mixed Hessian valuations, let us show a compatibility property for the two versions of the Goodey-Weil embedding. As discussed before, any $\phi \in C_c^\infty(V)$ defines a 1-homogeneous function $\tilde{\phi} \in C(V \times \mathbb{R})$ by setting

$$\tilde{\phi}(y,s) := \begin{cases} 0 & \text{for } s \geq 0, \\ -s\phi(\frac{y}{s}) & \text{for } s < 0. \end{cases}$$

In addition, $\tilde{\phi}$ is smooth outside of $(0,0) \in V \times \mathbb{R}$ and can thus be considered as an element of $C^\infty(\mathbb{P}_+(V \times \mathbb{R}), L)$, which we will denote by $\tilde{\phi}$ again. Also note that $\tilde{\phi}(\cdot,-1) = \phi$ by construction. We thus obtain a continuous inclusion

$$i : C_c^\infty(V^{k+1}) \to C^\infty(\mathbb{P}_+(V \times \mathbb{R})^{k+1}, L^{k+1}).$$

The image of i consists precisely of all sections with support contained in $\mathbb{P}_+(V \times \mathbb{R})_-$.

Proposition 8.3.1. *For $\mu \in \mathrm{VConv}_k(V)$, $\phi_1, \dots, \phi_k \in C_c^\infty(V)$ the following holds:*

$$\mathrm{GW}(T(\mu))\left(\tilde{\phi}_1 \otimes \dots \otimes \tilde{\phi}_k\right) = \overline{\mathrm{GW}}(\mu)\left(\phi_1 \otimes \dots \otimes \phi_k\right).$$

Proof. Without loss of generality, let $V = \mathbb{R}^n$. Consider the unit ball B in $\mathbb{R}^n \times \mathbb{R}$. Then

$$h_B(y,-1) = \sqrt{1+|y|^2} \quad \text{for } y \in \mathbb{R}^n.$$

It is easy to see that there exists $\delta > 0$ such that $h_B(\cdot,-1) + \sum_{i=1}^k \delta_i \phi_i$ is convex for all $|\delta_i| \leq \delta$. In addition, $h_B + \sum_{i=1}^k \delta_i \tilde{\phi}_i$ is the support function of some smooth, strictly convex body $K_{\delta_1,\dots,\delta_k}$ in $\mathbb{R}^n \times \mathbb{R}$ for all δ_i small enough. The definitions of both versions of the Goodey-Weil embedding imply

$$\begin{aligned}
&\mathrm{GW}(T(\mu))\left(\tilde{\phi}_1 \otimes \dots \otimes \tilde{\phi}_k\right) = \frac{1}{k!}\frac{\partial}{\partial\delta_1}\Big|_0 \dots \frac{\partial}{\partial\delta_k}\Big|_0 T(\mu)\left[K_{\delta_1,\dots,\delta_k}\right] \\
=&\frac{1}{k!}\frac{\partial}{\partial\delta_1}\Big|_0 \dots \frac{\partial}{\partial\delta_k}\Big|_0 \mu\left(h_{K_{\delta_1,\dots,\delta_k}}(\cdot,-1)\right) = \frac{1}{k!}\frac{\partial}{\partial\delta_1}\Big|_0 \dots \frac{\partial}{\partial\delta_k}\Big|_0 \mu\left(h_B(\cdot,-1) + \sum_{i=1}^k \delta_i\tilde{\phi}_i(\cdot,-1)\right) \\
=&\frac{1}{k!}\frac{\partial}{\partial\delta_1}\Big|_0 \dots \frac{\partial}{\partial\delta_k}\Big|_0 \mu\left(h_B(\cdot,-1) + \sum_{i=1}^k \delta_i\phi_i\right) = \overline{\mathrm{GW}}(\mu)(\phi_1 \otimes \dots \otimes \phi_k).
\end{aligned}$$

□

Proposition 8.3.2. *The map*

$$\begin{aligned}
\tilde{\tilde{\Theta}}_{k+1} : C_c(V) \times C_c^2(V)^k &\to \mathrm{VConv}_{n-k}(V) \\
(\phi_0, \psi_1 \dots, \psi_k) &\mapsto [f \mapsto \int_V \phi_0 d\,\mathrm{Hess}_n(f[n-k], \psi_1, \dots, \psi_k)]
\end{aligned}$$

is continuous. More precisely, the following holds: For every compact subset $K \subset$ $\mathrm{Conv}(V,\mathbb{R})$ and every compact set $A \subset V$, there is a constant $C_{A,K} > 0$ such that

$$\sup_{f \in K}\left|\tilde{\tilde{\Theta}}_{k+1}(\phi_0, \psi_1, \dots, \psi_k)[f]\right| \leq C_{A,K}\|\phi_0\|_{C(V)} \cdot \prod_{i=1}^k \|\psi_i\|_{C^2(V)}$$

for all $\psi_1, \dots, \psi_k \in C_c^\infty(V)$ and $\phi_0 \in C_c(V)$ with $\operatorname{supp}\phi_0 \subset A$.

Proof. Let $A\subset V$, $K\subset \mathrm{Conv}(V,\mathbb{R})$ be fixed compact subsets.
For $\phi_0\in C_c(V)$ with $\operatorname{supp}\phi_0\subset A$ and $\psi_1,\dots,\psi_k\in C^2_c(V)$, let $\mu\in \mathrm{VConv}(V)$ be given by

$$\mu(f):=\int_V \phi_0 d\operatorname{Hess}_n(f[n-k],\psi_1,\dots,\psi_k).$$

Let $\mathcal{M}(V)$ denote the space of signed Radon measures on V equipped with the vague topology, and consider $\Phi\in\mathrm{VConv}(V,\mathcal{M}(V))$ defined by

$$\Phi(f):=\operatorname{Hess}_n(f[n-k],\psi_1,\dots,\psi_k).$$

Then $|\mu(f)|=|\Phi(f)|_{\phi_0}$, where we have used the continuous semi-norm $|\nu|_{\phi_0}:=\left|\int_V\phi_0 d\nu\right|$ on $\mathcal{M}(V)$. By Lemma 5.4.1, there exists a constant $C(B)>0$ for each compact subset $B\subset V$ such that we can find convex functions $f_i,h_i\in\mathrm{Conv}(V,\mathbb{R})$ with $f_i=h_i+\psi_i$ and $\|f_i|_B\|_\infty,\|h_i|_B\|_\infty\le C(B)\|\psi_i\|_{C^2(V)}$. By Equation (5.2),

$$\begin{aligned}&\operatorname{Hess}_n(f[n-k],\phi_1,\dots,\psi_k)\\ =&\sum_{l=1}^k(-1)^{k-l}\frac{1}{l!(k-l)!}\sum_{\sigma\in S_k}\operatorname{Hess}_n(f[n-k],f_{\sigma(1)},\dots,f_{\sigma(l)},h_{\sigma(l+1)},\dots,h_{\sigma(k)}).\end{aligned}$$

Setting $\tilde{f}_i:=\frac{f_i}{\|\psi_i\|_{C^2(V)}}$, $\tilde{h}_i:=\frac{h_i}{\|\psi_i\|_{C^2(V)}}$,

$$\begin{aligned}&|\mu(f)|=|\Phi(f)|_{\phi_0}\\ \le&\sum_{l=1}^k\frac{1}{l!(k-l)!}\sum_{\sigma\in S_k}\left|\operatorname{Hess}_n(f[n-k],f_{\sigma(1)},\dots,f_{\sigma(l)},h_{\sigma(l+1)},\dots,h_{\sigma(k)})\right|_{\phi_0}\\ =&\sum_{l=1}^k\frac{1}{l!(k-l)!}\sum_{\sigma\in S_k}\left|\operatorname{Hess}_n(f[n-k],\tilde{f}_{\sigma(1)},\dots,\tilde{f}_{\sigma(l)},\tilde{h}_{\sigma(l+1)},\dots,\tilde{h}_{\sigma(k)})\right|_{\phi_0}\prod_{i=1}^k\|\psi_i\|_{C^2(V)}.\end{aligned}$$

Let $C\subset\mathrm{Conv}(V,\mathbb{R})$ denote the subset of functions that are bounded by $C(B)$ on B for all compact subsets $B\subset V$. By Proposition 4.2.2, C is compact, so

$$\begin{aligned}&\sup_{f\in K}\left|\operatorname{Hess}_n(f[n-k],\tilde{f}_{\sigma(1)},\dots,\tilde{f}_{\sigma(l)},\tilde{h}_{\sigma(l+1)},\dots,\tilde{h}_{\sigma(k)})\right|_{\phi_0}\\ \le&\sup_{f_1,\dots,f_n\in K\cup C}|\operatorname{Hess}_n(f_1,\dots,f_n)|_{\phi_0}\\ \le&C_n\sup_{f\in K'}|\operatorname{Hess}_n(f)|_{\phi_0}\end{aligned}$$

by Lemma 5.3.12 for some compact subset $K'\subset\mathrm{Conv}(V,\mathbb{R})$ and some constant $C_n>0$ depending on n only. By Lemma 5.2.3, $\operatorname{Hess}_n(f)$ is a non-negative measure for each $f\in\mathrm{Conv}(V,\mathbb{R})$. Let U be an open neighborhood of the compact set A and take a non-negative function $\phi_A\in C_c(U)$ with $\phi_A=1$ on A. Then $\phi_A=1$ on $\operatorname{supp}\phi_0$, so

$$|\operatorname{Hess}_n(f)|_{\phi_0}=\left|\int_V\phi_0 d\operatorname{Hess}_n(f)\right|\le\int_V|\phi_0|d\operatorname{Hess}_n(f)\le\|\phi_0\|_{C(V)}\cdot\int_V\phi_A d\operatorname{Hess}_n(f).$$

Using our estimates for every term in the sum, we obtain

$$\begin{aligned}\sup_{f\in K}|\mu(f)| \leq& 2^k\cdot C_n \sup_{f\in K'}|\operatorname{Hess}_n(f)|_{\phi_0}\prod_{i=1}^k\|\psi_i\|_{C^2(V)}\\ \leq& 2^k\cdot C_n\cdot\left[\sup_{f\in K'}\int_V\phi_A d\operatorname{Hess}_n(f)\right]\cdot\|\phi_0\|_{C(V)}\cdot\prod_{i=1}^k\|\psi_i\|_{C^2(V)}.\end{aligned}$$

Setting $C_{A,K}:=2^k\cdot C_n\cdot\left[\sup_{f\in K'}\int_V\phi_A d\operatorname{Hess}_n(f)\right]<\infty$ for $A\subset V$ compact, we obtain the inequality

$$\sup_{f\in K}\left|\int_V\phi_0 d\operatorname{Hess}_n(f[n-k],\psi_1,\dots,\psi_k)\right|\leq C_{A,K}\|\phi_0\|_{C(V)}\cdot\prod_{i=1}^k\|\psi_i\|_{C^2(V)}$$

for all $\psi_1,\dots,\psi_k\in C^\infty_c(V)$ and $\phi_0\in C_c(V)$ with $\operatorname{supp}\phi_0\subset A$. □

Using the L. Schwartz kernel theorem 2.2.1, we extend $\tilde{\bar{\Theta}}_{k+1}$ to a continuous linear functional

$$\bar{\Theta}_{k+1}:C^\infty_c(V^{k+1})\to\operatorname{VConv}_{n-k}(V).$$

Corollary 8.3.3. *The order of $\bar{\Theta}_k$ is uniformly bounded: There exists $N\in\mathbb{N}$ such that for every compact subset $K\subset\operatorname{Conv}(V,\mathbb{R})$ and every compact subset $A\subset V^{k+1}$ there exists a constant $C>0$ such that*

$$\sup_{f\in K}|\Theta_k(f)|\leq C\|f\|_{C^N(V^{k+1})}\quad\forall f\in C^\infty_c(V^{k+1})\text{ with }\operatorname{supp}f\subset A.$$

Proof. Combine the estimate from Proposition 8.3.2 with Theorem 2.2.1. □

Let us once again remark that we have fixed a Euclidean structure on V. This induces a trivialization $\operatorname{Dens}(V^*\times\mathbb{R})\cong\mathbb{R}$ using the induced metric on $V^*\times\mathbb{R}$. We will thus consider $\Theta_k:C^\infty_c(\mathbb{P}_+(V^*\times\mathbb{R})^{k+1}_-,L^{\boxtimes k+1})\to\operatorname{Val}_{n-k}(V\times\mathbb{R})^{sm}$.

Proposition 8.3.4. *The diagram*

$$\begin{tikzcd} C^\infty_c(V^{k+1}) \arrow[r, "i"] \arrow[d, "\frac{1}{n+1}\frac{n!}{(k+1)!}\bar{\Theta}_{k+1}"'] & C^\infty_c(\mathbb{P}_+(V\times\mathbb{R})^{k+1}_-,L^{\boxtimes k+1}) \arrow[d, "\Theta_{k+1}"] \\ \operatorname{VConv}_{n-k}(V)^{sm} \arrow[r, "T"] & \operatorname{Val}_{n-k,\mathbb{P}_+(V\times\mathbb{R})_-}(V^*\times\mathbb{R})^{sm} \end{tikzcd}$$

commutes. In particular, $\bar{\Theta}_{k+1}:C^\infty_c(V^{k+1})\to\operatorname{VConv}_{n-k}(V)^{sm}$ is well defined and surjective.

Proof. T is surjective by the characterization of $\mathrm{VConv}(V)^{sm}$ in Proposition 7.3.4 and the description of the image of T in Theorem 6.3.2. To see that Θ_{k+1} is surjective, it is enough to apply Proposition 3.2.3 to a compact neighborhood of the support of any valuation $\mu \in \mathrm{Val}_{n-k}(V^* \times \mathbb{R})$ with vertical support compactly contained in $\mathbb{P}_+(V \times \mathbb{R})_-$ that is also contained in $\mathbb{P}_+(V \times \mathbb{R})_-$. i is bijective, so $\bar{\Theta}_{k+1}$ is surjective and defines smooth valuations if the diagram commutes.
As functions of the form $\phi_0 \otimes \cdots \otimes \phi_k$ for $\phi_i \in C_c^\infty(V)$ span a dense subspace of $C_c^\infty(V^{k+1})$ and all maps are continuous, it is enough to show

$$\frac{1}{n+1}\frac{n!}{(k+1)!}T\left(\bar{\Theta}_{k+1}(\phi_0 \otimes \cdots \otimes \phi_k)\right) = \Theta_{k+1}(\tilde{\phi}_0 \otimes \cdots \otimes \tilde{\phi}_k).$$

Applying GW to both sides and using Proposition 8.3.1, this is equivalent to

$$\begin{aligned}&\frac{1}{n+1}\frac{n!}{(k+1)!}\overline{\mathrm{GW}}\left(\Theta_{k+1}\left(\phi_0 \otimes \cdots \otimes \phi_k\right)\right)\left[\psi_{k+1} \otimes \cdots \otimes \psi_n\right]\\ =&\,\mathrm{GW}\left(\Theta_{k+1}\left(\tilde{\phi}_0 \otimes \cdots \otimes \tilde{\phi}_k\right)\right)\left[\tilde{\psi}_{k+1} \otimes \cdots \otimes \tilde{\psi}_n\right]\end{aligned}$$

for $\psi_{k+1}, \dots, \psi_n \in C_c^\infty(V)$.
We will use the metric on $V \times \mathbb{R}$ to identify $C^\infty(\mathbb{P}_+(V \times \mathbb{R}), L) \cong C^\infty(S(V \times \mathbb{R}))$. From Proposition 8.1.2 we deduce

$$\Theta_1(\tilde{\phi}_0) = \frac{1}{n+1}\int_{S(V\times\mathbb{R})} \tilde{\phi}_0 dS_n = \frac{1}{n+1}T\left(\int_V \phi_0 d\,\mathrm{Hess}_n\right),$$

where the last equality follows from Lemma 8.1.4. Using the compatibility of the maps Θ_i and GW from Proposition 3.2.2, we obtain

$$\begin{aligned}&\mathrm{GW}\left(\Theta_{k+1}\left(\tilde{\phi}_0 \otimes \cdots \otimes \tilde{\phi}_k\right)\right)\left[\tilde{\psi}_{k+1} \otimes \cdots \otimes \tilde{\psi}_n\right]\\ =&\binom{n+1}{k+1}\Theta_n\left(\tilde{\phi}_0 \otimes \cdots \otimes \tilde{\phi}_k \otimes \tilde{\psi}_{k+1} \otimes \cdots \otimes \tilde{\psi}_n\right)(\{0\})\\ =&\binom{n+1}{k+1}\frac{1}{n+1}\mathrm{GW}\left(\Theta_1\left(\tilde{\phi}_0\right)\right)\left[\tilde{\phi}_1 \otimes \cdots \otimes \tilde{\phi}_k \otimes \tilde{\psi}_{k+1} \otimes \cdots \otimes \tilde{\psi}_n\right]\\ =&\binom{n+1}{k+1}\frac{1}{(n+1)^2}\mathrm{GW}\left(T\left(\int_V \phi_0 d\,\mathrm{Hess}_n\right)\right)\left[\tilde{\phi}_1 \otimes \cdots \otimes \tilde{\phi}_k \otimes \tilde{\psi}_{k+1} \otimes \cdots \otimes \tilde{\psi}_n\right]\\ =&\binom{n+1}{k+1}\frac{1}{(n+1)^2}\overline{\mathrm{GW}}\left(\int_V \phi_0 d\,\mathrm{Hess}_n\right)\left[\phi_1 \otimes \cdots \otimes \phi_k \otimes \psi_{k+1} \otimes \cdots \otimes \psi_n\right],\end{aligned}$$

where we have again used Proposition 8.3.1 in the last step. Thus

$$\begin{aligned}
&\mathrm{GW}\left(\Theta_{k+1}\left(\tilde{\phi}_0\otimes\cdots\otimes\tilde{\phi}_k\right)\right)\left[\tilde{\psi}_{k+1}\otimes\cdots\otimes\tilde{\psi}_n\right]\\
=&\binom{n+1}{k+1}\frac{1}{(n+1)^2}\int_V\phi_0 d\,\mathrm{Hess}_n\left(\phi_1,\dots,\phi_k,\psi_{k+1},\dots,\psi_n\right)\\
=&\frac{1}{(n+1)(k+1)}\binom{n}{k}\int_V\phi_0 d\,\mathrm{Hess}_n\left(\phi_1,\dots,\phi_k,\psi_{k+1},\dots,\psi_n\right)\\
=&\frac{n!}{(n+1)(k+1)!}\overline{\mathrm{GW}}\left(\int_V\phi_0 d\,\mathrm{Hess}_n\left(\cdot[n-k],\phi_1,\dots,\phi_k\right)\right)\left[\psi_{k+1}\otimes\cdots\otimes\psi_n\right]\\
=&\frac{n!}{(n+1)(k+1)!}\overline{\mathrm{GW}}\left(\bar{\Theta}_{k+1}\left(\phi_0\otimes\cdots\otimes\phi_k\right)\right)\left[\psi_{k+1}\otimes\cdots\otimes\psi_n\right].
\end{aligned}$$

□

Theorem 8.3.5. *For every $\mu\in\mathrm{VConv}_{n-k}(V)^{sm}$ and every open neighborhood U of $\operatorname{supp}\mu$ there exist functions $\phi_i^j\in C_c^\infty(U)$ for $0\le i\le k$, $j\in\mathbb{N}$ such that*

$$\mu(f)=\sum_{j=1}^\infty\int_V\phi_0^j d\,\mathrm{Hess}_n(f[n-k],\phi_1^j,\dots,\phi_k^j).$$

In particular, the space generated by smooth mixed Hessian valuations is a dense subspace of $\mathrm{VConv}(V)^{sm}$ and $\mathrm{VConv}(V)$.

Proof. Let $\mu\in\mathrm{VConv}_{n-k}(V)^{sm}$. Fix a compact neighborhood A of $\operatorname{supp}\mu$ such that $A\subset U$ for the desired open neighborhood U of $\operatorname{supp}\mu$. By Proposition 3.2.3, there exists a function $\tilde{f}\in C_{P(A)}(\mathbb{P}_+(V\times\mathbb{R})^k,L^{\boxtimes k})$ such that $\Theta_{k+1}(\tilde{f})=T(\mu)$. Using the inverse of i and Proposition 8.3.4, we obtain a function $f\in C_c^\infty(V^{k+1})$ with support contained in A^{k+1} such that $\frac{1}{n+1}\frac{n!}{(k+1)!}\bar{\Theta}_{k+1}(f)=\mu$.
As the order of $\bar{\Theta}_{k+1}$ is uniformly bounded by some $N\in\mathbb{N}$ by Corollary 8.3.3, we can extend $\bar{\Theta}_{k+1}$ to a continuous linear functional on $C^N(V^{k+1})$. Increasing N if necessary, we can apply Proposition 2.2.3 to the neighborhood U^{k+1} of A^{k+1} to obtain functions $\phi_0^j,\dots,\phi_k^j\in C_c^\infty(U)$ with

$$f=\sum_{j=1}^\infty\phi_0^j\otimes\cdots\otimes\phi_k^j$$

in the C^N-topology. Then

$$\begin{aligned}
\mu=&\frac{1}{n+1}\frac{n!}{(k+1)!}\bar{\Theta}_{k+1}(f)=\frac{1}{n+1}\frac{n!}{(k+1)!}\sum_{j=1}^\infty\bar{\Theta}_{k+1}\left(\phi_0^j\otimes\cdots\otimes\phi_k^j\right)\\
=&\frac{1}{n+1}\frac{n!}{(k+1)!}\sum_{j=1}^\infty\int_V\phi_0^j d\,\mathrm{Hess}_n\left(\cdot[n-k],\phi_1^j,\dots,\phi_k^j\right).
\end{aligned}$$

Rescaling the functions by an appropriate constant, we obtain the desired expression. □

9 Smooth rotation invariant valuations

Let us consider $V = \mathbb{R}^n$ equipped with its standard scalar product. We will identify the differential cycle of $f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ with its image in $T\mathbb{R}^n$ under the natural identification of $\mathbb{R}^n$ with its dual space. We will be interested in smooth valuations that are invariant under the operation of $\mathrm{SO}(n)$. Combining Theorem 7.2.12 and Corollary 7.3.6, we are lead to a classification of all $\mathrm{SO}(n)$-invariant and vertically translation invariant differential forms. Note that $\mathrm{SO}(n)$ acts diagonally by symplectomorphisms on $T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$, and translations in the second component also preserve the symplectic form on $T\mathbb{R}^n$. As the Lefschetz decomposition (Proposition 7.2.1) is compatible with symplectomorphisms, we thus only need to classify *primitive* invariant forms.

Section 9.1 introduces a class of differential forms related to the Hessian measures defined in [14], which were also used in [20] to construct valuations on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$. The whole array of all invariant forms needed for the classification is presented in Section 9.2. In Section 9.3 we classify the relevant primitive differential forms, which will be used to derive the desired representation formulas for smooth $\mathrm{SO}(n)$-invariant valuations in Section 9.4.

9.1 Representation of the Hessian measures by differential forms

In [20] a class of rotation invariant valuations on $\mathrm{Conv}(\mathbb{R}^n, \mathbb{R})$ was constructed using the so called *Hessian measures* from [14]. In this section, we will consider a version of these measures in terms of differential forms. The connection with the functionals from [20] is established in Proposition 9.1.4 below.

Throughout this chapter, we will denote the standard coordinates on $\mathbb{R}^n$ by $(x_1, \dots, x_n)$, with induced coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on $T\mathbb{R}^n$. We start with the following well known result.

Proposition 9.1.1. *The forms*

$$\omega_s = \sum_{i=1}^n dx_i \wedge dy_i,$$
$$\kappa_k = \frac{1}{k!(n-k)!} \sum_{\pi \in S_n} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)} \quad 0 \leq k \leq n,$$

generate the algebra of all $\mathrm{SO}(n)$*-invariant forms in* $\Lambda^*(\mathbb{R}^n \times \mathbb{R}^n)$.

For $t \in \mathbb{R}$ let $G_t : T\mathbb{R}^n \to T\mathbb{R}^n$ be given by $G_t(x,y) = (x, y + tx)$.

Lemma 9.1.2.

$$G_t^* \kappa_n = \sum_{k=0}^n t^{n-k} \kappa_k.$$

Proof. Let us show

$$G_t^*(dy_1 \dots dy_m) = \sum_{k=0}^m \frac{t^{m-k}}{k!(m-k)!} \sum_{\pi \in S_m} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(m-k)} \wedge dy_{\pi(m-k+1)} \dots dy_{\pi(m)}$$

for $1 \leq m \leq n$ by induction. For $m = 1$, this is trivial. Assuming the equation holds for m, we calculate

$$\begin{aligned}
&G_t^*(dy_1 \dots dy_{m+1}) = G_t^*(dy_1 \dots dy_m) \wedge G_t^* dy_{m+1} \\
=&G_t^*(dy_1 \dots dy_m) \wedge dy_{m+1} + t G_t^*(dy_1 \dots dy_m) \wedge dx_{m+1} \\
=&\sum_{k=0}^m \frac{t^{m-k}}{k!(m-k)!} \sum_{\pi \in S_m} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(m-k)} \wedge dy_{\pi(m-k+1)} \dots dy_{\pi(m)} \wedge dy_{m+1} \\
&+ \sum_{k=0}^m \frac{t^{m+1-k}}{k!(m-k)!} \sum_{\pi \in S_m} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(m-k)} \wedge dy_{\pi(m-k+1)} \dots dy_{\pi(m)} \wedge dx_{m+1} \\
=&\sum_{k=0}^m \frac{t^{m-k}}{k!(m-k)!} \sum_{\substack{\pi \in S_{m+1} \\ \pi(m+1)=m+1}} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(m-k)} \wedge dy_{\pi(m-k+1)} \dots dy_{\pi(m+1)} \\
&+ \sum_{k=0}^m \frac{t^{m+1-k}}{k!(m-k)!} \sum_{\substack{\pi \in S_{m+1} \\ \pi(m+1-k)=m+1}} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(m+1-k)} \wedge .dy_{\pi(m+1-k+1)} \dots dy_{\pi(m+1)}
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{\substack{\pi \in S_{m+1} \\ \pi(m+1)=m+1}} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(m-k)} \wedge dy_{\pi(m-k+1)} \dots dy_{\pi(m+1)} \\
=&\frac{1}{k+1} \sum_{\substack{\pi \in S_{m+1} \\ m+1 \in \pi(\{m-k+1,\dots,m+1\})}} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(m-k)} \wedge dy_{\pi(m-k+1)} \dots dy_{\pi(m+1)}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\sum_{\substack{\pi\in S_{m+1}\\ \pi(m+1-k)=m+1}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m+1-k)}\wedge dy_{\pi(m+1-k+1)}\dots dy_{\pi(m+1)}\\
&=\frac{1}{(m+1-k)}\sum_{\substack{\pi\in S_{m+1}\\ m+1\in\pi(\{1,\dots,m+1-k\})}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m+1-k)}\wedge dy_{\pi(m+1-k+1)}\dots dy_{\pi(m+1)},
\end{aligned}$$

so

$$\begin{aligned}
&G_t^*(dy_1\dots dy_{m+1})\\
=&\sum_{k=0}^{m}\frac{t^{m-k}}{(k+1)!(m-k)!}\sum_{\substack{\pi\in S_{m+1}\\ m+1\in\pi(\{m-k+1,\dots,m+1\})}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m-k)}\wedge dy_{\pi(m-k+1)}\dots dy_{\pi(m+1)}\\
&+\sum_{k=0}^{m}\frac{t^{m+1-k}}{k!(m+1-k)!}\sum_{\substack{\pi\in S_{m+1}\\ m+1\in\pi(\{1,\dots,m+1-k\})}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m+1-k)}\wedge dy_{\pi(m+1-k+1)}\dots dy_{\pi(m+1)}\\
=&dy_1\dots dy_{m+1}+t^{m+1}dx_1\dots dx_{m+1}\\
&+\sum_{k=0}^{m-1}\frac{t^{m-k}}{(k+1)!(m-k)!}\sum_{\substack{\pi\in S_{m+1}\\ m+1\in\pi(\{m-k+1,\dots,m+1\})}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m-k)}\wedge dy_{\pi(m-k+1)}\dots dy_{\pi(m+1)}\\
&+\sum_{k=1}^{m}\frac{t^{m+1-k}}{k!(m+1-k)!}\sum_{\substack{\pi\in S_{m+1}\\ m+1\in\pi(\{1,\dots,m+1-k\})}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m+1-k)}\wedge dy_{\pi(m+1-k+1)}\dots dy_{\pi(m+1)}\\
=&dy_1\dots dy_{m+1}+t^{m+1}dx_1\dots dx_{m+1}\\
&+\sum_{k=0}^{m-1}\frac{t^{m-k}}{(k+1)!(m-k)!}\sum_{\substack{\pi\in S_{m+1}\\ m+1\in\pi(\{m-k+1,\dots,m+1\})}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m-k)}\wedge dy_{\pi(m-k+1)}\dots dy_{\pi(m+1)}\\
&+\sum_{k=0}^{m-1}\frac{t^{m-k}}{(k+1)!(m-k)!}\sum_{\substack{\pi\in S_{m+1}\\ m+1\in\pi(\{1,\dots,m-k\})}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m-k)}\wedge dy_{\pi(m-k+1)}\dots dy_{\pi(m+1)}\\
=&dy_1\dots dy_{m+1}+t^{m+1}dx_1\dots dx_{m+1}\\
&+\sum_{k=0}^{m-1}\frac{t^{m-k}}{(k+1)!(m-k)!}\sum_{\pi\in S_{m+1}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m-k)}\wedge dy_{\pi(m-k+1)}\dots dy_{\pi(m+1)}\\
=&dy_1\dots dy_{m+1}+t^{m+1}dx_1\dots dx_{m+1}\\
&+\sum_{k=1}^{m}\frac{t^{m+1-k}}{k!(m+1-k)!}\sum_{\pi\in S_{m+1}} \operatorname{sign}(\pi) dx_{\pi(1)}\dots dx_{\pi(m+1-k)}\wedge dy_{\pi(m+1-k+1)}\dots dy_{\pi(m+1)},
\end{aligned}$$

which reduces to

$$\sum_{k=0}^{m+1}\frac{t^{m+1-k}}{k!(m+1-k)!}\sum_{\pi\in S_{m+1}}\operatorname{sign}(\pi)dx_{\pi(1)}\dots dx_{\pi(m+1-k)}\wedge dy_{\pi(m+1-k+1)}\dots dy_{\pi(m+1)}.$$

For $m=n$ we obtain the desired result. □

Definition 9.1.3. *For $f\in\operatorname{Conv}(\mathbb{R}^n,\mathbb{R})$ we call $\Phi_k(f):=\pi_*[D(f)\llcorner\kappa_k]$ the k-th Hessian measure of f, i.e. we consider the measures given by*

$$\phi\mapsto D(f)[\pi^*\phi\wedge\kappa_k]\quad \text{for } \phi\in C_c(\mathbb{R}^n).$$

Note that the pushforward is well defined, as the support of $D(f)$ is locally vertically bounded. Moreover, $D(f)$ is an integral current and so in particular a current of locally finite mass, so we obtain signed measure on $\mathbb{R}^n$ for each $f\in\operatorname{Conv}(\mathbb{R}^n,\mathbb{R})$. With this terminology, $\Phi_n=\operatorname{Hess}_n$.

Proposition 9.1.4. *For $f\in\operatorname{Conv}(\mathbb{R}^n,\mathbb{R})\cap C^2(\mathbb{R}^n)$ and $\phi\in C_c(\mathbb{R}^n)$:*

$$\int_{\mathbb{R}^n}\phi d\Phi_k(f)=\int_{\mathbb{R}^n}\phi(x)[D^2f(x)]_k dx,$$

where $[D^2f(x)]_k$ denotes the k-th elementary symmetric polynomial in the eigenvalues of the Hessian of f.

Proof. As $D(f)$ is an integral current, it is sufficient to show this equation for $\phi\in C_c^\infty(\mathbb{R}^n)$. Then $f\mapsto\int_{\mathbb{R}^n}\phi d\Phi_k(f)$ is a smooth dually epi-translation invariant valuation, represented by the differential form $\pi^*\phi\wedge\kappa_k$, so by continuity it is sufficient to show the equation for $f\in\operatorname{Conv}(\mathbb{R}^n,\mathbb{R})\cap C^\infty(\mathbb{R}^n)$ (note that the right hand side is continuous in the C^2-topology). Let $h\in\operatorname{Conv}(\mathbb{R}^n,\mathbb{R})$ be given by $h(x)=\frac{|x|^2}{2}$. Then

$$\int_{\mathbb{R}^n}\phi d\Phi_n(f+th)=D(f+th)[\pi^*\phi\wedge\kappa_n]=G_{t*}D(f)[\pi^*\phi\wedge\kappa_n],$$

for $t\geq 0$ by Proposition 7.1.3. Thus

$$\begin{aligned}\int_{\mathbb{R}^n}\phi d\Phi_n(f+th)=&D(f)[\pi^*\phi\wedge G_t^*\kappa_n]=\sum_{k=0}^n t^{n-k}D(f)[\pi^*\phi\wedge\kappa_k]\\ =&\sum_{k=0}^n t^{n-k}\int_{\mathbb{R}^n}\phi d\Phi_k(f)\end{aligned}$$

by Lemma 9.1.2. On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^n}\phi d\Phi_n(f+th)=&\int_{\mathbb{R}^n}\phi(x)\det(H_f(x)+tId_n)dx=\int_{\mathbb{R}^n}\phi(x)\sum_{k=0}^n t^{n-k}[D^2f(x)]_k dx\\ =&\sum_{k=0}^n t^{n-k}\int_{\mathbb{R}^n}\phi(x)[D^2f(x)]_k dx.\end{aligned}$$

Comparing coefficients, we obtain the desired formula. □

9.2 Basic invariant differential forms

Definition 9.2.1. *We define the following* $\mathrm{SO}(n)$*-invariant differential forms on* $T\mathbb{R}^n$ *in degree* 1

$$\begin{aligned}\alpha :=& \sum_{i=1}^n y_i dx_i,\\ \beta :=& \sum_{i=1}^n x_i dx_i,\\ \gamma :=& \sum_{i=1}^n x_i dy_i = d(x \cdot y) - \alpha,\end{aligned}$$

and for $1 \leq k \leq n-1$ *in degree* $n-1$

$$\kappa_k' := \frac{1}{k!(n-k)!} \sum_{\pi \in S_n} \operatorname{sign}(\pi) x_{\pi(1)} dx_{\pi(2)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}.$$

With this notation, the forms κ_k' are k-homogeneous in the sense of Section 7.2. Note that $d\kappa_k' = \kappa_k$ are exactly the invariant forms from Proposition 9.1.1 and α is the canonical 1-form on $T\mathbb{R}^n$, i.e. the symplectic form is given by $\omega_s = -d\alpha = \sum_{i=1}^n dx_i \wedge dy_i$. Let $r : T\mathbb{R}^n \to \mathbb{R}$ be the function given by $r(x, y) = |x|$. Then $r^2 : T\mathbb{R}^n \to \mathbb{R}$ is smooth and $dr^2 = 2\beta$.

Proposition 9.2.2. *For* $1 \leq k \leq n-1$,

$$r^2 \kappa_k = (n-k)\beta \wedge \kappa_k' + (n-k+1)\gamma \wedge \kappa_{k-1}'.$$

In addition,

$$\begin{aligned}r^2 \kappa_0 =& \beta \wedge \kappa_0',\\ r^2 \kappa_n =& \gamma \wedge \kappa_{n-1}'.\end{aligned}$$

Proof. As both sides are $\mathrm{SO}(n)$-invariant as well as vertically translation invariant, we only need to examine the equation in one point of each orbit, i.e. in the point $(x, y) = (te_1, 0)$, $t \geq 0$. Here

$$\begin{aligned}\beta &= t dx_1,\\ \gamma &= t dy_1,\\ \kappa_k' &= t \frac{1}{k!(n-k)!} \sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) dx_{\pi(2)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)},\\ r^2 \kappa_k &= t^2 \frac{1}{k!(n-k)!} \sum_{\pi \in S_n} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}.\end{aligned}$$

Therefore

$$\beta \wedge \kappa_k' = t^2 \frac{1}{k!(n-k)!} \sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)},$$

while $\gamma \wedge \kappa_{k-1}'$ is given by

$$\begin{aligned}
&\frac{t^2(-1)^{n-k}}{(k-1)!(n-k+1)!} \sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) dx_{\pi(2)} \dots dx_{\pi(n-k+1)} \wedge dy_1 \wedge dy_{\pi(n-k+2)} \dots dy_{\pi(n)} \\
=&\frac{t^2}{(k-1)!(n-k+1)!} \sum_{\pi \in S_n, \pi(k+1)=1} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}.
\end{aligned}$$

Thus

$$\begin{aligned}
r^2 \kappa_k =& t^2 \frac{1}{k!(n-k)!} \sum_{\pi \in S_n} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)} \\
=& t^2 \frac{1}{k!(n-k)!} \sum_{\pi \in S_n, 1 \in \pi(\{1,\dots,n-k\})} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)} \\
&+ t^2 \frac{1}{k!(n-k)!} \sum_{\pi \in S_n, 1 \in \pi(\{n-k+1,\dots,n\})} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)} \\
=& t^2 \frac{1}{k!(n-k)!} (n-k) \sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)} \\
&+ t^2 \frac{1}{k!(n-k)!} k \sum_{\pi \in S_n, \pi(n-k+1)=1} \operatorname{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)} \\
=& (n-k) \beta \wedge \kappa_k' + (n-k+1) \gamma \wedge \kappa_{k-1}'.
\end{aligned}$$

The cases $r^2 \kappa_0 = \beta \wedge \kappa_0'$ and $r^2 \kappa_n = \gamma \wedge \kappa_{n-1}'$ follow with a similar calculation. □

Definition 9.2.3. *For $1 \leq k \leq n-1$ we set*

$$\begin{aligned}
\tau_{k,1} :=& (n-k) \beta \wedge \kappa_k', \\
\tau_{k,2} :=& (n-k+1) \gamma \wedge \kappa_{k-1}'.
\end{aligned}$$

Note that these forms are linearly independent in each point of $T\mathbb{R}^n \setminus T_0\mathbb{R}^n$. Moreover

$$r^2 \kappa_k = \tau_{k,1} + \tau_{k,2} \quad \text{for } 1 \leq k \leq n-1$$

by Proposition 9.2.2.

Lemma 9.2.4. *For $1 \leq k \leq n-1$,*

$$\begin{aligned}
d\tau_{k,1} =& -(n-k) \beta \wedge \kappa_k, \\
d\tau_{k,2} =& (n-k+2) \beta \wedge \kappa_k.
\end{aligned}$$

In particular,

$$(n-k+2) d\tau_{k,1} = -(n-k) d\tau_{k,2}.$$

Proof. By definition,

$$d\tau_{k,1} = (n-k)d(\beta \wedge \kappa'_k) = -(n-k)\beta \wedge d\kappa'_k = -(n-k)\beta \wedge \kappa_k.$$

Thus Proposition 9.2.2 implies

$$2\beta \wedge \kappa_k = d(r^2\kappa_k) = d\tau_{k,1} + d\tau_{k,2} = -(n-k)\beta \wedge \kappa_k + d\tau_{k,2},$$

i.e. $(n-k+2)\beta \wedge \kappa_k = d\tau_{k,2}$. □

Lemma 9.2.5. *For $1 \leq k \leq n-1$,*

$$\beta \wedge \gamma \wedge \kappa'_k = r^2\omega_s \wedge \kappa'_k.$$

Proof. As in the proof of Proposition 9.2.2, it is sufficient to examine the equation in the point $(te_1, 0)$, $t \geq 0$. We compute

$$\begin{aligned}
&\beta \wedge \gamma \wedge \kappa'_k \\
=&t dx_1 \wedge t dy_1 \wedge \frac{1}{k!(n-k)!} \sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) t dx_{\pi(2)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}, \\
=&\frac{t^3}{k!(n-k)!} \sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) dx_1 \wedge dy_1 \wedge dx_{\pi(2)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)},
\end{aligned}$$

as well as

$$\begin{aligned}
&r^2\omega_s \wedge \kappa'_k \\
=&t^2 \sum_{i=1}^n dx_i \wedge dy_i \wedge \frac{1}{k!(n-k)!} \sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) t dx_{\pi(2)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)} \\
=&t^3 \frac{1}{k!(n-k)!} \sum_{\pi \in S_n, \pi(1)=1} \operatorname{sign}(\pi) \sum_{i=1}^n dx_i \wedge dy_i \wedge dx_{\pi(2)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}.
\end{aligned}$$

Now notice that the term $dx_i \wedge dy_i \wedge dx_{\pi(2)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}$ vanishes unless $i = 1$. The claim follows. □

Corollary 9.2.6. *For $1 \leq k \leq n-1$,*

$$\begin{aligned}
\beta \wedge \kappa_k =&(n-k+1)\omega_s \wedge \kappa'_{k-1}, \\
\gamma \wedge \kappa_k =&-(n-k)\omega_s \wedge \kappa'_k.
\end{aligned}$$

In particular

$$\begin{aligned}
d\tau_{k,1} =&-(n-k)(n-k+1)\omega_s \wedge \kappa'_{k-1}, \\
d\tau_{k,2} =&(n-k+2)(n-k+1)\omega_s \wedge \kappa'_{k-1}.
\end{aligned}$$

Proof. Let us multiply the equation

$$\kappa_k = \frac{n-k}{r^2}\beta \wedge \kappa_k' + \frac{n-k+1}{r^2}\gamma \wedge \kappa_{k-1}'$$

from Proposition 9.2.2 by β:

$$\beta \wedge \kappa_k = \frac{n-k+1}{r^2}\beta \wedge \gamma \wedge \kappa_{k-1}'.$$

The term on the right is equal to $(n-k+1)\omega_s \wedge \kappa_{k-1}'$ by Lemma 9.2.5.
On the other hand, multiplying the equation by γ, we obtain

$$\gamma \wedge \kappa_k = -\frac{n-k}{r^2}\beta \wedge \gamma \wedge \kappa_k' = -(n-k)\omega_s \wedge \kappa_k'.$$

The formulas for $d\tau_{k,1}$ and $d\tau_{k,2}$ follow from Lemma 9.2.4 and the following calculation:

$$\begin{aligned} d\tau_{k,1} &= = -(n-k)\beta \wedge \kappa_k = -(n-k)(n-k+1)\omega_s \wedge \kappa_{k-1}', \\ d\tau_{k,2} &=(n-k+2)\beta \wedge \kappa_k = (n-k+2)(n-k+1)\omega_s \wedge \kappa_{k-1}'. \end{aligned}$$

□

Lemma 9.2.7. *The forms κ_0, κ_n, as well as $\tau_{k,1}$ and $\tau_{k,2}$, for $1 \leq k \leq n-1$, are primitive.*

Proof. We need to show that the product of these forms with the symplectic form ω_s vanishes. For κ_0 and κ_n this is obviously true.
Let $1 \leq k \leq n-1$. As ω_s, as well as $\tau_{k,1}$ and $\tau_{k,2}$ are $\mathrm{SO}(n)$- and vertically translation invariant, it is sufficient to check this for one point of each orbit, i.e. in the point $(te_1, 0)$, $t \geq 0$. As in the proof of Proposition 9.2.2, we obtain

$$\begin{aligned} \tau_{k,1} &=t^2 \frac{1}{k!(n-k-1)!} \sum_{\pi \in S_n, \pi(1)=1} \mathrm{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}, \\ \tau_{k,2} &=t^2 \frac{1}{(k-1)!(n-k)!} \sum_{\pi \in S_n, \pi(k+1)=1} \mathrm{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}. \end{aligned}$$

Note that every term in the sums contains either dx_i or dy_i for $i \in \{1, \dots, n\}$, so the product of each term with $\omega_s = \sum_{i=1}^n dx_i \wedge dy_i$ vanishes and we obtain $\omega_s \wedge \tau_{k,1} = \omega_s \wedge \tau_{k,2} = 0$. □

Corollary 9.2.8. *The forms κ_k, $0 \leq k \leq n$, are primitive.*

Proof. For $1 \leq k \leq n-1$, this follows from the relation $\kappa_k = \frac{1}{r^2}\tau_{k,1} + \frac{1}{r^2}\tau_{k,2}$ on $T\mathbb{R}^n \backslash T_0\mathbb{R}^n$, from which the general case follows by continuity. For $k = 0, n$ this is trivial. □

9.3 Primitive SO(n)-invariant forms

To give a characterization of all primitive, SO(n)- and vertically translation invariant differential n-forms on $T\mathbb{R}^n$, we will need the following technical lemma.

Lemma 9.3.1. *Let $\phi \in C^\infty((0,\infty))$ be a function such that $f : \mathbb{R} \to \mathbb{R}$,*

$$f(r) := \begin{cases} \phi(r^2)r^2 & r \neq 0, \\ 0 & r = 0, \end{cases}$$

is a smooth function on $\mathbb{R}$. Then ϕ extends to an element of $C^\infty([0,\infty))$.

Proof. As f is a smooth, even function, its odd derivatives vanish in $r = 0$. Applying de L'Hospital's rule repeatedly, we obtain

$$\begin{aligned} f^{(2k)}(0) &= \lim_{s\to 0}(2k)! \frac{f(s) + f(-s) - 2\sum\limits_{i=0}^{k-1} \frac{s^{2i}}{(2i)!} f^{(2i)}(0)}{2s^{2k}} \\ &= \lim_{s\to 0}(2k)! \frac{f(s) - \sum\limits_{i=1}^{k-1} \frac{s^{2i}}{(2i)!} f^{(2i)}(0)}{s^{2k}}. \end{aligned} \tag{9.1}$$

For $k = 1$ this implies

$$f''(0) = \lim_{s\to 0} 2! \frac{f(s) - f(0)}{s^2} = \lim_{s\to 0} 2! \frac{\phi(s^2)s^2}{s^2} = \lim_{s\to 0} 2!\phi(s^2).$$

Thus ϕ has a continuous extension to $[0,\infty)$, which we will denote by ϕ again. We will now show the following by induction: For $k \in \mathbb{N}$, $\phi^{(k-1)}$ is differentiable in 0 with

$$\phi^{(k)}(0) = \frac{k!}{(2(k+1))!} f^{(2(k+1))}(0),$$

i.e. $f^{(2(k+1))}(0) = \frac{(2(k+1))!}{k!}\phi^{(k)}(0)$. Note that this implies that $\phi^{(k-1)}$ is continuous. For $k = 1$ Equation (9.1) implies

$$\begin{aligned} f^{(4)}(0) &= \lim_{s\to 0} 4! \frac{f(s) - \frac{s^2}{2!} f^{(2)}(0)}{s^4} \\ &= \lim_{s\to 0} 4! \frac{\phi(s^2)s^2 - \frac{s^2}{2!} 2!\phi(0)}{s^4} \\ &= \lim_{s\to 0} 4! \frac{\phi(s^2) - \phi(0)}{s^2} \\ &= \lim_{s\to 0} 4! \frac{\phi(s) - \phi(0)}{s}. \end{aligned}$$

Thus ϕ is differentiable in 0 and the derivative is given by the desired formula.
Assume that we have shown this relation for all derivatives of order less or equal to $k-1$. Then

$$\begin{aligned} f^{(2(k+1))}(0) &= \lim_{s\to 0}(2(k+1))! \frac{f(s) - \sum\limits_{i=1}^{k} \frac{s^{2i}}{(2i)!} f^{(2i)}(0)}{s^{2k+2}} \\ &= \lim_{s\to 0}(2(k+1))! \frac{\phi(s^2)s^2 - \sum\limits_{i=1}^{k} \frac{s^{2i}}{(2i)!}\frac{(2i)!}{(i-1)!}\phi^{(i-1)}(0)}{s^{2k+2}} \\ &= \lim_{s\to 0}(2(k+1))! \frac{\phi(s)s - \sum\limits_{i=1}^{k} \frac{s^{i}}{(i-1)!}\phi^{(i-1)}(0)}{s^{k+1}} \\ &= \lim_{s\to 0}(2(k+1))! \frac{\phi(s) - \sum\limits_{i=1}^{k} \frac{s^{i-1}}{(i-1)!}\phi^{(i-1)}(0)}{s^{k}} \\ &= \lim_{s\to 0}(2(k+1))! \frac{\phi(s) - \sum\limits_{i=0}^{k-1} \frac{s^{i}}{i!}\phi^{(i)}(0)}{s^{k}}. \end{aligned}$$

Applying de L'Hospital's rule $(k-1)$-times, we obtain

$$f^{(2(k+1))}(0) = \lim_{s\to 0}(2(k+1))! \frac{\phi^{(k-1)}(s) - \phi^{(k-1)}(0)}{k!s},$$

i.e. the limit $\lim\limits_{s\to 0} \frac{\phi^{(k-1)}(s)-\phi^{(k-1)}(0)}{s} = \frac{k!}{(2(k+1))!} f^{(2(k+1))}(0)$ exists. Thus $\phi^{(k-1)}$ is differentiable in 0 and the value of its derivative in 0 is given by desired expression. □

Proposition 9.3.2. *Let $\tau \in \Omega^n(T\mathbb{R}^n)$ be a vertically translation invariant differential form of bidegree $(n-k,k)$ that is also* SO(n)*-invariant and primitive. For $1 \le k \le n-1$ there exists a unique $c \in \mathbb{R}$ and two unique functions $\phi_1, \phi_2 \in C^\infty([0,\infty))$ such that*

$$\tau = c\kappa_k + \phi_1(r^2)\tau_{k,1} + \phi_2(r^2)\tau_{k,2}.$$

For $k = 0, n$ there exists a unique function $\phi \in C^\infty([0,\infty))$ such that

$$\tau = \phi(r^2)\kappa_k.$$

Proof. We will only show this for $1 \le k \le n-1$, the two other cases are similar and simpler.
Let us show that such a decomposition exists. As τ is vertically translation invariant, we only need to consider points of the form $(x,y) = (x,0)$.
There are two orbits of SO(n) on $\mathbb{R}^n$: The origin $x = 0$ and its complement. The stabilizer of $x = 0$ is SO(n), so by evaluating τ in $(x,0) = (0,0)$, we obtain an SO(n)-invariant element of $\Lambda^n T_{(0,0)}T\mathbb{R}^n \cong \Lambda^n(\mathbb{R}^n \times \mathbb{R}^n)$. By Proposition 9.1.1, the space of

SO(n)-invariant elements is generated by the forms ω_s, κ_j for $0 \leq j \leq n$. As τ is primitive, the Lefschetz decomposition 7.2.1 shows that τ evaluated in 0 is given by a linear combination of the forms κ_j, $0 \leq j \leq n$. Now the degree of homogeneity forces $\tau|_{(0,0)} = c\kappa_k|_{(0,0)}$ for some $c \in \mathbb{R}$.
Consider the form $\tilde{\tau} := \tau - c\kappa_k$, which vanishes in $x = 0$. We evaluate this form in $x \neq 0$. The stabilizer of this point can be identified with SO($n-1$), and the tangent space splits into the equivariant direct sum

$$T_{(x,0)} \cong \mathbb{R}^n \oplus \mathbb{R}^n = (\mathbb{R}x \oplus V) \oplus (\mathbb{R}x \oplus V) \cong (\mathbb{R}x \oplus \mathbb{R}x) \oplus (V \oplus V),$$

where $V = x^\perp$. The stabilizer $\mathrm{Stab}(x) \subset \mathrm{SO}(n)$ of $(x,0)$ acts trivially on $\mathbb{R}x \oplus \mathbb{R}x$ and by the usual diagonal action on $V \oplus V \cong \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1}$. The evaluation of $\tilde{\tau}$ in $(x,0)$ thus leads to an element of

$$(\Lambda^n T_{(x,0)}\mathbb{R}^n)^{\mathrm{Stab}(x)} \cong \bigoplus_{a+b+c=n} \Lambda^a(\mathbb{R}x) \otimes \Lambda^b(\mathbb{R}x) \otimes \Lambda^c(\mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1})^{\mathrm{SO}(n-1)},$$

where $a, b \in \{0,1\}$, $c \in \{n-1, n\}$. Now consider the invariant forms from Definition 9.2.1. The forms β and γ span the first two factors, while the last space is generated by the forms κ'_j, $0 \leq k \leq n-1$, and the restriction of the symplectic form to $V \oplus V$, where we have used Proposition 9.1.1 again. Note that the symplectic form of $V \oplus V$ is given by a multiple of $r^2\omega_s - \beta \wedge \gamma$. Thus the space on the right is contained in the space generated by β, γ, ω_s, and κ'_j for $0 \leq j \leq n-1$. The only n-forms are thus the forms $\tau_{j,1}$, $\tau_{j,2}$, for $1 \leq j \leq n-1$, and κ_0, κ_n, as well as multiples of ω_s. The only primitive forms are thus $\tau_{j,1}$, $\tau_{j,2}$, and κ_j, compare Lemma 9.2.7 and Corollary 9.2.6, so $\tilde{\tau}$ is a linear combination of these forms. Now note that κ_j is a linear combination of $\tau_{j,1}$ and $\tau_{j,2}$ for $x \neq 0$ by Proposition 9.2.2, so $\tilde{\tau}$ is in fact a linear combinations of the forms $\tau_{j,1}$, $\tau_{j,2}$ for $1 \leq j \leq n-1$. Comparing the degree of homogeneity, we obtain

$$\tilde{\tau} = \phi_1(x)\tau_{k,1} + \phi_2(x)\tau_{k,2}$$

for two smooth functions $\phi_1, \phi_2 : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. Note that ϕ_1 and ϕ_2 are uniquely determined by this equation, as the forms $\tau_{k,1}$ and $\tau_{k,2}$ are linearly independent at each point $x \neq 0$. In particular, the SO(n)-invariance forces these functions to be rotation invariant, i.e. we can assume that

$$\tilde{\tau} = \phi_1(r^2)\tau_{k,1} + \phi_2(r^2)\tau_{k,2}$$

for $r \neq 0$, where $\phi_1, \phi_2 \in C^\infty((0,\infty))$. Evaluating these forms along the line $\mathbb{R}(e_1, 0)$, we obtain, using the formula for $\tau_{k,1}$, $\tau_{k,2}$ from the proof of Proposition 9.2.2,

$$\begin{aligned}
&\tilde{\tau}|_{(te_1,0)} \\
=&\phi_1(t^2)t^2 \frac{1}{k!(n-k-1)!} \sum_{\pi \in S_n, \pi(1)=1} \mathrm{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)} \\
&+ \phi_2(t^2)t^2 \frac{1}{(k-1)!(n-k)!} \sum_{\substack{\pi \in S_n, \\ \pi(n-k+1)=1}} \mathrm{sign}(\pi) dx_{\pi(1)} \dots dx_{\pi(n-k)} \wedge dy_{\pi(n-k+1)} \dots dy_{\pi(n)}.
\end{aligned}$$

The two sums on the right are linearly independent on the whole tangent space, so we see that the limits $\lim_{t\to 0}\phi_i(t^2)t^2$ exist. Furthermore, $\tilde{\tau}$ vanishes for $t=0$. Thus the functions

$$f_i(r) := \begin{cases}\phi_i(r^2)r^2 & r\neq 0\\ 0 & r=0\end{cases}$$

are smooth functions on $\mathbb{R}$, so we can apply Lemma 9.3.1 to extend ϕ_1 and ϕ_2 to elements of $C^\infty([0,\infty))$. In total, we obtain the desired decomposition on $T\mathbb{R}^n$:

$$\tau = c\kappa_k + \tilde{\tau} = c\kappa_k + \phi_1(r^2)\tau_{k,1} + \phi_2(r^2)\tau_{k,2}.$$

To see that this decomposition is unique, assume that $c\kappa_k + \phi_1(r^2)\tau_{k,1} + \phi_2(r^2)\tau_{k,2} = 0$. As $\tau_{k,1}$ and $\tau_{k,2}$ vanish in $x=0$, we obtain $c=0$ by evaluating this expression in $(x,y)=(0,0)$. As the forms $\tau_{k,1}$ and $\tau_{k,2}$ are linearly independent outside of $x=0$, this implies $\phi_1=\phi_2=0$ on $(0,\infty)$, so they have to vanish identically on $[0,\infty)$ by continuity. □

Corollary 9.3.3. *If $\tau\in\Omega^{n-k,k}$ is $\mathrm{SO}(n)$-invariant and primitive, then there exist $\phi_0,\phi_1,\phi_2\in C_c^\infty([0,\infty))$ such that*

$$\begin{aligned}\tau =&\phi_0(r^2)\kappa_k + \phi_1(r^2)\tau_{k,1}+\phi_2(r^2)\tau_{k,2} && \textit{for } 1\le k\le n-1,\\ \tau =&\phi_0(r^2)\kappa_k && \textit{for } k=0,n.\end{aligned}$$

Proof. Let $R>0$ be such that $\operatorname{supp}\tau\subset\pi^{-1}(B_R(0))$ and let $\psi\in C_c^\infty([0,\infty))$ be a function with $\psi=1$ on $[0,R^2]$. Then multiply the decomposition in Proposition 9.3.2 with $\psi(r^2)$. □

9.4 Classification of smooth rotation invariant valuations

Corollary 9.3.3 lets us define three families of smooth rotation invariant valuations in degree $1\le k\le n-1$, each obtained from $\tau_{k,1}$, $\tau_{k,2}$ and κ_k by multiplying the form with $\phi(r^2)$ for $\phi\in C_c^\infty([0,\infty))$. For $k=0,n$ we only obtain one family of differential forms: They are given by $\tau=\phi_k(r^2)\kappa_k$, i.e. we are considering the valuations

$$\begin{aligned}\mu_n(f) =&D(f)[\phi_n(r^2)dy_1\wedge\dots\wedge dy_n] = \int_{\mathbb{R}^n}\phi_n(r^2)d\operatorname{Hess}_n(f) && \text{for } k=n,\\ \mu_0(f) =&D(f)[\phi_0(r^2)dx_1\wedge\dots\wedge dx_n] = \int_{\mathbb{R}^n}\phi_0(|x|^2)dx && \text{for } k=0,\end{aligned}$$

for $f\in\mathrm{Conv}(\mathbb{R}^n,\mathbb{R})$. In particular, the function ϕ_n is uniquely determined by μ_n and can be recovered from the characteristic function of μ_n, see Section 5.2.
The goal of this section is to show that, for $1\le k\le n-1$, every k-homogeneous valuation

can be represented by a differential form $\tau = \phi(r^2)\kappa_k$. As the integral condition in Theorem 7.2.5 is always satisfied if $k > 0$, we thus have to compare the symplectic Rumin differential of forms belonging to the three families. We begin with the families belonging to $\tau_{k,1}$ and $\tau_{k,2}$.

Proposition 9.4.1. *For $1 \leq k \leq n-1$*

$$\begin{aligned}
d(\phi(r^2)\tau_{k,1}) =& -(n-k)(n-k+1)\phi(r^2)\omega_s \wedge \kappa'_{k-1},\\
d(\phi(r^2)\tau_{k,2}) =& (n-k+1)[2\phi'(r^2)r^2 + \phi(r^2)(n-k+2)]\omega_s \wedge \kappa'_{k-1}.
\end{aligned}$$

Thus for $2 \leq k \leq n-1$

$$\begin{aligned}
\bar{\mathrm{D}}(\phi(r^2)\tau_{k,1}) =& -2(n-k)\phi'(r^2)\tau_{k-1,1} - (n-k)(n-k+1)\phi(r^2)\kappa_{k-1},\\
\bar{\mathrm{D}}(\phi(r^2)\tau_{k,2}) =& 2[2\phi''(r^2)r^2) + \phi'(r^2)(n-k+4)]\tau_{k-1,1}\\
&+ (n-k+1)[2\phi''(r^2)r^2 + 2\phi'(r^2) + \phi'(r^2)(n-k+2)]\kappa_{k-1},
\end{aligned}$$

and for $k = 1$

$$\begin{aligned}
\bar{\mathrm{D}}(\phi(r^2)\tau_{1,1}) =& -(n-1)n[2\phi'(r^2)r^2 + \phi(r^2)]\kappa_0,\\
\bar{\mathrm{D}}(\phi(r^2)\tau_{1,2}) =& n[4\phi''(r^2)r^4 + 2\phi'(r^2)r^2(n+4) + \phi(r^2)(n+1)]\kappa_0.
\end{aligned}$$

Proof. The differentials of $\tau_{k,1}$ and $\tau_{k,2}$ were already calculated in Proposition 9.4.1. Thus, using $dr^2 = 2\beta$,

$$\begin{aligned}
d(\phi(r^2)\tau_{k,1}) =& \phi'(r^2)2\beta \wedge \tau_{k,1} + \phi(r^2)d\tau_{k,1}\\
=& 0 - \phi(r^2)(n-k)\beta \wedge \kappa_k\\
=& -\phi(r^2)(n-k)(n-k+1)\omega_s \wedge \kappa'_{k-1},
\end{aligned}$$

where we have used Corollary 9.2.6 in the last step. Similarly,

$$\begin{aligned}
d(\phi(r^2)\tau_{k,2}) =& \phi'(r^2)2\beta \wedge \tau_{k,2} + \phi(r^2)d\tau_{k,2}\\
=& \phi'(r^2)2(n-k+1)\beta \wedge \gamma \wedge \kappa'_{k-1} + \phi(r^2)(n-k+2)\beta \wedge \kappa_k\\
=& \phi'(r^2)2(n-k+1)r^2\omega_s \wedge \kappa'_{k-1} + \phi(r^2)(n-k+2)(n-k+1)\omega_s \wedge \kappa'_{k-1}\\
=& (n-k+1)[2\phi'(r^2)r^2 + \phi(r^2)(n-k+2)]\omega_s \wedge \kappa'_{k-1}
\end{aligned}$$

by Lemma 9.2.5 and Corollary 9.2.6. Dividing by ω_s and applying d, we obtain the symplectic Rumin differential of these forms:

$$\begin{aligned}
\bar{\mathrm{D}}(\phi(r^2)\tau_{k,1}) =& -(n-k)(n-k+1)d[\phi(r^2)\kappa'_{k-1}]\\
=& -(n-k)(n-k+1)[\phi'(r^2)2\beta \wedge \kappa'_{k-1} + \phi(r^2)d\kappa'_{k-1}],\\
\bar{\mathrm{D}}(\phi(r^2)\tau_{k,2}) =& (n-k+1)d[[2\phi'(r^2)r^2 + \phi(r^2)(n-k+2)]\kappa'_{k-1}]\\
=& (n-k+1)[2\phi''(r^2)r^2 + 2\phi'(r^2) + \phi'(r^2)(n-k+2)]2\beta \wedge \kappa'_{k-1}\\
&+ (n-k+1)[2\phi'(r^2)r^2 + \phi(r^2)(n-k+2)]d\kappa'_{k-1}.
\end{aligned}$$

For $k \neq 1$ this reduces to

$$\begin{aligned}\bar{\mathrm{D}}(\phi(r^2)\tau_{k,1}) =& -2(n-k)\phi'(r^2)\tau_{k-1,1} - (n-k)(n-k+1)\phi(r^2)\kappa_{k-1},\\ \bar{\mathrm{D}}(\phi(r^2)\tau_{k,2}) =& 2[2\phi''(r^2)r^2 + \phi'(r^2)(n-k+4)]\tau_{k-1,1}\\ &+ (n-k+1)[2\phi'(r^2)r^2 + \phi(r^2)(n-k+2)]\kappa_{k-1},\end{aligned}$$

while we obtain

$$\begin{aligned}\bar{\mathrm{D}}(\phi(r^2)\tau_{1,1}) =& -(n-1)n[\phi'(r^2)2\beta \wedge \kappa_0' + \phi(r^2)\kappa_0]\\ =& -(n-1)n[2\phi'(r^2)r^2 + \phi(r^2)]\kappa_0,\\ \bar{\mathrm{D}}(\phi(r^2)\tau_{1,2}) =& n[2\phi''(r^2)r^2 + \phi'(r^2)(n+3)]2r^2\kappa_0 + n[2\phi'(r^2)r^2 + \phi(r^2)(n+1)]\kappa_0\\ =& n[4\phi''(r^2)r^4 + 2\phi'(r^2)r^2(n+3) + 2\phi'(r^2)r^2 + \phi(r^2)(n+1)]\kappa_0\\ =& n[4\phi''(r^2)r^4 + 2\phi'(r^2)r^2(n+4) + \phi(r^2)(n+1)]\kappa_0\end{aligned}$$

for $k = 1$. □

Proposition 9.4.2.

$$\begin{aligned}\bar{\mathrm{D}}(\phi(r^2)\kappa_k) =& 4\phi''(r^2)\tau_{k-1,1} + 2(n-k+1)\phi'(r^2)\kappa_{k-1} && \text{for } k \neq 1,\\ \bar{\mathrm{D}}(\phi(r^2)\kappa_1) =& 2n[2\phi''(r^2)r^2 + \phi'(r^2)]\kappa_0 && \text{for } k = 1.\end{aligned}$$

In particular,

$$(n-k)\,\bar{\mathrm{D}}(\phi(r^2)\kappa_k) = -2\,\bar{\mathrm{D}}(\phi'(r^2)\tau_{k,1}).$$

Proof. By Corollary 9.2.6,

$$d(\phi(r^2)\kappa_k) = 2\phi'(r^2)\beta \wedge \kappa_k = 2(n-k+1)\phi'(r^2)\omega_s \wedge \kappa_{k-1}'.$$

Dividing by ω_s and applying d, we obtain

$$\begin{aligned}\bar{\mathrm{D}}(\phi(r^2)\kappa_k) =& 2(n-k+1)d[\phi'(r^2)\kappa_{k-1}']\\ =& 2(n-k+1)[2\phi''(r^2)\beta \wedge \kappa_{k-1}' + \phi'(r^2)d\kappa_{k-1}'].\end{aligned}$$

For $k \neq 1$, this implies

$$\bar{\mathrm{D}}(\phi(r^2)\kappa_k) = 4\phi''(r^2)\tau_{k-1,1} + 2(n-k+1)\phi'(r^2)\kappa_{k-1}',$$

while we obtain

$$\bar{\mathrm{D}}(\phi(r^2)\kappa_1) = 2n[2\phi''(r^2)r^2 + \phi'(r^2)]\kappa_0$$

for $k = 1$. The last equation follows by comparing these formulas with Proposition 9.4.1. □

Corollary 9.4.3.

$$(n-k)\,\bar{\mathrm{D}}(\phi(r^2)\tau_{k,2}) = -\bar{\mathrm{D}}([2\phi'(r^2)r^2 + (n-k+2)\phi(r^2)]\tau_{k,1})$$

Proof. By Proposition 9.2.2, $r^2\kappa_k = \tau_{k,1} + \tau_{k,2}$, so

$$\begin{aligned}(n-k)\,\bar{\mathrm{D}}(\phi(r^2)\tau_{k,2}) =&(n-k)[\bar{\mathrm{D}}(\phi(r^2)r^2\kappa_k) - \bar{\mathrm{D}}(\phi(r^2)\tau_{k,1})]\\ =&-2\,\bar{\mathrm{D}}([\phi'(r^2)r^2+\phi]\tau_{k,1}) - (n-k)\,\bar{\mathrm{D}}(\phi(r^2)\tau_{k,1})\\ =&-\bar{\mathrm{D}}([2\phi'(r^2)r^2+(n-k+2)\phi(r^2)]\tau_{k,1})\end{aligned}$$

by Proposition 9.4.2. □

Theorem 9.4.4. *For every $\mu \in \mathrm{VConv}_k(\mathbb{R}^n)^{sm} \cap \mathrm{VConv}(\mathbb{R}^n)^{\mathrm{SO}(n)}$ of degree $k > 0$, there exists a unique function $\phi \in C_c^\infty([0,\infty))$ such that*

$$\mu(f) = D(f)[\phi(r^2)\kappa_k] \quad \text{for all } f \in \mathrm{Conv}(\mathbb{R}^n,\mathbb{R}).$$

If $f \in \mathrm{Conv}(\mathbb{R}^n,\mathbb{R}) \cap C^2(\mathbb{R}^n)$, then

$$\mu(f) = \int_{\mathbb{R}^n} \phi(|x|^2)[D^2 f]_k(x)dx,$$

where $[D^2 f(x)]_k$ denotes the k-th elementary symmetric polynomial in the eigenvalues of the Hessian of f.

Proof. For $k = n$, this follows directly from the properties of the characteristic function, so we can focus on the case $1 \leq k \leq n-1$.
Any such valuation can be represented by a vertically translation invariant differential form τ of bidegree $(n-k,k)$ that has horizontally compact support and is $\mathrm{SO}(n)$-invariant as well as primitive. By Corollary 9.3.3, any such form τ is given by

$$\tau = \phi_0(r^2)\kappa_k + \phi_1(r^2)\tau_{k,1} + \phi_2(r^2)\tau_{k,2}$$

for some functions $\phi_0, \phi_1, \phi_2 \in C_c^\infty([0,\infty))$. Combining Proposition 9.4.2 and Theorem 7.2.5, we see that $-(n-k)\phi_0(r^2)\kappa_k$ and $2\phi_0'(r^2)\tau_{k,1}$ induce the same valuation. By Corollary 9.4.3, the same holds for $[2\phi_2'(r^2)r^2 + (n-k+2)\phi_2(r^2)]\tau_{k,1}$ and $\phi_2(r^2)\tau_{k,2}$. Replacing these terms and combining everything, we obtain $\psi \in C_c^\infty([0,\infty))$ such that $\psi(r^2)\tau_{k,1}$ and τ induce the same valuation. Now set

$$\phi(s) := \int_0^s \psi(t)dt - \int_0^\infty \psi(t)dt,$$

i.e. $\phi'(s) = \psi(s)$ and $\phi \in C_c^\infty([0,\infty))$. By Proposition 9.4.2, $(n-k)\,\bar{\mathrm{D}}(\phi(r^2)\kappa_k) = -2\,\bar{\mathrm{D}}(\psi(r^2)\tau_{k,1}) = -2\,\bar{\mathrm{D}}\,\tau$, so $(n-k)\phi(r^2)\kappa_k$ and -2τ induce the same valuation. Rescaling ϕ by an appropriate constant, we obtain the desired representation.
To see that ϕ is uniquely determined, observe that μ is uniquely determined by $\bar{\mathrm{D}}(\phi(r^2)\kappa_k)$ by Theorem 7.2.5 (or Theorem 7.2.12). By Proposition 9.4.2, this differential is given by

$$\bar{\mathrm{D}}(\phi(r^2)\kappa_k) = 4\phi''(r^2)\tau_{k-1,1} + 2(n-k+1)\phi'(r^2)\kappa_{k-1}$$

for $k \neq 1$ and by

$$\bar{\mathrm{D}}(\phi(r^2)\kappa_1) = 2n[2\phi''(r^2)r^2 + \phi'(r^2)]\kappa_0$$

for $k = 1$. As $\tau_{k,1}$ and κ_k are linearly independent on the complement of $x = 0$, $\bar{\mathrm{D}}(\phi(r^2)\kappa_k) = 0$ implies $\phi'' = 0$ on $(0, \infty)$ for $k \neq 1$. Thus ϕ is the restriction of an affine function to $[0, \infty)$. As ϕ has compact support, this implies $\phi \equiv 0$.
For $k = 1$, take $R > 0$ such that $\phi(s) = 0$ for all $s \geq R$. $\bar{\mathrm{D}}(\phi(r^2)\kappa_1) = 0$ implies that ϕ is a solution of the ordinary differential equation

$$2u''(s)s + u'(s) = 0 \quad \text{on } (0, \infty).$$

Given $a, b \in \mathbb{R}$, this ordinary differential equation has a unique maximal solution with $u(R) = a$, $u'(R) = b$. In our case, ϕ is a solution on $(0, \infty)$ for $a = b = 0$. The same holds true for the trivial solution $u \equiv 0$, so $\phi \equiv 0$ on $(0, \infty)$ by uniqueness. Thus ϕ has to vanish identically on $[0, \infty)$ by continuity.
In both cases, ϕ is uniquely determined by μ. The last statement is a reformulation of Proposition 9.1.4. □

Corollary 9.4.5. *Every* $\mathrm{SO}(n)$*-invariant valuation in* $\mathrm{VConv}(\mathbb{R}^n)$ *is* $\mathrm{O}(n)$*-invariant.*

Proof. As the operation of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ is continuous on $\mathrm{VConv}(V)$, it is enough to verify this claim for the subspace of smooth $\mathrm{SO}(n)$-invariant valuations, which is dense on $\mathrm{VConv}(\mathbb{R}^n)^{\mathrm{SO}(n)}$ by Corollary 7.3.6. Thus the claim follows from the previous characterization of smooth $\mathrm{SO}(n)$-invariant valuations, as these functionals are $\mathrm{O}(n)$-invariant. □

10 Deutsche Zusammenfassung

Für einen reellen Vektorraum V der Dimension $\dim(V) = n$ bezeichne $\mathcal{K}(V)$ die Menge der konvexen Körper in V, d.h. die Menge aller kompakten, konvexen Teilmengen. Versieht man $\mathcal{K}(V)$ mit der sogenannten Hausdorff-Metrik, so erhält man einen lokal kompakten metrischen Raum. Ist $(G, +)$ eine abelsche Gruppe, so wird eine Abbildung $\mu : \mathcal{K}(V) \to (G, +)$ als *Bewertung* bezeichnet, falls sie

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

für alle $K, L \in \mathcal{K}(V)$ mit $K \cup L \in \mathcal{K}(V)$ erfüllt. Die Klasse der Bewertungen enthält viele geometrische Funktionale wie beispielsweise das Lebesgue-Maß, sowie Oberflächenmaß und Eulercharakteristik. Die Theorie der Bewertungen auf konvexen Körper ist daher seit jeher ein wichtiger Teil der Konvexgeometrie mit vielen Anwendungen im Kontext von geometrischen Ungleichungen und Integralgeometrie. Dies trifft insbesondere auf den Raum $\mathrm{Val}(V)$ der reellwertigen, stetigen, translationsinvarianten Bewertungen auf konvexen Körpern zu. Eine der wichtigesten Eigenschaften dieses Raums ist die Existenz einer homogenen Zerlegung. Es bezeichne $\mathrm{Val}_k(V)$ den Unterraum von $\mathrm{Val}(V)$ der k-homogenen Bewertungen, d.h. aller $\mu \in \mathrm{Val}(V)$, welche $\mu(tK) = t^k\mu(K)$ für alle $K \in \mathcal{K}(V)$ und $t \geq 0$ erfüllen.

Theorem 10.0.1 (McMullen Zerlegung [43])**.**

$$\mathrm{Val}(V) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(V)$$

Mit anderen Worten ist die Abbildung $t \mapsto \mu(tK)$ ein Polynom, dessen Grad durch die Dimension des Raumes beschränkt ist. Definiert man für eine homogene Bewertung $\mu \in \mathrm{Val}_k(V)$ die *Polarisierung* $\bar{\mu} : \mathcal{K}(V)^k \to \mathbb{R}$ von μ durch

$$\bar{\mu}(K_1, \dots, K_k) := \frac{1}{n!} \frac{\partial}{\partial \lambda_1}\Big|_0 \dots \frac{\partial}{\partial \lambda_k}\Big|_0 \mu(\lambda_1 K_1 + \dots + \lambda_k K_k),$$

wobei $A + B := \{a + b : a \in A, b \in V\}$ die Minkowski-Summe zweier Teilmengen von V bezeichnet, so erhält man ein symmetrisches Funktional, welches in jedem Argument eine *additive* Bewertung ist: Für $K, L, K_2, \dots, K_k$ gilt

$$\bar{\mu}(K + L, K_2, \dots, K_k) = \bar{\mu}(K, K_2, \dots, K_k) + \bar{\mu}(L, K_2, \dots, K_k).$$

Wendet man diese Konstruktion auf ein Lebesgue-Maß an, so erhält man Minkowskis gemischte Volumina $V(K_1, \dots, K_n)$, aus denen sich eine wichtige Klasse von Bewertungen konstruieren lässt: Fixiert man $K_{k+1}, \dots, K_n \in \mathcal{K}(V)$, so definiert die Abbildung

$K \to V(K[k], K_{k+1}, \ldots, K_n)$ eine stetige, translationsinvariante und k-homogene Bewertungen. Tatsächlich bilden Linearkombinationen dieser Bewertungen einen dichten Teilraum von $\mathrm{Val}_k(V)$ bezüglich der Topologie der lokal gleichmäßigen Konvergenz. Diese Aussage ist als McMullen-Vermutung bekannt und wurde von Alesker im Jahr 2001 bewiesen. Tatsächlich ist das von ihm gezeigte Resultat deutlich allgemeiner.

Theorem 10.0.2 (Aleskers Irreduzibilitätstheorem [2]). *Die natürliche Darstellung von* $\mathrm{GL}(V)$ *auf* $\mathrm{Val}_k^{\pm}(V) := \{\mu \in \mathrm{Val}_k(V) : \mu(-K) = \pm\mu(K)\ \forall K \in \mathcal{K}(V)\}$ *ist topologisch irreduzibel.*

Die Operation von $\mathrm{GL}(V)$ ist hierbei gegeben durch $\pi(g)\mu(K) := \mu(g^{-1}K)$ für $\mu \in \mathrm{Val}(V)$ und $g \in \mathrm{GL}(V)$ und eine Darstellung einer Lie-Gruppe auf einem topologischen Vektorraum wird als topologisch irreduzibel bezeichnet, wenn die einzigen abgeschlossenen invarianten Unterräume der Nullraum oder der gesamte Raum sind. Mit anderen Worten ist ein invarianter Unterraum entweder trivial oder dicht. Indem man geeignete Linearkombinationen von gemischten Volumina betrachtet, lässt sich leicht zeigen, dass der von diesen erzeugte Unterraum mit jedem der obigen irreduziblen Teilräume nichtleeren Schnitt besitzt und somit dicht liegt.
Für $k = 1$ wurde die McMullen-Vermutung bereits vorher von Goodey und Weil bewiesen. Sie betrachteten die Stützfunktion $h_K : V^* \to \mathbb{R}$ eines konvexen Körpers $K \in \mathcal{K}(V)$, welche gegeben ist durch

$$h_K(y) = \sup_{x \in K} \langle y, x \rangle \quad \text{für } y \in V^*,$$

wobei die Klammern die natürliche Paarung zwischen V und seinem Dualraum V^* bezeichnen. Da $h_{K+L} = h_K + h_L$ für all $K, L \in \mathcal{K}(V)$ gilt, kann man die Polarisierung als multilineares Funktional auf dem Kegel der Stützfunktionen ansehen.

Theorem 10.0.3. *Für jede Bewertung* $\mu \in \mathrm{Val}_k(V)$ *existiert eine eindeutige Distribution* $\mathrm{GW}(\mu) \in \mathcal{D}'(\mathbb{P}_+(V^*)^k, L^{\boxtimes k})$*, welche als Goodey-Weil Distribution von* μ *bezeichnet wird, welche*

$$\mathrm{GW}(\mu)\,[h_{K_1} \otimes \cdots \otimes h_{K_k}] = \bar{\mu}(K_1, \ldots, K_k)$$

für alle glatten und strikt konvexen $K \in \mathcal{K}(V)$ *erfüllt.*
Zusätzlich ist der Träger von $\mathrm{GW}(\mu)$ *in der Diagonale von* $\mathbb{P}_+(V^*)^k$ *enthalten.*

Hier bezeichnet $L \to \mathbb{P}_+(V^*)$ ein gewisses Linienbündel über dem Raum der orientierten Geraden in V^*, dessen Schnitte kanonisch mit 1-homogenen Funktionen auf V^* identifiziert werden können. Insbesondere definiert die Stützfunktion eines glatten, strikt konvexen Körpers einen glatten Schnitt von L. Die Abbildung $\mathrm{GW} : \mathrm{Val}(V) \to \mathcal{D}'(\mathbb{P}_+(V^*)^k, L^{\boxtimes k})$ wurde ursprünglich von Goodey und Weil für Funktionen auf der Sphäre definiert [30], die obige invariante Version stammt von Alesker [1], welcher auch die Einschränkungen an den Träger bemerkte, was sich als kritisch für den Beweis des Irreduzibilitätstheorems herausstellte.

Diese Arbeit beschäftigt sich mit funktionalen Versionen der obigen Resultate. Der Begriff der Bewertung überträgt sich wie folgt: Eine Abbildung $\mu : X \to G$, definiert auf einem Raum X reellwertiger Funktionen, wird als Bewertung bezeichnet, wenn

$$\mu(f) + \mu(h) = \mu(f \vee h) + \mu(f \wedge h)$$

für alle $f, h \in X$ gilt, für die das punktweise Maximum $f \vee h$, beziehungsweise Minimum $f \wedge h$ ebenfalls in X liegt. Während dieser Begriff im Kontext von Funktionen relativ jung ist, ist die dahinterstehende Idee sehr alt: Um funktionale Versionen geometrisch interessanter Bewertungen auf einer Klasse von Mengen zu erhalten, betrachtet man zunächst Funktionen, die aus einfachen Grundbausteinen zusammengesetzt werden, und verwendet die Bewertungseigenschaft, um eine kombinatorische Fortsetzung auf dieser Klasse von Funktionen zu erhalten. Anschließend stellt sich die Frage, ob sich diese Definition durch Ausnutzung von Stetigkeitseigenschaften auf eine größere Klasse von Funktionen fortsetzen lässt.
In diesem Sinne ist das Lebesgue-Integral eine Fortsetzung des Volumens $\mathrm{vol}_n \in \mathrm{Val}_n(\mathbb{R}^n)$ zu einer Bewertung I auf den Raum $\mathrm{L}^1(\mathbb{R}^n)$ der Lebesgue-integrierbaren Funktionen, welche $I(\alpha 1_K) = \alpha \,\mathrm{vol}_n(K)$ für $\alpha \in \mathbb{R}$ und $K \in \mathcal{K}(V)$ erfüllt. Dies ist bei weitem jedoch nicht die einzige Möglichkeit. Nimmt man eine stetige Funktion $F : \mathbb{R} \to \mathbb{R}$, welche $|F(t)| \leq C|t|$ für alle $t \in \mathbb{R}$ und ein $C > 0$ erfüllt, so definiert

$$\tilde{I}(f) := \int_{\mathbb{R}^n} F(f(x))dx \quad \text{für } f \in \mathrm{L}^1(\mathbb{R}^n)$$

ebenfalls eine Bewertung auf $\mathrm{L}^1(\mathbb{R}^n)$, welche durch $\tilde{I}(\alpha 1_K) = F(\alpha) \,\mathrm{vol}_n(K)$ charakterisiert ist. Schränkt man diese beiden Funktionale jedoch auf Indikatorfunktionen ein, so erhält man in beiden Fällen das Lebesgue-Maß - mit anderen Worten verfügen Bewertungen auf Funktionenräumen üblicherweise über mehr Freiheitsgrade als die zugrundeliegenden geometrischen Funktionale. Hierbei muss man allerdings erwähnen, dass diese Aussage je nach Kontext auch komplett falsch sein kann, siehe beispielsweise [22]. Auch die vorliegende Arbeit verfolgt in Teilen einen gegensätzlichen Ansatz.
Nichts desto trotz verbleibt die Problematik, geometrisch relevante Bewertungen auf Funktionen zu finden. Ein großer Teil der Arbeiten zu Bewertungen auf Funktionen beschäftigt sich daher mit der Klassifikation von Bewertungen, die bestimmte Invarianzeigenschaften besitzen. Die obigen Beispiele ergeben beispielsweise eine vollständige Charakterisierung aller stetigen, translationsinvarianten Bewertungen auf $\mathrm{L}^1(\mathbb{R}^n)$ [56]. Ähnliche Resultate existieren für Sobolev-Räume [38, 39, 42], L^p-Räume [40, 50, 56, 57], quasi-konkave Funktionen [10, 15, 16], Orlicz-Räume [36], Lipschitzfunktionen [22, 23], definierbare Funktionen [10], Funktionen mit beschränkter Variation [59] und konvexe Funktionen [12, 17, 18, 48, 49]. Daneben gibt es auch einige Resultate zu analytischen Eigenschaften von Bewertungen auf allgemeinen Banach-Verbänden [55].

Diese Arbeit strebt ebenfalls an, einen speziellen Raum von Bewertungen auf Funktionen zu beschreiben. Der Fokus liegt dabei auf der Konstruktion gewisser dichter Teilräume. Eine vollständige Charakterisierung scheint zu diesem Zeitpunkt noch in

weiter Ferne zu stehen - die betrachteten Bewertungen lassen sich in einem sehr präzisen Sinne als translationsinvariante Bewertungen auf konvexen Körpern auffassen, für die keine solche Charakterisierung bekannt ist.
Im Folgenden betrachten wir vor allem Unterräume von

$$\mathrm{Conv}(V) := \{f : V \to \mathbb{R} \cup \{+\infty\} \ : \ f \text{ konvex, unterhalbstetig}, f \not\equiv +\infty\}.$$

Dieser Raum trägt eine metrisierbare Topologie, welche durch *Epi-Konvergenz* von Folgen konvexer Funktionen charakterisiert ist. Für die nachfolgende Diskussion wird die genaue Definition dieser Begriffe nicht benötigt, es sei jedoch gesagt, dass Epikonvergenz im Fall des Unterraums $\mathrm{Conv}(V,\mathbb{R}) := \{f \in \mathrm{Conv}(V) : f < +\infty\}$ der endlichen konvexen Funktionen äquivalent zu sowohl punktweiser, als auch lokal gleichmäßiger Konvergenz ist. Wir nehmen im Folgenden zusätzlich an, dass V mit einem Skalarprodukt ausgestattet ist.

Beispiel 10.0.4. *Es sei V ein euklidischer Vektorraum. In [7] betrachtete Alesker Abbildungen der Form*

$$\begin{aligned} \mathrm{Conv}(V,\mathbb{R}) \cap C^2(V) \to& \mathbb{R} \\ f \mapsto& \mu(f) := \int_V B(x) \det(H_f(x)[k], A_1(x), \dots, A_{n-k}(x)) dx, \end{aligned}$$

wobei $B \in C_c(V)$ und $A_1, \dots, A_{n-k} \in C_c(V, \mathcal{H}(V))$ stetige Funktionen mit Werten im Raum der symmetrischen Endomorphismen von V sind, H_f die Hesse-Matrix von $f \in C^2(V)$, und det *die gemischte Determinante von n symmetrischen Endomorphismen bezeichnet. Er zeigte, dass sich diese zu stetigen Bewertungen auf* $\mathrm{Conv}(V,\mathbb{R})$ *fortsetzen lassen und dass Bewertungen dieser Form über*

$$K \mapsto \mu(h_K)$$

einen dichten Teilraum von $\mathrm{Val}(V^*)$ *definieren.*

Sein Hauptresult beruht auf der Beobachtung, dass die Stützfunktion eines konvexen Körpers $K \in \mathcal{K}(V^*)$ die Bedingung $h_{K+x}(y) = h_K + \langle y, x\rangle$ für alle $x \in V^*$, $y \in V$ erfüllt. Da die obigen Funktionale invariant bezüglich der Addition linearer Funktionen sind, definiert $K \mapsto \mu(h_K)$ eine stetige, translationsinvariante Bewertung auf $\mathcal{K}(V^*)$. Durch Betrachtung geeigneter Beispiele folgt mithilfe des Irreduzibilitätstheorems, dass Bewertungen dieser Art einen dichten Teilraum von $\mathrm{Val}(V^*)$ aufspannen.
Gemäßg der McMullen-Zerlegung zerfallen die so konstruierten Bewertungen auf $\mathcal{K}(V^*)$ in homogenen Komponenten. Tatsächlich sind die obigen Funktionale auf $\mathrm{Conv}(V,\mathbb{R})$ ebenfalls k-homogen, d.h. sie erfüllen $\mu(tf) = t^k\mu(f)$ für alle $t > 0$ und $f \in \mathrm{Conv}(V,\mathbb{R})$. Es drängt sich somit die Frage auf, ob für den Raum der stetigen Bewertungen auf $\mathrm{Conv}(V,\mathbb{R})$, die invariant bezüglich der Addition linearer Funktionen sind, ebenfalls eine solche Zerlegung existiert. Bedauerlicherweise ist dies nicht nicht der Fall, wie folgendes Beispiel zeigt.

Beispiel 10.0.5. *Für $f \in \mathrm{Conv}(V,\mathbb{R})$ definiert $\mu(f) := |f(0)|^p$ eine stetige, p-homogene Bewertung, welche invariant bezüglich der Addition linearer Funktionen ist.*

Tatsächlich liefert die Zuweisung $\mu \mapsto [K \mapsto \mu(h_K)]$ zwar einen dichten Teilraum von $\mathrm{Val}(V^*)$, jedoch auch einen unendlich dimensionalen Kern von Bewertungen auf $\mathrm{Conv}(V,\mathbb{R})$.

Die obigen Funktionale besitzen jedoch noch eine weitere Invarianzeigenschaft: Sie sind invariant bezüglich der Addition von Konstanten.

Definition 10.0.6. *Für $C \subset \mathrm{Conv}(V)$ und einen reellen Hausdorff topologischen Vektorraum F bezeichne* $\mathrm{VConv}(C;V,F)$ *den Raum der stetigen,* dual epi-translationsinvarianten *Bewertungen, d.h. aller stetigen Bewertungen $\mu : C \to F$, die zusätzlich*

$$\mu(f+\lambda+c) = \mu(f) \quad \textit{für alle } f \in C, \lambda \in V^*, c \in \mathbb{R}$$

mit $f+\lambda+c \in C$ erfüllen.

Für die geometrische Bedeutung dieses Invarianzbegriffs müssen wir noch einmal zur Definition einer Bewertung auf Funktionen zurückkehren. Für $f \in \mathrm{Conv}(V)$ ist der *Epigraph* $\mathrm{epi}(f) := \{(x,t) \in V \times \mathbb{R} : f(x) \leq t\}$ eine nichtleere, abgeschlossene konvexe Teilmenge von $V \times \mathbb{R}$. Zusätlich gilt

$$\mathrm{epi}(f \vee h) = \mathrm{epi}(f) \cap \mathrm{epi}(h), \qquad \mathrm{epi}(f \wedge h) = \mathrm{epi}(f) \cup \mathrm{epi}(h).$$

Mit anderen Worten lässt sich eine Bewertung auf Funktionen auch als mengentheoretische Bewertung auf Epigraphen interpretieren. Für $f \in \mathrm{Conv}(V)$ können wir die Legendre-Transformierte $f^* \in \mathrm{Conv}(V^*)$ gegeben durch

$$f^*(y) = \sup_{x\in V}\langle y,x\rangle - f(x) = \sup_{(x,t)\in\mathrm{epi}(f)} \langle y,x\rangle - t = h_{\mathrm{epi}(f)}(y,-1)$$

betrachten. Wegen $f^{**} = f$ lässt sich $f = h_{\mathrm{epi}(f^*)}(\cdot,-1)$ somit als Stützfunktion einer konvexen Menge in $V^* \times \mathbb{R}$ auffassen. Die obige Invarianzeigenschaft ist dann nichts anderes als die Invarianz der Bewertung bezüglich Translationen dieser Mengen in $V^* \times \mathbb{R}$. Insbesondere können wir die Einschränkung dieser Bewertungen auf Funktionen der Form $h_K(\cdot,-1)$ für $K \in \mathcal{K}(V^* \times \mathbb{R})$ betrachten; mit anderen Worten können wir Bewertungen auf konvexen Funktionen als Bewertungen auf höherdimensionalen konvexen Körpern interpretieren.

Theorem 10.0.7 (Theorem 5.2.5)**.** *Es sei $C \subset \mathrm{Conv}(V)$ eine Teilmenge, die $\mathrm{Conv}(V,\mathbb{R})$ enthält, F ein topologischer Vektorraum. Die Abbildung*

$$\begin{aligned} T : \mathrm{VConv}(C;V,F) &\to \mathrm{Val}(V^* \times \mathbb{R}, F) \\ \mu &\mapsto [K \mapsto \mu(h_K(\cdot,-1))] \end{aligned}$$

ist wohldefiniert, injektiv und stetig. Hierbei bezeichnet $\mathrm{Val}(V^* \times \mathbb{R}, F)$ den Raum der stetigen, translationsinvarianten Bewertungen mit Werten in F und beide Räume sind mit der Kompakt-Offen-Topologie ausgestattet.

Wir fassen $\mathrm{VConv}(C;V,F)$ mithilfe dieser Abbildung alsTeilraum von $\mathrm{Val}(V^*\times\mathbb{R},F)$ auf. Die McMullen-Zerlegung für diesen Raum impliziert direkt die folgende homogene Zerlegung

Theorem 10.0.8 (Theorem 5.3.4)**.** *Es sei $C\subset \mathrm{Conv}(V)$ eine Teilmenge mit $tf\in C$ für alle $f\in C$, $t>0$, welche $\mathrm{Conv}(V,\mathbb{R})$ enthält. Zusätzlich sei F ein reeller Hausdorff topologischer Vektorraum. Dann gilt*

$$\mathrm{VConv}(C;V,F)=\bigoplus_{k=0}^{n}\mathrm{VConv}_k(C;V,F),$$

wobei $\mathrm{VConv}_k(C;V,F):=\{\mu\in\mathrm{VConv}(C;V,F):\mu(tf)=t^k\mu(f)$ *für alle* $f\in C,t>0\}$.

Für reellwertige Funktionen und $C=\mathrm{Conv}(V,\mathbb{R})$ wurde dies bereits von Colesanti, Ludwig und Mussnig in [21] gezeigt. Ihr Beweis verwendet eine andere Art der Einbettung, beruht jedoch auch auf der McMullen-Zerlegung. Mit ihrer Methode lässt sich die obige Version ebenfalls zeigen, da die Einbettung $T:\mathrm{VConv}(C;V,F)\to\mathrm{Val}(V^*\times\mathbb{R},F)$ jedoch im weiteren Verlauf dieser Arbeit eine herausragende Rolle spielen wird, habe ich mich entschlossen, einen kurzen alternativen Beweis zu präsentieren.

Wie auch für Bewertungen auf konvexen Körpern erlaubt es uns diese Zerlegung, die Polarisierung einer homogenen Bewertung $\mu\in\mathrm{VConv}_k(C;V,F)$ zu definieren, falls es sich bei $C\subset\mathrm{Conv}(V)$ um einen Kegel handelt, d.h. falls $f+th\in C$ für all $f,h\in C$, $t>0$ gilt. Wir werden dabei eine zusätzliche Regularitätsannahme treffen müssen. Dafür bezeichnen wir einen Kegel $C\subset\mathrm{Conv}(V)$ als *regulär*, wenn $\mathrm{dom}(f):=\{x\in V: f(x)<+\infty\}$ für alle $f\in C$ nichtleeres Innere besitzt. Dies ist hinreichend für die Stetigkeit der Additionsabbildung $+:C^2\to C$, was notwendig für die Konstruktion der Polarisierung ist. Wir erhalten in diesem Fall eine stetige Abbildung $\bar{\mu}:C^k\to F$ für jedes $\mu\in\mathrm{VConv}_k(C;V,F)$, welche folgenden Eigenschaften besitzt (siehe Theorem 5.3.9):

1. $\bar{\mu}$ ist symmetrisch.
2. $\bar{\mu}$ ist additiv und 1-homogen in jedem Argument:
$$\bar{\mu}(f+th,f_2,\dots,f_k)=\bar{\mu}(f,f_2,\dots,f_k)+t\bar{\mu}(h,f_2,\dots,f_k)$$
für $f,h,f_2,\dots,f_k\in C$, $t>0$.
3. $\bar{\mu}(f,\dots,f)=\mu(f)$ für all $f\in C$.

Indem man Differenzen von konvexen Funktionen betrachtet, kann man $\bar{\mu}$ zu einem multilinearen Funktional auf eben diesen fortsetzen. Es stellt sich heraus, dass die Einschränkung dieses Funktionals auf $C^2_c(V)$ stetig ist, welches sich somit gemäß des Kerntheorems von Schwartz zu einer eindeutigen Distribution fortsetzen lässt. Dies führt auf folgende Version der Goodey-Weil Einbettung für $\mathrm{VConv}(C;V,F)$, sofern es sich bei F um einen lokal konvexer Vektorraum handelt, d.h. einen reellen Hausdorff topologischen Vektorraum, dessen Topologie von einer Familie von Halbnormen erzeugt wird.

Theorem 10.0.9 (Theorem 5.4.9 und 5.4.7). *Es sei $\mu \in \mathrm{VConv}_k(C; V, F)$ und F ein lokal konvexer Vektorraum, der mit einer stetigen Norm versehen werden kann. Dann existiert eine eindeutige Distribution $\overline{\mathrm{GW}}(\mu) \in \mathcal{D}'(V^k, \bar{F})$ mit kompaktem Träger, welche*

$$\overline{\mathrm{GW}}(f_1 \otimes \cdots \otimes f_k) = \bar{\mu}(f_1, \ldots, f_K) \quad \text{für alle } f_1, \ldots, f_k \in \mathrm{Conv}(V, \mathbb{R}) \cap C^\infty(V) \tag{10.1}$$

erfüllt und als Goodey-Weil Distribution von μ bezeichnet wird. Zusätzlich ist der Träger dieser Distribution in der Diagonalen in V^k enthalten.

Hier bezeichnet $\bar{F}$ die Vervollständigung von F. Wir definieren ebenfalls eine Version der Goodey-Weil Distributionen für allgemeine lokal konvexe Vektorräume. In diesem Fall ist der Träger zwar noch immer in der Diagonale enthalten, jedoch im Allgemeinen nicht kompakt, weswegen sich die Distribution nicht kanonisch auf alle glatten Funktionen fortsetzen lässt. In beiden Fällen ist die Bewertung jedoch durch ihre zugehörige Goodey-Weil Distribution eindeutig bestimmt.
Dies wirft insbesondere die Frage auf, welche Eigenschaften der Goodey-Weil Distribution sich in korrespondierende Eigenschaften der zugrunde liegenden Bewertung übersetzen lassen. Schaut man sich die charakterisierende Eigenschaft der Goodey-Weil Distribution in Gleichung (10.1) an, so scheint der Träger der Distribution ein geeigneter Kandidat zu sein, schließlich lässt sich dieser durch reines Auswerten auf passenden Funktionen charakterisieren. Da der Träger von $\overline{\mathrm{GW}}(\mu)$ in der Diagonalen enthalten ist, können wir ihn als Bild des *Trägers* $\operatorname{supp}\mu$ der Bewertung μ unter der Diagonaleinbettung $\Delta : V \to V^k$ auffassen. Ist $\mu \in \mathrm{VConv}(C; V, F)$ eine beliebige Bewertung, so definieren wir $\operatorname{supp}\mu$ durch die Vereinigung der Träger der homogenen Komponenten von μ. Während diese Definition zunächst auf der Goodey-Weil Distribution von μ beruht, ist es nicht schwer zu sehen, dass der Träger die folgende Eigenschaft hat: Sind $f, h \in \mathrm{Conv}(V, \mathbb{R})$ zwei Funktionen, die $f = h$ auf einer Umgebung des Trägers erfüllen, so gilt $\mu(f) = \mu(h)$. Tatsächlich lässt sich der Träger durch diese Eigenschaft auch ohne Bezugnahme auf die Goodey-Weil Distribution charakterisierern, siehe Proposition 6.1.3.

Durch den Träger unterliegen die Bewertungen einer Reihe von Restriktionen, insbesondere reellwertige Bewertungen, die invariant unter nicht-kompakten Untergruppen der affinen Gruppe sind. So zeigt Korollar 6.3.4 beispielsweise, dass es außer konstanten Bewertungen keine translations- bzw. $\mathrm{SL}(V)$-invarianten Bewertungen gibt (im letzten Fall für $\dim V \geq 2$). Es besteht auch ein interessanter Zusammenhang zwischen dem Kegel $C \subset \mathrm{Conv}(V)$ und dem Träger der Bewertungen in $\mathrm{VConv}(C; V, F)$.

Theorem 10.0.10 (Theorem 6.3.5). *Es sei $C \subset \mathrm{Conv}(V)$ ein regulärer Kegel, der $\mathrm{Conv}(V, \mathbb{R})$ enthält. Setzt man $\mathrm{dom}(C) := \bigcap_{f \in C} \overline{\mathrm{dom}\, f}$, so gilt:*

1. *Der Träger jeder Bewertung in $\mathrm{VConv}(C; V, F)$ ist in $\mathrm{dom}(C)$ enthalten.*
2. *Lässt sich F mit einer stetigen Norm versehen, so lässt sich jede Bewertung in $\mathrm{VConv}(V, F)$, deren Träger im Inneren von $\mathrm{dom}(C)$ enthalten ist, eindeutig zu einem Element von $\mathrm{VConv}(C; V, F)$ fortsetzen.*

Im letzten Fall haben wir somit Inklusionen

$$\mathrm{VConv}_{\mathrm{int}\,\mathrm{dom}(C)}(V,F) \hookrightarrow \mathrm{VConv}(C;V,F) \hookrightarrow \mathrm{VConv}_{\mathrm{dom}(C)}(V,F).$$

Beide Inklusionen sind im Allgemeinen strikt, wie wir an zwei Beispielen diskutieren werden, doch in bestimmten Fällen ist die erste Inklusion bijektiv. Dazu betrachten wir für eine offene und konvexe Menge $U \subset V$ den regulären Kegel $C_U := \{f \in \mathrm{Conv}(V) : f|_U < +\infty\}$. Zusätzlich setzen wir $\mathrm{Conv}(U,\mathbb{R}) := \{f : U \to \mathbb{R} : f \text{ konvex}\}$. Ausgestattet mit der Topologie der lokal gleichmäßigen Konvergenz ist dies ein metrisierbarer Raum. Wir bezeichnen den Raum der stetigen, dual epi-translationsinvarianten Bewertungen auf $\mathrm{Conv}(U,\mathbb{R})$ mit Werten in F mit $\mathrm{VConv}(U,F)$ und statten diesen Raum mit der Kompakt-Offen-Topologie aus.

Theorem 10.0.11 (Theorem 6.3.12)**.** *Ist $U \subset V$ offen und konvex und F ein lokal konvexer Raum, der mit einer stetigen Norm ausgestattet werden kann, so ist die Abbildung*

$$\begin{aligned}\mathrm{res}^* : \mathrm{VConv}(U,F) &\to \mathrm{VConv}(C_U;V,F)\\ \mu &\mapsto [f \mapsto \mu(f|_U)]\end{aligned}$$

wohldefiniert und ein topologischer Isomorphismus.

Wir betrachten ebenfalls Teilräume von $\mathrm{VConv}(V,F)$ von Bewertungen mit kompaktem Träger. Fixiert man eine kompakte Menge $A \subset V$, so hängt der Wert jeder Bewertung im Raum $\mathrm{VConv}_A(V,F)$ der Bewertungen, deren Träger in A enthalten ist, nur von den Werten ihres Arguments auf einer beliebigen offenen Umgebung von A ab. Das unterschiedliche Divergenzverhalten verschiedener Klassen konvexer Funktionen im Unendlichen spielt für diese Teilräume somit nur eine sehr untergeordnete Rolle. Wir werden dies dahingehend nutzen, die Halbnormen, welche die Kompakt-Offen-Topologie auf $\mathrm{VConv}(V,F)$ erzeugen, auf den Teilräumen $\mathrm{VConv}_A(V,F)$ durch einfachere Halbnormen zu ersetzen. Diese erzeugen die Unterraumtopologie, jedoch erhalten wir für jede stetige Halbnorm auf F genau eine zugehörige Halbnorm auf $\mathrm{VConv}_A(V,F)$. Dies impliziert insbesondere, dass der Raum $\mathrm{VConv}_A(V)$ der reellwertigen Bewertungen mit Träger in A ein Banachraum ist.

Kehren wir nun zurück zu unserer Einbettung $T : \mathrm{VConv}(V,F) \to \mathrm{Val}(V^* \times \mathbb{R}, F)$. Die Konstruktionen der Goodey-Weil Einbettungen für beide Räume legen nahe, die zugehörigen Distributionen mithilfe von T miteinander in Beziehung zu setzen, schließlich interpretieren wir die Elemente von $\mathrm{VConv}(C;V,F)$ mittels T als Funktionale auf Stützfunktionen, analog zu den Goodey-Weil Distributionen. Insbesondere können wir über den Träger der Goodey-Weil Distribution einer homogenen Bewertung in $\mathrm{Val}(V^* \times \mathbb{R}, F)$ den *vertikalen Träger* der zugehörigen Bewertung definieren, analog zum Träger von Bewertungen in $\mathrm{VConv}(V,F)$. Dies erlaubt die folgende Charakterisierung des Bildes der obigen Einbettung.

Theorem 10.0.12 (Theorem 6.3.2)**.** *Es sein F ein lokal konvexer Raum, der mit einer stetigen Norm ausgestattet werden kann. Dann besteht das Bild von $T : \mathrm{VConv}_k(V,F) \to$*

$\mathrm{Val}_k(V^* \times \mathbb{R}, F)$ *genau aus den Bewertungen in* $\mathrm{Val}_k(V^* \times \mathbb{R}, F)$*, deren vertikaler Träger in der unteren Halbsphäre* $\mathbb{P}_+(V \times \mathbb{R})_- := \{[(y,s)] \in \mathbb{P}_+(V \times \mathbb{R}) : s < 0\}$ *enthalten ist. Ist* F *ein Fréchetraum, so ist* $T : \mathrm{VConv}_A(V, F) \to \mathrm{Val}_{P(A)}(V^* \times \mathbb{R}, F)$ *ein topologischer Isomorphismus für alle kompakten Teilmengen* $A \subset V$*. Hierbei ist*

$$\begin{aligned} P : V &\to \mathbb{P}_+(V \times \mathbb{R}) \\ x &\mapsto [(x, -1)]. \end{aligned}$$

Dies erlaubt es uns, Approximationsprobleme für dual epi-translationsinvariante Bewertungen zunächst für translationsinvariante Bewertungen auf konvexen Körpern zu behandeln, sofern diese im folgenden Sinne mit dem Träger verträglich sind: Hat man eine gewisse Klasse von Bewertungen auf Funktionen gegeben, so kann man die korrespondierende Klasse von Bewertungen auf konvexen Körpern betrachten, mit anderen Worten das Bild dieser Funktionale unter der Abbildung T. Lässt sich nun eine gegebene Bewertung im Bild von T dahingehend durch Bewertungen in der untersuchten Klasse approximieren, dass die vertikalen Träger einer approximierenden Folge ab einem gewissen Folgenglied stets in einer passenden Umgebung des vertikalen Trägers des Grenzwerts enthalten sind, so impliziert das obige Theorem direkt die Konvergenz dieser Folge in $\mathrm{VConv}(V, F)$.

Dies lässt sich auf auf *glatte*, translationsinvariante Bewertungen in $\mathrm{Val}(V)$ anwenden. Eine Bewertung $\mu \in \mathrm{Val}(V)$ wird dabei als glatt bezeichnet, wenn die Abbildung

$$\begin{aligned} \mathrm{GL}(V) &\to \mathrm{Val}(V) \\ g &\mapsto \pi(g)\mu \end{aligned}$$

glatt ist. Ein Standardresultat aus der Darstellungstheorie besagt, dass glatten Bewertungen einen dichten Teilraum von $\mathrm{Val}(V)$ bilden, was üblicherweise dadurch gezeigt wird, dass eine gegebene Bewertung mit einer glatten Approximation der δ-Distribution im neutralen Element gefaltet wird. Betrachtet man diese Faltungsoperation, so sieht man schnell, dass diese im obigen Sinne kompatibel mit dem vertikalen Träger der Bewertungen ist. Wir erhalten somit einen dichten Teilraum von $\mathrm{VConv}(V)$.
Um diesen zu beschreiben, verwenden wir den *differentiellen Zykel* $D(f)$ einer konvexen Funktion $f \in \mathrm{Conv}(V, \mathbb{R})$. Dabei handelt es sich um einen von Fu in [26] eingeführten integralen n-Strom im Kotangentialbündel T^*V, der mit der Integration von Differentialformen über den Graphen des Differentials $df : V \to V^*$ übereinstimmt, sofern f zweimal stetig differenzierbar ist. Zusätzlich erfüllt der differentielle Zykel die Bewertungseigenschaft. Durch Einsetzen geeigneter Differentialformen erhält man somit reellwertige Bewertungen. Wir betrachten dazu den Raum $\Omega^n_{hc}(T^*V)$ der glatten n-Formen mit *horizontal kompaktem Träger*, d.h. alle Formen $\tau \in \Omega^n(T^*M)$, die $\operatorname{supp}\tau \subset \pi^{-1}(A)$ für eine kompakte Teilmenge $A \subset V$ erfüllen. Hierbei ist $\pi : T^*V \to V$ die natürliche Projektion. Wir bezeichnen eine Bewertung $\mu : \mathrm{Conv}(V, \mathbb{R}) \to \mathbb{R}$ als *glatt*, wenn es eine Form $\tau \in \Omega^n_{hc}(T^*V)$ gibt, sodass μ gegeben ist durch

$$\mu(f) = D(f)[\tau] \quad \text{für alle } f \in \mathrm{Conv}(V, \mathbb{R}).$$

Die Form τ in dieser Darstellung ist hochgradig uneindeutig. Um den Kern dieser Zuweisung zu beschreiben, benötigen wir einen gewissen Differentialoperator $\bar{\mathrm{D}}$ zweiter Ordnung, der in dieser Arbeit als *symplektischer Ruminoperator* bezeichnet wird.

Theorem 10.0.13 (Theorem 7.2.5). *$\tau \in \Omega^n_{hc}(T^*V)$ erfüllt genau dann $D(f)[\tau] = 0$ für alle $f \in \mathrm{Conv}(V, \mathbb{R})$, wenn*

1. *$\bar{\mathrm{D}}\,\tau = 0$,*
2. *$\int_V \tau = 0$, wobei wir den Nullschnitt $V \hookrightarrow T^*V$ als Untermannigfaltigkeit auffassen.*

Wir werden den Raum der glatten Bewertungen in $\mathrm{VConv}(V)$ und $\mathrm{VConv}_k(V)$ mit $\mathrm{VConv}(V)^{sm}$ und $\mathrm{VConv}_k(V)^{sm}$ bezeichnen. Nachdem wir diesen Raum mit $\mathrm{Im}T \cap \mathrm{Val}(V^* \times \mathbb{R})^{sm}$ indentifiziert haben, liefert das obige Approximationsargument die folgende Aussage:

Theorem 10.0.14 (Theorem 7.3.5). *Der Unterraum $\mathrm{VConv}(V)^{sm}$ der glatten Bewertungen ist dicht in $\mathrm{VConv}(V)$.*

Indem man eine approximierende Folge glatter Bewertungen bezüglich des Haarmaßes mittelt, erhält man folgendes Korollar über invariante Bewertungen:

Korollar 10.0.15 (Corollary [7.3.6). *Ist $G \subset \mathrm{GL}(V)$ eine kompakte Untergruppe, so ist der Raum der glatten, G-invarianten Bewertungen dicht im Raum $\mathrm{VConv}(V)^G$ der stetigen, G-invarianten Bewertungen in $\mathrm{VConv}(V)$.*

Für eine Beschreibung dieser Bewertungen wird lediglich eine Klassifikation der relevanten Differentialformen benötigt. Wir führen dies exemplarisch für den Fall $G = \mathrm{SO}(n)$ durch. Dies führt auf die folgene Klassifikation.

Theorem 10.0.16 (Theorem 9.4.4). *Für jede glatte Bewertung $\mu \in \mathrm{VConv}_k(\mathbb{R}^n)^{sm} \cap \mathrm{VConv}(\mathbb{R}^n)^{\mathrm{SO}(n)}$ vom Grad $k > 0$ existiert eine eindeutige Funktion $\phi \in C^\infty_c([0, \infty))$, sodass*

$$\mu(f) = \int_{\mathbb{R}^n} \phi(|x|^2)[D^2 f(x)_k]dx \quad \textit{für all } f \in \mathrm{Conv}(\mathbb{R}^n, \mathbb{R}) \cap C^2(\mathbb{R}^n) \textit{ gilt,}$$

wobei $[D^2 f(x)]_k$ das k-te elementarsymmetrische Polynom in den Eigenwerten der Hesseschen von f bezeichnet.

Als Korollar dieser Aussage erhalten wir, dass jede $\mathrm{SO}(n)$-invariante Bewertung in $\mathrm{VConv}(\mathbb{R}^n)$ bereits $\mathrm{O}(n)$-invariant ist.

Die Dichtheitsaussage über glatte Bewertungen in $\mathrm{VConv}(V)$ beruht auf dem Zusammenhang zwischen dem sogenannten Konormalenzykel eines konvexen Körpers $K \in \mathcal{K}(V^* \times \mathbb{R})$ und dem differentiellen Zykel seiner eigenschränkten Stützfunktion $h_K(\cdot, -1)$, sowie der Repräsentation glatter Bewertungen in $\mathrm{Val}(V^* \times \mathbb{R})$ mithilfe des Konormalenzykels. Eine zweite Darstellung wurde von Alesker in [3] verwendet. Dazu betrachtete er

die Abbildung $\mathrm{Dens}(V)\times C^\infty(\mathbb{P}_+(V^*),L)^k \to \mathrm{Val}_{n-k}(V)$, welche einem Tupel von Stützfunktionen das zugehörige gemischte Volumen zuordnet. Mithilfe des Kerntheorems von Schwartz erhält man eine *surjektive Abbildung* $\mathrm{Dens}(V)\otimes C^\infty(\mathbb{P}_+(V^*)^k, L^{\boxtimes k}) \to \mathrm{Val}_{n-k}(V)^{sm}$, welche eine glatte Bewertung als konvergente Reihe gemischter Volumina darstellt.
Um eine Version dieser Abbildung für Bewertungen auf Funktionen zu erhalten, betrachten wir die von Alesker in [7] untersuchten Bewertungen. Diese lassen sich auf folgende Konstruktion zurückführen: Ist $f:\mathrm{Conv}(V,\mathbb{R})\cap C^2(V)$ eine *strikt* konvexe Funktion, so ist das Differential $df: V\to V^*$ ein Homeomorphismus. Für ein Lebesgue-Maß $\mathrm{vol}_{V^*}\in\mathrm{Dens}(V^*)$ auf V^* können wir somit den Pushforward $(df^{-1})_*\mathrm{vol}_{V^*}$ betrachten, welcher ein Maß auf V definiert. Wählt man ein Skalarprodukt auf V, welches über die induzierte Identifikation $V\cong V^*$ das Lebesgue-Maß vol_{V^*} induziert, so ist dieses Maß gegeben durch

$$(df^{-1})_*\mathrm{vol}_{V^*}(U)=\int_U \det(H_f(x))dx \quad \text{für alle Borelmengen } U\subset V.$$

Diese Abbildung lässt sich zu einer Bewertung $\mathrm{Hess}_n\in\mathrm{VConv}_n(V,\mathcal{M}(V))$ mit Werten im Raum $\mathcal{M}(V)$ der signierten Radonmaße fortsetzen, welche bezüglich der vagen Topologie stetig ist (siehe [14], [20]). Betrachtet man die Polarisierung dieser als *Hessesches Maß* bezeichneten Bewertung Hess_n und setzt diese auf Differenzen von konvexen Funktionen fort, so erhält man eine Abbildung

$$\begin{aligned} C_c(V)\times C^2_c(V)^k &\to \mathrm{VConv}_{n-k}(V)\\ (\phi_0,\phi_1,\dots,\phi_k)&\mapsto \left[f\mapsto \int_V \phi_0 d\,\mathrm{Hess}_n(f[n-k],\phi_1,\dots,\phi_k)\right], \end{aligned}$$

wobei wir hier die Polarisierung von Hess_n wieder mit dem selben Ausdruck bezeichnet haben. Bewertungen dieser Art sind Beispiele für sogenannte *gemischte Hessesche Bewertungen*. Durch Einschränkung dieser Abbildung auf glatte Funktionen erhalten wir eine stetige, multilineare Abbildung, welche wir mithilfe des Kerntheorems von Schwartz zu einer Abbildung $C^\infty(V^{k+1})\to\mathrm{VConv}_{n-k}(V)$ fortsetzen. Im Anschluss vergleichen wir dieses Funktional mithilfe der Einbettung T mit Aleskers Konstruktion. Der kritische Schritt besteht dabei darin, das Oberflächenmaß eines konvexen Körpers $K\in\mathcal{K}(V^*\times\mathbb{R})$ und das Hessesche Maß seiner Stützfunktion $h_K(\cdot,-1)$ miteinander in Beziehung zu setzen. Nachdem wir die notwendigen Kompatibilitätsbedingungen dieser Abbildungen überprüft haben, erhalten folgende Version der McMullen-Vermutung für $\mathrm{VConv}(V)$:

Theorem 10.0.17 (Theorem 8.3.5)**.** *Für jede glatte Bewertung* $\mu\in\mathrm{VConv}_{n-k}(V)^{sm}$ *und jede offene Umgebung* U *von* $\mathrm{supp}\,\mu$ *existieren Funktionen* $\phi_i^j\in C_c^\infty(U)$ *für* $0\le i\le k$, $j\in\mathbb{N}$, *sodass* μ *gegeben ist durch*

$$\mu(f)=\sum_{j=1}^\infty\int_V \phi_0^j d\,\mathrm{Hess}_n(f[n-k],\phi_1^j,\dots,\phi_k^j).$$

Insbesondere spannen glatte gemischte Hessesche Bewertungen einen dichten Unterraum von $\mathrm{VConv}(V)^{sm}$ *und* $\mathrm{VConv}(V)$ *auf.*

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