First passage percolation in the mean field limit

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Dekan:	Prof. Dr.-Ing. Lars Hedrich
Gutachter:	Prof. Dr. Nicola Kistler Prof. Dr. Lisa Hartung Prof. Dr. Louigi Addario-Berry

Datum der Disputation:

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Chapter 1

Introduction

1.1 Introduction: the first passage percolation problem

This dissertation deals with two classical problems in statistical mechanics: the *first passage percolation* on Euclidean spaces, FPP for short, in both directed and undirected settings. Unfortunately, a first warning concerning the wordings is already at place: indeed, depending on the scientific community/type of studies one performs, the FPP problem is often also referred to as the problem of *random polymers in random environment*, RPRE for short. This thesis will be no exception: we will switch from the FPP to the polymer terminology at our discretion, i.e. whenever we feel one terminology more suited to convey the type of results discussed.

So what is the FPP? The abstract formulation goes as follows: denote by $G = (V, E)$ a graph, where V is a set of vertices, and E the set of edges connecting vertices. To each edge, we assign *positive* random weights. There are various intepretations for these random weights:

- these can be seen as e.g. the **time** it takes for a bit of information to pass from one vertex (node) to the neighbour connected by the considered bond. Information here can be quite anything: in mathematical biology, for instance, one assumes that a vertex is infected, and this will infect its neighbors in the amount of time given by the random weight associated to the connecting edges.
- In case of random polymers, nodes connected by an edge correspond to atoms, whereas the random weights are the **energy** of the chemical bond.

With these interpretations in mind, let us consider a vertex $v_0 \in G$ which is infected – we may think of this as "patient zero": after the random times modelled by the weights, v_0 will infect its neighbours, and these will then infect after some random times *their* still uninfected neighbours, and so forth. Given $v \in G$ another vertex, the *first passage time* between v_0 and v corresponds to the time it takes for the disease to spread to the point of infecting also v. Note that, in general, there might exist several nearest neighbors paths connecting these two specified vertices, in which case

the FPP between v_0 and v is the minimal time over all paths for the infection to spread between the considered vertices.

In the polymer terminology, the FPP therefore corresponds to the socalled *ground state*, i.e. the polymer's configuration which attains the minimal energy. Such configuration is physically the most stable, and the one any chemical system at *equilibrium* will reach.

From a physical standpoint, the most important FPP models are definitely those on the n-dimensional integer lattice $\mathbb{Z}^n$ where $n = 2, 3$. These type of questions are however notoriously difficult: with the exception of very few cases in two dimensions where one can rely on the complete integrability of the system, and some deep yet mysterious connections with interacting particle systems / random matrices, see [33, 46] and references therein, our understanding of FPP in low dimensions is virtually non-existent.

The ultimate reason as to why the FPP is, in general, an extremely challenging (and vastly open) problem is quickly explained: the problem concerns a so called *extremal event* - the minimum of a random field - and this puts the FPP in the realm of the so called extreme value theory, EVT for short.

EVT is a well established field of mathematics and probability theory whose foundations go back to the last century, with groundbreaking works by Kolmogorov, Fréchet, Weibull, Gumbel, Gnedenko, and many others. In case of *indepedendent* random fields, the answer to the question of its minimum has been settled more than 60 years ago, mostly by Fisher-Tippett and Gnedenko: in the limit of large cardinalities, only three extremal distributions arise as possible candidates for the weak limit of the (possibly recentered and rescaled) minimum – the celebrated Gumbel-, Fréchet- or Weibull-universality classes. It is furthermore known that the extremal process of a random field consisting of *independent* random variables converges to a Poisson Point Process – a result which is equivalent to the central limit theorem for sums of independent random variables, see e.g. [41].

We emphasize that the fundamental assumption in classical EVT is that of *independence*, which is by far not met for the FPP on the n-dimensional integer lattice $\mathbb{Z}^n$: in fact, paths (polymers) connecting any two vertices v_0 and v_1 can share common edges; this implies that the passage times (energies) are in fact *correlated*: the more edges two paths share, the stronger the correlations. The beautiful and powerful theorems from classical EVT therefore no longer apply.

To make some headway in this challenging field, in this Ph.D. thesis we rely on some simplifications which set in when one considers the *mean field limit*, i.e. when the dimension of the underlying Euclidean lattice is sent to infinity. The simplification is most easily explained by means of the following back-of-the-envelope computation: let us consider the n-dimensional hypercube $\{0,1\}^n \subset \mathbb{Z}^n$; the reference vertices will be any two opposite vertices, without loss of generality we assume these to be $v_0 := \mathbf{0} = (0,0,\dots,0)$ and $v := \mathbf{1} = (1,1,\dots,1)$. In order to formulate the fundamental observation we consider the case of *oriented* FPP [1]: we say a path from $\mathbf{0}$ to $\mathbf{1}$ is *oriented* if it doesn't "backtrack", i.e. the path is encoded by n distinct steps at which any of the remaining 0-coordinates is switched into 1; in contrast, a backtracking path (the $1's$ are allowed to be switched back to $0's$) is called *unoriented*. In other words: an oriented path can only move "forward", in which case there are clearly $n!$-many such paths connecting $\mathbf{0}$ and $\mathbf{1}$. We then pick $\alpha \in [0,1]$ and ask the following question:

how many pairs of paths meet in the core of the hypercube, i.e. after αn steps?

[1] The observation we are about to make plays a key role also in the treatment of the *unoriented* setting, although the difficulty in a rigorous mathematical treatment will be increased by orders of magnitude.

Note that meeting *somewhere* in the hypercube is a necessary requirement for paths to share edges, hence for the passage times (in FPP) or the energies (RPRE) to be correlated. It is therefore clear that an answer to the above question will provide key insights into the possible correlation structures to be expected in the FPP. Simple counting, and an elementary application of Stirling's approximation shows that there are

$$n!(\alpha n)!((1-\alpha)n)! \approx (n!)^2 \exp\left(-nI(\alpha)\right), \qquad (n \uparrow \infty) \tag{1.1.1}$$

many pairs of such paths, where

$$I(\alpha) \equiv -\alpha \log(\alpha) - (1-\alpha)\log(1-\alpha).$$

Since for $0 < \alpha < 1$ it holds that $I(\alpha) > 0$, and due to the factor $e^{-nI(\alpha)}$ in (1.1.1), we see that requiring two paths to meet in the core of the hypercube comes at an *exponential cost*. In other words, any two polymers connecting $\mathbf{0}$ to $\mathbf{1}$ will hardly meet, in high dimensions, in the core of the hypercube : "loops" should therefore be unlikely/negligible. Graphs with no loops are however *trees*, hence the above consideration suggests that

in high dimensions, the FPP on the hypercube bears strong resemblance with the FPP on trees.

This insight, which we may refer to as the *hierarchical approximation of the FPP in large dimensions*, turns out to be correct, and plays an absolutely fundamental role in this dissertation. This is due to the fact that the FPP on trees / hierarchical structures is amenable to a rigorous analysis thanks to the recent advances in the field of spin glasses and disordered systems [35]: the key aspects of such a treatment and the ensuing results will be discussed in Section 1.2. In Section 1.3 we will formulate precisely the setting for FPP on the hypercube as dealt with in this thesis. In Section 1.3.1 we will then present our results for the *oriented* FPP, whereas in Section 1.3.2 we will discuss our contribution in the (way more challenging) *unoriented* setting.

1.2 First passage percolation on trees

The FPP on trees is defined as follows:

- Pick a rooted tree.
- Given the tree, attach independent random variables to the edges.
- Consider the field indexed by the leaves that is obtained by associating each leaf to the sum of the random variables from the root to the leaf.

The ensuing hierarchical fields are instances of the (G)REM-models [23, 24]: these are random fields introduced by Bernard Derrida in the context of mean field spin glasses in the 1980s and which have played a fundamental role in the study of disordered systems ever since. The GREMs are typically defined via Gaussian random variables as edge-weights, but with our applications to the FPP problem in mind, we will instead attach independent gamma random variables. The passage time from the root to any vertex at depth n is Gamma$(n, 1)$-distributed random variable, i.e. the sum of n independent exponentials.

The REM-model is the simplest of all possible trees: it consists of a root with $n!$ leaves, see Figure 1.1 below for a graphical rendition. To each leaf we thus associate

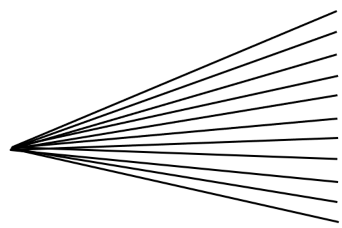

Figure 1.1: Random energy model

independent Gamma$(n, 1)$-distributed random energies, denoted by $(X_i)_{i\in\{1,n!\}}$. The FPP then concerns their minimal value, to wit

$$m_n[REM] \equiv \min_{i\in\{1,\ldots,n!\}} X_i, \tag{1.2.1}$$

in the large-n limit.

Since the REM-tree consists of a single scale (a feature which leads to a collection of *independent* random variables), the associated FPP falls in the realm of *classical* EVT, and it holds:

Theorem 1 (Minimum of the exponential REM)**.** *With the above notations, it holds*

$$\lim_{n\to\infty} m_n[\text{REM}] = 1, \tag{1.2.2}$$

in probability.

We will not give a proof of this simple fact as it can be found (or derived with the tools explained) in any book on classical EVT, see e.g. [41]. For later purposes, we state an extension of the above classical result which settles the full statistics of extreme values (not only the smallest, but also the second, third etc. smallest). This is most efficiently done through the extremal process, the random Radon measure

$$\Xi_n \equiv \sum_{i\leq n!} \delta_{n(X_i-1)}\,. \tag{1.2.3}$$

It is again a well known fact from classical EVT that in case of independent random variables the extremal process converges to a Poisson point process:

Theorem 2 (Extremal process REM). *With Ξ a Poisson process with intensity $e^{x-1}dx$, it holds*

$$\lim_{n\to\infty} \Xi_n = \Xi, \tag{1.2.4}$$

weakly. In particular, it follows that the recentered and rescaled first passage time converges weakly to a Gumbel distribution, to wit

$$\lim_{n\to\infty} \mathbb{P}(n(m_n[REM]-1) \le x) = 1 - e^{-e^{x-1}}. \tag{1.2.5}$$

As already mentioned, these results are truly classical, and rely on the fact that we are dealing with random fields of independent variables. In order to address the FPP in case of more interesting (correlated) fields, we will define a generalized REM, a GREM with two levels. To do so, we consider a tree with depth 2 where the root has $(\frac{n}{2})!\binom{n}{\frac{n}{2}}$ children, each of which have $(\frac{n}{2})!$ offspring, see Fig. 1.2 below. To each edge, we attach $\text{Gamma}\left(\frac{n}{2},1\right)$ independent random variables (i.e. the sum of $\frac{n}{2}$ independent exponentials). The mystery for considering a tree with such branching factor and edge-weights is quickly

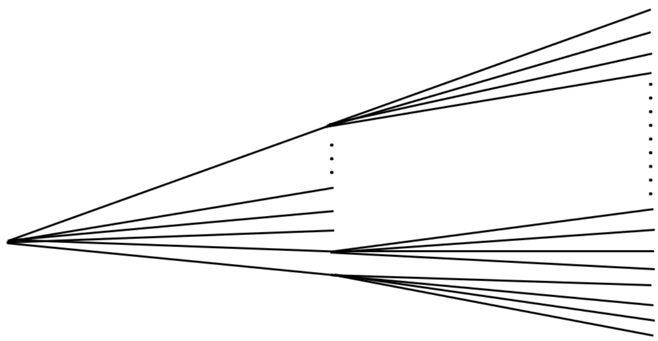

Figure 1.2: Generalized random energy model

lifted:

i) first we remark that with this choices both REM and the GREM have the same number of leaves.

ii) Furthermore, since the sum of two independent $\text{Gamma}(k_1,1)$ and $\text{Gamma}(k_2,1)$ random variables is again a $\text{Gamma}(k_1+k_2,1)$ random variable for all $k_1,k_2 \in \mathbb{N}$, a path from the root to a leaf in the REM has the same distribution as a path in the GREM(2).

In this sense, the exponential REM and GREM(2) are comparable. There is, however, a fundamental twist: contrary to the REM-case, paths in the GREM(2) can overlap. As already mentioned, this feature leads to the fact that the energies attached to the leaves in the GREM(2) are no longer independent random variables. The GREM(2) model described here therefore falls out of the classical EVT setting. Somewhat surprisingly, one can however prove that Theorem 1 and 2 are still correct in spite of the inbuilt correlations. We will restrain from giving a precise statement of this fact and its proof, as this is by now a fairly standard application of the tools developed by Derrida [24]. Instead, we will present a further generalization of the GREM(2)-model which is not classical, and which allows for even more severe correlations: *the exponential GREM with n levels*. To this end, consider the rooted tree with depth n where each vertex at depth i has $(n-i)$ children. A graphical rendition is given by Figure 1.3 below. To each edge, we

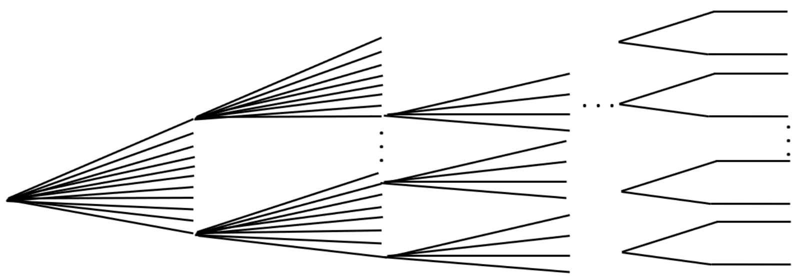

Figure 1.3: Generalized random energy model with n levels

attach independent standard exponentials. Due to our choice of the branching factors, one easily checks that there are $n!$ paths from the root to the leaves at depth n. Sticking to the above notation, we denote also in this case by $(X_i)_{i\in\{1,\ldots,n!\}}$ the passage times of the paths (energies of the polymer): being sums of n independent standard exponentials, these weights are again Gamma$(n,1)$-distributed; they are, however, *strongly correlated*. As before we are interested in the first passage time

$$m_n[\text{GREM}(n)] \equiv \min_{i\in\{1,\ldots,n!\}} X_i, \tag{1.2.6}$$

in the limit of large dimensions, i.e. for $n\uparrow\infty$.

In spite of the strong, inbuilt correlations, one can show that to *leading* order, the GREM(n) still behaves like its REM-counterpart, to wit

Theorem 3 (Minimum of the exponential GREM(n))**.** *It holds:*

$$\lim_{n\to\infty} m_n[\text{GREM}(n)] = 1, \tag{1.2.7}$$

in probability.

Contrary to the GREM(2)-case, a proof of Theorem 3 is rather involved, and relies on a tool which has only recently crystallysed in the field of disordered systems, the *multiscale refinement of the second moment method* [35]. This is a flexible tool which has played a major role in the study of the extreme values of highly correlated random structures, such as the issue of cover times [10, 45, 11], the extreme values of the Riemann zeta function on the critical line [7, 6], the maximum of the characteristic polynomial of random matrices [5, 21], the Ginzburg-Landau model [12], and much more, see also [4] and references therein.

We will not give a proof of Theorem 3 neither. First, this would take up too much space, and second, it would even be quite redundant: as a matter of fact, the method of proof of this result (the multiscale refinement of the second moment method) is the backbone of our treatment of the FPP problem in the technically way more challenging case of the hypercube in both oriented and unoriented settings. Our aim in this introduction is rather to canvass a mental picture based on the behavior of the FPP on trees which will guide us in the treatment of the FPP on Euclidean lattices: indeed, we will see that Theorem 3 remains true even in the case of the oriented FPP in the mean field limit, up to a minor twist.

Coming back to the exponential GREM(n), we see that, perhaps somewhat surprisingly, correlations have no impact on the leading order of the first passage times. This naturally raises the question about subleading corrections and fluctuations, more precisely whether correlations are strong enough to be detectable at such scales. As it turns out, this is indeed the case: the following theorem states that the extremal process of the exponential GREM(n) converges, in the mean field limit, to a *Cox process* with exponential intensity:

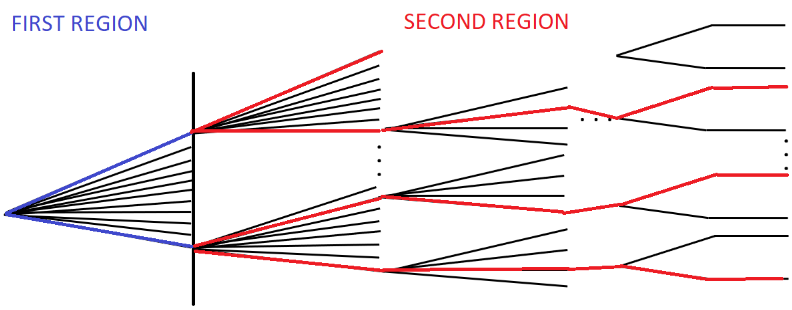

Figure 1.4: Four optimal paths in the GREM(n). Remark that two of these paths share the first edge: their energies are therefore correlated. This feature is eventually responsible for the random intensity in the limiting extremal process.

Theorem 4 (Extremal process GREMn)**.** *Let* Ξ *be a Cox process with intensity* $Ze^{x-1}dx$*,* *where Z is a standard exponential random variable. Then*

$$\lim_{n\to\infty} \Xi_n = \Xi, \tag{1.2.8}$$

weakly. In particular, it follows that the first passage time converges weakly to a random shift *of the Gumbel distribution:*

$$\lim_{n\to\infty} \mathbb{P}(n(m_n[GREM(n)] - 1) \leq t) = 1 - \mathbb{E}\left[e^{-Ze^{t-1}}\right] \tag{1.2.9}$$

Remark 5. *The integral in* (1.2.9) *can be explicitely computed. It holds*

$$1 - \mathbb{E}\left[e^{-Ze^{-t+1}}\right] = \frac{e^{t-1}}{1+e^{t-1}}. \tag{1.2.10}$$

A comparison of Theorem 2 and 4 shows a radical difference: the extremal process of the GREM(n)-model is not a simple Poisson point process anymore, but a Poisson point process with *random intensity.* Without entering into the details of a proof of such statement, we mention that the additional random component finds its origin in the weak, but persistent, correlations close to the *root* of the tree, a phenomenon which has been first identified in the extremal process of branching Brownian motion [1, 8]. Precisely, one can prove with the multiscale refinement of the second moment method [35], that extremal paths/optimal polymers (i.e. those with weights approximately equal to 1) in the GREM(n) are not completely disjoint: they can share a finite number of edges close to the root, see Figure 1.4 for a graphical rendition. It is this overlapping which leads to the random intensity.

In the next sections we will give an overview of the new results obtained in this dissertation. In particular, we will see to which extent the hierarchical approximation (via GREM-models) of the FPP on large dimensional Euclidean spaces holds true. We anticipate that in case of *oriented* FPP, this approximation is absolute, as the very same picture depicted by Theorem 4 for the exponential GREM(n)-model, holds true for the hypercube in the mean field limit. In the way more challenging *unoriented* setting our understanding is still somewhat incomplete, but our results strongly suggest that also in this case a picture akin to Theorem 4 remains valid.

1.3 First passage percolation on the hypercube

We begin this section with a precise definition of the FPP on the hypercube. To this end, we denote by $G_n = (V_n, E_n)$ the n-dimensional hypercube. $V_n = \{0,1\}^n$ is thus the set of vertices, and E_n the set of edges connecting nearest neighbours. We write $\mathbf{0} = (0,0,...,0)$ and $\mathbf{1} = (1,1,...,1)$ for diametrically opposite vertices. For $l \in \mathbb{N}$ we let

$$\Pi_{n,l} \equiv \text{the set of polymers, i.e. paths from } \mathbf{0} \text{ to } \mathbf{1} \text{ of length } l\,,$$

as well as

$$\Pi_n \equiv \bigcup_{l=1}^{\infty} \Pi_{n,l}.$$

Every edge of the n-hypercube is parallel to some unit vector $e_j \in \mathbb{R}^n$, where e_j connects

$$(0,\dots,0) \text{ and } (0,\dots,0,\underbrace{1}_{j^{th}-\text{coordinate}},0,\dots,0)\,.$$

We write $e_{-j} \equiv -e_j$. The quantity $\pi_j \in \{1,\dots,n\} \cup \{-1,\dots,-n\}$ then specifies the direction of a π-path at step j. Remark that the endpoint of the (sub)path $\pi_1\pi_2\dots\pi_i$ coincides with the vertex given by $\sum_{j\le i} e_{\pi_j}$. The edge traversed in the j-th step by the π-path will be denoted $[\pi]_j$.

To each edge we attach independent, standard (mean one) exponential random variables ξ, the random environment, (this choice represents no loss of generality: only the behavior for small values matters) and assign to a polymer $\pi \in \Pi_{n,l}$ its *weight* according to

$$X_\pi \equiv \sum_{j=1}^{l} \xi_{[\pi]_j}.$$

The two models mentioned at the beginning of this section are :

- The oriented FPP on the hypercube concerns the minimal weight

$$m_n[\text{dir}] \equiv \min_{\pi\in\Pi_{n,n}} X_\pi, \tag{1.3.1}$$

- The unoriented FPP, also known as the undirected polymers in random environment, concerns

$$m_n[\text{undir}] \equiv \min_{\pi\in\Pi_n} X_\pi. \tag{1.3.2}$$

In both cases, the limit we are interested in is that of large dimensions, i.e. as $n \to \infty$.

It goes without saying, the unoriented FPP is, due to the additional "degrees of freedom", a much harder problem than its oriented counterpart. We will thus begin with the latter, also because this model plays a role in our treatment of the former.

1.3.1 Oriented FPP on the hypercube

The *leading* order of the ground state has been conjectured by Aldous and settled by Fill and Pemantle more three decades ago:

Theorem 6 (Aldous [2], Fill-Pemantle [29])**.** *For the directed FPP on the hypercube,*

$$\lim_{n\to\infty} m_n[\mathrm{dir}] = 1, \tag{1.3.3}$$

in probability.

To shed some light on the above result, remark that $|\Pi_{n,n}| = n!$, and that oriented paths from $\mathbf{0}$ to $\mathbf{1}$ have length n. The oriented FPP on the hypercube thus manages to reach the same value as in the case of independent FPP: exactly as in the exponential REM discussed in Theorem 1 above. Due to the "unlikeliness" of loops in high dimensions discussed in the previous section, and the ensuing hierarchical approximation, the Fill-Pemantle result hardly comes as a suprise.

In a first part of this Ph.D. thesis, which has lead to the publication [37], we propose a streamlined proof of the Fill-Pemantle theorem by means of the multiscale refinement of the second moment method [35], which, most importantly, establishes a neat point of contact between the FPP on the hypercube and hierarchical structures. This fresh take on the subject has been absolutely instrumental in our second contribution to the subject [38] which settles the full limiting picture / extremal process. Towards this goal, the simple yet key observation which provides a link / "dictionary" between the FPP on the hypercube and GREM-like structures concerns the evolution of the paths in a directed setting: indeed, after k steps, the polymer has $(n-k)$ possible choices (the nearest neighbors) for the next step; this is of course reminiscent of the branching factor in a GREM(n)-model (and eventually the reason for discussing the latter model first). This powerful point of contact, together with the ensuing mental pictures and the arsenal of technical tools borrowed from the study of disordered systems, leads to a complete control of the extremal process

$$\Xi_n \equiv \sum_{\pi\in\Pi_{n,n}} \delta_{n(X_\pi - 1)}\,. \tag{1.3.4}$$

Indeed, we proved in [38] the following:

Theorem 7 (Extremal process of oriented FPP on the hypercube)**.** *Let Ξ be a Cox process with intensity $Ze^{x-1}dx$, where Z is distributed like the product of* two *independent standard exponentials. Then*

$$\lim_{n\to\infty} \Xi_n = \Xi, \tag{1.3.5}$$

weakly. In particular, it follows for the first passage time m_n that

$$\lim_{n\to\infty} \mathbb{P}(n(m_n[\mathrm{dir}]-1) \le t) = \int_0^\infty \frac{x}{e^{1-t}+x} e^{-x} dx\,. \tag{1.3.6}$$

We emphasize that the picture established by Theorem 7 is the very same as for the GREM(n)-model, up to a minor twist which can be easily anticipated: due to the inherent symmetry, the hypercube has two "roots" (the opposite vertices **0** and **1**): rather than one tree only, *we are thus dealing with two trees patched together*, see Figure 1.5 below for a graphical, and suggestive rendition of the emerging phenomena. Exactly as in the case of the GREM(n)-model, the random intensity of the limiting point process is due to the fact that optimal polymers can share a finite number of edges: quite simply, in the case of the hypercube this can happen both at the beginning of the polymers' evolution, and towards the end.

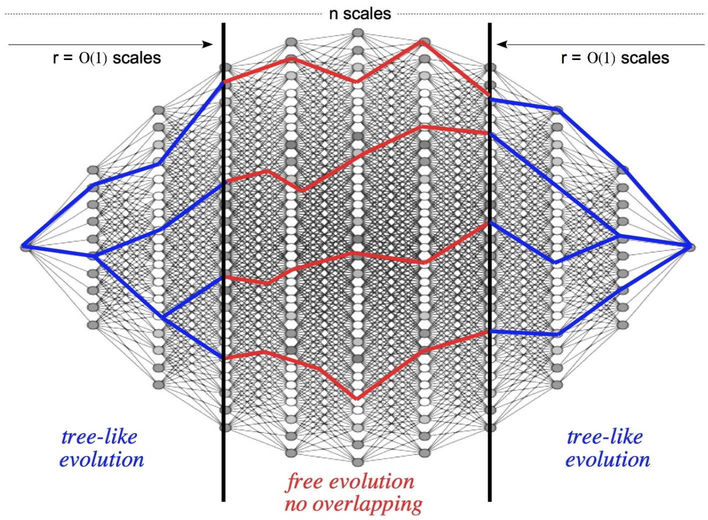

Figure 1.5: Close to 0 and 1, optimal paths perform a tree like evolution (blue edges). In the core of the hypercube (red) paths share no edges: this is nothing but a neat formulation of the aforementioned fact that "loops" (in particular mesoscopic loops in the core of the hypercube) are, in high dimensions, extremely unlikely.

1.3.2 Unoriented FPP on the hypercube

The third chapter of this dissertation deals with the unoriented FPP on the hypercube. The leading order in the mean field limit has been recently identified by Anders Martinsson, who settled a conjecture by Fill and Pemantle [29] in a series of papers:

Theorem 8. *[Martinsson, [43, 44]] With* $\mathsf{E} \equiv \ln(1+\sqrt{2})$, *it holds*

$$\lim_{n\to\infty} m_n[\text{undir}] = \mathsf{E}, \tag{1.3.7}$$

in probability.

In order to establish this result, Martinsson proposes in fact two radically different proofs, which are both however quite implicit, and suffer from certain shortcomings.

In the chronologically first proof, Martinsson considers a model which stochastically dominates the unoriented FPP, the socalled Branching Translation Process, BTP for short, which had been introduced by Durrett [26] and advertised, with applications to the FPP in mind, by Fill and Pemantle [29].

The BTP process is constructed as follows: at time zero, a particle sits at the origin $\mathbf{0}$, and generates offspring at rate n. Each offspring will then jump ("populate") to a neighbouring vertex chosen uniformly among all possible choices. At the next step, these will then continue the procedure independently: they will produce their own offspring, which will then populate their neighbors, and so forth. Here is the key observation: one can easily show that the time when the target $\mathbf{1}$ is populated stochastically dominates the first passage time of the unoriented FPP. Precisely, the time it takes a BTP to populate $\mathbf{1}$ is *smaller* than the FPP: since the population time of the BTP can be analytically (and explicitely) computed, this gives a lower bound for the latter which, somewhat myraculously, turns out to be tight. The upper bound proceeds along a similar line of reasoning, except that for this Martinsson considers an ingeniously chosen subprocess of the BTP, i.e. a subset of the population alive at any given time. Since both bounds obtained in this way are, *to leading order*, tight, Theorem 8 thus follows. Martinsson's approach through the BTP process is charming, but it suffers from a number of shortcomings. The first one is chiefly technical: indeed, the stochastic domination is known to hold true only in case of exponentially distributed random weight, it is thus a non-universal result; the second shortcoming is more conceptual: it is by far not obvious why the population time in the BTP gives, to leading order, a tight estimate for the first passage of the unoriented FPP. Even more mysterious/delicate would be the behavior to subleading order and the weak limits, where the validity of such strong approximation is highly questionable.

In a second paper, Martinsson manages to identify the leading order of the unoriented FPP on Cartesian power graphs, of which $\mathbb{Z}^n$ is only a representative. Even more so, the result is not restricted to exponential random weights, but holds true for any "reasonable" positive random variables[2]. The approach is quickly summarised: it consists of upper and lower bounds to the first passage time. Precisely, one can show through a simple application of the Markov inequality that for $\epsilon > 0$ and with the above notations,

$$\lim_{n\to\infty} \mathbb{P}\left(m_n[\text{undir}] \le \mathsf{E} - \epsilon\right) = 0. \tag{1.3.8}$$

In other words, E is a lower bound to the ground state of unoriented polymers. The upper bound, i.e. proving that

$$\lim_{n\to\infty} \mathbb{P}\left(m_n[\text{undir}] \le \mathsf{E} + \epsilon\right) = 1, \tag{1.3.9}$$

[2]Such a universality phenomenon is to be expected due to some simple heuristics which will in fact become apparent in our treatment.

is much more delicate. To achieve this result Martinsson first proves that

$$\lim_{n\to\infty} \mathbb{P}\left(m_n[\text{undir}] \leq \mathsf{E} + \epsilon\right) > 0, \tag{1.3.10}$$

strictly, and in a second step manages to improve the r.h.s. of (1.3.10) to $1 - \delta$ (for $0 < \delta < 1$ arbitrary) by means of a bootstrapping argument which relies on self-similar properties of Cartesian power graphs and some monotonicities related to the positivity of random weights. The approach of Martinsson therefore stands and falls with (1.3.10), which is proved with a clever use of the FKG inequality. As is often the case, the FKG, when applicable, leads to an efficient treatment of the problem at hand: the case of unoriented FPP is not an exception. Unfortunately, the FKG inequality suffers itself from a number of shortcomings: first, it hardly identifies the underlying physical mechanisms which lead to the all-important "decoupling" on which the FKG itself is based; second, it is seldom clear to which level of precision this decoupling holds true. Indeed, the FKG inequality is eventually a local argument which fails to give any information whatsoever in case the "strategies" implemented (in this case by the random polymers) to achieve a ground state are global, see [35] for more on this issue in disordered systems.. All in all, the use of the FKG inequality, although marvellously efficient on a technical level, leads to an opaque analysis, which more often than not cannot be improved.

In order to fill the gaps left by Martinsson's proofs, and to open a gateway towards the unsettled issue of fluctuations (see Section 2.2.3 below for more on this), in the preprint [36] we implement the multiscale refinement of the second moment method [35]. This approach is truly constructive: it requires as "input" a detailed identification of the underlying mechanisms/strategies adopted by the polymers to reach minimal values; this in turns leads to a precise geometrical description of optimal paths, and yields Martinsson's theorem as a simple corollary. Here are the main steps/results of our treatment.

The first step in the multiscale refinement of the second moment method is that of *coarse graining*: with $K \in \mathbb{N}$ a large but finite constant, we split the hypercube into K slabs, and consider the ensuing separating hyperplanes

$$H_i \equiv \left\{ v \in V_n, d(0,v) = i\frac{n}{K} \right\}, \qquad i = 1 \dots K. \tag{1.3.11}$$

For any two vertices $\boldsymbol{v} \in H_{i-1}$ and $\boldsymbol{w} \in H_i$ lying on successive H-planes, we introduce the following concepts:

- The *effective forward steps* are given by

$$\mathsf{ef}(\boldsymbol{v}, \boldsymbol{w}) \equiv \# \left\{ 0's \text{ in } \boldsymbol{v} \text{ which switch into } 1's \text{ in } \boldsymbol{w} \right\}$$

- The *effective backsteps* are given by

$$\mathsf{eb}(\boldsymbol{v}, \boldsymbol{w}) \equiv \# \left\{ 1's \text{ in } \boldsymbol{v} \text{ which switch into } 0's \text{ in } \boldsymbol{w} \right\} .$$

- For a path π connecting $\boldsymbol{v}$ and $\boldsymbol{w}$, the *detours* are given by

$$\gamma_\pi(\boldsymbol{v},\boldsymbol{w}) \equiv \{l_\pi(\boldsymbol{v},\boldsymbol{w}) - d(\boldsymbol{v},\boldsymbol{w})\}\,.$$

Note that $d(\boldsymbol{v},\boldsymbol{w}) = \mathsf{ef}(\boldsymbol{v},\boldsymbol{w}) + \mathsf{eb}(\boldsymbol{v},\boldsymbol{w})$. For the readers convenience, we emphasize that the *effective forward steps* encode the number of steps forward which are not undone by backsteps in the reverse direction; similarly, the *effective backsteps* encode the number of backsteps which are not undone by steps forward in the reverse direction (or vice versa). Finally, the *detours* capture the amount of forward steps in a path π which are cancelled by backsteps in the reverse direction (or vice versa): the smaller γ_π, the higher the so called *tension* of the substrand. For this reason, we call a substrand *stretched* if the detours vanish. A stretched path is, in fact, a *geodesic* (with respect to the Hamming metric).

The picture which underlies/emerges from the multi-scale analysis implemented in [36] can be summarized as follows. The energy of the polymer is, to first approximation, uniformly spread along the strand. The polymer's bonds carry however a lower energy than in the directed setting, and are reached through the following geometrical evolution. Close to the origin, the polymer proceeds in oriented fashion. The tension of the strand decreases however gradually, with the polymer allowing for more and more backsteps as it enters the core of the hypercube. (The tension of the strand is encoded by $\gamma_\pi(\mathbf{0},\boldsymbol{w})$, where $\boldsymbol{w}$ is taken along a path π).

As a matter of fact, the properties of optimal polymers mentioned in the previous paragraph are all deduced from a key observation which relates (fraction of) the length l of the substrand and its position, i.e. (fraction of) its distance d from the origin w.r.t. the Hamming metric. This key relation is given by the simple formula

$$d(l) = \sinh(\frac{l}{\sqrt{2}})\cosh(\mathsf{E} - \frac{l}{\sqrt{2}}). \tag{1.3.12}$$

This new formula gives the typical geometrical evolution of the polymer (at first order). A graphical rendition which compares the dynamical evolution of oriented vs. unoriented case is given in Figure 1.6 below.

The connection between directed and undirected polymers, is absolutely key in our treatment, and goes way beyond the similarity between the evolution close to origin and target. There is in fact an additional, and much deeper, point of contact between these models, which again hides behind (1.3.12). To see how this comes about, we first go back to one essential feature of undirected polymers, namely that they *do perform backsteps*. These naturally increase the length of the strand, but also allow the polymer to reach energetically favorable edges which are otherwise unattainable in a fully directed regime. As it turns out, and somewhat surprisingly, optimal polymers manage to connect such reservoirs through approximate geodesics with respect to the Hamming metric! How exactly this phenomenon sets in is best explained through a picture, see Figure 1.7 below.

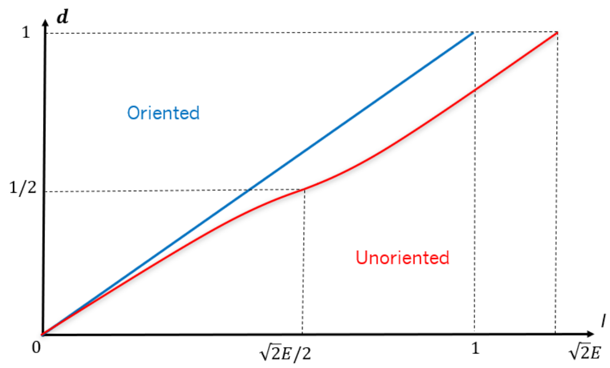

Figure 1.6: Hamming-depth as a function of the length: directed (blue) vs. undirected (red curve, which is but a plot of the function (1.3.12)) polymers. For small lengths, the depths are comparable: close to the origin, the undirected polymer is thus as directed as possible. The slope of the red curve decreases however gradually as the polymer approaches the core of the hypercube: the further the polymer goes, the ”loser” it becomes. Due to the inherent symmetry of the hypercube, a mirror picture sets in, of course, at half-length.

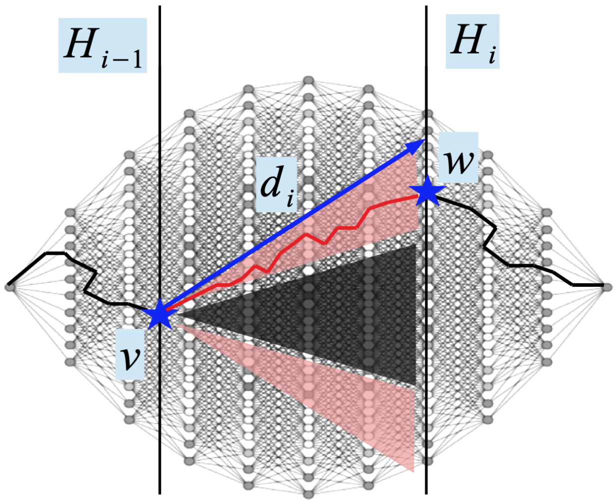

Figure 1.7: The black-shaded cone corresponds to the region where a fully directed polymer would lie while connecting the H-planes. Undirected polymers evolve however in the red-shaded cones, thereby reaching vertices which are at a larger Hamming distance d_i than their directed counterparts (d_i depending only on the position of H_i). However, and crucially: the substrands (in red) of optimal polymers are, in first approximation, geodesics: they reach energetically favorable edges (unattainable in a fully directed regime) with the least possible amount of steps.

The property according to which polymers connect successive hyperplanes through (approximate) geodesics gives profound insights into their geometrical evolution which are not available from Martinsson implicit analysis; furthermore, this key property identifies a most effective point of contact between the unoriented and oriented models, eventually allowing the use of all the machinery available in the oriented case, and dramatically simplifying the rigorous manipulation of some otherwise daunting combinatorial objects.

1.3.3 What's next? Towards the weak limit of unoriented polymers

Analogously to the developments on the oriented FPP on the hypercube, the line of research initiated in [36] opens a gateway towards the weak limits of the unoriented FPP, for which not even a conjecture seems to have been made in the literature. Thanks to the insights gathered in [36], it seems reasonable to put forward the conjecture that also in the unoriented case correlations will hardly play a role in the core of the hypercube in the large n-limit. In other words, we expect a picture much reminiscent of Theorem 7 to still hold.

It goes without saying, dropping the orientedness assumption increases the difficulties by orders of magnitude. Let us single out one key conceptual issue which must be addressed and which is not even present in the oriented case: the unoriented FPP has an additional "degree of freedom", namely the *length of the polymer*. In [36] we proved that the optimal length strongly concentrates, to leading order, around its mean. This phenomenon will naturally no longer hold when one addresses the weak limits, so - as a first step - one will also need to identify the fluctuations of the length. Again in virtue of some insights gathered in [36], we believe that the fluctuations of the length are, in fact, *Gaussian*.

Chapter 2

Zusammenfassung (German summary)

2.1 Einleitung: *the first passage percolation problem*

Diese Dissertation beschäftigt sich mit zwei klassischen Problemen der statistischen Mechanik: Wir interessieren uns für die sogenannte *first passage percolation* [FPP] auf euklidischen Räumen sowohl im *gerichteten* als auch im *ungerichteten* Fall. Zu beachten ist, dass in der Literatur abhängig vom wissenschaftlichen Hintergrund der jeweiligen Autoren das FPP-Problem häufig auch als das Problem von *random polymers in random environment* [RPRE] bezeichnet wird. Auch in dieser Arbeit werden wir nach eigenem Ermessen zwischen den beiden Terminologien wechseln, je nachdem, welche davon besser geeignet ist, die Art der diskutierten Ergebnisse zu vermitteln.

Was ist die FPP? Die abstrakte Formulierung lautet wie folgt: Bezeichne mit $G = (V, E)$ einen Graphen, wobei V eine Menge von Knoten ist und E die Menge von Kanten, die die Nachbarn verbinden. Jeder Kante weisen wir zufällige Gewichte zu. Es gibt verschiedene Interpretationen für diese zufälligen Gewichte:

- Man könnte die Gewichte beispielsweise als die **Zeit** auffassen, die benötigt wird, um eine Information/Eigenschaft von einem Punkt (Knoten) zu einem Nachbarn durch die betrachtete Bindung zu übertragen. Informationen können hier alles sein: In der mathematischen Biologie wird beispielsweise angenommen, dass ein Punkt infiziert ist, und dieser infiziert wiederum einen Nachbarn in der Zeit, die durch das zufällige Gewicht der Verbindungskante gegeben ist.

- Im Falle zufälliger Polymeren entsprechen die Knoten Atomen die durch Kanten verbunden sind, wobei die zufälligen Gewichte die **Energie** der chemischen Bindung quantifizieren.

Im Lichte der ersten Intrepretation betrachten wir nun einen Punkt $v_0 \in G$, der infiziert ist – wir bezeichnen diesen als "Patienten *Null*". Dieser infiziert dann innerhalb der durch die Gewichte gegebenen zufälligen Zeiten seine Nachbarn, welche wiederum ihrerseits ihre noch nicht infizierte Nachbarn infizieren. Sei $v \in G$ ein weiterer Knoten: Die *first passage percolation* zwischen v_0 und v entspricht der Zeit, die die Krankheit benötigt, um sich bis zur Infektion von v auszubreiten. Man beachte, dass im Allgemeinen mehrere verschiedene Pfade existieren, die die beiden Punkte v_0 und v verbinden. In diesem Fall gilt

Die *first passage percolation* zwischen v_0 und v ist die minimale Zeit über alle Pfade, die v und v_0 verbinden.

In der Polymerterminologie entspricht die FPP daher dem sogenannten *ground state*, d.h. der Konfiguration des Polymers mit minimaler Energie. Eine solche Konfiguration ist physikalisch die stabilste und diejenige, die jedes chemische System im Gleichgewicht einnimmt.

Aus physikalischer Sicht sind die wichtigsten FPP-Modelle auf $\mathbb{Z}^n$ mit $n = 2, 3$ definiert. Es ist jedoch bekannt, dass die mathematische Behandlung dieser Modelle sehr schwierig ist. Mit Ausnahme von sehr wenigen Fällen in zwei Dimensionen, in denen man sich auf die vollständige Integrierbarkeit des Systems oder auf tiefliegende Verbindungen zu interagierenden Partikelsystemen/ Zufallsmatrizen verlassen kann, siehe [33, 46] und die dort genannten Referenzen, ist die FPP in niedrigen Dimensionen praktisch nicht verstanden.

Der letztendliche Grund, warum das FPP-Problem im Allgemeinen ein äußerst herausforderndes (und weiterhin offenes) Problem ist, ist schnell erklärt: Das Problem betrifft ein sogenanntes *Extremereignis* - das Minimum eines zufälligen Feldes - damit ist das FPP-Problem im Bereich der sogenannten Extremwerttheorie [EWT] angesiedelt. EWT ist ein gut etabliertes Gebiet der Mathematik bzw. genauer der Wahrscheinlichkeitstheorie, dessen Grundlagen bis ins letzte Jahrhundert zurückreichen, mit zahlreichen bahnbrechenden Arbeiten zum Beispiel von Kolmogorov, Fréchet, Weibull, Gumbel, Gnedenko. Bei *unabhängigen* Zufallsfeldern wurde die Antwort auf die Frage nach dem Minimum vor mehr als 60 Jahren geklärt, hauptsächlich von Fisher-Tippett und Gnedenko: Im Limes großer Kardinalitäten ergeben sich nur drei extreme Verteilungen als Kandidaten für die schwache Grenzverteilung des (möglicherweise neu zentrierten und neu skalierten) Minimums - die berühmten Gumbel-, Fréchet- oder Weibull-Universalitätsklassen. Es ist ferner bekannt, dass der extremale Prozess eines Zufallsfeldes, das aus *unabhängigen* Zufallsvariablen besteht, gegen einen Poisson-Punkt-Prozess konvergiert - ein Ergebnis, das zu dem zentralen Grenzwertsatz für Summen unabhängiger Zufallsvariablen äquivalent ist, siehe z.B. [41].

Wir betonen, dass die Grundannahme der klassischen EWT die Unabhängigkeit ist, die für das FPP-Problem auf $\mathbb{Z}^n$ bei weitem nicht erfüllt ist: Tatsächlich verbinden mehrere Pfade (Polymere) zwei beliebige Punkte v_0 und v_1, und diese können gemeinsame Kanten haben. Dies impliziert, dass die Durchgangszeiten (Energien) tatsächlich *korreliert* sind: Je mehr Kanten zwei Pfade gemeinsam haben, desto stärker sind die Korrelationen. Die schönen und starken Theoreme der klassischen EWT gelten daher nicht mehr.

Um in diesem herausfordernden Bereich Fortschritte zu erzielen, stützen wir uns auf einige Vereinfachungen, die auftreten, wenn man den *mean field limit* betrachtet, d.h. wenn die Dimension des zugrunde liegenden euklidischen Gitters ins Unendliche geschickt wird. Präziser werden wir den n-dimensionalen Hyperwürfel $\{0,1\}^n \subset \mathbb{Z}^n$ betrachten. Wir schreiben $\mathbf{0} \equiv (0,..,0)$ und $\mathbf{1} \equiv (1,..,1)$ für die Knoten, welche diametral gegenüber liegen. Wir interessieren uns für die FPP zwischen diesen zwei Knoten, wenn n gegen Unendlich geht. Die resultierende Vereinfachung ist leicht zu erklären: Typischerweise, wenn sich zwei Pfade in hohen Dimensionen trennen, werden sie sich nicht mehr treffen (außer an den Rändern im Fall des Hyperwürfels). Dementsprechend sind die Energien weniger korreliert! Graphen ohne Schleifen sind jedoch *Bäume*, daher liegt die folgende Überlegung nahe:

In hohen Dimensionen trägt das FPP-Problem auf dem Hyperwürfel starke Ähnlichkeit mit dem FPP-Problem auf Bäumen.

Diese Einsicht, die wir als *hierarchische Approximation des FPP-Problems in großen Dimensionen* bezeichnen, erweist sich als richtig und spielt in dieser Dissertation eine fundamentale Rolle. Dies liegt an der Tatsache, dass das FPP-Problem für Bäume/ hierarchische Strukturen dank der jüngsten Fortschritte auf dem Gebiet der Spingläser und ungeordneten Systeme einer rigorosen Analyse zugänglich ist (siehe [35]). In Abschnitt 2.2 werden wir genau dieses Szenario für das FPP-Problem auf dem Hyperwürfel formulieren. In Abschnitt 2.2.1 werden wir dann unsere Ergebnisse für das *-gerichtete* FPP-Problem präsentieren, während wir in Abschnitt 2.2.2 unseren Beitrag im (viel anspruchsvolleren) *ungerichteten* Fall diskutieren werden.

2.2 First passage percolation auf dem Hyperwürfel

Wir beginnen dieses Kapitel mit einer genauen Definition des FPP-Problems auf dem Hyperwürfel. Zu diesem Zweck bezeichnen wir mit $G_n = (V_n, E_n)$ den n-dimensionalen Hyperwürfel. $V_n = \{0,1\}^n$ ist somit die Menge der Punkte und E_n die Menge der Kanten, die die nächsten Nachbarn verbinden. Wir schreiben $\mathbf{0} = (0,0,...,0)$ und $\mathbf{1} = (1,1,...,1)$ für diametral gegenüberliegende Knoten. Für $l \in \mathbb{N}$, sei

$$\Pi_{n,l} \equiv \text{die Menge der Polymere, d.h. Pfade von } \mathbf{0} \text{ zu } \mathbf{1} \text{ der Länge } l\,,$$

und

$$\Pi_n \equiv \bigcup_{l=1}^{\infty} \Pi_{n,l}.$$

Jede Kante des n-Hyperwürfels ist parallel zu einem Einheitsvektor $e_j \in \mathbb{R}^n$, wobei e_j

$$(0,\ldots,0) \text{ und } (0,\ldots,0, \underbrace{1}_{j-\text{ten}-\text{Koordinate}}, 0,\ldots,0) \text{ verbindet.}$$

Wir schreiben $e_{-j} \equiv -e_j$. Die Menge $\pi_j \in \{1,\ldots,n\} \cup \{-1,\ldots,-n\}$ gibt dann die Richtung eines π-Pfads im j-ten Schritt an. Man beachte, dass der Endpunkt des (Unter-)Pfads $\pi_1\pi_2\ldots\pi_i$ mit dem durch $\sum_{j\le i} e_{\pi_j}$ angegebenen Knoten übereinstimmt. Die Kante, die im j-ten Schritt vom Pfad π durchlaufen wird, wird mit $[\pi]_j$ bezeichnet.

Die Kanten seien mit unabhängigen, standardexponentialverteilten Zufallsvariablen ξ gewichtet, die nun zusammen die zufällige Umgebung definieren. Diese Auswahl stellt keinen Verlust an Allgemeinheit dar: Nur das Verhalten für kleine Werte ist von Bedeutung. Wir weisen einem Polymer $\pi \in \Pi_{n,l}$ sein *Gewicht* über

$$X_\pi \equiv \sum_{j=1}^{l} \xi_{[\pi]_j}$$

zu. In dieser Arbeit betrachten wir das *-gerichtete* FPP-Problem und *-ungerichtete* FPP-Problem:

- Das -gerichtete FPP-Problem auf dem Hyperwürfel betrifft das minimale Gewicht

$$m_n[\mathrm{dir}] \equiv \min_{\pi \in \Pi_{n,n}} X_\pi, \tag{2.2.1}$$

- Das ungerichtete FPP-Problem, auch bekannt als random polymers in random environment, betrifft

$$m_n[\mathrm{undir}] \equiv \min_{\pi \in \Pi_n} X_\pi. \tag{2.2.2}$$

In beiden Fällen ist die Asymptotik, an der wir interessiert sind, die der großen Dimensionen, d.h. wenn n gegen ∞ strebt.
Es versteht sich von selbst, dass das ungerichtete FPP-Problem aufgrund der zusätzlichen "Freiheitsgrade" viel schwieriger als das gerichtete FPP-Problem ist. Wir werden daher mit letzterem beginnen, auch weil dieses Modell bei unserer Behandlung des ersteren eine Rolle spielt.

2.2.1 Gerichtete FPP auf dem Hyperwürfel

Die *führende* Ordnung des Grundzustands wurde von Aldous vermutet und vor mehr als drei Jahrzehnten von Fill und Pemantle bewiesen:

Theorem 9 (Aldous [2], Fill-Pemantle [29]). *Für das gerichtete FPP auf dem Hyperwürfel gilt*

$$\lim_{n\to\infty} m_n[\mathrm{dir}] = 1, \tag{2.2.3}$$

in Wahrscheinlichkeit.

Im ersten Teil dieser Arbeit schlagen wir einen einfacheren und intuitiveren Beweis des Fill-Pemantle-Theorems durch die *multiscale refinement of the second moment method* aus [35] vor, die in [37] erschienen ist. Vor allem hat diese Arbeit eine Verbindung zwischen der FPP auf dem Hyperwürfel und hierarchischen Strukturen hergestellt. Dieser kraftvolle Verbindung führt zusammen mit den daraus resultierenden mentalen Bildern und dem Arsenal an technischen Werkzeugen, die der Untersuchung ungeordneter Systeme entlehnt sind, zu einer vollständigen Verständnis des extremalen Prozesses

$$\Xi_n \equiv \sum_{\pi \in \Pi_{n,n}} \delta_{n(X_\pi - 1)} \,. \tag{2.2.4}$$

In der Tat haben wir in [38] folgendes bewiesen:

Theorem 10 (Extremer Prozess der gerichteten FPP auf dem Hyperwürfel). *Sei Ξ ein Cox Prozess der Intensität $Ze^{x-1}dx$, wobei Z wie das Produkt von* zwei *unabhängigen standardexponentialverteilten Zufallsvariablen verteilt sei. Dann gilt*

$$\lim_{n\to\infty} \Xi_n = \Xi, \tag{2.2.5}$$

schwach. Insbesondere folgt, dass

$$\lim_{n\to\infty} \mathbb{P}(n(m_n[\text{dir}] - 1) \le t) = \int_0^\infty \frac{x}{e^{1-t} + x} e^{-x} dx\,. \tag{2.2.6}$$

Aufgrund der inhärenten Symmetrie hat der Hyperwürfel zwei "Wurzeln" (die entgegengesetzten Eckpunkte **0** und **1**) anstelle von nur einem Baum, *es handelt sich also um zwei zusammengeflickte Bäume.* In Abbildung 2.1 unten finden Sie eine grafisch anschauliche Darstellung der dabei entstehenden Phänomene.

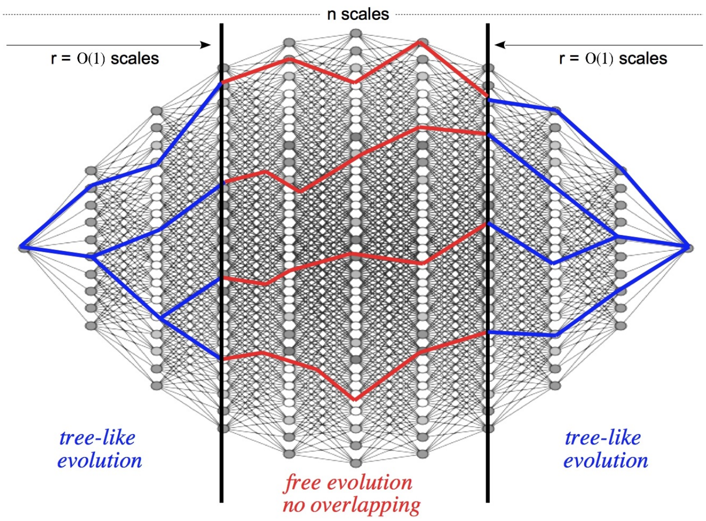

Figure 2.1: Nahe 0 und 1 haben optimale Pfade eine baumähnliche Entwicklung (blaue Pfade). Im Kern des Hyperwürfels (rot) teilen sich Pfade keine Kanten: Das ist nichts anderes als die oben genannten Tatsache, dass "Schleifen" in hohen Dimensionen äußerst unwahrscheinlich sind.

2.2.2 Ungerichtete FPP auf dem Hyperwürfel

Das dritte Kapitel dieser Dissertation behandelt das ungerichtete FPP-Problem auf dem Hyperwürfel. Die *führende* Ordnung des Grundzustands wurde kürzlich von Anders Martinsson identifiziert, der eine Vermutung von Fill und Pemantle [29] in einer Reihe von Artikeln gelöst hat:

Theorem 11. *[Martinsson, [43, 44]] Sei* $\mathsf{E} \equiv \ln(1+\sqrt{2})$, *es gilt*

$$\lim_{n\to\infty} m_n[\text{undir}] = \mathsf{E}, \tag{2.2.7}$$

in Wahrscheinlichkeit.

Martinsson schlägt tatsächlich zwei radikal unterschiedliche Beweise vor, die jedoch beide ziemlich implizit sind und an gewissen Mängeln leiden. Zum einen bleibt unklar, aufgrund welcher physikalischen Phänomene bestimmte Pfade optimale Energien erreichen. Zum anderen lassen sich mit diesen Methoden nicht die Schwankungen des (zentralisierten und reskalierten) extremalen Prozesses analysieren.

Um die Lücken zu schließen, die Martinssons Beweise hinterlassen haben, und um eine Brücke zum ungeklärten Problem der Schwankungen zu schlagen (siehe Abschnitt 2.2.3 weiter unten), implementieren wir in [36] *the multiscale refinement of the second moment method* aus [35]. Dieser Ansatz erweist sich tatsächlich als konstruktiv: Er erfordert als "Input" eine detaillierte Identifizierung der zugrunde liegenden Mechanismen/Strategien, die von den Polymeren angewendet werden, um minimale Werte zu erreichen. Dies führt wiederum zu einer genauen geometrischen Beschreibung der optimalen Pfade und liefert den Satz von Martinsson als einfaches Korollar. Hier sind die wichtigsten Ergebnisse unserer Behandlung.

Die Energie des Polymers ist in erster Näherung gleichmäßig entlang der Pfade verteilt. Die Bindungen des Polymers tragen jedoch eine geringere Energie als in der gerichteten Einstellung und werden durch die folgende geometrische Entwicklung erreicht. In der Nähe des Ursprungs verläuft das Polymer gerichtet. Die Spannung der Pfade nimmt jedoch allmählich ab, wobei das Polymer immer mehr Rückschritte zulässt, wenn es in den Kern des Hyperwürfels eintritt.

Tatsächlich werden die Eigenschaften der im vorhergehenden Absatz erwähnten optimalen Polymere alle aus einer Schlüsselbeobachtung abgeleitet. Die (anteilige) Länge l des Substrats und seiner Position, d.h. der (anteilige) Abstand d vom Ursprung bezüglich der Hamming-Metrik, werden durch die folgende einfache Formel in Beziehung gesetzt:

$$d(l) = \sinh(\frac{l}{\sqrt{2}}) \cosh(\mathsf{E} - \frac{l}{\sqrt{2}}). \tag{2.2.8}$$

Diese neue Formel gibt die typische geometrische Entwicklung des Polymers (erster Ordnung) an. Eine grafische Darstellung, die die dynamische Entwicklung von gerichtetem und nicht gerichtetem Fall vergleicht, ist in Abbildung 2.2 unten dargestellt.

Die Verbindung zwischen gerichteten und ungerichteten Polymeren ist für unser Vorgehen von entscheidender Bedeutung und geht weit über die Ähnlichkeit zwischen der Entwicklung in der Nähe von Ursprung und Ziel hinaus. Tatsächlich gibt es einen zusätzlichen und viel tieferen Gegensatz zwischen diesen Modellen, der sich wiederum hinter (2.2.8) verbirgt. Um zu sehen, wie dies zustande kommt, kehren wir zunächst zu einem wesentlichen Merkmal ungerichteter Polymere zurück, nämlich der Tatsache, dass sie Rückschritte ausführen. Diese erhöhen natürlich die Länge der Pfade, ermöglichen es dem Polymer aber auch, energetisch günstige Kanten zu erreichen, die ansonsten in einem vollständig gerichteten Zustand nicht erreichbar sind. Im mikroskopischen Maßstab, gelingt es optimalen Polymeren, solche Reservoire durch ungefähre *Geodätische* bezüglich der Hamming-Metrik zu verbinden. Wie genau dieses Phänomen entsteht, lässt sich am

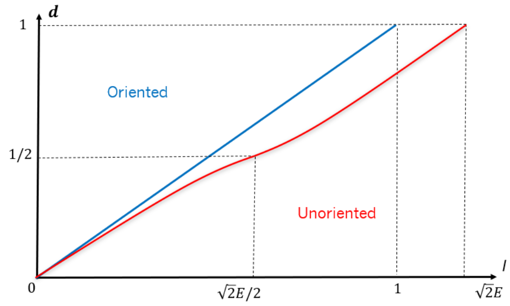

Figure 2.2: Hamming-Tiefe als Funktion der Länge: Gerichtete (blau) vs. ungerichtete (rote Kurve, die nur ein Diagramm der Funktion (2.2.8) ist) Polymere. Bei kleinen Längen sind die Tiefen vergleichbar: In der Nähe des Ursprungs ist das ungerichtete Polymer somit so gerichtet wie möglich. Die Steigung der roten Kurve nimmt jedoch allmählich ab, wenn sich das Polymer dem Kern des Hyperwürfels nähert: Je weiter das Polymer vorrückt, desto mehr kommt es zum "Erschlaffen". Aufgrund der inhärenten Symmetrie des Hyperwürfels setzt ein Spiegelbild natürlich in halber Länge ein.

besten anhand des folgenden Bildes erklären (siehe Abbildung 2.3 unten).

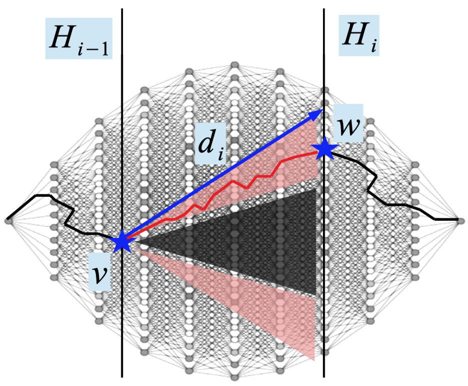

Figure 2.3: Sei $K \in \mathbb{N}$ eine große, aber endliche Konstante. Wir teilen den Hyperwürfel in K Hyperebenen: $H_i \equiv \left\{v \in V_n, d(0,v) = i\frac{n}{K}\right\}$, $i = 1 \dots K$. Der schwarz schattierte Kegel entspricht dem Bereich, in dem ein vollständig gerichtetes Polymer liegen würde, während die H-Ebenen verbunden werden. Ungerichtete Polymere entwickeln sich jedoch in den rot schattierten Kegeln und erreichen dadurch Punkte, die sich in einem größeren Hamming-Abstand d_i befinden als ihre gerichteten Gegenstücke (d_i hängt nur von der Position von H_i ab). Entscheidend ist jedoch, dass die Substrate (in Rot) optimaler Polymere in erster Näherung *Geodätische* sind: Sie erreichen energetisch günstige Kanten (in einem vollständig gerichteten Zustand nicht erreichbar) mit möglichst wenigen Schritten.

Die Eigenschaft, nach der Polymere aufeinanderfolgende Hyperebenen (siehe grafik 2.3) durch (ungefähre) *Geodätische* miteinander verbinden, gibt tiefe Einblicke in ihre geometrische Entwicklung, die aus der impliziten Analyse von Martinsson nicht hervorgehen. Darüber hinaus macht diese Schlüsseleigenschaft einen äußerst effektiven Anknüpfungspunkt zwischen dem nicht gerichteten und dem gerichteten Modell aus, der schließlich die Verwendung aller im gerichteten Fall verfügbaren Werkzeuge ermöglicht und die rigorose Manipulation einiger ansonsten entmutigender kombinatorischer Objekte dramatisch vereinfacht.

2.2.3 Vermutung

Analog zu den Entwicklungen bei der gerichteten FPP auf dem Hyperwürfel baut die in [36] initiierte Forschungslinie eine Brücke zu den schwachen Grenzen der ungerichteten FPP, für die es in der Literatur bislang noch nicht einmal keine Vermutung zu geben scheint. Dank der in [36] gesammelten Erkenntnisse erscheint es nun vernünftig, die Vermutung aufzustellen, dass Korrelationen auch im ungerichteten Fall im Kern des Hyperwürfels im großen n -Grenzwert kaum eine Rolle spielen werden. Mit anderen Worten erwarten wir, dass sich ein ähnliches Bild wie in Theorem 10 ergeben wird.

Es versteht sich von selbst, dass das Weglassen der Orientierungsannahme die Schwierigkeiten um Größenordnungen erhöht. Erwähnt sei hier das folgende zentrale, konzeptionelle Problem, welches im gerichteten Fall so keine Entsprechung hat: Das ungerichtete FPP-Problem hat einen zusätzlichen "Freiheitsgrad", nämlich die *Länge des Polymers*. In [36] haben wir bewiesen, dass sich die optimale Länge in erster Ordnung stark um ihren Mittelwert konzentriert. Dieses Phänomen taucht natürlich so nicht mehr auf, wenn man die schwachen Grenzwerte behandelt, sodass man als ersten Schritt auch die Schwankungen der Länge identifizieren muss. Aufgrund einiger in [36] gesammelter Erkenntnisse glauben wir wiederum, dass die Schwankungen der Länge sich tatsächlich *Gaußsch* verhalten.

Chapter 3

Oriented FPP I: The first order

This chapter is up to minor changes published in [37]. The *Poisson clumping heuristic* has lead Aldous to conjecture the value of the oriented first passage percolation on the hypercube in the limit of large dimensions. Aldous' conjecture has been rigorously confirmed by Fill and Pemantle [*Annals of Applied Probability* **3** (1993)] by means of a variance reduction trick. We present here a streamlined and, we believe, more natural proof based on ideas emerged in the study of Derrida's random energy models.

3.1 Introduction

We consider the following (oriented) first passage percolation (FPP) problem. We first recall the notations taken in the introduction of this dissertation. Denote by $G_n = (V_n, E_n)$ the n-dimensional hypercube. $V_n = \{0,1\}^n$ is thus the set of vertices, and E_n the set of edges connecting nearest neighbours. To each edge we attach independent, identically distributed random variables ξ. We assume these to be standard (mean one) exponentials. (As will become clear in the treatment, this choice represents no loss of generality: only the behavior for small values matters). We write $\mathbf{0} = (0,0,...,0)$ and $\mathbf{1} = (1,1,...,1)$ for diametrically opposite vertices, and denote by $\Pi_{n,n}$ the set of paths of length n from $\mathbf{0}$ to $\mathbf{1}$. Remark that $\sharp\Pi_{n,n} = n!$, and that any $\pi \in \Pi_{n,n}$ is of the form $\mathbf{0} = v_0, v_1, ..., v_n = \mathbf{1}$, with the $v's \in V_n$. To each path π we assign its *weight*

$$X_\pi \equiv \sum_{(v_j, v_{j-1})\in\pi} \xi_{v_{j-1},v_j}.$$

The FPP on the hypercube concerns the minimal weight

$$m_n[\text{dir}] \equiv \min_{\pi\in\Pi_{n,n}} X_\pi, \tag{3.1.1}$$

in the limit of large dimensions, i.e. as $n \to \infty$. The *leading* order has been conjectured by Aldous [2], and rigorously established by Fill and Pemantle [29]:

Theorem 12 (Fill and Pemantle)**.** *For the FPP on the hypercube,*

$$\lim_{n\to\infty} m_n[\text{dir}] = 1, \tag{3.1.2}$$

in probability.

The result is surprising, but then again not. On the one hand, it can be readily checked that (3.1.2) coincides with the large-n minimum of $n!$ independent sums, each consisting of n independent, standard exponentials. The FPP on the hypercube thus manages to reach the same value as in the case of *independent FPP*. In light of the severe correlations among the weights (eventually due to the tendency of paths to overlap), this is indeed a notable feat. On the other hand, the asymptotics involved is that of large dimensions, in which case (and perhaps according to some folklore) a *mean-field trivialization* is expected, in full agreement with Theorem 12. The situation is thus reminiscent of Derrida's generalized random energy models, the GREMs [23, 24, 35], which are hierarchical Gaussian fields playing a fundamental role in the Parisi theory of mean field spin glasses. Indeed, for specific choice of the underlying parameters, the GREMs undergo a *REM-collapse* where the geometrical structure is no longer detectable in the large volume limit, see also [15, 20]. Mean field trivialization and REM-collapse are two sides of the same coin.

The proof of Theorem 12 by Fill and Pemantle implements a *variance reduction trick* which is ingenious but, to our eyes, slightly opaque. The purpose of the present notes is to provide a more natural proof which relies, first and foremost, on neatly exposing the aforementioned point of contact between the FPP on the hypercube and the GREMs. The key observation (already present in [29], albeit perhaps somewhat implicitly) is thereby the following well-known, loosely formulated property:

in high-dimensional spaces, two walkers which depart from
one another are unlikely to ever meet again. (3.1.3)

Underneath the FPP thus lies an *approximate hierarchical structure*, whence the point of contact with the GREMs. Such a connection then allows to deploy the whole arsenal of mental pictures, insights and tools recently emerged in the study of the *REM-class*: specifically, we use the multi-scale refinement of the 2nd moment method introduced in [35], a flexible tool which has proved useful in a variety of models, most notably the log-correlated class, see e.g. [4] and references therein. (It should be however emphasized that the FPP at hand is not, strictly speaking, a log-correlated field).

Before addressing a model in the REM-class, it is advisable to first work out the details for the associated GREM, i.e. on a suitably constructed tree. In the specific case of the hypercube, one should rather think of two trees patched together, the vertices $\mathbf{0}$ and $\mathbf{1}$ representing the respective roots, see Figure 3.1 below. For brevity, we restrain from giving the details for the tree(s), and tackle right away the FPP on the hypercube. Indeed, it will become clear below that once the connection with the GREMs is established, the problem on the hypercube reduces essentially to a delicate *path counting*, requiring in particular combinatorial estimates, many of which have however already been established in [29].

The route taken in these notes neatly unravels, we believe, the physical mechanisms eventually responsible for the mean field trivialization. What is perhaps more, the point of contact with the REMs opens the gate towards some interesting and to date unsettled issues, such as the corrections to subleading order, or the weak limit. These aspects will be addressed elsewhere. To what extend our approach applies to related models like unoriented FPP [43] or accessibility percolation models [31] is an interesting question, which we cannot answer.

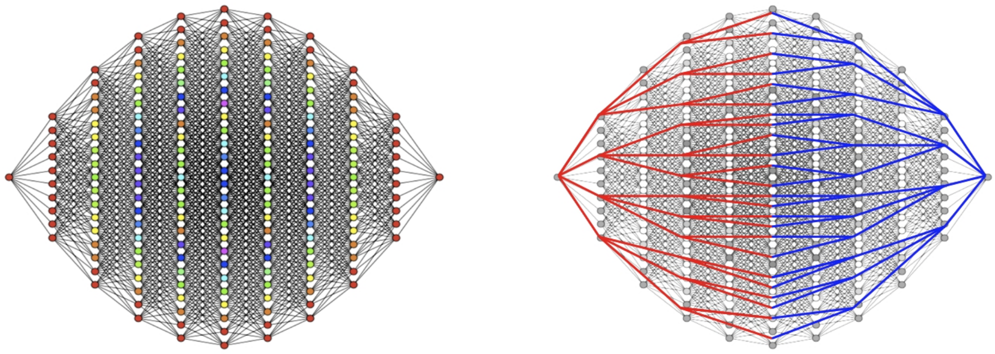

Figure 3.1: A rendition of the 10-dim hypercube, and the associated trees patched together. Observe in particular how the branching factor decreases when wandering into the core of the hypercube: this is due to the fact that a walker starting out in **0** and heading to **1** has, after k steps, $(N - k)$ possible choices for the next step. (The walker's steps correspond to the scales; the underlying trees are thus non-homogenuous, a fact already pointed out in [2]). The figure should be taken *cum grano*: in the FPP, trees simply capture the aforementioned property of high-dimensional spaces, see (3.1.3) above, modulo the constraint that paths must start and end at prescribed vertices.

In the next section we sketch the main steps behind the new approach to Theorem 12. The proofs of all statements are given in a third and final section.

3.2 The multi-scale refinement of the 2nd moment method

We will provide (asymptotically) matching lower and upper bounds following the recipe laid out in [35, Section 3.1.1]. The lower bound, which is the content of the next Proposition, will follow seamlessly from Markov's inequality and some elementary path-counting.

Proposition 13. *For the FPP on the hypercube,*

$$\lim_{n\to\infty} m_n[\text{dir}] \geq 1, \tag{3.2.1}$$

almost surely.

In order to state the main steps behind the upper bound, we need to introduce some additional notation. First, remark that the vertices of the n-hypercube stand in one to one correspondence with $\{0,1\}^n$. Indeed, every edge is parallel to some unit vector e_j, where e_j connects $(0,\ldots,0)$ to $(0,\ldots,0,1,0,\ldots,0)$ with a 1 in position j. We identify a path π of length n from **0** to **1** by a permutation of $12\ldots n$ say $\pi_1\pi_2\ldots\pi_n$. π_l is giving the direction the path π goes in step l, hence after i steps the path $\pi_1\pi_2\dots\pi_n$ is at vertex $\sum_{j\leq i} e_{\pi_j}$. We denote the edge traversed in the i-th step of π by $[\pi]_i$ and define the weight of path π by

$$X_\pi = \sum_{i\leq n} \xi_{[\pi]_i}$$

where $\{\xi_e, e\in E_n\}$ are independent standard exponentials and T_n the space of permutations of $12\dots n$. Note that $[\pi]_i = [\pi']_j$ if and only if $i=j$, $\pi_i=\pi'_j$ and $\pi_1\pi_2\dots\pi_{i-1}$ is a permutation of $\pi'_1\pi'_2\dots\pi'_{j-1}$.

As mentioned, we will implement the multiscale refinement of the 2nd moment method from [35], albeit with a number of twists. In the multiscale refinement, the first step is "to give oneself an epsilon of room": we will indeed consider $\epsilon > 0$ and show that

$$\lim_{n\to\infty} \mathbb{P}\left(\#\{\pi\in T_n, \sum_{i=1}^n \xi_{[\pi]_i} \leq 1+\epsilon\} > 0\right) = 1. \tag{3.2.2}$$

The natural attempt to prove the above via the Paley-Zygmund inequality is bound to fail due to the severe correlations. We bypass this obstacle partitioning the hypercube into three regions which we refer to as 'first', 'middle' and 'last', see Fig. 3.2 below, and handling on separate footings. (This step slightly differs from the recipe in [35]).

We then address the first region, proving that one finds a *growing* number of edges outgoing from $\mathbf{0}$ with weight less than $\epsilon/3$. (By symmetry, the same then holds true for the last region). We will refer to these edges with low weights as *ϵ-good*, or simply *good*. The existence of a positive fraction of good edges is the content of Proposition 14 below.

Proposition 14. *With*

$$A_n^{\mathbf{0}} \equiv \{v\leq n : (\mathbf{0}, e_v)\in E_n \text{ is } \epsilon\text{-good}\}, \quad A_n^{\mathbf{1}} \equiv \{v\leq n : (\mathbf{1}-e_v, \mathbf{1})\in E_n \text{ is } \epsilon\text{-good}\}, \tag{3.2.3}$$

there exists $C = C(\epsilon) > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(|A_n^{\mathbf{0}}\setminus A_n^{\mathbf{1}}| \geq Cn\right),\ \mathbb{P}\left(|A_n^{\mathbf{1}}\setminus A_n^{\mathbf{0}}| \geq Cn\right) = 1\,. \tag{3.2.4}$$

Proof. Consider independent exponentially (mean one) distributed random variables $\{\xi_i\}, \{\xi'_i\}$. We have:

$$\frac{|A_n^{\mathbf{0}}\setminus A_n^{\mathbf{1}}|}{n} \overset{d}{=} \frac{1}{n}\sum_{i=1}^n \mathbf{1}_{\{\xi_i\leq\frac{\epsilon}{3}, \xi'_i>\frac{\epsilon}{3}\}} \xrightarrow[n\to\infty]{a.s.} p(\epsilon), \tag{3.2.5}$$

by the law of large numbers, where $p(\epsilon) = \mathbb{P}(\xi_1\leq\epsilon)\mathbb{P}(\xi_1>\epsilon) > 0$. The claim thus holds true for any $C\in(0, p(\epsilon))$. The second claim is fully analogous.

□

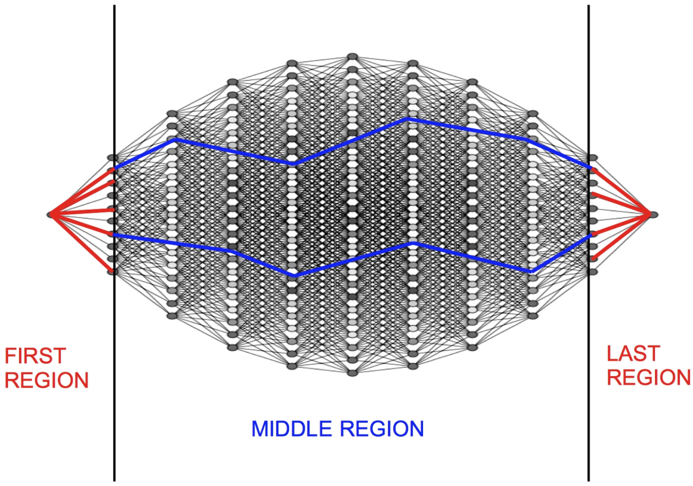

Figure 3.2: Partitioning the hypercube into the three regions. Red edges are ϵ-*good*: their weight is smaller than $\epsilon/3$. Blue paths connecting first and last level have weights smaller than $1+\epsilon/3$. The total weight of a path consisting of one red edge outgoing from $\mathbf{0}$, a connecting blue path, and a final red edge going into $\mathbf{1}$ is thus less than $1+\epsilon$. These are the relevant paths leading to tight upper bounds for the FPP.

By the above, the missing ingredient in the proof of (3.2.2) is thus the existence of (at least) one path in the middle region with weight less than $1+\epsilon/3$, and which connects an ϵ-good edge in the first region to one in the last. This will be eventually done in Proposition 15 by means of a full-fledged multiscale analysis. Towards this goal, consider the random variable accounting for *good paths* connecting $\mathbf{0}$ and $\mathbf{1}$ whilst going through good edges in first and last region, to wit:

$$\mathcal{N}_n = \#\left\{\pi \in T_n : \pi_1 \in A_n^{\mathbf{0}} \setminus A_n^{\mathbf{1}}, \pi_n \in A_n^{\mathbf{1}} \setminus A_n^{\mathbf{0}} \text{ and } \sum_{i=2}^{n-1} \xi_{[\pi]_i} \leq 1+\frac{\epsilon}{3}\right\}, \tag{3.2.6}$$

We now claim that

$$\lim_{n\to\infty} \mathbb{P}\left(\mathcal{N}_n > 0\right) = 1, \tag{3.2.7}$$

which would naturally imply (3.2.2). To establish (3.2.7), we exploit the existence of a wealth of good edges,

$$\mathbb{P}\left(\mathcal{N}_n > 0\right) \geq \mathbb{P}\left(\mathcal{N}_n > 0, |A_n^{\mathbf{0}} \setminus A_n^{\mathbf{1}}| \geq Cn, |A_n^{\mathbf{1}} \setminus A_n^{\mathbf{0}}| \geq Cn\right) \tag{3.2.8}$$

Using that the weights involved in $A_n^{\mathbf{0}}$ and $A_n^{\mathbf{1}}$ are independent of all other weights and that considering more potential paths increases the probability of there beeing a path with specific properties we have that

$$\mathbb{P}\left(\mathcal{N}_n > 0 \mid |A_n^{\mathbf{0}} \setminus A_n^{\mathbf{1}}| = j, |A_n^{\mathbf{1}} \setminus A_n^{\mathbf{0}}| = k\right)$$

is monotonically growing in j and k as long as the probability is well defined, i.e. as long as $j + k \leq n$. Therefore

$$\begin{aligned}(3.2.8) &\geq \mathbb{P}\left(\mathcal{N}_n > 0 \mid |A_n^{\mathbf{0}} \setminus A_n^{\mathbf{1}}| = \lceil Cn \rceil, |A_n^{\mathbf{1}} \setminus A_n^{\mathbf{0}}| = \lceil Cn \rceil\right) \mathbb{P}\left(|A_n^{\mathbf{0}} \setminus A_n^{\mathbf{1}}| \geq Cn, |A_n^{\mathbf{1}} \setminus A_n^{\mathbf{0}}| \geq Cn\right) \\ &= \mathbb{P}\left(\mathcal{N}_n > 0 \mid |A_n^{\mathbf{0}} \setminus A_n^{\mathbf{1}}| = \lceil Cn \rceil, |A_n^{\mathbf{1}} \setminus A_n^{\mathbf{0}}| = \lceil Cn \rceil\right) - o(1)\end{aligned} \tag{3.2.9}$$

in virtue of Proposition 14 for properly chosen $C = C(\epsilon) > 0$. This in turn equals

$$= \mathbb{P}\left(\mathcal{N}_n > 0 \mid A_n^{\mathbf{0}} \setminus A_n^{\mathbf{1}} = A, A_n^{\mathbf{1}} \setminus A_n^{\mathbf{0}} = A'\right) - o(1)$$

for any admissible choice A, A' with $|A| = |A'| = \lceil Cn \rceil$, say $A \equiv \{j : j \leq Cn\}$ and $A' \equiv \{j : j \geq (1 - C)n\}$. Claim (3.2.7) will steadily follow from

Proposition 15. *(Connecting first and last region) Let*

$$T_n^{(1,n)} \equiv \{\pi \in T_n : \pi_1 \in A, \pi_n \in A'\}. \tag{3.2.10}$$

It then holds:

$$\lim_{n \to \infty} \mathbb{P}\left(\# \left\{\pi \in T_n^{(1,n)} : \sum_{i=2}^{n-1} \xi_{[\pi]_i} \leq 1 + \epsilon/3\right\} > 0\right) = 1.$$

Since (3.2.7) implies (3.2.2), the upper bound for the main theorem immediately follows from Propositions 13 and 15. It thus remains to provide the proofs of these two propositions: this is done in the next, and last section.

3.3 Proofs

3.3.1 Tail estimates, and proof of the lower bound

We first state a useful

Lemma 16. *(Tail estimates.) Consider independent exponentially (mean one) distributed random variables $\{\xi_i\}, \{\xi'_i\}$. With $X_n \equiv \sum_{i=1}^n \xi_i$ and $x > 0$, it then holds:*

$$\mathbb{P}\left(X_n \leq x\right) = \left(1 + K(x, n)\right) \frac{e^{-x} x^n}{n!}, \tag{3.3.1}$$

with $0 \leq K(x, n) \leq e^x x/(n + 1)$.

Furthermore, consider $X'_n \equiv \sum_{i=1}^n \xi'_i$*, and assume that* X'_n *shares exactly* k edges *(meaning here* k *exponential random variables) with* X_n*: without loss of generality we may write this as*

$$X'_n = \sum_{i=1}^{k} \xi_i + \sum_{i=k+1}^{n} \xi'_i.$$

Then

$$\mathbb{P}\left(X_n \leq x, X'_n \leq x\right) \leq \mathbb{P}\left(X_n \leq x\right) \mathbb{P}\left(X_{n-k} \leq x\right). \tag{3.3.2}$$

Proof. One easily checks (say through characteristic functions) that X_n is a Gamma$(n,1)$-distributed random variable, in which case

$$\mathbb{P}\left(X_n \leq x\right) = \frac{1}{(n-1)!} \int_0^x t^{n-1} e^{-t} dt = 1 - e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}\,, \tag{3.3.3}$$

the second step by partial integration. We write the r.h.s. above as

$$e^{-x} \sum_{k=n}^{\infty} \frac{x^k}{k!} = e^{-x} \frac{x^n}{n!} \left(1 + \frac{n!}{x^n} \sum_{k=n+1}^{\infty} \frac{x^k}{k!}\right). \tag{3.3.4}$$

By Taylor expansions,

$$\sum_{k=n+1}^{\infty} \frac{x^k}{k!} \leq \frac{e^x x^{n+1}}{(n+1)!}, \tag{3.3.5}$$

hence (3.3.1) holds with

$$K(x,n) := \frac{n!}{x^n} \sum_{k=n+1}^{\infty} \frac{x^k}{k!} \leq \frac{e^x x}{(n+1)} \tag{3.3.6}$$

As for the second claim, by positivity of exponentials,

$$\mathbb{P}\left(X_n \leq x, X'_n \leq x\right) \leq \mathbb{P}\left(\sum_{i=1}^{n} \xi_i \leq x, \sum_{i=k+1}^{n} \xi'_i \leq x\right). \tag{3.3.7}$$

Claim (3.3.2) thus follows from the independence of the ξ, ξ' random variables. □

Armed with these estimates, we can move to the

Proof of Proposition 13 (the lower bound). With $\mathcal{N}_n^x = \#\{\pi \in T_n, X_\pi \leq x\}$, it holds:

$$\begin{aligned} \mathbb{P}\left(m_n[\text{dir}] \leq x\right) = \mathbb{P}\left(\mathcal{N}_n^x \geq 1\right) &\leq \mathbb{E}\mathcal{N}_n^x \\ &= n!\mathbb{P}\left(X_\pi \leq x\right) \\ &\overset{(3.3.2)}{=} \left(1 + o_n(1)\right) e^{-x} x^n\,, \end{aligned} \tag{3.3.8}$$

the second step by Markov inequality. Remark that (3.3.8) vanishes exponentially fast for any $x < 1$; an elementary application of the Borel-Cantelli Lemma thus yields (3.2.1) and "half of the theorem", the *lower* bound, is proven. □

3.3.2 Combinatorial estimates

The proof of the upper bounds relies on a somewhat involved path-counting procedure. The required estimates are a variant of [29, Lemma 2.4] and are provided by the following

Lemma 17 (Path counting)**.** *Let π' be any reference path on the n-dim hypercube connecting $\mathbf{0}$ and $\mathbf{1}$, say $\pi' = 12...n$. Denote by $f(n,k)$ the number of paths π that share precisely k edges ($k \geq 1$) with π' without considering the first and the last edge. Finally, shorten $\mathfrak{n}_\mathfrak{c} \equiv n - 5e(n+3)^{2/3}$.*

- *For any $K(n) = o(n)$ as $n \to \infty$,*

$$f(n,k) \leq (1+o(1))(k+1)(n-k-1)! \tag{3.3.9}$$

  uniformly in k for $k \leq K(n)$.

- *Suppose $k + 2 \leq \mathfrak{n}_\mathfrak{c}$. Then, for n large enough,*

$$f(n,k) \leq 2n^6(n-k)!\,. \tag{3.3.10}$$

- *Suppose $k \geq \mathfrak{n}_\mathfrak{c} - 1$. Then, for n large enough,*

$$f(n,k) \leq \frac{1}{k!}(n-2)!(n-k-1)\,. \tag{3.3.11}$$

Proof of Lemma 17. To see (3.3.9), consider a path π which shares precisely k edges with the reference path $\pi' = 12\dots n$. We set $r_i = l$ if the l-th traversed edge by π is the i-th shared edge of π and π'. (We set by convention $r_0 = 0$ and $r_{k+1} = n+1$). Shorten $\mathbf{r} \equiv \mathbf{r}(\pi) = (r_0, ..., r_{k+1})$, and $s_i \equiv r_{i+1} - r_i$, $i = 0, ..., k$. For any sequence $\mathbf{r}_0 = (r_0, ..., r_{k+1})$ with $0 = r_0 < r_1 < ... < r_k < r_{k+1} = n+1$, let $C(\mathbf{r}_0)$ denote the number of paths π with $\mathbf{r}(\pi) = \mathbf{r}_0$. Since the values $\pi_{r_i+1}, ..., \pi_{r_i+s_i-1}$ must be a permutation of $\{r_i+1, ..., r_i+s_i-1\}$, one easily sees that $C(\mathbf{r}) \leq G(\mathbf{r})$, where

$$G(\mathbf{r}) = \prod_{i=0}^{k}(s_i - 1)!\,. \tag{3.3.12}$$

Let now $j = j(\mathbf{r}) \equiv \max_i(s_i - 1)$. We will consider separately the cases $j < n - 4k$ and $j \geq n - 4k$, the underlying idea being that $G(\mathbf{r})$ is small in the first case, and while not small in the second, there are only few sequences with such large j-value.
Denote by $f_{\{j<n-4k\}}(n,k)$ resp. $f_{\{j\geq n-4k\}}(n,k)$ the number of paths π that share precisely k edges with π' not counting the first and the last edge, where $j < n - 4k$ for the first function and $j \geq n - 4k$ for the second one. It holds:

$$f(n,k) = f_{\{j<n-4k\}}(n,k) + f_{\{j\geq n-4k\}}(n,k)\,. \tag{3.3.13}$$

Case $j < n - 4k$. We claim that

$$G(\mathbf{r}) \le (n - 4k - 1)!(3k + 1)!\,. \tag{3.3.14}$$

In fact, for $j \le n-4k-1$, and by log-convexity, the product in (3.3.12) is maximized at $\mathbf{r}'s$ such that $j(\mathbf{r}) = n - 4k - 1$. It thus follows that

$$\begin{aligned} G(\mathbf{r}) &\le \left(\max_i(s_i - 1)\right)! \left(\sum_i (s_i - 1) - \max_i(s_i - 1)\right)! \\ &\le (n - 4k - 1)!(3k + 1)!\,, \end{aligned} \tag{3.3.15}$$

the last step since $\sum_i(s_i-1) = n-k$. On the other hand, the number of $\mathbf{r}$-sequences under consideration is at most $\binom{n-2}{k}$: combining with (3.3.15),

$$\begin{aligned} f_{\{j<n-4k\}}(n,k) &\le \frac{(n-4k-1)!(3k+1)!(n-2)!}{(k)!(n-2-k)!} \\ &= (n-k-1)!\frac{(n-4k-1)!}{(n-k-1)!}\frac{(3k+1)!}{(k)!}\frac{(n-2)!}{(n-2-k)!} \\ &\le (n-k-1)!\frac{1}{(n-4k)^{3k}}(3k+1)(3k)^{2k}(n-2)^k \\ &\le (n-k-1)!3(k+1)\left[\frac{(3k)^2(n-2)}{(n-4k)^3}\right]^k\,, \end{aligned} \tag{3.3.16}$$

by simple bounds. The term in square brackets converges to 0 as n $\to \infty$ uniformly in k as long as $k \le K(n) = o(n)$, hence the contribution from the first case is $o((k+1)(n-k-1)!)$, uniformly in such $k's$.

Case $j \ge n - 4k$. Again by log-convexity of factorials,

$$G(\mathbf{r}) \le j(\mathbf{r})!(n - k - j(\mathbf{r}))!\,. \tag{3.3.17}$$

The number of $\mathbf{r}$-sequences for which $j(\mathbf{r}) = j_0$ is at most $(k+1)$ times the number of $\mathbf{r}$-sequences with $s_0 - 1 = j_0$; since the $k-1$ common edges have to be placed before the last edge in our definition of $f(n,k)$, the latter is thus at most $\binom{n-1-j_0-1}{k-1}$. For fixed j_0, the contribution is therefore at most

$$\begin{aligned} &\frac{(n-1-j_0-1)!(k+1)j_0!(n-j_0-k)!}{(k-1)!(n-1-j_0-k)!} \\ &\qquad = \frac{(n-j_0-2)!(k+1)j_0!(n-j_0-k)}{(k-1)!}\,. \end{aligned} \tag{3.3.18}$$

Summing (3.3.18) over all possible values $n-4k \leq j_0 \leq n-k-1$, we get

$$\begin{aligned}
f_{\{j\geq n-4k\}}(n,k) &\leq (k+1)(n-k-1)! \sum_{j_0=n-4k}^{n-k-1} \frac{(n-j_0-2)!j_0!(n-j_0-k)}{(k-1)!(n-k-1)!} \\
&= (k+1)(n-k-1)! \sum_{i=1}^{3k} \frac{(k+i-2)!}{(k-1)!} \frac{(n-k-i)!}{(n-k-1)!} i \\
&\leq (k+1)(n-k-1)! \sum_{i=1}^{3k} (4k)^{i-1} \frac{1}{(n-4k)^{i-1}} i \\
&\leq (k+1)(n-k-1)! \sum_{i=1}^{3K(n)} \left(\frac{n}{4K(n)} - 1\right)^{1-i} i \\
&\leq (k+1)(n-k-1)! \left(1 + \sum_{i=1}^{3K(n)} 2^i \left(\frac{n}{4K(n)} - 1\right)^{-i}\right) \\
&= (k+1)(n-k-1)!(1+o_n(1)) \,.
\end{aligned} \tag{3.3.19}$$

Using the upperbounds (3.3.16) and (3.3.19) in (3.3.13) settles the proof of (3.3.9).

The second claim of the Lemma relies on estimates established by Fill and Pemantle, and which we now recall for completeness. Denote by $f_1(n,k)$ the number of paths π that share precisely k edges with the reference path $\pi' = 12\cdots n$. (Contrary to $f(n,k)$, first and last edge do matter here!) By [29, Lemma 2.4] the following holds

$$f_1(n,k) \leq n^6 (n-k)! \,, \tag{3.3.20}$$

as soon as $k \leq \mathfrak{n}_\mathfrak{c}$ and n is large enough. It then holds:

$$\begin{aligned}
f(n,k) &\leq f_1(n,k) + f_1(n,k+1) + f_1(n,k+2) \\
&\overset{(3.3.20)}{\leq} n^6(n-k)!(1 + \frac{1}{(n-k)} + \frac{1}{(n-k)(n-k-1)}) \\
&\leq 2n^6(n-k)! \,,
\end{aligned} \tag{3.3.21}$$

yielding (3.3.10).

It remains to address the third claim of the Lemma, which we recall reads

$$f(n,k) \leq \frac{1}{k!}(n-2)!(n-k-1)! \,, \tag{3.3.22}$$

for $\mathfrak{n}_\mathfrak{e} - 1 \leq k \leq n$. For this, it is enough to proceed by worst-case: there are at most $(n-k-1)!$ paths sharing k edges with the reference-path π' for given $\mathbf{r}$, and $\binom{n-2}{k}$ ways to choose such $\mathbf{r}$-sequences. All in all, this leads to

$$f(n,k) \leq \binom{n-2}{k}(n-k-1)! = \frac{(n-2)!(n-k-1)}{k!}, \tag{3.3.23}$$

settling the proof of (3.3.22). □

3.3.3 Proof of the upper bound

Proof of Proposition 15 (Connecting first and last region). The claim is that

$$\lim_{n\to\infty} \mathbb{P}\left(\mathcal{N}_n^{(1)} > 0\right) = 1, \tag{3.3.24}$$

where $\mathcal{N}_n^{(1)} = \#\{\pi \in T_n^{(1,n)}, \sum_{i=2}^{n-1} \xi_{[\pi]_i} \leq 1 + \frac{\epsilon}{3}\}$. This will now follow from the Paley-Zygmund inequality, which requires control of 1st- and 2nd-moment estimates. As for the 1st moment, by simple counting and with C as in Proposition 14,

$$\begin{aligned} \mathbb{E}\mathcal{N}_n^{(1)} &= C^2 n^2 (n-2)! \times \mathbb{P}\left(\sum_{i=2}^{n-1} \xi_{[\pi]_i} \leq 1 + \frac{\epsilon}{3}\right) \\ &= \kappa n^2 \left(1 + \frac{\epsilon}{3}\right)^{n-2} [1 + o(1)] \qquad (n \to \infty), \end{aligned} \tag{3.3.25}$$

(the last step by Lemma 16) for some numerical constant $\kappa > 0$.

Now shorten $B \equiv \{\pi, \pi'' \in T_n$ *have no edges in common in the middle region*$\}$. For the 2nd moment, it holds:

$$\begin{aligned} \mathbb{E}\left[\mathcal{N}_n^{(1)^2}\right] = &\sum_{(\pi,\pi'')\in B} \mathbb{P}\left(\sum_{i=2}^{n-1} \xi_{[\pi]_i} \leq 1 + \frac{\epsilon}{3}\right)^2 \\ &+ \sum_{(\pi,\pi'')\in B^c} \mathbb{P}\left(\sum_{i=2}^{n-1} \xi_{[\pi]_i} \leq 1 + \frac{\epsilon}{3}, \sum_{i=2}^{n-1} \xi_{[\pi'']_i} \leq 1 + \frac{\epsilon}{3}\right) \\ =: &(\Sigma_B) + (\Sigma_{B^c}), \quad \text{say.} \end{aligned} \tag{3.3.26}$$

But by independence,

$$(\Sigma_B) \leq \left(\mathbb{E}\mathcal{N}_n^{(1)}\right)^2, \tag{3.3.27}$$

hence it steadily follows from (3.3.26) that

$$1 \leq \frac{\mathbb{E}\left[\mathcal{N}_n^{(1)^2}\right]}{\left(\mathbb{E}\mathcal{N}_n^{(1)}\right)^2} \leq 1 + \frac{(\Sigma_{B^c})}{\left(\mathbb{E}\mathcal{N}_n^{(1)}\right)^2}. \tag{3.3.28}$$

It thus remains to prove that

$$(\Sigma_{B^c}) = o\left(\mathbb{E}\left[\mathcal{N}_n^{(1)}\right]^2\right) \qquad (n \to \infty)\,. \tag{3.3.29}$$

To see (3.3.29), by symmetry it suffices to consider the case where π'' is any reference path, say $\pi'' = \pi' = 12\cdots n$. By the second claim of Lemma 16, and with X_n denoting a Gamma$(n,1)$-distributed random variable, it holds:

$$\begin{aligned}(\Sigma_{B^c}) \leq (Cn)^2(n-2)! \sum_{k=1}^{n-3} f(n,k)\mathbb{P}\left(X_{n-2} \leq 1+\frac{\epsilon}{3}\right)\mathbb{P}\left(X_{n-2-k} \leq 1+\frac{\epsilon}{3}\right) \\ + (Cn)^2(n-2)!\mathbb{P}\left(X_{n-2} \leq 1+\frac{\epsilon}{3}\right)\,,\end{aligned} \tag{3.3.30}$$

hence

$$\frac{(\Sigma_{B^c})}{\left(\mathbb{E}\mathcal{N}_n^{(1)}\right)^2} \leq \frac{1}{(Cn)^2(n-2)!}\left(\sum_{k=1}^{n-3} f(n,k)\frac{\mathbb{P}\left(X_{n-2-k} \leq 1+\frac{\epsilon}{3}\right)}{\mathbb{P}\left(X_{n-2} \leq 1+\frac{\epsilon}{3}\right)} + \frac{1}{\mathbb{P}\left(X_{n-2} \leq 1+\frac{\epsilon}{3}\right)}\right) \tag{3.3.31}$$

By Lemma 16,

$$\frac{\mathbb{P}\left(X_{n-2-k} \leq 1+\frac{\epsilon}{3}\right)}{\mathbb{P}\left(X_{n-2} \leq 1+\frac{\epsilon}{3}\right)} \leq \frac{2(n-2)!}{(n-k-2)!(1+\frac{\epsilon}{3})^k}\,, \tag{3.3.32}$$

and therefore, up to the irrelevant $o(1)$-term,

$$\begin{aligned}(3.3.31) &\leq \frac{2}{(Cn)^2}\sum_{k=1}^{n-3}\frac{f(n,k)}{(n-k-2)!(1+\frac{\epsilon}{3})^k} \\ &= \frac{2}{(Cn)^2}\left(\sum_{k=1}^{K(n)} + \sum_{k=K(n)+1}^{\mathfrak{n}_\mathfrak{e}-2} + \sum_{k=\mathfrak{n}_\mathfrak{e}-1}^{n-3}\right)\frac{f(n,k)}{(n-k-2)!(1+\frac{\epsilon}{3})^k}\,,\end{aligned} \tag{3.3.33}$$

where $K(n) \equiv n^{1/4}$ and $\mathfrak{n}_\mathfrak{e} = n - 5e(n+3)^{2/3}$. By Lemma 17 the first sum on the r.h.s. of (3.3.33) is at most

$$\begin{aligned}\frac{2}{(Cn)^2}\sum_{k=1}^{n^{1/4}}\frac{f(n,k)}{(n-k-2)!(1+\frac{\epsilon}{3})^k} &\overset{(3.3.9)}{\leq} \frac{2}{(Cn)^2}\sum_{k=1}^{n^{1/4}}\frac{2(k+1)(n-k-1)!}{(n-k-2)!(1+\frac{\epsilon}{3})^k} \\ &\leq \frac{4(n^{1/4}+1)}{C^2 n}\sum_{k=1}^{n^{1/4}}\frac{1}{(1+\frac{\epsilon}{3})^k} = \frac{12}{C^2\epsilon}n^{-3/4}[1+o(1)],\end{aligned} \tag{3.3.34}$$

which vanishes for $n \to \infty$. As for the second sum on the r.h.s. of (3.3.31),

$$\begin{aligned}
&\frac{2}{(Cn)^2} \sum_{k=n^{1/4}+1}^{\mathfrak{n}_e-2} \frac{f(n,k)}{(n-k-2)!(1+\frac{\epsilon}{3})^k} \\
&\stackrel{(3.3.10)}{\leq} \frac{4n^6}{(Cn)^2} \sum_{k=n^{1/4}+1}^{\mathfrak{n}_\mathfrak{e}-2} \frac{(n-k)!}{(n-k-2)!(1+\frac{\epsilon}{3})^k} \\
&\leq \frac{4n^6}{C^2} \sum_{k=n^{1/4}+1}^{\mathfrak{n}_\mathfrak{e}-2} \left(1+\frac{\epsilon}{3}\right)^{-k} \leq \frac{12n^6}{\epsilon C^2} \left(1+\frac{\epsilon}{3}\right)^{-n^{1/4}} [1+o(1)],
\end{aligned} \tag{3.3.35}$$

which is thus also vanishing in the large-n limit. It thus remains to check that the same is true for the third and last term on the r.h.s. of (3.3.33):

$$\begin{aligned}
\frac{2}{(Cn)^2} \sum_{\mathfrak{n}_e-1}^{n-3} \frac{f(n,k)}{(n-k-2)!(1+\frac{\epsilon}{3})^k} &\stackrel{(3.3.10)}{\leq} \frac{2}{(Cn)^2} \sum_{k=\mathfrak{n}_\mathfrak{e}-1}^{n-3} \frac{(n-k-1)}{k!} \frac{(n-2)!}{(n-2-k)!(1+\frac{\epsilon}{3})^k} \\
&\leq \frac{2}{C^2 n} \sum_{k=\mathfrak{n}_\mathfrak{e}-1}^{n-3} \binom{n-2}{\mathfrak{n}_\mathfrak{e}-1} \left(1+\frac{\epsilon}{3}\right)^{-k},
\end{aligned} \tag{3.3.36}$$

the last inequality by simple estimates on the binomial coefficients (using $\mathfrak{n}_\mathfrak{e} - 1 \geq n/2$). Remark that

$$(3.3.36) \leq \frac{6}{\epsilon C^2} \binom{n-2}{\mathfrak{n}_\mathfrak{e}} (1+\frac{\epsilon}{3})^{2-\mathfrak{n}_\mathfrak{e}} \tag{3.3.37}$$

By Stirling's formula, one plainly checks that

$$\binom{n-2}{\mathfrak{n}_\mathfrak{e}} \leq \frac{n!}{\mathfrak{n}_\mathfrak{e}!} = \frac{n^n e^{\mathfrak{n}_\mathfrak{e}-n}}{(\mathfrak{n}_\mathfrak{e})^{\mathfrak{n}_\mathfrak{e}}} [1+o(1)] \,. \tag{3.3.38}$$

Plugging this estimate into (3.3.39) we thus get for some numerical constant $\kappa > 0$ that

$$(3.3.36) \leq \kappa \frac{n^n}{\left(\mathfrak{n}_\mathfrak{e} \left(1+\frac{\epsilon}{3}\right)\right)^{\mathfrak{n}_\mathfrak{e}}} \xrightarrow[n\to\infty]{} 0, \tag{3.3.39}$$

and (3.3.29) follows. An elementary application of the Paley-Zygmund inequality then settles the proof of Proposition 15. □

Chapter 4

Oriented FPP II: The extremal process

This chapter is, up to minor changes, published in [38]. We address the behavior of oriented first passage percolation on the hypercube in the limit of large dimensions. We prove here that the extremal process converges to a Cox process with exponential intensity. This entails, in particular, that the first passage time converges weakly to a random shift of the Gumbel distribution. The random shift, which has an explicit, universal distribution related to modified Bessel functions of the second kind, is the sole manifestation of correlations ensuing from the geometry of Euclidean space in infinite dimensions. The proof combines the multiscale refinement of the second moment method with a conditional version of the Chen-Stein bounds, and a contraction principle.

4.1 Introduction and main results

We first recall the notations taken in the introduction of this dissertation. We first embed the n-dimensional hypercube in $\mathbb{R}^n$, for $e_1, .., e_n$ the standard basis, we identify the hypercube as the graph $G_n \equiv (V_n, E_n)$, where $V_n = \{0,1\}^n$ and $E_n \equiv \{(v, v+e_j) : v, v+e_j \in V_n, j \leq n\}$. The set of shortest (directed) paths connecting diametrically opposite vertices, say $\mathbf{0} \equiv (0, .., 0)$ and $\mathbf{1} \equiv (1, .., 1)$, is given by

$$\Pi_{n,n} \equiv \{\pi \in V_n^{n+1} : \pi_1 = \mathbf{0}, \pi_{n+1} = \mathbf{1}, (\pi_i, \pi_{i+1}) \in E_n, \forall i \leq n\}. \tag{4.1.1}$$

A graphical rendition is given in Figure 4.1 below.

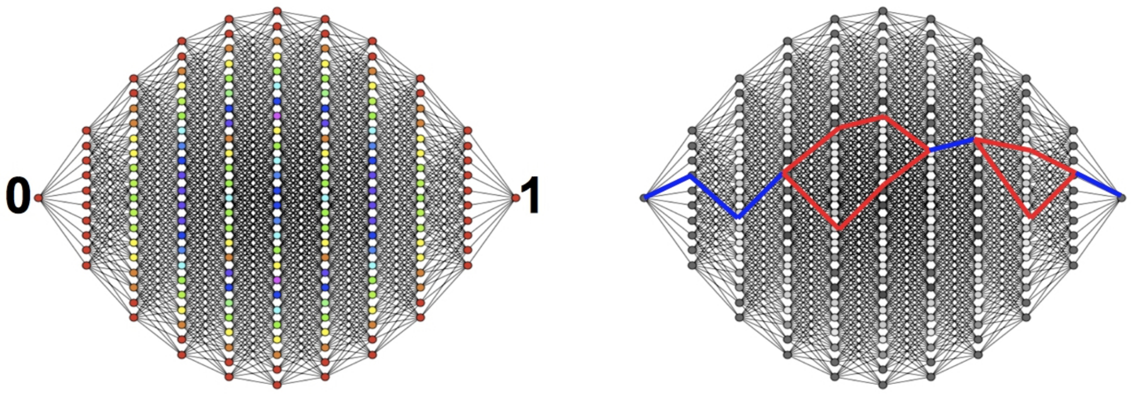

Figure 4.1: The 10-dimensional hypercube (left), and two oriented connecting paths (right). Blue edges are common to both paths, whereas paths do not overlap on red edges.

Let now $(\xi_e)_{e \in E}$ be a family of independent standard exponentials, i.e. exponentially distributed random variables with parameter 1, and assign to each oriented path $\pi \in \Pi_{n,n}$ its *weight*

$$X_\pi \equiv \sum_{k \leq n} \xi_{[\pi]_k},$$

where $[\pi]_i = (\pi_i, \pi_{i+1})$ is the i-th edge of the path.

A key question in first passage percolation, FPP for short, concerns the so-called *first passage time*,

$$m_n[\text{dir}] \equiv \min_{\pi \in \Pi_{n,n}} X_\pi \,, \tag{4.1.2}$$

namely the smallest weight of connecting paths. The limiting value of $m_n[\text{dir}]$ *to leading order* has been settled by Fill and Pemantle [29], who proved that

$$\lim_{n\to\infty} m_n[\text{dir}] = 1, \tag{4.1.3}$$

in probability.

The "law of large numbers" (4.1.3) naturally raises questions on fluctuations and weak limits, and calls for a description of the paths with minimal weight. As a first step towards this goal we presented in [37] an alternative, "modern" approach to (4.1.3) much inspired by the recent advances in the study of Derrida's random energy models (see [35] and references therein) and which relies on the hierarchical approximation to the FPP. In this companion paper we bring the approach to completion by establishing the full limiting picture, i.e. identifying the weak limit of the *extremal process*

$$\Xi_n \equiv \sum_{\pi \in \Pi_{n,n}} \delta_{n(X_\pi - 1)} \,.$$

Theorem 18 (Extremal process)**.** *Let Ξ be a Cox process with intensity $Ze^{x-1}dx$, where Z is distributed like the product of two independent standard exponentials. Then*

$$\lim_{n\to\infty} \Xi_n = \Xi, \tag{4.1.4}$$

weakly. In particular, it follows for the first passage time $m_n[\text{dir}]$ that

$$\lim_{n\to\infty} \mathbb{P}(n(m_n[\text{dir}] - 1) \le t) = \int_0^\infty \frac{x}{e^{1-t} + x} e^{-x} dx \,. \tag{4.1.5}$$

It will become clear below, see in particular Remark 23, that the assumption on the distribution of the edge-weights is no restriction, any distribution in the same extremality class of the exponentials (i.e. any distribution with similar behavior for small values, to leading order) will lead to the same limiting picture and weak limits. Although not needed, we also point out that the distribution of the mixture is given by $f(z) = 2z^2 K_0(2\sqrt{z})$, with K_0 a modified Bessel function of the second kind.

What lies behind the onset of the Cox processes is a *decoupling* whose origin can be traced back to the high-dimensional nature of the problem at hand. Indeed, the following mechanism, depicted in Figure 4.2 below, holds with overwhelming probability in the limit $n \to \infty$ first, and $r \to \infty$ next: *Walkers connecting* $\mathbf{0}$ *to* $\mathbf{1}$ *through paths of minimal weight may share at most the first r steps of their journey. Yet, and crucially, whenever they depart from one another ('branch off'), they cannot meet again until they lie at distance at most r from the target. If meeting happens, they must continue on the same path (no further branching is possible).* The long stretches during which optimal paths do not overlap are eventually responsible for the Poissonian component of the extremal process, whereas the mixing is due to the relatively short stretches of tree-like (early and late) evolution of which the system keeps persistent memory. The picture is thus very reminiscent of the extremes of branching Brownian motion [BBM], see [17] and references therein. More specifically, the extremal process of FPP on the hypercube can be (partly) seen as the "gluing together" of two extremal processes of BBM in the weak correlation regime as studied by Bovier and Hartung [18, 19], see also [25, 27, 28].

The backbone of the proof of Theorem 18 will be presented in Section 4.2 below. We anticipate that we will check the assumptions of a well-known theorem by Kallenberg by means of the Chein-Stein method [9]. This is arguably the classical route for this type of problems, see e.g. [16, 20, 22, 39]. Contrary to these works, we will however need here a *conditional* version of the Chen-Stein method which we haven't found in the literature, and which may be of independent interest. Section 4.3 and the Appendix are devoted to the proofs.

4.2 Strategy of proof

We lighten notation by setting, for $A \subset \mathbb{R}$ a generic subset and π an oriented path,

$$I_\pi(A) \equiv \delta_{n(X_\pi - 1)}(A), \quad \text{and} \quad \Xi_n(A) \equiv \sum_{\pi \in \Pi_{n,n}} I_\pi(A)\,.$$

We then claim that with Z as in Theorem 18, and A a finite union of bounded intervals, one has

- Convergence of the intensity:

$$\lim \mathbb{E}\,\Xi_n(A) \underset{n\to\infty}{\longrightarrow} \mathbb{E} \int_A Z e^{x-1} dx = \int_A e^{x-1} dx\,. \tag{4.2.1}$$

- Convergence of the avoidance function:

$$\mathbb{P}\left(\Xi_n(A) = 0\right) \underset{n\to\infty}{\longrightarrow} \mathbb{P}\left(\Xi(A) = 0\right) = \mathbb{E}\left[\exp\left(-Z \int_A e^{x-1} dx\right)\right]. \tag{4.2.2}$$

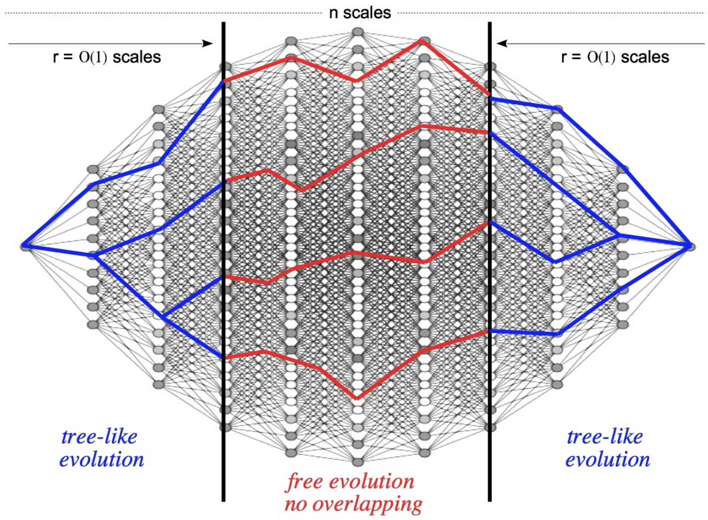

Figure 4.2: Four extremal paths. Remark in particular the tree-like evolution close to **0** and **1** (blue edges) and the (comparatively) longer stretch where paths share no common edge (red). This should be contrasted with the low-dimensional scenario: "loops" in the core of the hypercube, as depicted in Figure 4.1, become less and less likely with growing dimension.

Theorem 18 then immediately follows in virtue of Kallenberg's Theorem [34, Theorem 4.15].

In the remaining part of this Section we provide a bird's eye view of the main steps involved in the analysis of intensity and avoidance functions. The former is rather straightforward: it only requires tail-estimates which we now state for they will be constantly used throughout the paper. (The simple proof may be found in [37, Lemma 5]).

Lemma 19. *Let $\{\xi_i\}_{i\leq n}$ be independent standard exponentials, and set $X_n \equiv \sum_{i=1}^n \xi_i$. Then*

$$\mathbb{P}\left(X_n \leq x\right) = \left(1 + K(x,n)\right) \frac{e^{-x}x^n}{n!}, \tag{4.2.3}$$

for $x > 0$ and with the error-term satisfying $0 \leq K(x,n) \leq e^x x/(n+1)$.

Armed with these estimates we can proceed to the short proof of (4.2.1). Here and below, we will always consider sets of the form $A = (-\infty, a]$, $a \in \mathbb{R}$. This is enough for our purposes since the general case follows by additivity. It holds that

$$
\begin{aligned}
\mathbb{E}\Xi_n(A) &= \sum_{\pi\in\Pi_{n,n}} \mathbb{P}\left(n(X_\pi - 1) \leq a\right) \\
&= n!\mathbb{P}\left(n(X_{\pi^*} - 1) \leq a\right) \qquad \text{(symmetry, } \pi^* \in \Pi_{n,n} \text{ is arbitrary)} \\
&= n!\left\{1 + K\left(1+\frac{a}{n}, n\right)\right\} e^{-1-\frac{a}{n}} \left(\left(1+\frac{a}{n}\right)^+\right)^n (n!)^{-1} \qquad \text{(Lemma 19)} \qquad (4.2.4) \\
&= (1+o_n(1))e^{-1+a} \\
&= (1+o_n(1)) \int_A e^{x-1} dx,
\end{aligned}
$$

as claimed. Convergence of the intensity (4.2.1) is thus already settled.

Contrary to convergence of the intensity, convergence of avoidance functions (4.2.2) will require a fair amount of work. This will be split in a number of intermediate steps. The main ingredient is a conditional version of the Chen-Stein bounds, a variant of the classical Chen-Stein method [9] which is tailor-suited to our purposes. Since we haven't found in the literature any similar statement, we provide the proof in the appendix for completeness.

Theorem 20 (Conditional Chen-Stein Method)**.** *Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, a sigma-algebra $\mathcal{F} \subset \mathscr{F}$, a finite set I, and a family $(X_i)_{i\in I}$ of Bernoulli random variables issued on this space. Let furthermore*

$$
W = \sum_{i\in I} X_i \quad \text{and} \quad \lambda = \sum_{i\in I} \mathbb{E}(X_i|\mathcal{F}).
$$

Denote by $N_i, i \in I$ a collection of conditionally dissociating neighborhoods, i.e. with the property that X_i and $\{X_j : j \in (N_i \cup \{i\})^c\}$ are independent, conditionally upon $\mathcal{F}$. Finally, consider a random variable $\widehat{W}$ with the property that its law conditionally upon $\mathcal{F}$ is Poisson, i.e. $\mathcal{L}(\widehat{W}|\mathcal{F}) = \mathrm{Poi}(\lambda)$. It then holds:

$$
d_{TV|\mathcal{F}}(W, \widehat{W}) \leq \sum_{i\in I} \mathbb{E}(X_i|\mathcal{F})^2 + \sum_{i\in I}\sum_{j\in N_i} \left(\mathbb{E}(X_i|\mathcal{F})\mathbb{E}(X_j|\mathcal{F}) + \mathbb{E}(X_iX_j|\mathcal{F})\right), \qquad (4.2.5)
$$

where

$$
d_{TV|\mathcal{F}}(W, \widehat{W}) \equiv \sup_{A\in\mathscr{F}} \left(\mathbb{P}_W(A|\mathcal{F}) - \mathbb{P}_{\widehat{W}}(A|\mathcal{F})\right)
$$

is the total variation distance conditionally upon $\mathcal{F}$.

We will apply Theorem 20 by conditioning on the left- and rightmost regions of Figure 4.2, namely those regions where tree-like evolutions eventually kick in. Specifically, we make the following choices:

a) $I \equiv \Pi_{n,n}$, the set of admissible (oriented) paths connecting $\mathbf{0}$ to $\mathbf{1}$.

b) $\mathcal{F}$ is the sigma-algebra generated by the weights of edges at distance at most r from $\mathbf{0}$ or $\mathbf{1}$, to wit

$$\mathcal{F} = \mathcal{F}_{r,n} \equiv \sigma(\xi_e : e = (u,v) \in E, \min\{d(u,\mathbf{0}), d(v,\mathbf{0})\} \in [0,r) \cup [n-r,n))\,.$$

c) The family of Bernoulli random variables is given by $(I_\pi(A))_{\pi\in\Pi_{n,n}}$.

d) The (random) Poisson-parameter is

$$\lambda = \lambda_{r,n}(A) \equiv \sum_{\pi\in\Pi_{n,n}} \mathbb{E}\left[I_\pi(A) \mid \mathcal{F}_{r,n}\right]$$

e) The dissociating neighborhoods are given, for $\pi \in \Pi_{n,n}$, by

$$N_\pi \equiv \{\pi' \in \Pi_{n,n} \setminus \{\pi\} : \exists i \in \{r+1,..,n-r\} \text{ s.t. } [\pi]_i = [\pi']_i\}$$

A first, fundamental observation concerns item d), namely the weak convergence of the Poisson-parameter in the double limit $n \to \infty$ first and $r \to \infty$ next.

Proposition 21. *(The double weak-limit). For $\pi_1,..,\pi_{i-1} \in \mathbb{N}$ and $i \le r$, denote by*

$$\left(\eta_{\pi_1,..,\pi_{i-1},\pi_i}\right)_{\pi_i\in\mathbb{N}}, \quad \textit{and} \quad \left(\tilde\eta_{\pi_1,..,\pi_{i-1},\pi_i}\right)_{\pi_i\in\mathbb{N}}$$

independent Poisson point processes with intensity $\mathbb{1}_{\mathbb{R}^+}dx$, and set

$$Z_r \equiv \sum_{\pi\in\mathbb{N}^r} \exp\left(-\sum_{j=1}^r \eta_{\pi_1\pi_2\ldots\pi_j}\right), \quad \widetilde{Z}_r \equiv \sum_{\pi\in\mathbb{N}^r} \exp\left(-\sum_{j=1}^r \tilde\eta_{\pi_1\pi_2\ldots\pi_j}\right). \tag{4.2.6}$$

For $A \subset \mathbb{R}$, the following "n-convergence" then holds:

$$\lim_{n\to\infty} \lambda_{r,n}(A) = Z_r \times \widetilde{Z}_r \int_A e^{x-1}dx,$$

weakly. Furthermore, Z_r and $\widetilde{Z}_r$ weakly converge, as $r \to \infty$, to independent standard exponentials.

The proof of the double weak-limit goes via a contraction argument which is given in Section 4.3.1. Here we shall only point out that both limits Z_r and $\widetilde{Z}_r$ are constructed outgoing from *hierarchical*[1] superpositions of Poisson point processes (PPP for short),

[1] Superpositions of PPP such as those involved in (4.2.6) are ubiquitous in the Parisi theory of mean field sping glasses, see [35] and references, where they are referred to as *Derrida-Ruelle cascades.* Although no knowledge of the Parisi theory is assumed/needed, our approach to the oriented FPP in the limit of large dimensions heavily draws on ideas which have recently crystallised in that field.

and this accounts for the somewhat surprising fact that close to **0** and **1** only tree-like structures contribute to the extremal process in the mean field limit, as depicted in Figure 4.2.

Most of the technical work will go into the proof of (4.2.2), which addresses the convergence of avoidance functions. The line of reasoning here goes as follows: recalling that $\Xi_n(A) = \sum_{\pi\in\Pi_{n,n}} I_\pi(A)$, we write

$$\begin{aligned}|\mathbb{P}(\Xi_n(A)=0) - \mathbb{P}(\Xi(A)=0)| &= |\mathbb{E}\mathbb{P}(\Xi_n(A)=0 \mid \mathcal{F}_{r,n}) - \mathbb{E}\mathbb{P}(\Xi(A)=0 \mid Z)| \\ &\le |\mathbb{E}\mathbb{P}(\Xi_n(A)=0 \mid \mathcal{F}_{r,n}) - \mathbb{P}(\text{Poi}(\lambda_{r,n}(A))=0 \mid \mathcal{F}_{r,n})| \\ &\quad + |\mathbb{E}\mathbb{P}(\text{Poi}(\lambda_{r,n}(A))=0 \mid \mathcal{F}_{r,n}) - \mathbb{E}\mathbb{P}(\Xi(A)=0 \mid Z)|,\end{aligned} \tag{4.2.7}$$

by the triangle inequality. Furthermore, by convexity,

$$\begin{aligned}&|\mathbb{E}\mathbb{P}(\Xi_n(A)=0 \mid \mathcal{F}_{r,n}) - \mathbb{P}(\text{Poi}(\lambda_{r,n}(A))=0 \mid \mathcal{F}_{r,n})| \\ &\quad\le \mathbb{E}\,|\mathbb{P}(\Xi_n(A)=0 \mid \mathcal{F}_{r,n}) - \mathbb{P}(\text{Poi}(\lambda_{r,n}(A))=0 \mid \mathcal{F}_{r,n})| \\ &\quad\le \mathbb{E} d_{TV,\mathcal{F}_{r,n}}(\Xi_n(A), \text{Poi}(\lambda_{r,n}(A))),\end{aligned} \tag{4.2.8}$$

and therefore

$$\begin{aligned}|\mathbb{P}(\Xi_n(A)=0) - \mathbb{P}(\Xi(A)=0)| \le& \mathbb{E} d_{TV,\mathcal{F}_{r,n}}(\Xi_n(A), \text{Poi}(\lambda_{r,n}(A))) \\ &+ |\mathbb{E}\mathbb{P}(\text{Poi}(\lambda_{r,n}(A))=0 \mid \mathcal{F}_{r,n}) - \mathbb{E}\mathbb{P}(\Xi(A)=0 \mid Z)|.\end{aligned} \tag{4.2.9}$$

Concerning the second term on the right-hand side above, it follows from Proposition 21 that

$$|\mathbb{E}\mathbb{P}(\text{Poi}(\lambda_{r,n}(A))=0 \mid \mathcal{F}_{r,n}) - \mathbb{E}\mathbb{P}(\Xi(A)=0 \mid Z)| = \left|\mathbb{E}\left(e^{-\lambda_{r,n}(A)} - e^{-Z\int_A e^{x-1}dx}\right)\right| \longrightarrow 0, \tag{4.2.10}$$

in the double-limit $n\to\infty$ first, and $r\to\infty$ next.

We finally claim that the first term on the right-hand side of (4.2.9), the "Chen-Stein term", also vanishes in the considered double-limit,

$$\lim_{r\to\infty}\lim_{n\to\infty} \mathbb{E} d_{TV,\mathcal{F}_{r,n}}(\Xi_n(A), \text{Poi}(\lambda_{r,n}(A))) = 0. \tag{4.2.11}$$

This is, in fact, the key claim, and its proof is given in Section 4.3.2 as an application of the conditional Chen-Stein method, Theorem 20. Assuming this for the time being, by combining (4.2.10) and (4.2.11), we obtain convergence of the avoidance function and the main Theorem 18 therefore follows.

4.3 Proofs

4.3.1 The double weak-limit

The goal of this section is to prove Proposition 21. To see how the limiting objects come about, we lighten notation by setting $V \equiv V_n$, and denote the set of all pairs of paths leading r-steps away from the start/end respectively, and which can be part of an oriented path from $\mathbf{0}$ to $\mathbf{1}$ by

$$\begin{aligned}\mathcal{V}_{r,n} =&\{(x,y) \in V^{r+1} \times V^{r+1} : x_1 = \mathbf{0}, d(x_{r+1}, \mathbf{0}) = r, d(y_1, \mathbf{1}) = r, y_{r+1} = \mathbf{1}, \\ &y_1 - x_{r+1} \in V, (x_i, x_{i+1}), (y_i, y_{i+1}) \in E, \forall i \leq r\}.\end{aligned} \tag{4.3.1}$$

Note that $y_1 - x_{r+1} \in V$ is equivalent to there being a directed path from $\mathbf{0}$ to $\mathbf{1}$ containing x and y. For $(x,y) \in \mathcal{V}_{r,n}$ we define the set of paths connecting x and y by

$$\begin{aligned}\Sigma_{x,y} \equiv&\{\pi' \in V^{n-2r+1} : \exists \pi \in \Pi_{n,n} \text{ s.t. } ([\pi]_i)_{i\leq r} = ([x]_i)_{i \leq r} \quad \text{and} \\ &([\pi]_i)_{r<i\leq n-r} = ([\pi']_i)_{r<i\leq n-r}, ([\pi]_i)_{i>n-r} = ([y]_i)_{i>n-r}\}.\end{aligned} \tag{4.3.2}$$

By definition,

$$\begin{aligned}\lambda_{r,n}(A) &= \sum_{\pi \in \Pi_{n,n}} \mathbb{P}\left(n(X_\pi - 1) \leq a \middle| \mathcal{F}_{r,n}\right) \\ &= \sum_{(x,y)\in\mathcal{V}_{r,n}} \sum_{\pi' \in \Sigma_{x,y}} \mathbb{P}\left(\sum_{i=1}^{n-2r} \xi_{[\pi']_i} \leq 1 + \frac{a}{n} - \sum_{i=1}^{r} \xi_{[x]_i} + \xi_{[y]_i} \middle| \mathcal{F}_{r,n}\right).\end{aligned} \tag{4.3.3}$$

Shorten

$$X_{x,y} \equiv \sum_{i=1}^{r} \xi_{[x]_i} + \xi_{[y]_i}.$$

By Lemma 19, and since $|\Sigma_{x,y}| = (n-2r)!$, the right-hand side of (4.3.3) equals

$$\sum_{(x,y)\in\mathcal{V}_{r,n}} \left(1 + K(1 + \frac{a}{n} - X_{x,y}, n-2r)\right) \exp\left(-1 - \frac{a}{n} + X_{x,y}\right) \left(\left(1 + \frac{a}{n} - X_{x,y}\right)^+\right)^{n-2r}. \tag{4.3.4}$$

By the tail-estimates from Lemma 19, the following holds

$$K\left(1 + \frac{a}{n} - X_{x,y}, n-2r\right) \leq \frac{2e^2}{n-2r},$$

for all non-zero summands, and $n \geq a$. Remark that there are $O(n^{2r})$ such summands, while r and a are fixed. One easily checks that dropping all summands where $X_{x,y} >$

$(\ln n)^2/n$ only causes a deterministically vanishing error, hence

$$\begin{aligned}(4.3.4) &= (1+o_n(1))\left(o_n(1)+e^{-1}\sum_{(x,y)\in\mathcal{V}_{r,n}}\mathbb{1}_{\{X_{x,y}\le\frac{(\ln n)^2}{n}\}}\exp\left((n-2r)\ln\left(1+\frac{a}{n}-X_{x,y}\right)^+\right)\right)\\ &= (1+o_n(1))\left(o_n(1)+e^{-1+a}\sum_{(x,y)\in\mathcal{V}_{r,n}}\exp\left(-nX_{x,y}\right)\right)\\ &= (1+o_n(1))\left(o_n(1)+e^{-1+a}\sum_{(x,y)\in\mathcal{V}_{r,n}}\exp -n\sum_{i=1}^{r}\left(\xi_{[x]_i}+\xi_{[y]_i}\right)\right),\end{aligned} \tag{4.3.5}$$

the second step follows by Taylor-expanding the logarithm around 1 to first order, and the third by definition. We now address the sum on the right-hand side of (4.3.5), on which we perform the aforementioned double limit $n\to\infty$ first and $r\to\infty$ next.

We first address the n-convergence, which states that

$$\lim_{n\to\infty}\sum_{(x,y)\in\mathcal{V}_{r,n}}\exp\left(-n\sum_{l=1}^{r}\xi_{[x]_l}+\xi_{[y]_l}\right)=Z_r\times\widetilde{Z}_r, \tag{4.3.6}$$

weakly, where $Z_r, \widetilde{Z}_r$ are defined in (4.2.6). The idea here is to enlarge the set of paths over which the sum is taken, as this enables a useful decoupling, see (4.3.12) below. Precisely, consider the set of directed paths of length r *from* $\mathbf{0}$,

$$\mathcal{V}_{r,n}^{\leftarrow}=\left\{x\in V^{r+1}: x_1=\mathbf{0}, d(x_{r+1},\mathbf{0})=r, [x]_i\in E, \forall i\le r\right\}, \tag{4.3.7}$$

and respectively *to* $\mathbf{1}$,

$$\mathcal{V}_{r,n}^{\rightarrow}=\left\{y\in V^{r+1}: y_{r+1}=\mathbf{1}, d(y_1,\mathbf{1})=r, [y]_i\in E, \forall i\le r\right\}. \tag{4.3.8}$$

One easily checks that

$$\left|\mathcal{V}_{r,n}^{\leftarrow}\times\mathcal{V}_{r,n}^{\rightarrow}\setminus\mathcal{V}_{r,n}\right|=O(n^{2r-1}). \tag{4.3.9}$$

We split the sum over the larger subset into a sum over $\mathcal{V}_{r,n}$ and a "rest-term",

$$\begin{aligned}&\sum_{(x,y)\in\mathcal{V}_{r,n}^{\rightarrow}\times\mathcal{V}_{r,n}^{\leftarrow}}\exp\left(-n\sum_{l=1}^{r}\xi_{[x]_l}+\xi_{[y]_l}\right)=\\ &\quad=\sum_{(x,y)\in\mathcal{V}_{r,n}}\exp\left(-n\sum_{l=1}^{r}\xi_{[x]_l}+\xi_{[y]_l}\right)\\ &\qquad+\sum_{(x,y)\in(\mathcal{V}_{r,n}^{\rightarrow}\times\mathcal{V}_{r,n}^{\leftarrow})\setminus\mathcal{V}_{r,n}}\exp\left(-n\sum_{l=1}^{r}\xi_{[x]_l}+\xi_{[y]_l}\right).\end{aligned} \tag{4.3.10}$$

and claim that the term on the right-hand side vanishes in probability. Indeed, by a simple computation involving the moment generating function of the exponential distribution, we have

$$\mathbb{E}\left|\sum_{(x,y)\in(\mathcal{V}_{r,n}^{\rightarrow}\times\mathcal{V}_{r,n}^{\leftarrow})\setminus\mathcal{V}_{r,n}} \exp\left(-n\sum_{l=1}^{r}\xi_{[x]_l}+\xi_{[y]_l}\right)\right| = \left|(\mathcal{V}_{r,n}^{\rightarrow}\times\mathcal{V}_{r,n}^{\leftarrow})\setminus\mathcal{V}_{r,n}\right|(n+1)^{-2r}$$
$$= O\left(\frac{1}{n}\right)\underset{n\to\infty}{\longrightarrow} 0, \tag{4.3.11}$$

by (4.3.9). It thus follows from Markov's inequality that the contribution of paths in $(\mathcal{V}_{r,n}^{\rightarrow}\times\mathcal{V}_{r,n}^{\leftarrow})\setminus\mathcal{V}_{r,n}$ is irrelevant for our purposes. The weak limit when summing over $\mathcal{V}_{r,n}$, and that when summing over $\mathcal{V}_{r,n}^{\rightarrow}\times\mathcal{V}_{r,n}^{\leftarrow}$ coincide, provided one of them exists. On the other hand, the sum over the enlarged set of paths "decouples" into two independent identically distributed terms,

$$\sum_{(x,y)\in\mathcal{V}_{r,n}^{\rightarrow}\times\mathcal{V}_{r,n}^{\leftarrow}} \exp\left(-n\sum_{l=1}^{r}\xi_{[x]_l}+\xi_{[y]_l}\right) = \sum_{x\in\mathcal{V}_{r,n}^{\rightarrow}}\exp\left(-n\sum_{l=1}^{r}\xi_{[x]_l}\right)\sum_{y\in\mathcal{V}_{r,n}^{\leftarrow}}\exp\left(-n\sum_{l=1}^{r}\xi_{[y]_l}\right). \tag{4.3.12}$$

The n-convergence will therefore follow as soon as we show that

$$Z_{r,n}\equiv\sum_{x\in\mathcal{V}_{r,n}^{\rightarrow}}\exp\left(-n\sum_{l=1}^{r}\xi_{[x]_l}\right)\underset{n\to\infty}{\longrightarrow}\sum_{\pi\in\mathbb{N}^r}\exp\left(\sum_{l=1}^{r}-\eta_{\pi_1\pi_2\dots\pi_j}\right)\equiv Z_r \tag{4.3.13}$$

holds weakly. This will be done by induction on r. The base-case $r=1$ is addressed in

Lemma 22. *Consider $\eta\equiv\sum_{i\in\mathbb{N}}\delta_{\eta_i}$ a PPP$(\mathbb{1}_{\mathbb{R}^+}dx)$ and independent standard exponentials $(\xi_i)_{i\in\mathbb{N}}$. It then holds:*

$$\sum_{i=1}^{n}\delta_{\xi_i n}\underset{n\to\infty}{\longrightarrow}\eta \tag{4.3.14}$$

weakly. Furthermore, the following weak limit holds:

$$\sum_{i=1}^{n}\exp\left(-\xi_i n\right)\underset{n\to\infty}{\longrightarrow}\sum_{i\in\mathbb{N}}\exp\left(-\eta_i\right). \tag{4.3.15}$$

Proof of Lemma 22. Claim (4.3.14) is a classical result in extreme value theory. We thus omit its elementary proof. As for the second claim, it is steadily checked (e.g. by Markov's inequality) that the sum on the left-hand side of (4.3.15) is almost surely finite. In order to prove (4.3.15) it thus suffices to compute the Laplace transform of the two sums. For

$t \in \mathbb{R}^+$, since the $\xi's$ are independent, we have

$$\begin{aligned}\mathbb{E}\exp -t\sum_{i=1}^{n} e^{-\xi_i n} = \mathbb{E}\left(e^{te^{-\xi_1 n}}\right)^n &= \left(1+\int_0^{+\infty} e^{-x}(e^{te^{-xn}}-1)dx\right)^n \\ &= \left(1+\frac{1}{n}\int_0^{+\infty} e^{-u/n}(e^{te^{-u}}-1)du\right)^n .\end{aligned} \tag{4.3.16}$$

But $e^{-u/n}(e^{te^{-u}}-1) \leq (e^{te^{-u}}-1)$, which is integrable, hence by dominated convergence we have that the right-hand side of (4.3.16) converges, as $n \uparrow \infty$, to the limit

$$\exp\left(\int_0^{+\infty}(e^{-te^{-x}}-1)dx\right) = \mathbb{E}\exp -t\sum_{i\in\mathbb{N}} e^{-\eta_i}, \tag{4.3.17}$$

(4.3.15) is therefore settled. □

Remark 23. *In virtue of Lemma 22, Theorem 18 holds for any choice of edge-weights falling in the same universality class of the exponential distribution, i.e. for which* (4.3.14) *holds.*

For the n-convergence, we will work with the Prohorov metric, which we recall is defined as follows: for $\mu, \nu \in \mathcal{M}_1(\mathbb{R})$ two probability measures, the Prohorov distance is given by

$$\mathrm{d_P}\left(\mu,\nu\right) \equiv \inf\left\{\epsilon>0 : \mu(A)\leq \nu(A^\epsilon)+\epsilon,\ \forall A\subset \mathbb{R}\,\text{closed}\right\},$$

where $A^\epsilon \equiv \{x \in \mathbb{R} : \mathrm{d}(A,x) \leq \epsilon\}$ is the ϵ-neighborhood of the set A. It is a classical fact that the Prohorov distance metricizes weak convergence. We also recall the following implication, as it will be used at different occurences: for two random variables X, Y, slightly abusing notation, one has

$$\mathbb{P}(|X-Y|>\varepsilon)\leq\varepsilon \Rightarrow \mathrm{d_P}(X,Y)\leq\varepsilon\,. \tag{4.3.18}$$

We now proceed to the induction step, we thus assume that $Z_{r,n}$ converges weakly to Z_r for some $r \in \mathbb{N}$ and show how to deduce that $Z_{r+1,n}$ converges weakly to Z_{r+1}. First, we observe that by definition

$$\begin{aligned}Z_{r+1,n} &= \sum_{i\leq n}\exp\left(-n\xi_{(\mathbf{0},e_i)}\right)\sum_{x\in\mathcal{V}^{\rightarrow}_{r+1,n}:x_2=e_i}\exp\left(-n\sum_{l=2}^{r+1}\xi_{[x]_l}\right) \\ &= \sum_{i\leq n}\exp\left(-n\xi_{(\mathbf{0},e_i)}\right)\times Z^{e_i}_{r,n}\,,\end{aligned} \tag{4.3.19}$$

changing notation for the second sum to lighten exposition.

We claim that it suffices to consider small ξ-values in the first sum. Precisely, let $\varepsilon > 0$, set $K_\varepsilon = -2\ln\varepsilon$, and restrict the first sum to those $\xi's$ such that $\xi_{(0,e_i)} \leq K_\varepsilon/n$. We claim that this causes only an ε-error in Prohorov distance:

$$\sup_{n,r} \mathrm{d_P}\left(Z_{r+1,n}, \sum_{i\leq n} \mathbb{1}_{\{\xi_{(\mathbf{0},e_i)}\leq K_\varepsilon/n\}} e^{-n\xi_{(\mathbf{0},e_i)}} \times Z_{r,n}^{e_i}\right) \leq \varepsilon. \tag{4.3.20}$$

In fact, for the contribution of large $\xi's$ Markov inequality yields

$$\begin{aligned}\mathbb{P}&\left(\sum_{i\leq n} \mathbb{1}_{\{\xi_{(\mathbf{0},e_i)}>K_\varepsilon/n\}} e^{-n\xi_{(\mathbf{0},e_i)}} \times Z_{r,n}^{e_i} > \varepsilon\right) \\ &\leq \frac{1}{\varepsilon}\mathbb{E}\left[\sum_{i\leq n} \mathbb{1}_{\{\xi_{(\mathbf{0},e_i)}>K_\varepsilon/n\}} e^{-n\xi_{(\mathbf{0},e_i)}} \times Z_{r,n}^{e_i}\right] \\ &= \frac{n}{\varepsilon}\mathbb{E}\left[\mathbb{1}_{\{\xi_{(\mathbf{0},e_i)}>K_\varepsilon/n\}} e^{-n\xi_{(\mathbf{0},e_i)}}\right] \times \mathbb{E}\left[Z_{r,n}^{e_i}\right],\end{aligned} \tag{4.3.21}$$

the last step by independence. One furthermore checks that

$$\mathbb{E}\left[Z_{r,n}^{e_i}\right] = \frac{(n-1)!}{(n-r-1)!}\left(\int_0^\infty e^{-(n+1)x}dx\right)^r = \frac{(n-1)!}{(n-r-1)!}(n+1)^{-r}. \tag{4.3.22}$$

Thus the right-hand side of (4.3.21) is *at most*

$$\frac{n}{\varepsilon}\int_{K_\varepsilon/n}^\infty e^{-(n+1)x}dx \times \frac{(n-1)!}{(n-r-1)!}(n+1)^{-r} \leq \frac{\exp -K_\varepsilon}{\varepsilon} = \varepsilon, \tag{4.3.23}$$

since $K_\varepsilon = -2\ln\varepsilon$. This settles (4.3.20).

Consider now the permutation p of $\{1,..,n\}$ such that $(\xi_{p(i)})_{i\leq n}$ is increasing, and set $\hat{K}_\varepsilon \equiv \lceil K_\varepsilon/\varepsilon\rceil$. We clearly have

$$Z_{r+1,n} \geq \sum_{i\leq \hat{K}_\varepsilon} e^{-n\xi_{p(i)}} Z_{r,n}^{e_{p(i)}}. \tag{4.3.24}$$

On the other hand, using that $\mathbb{P}(A) \leq \mathbb{P}(A\cap B) + \mathbb{P}(B^c)$ with obvious identification of the events, we have

$$\begin{aligned}\mathbb{P}\left(Z_{r+1,n} \geq \sum_{i\leq \hat{K}_\varepsilon} e^{-n\xi_{p(i)}} Z_{r,n}^{e_{p(i)}} + \varepsilon\right) \\ \leq \mathbb{P}\left(Z_{r+1,n} \geq \sum_{i\leq n} \mathbb{1}_{\{\xi_{(\mathbf{0},e_i)}\leq K_\varepsilon/n\}} e^{-n\xi_{(\mathbf{0},e_i)}} \times Z_{r,n}^{e_i} + \varepsilon\right) \\ + \mathbb{P}\left(\sum_{i\leq \hat{K}_\varepsilon} e^{-n\xi_{p(i)}} Z_{r,n}^{e_{p(i)}} \leq \sum_{i\leq n} \mathbb{1}_{\{\xi_{(\mathbf{0},e_i)}\leq K_\varepsilon/n\}} e^{-n\xi_{(\mathbf{0},e_i)}} \times Z_{r,n}^{e_i}\right).\end{aligned} \tag{4.3.25}$$

While the first term on the right-hand side of (4.3.25) is at most ε by (4.3.21) and (4.3.23), the second term equals

$$\begin{aligned}\mathbb{P}\left(\#\{i \leq n : \xi_{(\mathbf{0},e_i)} \leq K_\varepsilon/n\} > \hat{K}_\varepsilon\right) \\ \leq n\mathbb{P}(\xi_{(\mathbf{0},e_1)} \leq K_\varepsilon/n)/\hat{K}_\varepsilon \leq K_\varepsilon/\hat{K}_\varepsilon \leq \varepsilon\,,\end{aligned} \tag{4.3.26}$$

the first estimate follows by Markov inequality and the second using $(1 - e^{-x}) \leq x$.

All in all, in virtue of (4.3.18), the above considerations imply that

$$\sup_{n,r} \mathrm{d_P}\left(Z_{r+1,n}, \sum_{i\leq \hat{K}_\varepsilon} e^{-n\xi_{p(i)}} Z_{r,n}^{e_{p(i)}}\right) \leq 2\varepsilon\,. \tag{4.3.27}$$

A fixed, finite number of paths therefore carries essentially all weight. We will now show that these paths are, with overwhelming probability, organised in a "tree-like fashion". Towards this goal, we go back to the original formulation

$$\sum_{i\leq \hat{K}_\varepsilon} e^{-n\xi_{p(i)}} Z_{r,n}^{e_i} = \sum_{i\leq \hat{K}_\varepsilon} e^{-n\xi_{p(i)}} \sum_{x\in\mathcal{V}^{\rightarrow}_{r+1,n}: x_2=e_{p(i)}} \exp\left(-n\sum_{l=2}^{r+1} \xi_{[x]_l}\right). \tag{4.3.28}$$

Note that any directed path of length $r+1$ with first step $(\mathbf{0}, e_i)$, can only share an edge with another path starting with $(\mathbf{0}, e_j)$, $i \neq j$ if it goes in the direction e_j at some point. By this observation for $i \neq j$ and $i, j \in \{1, .., n\}$

$$|\{x \in \mathcal{V}^{\rightarrow}_{r+1,n} : x_2 = e_i, \exists x' \in \mathcal{V}^{\rightarrow}_{r+1,n} \text{ s.t. } x'_2 = e_j \text{ and } x \cap x' \neq \emptyset\}| = O(n^{r-1}) \tag{4.3.29}$$

holds. Combining this fact with the observation

$$\mathbb{E}\exp\left(-n\sum_{l=2}^{r+1} \xi_{[x]_l}\right) = (n+1)^{-r} \tag{4.3.30}$$

we see that the total contribution of such paths converges in probability to zero, by Markov inequality, and swapping the intersecting summands for copies that are independent of paths with different start edge does not change the weak limit. The weak limit of (4.3.33) therefore coincides with the weak limit of

$$\sum_{i\leq\hat{K}_\varepsilon}\exp\left(-n\xi_{p(i)}\right)\sum_{x\in\mathcal{V}^{\rightarrow}_{r,n-1}}\exp\left(-n\sum_{l=1}^{r}\xi^{(p(i))}_{[x]_l}\right) \tag{4.3.31}$$

where $\xi^{(p(i))}_{[x]_l}=\xi_{[x]_l}$ if $[x]_l$ cannot be part of a path starting with $e_{p(j)}$ for some $j\neq i$ with $j\leq\hat{K}_\varepsilon$. On the other hand, the $\xi^{(p(i))}_{[x]_l}$'s are exponentially distributed and independent of each other for different $p(i)$ and or different $[x]_l$ as well as independent of all $(\xi_e)_{e\in\mathbb{E}_n}$. Finally, we realize that replacing

$$\exp\left(-n\sum_{l=1}^{r}\xi^{(p(i))}_{[x]_l}\right)\quad\text{by}\quad\exp\left(-(n-1)\sum_{l=1}^{r}\xi^{(p(i))}_{[x]_l}\right) \tag{4.3.32}$$

causes, by the restriction argument (4.3.5), an error which vanishes in probability. Collecting all changes and estimates, we have thus shown that the distribution of $Z_{r+1,n}$ is at most $2\varepsilon+o_n(1)$-Prohorov distance away from the weak limit of

$$\sum_{i\leq\hat{K}_\varepsilon}\exp\left(-n\xi_{p(i)}\right)Z^{(i)}_{r,n-1}, \tag{4.3.33}$$

where $Z^{(i)}_{r,n-1}, i\in\mathbb{N}$ are independent copies of $Z_{r,n-1}$. By assumption $Z_{r,n-1}$ converges weakly to Z_r and by Lemma 22 the smallest finitely many $n\xi$'s converge weakly to the first that many points of a PPP($\mathbb{1}_{\mathbb{R}^+}dx$). We conclude that the Prohorov distance of $Z_{r+1,n}$ and

$$\sum_{i\leq\hat{K}_\varepsilon}\exp\left(-\hat{\eta}_i\right)Z^{(i)}_{r}, \tag{4.3.34}$$

is at most by an in n vanishing sequence larger than 2ε. Checking using Markov inequality that the contibution of $i>\hat{K}_\varepsilon$ is vanishing in probability gives that

$$d_P\left(\mathcal{L}(Z_{r+1,n}),\mathcal{L}(Z_{r+1})\right)\rightarrow 0 \tag{4.3.35}$$

has to hold as $n\rightarrow\infty$. This finishes the induction, and the proof of the n-convergence is thus settled.

We move to the proof of the second claim of Proposition 21, the r-convergence. This will be done via a contraction argument on the space $\mathcal{P}_2$ of probability measures on $\mathbb{R}$

with finite second moment. To this end, denote again by $(\eta_i)_{i\in\mathbb{N}}$ a PPP$(\mathbb{1}_{\mathbb{R}^+}dx)$. Define

$$\begin{aligned} T : \mathcal{P}_2 &\to \mathcal{P}_2, \\ \mu &\mapsto \mathcal{L}\left(\sum_{i\in\mathbb{N}} e^{-\eta_i} X_i\right), \end{aligned} \tag{4.3.36}$$

where $(X_i)_{i\in\mathbb{N}}$ are independent and identically μ-distributed, and independent of η. Note that T is well-defined, i.e., we have that $T\mu$ has a finite second moment for all $\mu \in \mathcal{P}_2$ by applying the triangle inequality, $\mathbb{E}[\sum_{i\in\mathbb{N}} e^{-2\eta_i}] = 1/2$ and independence. Moreover, since $\mathbb{E}[\sum_{i\in\mathbb{N}} e^{-\eta_i}] = 1$ the map T does not change the first moment. Hence, for the subset

$$\mathcal{P}_{2,1} := \left\{\mu \in \mathcal{P}_2 : \int x \, d\mu = 1\right\}$$

the restriction of T to $\mathcal{P}_{2,1}$ maps to $\mathcal{P}_{2,1}$. By construction, it holds that

$$\mathcal{L}(Z_{r+1}) = T\mathcal{L}(Z_r). \tag{4.3.37}$$

We now endow $\mathcal{P}_2$ with the minimal L_2-distance ℓ_2, also called Wasserstein distance of order 2. For $\mu, \nu \in \mathcal{P}_2$ this is defined by

$$\ell_2(\mu, \nu) = \inf\{\|V - W\|_2 : \mathcal{L}(V) = \mu, \mathcal{L}(W) = \nu\},$$

where the infimum is taken over all probability distributions on $\mathcal{P}\times\mathcal{P}$ whose marginals are μ and ν respectively. Convergence in ℓ_2 implies weak convergence, $(\mathcal{P}_2, \ell_2)$ and $(\mathcal{P}_{2,1}, \ell_2)$ are complete metric spaces. For these topological properties and the existence of optimal couplings used below see, e.g., Ambrosio, Gigli and Savaré [3] or Villani [48]. Within the present setting, in order to prove the r-convergence it suffices to prove that

- The restriction of T to $\mathcal{P}_{2,1}$ is a strict ℓ_2-contraction.
- The standard exponential distribution is a fixed point of T restricted to $\mathcal{P}_{2,1}$.

We remark that T as a map on $\mathcal{P}_2$ has infinitely many fixed points and that our argument below also implies that these fixed points are exactly the exponential distributions with arbitrary parameter, their negatives, and the Dirac measure in 0. Uniqueness of the fixed point on $\mathcal{P}_{2,1}$ is immediate by Banach fixed point theorem and the strict contraction property.

Contractivity goes as follows. For $\mu, \nu \in \mathcal{P}_{2,1}$, let $(X_i, Y_i)_{i\in\mathbb{N}}$ be a sequence of independent *optimal* ℓ_2-couplings, which are also independent of η; optimal ℓ_2-couplings means here that the pair (X_i, Y_i) has marginal distributions μ and ν, and that it attains the infimum in the definition of ℓ_2. It then holds:

$$\ell_2(T\mu, T\nu)^2 \le \mathbb{E}\left[\left(\sum_{i\in\mathbb{N}} e^{-\eta_i}(X_i - Y_i)\right)^2\right]. \tag{4.3.38}$$

Remark that the off-diagonal terms on the right-hand side above vanish, since $X_i - Y_i$ has zero expectation. Using this, we thus obtain

$$\ell_2(T\mu, T\nu)^2 \leq \mathbb{E}\left[\sum_i e^{-2\eta_i}\right]\mathbb{E}\left[(X_1 - Y_1)^2\right] = \frac{1}{2}\ell_2(\mu,\nu)^2, \tag{4.3.39}$$

the last step follows by optimality of the coupling. This implies that the restriction of the map T to $\mathcal{P}_{2,1}$ is an ℓ_2-contraction.

It thus remains to prove that the standard exponential distribution is the fixed point of T in $\mathcal{P}_{2,1}$. This can be checked via Laplace transformation. Consider independent standard exponentials $X_1, X_2, ...$ which are also independent of η. For $t > 0$,

$$\begin{aligned}\mathbb{E}\left[\exp\left(-t\sum_{i=1}^{\infty} e^{-\eta_i}X_i\right)\right] &= \mathbb{E}\left[\exp\left(-\sum_{i=1}^{\infty}\ln\left(1+te^{-\eta_i}\right)\right)\right] \\ &= \exp\left(\int_0^{\infty}\frac{1}{1+te^{-x}} - 1dx\right) = \frac{1}{1+t},\end{aligned} \tag{4.3.40}$$

which is the Laplace transform of a standard exponential. This implies *ii)*. The r-convergence therefore immediately follows from Banach fixed point theorem.

□

4.3.2 Vanishing of the Chen-Stein term.

The goal here is to prove (4.2.11), namely that

$$\lim_{r\to\infty}\lim_{n\to\infty}\mathbb{E}d_{TV,\mathcal{F}_{r,n}}\left(\Xi_n(A), \mathrm{Poi}\left(\lambda_{r,n}(A)\right)\right) = 0. \tag{4.3.41}$$

This requires some additional notation. Let

$$\begin{aligned}\Sigma_{n,r} \equiv \Big\{ &(\pi,\pi') \in \Pi_{n,n}\times\Pi_{n,n} : \pi,\pi' \text{ have at least a common edge } e, \\ &e = (u,v) \in E, \{d(u,\mathbf{0}), d(v,\mathbf{0})\} \in [r, n-r)\Big\}.\end{aligned}$$

For paths $(\pi,\pi') \in \Pi_{n,n}\times\Pi_{n,n}$, we denote by $\pi\wedge\pi'$ their *overlap*, i.e. the number of edges shared by both paths. Working out the conditional Chen-Stein bound (4.2.5), we get

$$\begin{aligned}\mathbb{E}d_{TV,\mathcal{F}_{r,n}}\left(\Xi_n(A), \mathrm{Poi}\left(\lambda_n(A)\right)\right) \leq &\sum_{\pi\in\Pi_{n,n}}\mathbb{E}\left[\mathbb{E}[I_\pi(A)|\mathcal{F}_{r,n}]^2\right] \\ &+\sum_{\star}\mathbb{E}\left[\mathbb{E}[I_\pi(A)|\mathcal{F}_{r,n}]\mathbb{E}[I_{\pi'}(A)|\mathcal{F}_{r,n}]\right] \\ &+\sum_{\star}\mathbb{E}\left[\mathbb{E}[I_\pi(A)I_{\pi'}(A)|\mathcal{F}_{r,n}]\right],\end{aligned} \tag{4.3.42}$$

where $\sum_\star$ denotes summation over all $(\pi, \pi') \in \Sigma_{n,r} : 1 \leq \pi \wedge \pi' \leq n-2$.

We will prove that all three terms on the right-hand side of (4.3.42) vanish in the limit $n \to \infty$ first, and $r \to \infty$ next. As the proof is long and technical, we formulate the statements in the form of three Lemmata.

Lemma 24.

$$\lim_{r\to\infty}\lim_{n\to\infty}\sum_{\pi\in\Pi_{n,n}} \mathbb{E}\left[\mathbb{E}[I_\pi(A)|\mathcal{F}_{r,n}]^2\right] = 0\,.$$

Lemma 25.

$$\lim_{r\to\infty}\lim_{n\to\infty}\sum_{\star} \mathbb{E}\left[\mathbb{E}[I_\pi(A)|\mathcal{F}_{r,n}]\mathbb{E}[I_{\pi'}(A)|\mathcal{F}_{r,n}]\right] = 0\,.$$

Lemma 26.

$$\lim_{r\to\infty}\lim_{n\to\infty}\sum_{\star} \mathbb{E}\left[\mathbb{E}[I_\pi(A)I_{\pi'}(A)|\mathcal{F}_{r,n}]\right] = 0\,.$$

The first contribution is easily taken care of:

Proof of Lemma 24. By symmetry we have that

$$\sum_{\pi\in\Pi_{n,n}} \mathbb{E}[\mathbb{E}[I_\pi(A)|\mathcal{F}_{r,n}]^2] = n!\mathbb{E}[\mathbb{E}[I_{\pi^*}(A)|\mathcal{F}_{r,n}]^2]\,, \tag{4.3.43}$$

where $\pi^* \in \Pi_{n,n}$ is arbitrary. It thus follows from the tail-estimates of Lemma 19 that

$$\begin{aligned}(4.3.43) &= n!\int_0^{1+\frac{a}{n}}\Big(1+K(1+\frac{a}{n}-x,n-2r)\Big)^2\frac{e^{-2(1+\frac{a}{n})+x}(1+\frac{a}{n}-x)^{2n-4r}x^{2r-1}}{(n-2r)!^2(2r-1)!}dx\\ &\leq n!\int_0^{1+\frac{a}{n}}\left(1+e^{(1+\frac{a}{n})}\frac{(1+\frac{a}{n})}{n-2r}\right)^2\frac{e^{-(1+\frac{a}{n})}(1+\frac{a}{n})^{2n-4r}(1+\frac{a}{n})^{2r-1}}{(n-2r)!^2(2r-1)!}dx\\ &= \frac{n!e^{2a}}{(n-2r)!^2(2r-1)!}(1+o_n(1))\,.\end{aligned} \tag{4.3.44}$$

Since the right-hand side of (4.3.44) is vanishing in the large n-limit, the proof of Lemma 24 is concluded. □

Lemma 25 and 26 require more work. In particular, we will make heavy use of the following combinatorial estimates, which have been established by Fill and Pemantle [29] (see Lemma 2.3, 2.4 and 2.5 p. 598):

Proposition 27 (Path counting)**.** *Let π' be any reference path on the n-dim hypercube connecting $\mathbf{0}$ and $\mathbf{1}$. Denote by $f(n,k)$ the number of paths π that share precisely k edges $(k \geq 1)$ with π'. Finally, shorten $\mathfrak{n}_\mathfrak{e} \equiv n - 5e(n+3)^{2/3}$.*

- *For any $K(n) = o(n)$ as $n \to \infty$,*
$$f(n,k) \leq (1+o(1))(k+1)(n-k)! \tag{4.3.45}$$
uniformly in k for $k \leq K(n)$.

- *Suppose $k \leq \mathfrak{n}_\mathfrak{c}$. Then, for n large enough,*
$$f(n,k) \leq n^6 (n-k)!\,. \tag{4.3.46}$$

- *Suppose $k \geq \mathfrak{n}_\mathfrak{c}$. Then, for n large enough,*
$$f(n,k) \leq \left(2n^{\frac{7}{8}}\right)^{n-k} (n-k+1)\,. \tag{4.3.47}$$

Proof of Lemma 25. Here and below, $\kappa_a > 0$ will denote a universal constant not necessarily the same at different occurences, and which depends solely on a. By symmetry,

$$\sum_{\star} \mathbb{E}[I_\pi(A) I_{\pi'}(A)] = n! \sum_{\star,\star} \mathbb{E}[I_{\pi^*}(A) I_{\pi'}(A)] \tag{4.3.48}$$

where $\pi^* \in \Pi_{n,n}$ is arbitrary and $\sum_{\star,\star}$ standing for summation over

$$\pi' \in \Pi_{n,n} : (\pi^*, \pi') \in \Sigma_{n,r}, 1 \leq \pi^* \wedge \pi' \leq n-2.$$

Let $k \in \{1, n-2\}$ and $\pi' \in \Pi_{n,n}, \pi^* \wedge \pi' = k$. Splitting X_{π^*} and $X_{\pi'}$ into common/non-common edges, we obtain

$$\begin{aligned} \mathbb{E}[I_{\pi^*}(A) I_{\pi'}(A)] &= \mathbb{P}\left(X_{\pi^*} \leq 1 + \frac{a}{n}, X_{\pi'} \leq 1 + \frac{a}{n}\right) \\ &= \int_{\mathbb{R}} \mathbb{P}\left(x + X_{n-k} \leq 1 + \frac{a}{n}, x + X'_{n-k} \leq 1 + \frac{a}{n} \mid X_k = x\right) \mathbb{P}(X_k \in dx)\,. \end{aligned} \tag{4.3.49}$$

In the above, X_{n-k} and X'_{n-k} correspond to the compound weights of the non-common edges, these are Gamma$(n-k,1)$-distributed random variables; X_k corresponds to the weight of the common edges, this is a Gamma$(k,1)$-distributed random variable. By construction, X_{n-k}, X'_{n-k} and X_k are independent. All in all, with the tail-estimate of Lemma 19, one has

$$\begin{aligned} \mathbb{E}[I_{\pi^*}(A) I_{\pi'}(A)] &= \int_0^{+\infty} \mathbb{P}\left(x + X_{n-k} \leq 1 + \frac{a}{n}\right)^2 \frac{e^{-x} x^{k-1}}{(k-1)!} dx \\ &\leq \frac{\kappa_a}{(n-k)!^2} \int_0^{1+\frac{a}{n}} \left(1 + \frac{a}{n} - x\right)^{2(n-k)} \frac{x^{k-1}}{(k-1)!} dx\,. \end{aligned} \tag{4.3.50}$$

Integration by parts then yields

$$\int_0^{1+\frac{a}{n}} \left(1+\frac{a}{n}-x\right)^{2(n-k)} x^{k-1}dx \leq \kappa_a \frac{(k-1)!(2(n-k))!}{(2n-k)!}. \tag{4.3.51}$$

and therefore

$$E[I_{\pi^*}(A)I_{\pi'}(A)] \leq \kappa_a \frac{(2(n-k))!}{(2n-k)!(n-k)!^2}. \tag{4.3.52}$$

Denoting by $f(n,k,r)$ the number of paths π' that share precisely k edges ($1 \leq k \leq n-2$) with π^* and that satisfy $(\pi', \pi^*) \in \Sigma_{n,r}$, we thus have that

$$\begin{aligned} n! \sum_{\star,\star} \mathbb{E}[I_{\pi^*}(A)I_{\pi'}(A)] &= n! \sum_{k=1}^{n-2} f(n,k,r)\mathbb{E}[I_{\pi^*}(A)I_{\pi'}(A)] \\ &\overset{(4.3.54)}{\leq} \kappa_a \sum_{k=1}^{n-2} \frac{f(n,k,r)}{(n-k)!} \times \frac{n!(2(n-k))!}{(n-k)!(2n-k)!} \end{aligned} \tag{4.3.53}$$

hence

$$n! \sum_{\star,\star} \mathbb{E}[I_{\pi^*}(A)I_{\pi'}(A)] \leq \kappa_a \sum_{k=1}^{n-2} \frac{f(n,k,r)}{(n-k)!} \times \frac{(1-\frac{k}{n})^{n-k}}{2^k(1-\frac{k}{2n})^{2n-k}}, \tag{4.3.54}$$

by Stirling approximation. To lighten notation, remark that with $\gamma \equiv k/n \in [0,1]$, the second factor in the last sum above can be written as

$$\frac{(1-\frac{k}{n})^{n-k}}{2^k(1-\frac{k}{2n})^{2n-k}} = \left(\frac{(4(1-\gamma))^{(1-\gamma)}}{(2-\gamma)^{(2-\gamma)}}\right)^n \equiv g(\gamma)^n. \tag{4.3.55}$$

With this, (4.3.53) takes the form

$$n! \sum_{\star,\star} \mathbb{E}[I_{\pi^*}(A)I_{\pi'}(A)] \leq \kappa_a \sum_{k=1}^{n-2} \frac{f(n,k,r)}{(n-k)!} \times g\left(\frac{k}{n}\right)^n. \tag{4.3.56}$$

The following observation, whose elementary proof is postponed to the end of this section, will be useful.

Fact 1. *The function $g : [0,1] \to \mathbb{R}_+$ defined in* (4.3.55) *is increasing on* $[2/3, 1)$*. Furthermore,*

$$\forall \gamma \leq 2/3 : \; g(\gamma) \leq \left(\frac{3}{4}\right)^\gamma. \tag{4.3.57}$$

In view of Proposition 27, recalling that $\mathfrak{n}_{\mathfrak{e}} = n - 5e(n+3)^{2/3}$ and with

$$C \equiv \frac{7}{\ln(4/3)}\,, \tag{4.3.58}$$

we split the sum on the right-hand side of (4.3.56) into three regimes, to wit:

$$\left(\sum_{k=1}^{C\ln(n)} + \sum_{k=C\ln(n)+1}^{\mathfrak{n}_{\mathfrak{e}}} + \sum_{k=\mathfrak{n}_{\mathfrak{e}}+1}^{n-2} \right) \frac{f(n,k,r)}{(n-k)!} \times g\left(\frac{k}{n}\right)^n . \tag{4.3.59}$$

Concerning the first sum, by (4.3.57) it holds

$$\begin{aligned} \sum_{k=1}^{C\ln(n)} \frac{f(n,k,r)}{(n-k)!} g\left(\frac{k}{n}\right)^n &\leq \sum_{k=1}^{C\ln(n)} \frac{f(n,k,r)}{(n-k)!} \left(\frac{3}{4}\right)^k \\ &\leq \sum_{k=1}^{r-1} \frac{f(n,k,r)}{(n-r+1)!} \left(\frac{3}{4}\right)^k + \sum_{k=r}^{C\ln(n)} \frac{f(n,k)}{(n-k)!} \left(\frac{3}{4}\right)^k \\ &\leq \sum_{k=1}^{r-1} \frac{f(n,k,r)}{(n-r+1)!} \left(\frac{3}{4}\right)^k + \kappa_a \sum_{k=r}^{C\ln(n)} (k+1) \left(\frac{3}{4}\right)^k , \end{aligned} \tag{4.3.60}$$

by Proposition 27.

The function $f(n,k,r)$ counts the number of paths π' that share precisely k edges $(1 \leq k \leq n-2)$ with π^* and that satisfy $(\pi', \pi^*) \in \Sigma_{n,r}$: we claim that

$$f(n,k,r) \leq r!(n-r-1)!n. \tag{4.3.61}$$

To see this, recall that the vertices of the hypercube stand in correspondence with the standard basis of $\mathbb{R}^n$: every edge is parallel to some unit vector e_j, where e_j connects $(0,\ldots,0)$ to $(0,\ldots,0,1,0,\ldots,0)$ with a 1 in position j. We identify a directed path π from $\mathbf{0}$ to $\mathbf{1}$ by a permutation of $12\ldots n$, say $\pi_1\pi_2\ldots\pi_n$. π_l is giving the direction the path π goes in step l, hence after i steps the path $\pi_1\pi_2\ldots\pi_n$ is at vertex $\sum_{j\leq i} e_{\pi_j}$. (By a slight abuse of notation, π_1 will refer here below to a number between, 1 and n). Let now π^* be the reference path, say $\pi^* = 12...n$. We set $u_i = l$ if the l-th traversed edge by π' is the i-th shared edge of π' and π^*, setting by convention $r_0 = 0$ and $r_{k+1} = n+1$. Shorten then $\mathbf{u} \equiv \mathbf{u}(\pi') = (u_0, ..., u_{k+1})$, and $s_i \equiv u_{i+1} - u_i$, $i = 0, ..., k$. For any sequence $\mathbf{u}_0 = (u_0, ..., u_{k+1})$ with $0 = u_0 < u_1 < ... < u_k < u_{k+1} = n+1$, let $C(\mathbf{u}_0)$ denote the number of paths π' with $\mathbf{u}(\pi') = \mathbf{u}_0$. Since the values $\pi'_{u_i+1}, ..., \pi'_{u_i+s_i-1}$ must be a permutation of $\{u_i+1, ..., u_i+s_i-1\}$, one easily sees that $C(\mathbf{u}) \leq G(\mathbf{u})$, where

$$G(\mathbf{u}) = \prod_{i=0}^{k} (s_i - 1)!\,. \tag{4.3.62}$$

We also observe that two such paths must have a common edge in the *middle region* $(\pi', \pi^*) \in \Sigma_{n,r}$. Let e be such an edge: as it turns out, this is quite restrictive. Indeed, it implies that there exists $u_j \in \{r+1, n-r\}$ for $j \in \{1, ..., k\}$. In virtue of (4.3.62) and log-convexity of factorials, one has at most $r!(n-r-1)!$ paths π' sharing the edge e with the reference-path π^*, and at most $\binom{n}{1} = n$ ways to choose this edge: combining all this settles (4.3.61).

It follows that

$$\sum_{k=1}^{C\ln(n)} \frac{f(n,k,r)}{(n-k)!} g\left(\frac{k}{n}\right)^n \leq \sum_{k=1}^{r-1} \frac{r!(n-r-1)!n}{(n-r+1)!} \left(\frac{3}{4}\right)^k + \kappa_a \sum_{k=r}^{+\infty} (k+1) \left(\frac{3}{4}\right)^k. \tag{4.3.63}$$

The first sum above clearly tends to 0 as $n \to \infty$, whereas the second sum vanishes when $r \to \infty$: the first regime in (4.3.59) therefore yields no contribution in the double limit.

As for the second regime, by Proposition 27,

$$\begin{aligned} \sum_{k=C\ln(n)}^{\mathfrak{n}_\mathfrak{e}} \frac{f(n,k,r)}{(n-k)!} g\left(\frac{k}{n}\right)^n &\leq \sum_{k=C\ln(n)}^{\mathfrak{n}_\mathfrak{e}} \frac{f(n,k)}{(n-k)!} g\left(\frac{k}{n}\right)^n \\ &\leq n^6 \sum_{k=C\ln(n)}^{\mathfrak{n}_\mathfrak{e}} g\left(\frac{k}{n}\right)^n \\ &= n^6 \left(\sum_{k=C\ln(n)}^{2n/3} g\left(\frac{k}{n}\right)^n + \sum_{k=2n/3+1}^{\mathfrak{n}_\mathfrak{e}} g\left(\frac{k}{n}\right)^n \right). \end{aligned} \tag{4.3.64}$$

As pointed out in Fact 1, the g-function is increasing on $[2/3, 1)$, whereas on the "complement" (4.3.57) holds: these observations, together with (4.3.64) imply that

$$\begin{aligned} \sum_{k=C\ln(n)}^{\mathfrak{n}_\mathfrak{e}} \frac{f(n,k,r)}{(n-k)!} g\left(\frac{k}{n}\right)^n &\leq n^6 \left(\sum_{k=C\ln(n)}^{2n/3} \left(\frac{3}{4}\right)^k + \sum_{k=2n/3+1}^{\mathfrak{n}_\mathfrak{e}} g\left(\frac{\mathfrak{n}_\mathfrak{e}}{n}\right)^n \right) \\ &\leq 4n^6 \left(\frac{3}{4}\right)^{C\ln(n)} + n^7 g\left(\frac{\mathfrak{n}_\mathfrak{e}}{n}\right)^n \\ &= 4\exp\left\{(6 + C\ln(3/4))\ln(n)\right\} + n^7 g\left(\frac{\mathfrak{n}_\mathfrak{e}}{n}\right)^n. \end{aligned} \tag{4.3.65}$$

In virtue of the choice (4.3.58) we have that $6 + C\ln(3/4) = -1$, hence

$$(4.3.65) = o_n(1) + n^7 g\left(\frac{\mathfrak{n}_\mathfrak{e}}{n}\right)^n. \tag{4.3.66}$$

By definition of the g-function (4.3.55) and $\mathfrak{n}_\mathfrak{e}$, it holds:

$$\begin{aligned} g\left(\frac{\mathfrak{n}_\mathfrak{e}}{n}\right)^n &= \frac{(1-\frac{\mathfrak{n}_\mathfrak{e}}{n})^{n-\mathfrak{n}_\mathfrak{e}}}{2^{\mathfrak{n}_\mathfrak{e}}(1-\frac{\mathfrak{n}_\mathfrak{e}}{2n})^{2n-\mathfrak{n}_\mathfrak{e}}} \\ &= \left(\frac{5e(n+3)^{\frac{2}{3}}}{n}\right)^{5e(n+3)^{\frac{2}{3}}} 2^{10e(n+3)^{\frac{2}{3}}}\left(1+\frac{5e(n+3)^{\frac{2}{3}}}{n}\right)^{-n-5e(n+3)^{2/3}}. \end{aligned} \tag{4.3.67}$$

Notice that

$$1+\frac{5e(n+3)^{\frac{2}{3}}}{n} \geq 1 \text{ and } (n+3)^{\frac{2}{3}} \leq 2n^{\frac{2}{3}} \text{ for } n \geq 3, \tag{4.3.68}$$

thus

$$g\left(\frac{\mathfrak{n}_\mathfrak{e}}{n}\right)^n \leq \left(\frac{40e}{n^{1/3}}\right)^{10en^{\frac{2}{3}}} = o(n^{-7}), \tag{4.3.69}$$

implying that the second regime in (4.3.59) yields no contribution in the limit $n \to +\infty$.

As for the third, and last regime, by definition of the g-function,

$$\begin{aligned} \sum_{k=\mathfrak{n}_\mathfrak{e}+1}^{n-2} \frac{f(n,k,r)}{(n-k)!} g\left(\frac{k}{n}\right)^n &\leq \sum_{k=\mathfrak{n}_\mathfrak{e}+1}^{n-2} \frac{f(n,k)}{(n-k)!}\frac{(1-\frac{k}{n})^{n-k}}{2^k(1-\frac{k}{2n})^{2n-k}} \\ &\leq \sum_{k=\mathfrak{n}_\mathfrak{e}+1}^{n-2} \frac{\left(2n^{\frac{7}{8}}\right)^{n-k}(n-k+1)}{(n-k)!}\frac{(1-\frac{k}{n})^{n-k}}{2^k(1-\frac{k}{2n})^{2n-k}}, \end{aligned} \tag{4.3.70}$$

the last step in virtue of Proposition 27. By change of variable, $n-k \mapsto u$, we get

$$\begin{aligned} \sum_{k=\mathfrak{n}_\mathfrak{e}+1}^{n-2} \frac{\left(2n^{\frac{7}{8}}\right)^{n-k}(n-k+1)}{(n-k)!}\frac{(1-\frac{k}{n})^{n-k}}{2^k(1-\frac{k}{2n})^{2n-k}} &= \sum_{u=2}^{5e(n+3)^{\frac{2}{3}}-1}\left(\frac{8un^{\frac{7}{8}}}{n}\right)^u \frac{(u+1)}{(1+\frac{u}{n})^{n+u}u!} \\ &\leq \sum_{u=2}^{\infty}\left(\frac{8e}{n^{\frac{1}{8}}}\right)^u (u+1) \end{aligned} \tag{4.3.71}$$

by Stirling's approximation. It thus follows that the contribution of the third and last regime in (4.3.59) also vanishes as $n \to +\infty$. The proof of Lemma 25 is concluded. □

We finally provide the elementary

Proof of Fact 1. The sign of g' is given by the sign of

$$\frac{d}{d\gamma}\left(\ln(4-4\gamma)(1-\gamma) - \ln(2-\gamma)(2-\gamma)\right) = \ln\left(\frac{2-\gamma}{4-4\gamma}\right).$$

It follows that $g'(\gamma) \leq 0 \ \forall \gamma \leq 2/3$ and $g'(\gamma) \geq 0 \ \forall \gamma \geq 2/3$. Furthermore, since

$$1 - \gamma \leq \left(1 - \frac{\gamma}{2}\right)^2,$$

we have

$$g(\gamma) = \frac{(4(1-\gamma))^{(1-\gamma)}}{(2-\gamma)^{(2-\gamma)}} \leq (2-\gamma)^{-\gamma} \leq \left(\frac{3}{4}\right)^{\gamma}, \tag{4.3.72}$$

$\forall \gamma \leq 2/3$, settling (4.3.57). □

Proof of Lemma 26. Again by symmetry,

$$\begin{aligned} \sum_{\star} & \mathbb{E}[\mathbb{E}[I_\pi(A)|\mathcal{F}_{r,n}]\mathbb{E}[I_{\pi'}(A)|\mathcal{F}_{r,n}]] \\ &= n! \sum_{\star,\star} \mathbb{E}[\mathbb{E}[I_{\pi^*(A)}|\mathcal{F}_{r,n}]\mathbb{E}[I_{\pi'}(A)|\mathcal{F}_{r,n}]] \\ &= n! \sum_{\star,\star} \mathbb{E}\left[\mathbb{P}\left(X_{\pi^*} \leq 1 + \frac{a}{n}|\mathcal{F}_{r,n}\right)\mathbb{P}\left(X_{\pi'} \leq 1 + \frac{a}{n}|\mathcal{F}_{r,n}\right)\right], \end{aligned} \tag{4.3.73}$$

where $\pi^* \in \Pi_{n,n}$ and $\sum_{\star,\star}$ stands for summation over

$$\pi' \in \Pi_{n,n}, (\pi^*, \pi') \in \Sigma_{n,r} : 1 \leq \pi^* \wedge \pi' \leq n-2.$$

We split this sum into two parts, the first contribution will stem from paths π' which share less than $2r$ edges with π^*, in which case π' and π^* are almost independent when n tends to $+\infty$; the second contribution will come from the (fewer) paths which are more correlated with π^*. Precisely, we write

$$\begin{aligned} (4.3.73) =& n! \sum_{\star,\star,1} \mathbb{E}\left[\mathbb{P}\left(X_{\pi^*} \leq 1 + \frac{a}{n}|\mathcal{F}_{r,n}\right)\mathbb{P}\left(X_{\pi'} \leq 1 + \frac{a}{n}|\mathcal{F}_{r,n}\right)\right] \\ &+ n! \sum_{\star,\star,2} \mathbb{E}\left[\mathbb{P}\left(X_{\pi^*} \leq 1 + \frac{a}{n}|\mathcal{F}_{r,n}\right)\mathbb{P}\left(X_{\pi'} \leq 1 + \frac{a}{n}|\mathcal{F}_{r,n}\right)\right] \end{aligned} \tag{4.3.74}$$

while $\sum_{\star,\star,1}$ denotes summation over

$$\pi' \in \Pi_{n,n}, (\pi^*, \pi') \in \Sigma_{n,r} : 1 \leq \pi^* \wedge \pi' \leq 2r,$$

whereas $\sum_{\star,\star,2}$ stands for summation over

$$\pi' \in \Pi_{n,n}, (\pi^*, \pi') \in \Sigma_{n,r} : 2r+1 \leq \pi^* \wedge \pi' \leq n-2.$$

We now proceed to estimate these two sums: in the first case we will exploit the fact that the involved paths are almost independent. To see how this goes, let

$$\begin{aligned} C_{r,n,\pi'} \equiv \Big\{ & e = (u,v) \in E_n, \min\{d(u,\mathbf{0}), d(v,\mathbf{0})\} \in [0,r) \cup [n-r,n), \\ & \text{e is a common edge of } \pi' \text{ and } \pi^* \Big\}, \end{aligned} \tag{4.3.75}$$

and denote by $\#C \equiv |C_{r,n,\pi'}|$ the cardinality of this set. We now make the following observations:

- $\#C = 0$ (i.e. $C_{r,n,\pi'} = \emptyset$) implies that π' and π^* are, conditionally upon $\mathcal{F}_{r,n}$, independent.
- If $\#C > 0$, by positivity of exponentials,

$$\begin{aligned} \mathbb{P}\left(X_{\pi'} \le 1 + \frac{a}{n} | \mathcal{F}_{r,n}\right) &\le \mathbb{P}\left(X_{\pi'} - \sum_{e \in C_{r,n,\pi'}} \xi_e \le 1 + \frac{a}{n} \middle| \mathcal{F}_{r,n}\right) \\ &= \mathbb{P}\left(X_{n-\#C} \le 1 + \frac{a}{n} \middle| \mathcal{F}_{r,n}\right) , \end{aligned} \tag{4.3.76}$$

  where $X_{n-\#C}$ is a Gamma$(n-\#C, 1)$-distributed random variable which is, conditionally upon $\mathcal{F}_{r,n}$, independent of X_{π^*}.

Altogether,

$$\begin{aligned} n! \sum_{\star,\star,1} \mathbb{E}\left[\mathbb{P}\left(X_{\pi^*} \le 1 + \frac{a}{n} | \mathcal{F}_{r,n}\right) \mathbb{P}\left(X_{\pi'} \le 1 + \frac{a}{n} | \mathcal{F}_{r,n}\right)\right] \\ \le n! \mathbb{P}\left(X_{\pi^*} \le 1 + \frac{a}{n}\right) \sum_{\star,\star,1} \mathbb{P}\left(X_{n-\#C} \le 1 + \frac{a}{n}\right) . \end{aligned} \tag{4.3.77}$$

Convergence of the intensity functions (4.2.1), implies that the first term $n!\mathbb{P}\left(X_{\pi_1} \le 1 + \frac{a}{n}\right)$ converges; in particular, it remains bounded as $n \to \infty$. It therefore suffices to prove that $\sum_{\star,\star,1} \mathbb{P}\left(X_{n-\#C} \le 1 + \frac{a}{n}\right)$ tends to 0 in the double limit. To see this, denote by $f(n,k,r)$ the number of paths π' that share precisely k edges ($1 \le k \le n-2$) with π^* and with $(\pi', \pi^*) \in \Sigma_{n,r}$. We then have

$$\begin{aligned} \sum_{\star,\star,1} \mathbb{P}\left(X_{n-\#C} \le 1 + \frac{a}{n}\right) &= \sum_{k=1}^{2r} f(n,k,r) \mathbb{P}\left(X_{n-\#C} \le 1 + \frac{a}{n}\right) \\ &\le \sum_{k=1}^{2r} f(n,k) \mathbb{P}\left(X_{n-\#C} \le 1 + \frac{a}{n}\right) , \end{aligned} \tag{4.3.78}$$

where $f(n,k)$ is the number of paths π' that share precisely $k \ge 1$ edges with π^*. By the tail-estimates from Lemma 19,

$$\mathbb{P}\left(X_{n-\#C} \le 1 + \frac{a}{n}\right) \le \frac{\kappa_a}{(n-\#C)!} \le \frac{\kappa_a}{(n-k+1)!} . \tag{4.3.79}$$

The second inequality holds since two paths in $\Sigma_{n,r}$ must share an edge in the *complement* of $C_{r,n,\pi'}$. Using (4.3.79) and Proposition 27 we obtain

$$\sum_{\star,\star,1} \mathbb{P}\left(X_{n-\#C} \le 1 + \frac{a}{n}\right) \le \kappa_a \sum_{k=1}^{2r} \frac{(n-k)!(k+1)}{(n-k+1)!} , \tag{4.3.80}$$

which vanishes as $n \to \infty$, the first sum in (4.3.74) therefore yields a vanishing contribution. As for the second sum, by Cauchy-Schwarz,

$$n! \sum_{\star,\star,2} \mathbb{E}[\mathbb{E}[I_{\pi^*(A)}|\mathcal{F}_{r,n}]\mathbb{E}[I_{\pi'}(A)|\mathcal{F}_{r,n}]] \leq n! \sum_{\star,\star,2} \mathbb{E}\left[\mathbb{P}\left(X_{\pi'} \leq 1+\frac{a}{n}|\mathcal{F}_{r,n}\right)^2\right]. \tag{4.3.81}$$

By the tail-estimates from Lemma 19, for the expectation on the right-hand side above it holds

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{P}\left(X_{\pi'} \leq 1+\frac{a}{n}|\mathcal{F}_{r,n}\right)^2\right] \\
&\quad = \int_0^{1+\frac{a}{n}} \left(1+K(1+\frac{a}{n}-x, n-2r)\right)^2 \frac{e^{-2(1+\frac{a}{n})+x}(1+\frac{a}{n}-x)^{2n-4r}x^{2r-1}}{(n-2r)!^2(2r-1)!} dx \qquad (4.3.82)\\
&\quad \leq \frac{\kappa_a}{(n-2r)!^2(2r-1)!}\int_0^{1+\frac{a}{n}} \left(1+\frac{a}{n}-x\right)^{2n-4r} x^{2r-1} dx.
\end{aligned}$$

Integration by parts then yields

$$\int_0^{1+\frac{a}{n}} \left(1+\frac{a}{n}-x\right)^{2n-4r} x^{2r-1} dx \qquad \leq \kappa_a \frac{(2n-4r)!(2r-1)!}{(2n-2r)!}, \tag{4.3.83}$$

Using (4.3.82) and (4.3.83) we get

$$(4.3.81) \leq \kappa_a \sum_{k=2r+1}^{n-2} \frac{f(n,k,r)}{(n-2r)!} \frac{n!(2n-4r)!}{(n-2r)!(2n-2r)!}. \tag{4.3.84}$$

It clearly holds that

$$\frac{n!(2n-4r)!}{(n-2r)!(2n-2r)!} \leq 1, \tag{4.3.85}$$

hence

$$\begin{aligned}
(4.3.84) &\leq \sum_{k=2r+1}^{n-2} \frac{f(n,k,r)}{(n-2r)!} \\
&= \left(\sum_{k=2r+1}^{2r+7} + \sum_{k=2r+8}^{\mathfrak{n}_\mathfrak{c}} + \sum_{k=\mathfrak{n}_\mathfrak{c}+1}^{n-2}\right) \frac{f(n,k,r)}{(n-2r)!} \qquad (4.3.86)\\
&=: (A)+(B)+(C).
\end{aligned}$$

By Proposition 27, and worst-case estimates, the following upperbounds hold:

$$\begin{aligned}
(A) &\leq \sum_{k=2r+1}^{2r+7} \frac{(k+1)(n-k)!}{(n-2r)!} \leq \kappa_a \frac{7(2r+8)(n-2r-1)!}{(n-2r)!} \\
(B) &\leq \sum_{k=2r+8}^{\mathfrak{n}_\mathfrak{c}} \frac{n^6(n-k)!}{(n-2r)!} \leq n^6 \sum_{k=2r+8}^{\mathfrak{n}_\mathfrak{c}} \frac{(n-k)!}{(n-2r)!} \leq n^7 \frac{(n-2r-8)!}{(n-2r)!} \\
(C) &\leq \sum_{k=\mathfrak{n}_\mathfrak{c}+1}^{n-2} \frac{\left(2n^{7/8}\right)^{n-k}(n-k+1)}{(n-2r)!} \leq \frac{n^2\left(2n^{7/8}\right)^{5e(n+3)^{2/3}}}{(n-2r)!} \,.
\end{aligned} \tag{4.3.87}$$

All three terms are clearly vanishing in the limit $n \to \infty$. This implies that the second sum in (4.3.74) yields no contribution, and the proof of Lemma 26 is thus concluded. □

Appendix: the conditional Chein-Stein method

All random variables in the course of the proof are defined on the same probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathcal{F} \subset \mathscr{F}$ be a sigma algebra, I is a finite (deterministic) set, and $(X_i)_{i\in I}$ a family of Bernoulli random variables. We set

$$W \equiv \sum_{i\in I} X_i, \qquad \lambda \equiv \sum_{i\in I} \mathbb{E}(X_i|\mathcal{F})\,.$$

Since the claim is trivial for $\lambda = 0$ we assume $\lambda > 0$ from here onwards. Additionally we denote by $\widehat{W}$ a random variable which is, conditionally upon $\mathcal{F}$, Poi(λ)-distributed, i.e.

$$\mathbb{P}(\widehat{W} = k|\mathcal{F})(\omega) = \frac{\lambda(\omega)^k}{k!} e^{-\lambda(\omega)}. \tag{4.3.88}$$

(To lighten notation, we will omit henceforth the ω-dependence). Assume to be given a bounded, $\mathcal{F}$-measurable (possibly random) real-valued function f which satisfies

$$\mathbb{E}(f(\widehat{W})|\mathcal{F}) = 0,$$

and define $g_f : \mathbb{N} \to \mathbb{R}$ by

$$g_f(0) \equiv 0, \quad g_f(n) \equiv \frac{(n-1)!}{\lambda^n} \sum_{k=0}^{n-1} \frac{f(k)\lambda^k}{k!} \quad n > 0\,. \tag{4.3.89}$$

We claim that g_f is $\mathcal{F}$-measurable, bounded, and satisfies the following identities:

$$f(n) = \lambda g_f(n+1) - n g_f(n), \qquad n \geq 0\,, \tag{4.3.90}$$

and

$$g_f(n) = -\frac{(n-1)!}{\lambda^n} \sum_{k=n}^{\infty} \frac{f(k)\lambda^k}{k!} \qquad n > 0. \tag{4.3.91}$$

Measurability and first identity follow steadily from the definition. The second identity follows from the fact that $\mathbb{E}(f(\widehat{W})|\mathcal{F}) = 0$, whereas boundedness follows from the integral representation of the Taylor rest-term of the exponential function,

$$| g_f(n) | \leq \frac{(n-1)! \max_{k\in\mathbb{N}} | f(k) |}{\lambda^n} \int_0^\lambda \frac{t^{n-1}}{(n-1)!} e^t dt \leq \frac{\max_{k\in\mathbb{N}} | f(k) | e^\lambda}{n}. \tag{4.3.92}$$

Let now $A \subset \mathbb{N}_0$, and consider the function

$$f_{A,\lambda}(n) \equiv \mathbb{1}_{n\in A} - \mathbb{P}(\widehat{W} \in A|\mathcal{F}), \quad n \in \mathbb{N}. \tag{4.3.93}$$

This is clearly a bounded, $\mathcal{F}$-measurable function which satisfies $\mathbb{E}(f_{A,\lambda}(\widehat{W})|\mathcal{F}) = 0$. Therefore, by the above and in particular (4.3.90), there exists a bounded $\mathcal{F}$-measurable function, denoted by $g_{A,\lambda}$, which satisfies

$$\mathbb{1}_{n\in A} - \mathbb{P}(\widehat{W} \in A|\mathcal{F}) = \lambda g_{A,\lambda}(n+1) - n g_{A,\lambda}(n), \tag{4.3.94}$$

almost surely for any $n \in \mathbb{N}$. It follows that

$$\mathbb{1}_{W\in A} - \mathbb{P}(\widehat{W} \in A|\mathcal{F}) = \lambda g_{A,\lambda}(W+1) - W g_{A,\lambda}(W). \tag{4.3.95}$$

Taking conditional expectations thus yields

$$\begin{aligned} \mathbb{P}(W \in A|\mathcal{F}) - \mathbb{P}(\widehat{W} \in A|\mathcal{F}) &= \lambda \mathbb{E}(g_{A,\lambda}(W+1)|\mathcal{F}) - \mathbb{E}(W g_{A,\lambda}(W)|\mathcal{F}) \\ &= \sum_{i\in I} \mathbb{E}(X_i|\mathcal{F})\mathbb{E}(g_{A,\lambda}(W+1)|\mathcal{F}) - \mathbb{E}(X_i g_{A,\lambda}(W)|\mathcal{F}). \end{aligned} \tag{4.3.96}$$

Consider now the random subset

$$N_i \equiv \{j \in I \setminus \{i\} : \ X_j \text{ and } X_i \text{ are not conditionally independent given } \mathcal{F}\},$$

and denote by $S^{(i)}$ a random variable which is distributed like $\sum_{j\in N_i} X_j$ conditionally upon $\mathcal{F}$ and $\{X_i = 1\}$, i.e.

$$\mathbb{P}(S^{(i)} = k|\mathcal{F}) = \mathbb{P}\left(\sum_{j\in N_i} X_j = k, X_i = 1 \Big| \mathcal{F}\right) \Big/ \mathbb{P}(X_i = 1|\mathcal{F}). \tag{4.3.97}$$

if $\mathbb{P}(X_i = 1|\mathcal{F}) > 0$, and arbitrarily defined otherwise.

We remark that X_i and $(X_j)_{j\in(N_i\cup\{i\})^c}$ are conditionally on $\mathcal{F}$ independent. Therefore

$$\mathbb{E}(X_i g_{A,\lambda}(W)|\mathcal{F}) = \mathbb{P}(X_i=1|\mathcal{F})\mathbb{E}\left[g_{A,\lambda}\left(1+S^{(i)}+\sum_{j\in I\setminus(N_i\cup\{i\})} X_j\right)\middle|\mathcal{F}\right], \tag{4.3.98}$$

since X_i and X_j are conditionally independent given $\mathcal{F}$. Plugging this into the right-hand side of (4.3.96) yields

$$\begin{aligned}&\mathbb{P}(W\in A|\mathcal{F})-\mathbb{P}(\widehat{W}\in A|\mathcal{F})\\&\quad=\sum_{i\in I}\mathbb{E}(X_i|\mathcal{F})\mathbb{E}\left[g_{A,\lambda}(1+W)-g_{A,\lambda}(1+S^{(i)}+\sum_{j\in I\setminus(N_i\cup\{i\})}X_j)\middle|\mathcal{F}\right].\end{aligned} \tag{4.3.99}$$

Set now

$$M\equiv\sup\{|g_{A,\lambda}(n+1)-g_{A,\lambda}(n)| : n\in\mathbb{N}_0\}. \tag{4.3.100}$$

(Notice that M is $\mathcal{F}$-measurable). By the triangle inequality, and worstcase-scenario,

$$\begin{aligned}|\,\mathbb{P}(W\in A|\mathcal{F})-\mathbb{P}(\widehat{W}\in A|\mathcal{F})\,|&\le M\sum_{i\in I}\mathbb{E}(X_i|\mathcal{F})\mathbb{E}(X_i+S^{(i)}+\sum_{j\in N_i}X_j|\mathcal{F})\\&=M\sum_{i\in I}\left(E(X_i|\mathcal{F})^2+\sum_{j\in N_i}(\mathbb{E}(X_jX_i|\mathcal{F})+\mathbb{E}(X_j|\mathcal{F})E(X_i|\mathcal{F}))\right).\end{aligned} \tag{4.3.101}$$

It remains to prove that $M\le 1$. To this end we observe that additivity of $g_{\cdot,\lambda}$ is inherited from $f_{\cdot,\lambda}$, hence

$$g_{A,\lambda}=\sum_{j\in A}g_{\{j\},\lambda}. \tag{4.3.102}$$

Furthermore,

$$\sum_{j=0}^{\infty}g_{\{j\},\lambda}(n+1)-g_{\{j\},\lambda}(n)=0, \tag{4.3.103}$$

since by (4.3.102) it holds

$$\sum_{j=0}^{\infty}g_{\{j\},\lambda}(n)=g_{\mathbb{N}_0,\lambda}(n)=0\quad\forall n\in\mathbb{N}. \tag{4.3.104}$$

($f_{\mathbb{N}_0,\lambda}$ is the zero function). Therefore, for any $A\subset\mathbb{N}_0$,

$$|g_{A,\lambda}(n+1)-g_{A,\lambda}(n)|\le\sum_{j=0}^{\infty}(g_{\{j\},\lambda}(n+1)-g_{\{j\},\lambda}(n))^+. \tag{4.3.105}$$

By (4.3.89), the definition of f and elementary computations we have, for $0 < n \leq j$, that

$$g_{\{j\},\lambda}(n) = -\mathbb{P}(\widehat{W} = j|\mathcal{F}) \sum_{l=0}^{n-1} \frac{(n-1)!}{\lambda^{l+1}(n-1-l)!}. \tag{4.3.106}$$

This implies in particular that $g_{\{j\},\lambda}(n)$ is decreasing in n on $[0, j]$, hence all summands $j \geq n+1$ in (4.3.105) vanish. On the other hand, by (4.3.91), again the definition of f and elementary computations we have for $n > j$

$$g_{\{j\},\lambda}(n) = \mathbb{P}(\widehat{W} = j|\mathcal{F}) \sum_{l=0}^{\infty} \frac{\lambda^l (n-1)!}{(n+l)!}. \tag{4.3.107}$$

Since this is also decreasing in n, it follows that $j = n$ is the only non-zero summand in (4.3.105). All in all,

$$M = \sup_{n\in\mathbb{N}} \mid g_{A,\lambda}(n+1) - g_{A,\lambda}(n) \mid \leq \sup_{n\in\mathbb{N}} \mid g_{\{n\},\lambda}(n+1) - g_{\{n\},\lambda}(n) \mid . \tag{4.3.108}$$

Now, for $n > 0$, by (4.3.106) and (4.3.107),

$$\begin{aligned} &\mid g_{\{n\},\lambda}(n+1) - g_{\{n\},\lambda}(n) \mid = \\ &= \frac{\lambda^n e^{-\lambda}}{n!} \left(\sum_{l=0}^{\infty} \frac{\lambda^l (n-1)!}{(n+l)!} + \sum_{l=0}^{n-1} \frac{(n-1)!}{\lambda^{l+1}(n-1-l)!} \right) \\ &= \frac{e^{-\lambda}}{n} \left(\sum_{l=n}^{\infty} \frac{\lambda^l}{l!} + \sum_{l=0}^{n-1} \frac{\lambda^l}{l!} \right) = \frac{1}{n} \leq 1. \end{aligned} \tag{4.3.109}$$

On the other hand, for $n = 0$, we have

$$\mid g_{\{0\},\lambda}(1) - g_{\{0\},\lambda}(0) \mid = \frac{1}{\lambda}(1 - e^{-\lambda}) \leq 1 \tag{4.3.110}$$

by Taylor estimate. Using (4.3.109) and (4.3.110) in (4.3.108) shows that $M \leq 1$ as claimed, and concludes the proof of the conditional Chen-Stein method.

□

Chapter 5

Unoriented FPP: path properties

This chapter contains only unpublished results but it is up to minor changes in [36]. We consider the problem of undirected polymers (tied at the endpoints) in random environment, also known as the unoriented first passage percolation on the hypercube, in the limit of large dimensions. By means of the multiscale refinement of the second moment method we obtain a fairly precise geometrical description of optimal paths, i.e. of polymers with minimal energy. The picture which emerges can be loosely summarized as follows. The energy of the polymer is, to first approximation, uniformly spread along the strand. The polymer's bonds carry however a lower energy than in the directed setting, and are reached through the following geometrical evolution. Close to the origin, the polymer proceeds in oriented fashion – it is thus as stretched as possible. The tension of the strand decreases however gradually, with the polymer allowing for more and more backsteps as it enters the core of the hypercube. Backsteps, although increasing the length of the strand, allow the polymer to connect reservoirs of energetically favorable edges which are otherwise unattainable in a fully directed regime. These reservoirs lie at mesoscopic distance apart, but in virtue of the high dimensional nature of the ambient space, the polymer manages to connect them through approximate geodesics with respect to the Hamming metric: this is the key strategy which leads to an optimal energy/entropy balance. Around halfway, the mirror picture sets in: the polymer tension gradually builds up again, until full orientedness close to the endpoint. The approach yields, as a corollary, a constructive proof of the result by Martinsson [*Ann. Appl. Prob.* **26** (2016), *Ann. Prob.* **46** (2018)] concerning the leading order of the ground state.

5.1 Introduction

We recall the notations taken in the introduction of this dissertation. We denote by $G_n = (V_n, E_n)$ the n-dimensional hypercube. $V_n = \{0,1\}^n$ is thus the set of vertices, and E_n the set of edges connecting nearest neighbours. We write $\mathbf{0} = (0,0,...,0)$ and $\mathbf{1} = (1,1,...,1)$ for diametrically opposite vertices. For $l \in \mathbb{N}$ we let

$$\widetilde{\Pi}_{n,l} \equiv \text{the set of polymers, i.e. paths from } \mathbf{0} \text{ to } \mathbf{1} \text{ of length } l\,,$$

as well as

$$\widetilde{\Pi}_n \equiv \bigcup_{l=1}^{\infty} \widetilde{\Pi}_{n,l}.$$

For $\pi \in \tilde{\Pi}_n$ a polymer going through two vertices $\boldsymbol{v}, \boldsymbol{w}$ of the hypercube, we denote by $l_\pi(\boldsymbol{v}, \boldsymbol{w})$ the length of the connecting substrand, also shortening $l_\pi \equiv l_\pi(\mathbf{0}, \mathbf{1})$.

Every edge of the n-hypercube is parallel to some unit vector $e_j \in \mathbb{R}^n$, where e_j connects

$$(0,\dots,0) \text{ and } (0,\dots,0,\underbrace{1}_{j^{th}-\text{coordinate}},0,\dots,0)\,.$$

We write $e_{-j} \equiv -e_j$. The quantity $\pi_j \in \{1,\dots,n\} \cup \{-1,\dots,-n\}$ then specifies the direction of a π-path at step j. A *forward step* occurs if $\pi_j \in \{1,n\}$; if $\pi_j \in \{-1,-n\}$ we refer to this as a *backstep*.

Note that the endpoint of the (sub)path $\pi_1\pi_2\dots\pi_i$ coincides with the vertex given by $\sum_{j\le i} e_{\pi_j}$. The edge traversed in the j-th step by the π-path will be denoted $[\pi]_j$.

To each edge we attach independent, standard (mean one) exponential random variables ξ, the random environment, and assign to a polymer $\pi \in \widetilde{\Pi}_{n,l}$ its *weight/energy* according to

$$X_\pi \equiv \sum_{j=1}^{l} \xi_{[\pi]_j}\,.$$

The question we wish to address concerns the ground state of undirected polymers in random environment[1], to wit:

$$m_n[\text{undir}] \equiv \min_{\pi\in\widetilde{\Pi}_n} X_\pi, \tag{5.1.1}$$

in the mean field limit $n \uparrow \infty$, and the statistical/geometrical properties of optimal paths.

A first remark is in place: since polymers with loops cannot achieve the ground state (their energy can always be reduced by removing the loops), we will henceforth focus on the set of *loopless* paths of length $l \in \mathbb{N}$, denoted $\Pi_{n,l}$, and shortening, in full analogy,

$$\Pi_n \equiv \bigcup_{l=1}^{\infty} \Pi_{n,l},$$

for the set of all loopless paths.

Looplessness will be very useful: it guarantees, in particular, that the energy of a polymer of length, say, l, is indeed given by the sum of l independent standard exponentials. On the other hand, loopless paths are not necessarily directed, see Figure 5.1 below for a graphical rendition.

[1]This problem also appears in the literature under the name of unoriented first passage percolation, FPP for short. In mathematical biology it bears relevance to the issue of fitness landscapes. in which case it is dubbed accessibility percolation, see [13, 14, 31, 43, 44, 42, 32, 40] and references therein. We adopt here the polymer terminology since it is arguably more suitable to convey the type of results we obtain.

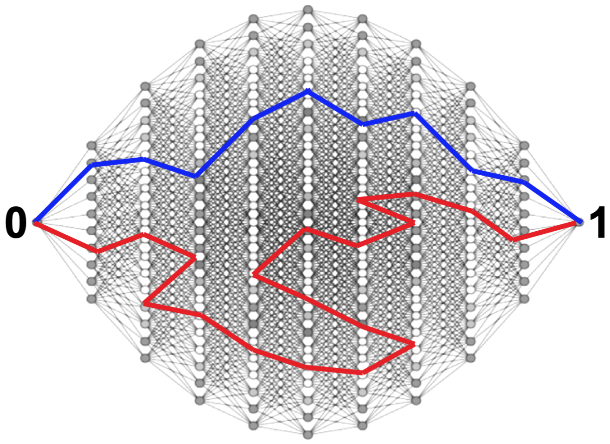

Figure 5.1: The 10-dim hypercube with two polymers. The blue polymer is *directed*: its length coincides with the dimension ($l = n = 10$), and it is thus as stretched as possible. The red polymer is *undirected*: it performs backsteps, which account for a lower "tension", and for the long excursions ($l = 20$).

It is clear that a major issue here will be that of *path counting*. For the hypercube, the following beautiful formula is available. We denote by $M_{n,l,d}$ the number of polymers of length l between two points at Hamming distance d, i.e. points thus disagree in exactly d coordinates. It then holds :

$$M_{n,l,d} = \frac{1}{2^n} \sum_{i=0}^{n} \sum_{j=0}^{d} \binom{d}{j} \binom{n-d}{i-j} (-1)^j (n-2i)^l \mathbb{1}_{j \leq i}. \tag{5.1.2}$$

(This formula concerns all paths of given length: loops, in particular, are also allowed). A proof of this formula, which relies on the classical approach via adjacency matrices, can be found in the monograph by Stanley [47]. Since we were not able to identify its first discoverer, we will refer to (5.1.2) as *Stanley's formula*.

No less remarkable is the following *Stanley's identity*, relating $M_{n,l,d}$ to hyperbolic functions. For $x \in \mathbb{R}$, it holds:

$$\sum_{l=0}^{\infty} M_{n,l,d} \frac{x^l}{l!} = \sinh(x)^d \cosh(x)^{n-d}. \tag{5.1.3}$$

Assuming the validity of (5.1.2), the proof of (5.1.3) only requires the binomial theorem and elementary Taylor expansions: it will be given in the Appendix for completeness. Lightening notations further by setting $M_{n,l} \equiv M_{n,l,n}$ for the number of polymers of length l between two opposite vertices on the hypercube, it thus follows from (5.1.3) that

$$\sum_{l=0}^{\infty} M_{n,l} \frac{x^l}{l!} = \sinh(x)^n \,. \tag{5.1.4}$$

This relation will allow for precise asymptotical analysis. Before seeing a first, key application, we shall recall yet another technical input concerning *tail estimates* for the distribution of the sum of independent standard exponentials as appearing in the problem at hand: denoting by $\{\xi_i\}_{i\in\mathbb{N}}$ a family of such random variables and with $X_l \equiv \sum_{i\leq l} \xi_i$, it then holds:

$$\mathbb{P}\left(X_l \leq x\right) = \left(1 + K(x,l)\right) \frac{e^{-x} x^l}{l!}, \tag{5.1.5}$$

for $x > 0$, and with $0 \leq K(x,l) \leq e^x x/(l+1)$. (The proof is truly elementary, but see e.g. [37, Lemma 5] for details).

Some notational convention: for $a_n, b_n \geq 0$ we write $a_n \lesssim b_n$ if $a_n \leq C b_n$ for some numerical constant $C > 0$ and $a_n \propto b_n$ if $a_n \lesssim b_n$ and $b_n \lesssim a_n$.

Armed with Stanley's formula and the tail estimates, we are now ready to make the aforementioned key observation concerning the ground state of undirected polymers: denoting by $N_{n,l,x} \equiv \#\{\pi \in \Pi_{n,l}, X_\pi \leq x\}$ the number of polymers of length l and energies at most x, by union bounds and Markov's inequality we have

$$\mathbb{P}\left(m_n[\text{undir}] \leq x\right) = \mathbb{P}\left(\cup_{l=0}^{\infty} \{N_{n,l,x} \geq 1\}\right) \leq \sum_{l=0}^{\infty} \mathbb{E}(N_{n,l,x}). \tag{5.1.6}$$

Note that we are considering polymers with no loops, in which case the energies are indeed sums of l independent random variables. Furthermore, it clearly holds that $\#\Pi_{n,l} \leq M_{n,l}$, since allowing loops can only increase the cardinality[2]. All in all, we have

$$\mathbb{E}(N_{n,l,x}) \leq M_{n,l} \mathbb{P}\left(X_l \leq x\right) \lesssim M_{n,l} \frac{x^l}{l!} \,, \tag{5.1.7}$$

the second inequality by the tail estimates.

Performing now the sum over all polymer-lengths in (5.1.6) and then using (5.1.3), we thus obtain

$$\mathbb{P}\left(m_n[\text{undir}] \leq x\right) \lesssim \sinh(x)^n \,. \tag{5.1.8}$$

The sinh-function is increasing, therefore, denoting by

$$\mathsf{E} \equiv \operatorname{arcsinh}(1) = \log(1 + \sqrt{2}), \tag{5.1.9}$$

[2]Here and henceforth we use Stanley's formula although we will be mostly considering loopless polymers: in hindsight, the error/overshooting will turn out to be negligible. This is course due to the high dimensionality of the problem at hand.

we deduce from (5.1.8), and the Borel-Cantelli lemma, a *lower bound* to the ground state, to wit:

$$\mathbb{P}\left(\lim_{n\to\infty} m_n[\text{undir}] \geq \mathsf{E}\right) = 1. \tag{5.1.10}$$

As it turns out, this bound is tight.

Martinsson's Theorem [43, 44]. *For undirected polymers on the hypercube, it holds*

$$\lim_{n\to\infty} m_n[\text{undir}] = \mathsf{E}, \tag{5.1.11}$$

in probability.

In other words, a "mean field trivialization" occurs in the limit of large dimensions, and the model of unoriented polymers in random environment thus falls in the so-called *REM class* [35]. Given the simple derivation of the lower bound, which eventually relies on the Markov inequality only, one is perhaps tempted to tackle the missing upper bound via the Second Moment Method. This is however not the route taken by Martinsson who, in fact, has found *two* rather distinct proofs.

The historically first proof has appeared in [43]. In that paper, Martinsson builds upon ideas of Durrett [26] and work by Fill and Pemantle [29], and settles the issue of the upper bound through a delicate comparison with the so-called Branching Translation Process, BTP for short. The BTP is a hierarchical model amenable to an explicit analysis and which, crucially, stochastically dominates the model of unoriented polymers.

In the second proof of the above theorem, Martinsson proceeds through some ingenious use of the FKG inequality, and (related) subadditivity/monotonicity properties of paths with optimal energies, see [44] for details.

Both proofs naturally come with their own strengths and weaknesses: the first one not only provides a solution of the problem at hand, but also insights into the structure of the BTP which are interesting in their own right, whereas the second proof settles the FPP on Cartesian power graphs, and thus applies in vast generality.

It seems however fair to say that, by their own nature, both approaches shed little light on the physical phenomena which eventually lead to the mean field trivialization. It is the purpose of this article to fill this gap by providing yet a third proof of the upper bound for the ground state, and hence of Martinsson's Theorem.

To this end, we will implement the *multiscale refinement of the second moment method* [35], a tool which forces us to identify the mechanisms allowing polymers to reach minimal energies. (As will become clear in the treatment, the choice of an exponentially distributed random environment presents no loss of generality). Unfortunately, the formulation of our main result, Theorem 2 below, requires some infrastructure: this will be provided in the next Section 5.2. In order the justify (and de-mystify) some otherwise odd looking formulas, concepts, *etc.* we will proceed gradually, increasing the amount of details concerning the geometry of optimal paths through simple observations and elementary computations. The upshot of these findings will be recorded in the form of Insights. A cautionary note is here due. The computations underlying Insight 28-32 below are rigorous yet *per se* not necessarily conclusive: indeed, they all rely on the *existence* of paths with the established geometric properties, but this will be, in fact, the content of Theorem 2 itself.

Our new approach leads to a proof of Martinsson's theorem which is much longer than those already available. It does however yield a detailed geometrical description of optimal polymers, and this in turn opens a gateway towards the unsettled issue of fluctuations and weak limits.

5.2 Drawing the picture

As we have seen, a reasonable candidate for the ground state eventually follows from an application of the Markov inequality. Albeit crucial, the ground state encodes however only some limited information. Another fundamental quantity is of course the *length* of an optimal polymer: as it turns out, a simple computation, allows to make an educated guess.

5.2.1 A candidate optimal length

Due to the high dimensionality of the problem, in order to identify the optimal length it seems natural to analyze the asymptotics of $\mathbb{E}(N_{n,l,x})$, the expected number of polymers with energies at most $x \in \mathbb{R}_+$, and prescribed length $l \in \mathbb{N}$. To this end, we recall Stanley's identity (5.1.4) which states that

$$\sum_{l=0}^{\infty} M_{n,l} \frac{x^l}{l!} = \sinh(x)^n. \tag{5.2.1}$$

Restricting to $x > 0$ implies that

$$M_{n,l} \frac{x^l}{l!} \leq \sinh(x)^n \,, \tag{5.2.2}$$

and therefore, by optimizing, we obtain,

$$M_{n,l} \leq \inf_{x>0} \left[\sinh(x)^n \frac{l!}{x^l} \right] \,. \tag{5.2.3}$$

Consistently with our terminology, we refer to (5.2.2) and (5.2.3) as *Stanley's M-bounds.*

Recall that $N_{n,l,E}$ is the number of paths of length l between two opposite vertices, and energy at most $\mathsf{E} = \log(1+\sqrt{2})$ as given in (5.1.9). By the tail estimates, and the above Stanley's M-bound, we thus have

$$\mathbb{E}(N_{n,l,\mathsf{E}}) \lesssim M_{n,l}\frac{\mathsf{E}^l}{l!} \le \mathsf{E}^l \inf_{x>0} \frac{\sinh(x)^n}{x^l} = \mathsf{E}^l \frac{\sinh(x^*)^n}{x^{*l}}, \tag{5.2.4}$$

where $x^* = x^*(l)$ is the minimizer of the r.h.s. above; taking the derivative of the target function, we see that this is the (unique) solution of

$$\frac{x}{\tanh(x)} = \frac{l}{n}. \tag{5.2.5}$$

At this point one is perhaps tempted to revert the line of reasoning: with the natural candidate for the optimal energy in mind, we choose $x^* \equiv \mathsf{E}$, in which case it follows from (5.2.5) that $l = \sqrt{2}\mathsf{E}n$, as an elementary computation shows. Changing the order of extremization is of course not quite justified[3], but the upshot turns out to be correct:

> **Insight 28.** *On the n-dim hypercube, the (candidate) length of optimal polymers is*
>
> $$\sqrt{2}\mathsf{E}n.$$

Henceforth, we will shorten

$$\mathsf{L} \equiv \sqrt{2}\mathsf{E}\,, \tag{5.2.6}$$

and always assume, without loss of generality, that $\mathsf{L}n \in \mathbb{N}$.

5.2.2 Uniform distribution of the energy

Having found natural candidates for the minimal energy and optimal length, a further question naturally arises:

how is an E-energy distributed along the polymer?

[3]One can prove that for all $l \in \mathbb{N}$, and x^* satisfying (5.2.5), it holds that

$$\sinh(x^*)^n \frac{\mathsf{E}^l}{x^{*l}} \le 1,$$

with the bound being saturated at $x^* = E$. As a matter of fact, we will prove an even stronger statement, namely that the length of optimal polymers indeed strongly concentrates on $\mathsf{L}n$, asymptotically in n. As we will see, this concentration follows from a key property of the power expansion (5.2.1), when evaluated at $x = \mathsf{E}$: in this case, the $(\mathsf{L}n)^{th}$ Taylor-term carries virtually the whole "mass" (whence the saturation). Such a result also provides intriguing clues about the issue of fluctuations, but since it is not instrumental for the rest of the discussion, we postpone the precise formulation, see Proposition 3 below.

To formalize, let us consider $\alpha \in [0,1]$, and shorten $\underline{\alpha} \equiv 1-\alpha$; furthermore let $\lambda \in [0,1]$ and similarly shorten $\underline{\lambda} = 1-\lambda$. We denote by

$$N_{n,\mathsf{L}n}^{\lambda,\alpha} := \# \left\{ \pi \in \Pi_{n,\mathsf{L}n} : \sum_{i=1}^{\alpha \mathsf{L}n} \xi_{[\pi]_i} \leq \lambda \mathsf{E}, \sum_{i=\underline{\alpha}\mathsf{L}n+1}^{\mathsf{L}n} \xi_{[\pi]_i} \leq \underline{\lambda} \mathsf{E} \right\} . \tag{5.2.7}$$

the number of polymers with the property that an λ-fraction of the energy E is carried by an α-fraction of the length (and similarly for the remaining part of the strand).

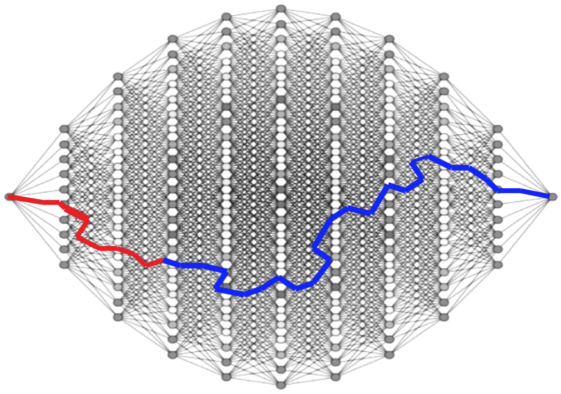

Figure 5.2: A polymer with (λ,α)-distribution of the energy E: the red strand has lenght $\alpha \mathsf{L}n$ and carries an energy $\lambda\mathsf{E}$, whereas the blue strand has length $\underline{\alpha}Ln$ and carries the remaining energy $\underline{\lambda}\mathsf{E}$.

Since polymers are loopless, and by independence, we have

$$\begin{aligned} \mathbb{E}\left(N_{n,\mathsf{L}n}^{\lambda,\alpha}\right) &\leq M_{n,\mathsf{L}n} \mathbb{P}\left(\sum_{i=1}^{\alpha \mathsf{L}n} \xi_{[\pi]_i} \leq \lambda \mathsf{E}, \sum_{i=\underline{\alpha}\mathsf{L}n+1}^{\mathsf{L}n} \xi_{[\pi]_i} \leq \underline{\lambda}\mathsf{E} \right) \\ &= M_{n,\mathsf{L}n} \mathbb{P}\left(\sum_{i=1}^{\alpha \mathsf{L}n} \xi_{[\pi]_i} \leq \lambda \mathsf{E} \right) \times \mathbb{P}\left(\sum_{i=\underline{\alpha}\mathsf{L}n+1}^{Ln} \xi_{[\pi]_i} \leq \underline{\lambda}\mathsf{E} \right) \\ &\lesssim M_{n,\mathsf{L}n} \frac{(\lambda \mathsf{E})^{\alpha \mathsf{L}n}}{(\alpha \mathsf{L}n)!} \times \frac{(\underline{\lambda}\mathsf{E})^{\underline{\alpha}\mathsf{L}n}}{(\underline{\alpha}\mathsf{L}n)!} , \end{aligned} \tag{5.2.8}$$

the last inequality by the usual tail estimates. By *Stanley's M-bound* (5.2.2), this time with $x = \mathsf{E}$, we have

$$M_{n,\mathsf{L}n} \leq \sinh(\mathsf{E})^n \frac{(\mathsf{L}n)!}{\mathsf{E}^{\mathsf{L}n}} = \frac{(\mathsf{L}n)!}{\mathsf{E}^{\mathsf{L}n}}, \tag{5.2.9}$$

the last step since $\sinh(\mathsf{E}) = 1$. Using this in (5.2.8) we thus get

$$\begin{aligned} \mathbb{E}\left(N_{n,\mathsf{L}n}^{\lambda,\alpha}\right) &\lesssim \frac{(\mathsf{L}n)!}{\mathsf{E}^{\mathsf{L}n}} \frac{(\lambda\mathsf{E})^{\alpha\mathsf{L}n}}{(\alpha\mathsf{L}n)!} \frac{(\underline{\lambda}\mathsf{E})^{\underline{\alpha}\mathsf{L}n}}{(\underline{\alpha}\mathsf{L}n)!} \\ &= \binom{\mathsf{L}n}{\alpha\mathsf{L}n} (\lambda)^{\alpha\mathsf{L}n} (\underline{\lambda})^{\underline{\alpha}\mathsf{L}n}, \end{aligned} \tag{5.2.10}$$

where in the last step we have used that $\mathsf{E}^{\alpha}\mathsf{E}^{\underline{\alpha}} = \mathsf{E}$, and simplified. By elementary Stirling approximation (to first order) of the binomial factor in (5.2.10), and again recalling that $\underline{\alpha} = 1 - \alpha$, and similarly for $\underline{\lambda}$, we thus arrive at the inequality

$$\mathbb{E}\left(N_{n,\mathsf{L}n}^{\lambda,\alpha}\right) \lesssim \left\{ \left(\frac{\lambda}{\alpha}\right)^{\alpha} \left(\frac{1-\lambda}{1-\alpha}\right)^{1-\alpha} \right\}^{\mathsf{L}n}. \tag{5.2.11}$$

Note that $x \mapsto x^y(1-x)^{1-y}$ is strictly concave with a unique critical point at $x = y$. Therefore, $\mathbb{E}N_{n,\mathsf{L}n}^{\lambda,\alpha}$ vanishes exponentially fast as soon as $\lambda \neq \alpha$. Borel-Cantelli then implies the following, loosely formulated summary of the current section:

Insight 29. *The energy* E *is spread* uniformly *along the polymer.*

This insight is of course in complete agreement with the phenomenon of mean field trivialization, see [35] for more on this issue.

5.2.3 Length vs. distance: the macroscopic picture

We address here the loosely formulated question:

at which Hamming distance from the origin
do we find a strand of prescribed length?

It is clear that the answer will yield profound insights into the geometry of optimal polymers. To formalize, consider as before $\alpha \in [0, 1]$. (We stick to the convention $\underline{\alpha} = 1 - \alpha$). For $d \in [0, 1]$, let $d_n = \lfloor dn \rfloor$ and denote by

$$H_{d_n} := \{\boldsymbol{v} \in V_n : \ d(\boldsymbol{0}, \boldsymbol{v}) = d_n\} \tag{5.2.12}$$

the *hyperplane* consisting of all vertices at Hamming distance d_n from the origin. (Remark that $\sharp H_{d_n} = \binom{n}{d_n}$: indeed, in order to specify a point on the hyperplane we simply need to switch d_n coordinates of $\boldsymbol{0} = (0, 0, \dots, 0)$ into 1).

For $\boldsymbol{w} \in H_{d_n}$ we denote by $\Pi^d_{\alpha\mathsf{L}n}[\boldsymbol{0} \to \boldsymbol{w}]$ the set of paths connecting $\boldsymbol{0}$ to $\boldsymbol{w}$ in αLn steps. In full analogy, $\Pi^d_{\underline{\alpha}\mathsf{L}n}[\boldsymbol{w} \to \boldsymbol{1}]$ stands for the set of path connecting $\boldsymbol{w}$ to $\boldsymbol{1}$ in $\underline{\alpha}\mathsf{L}n$

steps. Lastly, we denote by $\Pi^{d,\alpha}_{\mathsf{L}n}[\mathbf{0}\to\mathbf{1}]$ the set of paths of length $\mathsf{L}n$ from $\mathbf{0}$ to $\mathbf{1}$, which are in H_{d_n} after $\alpha\mathsf{L}n$ steps. (Note that these paths can cross the hyperplane multiple times, see Figure 5.3 below for a graphical rendition).

The goal is now to compute the expected number of these polymers after distributing the energy, in line with the Insight from the previous section, *uniformly* along the path. To this end, introduce the cardinalities

$$N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\boldsymbol{w}] = \#\left\{\pi\in\Pi^d_{\alpha\mathsf{L}n}[0\to\boldsymbol{w}] : \sum_{i=1}^{\alpha\mathsf{L}n}\xi_{[\pi]_i}\le\alpha\mathsf{E}\right\},$$

$$N^{d,\underline{\alpha}}_{n,\mathsf{L}n}[\boldsymbol{w}\to\mathbf{1}] = \#\left\{\pi\in\Pi^d_{\underline{\alpha}\mathsf{L}n}[\boldsymbol{w}\to 1], \sum_{i=1}^{\underline{\alpha}\mathsf{L}n}\xi_{[\pi]_i}\le\underline{\alpha}\mathsf{E}\right\},$$

and

$$N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\mathbf{1}] = \#\left\{\pi\in\Pi^{d,\alpha}_{\mathsf{L}n}[\mathbf{0}\to\mathbf{1}], \sum_{i=1}^{\mathsf{L}n}\xi_{[\pi]_i}\le\mathsf{E}\right\}.$$

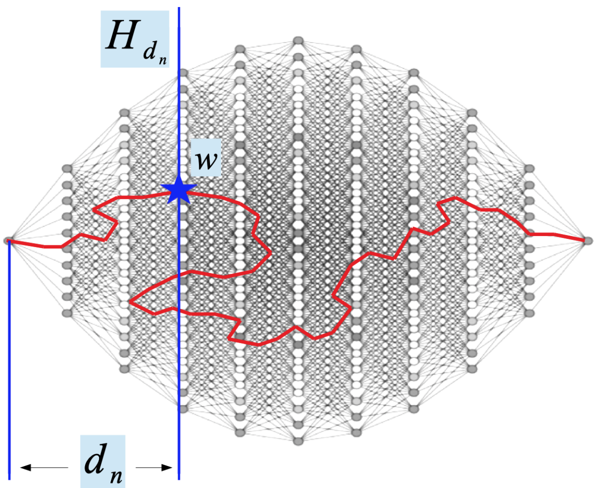

Figure 5.3: Path-decomposition with an hyperplane H_{d_n} at Hamming distance d_n from $\mathbf{0}$. The strand up to the first crossing of the hyperplane has an α-fraction of length, and carries an α-fraction of energy. The rest of the strand has length $\underline{\alpha}\mathsf{L}n$, and carries the remaining $\underline{\alpha}$-fraction of energy.

Since polymers are loopless, and by independence, it holds

$$\begin{aligned}
\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\mathbf{1}]\right) &= \sum_{\boldsymbol{w}\in H_{d_n}} \mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\boldsymbol{w}]\right)\mathbb{E}\left(N^{d,\underline{\alpha}}_{n,\mathsf{L}n}[\boldsymbol{w}\to\mathbf{1}]\right)\\
&= \binom{n}{d_n}\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\boldsymbol{w}]\right)\mathbb{E}\left(N^{d,\underline{\alpha}}_{n,\mathsf{L}n}[\boldsymbol{w}\to\mathbf{1}]\right) \qquad (5.2.13)\\
&\lesssim \binom{n}{d_n} M_{n,\alpha\mathsf{L}n,d_n}\frac{(\alpha\mathsf{E})^{\alpha\mathsf{L}n}}{(\alpha\mathsf{L}n)!}M_{n,\underline{\alpha}\mathsf{L}n,n-d_n}\frac{(\underline{\alpha}\mathsf{E})^{\underline{\alpha}\mathsf{L}n}}{(\underline{\alpha}\mathsf{L}n)!}\,,
\end{aligned}$$

the last inequality by the usual tail estimates.

In full analogy with (5.2.3), which is a consequence of Stanley's identity (5.1.4), the following *Stanley's M-bound* is a consequence of Stanley's identity (5.1.3): for $x > 0$, it holds

$$M_{n,l,d} \le \sinh(x)^d \cosh(x)^{n-d}\frac{l!}{x^l}\,. \qquad (5.2.14)$$

Using this for the r.h.s. of (5.2.13) we see that for arbitrary $y_1, y_2 > 0$, it holds:

$$\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\mathbf{1}]\right) \lesssim \binom{n}{d_n}\frac{\sinh(y_1)^{d_n}\cosh(y_1)^{n-d_n}}{\left(\frac{y_1}{\alpha\mathsf{E}}\right)^{\alpha L n}}\frac{\sinh(y_2)^{n-d_n}\cosh(y_2)^{d_n}}{\left(\frac{y_2}{\underline{\alpha}\mathsf{E}}\right)^{\underline{\alpha}\mathsf{L}n}}. \qquad (5.2.15)$$

Taking $y_1 = \alpha\mathsf{E}$ and $y_2 = \underline{\alpha}\mathsf{E}$, and by elementary Stirling approximation (to first order),

$$\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\mathbf{1}]\right) \lesssim \left(\frac{\cosh(\alpha E)\sinh(\underline{\alpha}E)}{1-\frac{d_n}{n}}\right)^{n-d_n}\left(\frac{\sinh(\alpha E)\cosh(\underline{\alpha}E)}{\frac{d_n}{n}}\right)^{d_n}. \qquad (5.2.16)$$

We will now slightly modify the form of the r.h.s. above. In order to do so, we recall that

$$\begin{aligned}
1 = \sinh(\mathsf{E}) &= \sinh\left(\alpha\mathsf{E}+\underline{\alpha}\mathsf{E}\right)\\
&= \cosh(\alpha E)\sinh(\underline{\alpha}\mathsf{E}) + \sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E})\,,
\end{aligned} \qquad (5.2.17)$$

the last step by the addition formula for hyperbolic functions, hence

$$\cosh(\alpha\mathsf{E})\sinh(\underline{\alpha}\mathsf{E}) = 1 - \sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E}) \qquad (5.2.18)$$

This allows to reformulate (5.2.16) as

$$\mathbb{E}\left(N^{d,\alpha}_{n,\mathsf{L}n}[\mathbf{0}\to\mathbf{1}]\right) \lesssim \left\{\left(\frac{1-\sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E})}{1-\frac{d_n}{n}}\right)^{1-\frac{d_n}{n}}\left(\frac{\sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E})}{\frac{d_n}{n}}\right)^{\frac{d_n}{n}}\right\}^n. \qquad (5.2.19)$$

One plainly checks that the function

$$[0,1]\ni\alpha\mapsto\sinh(\alpha\mathsf{E})\cosh(\underline{\alpha}\mathsf{E}) \qquad (5.2.20)$$

is bijective, whereas $x \mapsto (1-x)^{1-y}x^y$ is strictly concave with a unique critical point at $x = y$. It thus steadily follows that the r.h.s. of (5.2.19) is exponentially small if $\frac{d_n}{n} \neq \sinh(\alpha E)\cosh(\underline{\alpha}E)$. We may thus summarize these findings as follows:

Insight 30. *After an α-fraction of the total length, an optimal polymer finds itself at a typical (normalized) Hamming distance*

$$d = \sinh(\alpha \mathsf{E})\cosh((1-\alpha)\mathsf{E}) \tag{5.2.21}$$

from the origin.

The above Insight is both intriguing and delicate. Indeed, a polymer of length greater than the dimension can (must) cross multiple times certain hyperplanes, yet the map $\alpha \mapsto d(\alpha)$ as in (5.2.21) is increasing: for consistency, we must therefore deduce that excursions can only happen on mesoscopic (if not microscopic) scales. In other words, and loosely:

Insight 31. *Backsteps must be relatively rare, and spread out.*

Not surprisingly, this additional Insight will play a key role, and guide us through the next steps, but before proceeding any further, a comparison with the directed case is perhaps in place. To better visualize, we re-parametrize in terms of the (normalised) *length* of the polymer: with $\alpha E \hookrightarrow l$, and recalling that $\mathsf{L} = \sqrt{2}\mathsf{E}$, we see that the "Hamming depth" $d_{\text{un}}(l)$ reached by the unoriented polymer at length l is then given by

$$l \in [0, \mathsf{L}] \mapsto d_{\text{un}}(l) \equiv \sinh\left(\frac{l}{\sqrt{2}}\right)\cosh\left(\frac{\mathsf{L}-l}{\sqrt{2}}\right). \tag{5.2.22}$$

In case of oriented polymers, the Hamming depth as a function of the length is simply

$$l \in [0, 1] \mapsto d_{\text{or}}(l) \equiv l\,. \tag{5.2.23}$$

The two functions are plotted in Figure 5.4 below, whereas a rendition of the emerging picture at the level of the strands is given in Figure 5.5.

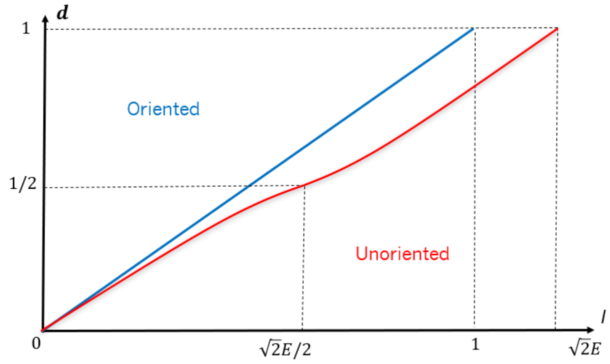

Figure 5.4: Hamming-depth as a function of the length: directed (blue) vs. undirected (red) polymers. For small lengths, the depths are comparable: close to the origin, the undirected polymer is thus as directed as possible. The slope of the red curve decreases however gradually as the polymer approaches the core of the hypercube: the further the polymer goes, the "loser" it becomes. Due to the inherent symmetry of the hypercube, a mirror picture sets in, of course, at half-length.

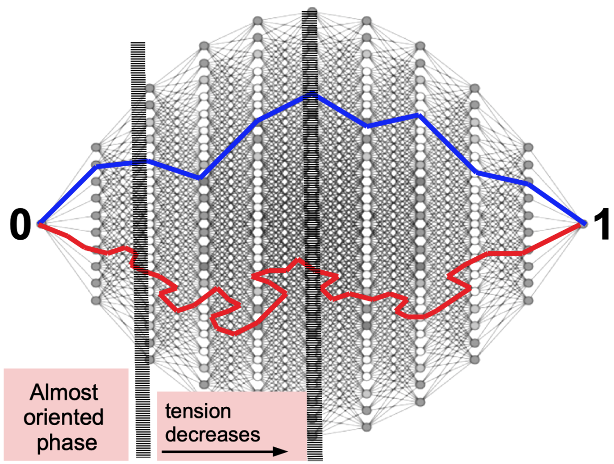

Figure 5.5: Directed (blue) vs. undirected (red) polymers. The red strand starts off as stretched as possible, but allows for more and more backsteps as it approaches the core of the hypercube. The phenomena are amplified for better visualisation only: in line with Insight 31, backsteps live on meso/microscopic scale only. In particular, long excursions as in Figure 5.1 above are, in fact, ruled out.

5.2.4 Length vs. distance: the mesoscopic picture

As mentioned in the introduction, our approach will eventually rest on a multiscale analysis: in this section, inspired by the previous Insights, we introduce the necessary *coarse* *graining* [35]. To see how this goes, we denote by $K \in \mathbb{N}$ the numbers of "scales", and shorten henceforth $\hat{n}_K \equiv n/K$ (assuming w.l.o.g. that $\hat{n}_K \in \mathbb{N}$). We then split the hypercube into K "slabs", i.e. hyperplanes equidistributed w.r.t. the Hamming distance: for $i = 1 \dots K$ we let

$$H_i \equiv \{v \in V_n, d(0, v) = i\hat{n}_K\} . \tag{5.2.24}$$

We will refer to these hyperplanes as *H-planes*. Accordingly, we split a polymer of length $\mathsf{L}n$ into K substrands of length $\alpha_i Ln$, for $i = 1 \dots K$, with the normalization $\sum_{i\leq K} \alpha_i = 1$. We shorten $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_K) \in [0, 1]^K$ for such a vector, $\overline{\boldsymbol{\alpha}}_i \equiv \sum_{j=1}^{i} \alpha_j$ for the (fraction of) length of the strand when the polymer crosses the i^{th} H-plane, and $\underline{\boldsymbol{\alpha}}_i \equiv 1 - \sum_{j=1}^{i} \alpha_j$ for the length of the remaining strand. A graphical rendition is given in Figure 5.6 below.

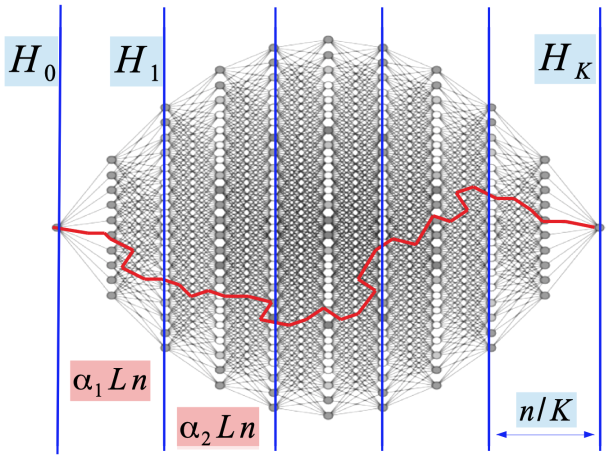

Figure 5.6: K-levels coarse graining: the Hamming distance between any two (successive) hyperplanes is $\hat{n}_K = n/K$. Remark that by (5.2.26)-(5.2.27), the length of the substrand from hyperplane to hyperplane is a function of E and K only.

By the above **Insight 30**, length of substrands and Hamming-depth must satisfy the fundamental relation

$$\sinh(\overline{\boldsymbol{\alpha}}_i \mathsf{E}) \cosh(\underline{\boldsymbol{\alpha}}_i \mathsf{E}) = \frac{i}{K}, \quad i = 1 \dots K. \tag{5.2.25}$$

The function $x \in [0, 1] \mapsto \sinh(x\mathsf{E}) \cosh((1 - x)\mathsf{E})$ is invertible, and one can even construct explicitely the solutions of the above equation: recalling that $\operatorname{arcsinh}(x) = \log(x + \sqrt{1 + x^2})$ one plainly checks that these are given by

$$\overline{\boldsymbol{\alpha}}_i = \frac{1}{2} \left\{ 1 + \frac{1}{\mathsf{E}} \operatorname{arcsinh} \left(2\frac{i}{K} - 1 \right) \right\} . \tag{5.2.26}$$

This also uniquely identifies the length of the substrands, to wit:

$$\alpha_i = \overline{\boldsymbol{\alpha}}_i - \overline{\boldsymbol{\alpha}}_{i-1}\,, \tag{5.2.27}$$

for $i = 1 \dots K$, see Figure 5.7 below for a plot.

In particular, it follows from (5.2.26) and (5.2.27) that

$$\alpha_i = \alpha_{K+1-i}, \tag{5.2.28}$$

which is in full agreement with the inherent symmetry of the problem at hand, and $\sum_{j\leq K} \alpha_j = 1$. Furthermore, since arcsinh is 1-Lipschitz we also immediately see that

$$\alpha_i \leq \frac{1}{K\mathsf{E}}. \tag{5.2.29}$$

In order to emphasize that the $\alpha's$ are no longer arbitrary, we will write henceforth $\mathsf{a} = \mathsf{a}(\mathsf{E}, K)$ for the solutions of the equations (5.2.26), (5.2.27).

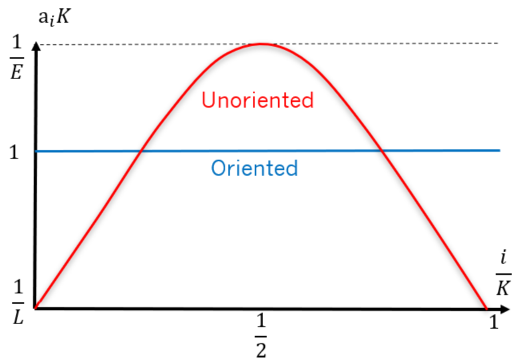

Figure 5.7: Substrand-length as function of the depth, $i \in \{1, \dots K\} \mapsto \mathsf{a}_i$. This plot simply restates the key property of optimal polymers: substrands between equidistant hyperplanes become longer as the polymer enters the core of the hypercube.

A straightforward large-K Taylor expansion (with $i/K = const.$) yields that

$$\mathsf{a}_{i+1} - \mathsf{a}_i = \frac{2}{K^2\mathsf{E}}\left(1 - \frac{2i}{K}\right) + O\left(\frac{1}{K^3}\right), \tag{5.2.30}$$

which is manifestly different from the case of directed polymers, where the differential would necessarily vanish. Thus a fundamental question immediately arises:

how do substrands of undirected polymers connect
the coarse graining-hyperplanes?

To shed light on this issue we consider $\boldsymbol{d} = (d_1, d_2, ..., d_K) \in [0,1]^K$ and introduce

$$\Pi_i^{\boldsymbol{d}}[\boldsymbol{v} \to \boldsymbol{w}] \equiv \textsf{all loopless paths connecting two vertices } \boldsymbol{v} \in H_{i-1},\ \boldsymbol{w} \in H_i \textsf{ which are at Hamming distance } d(\boldsymbol{v},\boldsymbol{w}) = d_i n\,, \tag{5.2.31}$$

and

$$\Pi_{\{1\ldots K\}}^{\boldsymbol{d}}[\boldsymbol{0} \to \boldsymbol{1}] \equiv \textsf{all loopless paths connecting } \boldsymbol{0} \textsf{ to } \boldsymbol{1}\,, \textsf{ and that cover a } d_i n\textsf{-Hamming distance while connecting the H-hyperplanes, } i = 1\ldots K. \tag{5.2.32}$$

A graphical rendition is given in Figure 5.8 below.

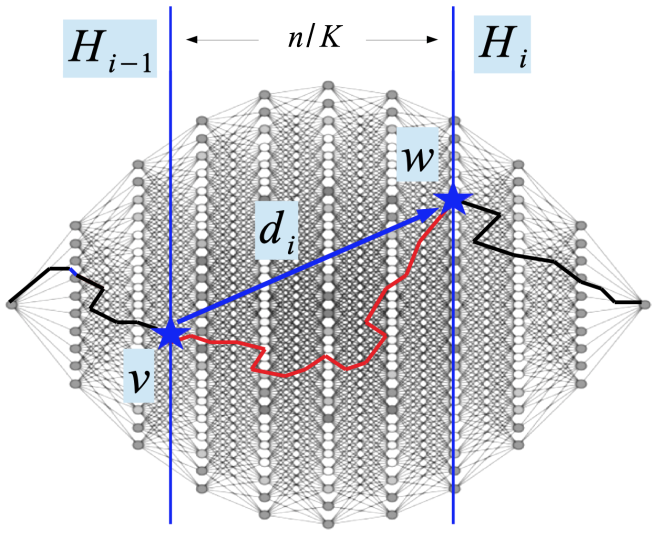

Figure 5.8: A polymer between two hyperplanes: the two vertices $\boldsymbol{v}$ and $\boldsymbol{w}$ are at a Hamming distance $d(\boldsymbol{v},\boldsymbol{w}) = d_i$. Remark that, in particular, $d(H_{i-1}, H_i) = 1/K \leq d_i \leq l_\pi(\boldsymbol{v},\boldsymbol{w})$.

For $\pi \in \Pi_{\{1\ldots K\}}^{\boldsymbol{d}}[\boldsymbol{0} \to \boldsymbol{1}]$, and two vertices $\boldsymbol{v} \in H_{i-1}, \boldsymbol{w} \in H_i$ (for some $i = 1\ldots K$), we furthermore shorten

$$X_\pi(\boldsymbol{v},\boldsymbol{w}) \equiv \text{energy of the substrand which connects } \boldsymbol{v},\ \boldsymbol{w}\ . \tag{5.2.33}$$

and denote by

$$N_i^{\boldsymbol{d}}[\boldsymbol{v} \to \boldsymbol{w}] = \#\left\{\pi \in \Pi_i^{\boldsymbol{d}}[\boldsymbol{v} \to \boldsymbol{w}], X_\pi(\boldsymbol{v},\boldsymbol{w}) \leq \mathsf{a}_i\mathsf{E}\right\}\,, \tag{5.2.34}$$

the number of substrands with energies at most $\mathsf{a}_i E$ connecting such vertices. Finally, let

$$N^{\boldsymbol{d}}_{\{1\ldots K\}}[\mathbf{0} \to \mathbf{1}] = \#\left\{\pi \in \Pi^{\boldsymbol{d}}_{\{1\ldots K\}}[\mathbf{0} \to \mathbf{1}],\ X_\pi(\mathbf{0},\mathbf{1}) \le \mathsf{E}\right\} \tag{5.2.35}$$

stand for the number of paths with prescribed evolutions[4]. The goal is to compute the expectation of this random set, as this will provide fundamental insights into the possible choices of $\boldsymbol{d}$, which are the only degrees of freedom left. As we will see shortly, there is only one reasonable choice. Before that we need however to introduce some key concepts.

Definition 5.2.1. *Let $\boldsymbol{v} \in H_{i-1}$ and $\boldsymbol{w} \in H_i$.*

- *The* effective forward steps *are given by*

$$\mathsf{ef}_i(\boldsymbol{v},\boldsymbol{w}) \equiv \frac{1}{n}\#\left\{0's \text{ in } \boldsymbol{v} \text{ which switch into } 1's \text{ in } \boldsymbol{w}\right\}$$

- *The* effective backsteps *are given by*

$$\mathsf{eb}_i(\boldsymbol{v},\boldsymbol{w}) \equiv \frac{1}{n}\#\left\{1's \text{ in } \boldsymbol{v} \text{ which switch into } 0's \text{ in } \boldsymbol{w}\right\}\ .$$

- *The* detours *are given by*

$$\gamma_\pi(\boldsymbol{v},\boldsymbol{w}) \equiv \frac{1}{n}\left\{l_\pi(\boldsymbol{v},\boldsymbol{w}) - d(\boldsymbol{v},\boldsymbol{w})\right\}.$$

Some comments concerning the above terminology are perhaps in place: we note that the *effective forward steps* encode the fraction of steps forward which are not undone by backsteps in the reverse direction; similarly, the *effective backsteps* encode the (fraction of) backsteps which are not undone by steps forward in the reverse direction (or vice versa). Finally, the *detours* capture the amount of forward steps in a path π which are cancelled by backsteps in the reverse direction (or vice versa): the smaller γ_π, the higher the "tension" of the substrand. For this reason, we call a substrand *stretched* if the detours vanish. A stretched path is, in fact, a *geodesic*.

The above quantities are all intertwined. Indeed, it holds:

$$d_i = \mathsf{ef}_i(\boldsymbol{v},\boldsymbol{w}) + \mathsf{eb}_i(\boldsymbol{v},\boldsymbol{w}) \quad \text{and} \quad \frac{1}{K} = \mathsf{ef}_i(\boldsymbol{v},\boldsymbol{w}) - \mathsf{eb}_i(\boldsymbol{v},\boldsymbol{w})\,. \tag{5.2.36}$$

[4] We shall perhaps emphasize that the above prescription of the evolution involves the Hamming-depths and energies, but *not* the length of the connecting substrands. This is because in (5.2.34) we are spreading the energies uniformly along the length of the polymer, very much in line with Insight 29: energies and optimal lengths are two sides of the same coin.

In particular, it follows from the above relations that

$$\mathsf{ef}_i(\boldsymbol{v},\boldsymbol{w}) = \frac{d_i}{2} + \frac{1}{2K} \quad \text{and} \quad \mathsf{eb}_i(\boldsymbol{v},\boldsymbol{w}) = \frac{d_i}{2} - \frac{1}{2K}\,. \tag{5.2.37}$$

In other words, effective forward- and backsteps along a substrand depend on the number of scales, and the remaining degrees of freedom $\boldsymbol{d}$ (which we are going to identify shortly), but *not* on the endpoints. An equally simple line of reasoning shows that detours, *as soon as the polymer-length is specified*, do not depend on the specific form of the π-path, neither: in fact, $\gamma_{i,\pi} n + d_i n = \mathsf{a}_i L n$.

As mentioned, the goal is to compute the expected number of paths connecting $\mathbf{0}$ to $\mathbf{1}$. Since polymers are loopless, and by independence, it holds:

$$\mathbb{E}\left(N^{\boldsymbol{d}}_{\{1\ldots K\}}[\mathbf{0}\to\mathbf{1}]\right) = \sum_{(\star)} \prod_{i=1}^{K} \mathbb{E}N_i^{\boldsymbol{d}}\left[\boldsymbol{v}^{(i-1)} \to \boldsymbol{v}^{(i)}\right], \tag{5.2.38}$$

where the $(\star)$-sum runs over all possible vertices $\boldsymbol{v}^{(i)} \in H_i$, $i = 1\ldots K$. But by (5.2.37), none of the expectations on the r.h.s. depend on the specific $\boldsymbol{v}$-choice. The cardinality of $(\star)$ is easily computed: shortening

$$[0,\infty) \ni x \mapsto \varphi(x) \equiv x^x, \tag{5.2.39}$$

one plainly checks that

$$\begin{aligned} \#(\star) &= \prod_{i=1}^{K} \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{\left(1-\frac{i-1}{K}\right)n}{\mathsf{ef}_i n} \\ &\lesssim \prod_{i=1}^{K}\left\{\frac{\varphi\left(\frac{i-1}{K}\right)\varphi\left(1-\frac{i-1}{K}\right)}{\varphi(\mathsf{eb}_i)\varphi\left(\frac{i-1}{K}-\mathsf{eb}_i\right)\varphi\left(\mathsf{ef}_i\right)\varphi\left(1-\frac{i-1}{K}-\mathsf{ef}_i\right)}\right\}^n, \end{aligned} \tag{5.2.40}$$

the last step by elementary Stirling-approximation to first order.

By the tail estimates, and Stanley's M-bound (5.2.14) with $x = \mathsf{a}_i\mathsf{E}$, it holds

$$\mathbb{E}N_i^{\boldsymbol{d}}\left[\boldsymbol{v}^{(i-1)} \to \boldsymbol{v}^{(i)}\right] \lesssim \sinh(\mathsf{a}_i\mathsf{E})^{d_i n}\cosh(\mathsf{a}_i\mathsf{E})^{(1-d_i)n}, \tag{5.2.41}$$

for $i = 1\ldots K$.

Plugging (5.2.40) and (5.2.41) into (5.2.38), and rearranging, we thus get the *upper-bound*

$$\mathbb{E}\left(N^{\boldsymbol{d}}_{\{1\ldots K\}}[\mathbf{0}\to\mathbf{1}]\right) \lesssim \mathcal{F}_{\mathsf{a},K}(\boldsymbol{d})^n, \tag{5.2.42}$$

where we have shortened

$$\mathcal{F}_{\mathsf{a},K}(\boldsymbol{d}) \equiv \prod_{i=1}^{K} \frac{\sinh(\mathsf{a}_iE)^{d_i}\cosh(\mathsf{a}_iE)^{(1-d_i)}\varphi\left(\frac{i-1}{K}\right)\varphi\left(1-\frac{i-1}{K}\right)}{\varphi(\mathsf{eb}_i)\varphi\left(\frac{i-1}{K}-\mathsf{eb}_i\right)\varphi\left(\mathsf{ef}_i\right)\varphi\left(1-\frac{i-1}{K}-\mathsf{ef}_i\right)}\,. \tag{5.2.43}$$

Since $\mathbf{a} = \mathbf{a}(\mathsf{E}, K)$ are solutions of (5.2.26)-(5.2.27), the $\boldsymbol{d}'s$ appearing in the $\mathcal{F}$-function are the only degrees of freedom left (By (5.2.37), we recall that ef_i and eb_i are function of d_i). The next result shows that even for these, there is in fact one reasonable choice only.

Theorem 1. (Optimal Hamming distance) *Let* $\mathbf{d} = (\mathsf{d}_1, \dots, \mathsf{d}_K)$, *with*

$$\mathsf{d}_i \equiv \sinh(\mathsf{a}_i\mathsf{E})\cosh((1-\mathsf{a}_i)\mathsf{E}). \tag{5.2.44}$$

It then holds:

$$\mathcal{F}_{\mathbf{a},K}(\mathbf{d}) = 1, \tag{5.2.45}$$

and

$$\mathcal{F}_{\mathbf{a},K}(\boldsymbol{d}) < 1, \quad \textit{for} \quad \boldsymbol{d} \neq \mathbf{d}. \tag{5.2.46}$$

By (5.2.42) and (5.2.46), the expected number of polymers connecting a sequence of prescribed vertices on the H-planes is thus exponentially small, *unless* the Hamming distance of the considered vertices satisfies (5.2.44): of course, the latter will henceforth be the value of our choice.

Theorem 1 is absolutely crucial for our approach. The proof, which requires a fair amount of work, is postponed. For the remaining part of this section we dwell rather informally on some of its far-reaching implications.

We anticipate that we will eventually consider a large (yet finite) number of scales for the coarse graining, in which case an elementary large-K Taylor expansion (together with the fact that $\mathsf{L} = \sqrt{2}\mathsf{E}$) shows that to first approximation, Hamming distance between two vertices on the H-planes and substrand-legth do, in fact, coincide:

$$\mathsf{d}_i = \sinh(\mathsf{a}_i\mathsf{E})\cosh((1-\mathsf{a}_i)\mathsf{E}) = \mathsf{a}_i\mathsf{L} + O\left(\frac{1}{K^2}\right). \tag{5.2.47}$$

A minute's thought suggests that the above may be reformulated as follows:

Insight 32. *Optimal polymers connect the coarse graining H-planes through essentially stretched paths.*

This is a somewhat surprising feature, which at first sight may even appear non-sensical. The devil is however in the details: by (5.2.26), and large-K Taylor expansions (again with $i/K = const$), one can check that

$$\mathsf{a}_i = \frac{1}{K\mathsf{E}\sqrt{1+(\frac{2i}{K}-1)^2}} + O\left(\frac{1}{K^2}\right), \tag{5.2.48}$$

which combined with (5.2.47), and recalling $\mathsf{L} = \sqrt{2}\mathsf{E}$, leads to

$$\mathsf{d}_i = \frac{\sqrt{2}}{K\sqrt{1+\left(\frac{2i}{K}-1\right)^2}} + O\left(\frac{1}{K^2}\right) . \tag{5.2.49}$$

From this we may evince that:

- for small i (say $i = sK$, and $s \ll 1/2$) it holds that

$$\mathsf{a}_i = \frac{1}{K\mathsf{E}\sqrt{2}} + O\left(\frac{1}{K^2}\right) = \frac{1}{K\mathsf{L}} + O\left(\frac{1}{K^2}\right) , \tag{5.2.50}$$

  as well as

$$\mathsf{d}_i = \frac{1}{K} + O\left(\frac{1}{K^2}\right) = d(H_{i-1}, H_i) + O\left(\frac{1}{K^2}\right) , \tag{5.2.51}$$

  the latter confirming that close to the origin, unoriented polymers proceed in almost directed fashion;

- for large i (say $i = sK$, and $s \uparrow 1/2$) it holds that $\mathsf{d}_i \approx \sqrt{2}/K \gg 1/K$, which is much larger than the Hamming distance between two successive H-planes. Substrands of optimal polymers close to the core of the hypercube therefore reach, through approximate geodesics, vertices which are otherwise unattainable in a fully directed regime. Although the length of the substrand is increased, this strategy allows undirected polymers to gain access to a reservoir of energetically favorable edges. A graphical rendition of this feature, which encodes the key strategy of optimal polymers, is given in Figure 5.9 below.

The feature according to which undirected polymers proceed through approximate geodesics is absolutely fundamental. On the one hand it neatly explains the deeper mechanisms eventually responsible for the onset of the mean field trivialization. On a more technical level, this property will lead to a dramatic simplification of some otherwise daunting combinatorial estimates, eventually enabling us to implement the second moment method. In fact, in a (fully) stretched regime, a backstep cannot be cancelled by a forward step (and vice versa). This entails, in particular, a natural representation of paths connecting say $\boldsymbol{v} \in H_{i-1}$ to $\boldsymbol{w} \in H_i$ in terms of permutations of the $\boldsymbol{v}$-coordinates which must be changed in order to obtain $\boldsymbol{w}$, see in particular Lemma 48 below for a clear manifestation of this feature.

5.2.5 Main result

We now specify a subset of polymers with path properties capturing all Insights gathered so far: our main result, which is at last formulated in this section, simply states that such

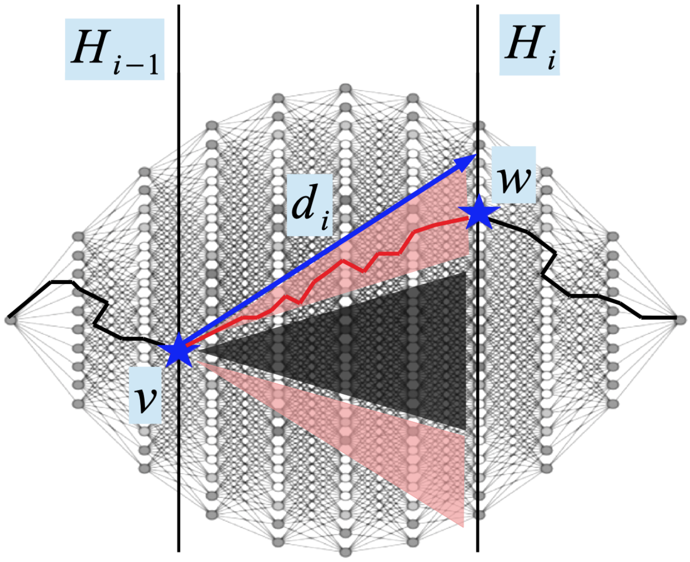

Figure 5.9: The black-shaded cone corresponds to the region where a fully directed polymer would lie. In virtue of Theorem 1, the optimal, undirected polymers evolve however in the red-shaded cones, thereby reaching vertices which are at larger Hamming distance. (Note also that black and red vertical boundaries of these cones are disjunct). For large hyperplane-density, the substrands (in red) of optimal polymers are, in first approximation, geodesics.

a subset is, in fact, non-empty. Towards this goal, some additional observations/notation is needed.

For arbitrary $\boldsymbol{d} = (d_1, \dots, d_K) \in [0,1)^K$ (the Hamming-depths) and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_K) \in [0, \infty)^K$ (the detours), consider the subset

$$\begin{aligned}\mathcal{P}_{n,K}\left\{\boldsymbol{d}, \boldsymbol{\gamma}\right\} \equiv &\ \textsf{all paths connecting } \mathbf{0} \textsf{ to } \mathbf{1}\ , \\ &\textsf{and that cover a normalized } d_i\textsf{-Hamming distance,} \\ &\textsf{with } \gamma_i \textsf{ detours,} \\ &\textsf{while connecting the H-hyperplanes, } i = 1 \dots K\ .\end{aligned} \tag{5.2.52}$$

We now make a specific choice of the free parameters, $\boldsymbol{d}$ and $\boldsymbol{\gamma}$, which is naturally justified by the picture canvassed in the above sections. As a matter of fact, we will force polymers to reflect an "extreme" version of the picture. Precisely:

- instead of considering polymers which are *essentially* directed close to the endpoints (recall in particular Figure 5.4) we will consider polymers which are *fully* directed in these regimes. We will achieve this by fixing a small $m = 205 \ll K$ (as already mentioned, we will choose K large enough). With $\mathsf{d} = (\mathsf{d}_1, \dots, \mathsf{d}_K)$ the optimal

Hamming distance as in (5.2.44) from Theorem 1 we then set

$$\mathbf{d}_{opt} = \Bigg(\underbrace{1/K, \ldots, 1/K}_{m-\text{times}}, \mathsf{d}_{m+1}, \mathsf{d}_{m+2}, ..., \mathsf{d}_{K-m}, \underbrace{1/K, \ldots, 1/K}_{m-\text{times}}\Bigg) , \qquad (5.2.53)$$

- instead of considering polymers which are *essentially* stretched between the coarse graining H-planes (recall in particular Insight 32), we will consider polymers which proceed through *exact geodesics*; this will be achieved by setting

$$\gamma_{opt} \equiv (0, \ldots, 0) . \qquad (5.2.54)$$

Denoting by L_{opt} the normalized length of paths in $\mathcal{P}_{n,K}\left\{\mathbf{d}_{opt}, \gamma_{opt}\right\}$, it holds that

$$\mathsf{L}_{opt} = \|\mathbf{d}_{opt}\|_1 . \qquad (5.2.55)$$

We then focus on the ensuing subset $\mathcal{P}_{n,K}\left\{\mathbf{d}_{opt}, \gamma_{opt}\right\} \subset \widetilde{\Pi}_{n,\mathsf{L}_{opt}n}$. A graphical rendition of these polymers, which are only marginally shorter than $\mathsf{L} = \sqrt{2}\mathsf{E}$ (see (5.2.59) below for more on this), is given in Figure 5.10.

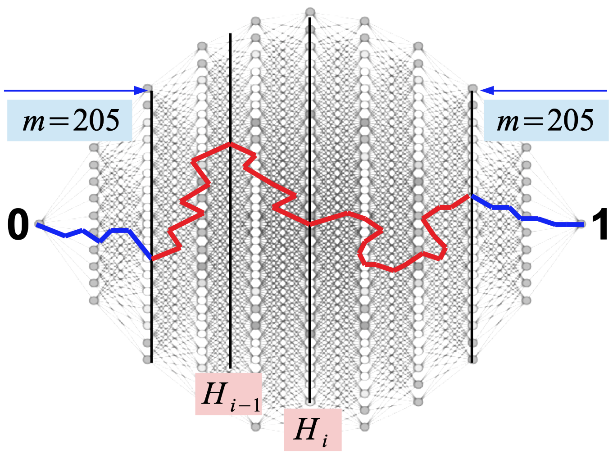

Figure 5.10: A polymer in $\mathcal{P}_{n,K}$: the blue substrand is fully directed. The red substrands connect the H-planes of the coarse graining through stretched paths, i.e. geodesics.

Since Hamming-depths and detours are specified, we lighten henceforth notation by

$$\mathcal{P}_{n,K} \equiv \mathcal{P}_{n,K}\left\{\mathbf{d}_{opt}, \gamma_{opt}\right\} . \qquad (5.2.56)$$

Let now $\epsilon > 0$, and consider the subset of polymers

$$\mathcal{E}^{\epsilon}_{n,K} \equiv \pi \in \mathcal{P}_{n,K} \text{ with energies } X_\pi \leq \mathsf{E} + \epsilon , \qquad (5.2.57)$$

namely those paths which *i)* are fully directed close to the endpoints, *ii)* connect the coarse graining H-planes in the core of the hypercube through geodesics, *iii)* and which reach an ϵ-neighborhood of the ground state energy. Our main result states that such polymers do, in fact, exist:

Theorem 2. (The geometry of optimal polymers). *For $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\# \mathcal{E}^{\epsilon}_{n,K} \geq 1\right) = 1. \tag{5.2.58}$$

The proof of Theorem 2, which eventually boils down to an application of the Paley-Zygmund inequality, is both technically demanding and long, and will be given in the next sections. Before seeing how this goes, some comments are in order.

First, we remark that the length of the substrands connecting the H-planes (which is related to the $\mathsf{a}'s$) does not appear explicitely in the statement of Theorem 2, and neither do the sub-energies. This is again due to the fact that, in line with Insight 29, uniformly spread lengths/energies will be hiding behind the optimal Hamming-depths.

Second, we point out that Theorem 2, when combined with the simple *lower bound* discussed in the Introduction, yields a constructive proof of Martinsson's Theorem.

Lastly, and with the unsettled issue of fluctuations in mind, we shall dwell on a conceptually intricate aspect of the theorem, namely the nature of the parameter K encoding the density of hyperplanes for the coarse graining. One perhaps expects that larger constants lead to more accurate pictures, but this is only to some extent correct. In fact, too large hyperplane-density would even lead to inconsistencies: higher and higher densities "unbend" the strands, ultimately to the point of complete directedness, but this, in turn, would starkly contradict the crucial feature of optimal polymers, namely that their length is *larger* than the dimension. A delicate balance must therefore be met. As we will see in the course of the second moment implementation, see (5.6.52), (5.6.82), (5.6.87) and (5.6.134) below, for the present purpose of analyzing the ground state to leading order, it indeed suffices to take a large but *finite* $K = \max\{2 \times 10^7, m\epsilon^{-2}\}$. How fast (in the dimension n) the hyperplane-density can be allowed to grow is an interesting, and important issue, which unfortunately eludes us.

We conclude this section with the aforementioned result concerning the concentration of the length of optimal polymers, as it provides a neat round-off of the picture. We emphasize that this result has first been proved by Martinsson via BTP-comparison [43], whereas our short proof will rely on Laplace method/saddle point analysis.

To formulate, remark that Theorem 2 involves paths of length L_{opt}; by a more detailed study of Taylor's remainder term in (5.2.47), (5.2.50) and (5.2.51), and recalling that $\mathsf{L} = \sqrt{2}\mathsf{E}$, it can be plainly checked that

$$0 \leq \mathsf{L} - \mathsf{L}_{opt} \leq \frac{m}{K}. \tag{5.2.59}$$

In other words, for large hyperplane density, the difference between L_{opt} and L is vanishing. Our second main result states that the length L is, in fact, optimal:

Theorem 3. (Concentration of the polymer's length). *For $\epsilon > 0$ and $a > \frac{\mathsf{E}}{2} + \sqrt{2}\mathsf{E} + \frac{1}{\sqrt{2}}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\#\left\{\pi \in \Pi_n:\ X_\pi \leq \mathsf{E} + \epsilon^2, \frac{1}{n}|l_\pi(\mathbf{0},\mathbf{1}) - \mathsf{L}n| \geq a\epsilon\right\} \geq 1\right) = 0\,. \tag{5.2.60}$$

Remark 33. *The proof of the above Theorem, which is given in Section 5.8 below, suggests (albeit feebly) that the (ϵ^2, ϵ)-scaling in* (5.2.60) *is, in fact, optimal, and this in turn suggests that a central limit theorem applies for the optimal length.*

The rest of the paper is organised as follows. In the next Section 5.3 we will provide a proof of Theorem 1. In Section 5.4, and for technical reasons which will become clear in the course of the treatment, some additional restrictions on the candidate optimal polymers will be specified: this will lead to the identification of a subset of $\mathcal{P}_{n,K}$ on which we will henceforth focus our attention. Specifying these additional requirements will have an impact on the first moment as controlled in Theorem 1, and these modifications will be dealt with in Section 5.5. Section 5.6 forms the main body of the paper: there we will set up the second moment approach, postponing, however, the highly technical issues concerning the required path-counting to Section 5.7. Finally, the proof of optimality of the length L is given in Section 5.8.

5.3 The optimal Hamming distance: proof of Theorem 1

Recall that $\varphi(x) = x^x$ for $x \geq 0$, with the convention $0^0 = 1$. We shorten

$$g_{j,K}(x) \equiv \frac{\sinh(\mathsf{a}_j\mathsf{E})^x \cosh(\mathsf{a}_j\mathsf{E})^{(1-x)} \varphi\left(\frac{j-1}{K}\right)\varphi\left(1-\frac{j-1}{K}\right)}{\varphi\left(\frac{x}{2}-\frac{1}{2K}\right)\varphi\left(\frac{j-1}{K}-\left(\frac{x}{2}-\frac{1}{2K}\right)\right)\varphi\left(\frac{x}{2}+\frac{1}{2K}\right)\varphi\left(1-\frac{j-1}{K}-\left(\frac{x}{2}+\frac{1}{2K}\right)\right)}\,, \tag{5.3.1}$$

in which case, in virtue of (5.2.37), we may represent the $\mathcal{F}$-function as

$$\mathcal{F}_{\mathsf{a},K}(\boldsymbol{d}) = \prod_{j=1}^{K} g_{j,K}(d_j)\,. \tag{5.3.2}$$

Since the terms in the product on the r.h.s. are non-interacting, we clearly have

$$\max_{\boldsymbol{d}}\{\mathcal{F}_{\mathsf{a},K}(\boldsymbol{d})\} = \prod_{j=1}^{K} \max_{x\geq 0}\{g_{j,K}(x)\}. \tag{5.3.3}$$

We now claim that

$$\prod_{j=1}^{K} \max_{x\geq 0}\{g_{j,K}(x)\} = 1, \tag{5.3.4}$$

and

$$\arg\max_{x\geq 0}\ \{g_{j,K}(x)\} = \mathsf{d}_j, \tag{5.3.5}$$

with d_j as in (5.2.44).

We will prove (5.3.5) first. We begin with the cases $j = 1, K$ and claim that

$$\arg\max_{x\geq 0} g_{1,K}(x) = \arg\max_{x\geq 0} g_{K,K}(x) = \frac{1}{K}\,, \tag{5.3.6}$$

and

$$\frac{1}{K} = \mathsf{d}_1 = \mathsf{d}_K. \tag{5.3.7}$$

In fact, $g_{1,K}(x)$ involves the terms

$$\begin{gathered} \varphi\left(\frac{x}{2} - \frac{1}{2K}\right), \\ \varphi\left(\frac{j-1}{K} - \left\{\frac{x}{2} - \frac{1}{2K}\right\}\right)\Big|_{j=1} = \varphi\left(\frac{1}{2K} - \frac{x}{2}\right), \end{gathered} \tag{5.3.8}$$

but for both to be properly defined it must hold

$$\frac{x}{2} - \frac{1}{2K} \geq 0, \quad \text{and} \quad \frac{1}{2K} - \frac{x}{2} \geq 0, \tag{5.3.9}$$

implying $x = \frac{1}{K}$. A similar reasoning applies to $g_{K,K}$, and (5.3.6) is settled. Claim (5.3.7) follows from (5.2.25) for the $j = 1$ case, whereas the $j = K$ case follows by symmetry, see in particular (5.2.28).

Concerning the other indices, we fix $j \in \{2, \ldots, K-1\}$ and shorten, for $x \geq 0$,

$$g_{j,K}(x) \equiv \frac{N_{j,K}(x)}{D_{j,K}(x)}, \tag{5.3.10}$$

where

$$N_{j,K}(x) \equiv \sinh(\mathsf{a}_j\mathsf{E})^x \cosh(\mathsf{a}_j\mathsf{E})^{(1-x)} \varphi\left(\frac{j-1}{K}\right)\varphi\left(1 - \frac{j-1}{K}\right), \tag{5.3.11}$$

and

$$D_{j,K}(x) \equiv \varphi\left(\frac{x}{2} - \frac{1}{2K}\right)\varphi\left(\frac{j}{K} - \frac{x}{2} - \frac{1}{2K}\right)\varphi\left(\frac{x}{2} + \frac{1}{2K}\right)\varphi\left(1 - \frac{j}{K} - \frac{x}{2} + \frac{1}{2K}\right). \tag{5.3.12}$$

Taking the x-derivative, we see that

$$g_{j,K}(x)' > 0 \iff N_{j,K}(x)' D_{j,K}(x) > N_{j,K}(x) D_{j,K}(x)'. \tag{5.3.13}$$

An elementary computation then yields

$$N_{j,K}(x)' = N_{j,K}(x) \log(\tanh(\mathsf{a}_j \mathsf{E})), \tag{5.3.14}$$

and

$$D_{j,K}(x)' = \frac{1}{2} D_{j,K}(x) \log \left\{ \frac{(\frac{x}{2} - \frac{1}{2K})(\frac{x}{2} + \frac{1}{2K})}{(\frac{j}{K} - \frac{x}{2} - \frac{1}{2K})(1 - \frac{j}{K} - \frac{x}{2} + \frac{1}{2K})} \right\}. \tag{5.3.15}$$

Combining (5.3.13), (5.3.14) and (5.3.15), we therefore get

$$g_{i,K}(x)' > 0 \iff \tanh(\mathsf{a}_j \mathsf{E})^2 > \frac{(\frac{x}{2} - \frac{1}{2K})(\frac{x}{2} + \frac{1}{2K})}{(\frac{j}{K} - \frac{x}{2} - \frac{1}{2K})(1 - \frac{j}{K} - \frac{x}{2} + \frac{1}{2K})}. \tag{5.3.16}$$

Consider now

$$\tanh(\mathsf{a}_j \mathsf{E})^2 = \frac{(\frac{x}{2} - \frac{1}{2K})(\frac{x}{2} + \frac{1}{2K})}{\left(\frac{j}{K} - \frac{x}{2} - \frac{1}{2K}\right)\left(1 - \frac{j}{K} - \frac{x}{2} + \frac{1}{2K}\right)}. \tag{5.3.17}$$

This is a quadratic equation (in x), whose unique positive solution is given by

$$\hat{x} \equiv -\sinh(\mathsf{a}_j \mathsf{E})^2 + \sqrt{\sinh(\mathsf{a}_j \mathsf{E})^4 + 4\sinh(\mathsf{a}_j \mathsf{E})^2 \left\{ \frac{2j-1}{2K} - \frac{j(j-1)}{K^2} \right\} + \frac{1}{K^2}}. \tag{5.3.18}$$

A straightforward analysis shows that the quotient on the r.h.s. of (5.3.16) is, in fact, increasing in x: in other words, the x-derivative $g_{i,K}'$ is positive for $x < \hat{x}$ and negative for $x > \hat{x}$, implying that $\hat{x}$ is indeed the extremal point. To finish the proof of (5.3.5) it thus remains to show that $\hat{x} = \mathsf{d}_j$, i.e. that $\hat{x} = \sinh(\mathsf{a}_j \mathsf{E}) \cosh((1 - \mathsf{a}_j)\mathsf{E})$. In order to do so, we will avoid the use of the explicit formulation (5.3.18), but rely rather on the expression (5.3.17) and the following

Lemma 34. *Let $d \in \mathbb{R}$ satisfy*

$$\frac{d}{2} - \frac{1}{2K} = \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) \sinh(\mathsf{a}_j \mathsf{E}) \sinh(\underline{\mathsf{a}}_j \mathsf{E}). \tag{5.3.19}$$

Then the above, and the following relations are all equivalent:

$$1 - \frac{j}{K} - \frac{d}{2} + \frac{1}{2K} = \cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) \cosh(\mathsf{a}_j \mathsf{E}) \sinh(\underline{\mathsf{a}}_j \mathsf{E}), \tag{5.3.20}$$

$$\frac{d}{2} + \frac{1}{2K} = \cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) \sinh(\mathsf{a}_j \mathsf{E}) \cosh(\underline{\mathsf{a}}_j \mathsf{E}), \tag{5.3.21}$$

$$\frac{j}{K} - \frac{d}{2} - \frac{1}{2K} = \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) \cosh(\mathsf{a}_j \mathsf{E}) \cosh(\underline{\mathsf{a}}_j \mathsf{E}). \tag{5.3.22}$$

It follows in particular, that for such d it holds $d = \hat{x}$, and $d = \mathsf{d}_j$.

Proof of Lemma 34. We first prove the equivalence of

$$(5.3.19) \iff (5.3.20) \iff (5.3.21) \iff (5.3.22), \tag{5.3.23}$$

Indeed, by (5.2.25) and the fact that

$$\sinh(\bar{\mathsf{a}}_j \mathsf{E})\cosh(\underline{\mathsf{a}}_j \mathsf{E}) + \cosh(\bar{\mathsf{a}}_j \mathsf{E})\sinh(\underline{\mathsf{a}}_j \mathsf{E}) = 1, \tag{5.3.24}$$

it holds:

$$\sinh(\underline{\mathsf{a}}_j \mathsf{E})\cosh(\bar{\mathsf{a}}_j \mathsf{E}) = 1 - \frac{j}{K} \tag{5.3.25}$$

for all $j = 1 \ldots K$. Relation (5.3.19) therefore implies that

$$\begin{aligned} 1 - \frac{j}{K} - \frac{d}{2} + \frac{1}{2K} &= 1 - \frac{j}{K} - \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j \mathsf{E})\sinh(\underline{\mathsf{a}}_j \mathsf{E}) \\ &= \{\cosh(\bar{\mathsf{a}}_j \mathsf{E}) - \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j \mathsf{E})\}\sinh(\underline{\mathsf{a}}_j \mathsf{E}) \\ &= \cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\mathsf{a}_j \mathsf{E})\sinh(\underline{\mathsf{a}}_j \mathsf{E}) \end{aligned} \tag{5.3.26}$$

the second equality with (5.3.25) and the last by the addition formula $\cosh(a+b) = \cosh(a)\cosh(b) + \sinh(a)\sinh(b)$. Thus,

$$(5.3.19) \iff (5.3.20). \tag{5.3.27}$$

A similar computation gives that

$$(5.3.21) \iff (5.3.22). \tag{5.3.28}$$

It remains to prove that

$$(5.3.19) \iff (5.3.21). \tag{5.3.29}$$

To see this we note that (5.3.19) yields

$$\frac{d}{2} + \frac{1}{2K} = \sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j \mathsf{E})\sinh(\underline{\mathsf{a}}_j \mathsf{E}) + \frac{1}{K} \tag{5.3.30}$$

but combining the fundamental r.h.s (5.2.25) and (5.3.25) gives that

$$\frac{1}{K} = \sinh(\underline{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) - \sinh(\underline{\mathsf{a}}_j \mathsf{E})\cosh(\bar{\mathsf{a}}_j \mathsf{E}) \tag{5.3.31}$$

Thus, by (5.3.31), we see that

$$\begin{aligned} (5.3.30) &= \sinh(\underline{\mathsf{a}}_j \mathsf{E})\left(\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j E) - \cosh(\bar{\mathsf{a}}_j \mathsf{E})\right) + \sinh(\underline{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) \\ &= -\sinh(\underline{\mathsf{a}}_j \mathsf{E})\cosh(\mathsf{a}_j \mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) + \sinh(\underline{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}), \end{aligned} \tag{5.3.32}$$

the last equality again by the addition formula $\cosh(a+b) = \cosh(a)\cosh(b)+\sinh(a)\sinh(b)$. Hence

$$\begin{aligned}(5.3.32) &= \big(-\sinh(\underline{\mathsf{a}}_j\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E}) + \sinh(\underline{\mathsf{a}}_j\mathsf{E}+\mathsf{a}_j\mathsf{E})\big)\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}) \\ &= \cosh(\underline{\mathsf{a}}_j\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E}).\end{aligned} \tag{5.3.33}$$

and (5.3.23) is established.

Let now d satisfy any of the equivalent (5.3.19)-(5.3.22). It holds:

$$\begin{aligned}&\frac{(\frac{d}{2}-\frac{1}{2K})(\frac{d}{2}+\frac{1}{2K})}{(\frac{j}{K}-\frac{d}{2}-\frac{1}{2K})(1-\frac{j}{K}-\frac{d}{2}+\frac{1}{2K})} = \\ &\quad= \frac{\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E})\times\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\mathsf{a}_j\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E})}{\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E})\times\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\mathsf{a}_j\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E})} \\ &\quad= \tanh(\mathsf{a}_j\mathsf{E})^2,\end{aligned} \tag{5.3.34}$$

hence, by uniqueness of the (positive) solution of (5.3.17), we deduce that $d = \hat{x}$.

Finally, it holds:

$$\begin{aligned}d &= \frac{d}{2}+\frac{1}{2K}+\frac{d}{2}-\frac{1}{2K} \\ &= \sinh(\mathsf{a}_i\mathsf{E})\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E}) + \sinh(\mathsf{a}_i\mathsf{E})\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E}),\end{aligned} \tag{5.3.35}$$

the last equality by (5.3.19) and (5.3.21), hence

$$\begin{aligned}d &= \sinh(\mathsf{a}_i\mathsf{E})\times\big\{\cosh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\cosh(\underline{\mathsf{a}}_j\mathsf{E})+\sinh(\bar{\mathsf{a}}_{j-1}\mathsf{E})\sinh(\underline{\mathsf{a}}_j\mathsf{E})\big\} \\ &= \sinh(\mathsf{a}_j\mathsf{E})\times\cosh((1-\mathsf{a}_j)\mathsf{E}),\end{aligned} \tag{5.3.36}$$

by the addition formula for hyperbolic functions (and using that $\bar{\mathsf{a}}_{j-1}+\underline{\mathsf{a}}_j = 1-\mathsf{a}_j$, by definition), settling the claim that $d = \mathsf{d}_j$.

□

The remaining Claim (5.3.4) is taken care of by the following Lemma, which tracks the evolution of the g-product while changing the hyperplane-index.

Lemma 35 (Evolution Lemma). *For any $i = 1\dots K$, it holds:*

$$\prod_{j=1}^{i} g_{j,K}(\mathsf{d_j}) = \left[\frac{\sinh(\bar{\mathsf{a}}_i\mathsf{E})}{\frac{i}{K}}\right]^{\frac{i}{K}}\left[\frac{\cosh(\bar{\mathsf{a}}_i\mathsf{E})}{1-\frac{i}{K}}\right]^{1-\frac{i}{K}}. \tag{5.3.37}$$

Furthermore,

$$\prod_{j=1}^{K} g_{j,K}(\mathsf{d_j}) = 1\,. \tag{5.3.38}$$

Proof. We will proceed by induction over i. The cases $K = 1, 2$ are trivial, so let $K \geq 3$. Recalling that $\mathsf{d}_1 = \frac{1}{K}$, we therefore have that

$$g_{1,K}(\mathsf{d}_1) = \left[\frac{\sinh(\mathsf{a}_1\mathsf{E})}{\frac{1}{K}}\right]^{\frac{1}{K}} \left[\frac{\cosh(\mathsf{a}_1\mathsf{E})}{1-\frac{1}{K}}\right]^{1-\frac{1}{K}}, \tag{5.3.39}$$

which settles the base case $i = 1$. We thus assume that (5.3.37) holds for an $i \in \{1, K-2\}$, and show that this implies the validity of the $(i+1)$-case, namely that

$$\left[\frac{\sinh(\bar{\mathsf{a}}_i\mathsf{E})}{\frac{i}{K}}\right]^{\frac{i}{K}} \left[\frac{\cosh(\bar{\mathsf{a}}_i\mathsf{E})}{1-\frac{i}{K}}\right]^{1-\frac{i}{K}} g_{i+1,K}(\mathsf{d}_{i+1}) = \left[\frac{\sinh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{\frac{i+1}{K}}\right]^{\frac{i+1}{K}} \left[\frac{\cosh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{1-\frac{i+1}{K}}\right]^{1-\frac{i+1}{K}}. \tag{5.3.40}$$

Remark that by (5.3.17),

$$\begin{aligned} \sinh(\mathsf{a}_{i+1}\mathsf{E})^{\mathsf{d}_{i+1}}\cosh(\mathsf{a}_{i+1}\mathsf{E})^{1-\mathsf{d}_{i+1}} &= \tanh(\mathsf{a}_{i+1}\mathsf{E})^{\mathsf{d}_{i+1}}\cosh(\mathsf{a}_{i+1}\mathsf{E}) \\ &= \left[\frac{(\frac{d_{i+1}}{2} - \frac{1}{2K})(\frac{d_{i+1}}{2} + \frac{1}{2K})}{\left(\frac{i+1}{K} - \frac{d_{i+1}}{2} - \frac{1}{2K}\right)(1 - \frac{i+1}{K} - \frac{d_{i+1}}{2} + \frac{1}{2K})}\right]^{\frac{d_{i+1}}{2}} \cosh(\mathsf{a}_{i+1}\mathsf{E})\,. \end{aligned} \tag{5.3.41}$$

By definition of $g_{i+1,K}$, the above, and simple rearrangements, we thus have

$$g_{i+1,K}(\mathsf{d}_{i+1}) = \frac{(\frac{\mathsf{d}_{i+1}}{2} - \frac{1}{2K})^{\frac{1}{2K}} \cosh(\mathsf{a}_{i+1}E)\left(\frac{i}{K}\right)^{\frac{i}{K}}(1-\frac{i}{K})^{1-\frac{i}{K}}}{(\frac{i+1}{K} - \frac{\mathsf{d}_{i+1}}{2} - \frac{1}{2K})^{\frac{i+1}{K}-\frac{1}{2K}}(\frac{\mathsf{d}_{i+1}}{2} + \frac{1}{2K})^{\frac{1}{2K}}(1 - \frac{i+1}{K} - \frac{\mathsf{d}_{i+1}}{2} + \frac{1}{2K})^{1-\frac{i+1}{K}+\frac{1}{2K}}}. \tag{5.3.42}$$

Thus (5.3.40) is equivalent to prove that

$$\begin{aligned} &\left[\frac{\frac{\mathsf{d}_{i+1}}{2} + \frac{1}{2K}}{\frac{\mathsf{d}_{i+1}}{2} - \frac{1}{2K}}\right]^{\frac{1}{2K}} \left[\frac{i+1}{K} - \frac{\mathsf{d}_{i+1}}{2} - \frac{1}{2K}\right]^{\frac{i+1}{K}-\frac{1}{2K}} \left[1 - \frac{i+1}{K} - \frac{\mathsf{d}_{i+1}}{2} + \frac{1}{2K}\right]^{1-\frac{i+1}{K}+\frac{1}{2K}} \\ &\qquad = \frac{\sinh(\bar{\mathsf{a}}_i\mathsf{E})^{\frac{i}{K}}\cosh(\bar{\mathsf{a}}_i\mathsf{E})^{1-\frac{i}{K}}\cosh(\mathsf{a}_{i+1}\mathsf{E})}{\left[\frac{\sinh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{\frac{i+1}{K}}\right]^{\frac{i+1}{K}}\left[\frac{\cosh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{1-\frac{i+1}{K}}\right]^{1-\frac{i+1}{K}}}\,. \end{aligned} \tag{5.3.43}$$

We now rewrite the term on the l.h.s. (5.3.43) as

$$\begin{aligned} &\left[\frac{(\frac{\mathsf{d}_{i+1}}{2} + \frac{1}{2K})(\frac{i+1}{K} - \frac{\mathsf{d}_{i+1}}{2} - \frac{1}{2K})}{(\frac{\mathsf{d}_{i+1}}{2} - \frac{1}{2K})(1 - \frac{i+1}{K} - \frac{\mathsf{d}_{i+1}}{2} + \frac{1}{2K})}\right]^{\frac{1}{2K}} \\ &\qquad\times \left[\frac{\frac{i+1}{K} - \frac{\mathsf{d}_{i+1}}{2} - \frac{1}{2K}}{1 - \frac{i+1}{K} - \frac{\mathsf{d}_{i+1}}{2} + \frac{1}{2K}}\right]^{\frac{i}{K}} \\ &\qquad\qquad\times \left[1 - \frac{i+1}{K} - \frac{d_{i+1}}{2} + \frac{1}{2K}\right], \end{aligned} \tag{5.3.44}$$

and the term on the r.h.s. of (5.3.43) as

$$\begin{aligned}
&\left[\left(\frac{\frac{i+1}{K}\cosh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{(1-\frac{i+1}{K})\sinh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}\right)^2\right]^{\frac{1}{2K}}\\
&\qquad\times\left[\frac{\frac{i+1}{K}\tanh(\bar{\mathsf{a}}_i\mathsf{E})\cosh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}{(1-\frac{i+1}{K})\sinh(\bar{\mathsf{a}}_{i+1}\mathsf{E})}\right]^{\frac{i}{K}}\\
&\qquad\qquad\times\frac{\cosh(\bar{\mathsf{a}}_i E)\cosh(\mathsf{a}_{i+1}E)(1-\frac{i+1}{K})}{\cosh(\bar{\mathsf{a}}_{i+1}E)} \qquad (5.3.45)\\
&=\left[\left(\frac{\cosh(\underline{\mathsf{a}}_{i+1}\mathsf{E})}{\sinh(\underline{\mathsf{a}}_{i+1}\mathsf{E})}\right)^2\right]^{\frac{1}{2K}}\\
&\qquad\times\left[\frac{\tanh(\bar{\mathsf{a}}_i\mathsf{E})\cosh(\underline{\mathsf{a}}_{i+1}\mathsf{E})}{\sinh(\underline{\mathsf{a}}_{i+1}\mathsf{E})}\right]^{\frac{i}{K}}\\
&\qquad\qquad\times\cosh(\bar{\mathsf{a}}_i\mathsf{E})\cosh(\mathsf{a}_{i+1}\mathsf{E})\sinh(\underline{\mathsf{a}}_{i+1}\mathsf{E}),
\end{aligned}$$

the last step by (5.2.25) and (5.3.25). But by (5.3.19), (5.3.20), (5.3.21) and (5.3.22), the terms raised to the same powers in (5.3.44) and the r.h.s. of (5.3.45) coincide, settling the induction step.

We now move to (5.3.38). It holds:

$$\begin{aligned}
\prod_{j=1}^{K} g_{j,K}(\mathsf{d}_j) &= \prod_{j=1}^{K-1} g_{j,K}(\mathsf{d}_j)\; g_{K,K}(\mathsf{d}_K)\\
&=\left[\frac{\sinh(\bar{\mathsf{a}}_{K-1}\mathsf{E})}{1-\frac{1}{K}}\right]^{1-\frac{1}{K}}\left[\frac{\cosh(\bar{\mathsf{a}}_{K-1}\mathsf{E})}{\frac{1}{K}}\right]^{\frac{1}{K}}\sinh(\mathsf{a}_K\mathsf{E})^{\frac{1}{K}}\cosh(\mathsf{a}_K\mathsf{E})^{1-\frac{1}{K}} \qquad (5.3.46)\\
&=\left[\frac{\sinh(\bar{\mathsf{a}}_{K-1}\mathsf{E})\cosh(\mathsf{a}_K\mathsf{E})}{1-\frac{1}{K}}\right]^{1-\frac{1}{K}}\left[\frac{\cosh(\bar{\mathsf{a}}_{K-1}\mathsf{E})\sinh(\mathsf{a}_K\mathsf{E})}{\frac{1}{K}}\right]^{\frac{1}{K}},
\end{aligned}$$

the second equality by the induction step, and the third by simple rearrangements. By the $\mathsf{a}'s$ symmetry (5.2.28), and the normalization $\sum_{i=1}^K \mathsf{a}_i = 1$, it thus holds

$$(5.3.46) = \left[\frac{\sinh(\bar{\mathsf{a}}_{K-1}\mathsf{E})\cosh(\underline{\mathsf{a}}_{K-1}\mathsf{E})}{1-\frac{1}{K}}\right]^{1-\frac{1}{K}}\left[\frac{\cosh((1-\mathsf{a}_1)\mathsf{E})\sinh(\mathsf{a}_1\mathsf{E})}{\frac{1}{K}}\right]^{\frac{1}{K}} = 1, \qquad (5.3.47)$$

the last equality by the fundamental (5.2.25). □

5.4 Taming optimal polymers

In order to prove our main result Theorem 2, we will show non-emptiness of a *subset* of $\mathcal{P}_{n,K}$, whose paths satisfy additional properties. As a matter of fact, we will introduce two additional restrictions: the first one, which is explained in Section 5.4.1, concerns the *geometry* of paths, i.e. their combinatorial properties. The second restriction, explained in Section 5.4.2, concerns the way *energies* are distributed along the paths. Both restrictions will be of course inspired by/in line with the above Insights. We emphasize that the reason for restricting the candidate polymers further is here chiefly technical: the additional requirements we are about to introduce will in fact lead to a considerable simplification of some otherwise daunting combinatorial estimates.

5.4.1 A sprinckle of microstructure

We introduce yet another coarse graining: for $i = 0 \dots K-1$, we split the region between two consecutive hyperplanes H_{i-1} and H_i further, into K' additional slabs:

$$H'_{i,j} \equiv \left\{ v \in V_n, d(0,v) = \left(i + \frac{j}{K'} \right) \hat{n}_K \right\}, \qquad j = 0 \dots K', \tag{5.4.1}$$

(remark that $H'_{i,0} = H_i$ and $H'_{i,K'} = H_{i+1}$), and focus henceforth on the subset

$$\begin{aligned} \mathcal{P}_{n,K,K'} \equiv \; & \text{all polymers } \pi \in \mathcal{P}_{n,K} \text{ which cover} \\ & \text{a (normalized) Hamming distance } (\mathsf{ef_i} + \mathsf{eb_i})/K' \\ & \text{while connecting the hyperplanes } H'_{i,j} \text{ and } H'_{i,j+1}, \\ & \text{for } j = 0 \dots K' \text{ and } i = 1 \dots K-1. \end{aligned} \tag{5.4.2}$$

The subset $\mathcal{P}_{n,K,K'}$ is of course motivated by Insight 31: adding an additional level of coarse graining and spreading the backsteps as evenly as possible among the K'-slabs, allows to rule out polymers where backsteps tend to accumulate, cfr. Figure 5.11 and 5.12 below.

Finally, we render the H-hyperplanes (of the coarser layer) *repulsive*, i.e. we force paths to cross them only once. As we will see shortly, see Lemma 36 below, this can be achieved by considering the following (sub)subset of polymers:

$$\begin{aligned} \mathcal{P}^{\mathsf{rep}}_{n,K,K'} \equiv \; & \text{all polymers } \pi \in \mathcal{P}_{n,K,K'} \text{ which connect the hyperplanes } H'_{i,0} \text{ and } H'_{i,1} \\ & \text{by first making } (\mathsf{ef}_i\, \hat{n}_{K'}) \text{ steps forward and only then } (\mathsf{eb}_i\, \hat{n}_{K'}) \text{ backsteps,} \\ & \text{and which connect the hyperplanes } H'_{i,K'-1} \text{ and } H'_{i,K'} \\ & \text{by first making } (\mathsf{eb}_i\, \hat{n}_{K'}) \text{ backsteps, and only then } (\mathsf{ef}_i\, \hat{n}_{K'}) \text{ steps forward}, \\ & \text{for } i = 1 \dots K. \end{aligned} \tag{5.4.3}$$

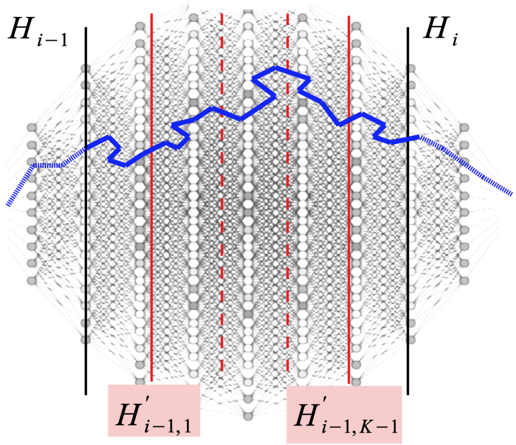

Figure 5.11: The backsteps (five in total) are spread as evenly as possible: one for each sublayer H'.

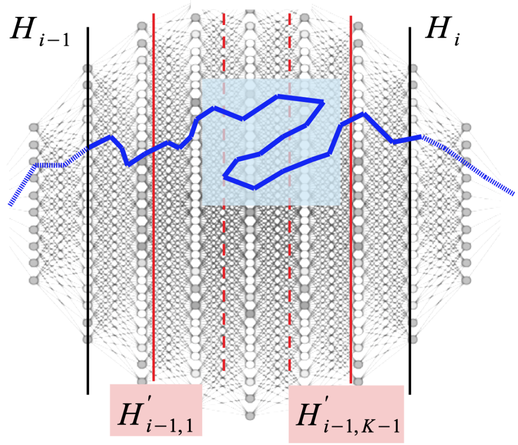

Figure 5.12: The five backsteps are lumped together: this polymer wouldn't belong to $\mathcal{P}_{n,K,K'}$.

Note that $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ is still a deterministic set. A graphical rendition is given in Figure 5.13 below.

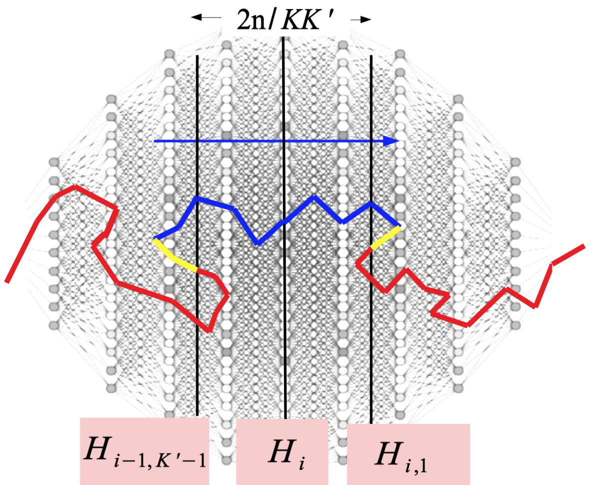

Figure 5.13: A path in $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$: red edges correspond to the free evolution of the path, yellow edges are backsteps, and blue edges are forward steps.

Remark that, by construction,

$$\mathcal{P}^{\mathsf{rep}}_{n,K,K'} \subset \mathcal{P}_{n,K,K'} \subset \mathcal{P}_{n,K}\left\{\mathbf{d}_{opt}, \gamma_{opt}\right\} . \tag{5.4.4}$$

Our main Theorem 2 will therefore follow as soon as we prove that one can find polymers in $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ which reach the ground state energy. Before seeing how this goes, here is the aforementioned result stating that H-hyperplanes are indeed repulsive:

Lemma 36. *For $K \geq 1$ the following holds true: a polymer $\pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ crosses the hyperplanes $H_1, \dots, H_K$ only once.*

Proof. The statement is trivial in the directed phase, so let $i \in \{m \dots K-m\}$.

There is of course a certain *directivity* in the polymers' evolution: this is captured by the fact that $\mathsf{ef}_i > \mathsf{eb}_i$ for all $i = 1 \dots K$ (see in particular the second relation in (5.2.36)), and graphically represented by evolutions "from the left to the right".

Sticking to this graphical convention, we begin with the case "to the right of the H_i-hyperplane": after crossing this hyperplane, a path $\pi \in \mathcal{P}_{n,K,K'}$ is bound to first make $(\mathsf{ef}_i\ \hat{n}_{K'})$ steps to the right (forward) and only then to make $(\mathsf{eb}_i\ \hat{n}_{K'})$ steps to the left (backwards). At this point, and by construction, the polymer will find itself on $H'_{i,1}$. Continuing its evolution, the polymer will eventually reach from there the next hyperplane $H'_{i,2}$, again through $(\mathsf{ef}_i\ \hat{n}_{K'})$ steps to the right, and $(\mathsf{eb}_i\ \hat{n}_{K'})$ steps to the left. Since in this phase no restriction is imposed on the *order* of back- and forwardsteps, it could thus happen that the polymer first performs all available steps to the left, in one fell swoop: this would increase the proximity of the polymer to H_i, with the hyperplane potentially even crossed for a second time. However, we claim that even in such worst case scenario, the polymer will find itself well to the right of H_i. In other words we claim that

$$\mathsf{ef}_i n_{K'} - 2\mathsf{eb}_i n_{K'} > 0, \tag{5.4.5}$$

or, which is the same, that

$$\mathsf{ef}_i - 2\mathsf{eb}_i > 0. \tag{5.4.6}$$

Indeed, it follows from (5.2.37) that

$$\begin{aligned} \mathsf{ef}_i - 2\mathsf{eb}_i &= \frac{\mathsf{d}_i}{2} + \frac{1}{2K} - 2\left(\frac{\mathsf{d}_i}{2} - \frac{1}{2K}\right) \\ &= \frac{1}{K} - \left(\frac{\mathsf{d}_i}{2} - \frac{1}{2K}\right) \\ &= \frac{1}{K} - \mathsf{eb}_i, \end{aligned} \tag{5.4.7}$$

the last step again by (5.2.37). Our new claim thus states that for large enough K,

$$\frac{1}{K} - \mathsf{eb}_i > 0. \tag{5.4.8}$$

To see this, we recall that by (5.3.19), the number of effective backsteps between hyperplanes in the stretched phase satisfies

$$\mathsf{eb}_i = \sinh(\bar{\mathsf{a}}_{i-1}\mathsf{E})\sinh(\mathsf{a}_i\mathsf{E})\sinh(\underline{\mathsf{a}}_i\mathsf{E}). \tag{5.4.9}$$

Real analysis shows that

$$\arg\max_{y\in[0,1]} \sinh(y\mathsf{E})\sinh((1-y)\mathsf{E}) = \frac{1}{2}\,. \tag{5.4.10}$$

Furthermore, by (5.2.29),

$$\mathsf{a}_i\mathsf{E} \le \frac{1}{K}\,, \tag{5.4.11}$$

which, together with an elementary large-K Taylor expansion, implies that

$$\sinh(\mathsf{a}_i\mathsf{E}) = \mathsf{a}_i\mathsf{E} + \frac{(\mathsf{a}_i\mathsf{E})^3}{6} \le \frac{1}{K} + \frac{1}{6K^3} \le \frac{2}{K}\,, \tag{5.4.12}$$

for $K \ge 1$. Using (5.4.12) in (5.4.9) we get

$$\begin{aligned} \mathsf{eb}_i &\le \sinh(\bar{\mathsf{a}}_i\mathsf{E})\sinh(\underline{\mathsf{a}}_i\mathsf{E}) \times \frac{2}{K} \\ &\le \sinh\left(\frac{\mathsf{E}}{2}\right)^2 \times \frac{2}{K}. \end{aligned} \tag{5.4.13}$$

the second inequality by (5.4.10). The first term on the r.h.s. above can be easily estimated:

$$\begin{aligned} \sinh\left(\frac{\mathsf{E}}{2}\right)^2 &= \frac{1}{4}\left(e^{\mathsf{E}/2} - e^{-\mathsf{E}/2}\right)^2 = \frac{1}{4}\left(e^{\mathsf{E}} - 2 + e^{-\mathsf{E}}\right) \\ &= \frac{1}{2}\left(\cosh(\mathsf{E}) - 1\right) = \frac{1}{2}\left(\sqrt{1+\sinh^2(\mathsf{E})} - 1\right) \\ &= \frac{1}{2}\left(\sqrt{2} - 1\right), \end{aligned} \tag{5.4.14}$$

the step before last by the Pythagorean's identity for hyperbolic functions, and the last since $\sinh(\mathsf{E}) = 1$ by definition. In particular, we see that

$$\sinh\left(\frac{\mathsf{E}}{2}\right)^2 \le \frac{1}{4}\,. \tag{5.4.15}$$

Using this in (5.4.13) we thus get $\mathsf{eb}_i \le \frac{1}{2K}$, hence

$$\frac{1}{K} - \mathsf{eb}_i \ge \frac{1}{2K} > 0, \tag{5.4.16}$$

settling claim (5.4.8), and therefore (5.4.6).

Summarizing the upshot of these considerations, we thus see that *after* crossing an H-plane for the first time, the polymer will forever remain "to its right". But by symmetry, a similar line of reasoning holds also for the case "to the left", i.e. for paths making $(\mathsf{eb}_i\hat{n}_{K'})$ steps to the left, and then $(\mathsf{ef}_i\hat{n}_{K'})$ steps to the right *before* reaching such hyperplane. Lemma 36 is therefore established. □

Remark 37. *Polymers in $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ are, in fact, loopless: this follows from Lemma 36, and the property that paths make no detours between H-planes.*

5.4.2 Partitioning the energy

We will eventually implement the multiscale refinement of the second moment method [35], a procedure which involves a number of steps. The first, and key, step is to break the self-similarity of the underlying random field: this can be achieved here by allowing the first and last edges of the polymers to carry an unusually large fraction of the energy, and handling these on different footing. This procedure has already been succesfully implemented for the problem of (directed) first passage percolation in [37], see also Remark 40 below for more on this issue.

We need some additional notation: since a path $\pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ consists of a set of edges which uniquely characterises the vertices visited by the polymer, by a a slight abuse of notation we will denote by $\pi \cap H_i$ the vertices that lie both in H_i and between two edges of the π-path.

For a polymer $\pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$, we begin by writing its energy as

$$X_\pi = \mathcal{F}_\pi + \left\{ X_m(\pi) + \left[\sum_{j=m+1}^{K-m} X_{j-1,j}(\pi) \right] + X_{K-m+1}(\pi) \right\} + \mathcal{L}_\pi \,, \tag{5.4.17}$$

with the following notational conventions:

- $\mathcal{F}_\pi \equiv X_{[\pi]_1}$ is the energy of the first edge of the path;
- $X_m(\pi) \equiv \sum_{j=2}^{m\hat{n}_K} \xi_{[\pi]_j}$ is the energy of the substrand connecting the second visited vertex to the m^{th}-hyperplane, i.e. $\mathbf{0}$ to the m^{th}-hyperplane, *but with the first edge excluded*;
- For $i = m+1 \dots K-m$,

$$X_{i-1,i}(\pi) \equiv X_\pi(\pi \cap H_{i-1}, \pi \cap H_i) \tag{5.4.18}$$

  is the energy of the substrand connecting consecutive H-hyperplanes;
- $X_{K-m+1}(\pi)$ is the energy of the substrand connecting the $(K-m)^{th}$-hyperplane to $\mathbf{1}$, *but with the last edge excluded*;
- $\mathcal{L}_\pi$ is the energy of the last edge of the path.

For $\epsilon > 0$, recalling $\{\mathsf{a}_i\}_{i=1}^K$ solutions of (5.2.26) and the convention $\bar{\mathsf{a}}_m = \sum_{i \le m} \mathsf{a}_i$, we set

$$\tilde{\mathsf{a}}_{m,\epsilon} \equiv \bar{\mathsf{a}}_m \left(\mathsf{E} + \frac{\epsilon}{5} \right) + \frac{\epsilon}{5} \,, \tag{5.4.19}$$

and

$$\tilde{\mathsf{a}}_{K-m+1,\epsilon} \equiv \tilde{\mathsf{a}}_{m,\epsilon}\,, \tag{5.4.20}$$

and for $i = m+1 \ldots K-m$,

$$\mathsf{a}_{i,\epsilon} \equiv \mathsf{a}_i \left(\mathsf{E} + \frac{\epsilon}{5} \right) . \tag{5.4.21}$$

We then introduce the following subsets of polymers:

$$\mathcal{E}^{1,\epsilon}_{n,K,K'} \equiv \pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'} \text{ such that } \mathcal{F}_\pi,\ \mathcal{L}_\pi \leq \epsilon/5. \tag{5.4.22}$$

$$\begin{aligned} \mathcal{E}^{2,\epsilon}_{n,K,K'} \equiv \pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'} & \text{ such that} \\ & X_m(\pi),\ X_{K-m+1}(\pi) \leq \tilde{\mathsf{a}}_{m,\epsilon}, \\ & X_{i-1,i}(\pi) \leq \mathsf{a}_{i,\epsilon} \text{ for } i = m+1 \ldots K-m\,. \end{aligned} \tag{5.4.23}$$

Recalling that $\overline{\mathsf{a}}_m + \sum_{i=m+1}^{K-m} \mathsf{a}_i + \underline{\mathsf{a}}_{K-m} = 1$, we emphasize that the newly constructed subset consists of polymers with sub-energies

$$\overline{X}_m^{K-m+1}(\pi) \equiv X_m(\pi) + \left[\sum_{j=m+1}^{K-m} X_{j-1,j}(\pi) \right] + X_{K-m+1}(\pi) \leq \mathsf{E} + \frac{3}{5}\epsilon\,, \tag{5.4.24}$$

and with first resp. last edges carrying unusually large an energy (potentially up to $\epsilon/5$). At last, we consider the sub-subset

$$\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \equiv \mathcal{E}^{1,\epsilon}_{n,K,K'} \cap \mathcal{E}^{2,\epsilon}_{n,K,K'}\,. \tag{5.4.25}$$

Thus, by definition, the polymers in $\overline{\mathcal{E}}^{\epsilon}_{n,K,K'}$ have energies less than $\mathsf{E} + \epsilon$. A graphical rendition of this set is given in Figure 5.14 below.

5.4.3 Connecting first and last region

By definition, and recalling the inclusions (5.4.4), it clearly holds that

$$\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \subset \mathcal{E}^{\epsilon}_{n,K}\,. \tag{5.4.26}$$

In particular, non-emptiness of $\overline{\mathcal{E}}^{\epsilon}_{n,K,K'}$ will immediately yield our main Theorem 2, and this is indeed the route we take. Precisely, we will show that one can connect the first and last edges through polymers satisfying the energy requirements in the directed/stretched phases. To see how this goes, we begin with the observation that

$$\begin{aligned} \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1\right) &\geq \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1,\ \#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor \right) \\ &= \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor \right) \mathbb{P}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor \right). \end{aligned} \tag{5.4.27}$$

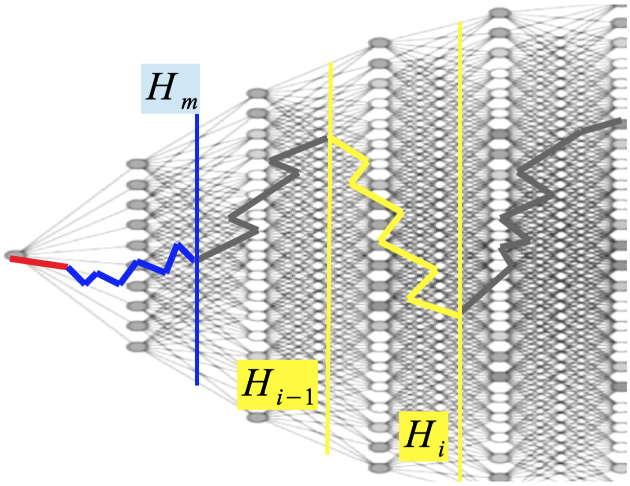

Figure 5.14: Distributing the energy in the first half of the hypercube. The first edge (red) has energy less than $\epsilon/5$. The blue strand is in the directed phase, and corresponds to $X_m(\pi) \leq \bar{\mathsf{a}}_m$. The yellow strand is in the stretched phase, it connects two consecutive H-hyperplanes with sub-energy less than $\mathsf{a}_{i,\epsilon}$. For the second half of the hypercube, an analogous (mirror) picture holds.

By independence, it clearly holds that

$$\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathbb{P}\left(\mathcal{F}_\pi \leq \frac{\epsilon}{5}\right)\mathbb{P}\left(\mathcal{L}_\pi \leq \frac{\epsilon}{5}\right)\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} = C(\epsilon)^2\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}, \tag{5.4.28}$$

where

$$C(\epsilon) \equiv 1-\exp(-\epsilon/5)\,. \tag{5.4.29}$$

We now claim that

$$\lim_{n\to\infty}\mathbb{P}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor\frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2}\right\rfloor\right) = 1. \tag{5.4.30}$$

Indeed, by Chebycheff's inequality, and for $\delta>0$,

$$\mathbb{P}\left(\left|\frac{\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{\mathbb{E}(\#\mathcal{E}^{1,\epsilon}_{n,K,K'})}-1\right| \geq \delta\right) \leq \frac{1}{\delta^2}\left\{\frac{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}{}^2\right)}{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}\right)^2}-1\right\}. \tag{5.4.31}$$

Let now $\pi \in \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ and denote by

$$f_\pi(n,k) \equiv \text{the number of paths in } \mathcal{P}^{\mathsf{rep}}_{n,K,K'} \text{ sharing } k \text{ weigthed edges with } \pi. \tag{5.4.32}$$

Since for paths in $\mathcal{E}^{1,\epsilon}_{n,K,K'}$ only the first and the last edges are weighted,

$$\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}{}^2\right) \leq \mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}\right)^2 + \#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}\left\{C(\epsilon)^3 f_\pi(n,1) + C(\epsilon)^2 f_\pi(n,2)\right\}, \tag{5.4.33}$$

the first term on the r.h.s. corresponding to the case of $k = 0$ shared edges. Using that $C(\epsilon) \leq 1$ and that $f_\pi(n,2) \leq f_\pi(n,1)$, the above becomes

$$\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}{}^2\right) \leq \mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}\right)^2 + 2\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} f_\pi(n,1)\,. \tag{5.4.34}$$

Therefore, for the r.h.s. of (5.4.31) we have

$$\frac{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}{}^2\right)}{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}\right)^2} - 1 \leq \frac{2\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} f_\pi(n,1)}{\mathbb{E}\left(\#\mathcal{E}^{1,\epsilon}_{n,K,K'}\right)^2} = \frac{2}{C(\epsilon)^4}\frac{f_\pi(n,1)}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}}\,. \tag{5.4.35}$$

Let now $f^l_\pi(n,1)$ be the number of paths which share one edge with π on the *left* of the hypercube. Clearly, $f^l_\pi(n,1) = 2f_\pi(n,1)$, hence

$$\frac{f_\pi(n,1)}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} = 2\frac{f^l_\pi(n,1)}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} \leq 2\frac{(m\hat{n}_K - 1)!}{(m\hat{n}_K)!} = \left(\frac{2K}{m}\right)\frac{1}{n}, \tag{5.4.36}$$

where for the key inequality we have used that there are $(m\hat{n}_K)!$ possibilities to reach a given (admissible) vertex on the H_m-plane, but specifying the first edge reduces such possibilities to $(m\hat{n}_K - 1)!$. Using (5.4.36) in (5.4.35) and then (5.4.31) we thus obtain

$$\mathbb{P}\left(\left|\frac{\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{\mathbb{E}(\#\mathcal{E}^{1,\epsilon}_{n,K,K'})} - 1\right| \geq \delta\right) \lesssim \frac{1}{n} \longrightarrow 0\,, \tag{5.4.37}$$

as $n \uparrow \infty$, which settles claim (5.4.30). Using the latter in (5.4.27) then yields

$$\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1\right) \geq \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor\right) - o_n(1). \tag{5.4.38}$$

Now, for any $\mathsf{J} \leq \#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$, it holds that

$$\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} \geq \mathsf{J}\right) \geq \mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right)\,, \tag{5.4.39}$$

since the more paths survive the "thinning procedure" via the energy condition on first and last edge, the higher the chance to find at least a connecting polymer which satisfies the imposed energy requirements. See Figure 5.15 for a graphical rendition.

Using (5.4.39) with

$$\mathsf{J} \equiv \left\lfloor \frac{\mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}}{2} \right\rfloor\,, \tag{5.4.40}$$

and by the Paley-Zygmund inequality, we thus get

$$\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right) \geq \frac{\mathbb{E}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right)^2}{\mathbb{E}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'}{}^2 \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right)}. \tag{5.4.41}$$

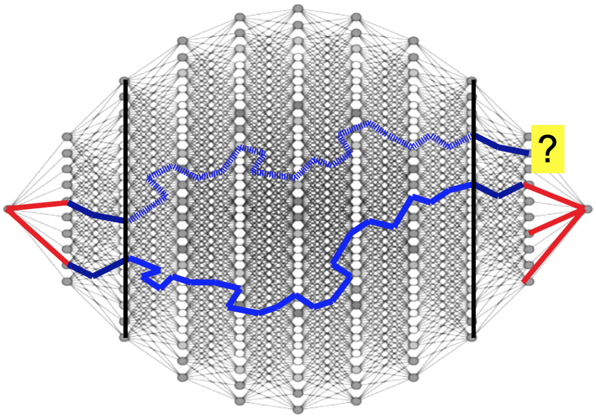

Figure 5.15: The first and last edges carrying an energy less than $\epsilon/5$ (hence surviving the thinning procedure) are drawn in red. The continuous blue strand manages to connect these edges while satisfying the energy constraints, whereas the dashed strand does not.

Consider now *any* deterministic set $\mathcal{J} \subset \mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ with cardinality $\#\mathcal{J} = \mathsf{J}$, and the subset

$$\mathcal{E}^{\epsilon}_{n,K,K'} \equiv \mathcal{E}^{2,\epsilon}_{n,K,K'} \cap \mathcal{J}\,, \tag{5.4.42}$$

which is obtained from $\mathcal{E}^{2,\epsilon}_{n,K,K'}$ via thinning procedure. We shorten $\#\mathcal{E}^{\epsilon}_{n,K,K'} \equiv \mathcal{N}^{\epsilon}_{n,K,K'}$. By independence of the sigma algebras issued from first and last edges, and the sigma algebra involving all other edges, we clearly have that

$$\mathbb{E}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right) = \mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \tag{5.4.43}$$

and

$$\mathbb{E}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'}{}^{2} \middle| \#\mathcal{E}^{1,\epsilon}_{n,K,K'} = \mathsf{J}\right) = \mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}{}^{2}\right). \tag{5.4.44}$$

Using (5.4.43) and (5.4.44) in (5.4.41), and by (5.4.38), we see that

$$\mathbb{P}\left(\#\overline{\mathcal{E}}^{\epsilon}_{n,K,K'} \geq 1\right) \geq \frac{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^{2}}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}{}^{2}\right)} - o_n(1). \tag{5.4.45}$$

Therefore, our main result Theorem 2, will be an immediate consequence of

Theorem 2′. *For $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^{2}}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}{}^{2}\right)} = 1, \tag{5.4.46}$$

for any $K' > 2\log(2)\mathsf{L}K^2$.

5.5 Π vs. $\mathcal{P}$, and a lower bound to the first moment

In Sections 5.4.1-5.4.2 we have altered the path-properties derived in Section 5.2, and this of course has relevant consequences. The following result precisely quantifies the changes to the first moment as given in Theorem 1 (which has been instrumental to all our considerations so far) once these modifications have been taken into account.

Theorem 1′. *For $\epsilon > 0$, shorten*

$$\epsilon_E \equiv \frac{\epsilon}{5\mathsf{E}}, \qquad \epsilon_{m,\mathsf{E}} \equiv \frac{\epsilon}{5\mathsf{E}} + \frac{\epsilon}{5\bar{\mathsf{a}}_m \mathsf{E}} . \tag{5.5.1}$$

Let furthermore

$$S_{n,K,m} \equiv \exp -n\left(\frac{1}{\sqrt{2}K} + \frac{\sqrt{2}m(m-1)}{K^2}\right), \qquad R_{n,K} \equiv \exp\left(-\frac{n}{K^2}\right), \tag{5.5.2}$$

and set

$$C_{n,K,m} \equiv R_{n,K} \times S_{n,K,m} . \tag{5.5.3}$$

Then for any $K' > 2\log(2)\mathsf{L}K^2$,

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \geq C_{n,K,m}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i}(1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K}\frac{Q_n}{P_n}, \tag{5.5.4}$$

where Q_n and P_n are finite degree polynomials.

Remark 38. *It will become clear in the course of the proof that the S-term in Theorem 1′ encodes the entropic cost for* stretching *the paths in Π in order to construct $\mathcal{P}_{n,K}$, whereas the R-term relates to the entropic cost for rendering the H-planes* repulsive*, i.e. in order to construct $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ out of $\mathcal{P}_{n,K}$.*

Proof of Theorem 1′. We begin by computing the cardinality of $\mathcal{P}_{n,K}$. To do so, we recall that paths in this set are *directed* in the m first (and last) H-planes: since there are $(m\hat{n}_K)!$ ways to reach a vertex on the m^{th}-hyperplane starting from $\mathbf{0}$, and

$$\binom{n}{m\hat{n}_K} \tag{5.5.5}$$

vertices on such hyperplane, we have, altogether,

$$(m\hat{n}_K)!\binom{n}{m\hat{n}_K} \tag{5.5.6}$$

subpaths connecting $\mathbf{0}$ to H_m. Furthermore, there are are

$$(m\hat{n}_K)! \tag{5.5.7}$$

subpaths connecting a given vertex in H_{K-m} to $\mathbf{1}$.

As for the *stretched* phase, we will heavily rely on the fact already mentioned in Figure 5.10, namely that a natural representation of paths in terms of permutations is available. First we remark that for any two vertices $\boldsymbol{v}, \boldsymbol{w}$ of the hypercube,

$$\#\text{ stretched paths between } \boldsymbol{v} \text{ and } \boldsymbol{w} = (nd(\boldsymbol{v},\boldsymbol{w}))!\,, \tag{5.5.8}$$

and therefore, by definition of $\mathcal{P}_{n,K}$,

$$\#\mathcal{P}_{n,K} = \underbrace{(m\hat{n}_K)!\binom{n}{m\hat{n}_K}}_{\text{directed}} \underbrace{\left(\sum_{(\star_m)} \prod_{i=m+1}^{K-m} (n\mathsf{d}_i)!\right)}_{\text{stretched}} \underbrace{(m\hat{n}_K)!}_{\text{directed}}, \tag{5.5.9}$$

where the $(\star_m)$-sum runs over all possible vertices $\boldsymbol{v} \in H_i$. By definition, the subpaths in $\mathcal{P}_{n,K}$ going through a given vertex of the H_{i-1}-plane can reach the same number of vertices on the H_i-plane as the subpaths in $\Pi^{\boldsymbol{d}}_{\{1\dots K\}}[\mathbf{0} \to \mathbf{1}]$: the $(\star_m)$-sum thus runs over the same vertices as the $(\star)$-sum in (5.2.40), hence

$$\#(\star_m) = \prod_{i=m+1}^{K-m} \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{\left(1-\frac{i-1}{K}\right)n}{\mathsf{ef}_i n}. \tag{5.5.10}$$

Combining (5.5.9) and (5.5.10) thus yields

$$\#\mathcal{P}_{n,K} = (m\hat{n}_K)!^2\binom{n}{m\hat{n}_K}\prod_{i=m+1}^{K-m} \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{\left(1-\frac{i-1}{K}\right)n}{\mathsf{ef}_i n}(n\mathsf{d}_i)!\,. \tag{5.5.11}$$

We now quantify the difference in cardinality between $\mathcal{P}_{n,K}$ and $\mathcal{P}_{n,K,K'}$, and then, in a second step, between $\mathcal{P}_{n,K,K'}$ and $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$. To do so, the following observation is helpful: in the stretched phase, since by (5.2.36) it holds that $\mathsf{eb}_i + \mathsf{ef}_i = \mathsf{d}_i$, we may re-write the r.h.s. of (5.5.8) as

$$(n\mathsf{d}_i)! = (n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{n\mathsf{d}_i}{n\mathsf{eb}_i}. \tag{5.5.12}$$

This elementary algebraic identity can be given an *interpretation* which proves useful for the purpose of computing the cardinality of $\mathcal{P}_{n,K,K'}$. To see this, let us assume that each step of the polymer is a ball which is both coloured *and* labeled: backsteps are red whereas forward steps are blue; the labels correspond to which coordinate switches its value during the considered step: there are thus $(n\mathsf{eb}_i)$ labels for the red balls, and $(n\mathsf{ef}_i)$ labels for the blue balls. The first factorial on the r.h.s. of (5.5.12) then stands for the number of possible ways of listing the red balls while discriminating according to the labels, and similarly for the second factorial corresponding to the blue balls. Finally, the binomial factor on the r.h.s. of (5.5.12) accounts for the number of ways to place the red and blue balls, but *without* discriminating among labels.

Now, the subset $\mathcal{P}_{n,K,K'}$ is constructed out of $\mathcal{P}_{n,K}$ by adding an additional layer of coarse graining, and modifying the order of appearance of balls while discriminating according to their colors, but disregarding the labels. Adapting the interpretation of (5.5.12) discussed in the previous paragraph, it is clear that there are now

$$(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^{K'}. \tag{5.5.13}$$

subpaths in $\mathcal{P}_{n,K,K'}$ connecting two vertices in H_{i-1} and H_i at Hamming distance $n\mathsf{d}_i$.

The subset $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ differs from $\mathcal{P}_{n,K,K'}$ in that the order of backsteps and forward steps between H_{i-1} and $H'_{i-1,1}$, and between $H'_{i-1,K'-1}$ and H_i, is totally specified. This evidently reduces the cardinality: instead of (5.5.13), there are only

$$(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^{K'-2}. \tag{5.5.14}$$

subpaths between any two given vertices connecting the H_{i-1} and H_i hyperplanes.
To compare quantitatively the cardinality of all these sets we write

$$\frac{\#\mathcal{P}_{n,K}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} = \frac{\#\mathcal{P}_{n,K}}{\#\mathcal{P}_{n,K,K'}} \times \frac{\#\mathcal{P}_{n,K,K'}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}}. \tag{5.5.15}$$

By (5.5.13), it holds that

$$\begin{aligned}\frac{\#\mathcal{P}_{n,K}}{\#\mathcal{P}_{n,K,K'}} &= \prod_{i=m+1}^{K-m} \frac{(n\mathsf{d}_i)!}{(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^{K'}} \\ &= \prod_{i=m+1}^{K-m} \frac{(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{n\mathsf{d}_i}{n\mathsf{eb}_i}}{(n\mathsf{eb}_i)!(n\mathsf{ef}_i)!\binom{\mathsf{d}_i\hat{n}_{K'}}{\mathsf{eb}_i\hat{n}_{K'}}^{K'}} \\ &\lesssim \prod_{i=m+1}^{K-m} \frac{\sqrt{2\pi n\mathsf{d}_i}}{\sqrt{2\pi n\mathsf{eb}_i}\sqrt{2\pi n\mathsf{ef}_i}} \left(\frac{\sqrt{2\pi\mathsf{eb}_i\hat{n}_{K'}}\sqrt{2\pi\mathsf{ef}_i\hat{n}_{K'}}}{\sqrt{2\pi\mathsf{d}_i\hat{n}_{K'}}}\right)^{K'},\end{aligned} \tag{5.5.16}$$

the last step by elementary Stirling approximation (this time including the lower order, polynomial terms). The r.h.s. of (5.5.16) is, up to irrelevant numerical constant, *at most*

$$(5.5.16) \lesssim \prod_{i=m+1}^{K-m} n^{\frac{K'-1}{2}} = n^{\frac{(K'-1)(K-2m)}{2}}. \tag{5.5.17}$$

Furthermore, one has

$$\begin{aligned}\frac{\#\mathcal{P}_{n,K,K'}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} &= \prod_{i=m+1}^{K-m} \binom{\mathsf{d}_i \hat{n}_{K'}}{\mathsf{eb}_i \hat{n}_{K'}}^2 \\ &\lesssim \left\{ \prod_{i=m+1}^{K-m} \left(1-\frac{\mathsf{ef}_i}{\mathsf{d}_i}\right)^{\mathsf{d}_i-\mathsf{ef}_i} \left(\frac{\mathsf{ef}_i}{\mathsf{d}_i}\right)^{\mathsf{ef}_i} \right\}^{-2\hat{n}_{K'}} \prod_{i=m+1}^{K-m} \frac{K'\mathsf{d}_i}{2\pi n \mathsf{eb}_i \mathsf{ef}_i},\end{aligned} \tag{5.5.18}$$

the last inequality again by Stirling approximation. Since the term in the curly bracket is raised to a negative power, we will use the following lower bound

$$\prod_{i=m+1}^{K-m} \left\{ \left(1-\frac{\mathsf{ef}_i}{\mathsf{d}_i}\right)^{1-\frac{\mathsf{ef}_i}{\mathsf{d}_i}} \left(\frac{\mathsf{ef}_i}{\mathsf{d}_i}\right)^{\frac{\mathsf{ef}_i}{\mathsf{d}_i}} \right\}^{\mathsf{d}_i} \geq \prod_{i=m+1}^{K-m} \left(\frac{1}{2}\right)^{\mathsf{d}_i} \geq \left(\frac{1}{2}\right)^{\mathsf{L}}, \tag{5.5.19}$$

where the second inequality holds true since the function $x \mapsto (1-x)^{1-x}x^x$ is convex, and attains its minimal value 1/2 in $x = 1/2$, as can be plainly checked. Plugging the bound (5.5.19) in (5.5.18) then yields

$$\frac{\#\mathcal{P}_{n,K,K'}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} \lesssim n^{-(K-2m)} \exp\left(\frac{\mathsf{L}n}{K'} 2\log 2\right). \tag{5.5.20}$$

Remark that for any $K' > 2K^2 L \log 2$, it holds that

$$\exp\left(\frac{\mathsf{L}n}{K'} 2\log 2\right) \leq \exp\left(\frac{n}{K^2}\right). \tag{5.5.21}$$

Combining (5.5.15), (5.5.17), (5.5.20) and (5.5.21) therefore implies that the entropic cost for rendering the hyperplanes repulsive is

$$\frac{\#\mathcal{P}_{n,K}}{\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'}} \lesssim n^{(\frac{K'-1}{2}-1)(K-2m)} \times \exp\left(\frac{n}{K^2}\right). \tag{5.5.22}$$

Using (5.5.11) in (5.5.22) then yields

$$\#\mathcal{P}^{\mathsf{rep}}_{n,K,K'} \gtrsim (m\hat{n}_K)!^2 \binom{n}{m\hat{n}_K} \prod_{i=m+1}^{K-m} (n\mathsf{d}_i)! \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n} \binom{(1-\frac{i-1}{K})n}{\mathsf{e}f_i n} \times \frac{R_{n,K}}{P_n}, \tag{5.5.23}$$

where we have shortened

$$P_n \equiv n^{(\frac{K'-1}{2}-1)(K-2m)}, \qquad R_{n,K} \equiv \exp\left(-\frac{n}{K^2}\right). \tag{5.5.24}$$

Recall that by Remark 37, polymers in $\mathcal{P}^{\mathsf{rep}}_{n,K,K'}$ are loopless: this property, the ensuing independence of the sub-energies, and (5.5.23) thus yield

$$\begin{aligned}\mathbb{E}(\#\mathcal{E}^{2,\epsilon}_{n,K,K'}) \gtrsim (m\hat{n}_K)!\mathbb{P}\left(X_m(\pi) \leq \mathsf{a}_{m,\epsilon}\right)\binom{n}{m\hat{n}_K} \\ \times \prod_{i=m+1}^{K-m}(n\mathsf{d}_i)!\mathbb{P}\left(X_{i-1,i} \leq \mathsf{a}_{i,\epsilon}\right)\binom{\frac{i-1}{K}n}{\mathsf{e}b_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{e}f_i n} \\ \times (m\hat{n}_K)!\mathbb{P}\left(X_{K-m+1} \leq \mathsf{a}_{m,\epsilon}\right)\frac{R_{n,K}}{P_n}.\end{aligned} \tag{5.5.25}$$

Further, recalling that by the thinning procedure, it holds

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) = \frac{C(\epsilon)^2}{2}E(\#\mathcal{E}^{2,\epsilon}_{n,K,K'})\,, \tag{5.5.26}$$

and by the usual tail estimates, we thus see that

$$\mathbb{E}(\mathcal{N}^{\epsilon}_{n,K,K'}) \gtrsim \prod_{i=m+1}^{K-m}(\mathsf{a}_{i,\epsilon})^{nd_i}\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef_i} n}\binom{n}{m\hat{n}_K}{\mathsf{a}_{m,\epsilon}}^{2m\hat{n}_K-2}(m\hat{n}_K)^2\frac{R_{n,K}}{P_n}\,. \tag{5.5.27}$$

But since $(m\hat{n}_K)^2{\mathsf{a}_{m,\epsilon}}^{-2} > 1$ for n large enough, we have, altogether, that

$$\begin{aligned}\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \gtrsim \prod_{i=m+1}^{K-m}(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}\times \\ \times\binom{n}{m\hat{n}_K}(\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m}n\mathsf{d}_i}(1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K}\frac{R_{n,K}}{P_n}\,.\end{aligned} \tag{5.5.28}$$

The first term on the r.h.s. of (5.5.28) is reminiscent of the expression appearing in Theorem 1, but contrary to the latter, we are facing here a product which runs over the indeces $i = m+1\dots K-m$ only. The natural idea is thus to modify and then extend this partial product to a full product in order to exploit the control already established in Theorem 1. To do so we first note that, since on the positive axis it holds that $x \geq \tanh(x)$,

$$(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i} \geq \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}, \quad \text{and} \quad (\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K} \geq \tanh(\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K}\,. \tag{5.5.29}$$

Using this in (5.5.28) yields

$$\begin{aligned}\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \gtrsim \prod_{i=m+1}^{K-m}\tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\binom{\frac{i-1}{K}n}{\mathsf{eb_i} n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n} \\ \times\binom{n}{m\hat{n}_K}\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m}n\mathsf{d}_i}(1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K}\frac{R_{n,K}}{P_n}\,.\end{aligned} \tag{5.5.30}$$

The new (partial) product is closer yet not quite the same as that appearing in Theorem 1, so we artificially introduce some cosh-terms which however leave the r.h.s. above as a whole unaltered. Precisely, we rewrite (5.5.30) as

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \gtrsim \prod_{i=m+1}^{K-m} \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\left(\frac{\cosh(\mathsf{a}_i\mathsf{E})}{\cosh(\mathsf{a}_i\mathsf{E})}\right)^n \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}\times$$
$$\times\binom{n}{m\hat{n}_K}\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{2m\hat{n}_K}\left(\frac{\cosh(\bar{\mathsf{a}}_m\mathsf{E})}{\cosh(\bar{\mathsf{a}}_m\mathsf{E})}\right)^{2n}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i}(1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K}\frac{R_{n,K}}{P_n}; \tag{5.5.31}$$

We can now move to the aforementioned procedure of extending the product to *all* indeces $i = 1 \dots K$. This naturally requires a good control of the missing terms, i.e. for $i \le m$ (a case which is referred to below as First), and for $i \ge K - m + 1$ (Second case).

First case. We begin noting that by the Evolution Lemma 35,

$$\begin{aligned}\left[\prod_{i=1}^m g_{i,K}(\mathsf{d}_i)\right]^n &= \left(\frac{\sinh(\bar{\mathsf{a}}_m\mathsf{E})}{\frac{m}{K}}\right)^{m\hat{n}_K}\left(\frac{\cosh(\bar{\mathsf{a}}_m\mathsf{E})}{1-\frac{m}{K}}\right)^{n-m\hat{n}_K}\\ &= \tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^n\frac{n^n}{(m\hat{n}_K)^{m\hat{n}_K}(n-m\hat{n}_K)^{n-m\hat{n}_K}},\end{aligned} \tag{5.5.32}$$

the second equality by elementary rearrangement. But by "reverse" Stirling-approximation,

$$\frac{n^n}{(m\hat{n}_K)^{m\hat{n}_K}(n-m\hat{n}_K)^{n-m\hat{n}_K}} \propto \sqrt{n}\binom{n}{m\hat{n}_K}, \tag{5.5.33}$$

and therefore

$$\left[\prod_{i=1}^m g_{i,K}(\mathsf{d}_i)\right]^n \propto \sqrt{n}\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^n\binom{n}{m\hat{n}_K}. \tag{5.5.34}$$

Furthermore, by definition of the g-functions, and taking into account the lower orders in the Stirling approximation of the binomial factors, one also plainly checks that

$$\left[\prod_{i=1}^m g_{i,K}(\mathsf{d}_i)\right]^n \propto \sqrt{n}n^{m-1}\prod_{i=1}^m \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}. \tag{5.5.35}$$

Equating (5.5.34) and (5.5.35) therefore yields the asymptotic identity

$$\begin{aligned}&\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^n\binom{n}{m\hat{n}_K}\\ &\qquad\propto n^{m-1}\prod_{i=1}^m \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n\binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}.\end{aligned} \tag{5.5.36}$$

Remark, in particular, that what lies behind the l.h.s. above (these are terms contributing to (5.5.31)) are thus the first m-terms (up to irrelevant, for our purposes below) polynomial factors, of the product analysed in Theorem 1.

Second case. Again by the Evolution Lemma 35 it holds that

$$1 = \prod_{i=1}^{K-m} g_{i,K}(\mathsf{d}_i) \times \prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i)\,, \tag{5.5.37}$$

and therefore

$$\begin{aligned}\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i) &= \left[\prod_{i=1}^{K-m} g_{i,K}(\mathsf{d}_i)\right]^{-1} \\ &= \left[\left(\frac{\sinh(\bar{\mathsf{a}}_{K-m}\mathsf{E})}{\frac{K-m}{K}}\right)^{\frac{K-m}{K}} \left(\frac{\cosh(\bar{\mathsf{a}}_{K-m}\mathsf{E})}{1-\frac{K-m}{K}}\right)^{1-\frac{K-m}{K}}\right]^{-1},\end{aligned} \tag{5.5.38}$$

the second equality in virtue of (5.3.37). In order to get a handle on the r.h.s. above we use the fundamental relation (5.2.25) which states that

$$\sinh\left(\bar{\mathsf{a}}_{K-m}\mathsf{E}\right)\cosh\left(\underline{\mathsf{a}}_{K-m}\mathsf{E}\right) = \frac{K-m}{K}\,, \tag{5.5.39}$$

implying, in particular, that

$$\left[\left(\frac{\sinh(\bar{\mathsf{a}}_{K-m}\mathsf{E})}{\frac{K-m}{K}}\right)^{\frac{K-m}{K}}\right]^{-1} = \cosh(\underline{\mathsf{a}}_{K-m}\mathsf{E})^{\frac{K-m}{K}}\,. \tag{5.5.40}$$

Furthermore, the following "mirror" version of (5.5.39) holds in virtue of the addition formula for hyperbolic functions (see (5.3.24) for the detailed derivation):

$$\cosh\left(\bar{\mathsf{a}}_{K-m}\mathsf{E}\right)\sinh\left(\underline{\mathsf{a}}_{K-m}\mathsf{E}\right) = 1-\frac{K-m}{K}\,, \tag{5.5.41}$$

hence

$$\left[\left(\frac{\cosh(\bar{\mathsf{a}}_{K-m}\mathsf{E})}{1-\frac{K-m}{K}}\right)^{1-\frac{K-m}{K}}\right]^{-1} = \sinh(\underline{\mathsf{a}}_{K-m}\mathsf{E})^{1-\frac{K-m}{K}}\,. \tag{5.5.42}$$

Using (5.5.40) and (5.5.42) in (5.5.38) we thus have

$$\begin{aligned}\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i) &= \cosh(\underline{\mathsf{a}}_{K-m}\mathsf{E})^{\frac{K-m}{K}}\sinh(\underline{\mathsf{a}}_{K-m}\mathsf{E})^{1-\frac{K-m}{K}} \\ &= \cosh(\bar{\mathsf{a}}_{m}\mathsf{E})^{\frac{K-m}{K}}\sinh(\bar{\mathsf{a}}_{m}\mathsf{E})^{1-\frac{K-m}{K}}\,,\end{aligned} \tag{5.5.43}$$

the second identity since $\sum_{i=1}^K \mathsf{a}_i = 1$ and by symmetry of the $\mathsf{a}'s$. Raising (5.5.43) to the n^{th}- power, and by simple rearrangement, we thus see that

$$\left[\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i)\right]^n = \tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^n\,. \tag{5.5.44}$$

Again by the definition of the g-functions, and taking into account the lower orders in the Stirling approximation of the binomial factors, one plainly checks that

$$\left[\prod_{i=K-(m-1)}^{K} g_{i,K}(\mathsf{d}_i)\right]^n \propto n^{m-1} \prod_{i=K-(m-1))}^{K} \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n \binom{\frac{i-1}{K}n}{\mathsf{eb_i}n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}\,, \tag{5.5.45}$$

and therefore, equating (5.5.44) and (5.5.45), we also obtain the following asymptotic equivalence

$$\begin{aligned}&\tanh(\bar{\mathsf{a}}_m\mathsf{E})^{m\hat{n}_K}\cosh(\bar{\mathsf{a}}_m\mathsf{E})^n \\ &\qquad \propto n^{m-1}\prod_{i=K-(m-1)}^{K} \tanh(\mathsf{a}_i\mathsf{E})^{nd_i}\cosh(\mathsf{a}_i\mathsf{E})^n \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}\,.\end{aligned} \tag{5.5.46}$$

In full analogy to (5.5.36), we therefore see that behind the l.h.s. above (these are also terms contributing to (5.5.31)) hide in fact the last m-terms of the product analysed in Theorem 1.

Thanks to both (5.5.36) and (5.5.46), we may now replace the corresponding terms on the r.h.s. of (5.5.31): this indeed allows to extend the product to all indeces $i = 1, \dots, K$, and seamlessly leads to the lower bound

$$\begin{aligned}\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \gtrsim & \prod_{i=1}^{K} \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n \binom{\frac{i-1}{K}n}{\mathsf{eb_i}n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n} \\ & \times \frac{Q_n R_{n,K}}{P_n\cosh(\bar{\mathsf{a}}_m\mathsf{E})^{2n}} \prod_{i=m+1}^{K-m}\frac{1}{\cosh(\mathsf{a}_i\mathsf{E})}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i}(1+\epsilon_{m,\mathsf{E}})^{2m\hat{n}_K}\,,\end{aligned} \tag{5.5.47}$$

where $Q_n \equiv n^{2(m-1)}$ is yet another polynomial term.

The full product in the first line of the r.h.s. of (5.5.47) is easily taken care of. In fact, by elementary rearrangement, it holds that

$$\begin{aligned}&\prod_{i=1}^{K} \tanh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^n \binom{\frac{i-1}{K}n}{\mathsf{eb_i}n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n} \\ &\qquad = \prod_{i=1}^{K} \sinh(\mathsf{a}_i\mathsf{E})^{n\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^{n(1-\mathsf{d}_i)} \binom{\frac{i-1}{K}n}{\mathsf{eb}_i n}\binom{(1-\frac{i-1}{K})n}{\mathsf{ef}_i n}\,,\end{aligned} \tag{5.5.48}$$

and by Stirling approximation to second order, the r.h.s. of (5.5.48) equals

$$\prod_{i=1}^{K}\left\{\frac{\sinh(\mathsf{a}_i\mathsf{E})^{\mathsf{d}_i}\cosh(\mathsf{a}_i\mathsf{E})^{1-\mathsf{d}_i}\varphi\left(\frac{i-1}{K}\right)\varphi\left(1-\frac{i-1}{K}\right)}{\varphi(\mathsf{eb}_i)\varphi\left(\frac{i-1}{K}-\mathsf{eb}_i\right)\varphi\left(\mathsf{ef}_i\right)\varphi\left(1-\frac{i-1}{K}-\mathsf{ef}_i\right)}\right\}^n \times S_{n,K}, \tag{5.5.49}$$

where $S_{n,K}$ corresponds to the lower order (polynomial) terms in the approximation. But by Theorem 1, the first term of (5.5.49), i.e. the full product, equals unity, whereas an elementary inspection of the polynomial terms further shows that

$$\begin{aligned} S_{n,K} &\gtrsim \frac{1}{\sqrt{\frac{2\pi n}{K}(1-\frac{1}{K})}}\prod_{i=2}^{K-1}\left(\frac{(2\pi n)^2(\frac{i-1}{K})(1-\frac{i-1}{K})}{(2\pi n)^4\mathsf{eb}_i(\frac{i}{K}-\mathsf{ef}_i)\mathsf{ef}_i(1-\frac{i}{K}-\mathsf{eb}_i)}\right)^{\frac{1}{2}} \\ &\gtrsim \frac{1}{n^{K-2+\frac{1}{2}}}\,. \end{aligned} \tag{5.5.50}$$

Using all this in (5.5.47) yields

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \geq \frac{Q_n R_{n,K}}{P_n\cosh(\bar{\mathsf{a}}_m\mathsf{E})^{2n}}\prod_{i=m+1}^{K-m}\frac{1}{\cosh(\mathsf{a}_i\mathsf{E})^n}(1+\epsilon_{\mathsf{E}})^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i}(1+\epsilon_{\bar{\mathsf{a}}_m,\mathsf{E}})^{2m\hat{n}_K}\,, \tag{5.5.51}$$

where $P_n \equiv P_n n^{K-2+\frac{1}{2}}$ is yet another polynomial term.

It thus remains to control the cosh-terms in (5.5.51). To see how this goes we observe that by Taylor expanding the cosh-function to second order,

$$\begin{aligned} \cosh(\bar{\mathsf{a}}_m\mathsf{E})^{-1} &= \exp\left[-\log\cosh(\bar{\mathsf{a}}_m\mathsf{E})\right] \\ &\geq \exp -\log\left\{1+\frac{(\bar{\mathsf{a}}_m\mathsf{E})^2\cosh(\bar{\mathsf{a}}_m\mathsf{E})}{2}\right\} \\ &\geq \exp\left(-\frac{(\bar{\mathsf{a}}_m\mathsf{E})^2}{\sqrt{2}}\right)\,, \end{aligned} \tag{5.5.52}$$

the second inequality since $\log(1+x) \leq x$, and using that $\cosh(\bar{\mathsf{a}}_m\mathsf{E}) \leq \cosh(\mathsf{E}) = \sqrt{2}$. Moreover, by (5.2.29) it holds that $\mathsf{a}_i\mathsf{E} \leq \frac{1}{K}$: summing over $i = 1\ldots m$ thus leads to $\bar{\mathsf{a}}_m\mathsf{E} \leq \frac{m}{K}$, which combined with (5.5.52) yields

$$\cosh(\bar{\mathsf{a}}_m\mathsf{E})^{-1} \geq \exp\left(-\frac{m^2}{\sqrt{2}K^2}\right)\,. \tag{5.5.53}$$

A similar reasoning evidently yields

$$\cosh(\mathsf{a}_i E)^{-1} \geq \exp\left(-\frac{1}{\sqrt{2}K^2}\right)\,, \tag{5.5.54}$$

for any $i = 1 \dots K$. By (5.5.53) and (5.5.54) we thus have that

$$\frac{1}{\cosh(\bar{\mathsf{a}}_m \mathsf{E})^{2n}} \times \prod_{i=m+1}^{K-m} \frac{1}{\cosh(\mathsf{a}_i \mathsf{E})^n} \geq \exp\left\{ -\frac{n}{\sqrt{2}K} - \frac{2nm(m-1)}{\sqrt{2}K^2} \right\}, \tag{5.5.55}$$

which we recognize as the $S_{n,K,m}$-term announced in (5.5.2): *the entropic cost for stretching the paths.* Using (5.5.55) in (5.5.51) finally yields

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) \geq (1+\epsilon_E)^{\sum_{i=m+1}^{K-m} n\mathsf{d}_i} (1+\epsilon_{\bar{\mathsf{a}}_m,\mathsf{E}})^{2m\hat{n}_K} \frac{S_{n,K,m} R_{n,K} Q_n}{P_n}, \tag{5.5.56}$$

and Theorem 1′ is thus settled. □

5.6 The second moment, and proof of Theorem 2'

The goal of this section is to provide a proof of Theorem 2′. We begin with a technical input, concerning tail estimates for the probability of two correlated sums of exponentials.

Lemma 39 (Overlap probability)**.** *Consider independent standard exponentials $\{\xi_i\}$, and let $X_l \equiv \sum_{i=1}^{l} \xi_i$. Denote by X_l' the sum of l such ξ-exponentials, and assume that X_l' shares exactly k edges with X_l. Then for $x > 0$, it holds:*

$$\mathbb{P}\left(X_l \leq x, X_l' \leq x\right) \propto \frac{x^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l. \tag{5.6.1}$$

where

$$\gamma \in [0,1] \mapsto g(\gamma) \equiv \frac{\left\{4(1-\gamma)\right\}^{1-\gamma}}{\left\{2-\gamma\right\}^{2-\gamma}}. \tag{5.6.2}$$

In particular, $\|g\|_\infty \leq 1$.

Proof. Without loss of generality we may write

$$X_l' = \sum_{i=1}^{k} \xi_i + \sum_{i=k+1}^{l} \xi_i', \tag{5.6.3}$$

for independent ξ''s, which are also independent of the ξ-family. Remark that the first sum, the common trunk, is a $\Gamma(k,1)$-distributed r.v., whereas the second sum is $\Gamma(l-k,1)$-distributed. By conditioning on the common trunk, and by independence, it thus holds:

$$\begin{aligned}
\mathbb{P}\left(X_l \leq x, X_l' \leq x\right) &= \int_0^{+\infty} \mathbb{P}\left(t + X_{l-k} \leq x\right)^2 \mathbb{P}(X_k \in dt) \\
&= \int_0^{+\infty} \mathbb{P}\left(X_{l-k} \leq x-t\right)^2 \frac{t^{k-1}e^{-t}}{(k-1)!} dt \\
&\propto \frac{1}{(l-k)!^2 (k-1)!} \int_0^x (x-t)^{2(l-k)} t^{k-1} dt,
\end{aligned} \tag{5.6.4}$$

the last step by the standard tail-estimates. Integration by parts then yields

$$\int_0^x (x-t)^{2(l-k)} t^{k-1} dt = \frac{(k-1)!(2(l-k))!}{(2l-k)!} x^{2l-k}, \tag{5.6.5}$$

and therefore

$$\begin{aligned} \mathbb{P}\left(X_l \le x, X_l' \le x\right) &\propto \frac{(2(l-k))!}{(2l-k)!(l-k)!^2} x^{2l-k} \\ &\propto \frac{x^{2l-k}}{(l-k)!l!} \frac{l!(2(l-k))!}{(2l-k)!(l-k)!}. \\ &\propto \frac{x^{2l-k}}{(l-k)!l!} \frac{(1-\frac{k}{l})^{l-k}}{2^k(1-\frac{k}{2l})^{2l-k}}, \end{aligned} \tag{5.6.6}$$

the last inequality by Stirling approximation.

Remark that with $\gamma \equiv k/l \in [0,1]$, the second factor in the last term above can be written as

$$\frac{(1-\frac{k}{l})^{l-k}}{2^k(1-\frac{k}{2l})^{2l-k}} = \left\{ \frac{(4(1-\gamma))^{(1-\gamma)}}{(2-\gamma)^{(2-\gamma)}} \right\}^l \equiv g(\gamma)^l, \tag{5.6.7}$$

and using this in (5.6.6) yields

$$\mathbb{P}\left(X_l \le x, X_l' \le x\right) \propto \frac{x^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l, \tag{5.6.8}$$

concluding the proof of the estimate for the overlap probability. □

We now address the second moment of $\mathcal{N}^{\epsilon}_{n,K,K'}$, as required for a proof of Theorem 2′. For this, some notation is needed: recall from (5.4.42) that $\mathcal{J}$ is a deterministic subset of polymers with cardinality $\#\mathcal{J} = \mathsf{J} = \lfloor \mathbb{E}\#\mathcal{E}^{1,\epsilon}_{n,K,K'}/2 \rfloor$. Given a path $\pi \in \mathcal{J}$, we shorten:

$$\begin{aligned} \mathcal{J}_\pi(n,k) \equiv\ & \textsf{all paths } \pi' \in \mathcal{J} \\ & \textsf{which share } k \textsf{ edges with } \pi, \\ & \textsf{whithout considering the first and the last edge,} \end{aligned} \tag{5.6.9}$$

and for its cardinality

$$f_\pi(n,k) \equiv \#\mathcal{J}_\pi(n,k). \tag{5.6.10}$$

Analogously we shorten

$$\begin{aligned} \mathcal{J}^{(d)}_\pi(n,k) \equiv\ & \textsf{all paths } \pi' \in \mathcal{J} \textsf{ which share } k \textsf{ edges with} \\ & \pi \textsf{ only in the directed phase, i.e between} \\ & \mathbf{0} \textsf{ and } H_m \textsf{ or } H_{K-m} \textsf{ and } 1, \\ & \textsf{but without considering first and last edge,} \end{aligned} \tag{5.6.11}$$

and let

$$f_\pi^{(d)}(n,k) \equiv \# \mathcal{J}_\pi^{(d)}(n,k)\,, \tag{5.6.12}$$

denote its cardinality.

And finally,

$$\begin{aligned} \mathcal{J}_\pi^{(s)}(n,k) \equiv & \text{ number of paths } \pi' \in \mathcal{J} \text{ which share } k \text{ edges with} \\ & \pi \text{ with at least one common edge in the stretched} \\ & \text{phase, i.e between } H_m \text{ and } H_{K-m}, \\ & \text{but without considering first and last edge,} \end{aligned} \tag{5.6.13}$$

analogously shortening for its cardinality

$$f_\pi^{(s)}(n,k) \equiv \sharp \mathcal{J}_\pi^{(s)}(n,k)\,. \tag{5.6.14}$$

Remark that

$$f_\pi(n,k) = f_\pi^{(d)}(n,k) + f_\pi^{(s)}(n,k)\,. \tag{5.6.15}$$

We will also need the "worst case scenarios"

$$\begin{aligned} f(n,k) &\equiv \sup_{\pi\in\mathcal{J}} f_\pi(n,k)\,, \\ f^{(d)}(n,k) &\equiv \sup_{\pi\in\mathcal{J}} f_\pi^{(d)}(n,k)\,, \\ f^{(s)}(n,k) &\equiv \sup_{\pi\in\mathcal{J}} f_\pi^{(s)}(n,k)\,. \end{aligned} \tag{5.6.16}$$

in which case it holds, in particular, that

$$f(n,k) \le f^{(d)}(n,k) + f^{(s)}(n,k). \tag{5.6.17}$$

For $i = m+1 \dots K-m$, and two polymers $\pi, \pi' \in \mathcal{J}$, we shorten

$$\mathbb{P}_i(\pi) \equiv \mathbb{P}\left(X_{i-1,i}(\pi) \le \mathsf{a}_{i,\epsilon}\right), \tag{5.6.18}$$

and

$$\mathbb{P}_i(\pi,\pi') \equiv \mathbb{P}\left(X_{i-1,i}(\pi) \le \mathsf{a}_{i,\epsilon},\ X_{i-1,i}(\pi') \le \mathsf{a}_{i,\epsilon}\right)\,. \tag{5.6.19}$$

Furthermore, we shorten

$$\begin{aligned} \mathbb{P}_m(\pi) &\equiv \mathbb{P}\left(X_m(\pi) \le \mathsf{a}_{m,\epsilon}\right), \\ \mathbb{P}_{K-m+1}(\pi) &\equiv \mathbb{P}\left(X_{K-m+1}(\pi) \le \mathsf{a}_{K-m+1,\epsilon}\right), \end{aligned} \tag{5.6.20}$$

and

$$\begin{aligned} \mathbb{P}_m(\pi,\pi') &\equiv \mathbb{P}\left(X_m(\pi), X_m(\pi') \le \mathsf{a}_{m,\epsilon}\right) \\ \mathbb{P}_{K-m+1}(\pi,\pi') &\equiv \mathbb{P}\left(X_{K-m+1}(\pi), X_{K-m+1}(\pi') \le \mathsf{a}_{K-m+1,\epsilon}\right)\,, \end{aligned} \tag{5.6.21}$$

as well as

$$\mathbb{P}(\pi) \equiv \mathbb{P}\Big(X_m(\pi) \leq \mathsf{a}_{m,\epsilon},\ X_{i-1,i}(\pi) \leq \mathsf{a}_{i,\epsilon}\ i = m+1\dots K-m,\ X_{K-m+1}(\pi) \leq \mathsf{a}_{K-m+1,\epsilon}\Big), \tag{5.6.22}$$

and

$$\begin{aligned}\mathbb{P}(\pi,\pi') \equiv & \mathbb{P}(X_m(\pi),\ X_m(\pi') \leq \mathsf{a}_{m,\epsilon},\ X_{i-1,i}(\pi), X_{i-1,i}(\pi') \leq \mathsf{a}_{i,\epsilon} \text{ for} \\ & i = m+1\dots K-m,\ X_{K-m+1}(\pi), X_{K-m+1}(\pi') \leq \mathsf{a}_{K-m+1,\epsilon}).\end{aligned} \tag{5.6.23}$$

Remark that for loopless paths the substrand-energies are independent, hence, and with the above notation,

$$\mathbb{P}(\pi) = \prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi), \qquad \mathbb{P}(\pi,\pi') = \prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi,\pi'), \tag{5.6.24}$$

In particular, it holds that

$$\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right) = \mathsf{J}\mathbb{P}(\pi) = \mathsf{J}\prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi). \tag{5.6.25}$$

Concerning the second moment, we write

$$\begin{aligned}\mathbb{E}\left({\mathcal{N}^{\epsilon}_{n,K,K'}}^2\right) &= \sum_{\pi,\pi'\in\mathcal{J}} \mathbb{P}(\pi,\pi') \\ &= \sum_{\pi\in\mathcal{J}} \sum_{k=0}^{\mathsf{L}_{opt}n-2} \sum_{\pi'\in\mathcal{J}_\pi(n,k)} \mathbb{P}(\pi,\pi'),\end{aligned} \tag{5.6.26}$$

by arranging the sum according to the possible overlap-regimes.

The case $k = 0$ is both crucial and easily taken care of by the following observations: first remark that the distribution of the energies of a pair of polymers depends solely on the number of common edges; furthermore the number of pairs of polymers with zero common edges is at most J^2. Therefore, for any $(\hat\pi,\tilde\pi) \in (\mathcal{J},\mathcal{J}_{\hat\pi}(n,0))$ it holds:

$$\sum_{\pi\in\mathcal{J}} \sum_{\pi'\in\mathcal{J}_\pi(n,0)} \mathbb{P}(\pi,\pi') \leq \mathsf{J}^2\mathbb{P}(\hat\pi,\tilde\pi) = \mathsf{J}^2\mathbb{P}(\hat\pi)^2, \tag{5.6.27}$$

the last equality holding true since in case of non-overlapping paths, the $\hat\pi,\tilde\pi$-energies are independent and identically distributed. Using (5.6.25) in (5.6.27) therefore yields

$$\sum_{\pi\in\mathcal{J}} \sum_{\pi'\in\mathcal{J}_\pi(n,0)} \mathbb{P}(\pi,\pi') \leq \mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2, \tag{5.6.28}$$

This settles the $k = 0$ regime.

Remark 40. *Recovering the first moment squared as in* (5.6.28) *is absolutely crucial for the whole approach, and the main reason for treating first and last edge on different footing. Without such different treatment, one would get the first moment squared* up to a constant only*, and this would nullify the proof of Theorem 2. This feature is common to virtually all models in the REM-class, see [35] for more on this delicate issue.*

As for the remaining overlap-regimes, we will distinguish between

- $1 \leq k \leq 200\hat{n}_K$: this corresponds to the case of weak correlations (the overlap between the two polymers is small);
- $k > 200\hat{n}_K$: this corresponds to the case of strong correlations (the two polymers strongly overlap).

We now rearrange the second moment according to the above dichotomy. Henceforth, given $\pi \in \mathcal{J}$, and with $k \in \mathbb{N}$, we denote by $\pi_k^{(d)} \in \mathcal{J}_\pi^{(d)}(n,k)$ a polymer which shares k edges with π, and in full analogy for $\pi_k^{(s)} \in \mathcal{J}_\pi^{(s)}(n,k)$ and $\pi_k \in \mathcal{J}_\pi(n,k)$. With this notation, again using that specifying the number of common edges fixes the distribution of the pair of paths, and by (5.6.28), we thus have

$$\begin{aligned} \mathbb{E}\left({\mathcal{N}^\epsilon_{n,K,K'}}^2\right) \leq \mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}\right)^2 + & \\ + \mathsf{J} \sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(d)}\right) & \\ + \mathsf{J} \sum_{k=1}^{200\hat{n}_K} f^{(s)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(s)}\right) & \\ + \mathsf{J} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} f(n,k)\mathbb{P}\left(\pi, \pi_k\right) . & \end{aligned} \tag{5.6.29}$$

On the other hand, by Jensen inequality it holds

$$\mathbb{E}\left({\mathcal{N}^\epsilon_{n,K,K'}}^2\right) \geq \mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}\right)^2 . \tag{5.6.30}$$

In order to establish Theorem 2′ it therefore suffices to show that the last three sums on the r.h.s. of (5.6.29) are of lower order when compared with the first moment squared. This is indeed our key claim: since its proof is long and technical, we formulate it in the form of three Propositions.

Proposition 41. *For any $K > m\epsilon^{-2}$, it holds*

$$\mathsf{J} \sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi, \pi_k^{(d)}\right) = o\left(\mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}\right)^2\right) , \tag{5.6.31}$$

for $n \to \infty$.

Proposition 42. *For any* $K > \max(2\times 10^7, m\epsilon^{-2})$ *and* $K' > 2\log(2)\mathsf{L}K^2$*, it holds*

$$\mathsf{J}\sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} f(n,k)\mathbb{P}\left(\pi,\pi_k\right) = o\left(\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2\right), \tag{5.6.32}$$

for $n\to\infty$.

Proposition 43. *For any* $K > 2\times 10^7$ *and* $K' > 2\log(2)\mathsf{L}K^2$*, it holds*

$$\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(s)}(n,k)\mathbb{P}\left(\pi,\pi^{(s)}_k\right) = o\left(\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2\right), \tag{5.6.33}$$

for $n\to\infty$.

The following three sections are devoted to the proofs of the above statements. We anticipate that each proposition/treatment will require a good control of the asymptotics of the $f^{(d)}$, f- and $f^{(s)}$-terms: these will be formulated in the form of Lemmata whose proofs, relying on extremely technical combinatorial estimates, are however postponed to Section 5.7.

The reason for tackling the f-regime before the $f^{(s)}$-one is that the treatment of the latter will require some technical inputs which are obtained in the analysis of the the former.

5.6.1 Proof of Proposition 41

The goal is to prove that

$$\lim_{n\to\infty}\frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi,\pi^{(d)}_k\right)}{\mathbb{E}\left(\mathcal{N}^{\epsilon}_{n,K,K'}\right)^2} = 0. \tag{5.6.34}$$

The combinatorial input here is the following

Lemma 44. *For all* $k\le 200\hat{n}_K$*, one has*

$$f^{(d)}(n,k) \le \frac{\mathsf{J}(m\hat{n}_K - \lfloor\frac{k}{2}\rfloor)!(n-1-\lceil\frac{k}{2}\rceil)!}{(m\hat{n}_K)!n!}l(k), \tag{5.6.35}$$

where

$$l(k) \equiv \begin{cases} 32(k+1)^3 & k\le n^{1/4} \\ 16n^{13}(k+1) & \text{otherwise.}\end{cases} \tag{5.6.36}$$

The proof of this Lemma is postponed to Section 5.7. Coming back to the task of proving (5.6.34), by (5.6.24) and (5.6.25) we write

$$\frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi,\pi_k^{(d)}\right)}{\mathbb{E}\left(\mathcal{N}_{n,K,K'}^{\epsilon}\right)^2} = \frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\prod_{i=m}^{K-m+1}\mathbb{P}_i\left(\pi,\pi_k^{(d)}\right)}{\mathsf{J}^2\prod_{i=m}^{K-m+1}\mathbb{P}_i(\pi)^2}, \tag{5.6.37}$$

In the considered regime, polymers share no edges but in the directed phase: the probabilities indexed by $i \in \{m+1,\dots,K-m\}$ therefore factor out in virtue of the ensuing independence, and the r.h.s. of (5.6.37) then takes the neater form

$$\sum_{k=1}^{200\hat{n}_K} \frac{f^{(d)}(n,k)\mathbb{P}_m\left(\pi,\pi_k^{(d)}\right)\mathbb{P}_{K-m+1}\left(\pi,\pi_k^{(d)}\right)}{\mathsf{J}\mathbb{P}_m(\pi)^2\,\mathbb{P}_{K-m+1}(\pi)^2}. \tag{5.6.38}$$

Now, for $\pi \in \mathcal{J}$ and $\pi_k^{(d)} \in \mathcal{J}_\pi^{(d)}(n,k)$, let us denote by k_l the number of common edges between $\mathbf{0}$ and H_m, and by k_r the number of common edges between H_{K-m} and $\mathbf{1}$ (in which case it evidently holds that $k = k_l + k_r$). By the estimates for the overlap probabilities from Lemma 39 (using the rough bound $\|g\|_\infty \le 1$), it steadily follows that

$$\mathbb{P}_m\left(\pi,\pi_k^{(d)}\right)\mathbb{P}_{K-m+1}\left(\pi,\pi_k^{(d)}\right) \lesssim \frac{\mathsf{a}_{m,\epsilon}{}^{4(m\hat{n}_K-1)-k}}{(m\hat{n}_K-1-k_l)!(m\hat{n}_K-1-k_r)!(m\hat{n}_K-1)!^2}. \tag{5.6.39}$$

We now proceed by worst case scenario and *maximize* the r.h.s. over all possible (k_l,k_r)-choices. This can be seamlessly identified thanks to the well-known log-convexity of factorials, which we recall is the property that for any $a \ge b \ge j \ge 0$ it holds

$$(a+j)!(b-j)! \ge a!b!\,. \tag{5.6.40}$$

Using (5.6.40) with

$$a \equiv m\hat{n}_K - 1 - \lfloor\frac{k}{2}\rfloor, \qquad \text{and} \qquad b \equiv m\hat{n}_K - 1 - \lceil\frac{k}{2}\rceil, \tag{5.6.41}$$

we see that the worst case on the r.h.s. of (5.6.39) is attained in $k_r \in \{\lfloor\frac{k}{2}\rfloor, \lceil\frac{k}{2}\rceil\}$, which is equivalent to $k_l \in \{\lfloor\frac{k}{2}\rfloor, \lceil\frac{k}{2}\rceil\}$ because $k = k_l + k_r$, hence

$$\mathbb{P}_m\left(\pi,\pi_k^{(d)}\right)\mathbb{P}_{K-m+1}\left(\pi,\pi_k^{(d)}\right) \le \frac{\mathsf{a}_{m,\epsilon}{}^{4(m\hat{n}_K-1)-k}}{(m\hat{n}_K-1-\lfloor\frac{k}{2}\rfloor)!(m\hat{n}_K-1-\lceil\frac{k}{2}\rceil)!(m\hat{n}_K-1)!^2}. \tag{5.6.42}$$

Using the latter in (5.6.38), and by the usual tail estimates, we obtain

$$\frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(d)}(n,k)\mathbb{P}\left(\pi,\pi_k^{(d)}\right)}{\mathbb{E}\left(\mathcal{N}_{n,K,K'}^{\epsilon}\right)^2} \lesssim \sum_{k=1}^{200\hat{n}_K} \frac{f^{(d)}(n,k)(m\hat{n}_K-1)!^2}{\mathsf{J}(m\hat{n}_K-1-\lfloor\frac{k}{2}\rfloor)!(m\hat{n}_K-1-\lceil\frac{k}{2}\rceil)!(\mathsf{a}_{m,\epsilon})^k}. \tag{5.6.43}$$

To get a handle on the factorials in the r.h.s. above we employ the bound

$$\frac{(m\hat{n}_K-1)!^2}{(m\hat{n}_K-1-\lfloor\frac{k}{2}\rfloor)!(m\hat{n}_K-1-\lceil\frac{k}{2}\rceil)!}\leq\frac{(m\hat{n}_K)!^2}{(m\hat{n}_K-\lfloor\frac{k}{2}\rfloor)!(m\hat{n}_K-\lceil\frac{k}{2}\rceil)!}, \tag{5.6.44}$$

which can be plainly checked by writing out, and simplifying. Using (5.6.44), and the combinatorial estimates of Lemma 44 for the $f^{(d)}$-term, yields

$$\begin{aligned}(5.6.43) &\lesssim \sum_{k=1}^{200\hat{n}_K}\frac{(m\hat{n}_K-\lfloor\frac{k}{2}\rfloor)!(n-1-\lceil\frac{k}{2}\rceil)!l(k)(m\hat{n}_K)!^2}{(m\hat{n}_K)!n!(m\hat{n}_K-\lfloor\frac{k}{2}\rfloor)!(m\hat{n}_K-\lceil\frac{k}{2}\rceil)!(\mathsf{a}_{m,\epsilon})^k}\\ &=\sum_{k=1}^{200\hat{n}_K}\frac{(n-1-\lceil\frac{k}{2}\rceil)!l(k)(m\hat{n}_K)!}{n!(m\hat{n}_K-\lceil\frac{k}{2}\rceil)!(\mathsf{a}_{m,\epsilon})^k}\,.\end{aligned} \tag{5.6.45}$$

the second step in virtue of elementary, term by term, simplifications.

Using $(a-1)!=a!/a$ for the first factorial-term in the numerator on the r.h.s. above yields

$$\begin{aligned}(5.6.45) &= \sum_{k=1}^{200\hat{n}_K}\frac{(n-\lceil\frac{k}{2}\rceil)!l(k)(m\hat{n}_K)!}{(n-\lceil\frac{k}{2}\rceil)n!(m\hat{n}_K-\lceil\frac{k}{2}\rceil)!(\mathsf{a}_{m,\epsilon})^k}\\ &\lesssim \sum_{k=1}^{200\hat{n}_K}\frac{l(k)}{(n-\lceil\frac{k}{2}\rceil)}\cdot\frac{(1-\frac{1}{n}\lceil\frac{k}{2}\rceil)^{n-\lceil\frac{k}{2}\rceil}(\frac{m}{K})^{m\hat{n}_K}}{(\frac{m}{K}-\frac{1}{n}\lceil\frac{k}{2}\rceil)^{(m\hat{n}_K-\lceil\frac{k}{2}\rceil)}}\cdot\frac{1}{(\mathsf{a}_{m,\epsilon})^k}\end{aligned} \tag{5.6.46}$$

the last inequality by Stirling's approximation.

We now focus on the middle term on the r.h.s. above. Omitting the rounding operation, and shortening

$$Q(x)\equiv(1-x)\log(1-x)-\frac{m}{K}(1-x\frac{K}{m})\log\left(1-x\frac{K}{m}\right), \tag{5.6.47}$$

we may rewrite this middle term as

$$\frac{(1-\frac{k}{2n})^{n-\frac{k}{2}}(\frac{m}{K})^{m\hat{n}_K}}{(\frac{m}{K}-\frac{k}{2n})^{(m\hat{n}_K-\frac{k}{2})}}=\left(\sqrt{\frac{m}{K}}\right)^k\exp nQ\left(\frac{k}{n}\right)\,. \tag{5.6.48}$$

It is plainly checked that, for $k/n\in[0,1]$, the Q-function is in fact *negative* (for $K>m$), hence

$$(5.6.48)\leq\left(\sqrt{\frac{m}{K}}\right)^k. \tag{5.6.49}$$

By definition,

$$(\mathsf{a}_{m,\epsilon})^k=(\bar{\mathsf{a}}_m(\mathsf{E}+\epsilon)+\epsilon)^k\geq\epsilon^k\,, \tag{5.6.50}$$

the inequality by elementary minorization: this, as well as the bound (5.6.49), imply that (5.6.46) is *at most*

$$\sum_{k=1}^{200\hat{n}_K} \frac{l(k)}{(n-\frac{k}{2})} \frac{1}{\left(\sqrt{\frac{K}{m}}\epsilon\right)^k} = \left(\sum_{k=1}^{n^{\frac{1}{4}}} + \sum_{k=n^{\frac{1}{4}}+1}^{200\hat{n}_K} \right) \frac{l(k)}{(n-\frac{k}{2})} \frac{1}{\left(\sqrt{\frac{K}{m}}\epsilon\right)^k}. \tag{5.6.51}$$

If we now take K large enough such that $\sqrt{\frac{K}{m}}\epsilon > 1$, to wit:

$$K > m\epsilon^{-2}, \tag{5.6.52}$$

and recalling the definition of $l(k)$ as in (5.6.36), we obtain

$$(5.6.52) \lesssim \frac{1}{(n-n^{\frac{1}{4}})} \sum_{k=1}^{n^{\frac{1}{4}}} \frac{(k+1)^3}{(\sqrt{\frac{K}{m}}\epsilon)^k} + \sum_{k=n^{\frac{1}{4}}+1}^{200\hat{n}_K} n^{13} \frac{(n+1)}{(\sqrt{\frac{K}{m}}\epsilon)^k}. \tag{5.6.53}$$

The first sum on the r.h.s is, in the large-n limit, obviously convergent: its contribution therefore vanishes in virtue of the $(n-n^{1/4})$-normalization. The second sum converges exponentially fast to 0. All in all, the r.h.s. of (5.6.53) tends to 0 as $n \to \infty$: this settles the proof of claim (5.6.34), and therefore of Proposition 41.

□

5.6.2 Proof of Proposition 42

We will need here two technical inputs. The first one is similar in nature to Lemma 39, and provides tail-estimates for the energies of overlapping polymers. As the proof is short and elementary, it will be given right away.

Lemma 45. *Consider independent standard exponentials $\{\xi_i\}$, and let $X_l \equiv \sum_{i=1}^{l} \xi_i$. Denote by X_l' the sum of l such ξ-exponentials, and assume that X_l' shares exactly k edges with X_l. Then, for $a, b > 0$, it holds:*

$$\mathbb{P}\left(X_l \le a+b,\ X_l' \le a+b\right) \lesssim \mathbb{P}\left(X_l \le a,\ X_l' \le a\right) \left(1+\frac{b}{a}\right)^{2l-k}. \tag{5.6.54}$$

Proof. Recalling that

$$g(\gamma) \equiv \frac{\{4(1-\gamma)\}^{1-\gamma}}{\{2-\gamma\}^{2-\gamma}}, \tag{5.6.55}$$

by Lemma 39, it holds

$$\mathbb{P}\left(X_l \le a+b,\ X_l' \le a+b\right) \lesssim \frac{(a+b)^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l. \tag{5.6.56}$$

Using that $(a+b)^{2l-k} = a^{2l-k}\left(1+\frac{b}{a}\right)^{2l-k}$, we rephrase the r.h.s. of (5.6.56), to wit

$$\mathbb{P}\left(X_l \le a+b,\ X_l' \le a+b\right) \lesssim \frac{a^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l \left(1+\frac{b}{a}\right)^{2l-k}. \tag{5.6.57}$$

Again by Lemma 39, for the first two terms on the r.h.s. above we have that

$$\frac{a^{2l-k}}{(l-k)!l!} g\left(\frac{k}{l}\right)^l \lesssim \mathbb{P}\left(X_l \le a,\ X_l' \le a\right), \tag{5.6.58}$$

and plugging this in (5.6.57) yields the claim of the Lemma. □

The second technical input concerns the asymptotic of the f-terms. Here and below, we will denote by P_n, Q_n finite degree polynomials, not necessarily the same at different occurences, and which depend on the hypercube dimension only.

Lemma 46. *For all $k \le \mathsf{L}_{opt} n$, it holds*

$$\begin{aligned} f(n,k) \le \tanh\left(\mathsf{E}\left\{1-\frac{k}{\mathsf{L}_{opt}n}\right\}\right)^{\max\left(n-k, \frac{\mathsf{L}_{opt}n-k}{4}\right)} \\ \cosh\left(\mathsf{E}\left\{1-\frac{k}{\mathsf{L}_{opt}n}\right\}\right)^{n}\left(\frac{\mathsf{L}_{opt}n}{e\mathsf{E}}\right)^{\mathsf{L}_{opt}n-k} n^{Kn^\alpha} P_n, \end{aligned} \tag{5.6.59}$$

where P_n is polynomial with finite degree and $\alpha \equiv \frac{5}{6}$.

The proof of this Lemma is also postponed to Section 5.7: here we will use it for the

Proof of Proposition 42. By (5.6.24), it holds that

$$\frac{\mathsf{J}\sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} f(n,k)\mathbb{P}\left(\pi,\pi_k\right)}{\mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}\right)^2} = \frac{\mathsf{J}\sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} f(n,k)\prod_{i=m}^{K-m+1}\mathbb{P}_i\left(\pi,\pi_k\right)}{\mathbb{E}\left(\mathcal{N}^\epsilon_{n,K,K'}\right)^2}. \tag{5.6.60}$$

We claim that the r.h.s. of (5.6.60) converges to 0 as $n \to \infty$. To see this, some notation is needed: given two paths $\pi, \pi' \in \mathcal{J}$ which share k edges, we denote by

- k_l the number of common edges between $\mathbf{0}$ and H_m,
- k_m the number of common edges between H_m and H_{K-m},
- k_r the number of shared edges between H_{K-m} and $\mathbf{1}$.

It clearly holds that $k = k_l + k_m + k_r$. Using Lemma 45, we obtain

$$\begin{aligned}\prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi,\pi_k) \lesssim \mathbb{P}\left(X_m(\pi), X_m(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}\right) \times \\ \times \prod_{i=m+1}^{K-m} \mathbb{P}\left(X_{i-1,i}(\pi), X_{i-1,i}(\pi_k) \le \mathsf{a}_i \mathsf{E}\right) \times \\ \times \mathbb{P}\left(X_{K-m+1}(\pi), X_{K-m+1}(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}\right) \times \\ \times (1+\epsilon_{\mathsf{E}})^{2\sum_{i=m+1}^{K-m} \mathsf{n} d_i - k_m} (1+\epsilon_{m,\mathsf{E}})^{4m\hat{n}_K - 2 - k_l - k_r}.\end{aligned} \tag{5.6.61}$$

By definition of $\epsilon_{m,\mathsf{E}}$ and ϵ_{E}, see (5.5.1), the following lower bound plainly holds

$$1 + \epsilon_{m,\mathsf{E}} \ge 1 + \epsilon_{\mathsf{E}}\,. \tag{5.6.62}$$

Using the independence of sub-energies we rewrite

$$\begin{aligned}&\mathbb{P}\left(X_m(\pi), X_m(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}\right) \times \\ &\quad \times \prod_{i=m+1}^{K-m} \mathbb{P}\left(X_{i-1,i}(\pi), X_{i-1,i}(\pi_k) \le \mathsf{a}_i \mathsf{E}\right) \times \\ &\quad \times \mathbb{P}\left(X_{K-m+1}(\pi), X_{K-m+1}(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}\right) \\ &= \mathbb{P}\Bigg(X_m(\pi), X_m(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E}, \\ &\qquad X_{i-1,i}(\pi), X_{i-1,i}(\pi_k) \le \mathsf{a}_i \mathsf{E},\ i = m+1 \dots K-m, \\ &\qquad X_{K-m+1}(\pi), X_{K-m+1}(\pi_k) \le \bar{\mathsf{a}}_m \mathsf{E} \Bigg).\end{aligned} \tag{5.6.63}$$

Since $\sum_{i=1}^{K} \mathsf{a}_i = 1$, and by monotonicity of the probabilities, the r.h.s. of (5.6.63) is *at most*

$$\mathbb{P}\left(\overline{X}_m^{K-m+1}(\pi), \overline{X}_m^{K-m+1}(\pi_k) \le \mathsf{E}\right). \tag{5.6.64}$$

Using (5.6.62) and (5.6.64) in (5.6.61) thus yields

$$\begin{aligned}\prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi,\pi_k) \le \mathbb{P}\left(\overline{X}_m^{K-m+1}(\pi), \overline{X}_m^{K-m+1}(\pi_k) \le \mathsf{E}\right) \times \\ \times \frac{(1+\epsilon_E)^{2\sum_{i=m+1}^{K-m} \mathsf{n} d_i} (1+\epsilon_{\bar{\mathsf{a}}_m,E})^{4m\hat{n}_K - 2}}{(1+\epsilon_{\mathsf{E}})^k},\end{aligned} \tag{5.6.65}$$

which no longer depends on k_l, k_r, k_m, but only on their total sum. Using Lemma 39 in (5.6.65) we thus obtain

$$\prod_{i=m}^{K-m+1} \mathbb{P}_i(\pi, \pi_k) \lesssim \frac{\mathsf{E}^{2\mathsf{L}_{opt}n-2-k} g(\frac{k}{\mathsf{L}_{opt}n-2})^{\mathsf{L}_{opt}n-2}}{(\mathsf{L}_{opt}n-2)!(\mathsf{L}_{opt}n-2-k)!} \frac{(1+\epsilon_{\mathsf{E}})^{2\sum_{i=m+1}^{K-m} \mathsf{nd}_i}(1+\epsilon_{m,\mathsf{E}})^{4m\hat{n}_K-2}}{(1+\epsilon_{\mathsf{E}})^k}. \tag{5.6.66}$$

We now come back to (5.6.60): using the lower bound to the first moment of $\mathcal{N}^{\epsilon}_{n,K,K'}$ established in Theorem 1′ for the denominator, and (5.6.66) for the numerator, we see that

$$(5.6.60) \le \frac{P_n^2}{Q_n^2} \mathsf{J} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} \frac{f(n,k)\mathsf{E}^{2\mathsf{L}_{opt}n-2-k} g(\frac{k}{\mathsf{L}_{opt}n-2})^{\mathsf{L}_{opt}n-2}}{(1+\epsilon_E)^k(\mathsf{L}_{opt}n-2)!(\mathsf{L}_{opt}n-2-k)!C^2_{n,K,m}}. \tag{5.6.67}$$

(Recall the convention that P_n stands for some finite degree polynomial, not necessarily the same at different occurences). It is immediate to check that the following inequality holds

$$g\left(\frac{k}{\mathsf{L}_{opt}n-2}\right)^{\mathsf{L}_{opt}n-2} < g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n} P_n\,. \tag{5.6.68}$$

Furthermore,

$$(\mathsf{L}_{opt}n-2)! = \frac{(\mathsf{L}_{opt}n)!}{(\mathsf{L}_{opt}n)(\mathsf{L}_{opt}n-1)} = \frac{(\mathsf{L}_{opt}n)!}{P_n}, \tag{5.6.69}$$

where P_n is a polynomial of finite (quadratic) degree, and analogously

$$(\mathsf{L}_{opt}n-2-k)! = \frac{(\mathsf{L}_{opt}n-k)!}{P_n}. \tag{5.6.70}$$

Using (5.6.68), (5.6.69), and (5.6.70), we thus see that

$$(5.6.67) \le \frac{P_n}{Q_n} \mathsf{J} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} \frac{f(n,k)\mathsf{E}^{2\mathsf{L}_{opt}n-k} g(\frac{k}{\mathsf{L}_{opt}n})^{\mathsf{L}_{opt}n}}{(1+\epsilon_E)^k(\mathsf{L}_{opt}n)!(\mathsf{L}_{opt}n-k)!C^2_{n,K,m}}, \tag{5.6.71}$$

for some (modified, but still finite degree) polynomials P_n, Q_n.

The inclusion $\mathcal{J} \subset \Pi_{n,\mathsf{L}_{opt}n}$ holds by construction, hence

$$\mathsf{J} \le M_{n,\mathsf{L}_{opt}n} \le \sinh(\mathsf{E})^n \frac{(\mathsf{L}_{opt}n)!}{\mathsf{E}^{\mathsf{L}_{opt}n}} = \frac{(\mathsf{L}_{opt}n)!}{\mathsf{E}^{\mathsf{L}_{opt}n}}, \tag{5.6.72}$$

the second inequality by Stanley's M-bound (5.2.14) with $x := \mathsf{E}$, and the last step since E satisfies $\sinh(\mathsf{E}) = 1$. Plugging (5.6.72) into (5.6.71), we obtain

$$\begin{aligned}(5.6.71) &\le \frac{P_n}{C^2_{n,K,m}Q_n} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n-2} \frac{f(n,k)\mathsf{E}^{\mathsf{L}_{opt}n-k} g(\frac{k}{\mathsf{L}_{opt}n})^{\mathsf{L}_{opt}n}}{(1+\epsilon_E)^k(\mathsf{L}_{opt}n-k)!} \\ &\le \frac{P_n}{C^2_{n,K,m}Q_n} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n} \frac{f(n,k)(e\mathsf{E})^{\mathsf{L}_{opt}n-k} g(\frac{k}{\mathsf{L}_{opt}n})^{\mathsf{L}_{opt}n}}{(1+\epsilon_E)^k(\mathsf{L}_{opt}n-k)^{\mathsf{L}_{opt}n-k}},\end{aligned} \tag{5.6.73}$$

the last inequality by Stirling's approximation, and extending the sum up to $\mathsf{L}_{opt}n$ (the terms are positive anyhow). The estimates of Lemma 46 applied to (5.6.73) yield

$$(5.6.73) \le \frac{n^{Kn^\alpha} P_n}{C^2_{n,K,m} Q_n} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n} \left[\tanh\left(\mathsf{E}\left\{ 1 - \frac{k}{\mathsf{L}_{opt}n} \right\} \right)^{\max\left(n-k, \frac{\mathsf{L}_{opt}n-k}{4}\right)} \times \right.$$
$$\left. \times \frac{\cosh\left(\mathsf{E}\left\{1-\frac{k}{\mathsf{L}_{opt}n}\right\}\right)^n g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n}}{(1+\epsilon_\mathsf{E})^k \left(1-\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n-k}} \right]. \tag{5.6.74}$$

Recalling the definition (5.6.2) of the g-function, one plainly checks that

$$\frac{g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n}}{\left(1-\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n-k}} = \left[\frac{4^{1-\frac{k}{\mathsf{L}_{opt}n}}}{\left(2-\frac{k}{\mathsf{L}_{opt}n}\right)^{2-\frac{k}{\mathsf{L}_{opt}n}}} \right]^{\mathsf{L}_{opt}n}. \tag{5.6.75}$$

We lighten notation by setting, for $x \in [0,1]$,

$$\widehat{\Theta}(x) \equiv \frac{4^{1-x}}{(2-x)^{2-x}} \tanh\left(\mathsf{E}\{1-x\}\right)^{\max\left(\frac{1}{\mathsf{L}_{opt}}-x, \frac{1-x}{4}\right)} \cosh\left(\mathsf{E}\{1-x\}\right)^{\frac{1}{\mathsf{L}_{opt}}}. \tag{5.6.76}$$

With this notation, the r.h.s. of (5.6.74) then reads

$$\begin{aligned} &\frac{n^{Kn^\alpha} P_n}{C^2_{n,K,m} Q_n} \sum_{k=200\hat{n}_K+1}^{\mathsf{L}_{opt}n} \frac{1}{(1+\epsilon_E)^k} \widehat{\Theta}\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n} \\ &\quad = \frac{n^{Kn^\alpha} P_n}{C^2_{n,K,m} Q_n} \left(\sum_{k=200\hat{n}_K+1}^{\frac{\mathsf{L}_{opt}n}{5}} + \sum_{k=\frac{\mathsf{L}_{opt}n}{5}+1}^{\mathsf{L}_{opt}n} \right) \frac{1}{(1+\epsilon_E)^k} \widehat{\Theta}\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n} \\ &\quad =: (A) + (B), \end{aligned} \tag{5.6.77}$$

say. In order to prove that these two terms vanish as $n \uparrow \infty$, we need the following

Lemma 47. *It holds:*

$$\sup_{x \le 1} \widehat{\Theta}(x) \le 1. \tag{5.6.78}$$

Furthermore, for $x \le \frac{1}{5}$,

$$\widehat{\Theta}(x) \le \exp\left(-\frac{x}{100}\right). \tag{5.6.79}$$

The proof of Lemma 47 is given at the end of this section. We first use it to conclude the proof of Proposition 42: using the bound (5.6.79) for the (A)-term yields

$$\begin{aligned}(A) &\leq \frac{n^{Kn^\alpha} P_n}{C_{n,K,m}^2 Q_n} \sum_{k=200\hat{n}_K+1}^{\frac{\mathsf{L}_{opt} n}{5}} \frac{\exp -\frac{k}{100}}{(1+\epsilon_E)^k} \\ &\leq \frac{\exp -\frac{200n}{100K}}{C_{n,K,m}^2} \frac{n^{Kn^\alpha} P_n}{Q_n} \sum_{k=200\hat{n}_K+1}^{\frac{\mathsf{L}_{opt} n}{5}} \frac{1}{(1+\epsilon_E)^k},\end{aligned} \tag{5.6.80}$$

since $x \mapsto \exp(-x)$ is decreasing. Furthemore using that the above sum is convergent we thus see that

$$(A) \lesssim \frac{\exp -2\frac{n}{K}}{C_{n,K,m}^2} \frac{n^{Kn^\alpha} P_n}{Q_n}, \tag{5.6.81}$$

Finally plugging the definition (5.5.3) of $C_{n,K,m}$ into (5.6.81), yields

$$(A) \leq \exp n \left[\frac{\sqrt{2}-2}{K} + \frac{2\sqrt{2}m(m-1)+2}{K^2} \right] \times \frac{n^{Kn^\alpha} P_n}{Q_n}. \tag{5.6.82}$$

But for $K > 10^7$, the exponent on the r.h.s. above is < 0, hence the (A)-term vanishes as $n \uparrow \infty$, settling the first claim.

As for the (B)-term , using (5.6.78) yields

$$\begin{aligned}(B) &\leq \frac{n^{Kn^\alpha} P_n}{C_{n,K,m}^2 Q_n} \times \sum_{k=\frac{\mathsf{L}_{opt} n}{5}+1}^{\mathsf{L}_{opt} n} \frac{1}{(1+\epsilon_{\mathsf{E}})^k} \\ &\leq \frac{n^{Kn^\alpha} P_n}{C_{n,K,m}^2 Q_n} \times \frac{\mathsf{L}_{opt} n}{(1+\epsilon_{\mathsf{E}})^{\frac{\mathsf{L}_{opt} n}{5}}},\end{aligned} \tag{5.6.83}$$

the last inequality majorizing with the largest term of the sum. Again plugging the definition (5.5.3) of $C_{n,K,m}$ in (5.6.83), and absorbing the n-factor in the P-polynomial, yields

$$(B) \leq \exp n \left[\frac{\sqrt{2}}{K} + \frac{2\sqrt{2}m(m-1)+2}{K^2} \right] \times \frac{1}{(1+\epsilon_{\mathsf{E}})^{\frac{\mathsf{L}_{opt} n}{5}}} \times \frac{n^{Kn^\alpha} P_n}{Q_n}. \tag{5.6.84}$$

By (5.2.59), it holds that $\mathsf{L}_{opt} > \mathsf{L} - \frac{m}{K}$, clearly implying that for any $K > 10^5$,

$$1.25 \geq \mathsf{L}_{opt} \geq 1.24\,. \tag{5.6.85}$$

Using this in (5.6.84) yields

$$
\begin{aligned}
(B) &\leq \exp n \left[\frac{\sqrt{2}}{K} + \frac{2\sqrt{2}m(m-1)+2}{K^2} \right] \times \frac{1}{(1+\epsilon_{\mathsf{E}})^{\frac{1.24n}{5}}} \times \frac{n^{Kn^\alpha} P_n}{Q_n} \\
&= \exp n \left[\frac{\sqrt{2}}{K} + \frac{2\sqrt{2}m(m-1)+2}{K^2} - \frac{1.24n}{5} \log(1+\epsilon_{\mathsf{E}}) \right] \times \frac{n^{Kn^\alpha} P_n}{Q_n}.
\end{aligned}
\tag{5.6.86}
$$

Using the lower bound $\log(1+x) \geq x - \frac{x^2}{2}$ in (5.6.86) finally yields

$$
(B) \leq \exp n \left[\frac{\sqrt{2}}{K} + \frac{2\sqrt{2}m(m-1)+2}{K^2} - \left(\epsilon_{\mathsf{E}} - \frac{\epsilon_{\mathsf{E}}^2}{2} \right) \frac{1.24}{5} \right] \times \frac{n^{Kn^\alpha} P_n}{Q_n}. \tag{5.6.87}
$$

But for $K > \max(10^7, \epsilon^{-2})$, the exponent is definitely strictly negative, hence the (B)-terms also vanishes as $n \uparrow \infty$, concluding the proof of the second claim.

In order to conclude the proof of Proposition 42 we therefore owe to the reader a

Proof of Lemma 47. We first address claim (5.6.79): since $\mathsf{L}_{opt} \leq \sqrt{2}\mathsf{E} \leq 1.25$, one plainly checks that for all $x \leq \frac{1}{5}$ it holds

$$
\max\left(\frac{1}{\mathsf{L}_{opt}} - x, \frac{1-x}{4} \right) = \frac{1}{\mathsf{L}_{opt}} - x\,, \tag{5.6.88}
$$

therefore

$$
\begin{aligned}
\widehat{\Theta}(x) &= \frac{4^{1-x}}{(2-x)^{2-x}} \tanh\left(\mathsf{E}\left\{1-x\right\}\right)^{\frac{1}{\mathsf{L}_{opt}}-x} \cosh\left(\mathsf{E}\left\{1-x\right\}\right)^{\frac{1}{\mathsf{L}_{opt}}} \\
&= \frac{4^{1-x}}{(2-x)^{2-x}} \sinh\left(\mathsf{E}\left\{1-x\right\}\right)^{\frac{1}{\mathsf{L}_{opt}}-x} \cosh\left(\mathsf{E}\left\{1-x\right\}\right)^{x}.
\end{aligned}
\tag{5.6.89}
$$

The following inequalities can be easily checked using the convexity of $x \mapsto \sinh(\mathsf{E}(1-x))$, and of $x \mapsto \cosh(\mathsf{E}(1-x))$, and constructing the corresponding chords between $x=0$ and $x=1$: it holds

$$
\sinh(\mathsf{E}(1-x)) \leq (1-x)\ , \text{ and } \cosh(\mathsf{E}(1-x)) \leq \sqrt{2} + (1-\sqrt{2})x\,, \tag{5.6.90}
$$

Combining (5.6.89) and (5.6.90), we obtain

$$
\begin{aligned}
\widehat{\Theta}(x) &\leq \frac{4^{1-x}}{(2-x)^{2-x}} (1-x)^{\frac{1}{\mathsf{L}_{opt}}-x} \left(\sqrt{2} + (1-\sqrt{2})x \right)^x \\
&= \frac{2^{2(1-x)-(2-x)} (1-x)^{\frac{1}{\mathsf{L}_{opt}}-x} \left(\sqrt{2} + (1-\sqrt{2})x\right)^x}{\left(1-\frac{x}{2}\right)^{(2-x)}},
\end{aligned}
\tag{5.6.91}
$$

the last step by rearrangement. Moreover, it holds that

$$1-x \leq \left(1-\frac{x}{2}\right)^2 . \tag{5.6.92}$$

Simplifying the exponent of the first term in the numerator on the r.h.s. of (5.6.91), and using (5.6.92) for the middle term, yields

$$\begin{aligned} \widehat{\Theta}(x) &\leq \frac{2^{-x}\left(1-\frac{x}{2}\right)^{2(\frac{1}{\mathsf{L}_{opt}}-x)}\left(\sqrt{2}+(1-\sqrt{2})x\right)^x}{\left(1-\frac{x}{2}\right)^{(2-x)}} \\ &= \left(\frac{1+\frac{(1-\sqrt{2})}{\sqrt{2}}x}{\sqrt{2}(1-\frac{x}{2})}\right)^x \times \frac{1}{\left(1-\frac{x}{2}\right)^{2(1-\frac{1}{\mathsf{L}_{opt}})}}, \end{aligned} \tag{5.6.93}$$

the last step again by simple rearrangements.

Elementary inspection of the first derivative shows that, on the interval $[0, 1/5]$, the function

$$x \mapsto \frac{1+\frac{(1-\sqrt{2})}{\sqrt{2}}x}{(1-\frac{x}{2})} \tag{5.6.94}$$

is, in fact, increasing: bounding the function with its largest value attained in $x = 1/5$, and plugging in (5.6.93), yields

$$\begin{aligned} \widehat{\Theta}(x) &\leq \left(\frac{1+\frac{(1-\sqrt{2})}{\sqrt{2}}\frac{1}{5}}{\sqrt{2}\frac{9}{10}}\right)^x \times \frac{1}{(1-\frac{x}{2})^{2(1-\frac{1}{\mathsf{L}_{opt}})}} \\ &\leq \left(\frac{3}{4}\right)^x \times \frac{1}{(1-\frac{x}{2})^{2(1-\frac{1}{\mathsf{L}_{opt}})}} , \end{aligned} \tag{5.6.95}$$

the second inequality by elementary numerical estimates. Exponentiating the second term on the r.h.s. above then leads to

$$\begin{aligned} \widehat{\Theta}(x) &\leq \left(\frac{3}{4}\right)^x \exp\left[-2(1-\frac{1}{\mathsf{L}_{opt}})\log\left(1-\frac{x}{2}\right)\right] \\ &\leq \left(\frac{3}{4}\right)^x \exp\left[x\left(1-\frac{1}{\mathsf{L}_{opt}}\right)2\log(2)\right], \end{aligned} \tag{5.6.96}$$

where in the second step we have used that

$$-\log(1-\frac{x}{2}) \leq x\log(2), \tag{5.6.97}$$

which is an immediate consequence of the convexity of $x \mapsto -\log(1-\frac{x}{2})$. Recalling (5.2.59), and the ensuing elementary estimate $\mathsf{L}_{opt} < \sqrt{2}\mathsf{E} < 1.25$, we thus see that

$$\widehat{\Theta}(x) \leq \exp x \left[\log\left(\frac{3}{4}\right) + \left(1 - \frac{1}{1.25}\right) 2\log(2)\right] \leq \exp\left[-\frac{x}{100}\right], \tag{5.6.98}$$

the second inequality by straightforward numerical evaluation: claim (5.6.79) is thus settled.

We now move to claim (5.6.78). We recall that

$$\begin{aligned}\widehat{\Theta}(x) &= \frac{4^{1-x}}{(2-x)^{2-x}} \tanh\left(\mathsf{E}\left\{1-x\right\}\right)^{\max\left(\frac{1}{\mathsf{L}_{opt}}-x, \frac{1-x}{4}\right)} \cosh\left(\mathsf{E}\left\{1-x\right\}\right)^{\frac{1}{\mathsf{L}_{opt}}} \\ &= \frac{4^{1-x}}{(2-x)^{2-x}} \sinh\left(\mathsf{E}\left\{1-x\right\}\right)^{\frac{1}{\mathsf{L}_{opt}}-x} \cosh\left(\mathsf{E}\left\{1-x\right\}\right)^{x} 1_{\{\max\left(\frac{1}{\mathsf{L}_{opt}}-x, \frac{1-x}{4}\right) = \frac{1}{\mathsf{L}_{opt}}-x\}} \\ &+ \frac{4^{1-x}}{(2-x)^{2-x}} \sinh\left(\mathsf{E}\left\{1-x\right\}\right)^{\frac{1-x}{4}} \cosh\left(\mathsf{E}\left\{1-x\right\}\right)^{\frac{1}{\mathsf{L}_{opt}}-\frac{1-x}{4}} 1_{\{\max\left(\frac{1}{\mathsf{L}_{opt}}-x, \frac{1-x}{4}\right) = \frac{1-x}{4}\}}.\end{aligned} \tag{5.6.99}$$

By (5.2.59), it holds that $\mathsf{L}_{opt} > \mathsf{L} - \frac{m}{K}$ and this implies that for any $K > 10^5$,

$$\frac{1}{1.24} \geq \frac{1}{\mathsf{L}_{opt}} \geq \frac{1}{1.25}\,. \tag{5.6.100}$$

Let now

$$g_1(x) \equiv \frac{4^{1-x}}{(2-x)^{2-x}} \sinh\left(\mathsf{E}(1-x)\right)^{\frac{1}{1.25}-x} \cosh\left(\mathsf{E}(1-x)\right)^{x}, \tag{5.6.101}$$

$$g_2(x) \equiv \frac{4^{1-x}}{(2-x)^{2-x}} \sinh\left(\mathsf{E}(1-x)\right)^{\frac{1-x}{4}} \cosh\left(\mathsf{E}(1-x)\right)^{\frac{1}{1.24}-\frac{1-x}{4}}. \tag{5.6.102}$$

In virtue of (5.6.100), g_1 is larger than the first term in (5.6.99), whereas g_2 ls larger than the second one. In particular, setting $g_3 \equiv \min(g_1, g_2)$, we see that in order to establish (5.6.78) it suffices to prove that

$$\sup_{x\in[0,1]} g_3(x) \leq 1\,, \tag{5.6.103}$$

which is our new claim. A plot of these two functions is given in Figure 5.16 below.

To see this, we first note that (5.6.79) already shows that

$$\sup_{x\in[0,1/5]} g_1(x) \leq 1\,. \tag{5.6.104}$$

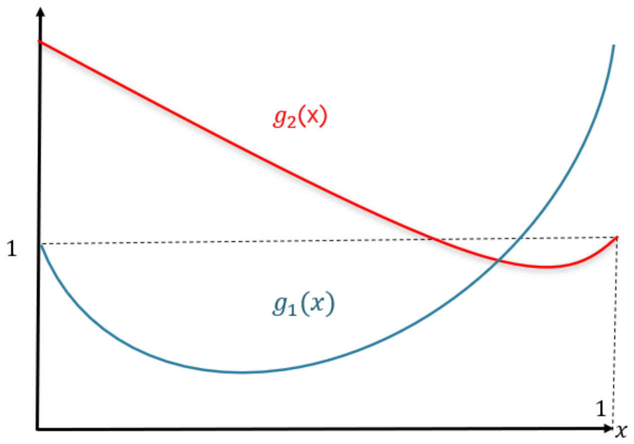

Figure 5.16: The functions g_1 and g_2. One clearly sees that the minimum of these functions is always below 1.

We now claim that

$$g_1 \text{ is convex on } [0.12, 0.73], \; g_1(0.12) \leq 1 \text{ and } g_1(0.73) \leq 1. \tag{5.6.105}$$

and that

$$g_2 \text{ is convex on } [0.71, 1], \; g_2(0.71) \leq 1 \text{ and } g_2(1) = 1. \tag{5.6.106}$$

Assuming the validity of these two claims for the time being, it follows that

$$g_1(x) \leq 1 \; \forall x \leq 0.73\,, \tag{5.6.107}$$

and

$$g_2(x) \leq 1 \; \forall x \geq 0.71\,. \tag{5.6.108}$$

Combining (5.6.107) and (5.6.108) thus yields

$$\sup_{x\in[0,1]} \; g_3(x) \leq 1, \tag{5.6.109}$$

and claim (5.6.78) is verified.

To conclude the proof of Lemma 47 it thus remains to prove (5.6.105) and (5.6.106). We begin with the convexity of g_1 on the interval $[0.12, 0.73]$. Since $g_1 > 0$,

$$\frac{d^2 \log(g_1)}{dx^2} = \frac{g_1'' g_1 - g_1'^2}{g_1^2} \geq 0 \Longrightarrow g_1''(x) \geq 0, \tag{5.6.110}$$

hence convexity of $\log(g_1)$ implies convexity of g_1: we will check the former by showing positivity of its second derivative. It holds:

$$\begin{aligned}\frac{d^2\log(g_1(x))}{dx^2} =& \frac{d^2}{dx^2}\Big[(1-x)\log(4)+(-2+x)\log(2-x)\Big]+\\ &+\frac{d^2}{dx^2}\Big[x\log(\cosh(\mathsf{E}(1-x)))\Big]+\\ &+\frac{d^2}{dx^2}\Big[\Big(-x+\frac{1}{1.25}\Big)\log\sinh(\mathsf{E}(1-x))\Big].\end{aligned} \tag{5.6.111}$$

By elementary computations, we see that:

$$\frac{d^2}{dx^2}\Big[(1-x)\log(4)+(-2+x)\log(2-x)\Big]=\frac{-1}{2-x}, \tag{5.6.112}$$

$$\begin{aligned}\frac{d^2}{dx^2}\Big[x\log(\cosh(\mathsf{E}(1-x)))\Big] &= \frac{d}{dx}\Big[\log(\cosh(\mathsf{E}(1-x))) - x\mathsf{E}\tanh(\mathsf{E}(1-x))\Big]\\ &= -2E\tanh(\mathsf{E}(1-x)) + \frac{x\mathsf{E}^2}{\cosh(\mathsf{E}(1-x))^2},\end{aligned} \tag{5.6.113}$$

and finally

$$\begin{aligned}&\frac{d^2}{dx^2}\Big[\Big(-x+\frac{1}{1.25}\Big)\log\sinh(\mathsf{E}(1-x))\Big]\\ &=\frac{d}{dx}\Big[-\log(\sinh(\mathsf{E}(1-x)))+\mathsf{E}\Big(x-\frac{1}{1.25}\Big)\coth(\mathsf{E}(1-x))\Big]\\ &=2\mathsf{E}\coth(\mathsf{E}(1-x))+\frac{\mathsf{E}^2(x-\frac{1}{1.25})}{\sinh(\mathsf{E}(1-x))^2}.\end{aligned} \tag{5.6.114}$$

Since $1/5 \geq 0.12$, say, by the previous considerations we see that $g_1(0.12) \leq 1$. We may thus restrict to to $x \in [0.12, 0.73]$: we first note that the first function on the r.h.s. of (5.6.112) is decreasing. In particular, it holds that

$$\frac{-1}{2-x} \geq \frac{-1}{2-0.73} \geq -0.8\,. \tag{5.6.115}$$

Plugging (5.6.112)-(5.6.114) in (5.6.111), and then using (5.6.115) and the fact that $\frac{x\mathsf{E}^2}{\cosh(\mathsf{E}(1-x))^2} \geq 0$, thus yields

$$(5.6.111) \geq -0.8 - 2\mathsf{E}\tanh(\mathsf{E}(1-x)) + 2\mathsf{E}\coth(\mathsf{E}(1-x)) + \frac{\mathsf{E}^2(x-\frac{1}{1.25})}{\sinh(\mathsf{E}(1-x))^2} \tag{5.6.116}$$

We now make two observations.

- First of all we note that the r.h.s. of (5.6.116) consists of three increasing functions.
- Furthermore, by Taylor expansions to fifth order, and some elementary yet tedious numerical estimates (which will be here omitted) one plainly checks that in $x = 0.12$ the r.h.s. of (5.6.116) is, in fact, *positive*, whereas $g_1(0.73) \leq 1$.

Combining the above items we see, in particular, that g_1 is indeed convex on $[0.12, 0.73]$, and the proof of claim (5.6.105) is therefore concluded.

We now move to the analysis of g_2. Simple computations show that

$$\begin{aligned} \frac{d^2}{dx^2}\Big[\log(g_2(x))\Big] &= \frac{-1}{2-x} - \frac{\mathsf{E}}{2}\tanh\left(\mathsf{E}(1-x)\right) + \frac{\mathsf{E}^2(\frac{1}{1.24}+\frac{x-1}{4})}{\cosh\left(\mathsf{E}(1-x)\right)^2} + \frac{\mathsf{E}}{2}\coth\left(\mathsf{E}(1-x)\right) \\ &\quad + \frac{\mathsf{E}^2(x-1)}{4\sinh\left(\mathsf{E}(1-x)\right)^2} \\ &\geq -1 - \frac{\mathsf{E}}{2}\tanh\left(\mathsf{E}(1-x)\right) + \frac{\mathsf{E}^2(\frac{1}{1.24}+\frac{x-1}{4})}{\cosh\left(\mathsf{E}(1-x)\right)^2} + \frac{\mathsf{E}}{2}\coth\left(\mathsf{E}(1-x)\right) \\ &\quad + \frac{\mathsf{E}^2(x-1)}{4\sinh\left(\mathsf{E}(1-x)\right)^2}, \end{aligned} \tag{5.6.117}$$

the last inequality using that $\frac{-1}{2-x} \geq -1$. We now proceed in full analogy to (5.6.116):

- First we note that the r.h.s. of (5.6.117) consists of four increasing functions.
- Furthermore, and again by some tedious yet elementary numerical estimates via Taylor expansions to fifth order (also omitted), one plainly checks that in $x = 0.71$, say, the r.h.s. of (5.6.117) is, in fact, *positive*, and $g_2(0.71) \leq 1$.

Since the above items clearly imply, in particular, that g_2 is convex on $[0.71, 1]$, the second claim (5.6.106) is also settled, and the proof of Lemma 47 is thus concluded. □

□

5.6.3 Proof of Proposition 43

We first state the technical input concerning the asymptotic of the $f^{(s)}$-terms. (As usual, P_n, Q_n stand for finite degree polynomials, not necessarily the same at different occurences).

Lemma 48. *For any $k \leq 200\hat{n}_K$, it holds*

$$\begin{aligned} f^{(s)}(n,k) \leq &\left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \tanh\left(\mathsf{E}(1-\frac{k}{\mathsf{L}_{opt}n})\right)^{n-k} \times \\ &\times \cosh\left(\mathsf{E}(1-\frac{k}{\mathsf{L}_{opt}n})\right)^{n}\left(\frac{\mathsf{L}_{opt}n}{e\mathsf{E}}\right)^{\mathsf{L}_{opt}n-k} n^{Kn^\alpha} P_n\,. \end{aligned} \tag{5.6.118}$$

The proof of this Lemma is also postponed to Section 5.7.

Proof of Proposition 43. . By (5.6.24), it holds that

$$\frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(s)}(n,k)\mathbb{P}\left(\pi,\pi_k^{(s)}\right)}{\mathbb{E}\left(\mathcal{N}_{n,K,K'}^{\epsilon}\right)^2} = \frac{\mathsf{J}\sum_{k=1}^{200\hat{n}_K} f^{(s)}(n,k)\prod_{i=m}^{K-m+1}\mathbb{P}_i\left(\pi,\pi_k^{(s)}\right)}{\mathbb{E}\left(\mathcal{N}_{n,K,K'}^{\epsilon}\right)^2}. \tag{5.6.119}$$

We claim that the r.h.s. of (5.6.119) converges to 0 as $n \to \infty$. To see this, we follow *exactly* the same steps which from (5.6.60) lead to (5.6.73), this time of course with $f^{(s)}$ instead of f. Omitting the details, the upshot is that the r.h.s. of (5.6.119) is *at most*

$$\frac{P_n}{C_{n,K,m}^2 Q_n}\sum_{k=1}^{200\hat{n}_K}\frac{f^{(s)}(n,k)(e\mathsf{E})^{\mathsf{L}_{opt}n-k}g(\frac{k}{\mathsf{L}_{opt}n})^{\mathsf{L}_{opt}n}}{(1+\epsilon_E)^k(\mathsf{L}_{opt}n-k)^{\mathsf{L}_{opt}n-k}}, \tag{5.6.120}$$

The estimates from Lemma 48 applied to (5.6.120) then yield

$$(5.6.120) \le \frac{(\frac{3}{4})^{(m-200)\hat{n}_K}n^{Kn^\alpha}P_n}{C_{n,K,m}^2 Q_n}\sum_{k=1}^{200\hat{n}_K}\frac{\tanh(\mathsf{E}(1-\frac{k}{\mathsf{L}_{opt}n}))^{n-k}\cosh(\mathsf{E}(1-\frac{k}{\mathsf{L}_{opt}n}))^{n}g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n}}{(1+\epsilon_{\mathsf{E}})^k\left(1-\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n-k}}. \tag{5.6.121}$$

As in (5.6.75), it holds that

$$\frac{g\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n}}{\left(1-\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n-k}} = \left[\frac{4^{1-\frac{k}{\mathsf{L}_{opt}n}}}{\left(2-\frac{k}{\mathsf{L}_{opt}n}\right)^{2-\frac{k}{\mathsf{L}_{opt}n}}}\right]^{\mathsf{L}_{opt}n}. \tag{5.6.122}$$

We lighten notation by setting, for $x \in [0, 1/\mathsf{L}_{opt}]$,

$$\Theta(x) \equiv \frac{4^{1-x}}{(2-x)^{2-x}}\tanh\left(\mathsf{E}(1-x)\right)^{\frac{1}{\mathsf{L}_{opt}}-x}\cosh\left(\mathsf{E}(1-x)\right)^{\frac{1}{\mathsf{L}_{opt}}}. \tag{5.6.123}$$

Using this, together with (5.6.122), the r.h.s. of (5.6.121) then takes the neater form

$$\left(\frac{3}{4}\right)^{(m-200)\hat{n}_K}\frac{n^{Kn^\alpha}P_n}{C_{n,K,m}^2 Q_n}\sum_{k=1}^{200\hat{n}_K}\frac{1}{(1+\epsilon_{\mathsf{E}})^k}\Theta\left(\frac{k}{\mathsf{L}_{opt}n}\right)^{\mathsf{L}_{opt}n} \tag{5.6.124}$$

We recall that

$$K > 2\times 10^7. \tag{5.6.125}$$

Thus, in the regime $k \le 200\hat{n}_K$, and since $\mathsf{L}_{opt} \ge 1$, we have

$$\frac{k}{\mathsf{L}_{opt}n} \le \frac{200n}{\mathsf{L}_{opt}Kn} \le \frac{200}{K} \le 10^{-5}. \tag{5.6.126}$$

We now claim that for all $x \leq 10^{-5}$,

$$\Theta(x) = \widehat{\Theta}(x)\,. \tag{5.6.127}$$

In fact, for any $x \leq 10^{-5}$,

$$\max\left(\frac{1}{\mathsf{L}_{opt}} - x, \frac{1-x}{4}\right) = \frac{1}{\mathsf{L}_{opt}} - x\,, \tag{5.6.128}$$

as a simple numerical inspection shows: this proves (5.6.127).

Combining Lemma 47 and (5.6.127), thus yields

$$\sup_{x \leq 10^{-5}} \Theta(x) \leq 1. \tag{5.6.129}$$

Using (5.6.129) in (5.6.124) then gives that

$$\begin{aligned}(5.6.124) &\leq \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \frac{n^{Kn^\alpha} P_n}{C^2_{n,K,m} Q_n} \sum_{k=1}^{200\hat{n}_K} \frac{1}{(1+\epsilon_{\mathsf{E}})^k} \\ &\lesssim \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \frac{n^{Kn^\alpha} P_n}{C^2_{n,K,m} Q_n}\,,\end{aligned} \tag{5.6.130}$$

since the sum is evidently convergent. Furthermore recalling the definition (5.5.3) of $C_{n,K,m}$, we thus see that

$$\begin{aligned}(5.6.130) &\lesssim \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K} \times \exp n\left[\frac{\sqrt{2}}{K} + \frac{2\sqrt{2}m(m-1)+2}{K^2}\right] \times \frac{n^{Kn^\alpha} P_n}{Q_n} \\ &= \exp n\left[\frac{1}{K}\left\{(m-200)\log\left(\frac{3}{4}\right) + \sqrt{2}\right\} + \frac{2\sqrt{2}m(m-1)+2}{K^2}\right] \times \frac{n^{Kn^\alpha} P_n}{Q_n}\end{aligned} \tag{5.6.131}$$

Since $m = 205$,

$$(m-200)\log\left(\frac{3}{4}\right) + \sqrt{2} < -\frac{1}{100}\,, \tag{5.6.132}$$

(this bound is, as a matter of fact, the reason for choosing m as we do), plugging (5.6.132) in (5.6.130), yields

$$(5.6.130) \lesssim \exp n\left[-\frac{1}{100K} + \frac{2\sqrt{2}m(m-1)+2}{K^2}\right] \times \frac{n^{Kn^\alpha} P_n}{Q_n}. \tag{5.6.133}$$

But again in virtue of (5.6.125), and with $m = 205$,

$$-\frac{1}{100K} + \frac{2\sqrt{2}m(m-1)+2}{K^2} < 0\,, \tag{5.6.134}$$

as can be immediately checked: the r.h.s. of (5.6.133) is therefore vanishing as $n \uparrow \infty$, and the proof of Proposition 43 is concluded. □

5.7 Combinatorial estimates

To control the asymptotics of the $f^{(d)}, f^{(s)}$ and f-terms requires some delicate path-counting.

5.7.1 Counting directed paths, and proof of Lemma 44

Key to the whole treatment are estimates for the number of pairs of *directed* paths with prescribed overlaps which are formulated in Lemma 49 below. We shall emphasize that the estimates (5.7.2) and (5.7.3) have been established by Fill and Pemantle [29, Lemma 2.2, 2.4], whereas (5.7.4) can be found in [38, Lemma 6].

Lemma 49 (Path counting directed, Fill and Pemantle). *Let π' be any reference path on the n-dim hypercube connecting $\mathbf{0}$ and $\mathbf{1}$, say $\pi' = 12...n$. For $k \geq 1$, denote by $F(n,k)$ the number of directed paths π that share* precisely *k edges with π', and by $F^*(n,k)$ the number of paths that share k edges with π', without considering the first and the last edge. Finally, shorten $\mathfrak{n}_\mathfrak{e} \equiv n - 5e(n+3)^{2/3}$. It holds:*

- *For all $k \geq \mathfrak{n}_\mathfrak{e}$, we have*
$$F(n,k) \leq (n-k)!\binom{n}{\mathfrak{n}_\mathfrak{e}}, \tag{5.7.1}$$
- *suppose $k \leq \mathfrak{n}_\mathfrak{e}$ for $n \geq 25$. Then, it holds*
$$F(n,k) \leq (n-k)!n^6. \tag{5.7.2}$$
- *For $k \leq n^{1/4}$, the stronger bounds hold*
$$F(n,k) \leq (n-k)!(k+1)(1+o_n(1)), \tag{5.7.3}$$
and
$$F^*(n,k) \leq (n-k-1)!(k+1)(1+o_n(1)), \tag{5.7.4}$$
as $n \uparrow \infty$, uniformly in k.

Proof. As mentioned, we only need to prove (5.7.1): to this end, consider a directed path π which shares precisely k edges with the reference path $\pi' = 12\ldots n$. We set $r_i = l$ if the l^{th} traversed edge by π is the i^{th} edge shared by π and π'. (We set by convention $r_0 \equiv 0$, and $r_{k+1} \equiv n+1$). Furthemore let $\mathbf{r} \equiv \mathbf{r}(\pi) = (r_0, ..., r_{k+1})$. For any sequence $\mathbf{r}_0 = (r_0, ..., r_{k+1})$ with $0 = r_0 < r_1 < ... < r_k < r_{k+1} = n+1$, let $C(\mathbf{r}_0)$ denote the number of paths π with $\mathbf{r}(\pi) = \mathbf{r}_0$. Since the values $\pi_{r_i+1}, ..., \pi_{r_{i+1}-1}$ must be a permutation of $\{r_i+1, ..., r_{i+1}-1\}$, it clearly holds that $C(\mathbf{r}) \leq G(\mathbf{r})$, where
$$G(\mathbf{r}) \equiv \prod_{i=0}^{k}(r_{i+1} - r_i - 1)!. \tag{5.7.5}$$

Iterating the log-convexity (5.6.40) of factorials in its simplest form: $a!b! \leq (a+b)!$, yields

$$G(\mathbf{r}) \leq \left(\sum_{i=0}^{k} r_{i+1} - r_i - 1 \right)! = (n+1-(k+1))! = (n-k)!\,, \tag{5.7.6}$$

which implies, in particular, that there are at most $(n-k)!$ paths sharing k edges with a reference-path π' for *given* $\mathbf{r}$-sequence. But since there are $\binom{n}{k}$ ways to choose such $\mathbf{r}$-sequences we obtain

$$F(n,k) \leq (n-k)! \binom{n}{k}. \tag{5.7.7}$$

Since the factorial term on the r.h.s. above is decreasing in k for $k \geq \lceil \frac{n}{2} \rceil$, we deduce that for $k \geq \mathfrak{n}_\mathfrak{c} \gg \frac{n}{2}$,

$$(n-k)! \binom{n}{k} \leq (n-k)! \binom{n}{\mathfrak{n}_\mathfrak{c}}, \tag{5.7.8}$$

settling the proof of (5.7.1). □

Armed with the above estimates on the number of directed paths with prescribed overlaps, we can move to the

Proof of Lemma 44. For $\pi \in \mathcal{J}$ and $\pi_k^{(d)} \in \mathcal{J}_\pi^{(d)}(n,k)$, let us denote by k_l the number of common edges between $\mathbf{0}$ and H_m, and by k_r the number of common edges between H_{K-m} and $\mathbf{1}$ (in which case it evidently holds that $k = k_l + k_r$). Furthermore, let

$$\begin{aligned} f_\pi^{(d)}(n,k,k_l) \equiv\ & \textsf{all paths } \pi' \in \mathcal{J} \textsf{ which share } k \textsf{ edges with} \\ & \pi \textsf{ only in the directed phase, i.e between} \\ & \mathbf{0} \textsf{ and } H_m \textsf{ or } H_{K-m} \textsf{ and } 1, \\ & \textsf{with } k_l \textsf{ edges in common between } \mathbf{0} \textsf{ and } H_m, \\ & \textsf{but without considering first and last edge.} \end{aligned} \tag{5.7.9}$$

We have

$$\begin{aligned} f_\pi^{(d)}(n,k) &= \sum_{k_l \geq k_r} f_\pi^{(d)}(n,k,k_l) + \sum_{k_l < k_r} f_\pi^{(d)}(n,k,k_l) \\ &\leq \sum_{k_l \geq k_r} f_\pi^{(d)}(n,k,k_l) + \sum_{k_l \leq k_r} f_\pi^{(d)}(n,k,k_l). \end{aligned} \tag{5.7.10}$$

We claim that

$$\sum_{k_l \geq k_r} f_\pi^{(d)}(n,k,k_l) = \sum_{k_l \leq k_r} f_\pi^{(d)}(n,k,k_l). \tag{5.7.11}$$

This claim is perhaps surprising at first sight, as k_l and k_r cannot be simply swapped. The idea is over to work through bijections relating the (pair) of paths appearing in the first sum to those in the second one.

Indeed, each vertex on the right side of the hypercube stands in one to one correspondence with a vertex on the left side: the (trivial) bijection here amounts to changing the $1's$ into $0's$ (and the $0's$ into $1's$).

Furthermore, by (5.2.28), backsteps and forward steps are symmetric around the center of the hypercube, meaning that for $i \in \{m+1, K-m\}$,

$$\mathsf{eb}_i = \mathsf{eb}_{K-i+1} \qquad \text{and} \qquad \mathsf{ef}_i = \mathsf{eb}_{K-i+1}\,. \tag{5.7.12}$$

This, together with the fact that polymers are stretched, implies that the number of subpaths reaching two given vertices between H_i and H_{i+1}, and the number of those between $H_{K-(i+1)}$ and H_{K-i} do in fact coincide.

Finally, we note that the "cone" of vertices in H_{i+1} which are attainable from a vertex in H_i in the first half of the hypercube is in one-to-one correspondence with the vertices in $H_{K-(i+1)}$ which lead to a given vertex in H_{K-i} (this can immediately seen by changing the 1-coordinates of a vertex into 0, or the other way around). Thus, for each cone on the left side of the hypercube, we find a cone on the right side which evolves in the opposite direction, settling claim (5.7.11).

Using (5.7.11) in (5.7.10) yields

$$f_\pi^{(d)}(n,k) \leq 2 \sum_{k_l \geq k_r} f_\pi^{(d)}(n,k,k_l). \tag{5.7.13}$$

We now make the following key observation: counting the number of directed subpaths which share k_l edges with π (disregarding the first edge) between $\mathbf{0}$ and any admissible point of H_m is equivalent to counting the number of directed subpaths π' that share k_l edges with the directed subpath of π, but on a hypercube of dimension $m\hat{n}_K$ (again disregarding the first edge). By symmetry, the same of course holds true for the number of subpaths between H_{K-m} and $\mathbf{1}$ (this time disregarding the last edge). The new goal is thus to solve the path-counting problem on these hypercubes of smaller dimensions. In order to do so, we focus on the rightmost edge shared by both polymers, and denote by

$$d_l \equiv d\left(\pi_{r_{k_l}}, H_m\right) \tag{5.7.14}$$

its Hamming distance to the H_m-plane. We now distinguish between two cases: the first case concerns the situation where $d_l = 0$, whereas the second case concerns $d_l > 0$.

If $d_l = 0$, the rightmost common edge leads directly into the H_m-plane. Any subpath sharing k_l edges with π can thus reach one vertex only on the target plane: counting the number of subpaths connecting $\mathbf{0}$ and this prescribed vertex, *while disregarding the first edge*, is therefore equivalent to estimating the number of directed paths which share $k_l - 1$ edges on a hypercube of dimension $m\hat{n}_K - 1$, also disregarding the first edge. We will solve the latter problem with the help of F, in which case a small detail must be taken into account. In fact, contrary to our current situation, the first edge does matter in the definition of F. We thus have to distinguish between the case whether the first edge is shared, respectively: not shared, by both paths. In both cases we need to specify $k_l - 1$ common edges disregarding first and "last" edge: in the first case the number of commond edges is, in fact, $(k_l - 1) + 1 = k_l$, and this leads to at most $F(m\hat{n}_K - 1, k_l)$ ways to choose them. In the second case the problem of the "hidden" (first) shared edge is not present, and we simply have *at most* $F(m\hat{n}_K - 1, k_l - 1)$ possibilities to choose the common edges. All in all, for the number of directed paths sharing k_l common edges (first one excluded), and $d_l = 0$, we have the rough bound

$$F(m\hat{n}_K - 1, k_l) + F(m\hat{n}_K - 1, k_l - 1) \leq 2F(m\hat{n}_K - 1, k_l - 1), \tag{5.7.15}$$

using for the inequality that $j \mapsto F(n, j)$ is decreasing.

We now move to the case $d_l > 0$ and first note that by definition of $f^{(d)}(n, k)$, neither first nor the last edges can be a common edge. The number of subpaths, which are sharing k_l edges between $\mathbf{0}$ and H_m with π whithout considering the first and the last edge is thus at most

$$(\# \text{ admissible vertices in } H_m) \times F^*(m\hat{n}_K, k_l). \tag{5.7.16}$$

We claim that

$$\# \text{ admissible vertices in } H_m = \binom{n - (m\hat{n}_K - d_l)}{d_l}. \tag{5.7.17}$$

Indeed, of the n possible 1-coordinates, $(m\hat{n}_K - d_l)$ many are already specified by the rightmost common edge; furthermore, in order to reach any of the admissible points on H_m we may switch, out of $n - (m\hat{n}_K - d_l)$ 0-coordinates, d_l many into $1's$: (5.7.17) thus follows by simple counting.

Next we claim that $j \mapsto \binom{n+j}{j}$ is increasing. To see this, we write

$$\binom{n+j}{j} = \frac{(n+j)\dots(j+1)}{n!}, \tag{5.7.18}$$

and observe that the term in the numerator on the r.h.s. above is increasing. It follows in particular, that the r.h.s. of (5.7.17) is maximized for $d_l = m\hat{n}_K - k_l - 1$ (recall that we are not considering the first edge), and therefore

$$(5.7.16) \leq \binom{n - k_l - 1}{m\hat{n}_K - k_l - 1} \times F^*(m\hat{n}_K, k_l). \tag{5.7.19}$$

Combining (5.7.15) and (5.7.19), we thus see that the overall number of subpaths sharing k_l edges on the "left side" of the hypercube (i.e. between $\mathbf{0}$ and H_m, but without considering the first edge) with a reference path π is less than

$$2F(m\hat{n}_K - 1, k_l - 1) + F^*(m\hat{n}_K, k_l) \times \binom{n - k_l - 1}{m\hat{n}_K - k_l - 1}. \tag{5.7.20}$$

We next move to the "right side" of the hypercube: in full analogy to the considerations leading to (5.7.15), one sees that the number of subpaths sharing k_r edges between a point on H_{K-m} and $\mathbf{1}$ with a given reference path π (disregarding, in this case, the last edge), is less than

$$F(m\hat{n}_K, k_r) + F(m\hat{n}_K, k_r + 1) \leq 2F(m\hat{n}_K, k_r). \tag{5.7.21}$$

The bounds (5.7.20) and (5.7.21) address "left" and "right" side of the hypercube on separate footing: for these bounds to be of any use in estimating the $f^{(d)}(n, k, k_l)$-terms appearing in (5.7.13), left and right side must be connected. We will do so by slightly "overshooting", insofar we do not take into account the fact that the number of subpaths connecting H_m and H_{K-m} is *reduced*, as soon as shared edges on the right region are specified. Recalling that $\mathsf{J} = \#\mathcal{J}$ takes the form

$$\mathsf{J} = \underbrace{(m\hat{n}_K)!\binom{n}{m\hat{n}_K}}_{\text{directed}} \times \underbrace{\mathsf{J}_s}_{\text{stretched}} \times \underbrace{(m\hat{n}_K)!}_{\text{directed}}, \tag{5.7.22}$$

with J_s denoting the number of subpaths between a given vertex on H_m and the H_{K-m}-plane, it follows from (5.7.20), (5.7.21) and the aforementioned overshooting, that

$$\begin{aligned} f_\pi^{(d)}(n, k, k_l) \leq \left(2F(m\hat{n}_K - 1, k_l - 1) + F^*(m\hat{n}_K, k_l) \times \binom{n - k_l - 1}{m\hat{n}_K - k_l - 1}\right) \times \\ \mathsf{J}_s \times 2F(m\hat{n}_K, k_r). \end{aligned} \tag{5.7.23}$$

The above is our fundamental estimate. Remark in particular, that it holds uniformly over $k = k_l + k_r$. To proceed further we will now distinguish two cases: either $k \leq n^{\frac{1}{4}}$ or $k > n^{\frac{1}{4}}$.

First case: $k \leq n^{\frac{1}{4}}$. We begin with an estimate for the terms in the large brackets of the r.h.s. of (5.7.23). In the considered k-regime, we may use the bounds provided by Lemma 49: display (5.7.3) yields the bound

$$F(m\hat{n}_K - 1, k_l - 1) \leq (k_l + 1)(m\hat{n}_K - k_l)!\,[1 + o_n(1)] \leq 2(k_l + 1)(m\hat{n}_K - k_l)!\,, \tag{5.7.24}$$

for n large enough, whereas display (5.7.4) of the same Lemma yields, for the F^*-term on the r.h.s. of (5.7.23) the bound

$$F^*(m\hat{n}_K, k_l) \leq (k_l + 1)(m\hat{n}_K - k_l - 1)!\,[1 + o_n(1)] \leq 2(k_l + 1)(m\hat{n}_K - k_l - 1)!\,, \tag{5.7.25}$$

which holds again for large enough n. Combining (5.7.24) and (5.7.25) we thus get that the terms in the large brackets of the r.h.s. of (5.7.23) are *at most*

$$\begin{aligned} 4(k_l + 1)(m\hat{n}_K - k_l)! + 2(k_l + 1)(m\hat{n}_K - k_l - 1)!\binom{n - k_l - 1}{m\hat{n}_K - k_l - 1} \\ \leq 4(k_l + 1)(m\hat{n}_K - k_l - 1)! \times \binom{n - k_l - 1}{m\hat{n}_K - k_l - 1}, \end{aligned} \tag{5.7.26}$$

the second inequality since $n - k_l - 1 \geq m\hat{n}_K - k_l - 1 \geq 5\hat{n}_K - 1$ (see $m = 205$ and $k \leq 200\hat{n}_K$) implies that the second term on the l.h.s. above is (exponentially) larger than the first one.

We may again use the bounds provided by Lemma 49, display (5.7.3), akin to (5.7.24), and we obtain

$$2F(m\hat{n}_K, k_r) \leq 4(k_r + 1)(m\hat{n}_K - k_r)!. \tag{5.7.27}$$

Plugging the estimates (5.7.26) and (5.7.27) into (5.7.23), we obtain

$$\begin{aligned} f_\pi^{(d)}(n, k, k_l) &\leq 16(k_l + 1)(m\hat{n}_K - k_l - 1)! \binom{n - k_l - 1}{m\hat{n}_K - k_l - 1} \mathsf{J}_s(k_r + 1)(m\hat{n}_K - k_r)! \\ &= 16(k_l + 1)(m\hat{n}_K - k_l - 1)! \binom{n - k_l - 1}{m\hat{n}_K - k_l - 1} \frac{\mathsf{J}(k_r + 1)(m\hat{n}_K - k_r)!}{(m\hat{n}_K)!^2 \binom{n}{m\hat{n}_K}}, \end{aligned} \tag{5.7.28}$$

the last equality expressing J_s as a function of J via the relation (5.7.22). Writing out the binomials, and after some elementary simplifications, (5.7.28) becomes

$$f_\pi^{(d)}(n, k, k_l) \leq 16(k_l + 1)(k_r + 1)(n - k_l - 1)!(m\hat{n}_K - k_r)! \frac{\mathsf{J}}{(m\hat{n}_K)!n!}. \tag{5.7.29}$$

In order to estimate the r.h.s. of (5.7.29), we recall that $k = k_r + k_l$, hence

$$(k_l + 1)(k_r + 1) \leq (k + 1)^2. \tag{5.7.30}$$

Furthermore, we claim that

$$(n - k_l - 1)!(m\hat{n}_K - k_r)! \leq \left(n - \lceil \frac{k}{2} \rceil - 1\right)! \left(m\hat{n}_K - \lfloor \frac{k}{2} \rfloor\right)!. \tag{5.7.31}$$

To see this, we will make use of the log-convexity (5.6.40) with $a \equiv \lceil n - k_l - 1 \rceil$ and $b \equiv \lfloor m\hat{n}_K - k_r \rfloor$, in which case it clearly holds that $a > b$ for any choice of $k_l = k - k_r$ with $k \leq 200\hat{n}_K$. By log-convexity we thus obtain

$$\begin{aligned} (n - k_l - 1)!(m\hat{n}_K - k_r)! &\leq (n - k_l - 1 + 1)!(m\hat{n}_K - k_r - 1)! \\ &= (n - k_l' - 1)!\,(m\hat{n}_K - k_r')!\,, \end{aligned} \tag{5.7.32}$$

where $k_l' \equiv k_l - 1$ and $k_r' \equiv k_r + 1$. Iterating the log-convexity as in (5.7.32) and taking into account that $k_l \geq k_r$ gives that the r.h.s. of (5.7.32) is maximized in $k_l = \lceil \frac{k}{2} \rceil$, settling the claim (5.7.31).

Plugging (5.7.30) and (5.7.31) into (5.7.29) then yields

$$f_\pi^{(d)}(n, k, k_l) \leq 16(k + 1)^2 \left(n - \lceil \frac{k}{2} \rceil - 1\right)! \left(m\hat{n}_K - \lfloor \frac{k}{2} \rfloor\right)! \frac{\mathsf{J}}{(m\hat{n}_K)!n!}. \tag{5.7.33}$$

All in all, using (5.7.13) and (5.7.33), we have

$$\begin{aligned} f_\pi^{(d)}(n,k) &\leq 2\sum_{k_l\geq k_r} 16(k+1)^2 \left(n-\lceil\frac{k}{2}\rceil-1\right)! \left(m\hat{n}_K-\lfloor\frac{k}{2}\rfloor\right)! \frac{\mathsf{J}}{(m\hat{n}_K)!n!} \\ &\leq 32(k+1)^3 \left(n-\lceil\frac{k}{2}\rceil-1\right)! \left(m\hat{n}_K-\lfloor\frac{k}{2}\rfloor\right)! \frac{\mathsf{J}}{(m\hat{n}_K)!n!}, \end{aligned} \tag{5.7.34}$$

the last inequality since $k_l+k_r=k$, implying that the sum consists at most of $k+1$ terms.

Second case: $k > n^{\frac{1}{4}}$. Note that we additionally require that $k \leq 200\hat{n}_K$. On the other hand, $200\hat{n}_K \leq \mathfrak{n}_\mathfrak{c}$, by definition. This implies, in particular, that $k \leq \mathfrak{n}_\mathfrak{c}$: we are thus in the (5.7.2)-regime. Recalling the definition of F^*, the upperbound clearly holds

$$\begin{aligned} F^*(m\hat{n}_K,k_l) &\leq F(m\hat{n}_K,k_l)+F(m\hat{n}_K,k_l+1)+F(m\hat{n}_K,k_l+2) \\ &\leq n^6(n-k_l)!\left(1+\frac{1}{(n-k_l)}+\frac{1}{(n-k_l)(n-k_l-1)}\right) \\ &\leq n^7(n-k_l-1)!2, \qquad\qquad \text{(n large enough)} \end{aligned} \tag{5.7.35}$$

the second inequality by (5.7.2). Following *exactly* the same steps which lead from (5.7.23) to (5.7.34), again using the Lemma 49 but this time with the estimate (5.7.2) and replacing (5.7.25) by (5.7.35), one immediately obtains

$$f_\pi^{(d)}(n,k) \leq 16n^{13}(k+1)\left(n-1-\lceil\frac{k}{2}\rceil\right)!\left(m\hat{n}_K-\lfloor\frac{k}{2}\rfloor\right)!\frac{\mathsf{J}}{(m\hat{n}_K)!n!}, \tag{5.7.36}$$

for all $\pi \in \mathcal{J}$, concluding the proof of Lemma 44. □

5.7.2 Counting undirected paths, and proofs of Lemmata 46 and 48

Thanks to the repulsive nature of the H-planes, if two paths share two edges between a different pair of H-planes, the common edge with the smaller Hamming distance to **0** is evidently crossed first. Given that paths eventually proceed according to the inherent directivity of the problem ("from left to right"), one may ask a similar question for the way two (or more) common edges between two successive H-planes (in the stretched phase) are crossed. To address this question, we will distinguish between two concepts: *i)* directionality, i.e. whether the path performs, while crossing the considered edge, a forward- or a backstep, and *ii)* order in which the considered edges are crossed[5].

[5] In hindsight, we only need two distinctions here: either the two paths cross the edges in the same, or in reverse order. We will avoid explicit definitions for this intuitive concept, but provide an example: assuming that the common edges are labeled a,b,c,d, etc., the order in which a path crosses them is simply the order of the labels: assume the path π crosses the edges in the order a-b-c-d; the path π' can cross the same edges either in exactly the same order a-b-c-d, or in reverse order d-c-b-a.

Lemma 50. *Let $\pi, \pi' \in \mathcal{J}$ share edges between the H_{i-1}- and the H_i-plane, for some $i \in \{m+1, \ldots, K-m\}$, and assume that the π-path crosses the common edges in a certain directionality and order. Then the π'-path has to cross the edges either*

- *in the same directionality* and *order,* *or*
- *in opposite directionality* and *reverse order.*

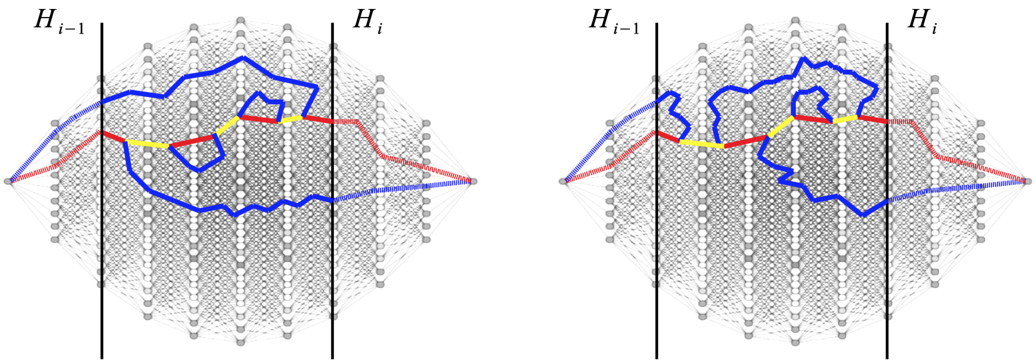

Figure 5.17: The yellow edges are shared by both polymers. The picture on the left satisfies the directionality: the red polymer crosses the yellow edges in graphical order "from left to right", while the blue polymer crosses the yellow edges in reversed order and opposite directionality. The picture on the right does not: the blue polymer first crosses the first common edge, but then reverts both order and directionality.

Proof of Lemma 50. Consider a path π, and the associated directionality/order in which it crosses the prescribed, common edges. A second path π' which does not follow such directionality and order (nor its complete reversal) will move away from one of the shared edges which are bound to be crossed in a future step. The second path will thus have to make up for this "departure", eventually, but this can only happen if it performs, during its evolution, a *detour*, i.e. if it goes through an edge (parallel to one of the unit vectors) in *both* directions. Since detours are not possible in the stretched phase at hand, the claim follows repeating the line of reasoning.

□

Proof of Lemma 46. Consider $\pi \in \mathcal{J}$, and $\boldsymbol{k} = (k_1, k_2, \ldots, k_K) \in \mathbb{N}^K$, such that $k_1 + k_2 + \cdots + k_K = k$. By a slight abuse of notation we denote by $f_\pi(n, \boldsymbol{k})$ the number of paths which share k_i edges with π between the hyperplanes H_{i-1} and H_i, $i \in \{1, \ldots, K\}$. It then holds

$$f_\pi(n, k) = \sum_{\boldsymbol{k}} f_\pi(n, \boldsymbol{k}). \tag{5.7.37}$$

If $k_i > 0$, let v_i^{fi} be the *first* vertex which π hits when crossing the first common edge between H_{i-1} and H_i, and v_i^{la} the *last* vertex from which π departs after crossing the last common edge (also between H_{i-1} and H_i). Furthermore, denote by

$$l_i^{\text{fi}}(\pi) \equiv d\left(H_{i-1} \cap \pi, v_i^{\text{fi}}\right), \qquad l_i^{\text{la}}(\pi) \equiv d\left(v_i^{\text{la}}, H_i \cap \pi\right), \tag{5.7.38}$$

the Hamming distance from (resp. to) the first (resp. last) vertex to the previous (resp. next) H-plane. If $k_i = 0$, we simply set $l_i^{\text{fi}}(\pi) \equiv d(H_{i-1} \cap \pi, H_i \cap \pi)$ and $l_i^{\text{la}} \equiv 0$.

Finally, consider the whole list (vector) of Hamming distances

$$\boldsymbol{l}(\pi) \equiv \left(l_1^{\text{fi}}(\pi), l_1^{\text{la}}(\pi), l_2^{\text{fi}}(\pi), l_2^{\text{la}}(\pi), \dots, l_K^{\text{fi}}(\pi), l_K^{\text{la}}(\pi)\right) \in \mathbb{N}^{2K}. \tag{5.7.39}$$

Let $f_\pi(n, \boldsymbol{k}, \boldsymbol{l})$ the number of paths sharing k_i edges with π between the hyperplanes H_{i-1} and H_i, $i = 1 = \dots K$, and with prescribed $\boldsymbol{l}$-vector. It then holds

$$f_\pi(n, k) = \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} f_\pi(n, \boldsymbol{k}, \boldsymbol{l}) \tag{5.7.40}$$

By Lemma 50, a path $\hat{\pi} \in \mathcal{J}_\pi(n, k)$ has two ways only to travel through the common edges between successive H-planes: either in identical, or opposite directionality/order. In order to keep track of this, we consider the $\boldsymbol{\sigma} \equiv (\sigma_1, \dots, \sigma_K) \in \{-1, 1\}^K$ with coordinates given by

$$\sigma_i \equiv +1, \qquad \text{if } k_i = 0, \tag{5.7.41}$$

and

$$\sigma_i \equiv \begin{cases} +1, & \text{if } \hat{\pi} \text{ crosses first } v_i^{\text{fi}}, \\ -1, & \text{if } \hat{\pi} \text{ crosses first } v_i^{\text{la}}. \end{cases} \qquad \text{and } k_i > 0. \tag{5.7.42}$$

We need some additional notation: if $k_i > 0$ and in case of identical directionality/order, i.e. $\sigma_i = +1$, we set

$$\begin{aligned} \hat{l}_i^{\text{fi}}(\hat{\pi}) &\equiv \text{length of the substrand connecting the vertices } H_{i-1} \cap \hat{\pi} \text{ and } v_i^{\text{fi}}, \\ \hat{l}_i^{\text{la}}(\hat{\pi}) &\equiv \text{length of the substrand connecting } v_i^{\text{la}} \text{ and } H_i \cap \hat{\pi}, \\ \hat{v}_i^{\text{fi}} &\equiv v_i^{\text{fi}}, \\ \hat{v}_i^{\text{la}} &\equiv v_i^{\text{la}}. \end{aligned} \tag{5.7.43}$$

If $k_i > 0$ and in case of reverse directionality/order, i.e. $\sigma_i = -1$, we set

$$\begin{aligned} \hat{l}_i^{\text{fi}}(\hat{\pi}) &\equiv \text{length of the substrand connecting the vertices } H_{i-1} \cap \hat{\pi} \text{ and } v_i^{\text{la}}, \\ \hat{l}_i^{\text{la}}(\hat{\pi}) &\equiv \text{length of the substrand connecting } H_i \cap \hat{\pi} \text{ and } v_i^{\text{fi}}, \\ \hat{v}_i^{\text{fi}} &\equiv v_i^{\text{la}}, \\ \hat{v}_i^{\text{la}} &\equiv v_i^{\text{fi}}. \end{aligned} \tag{5.7.44}$$

If $k_i = 0$, we simply set

$$\begin{aligned} \hat{l}_i^{\mathrm{fi}}(\hat{\pi}) &\equiv l_\pi(H_{i-1} \cap \pi, H_i \cap \pi), \\ \hat{l}_i^{\mathrm{la}}(\hat{\pi}) &\equiv 0. \end{aligned} \tag{5.7.45}$$

Furthermore, let

$$\begin{aligned} \hat{v}_0^{\mathrm{la}} &\equiv \mathbf{0}, \\ \hat{v}_{K+1}^{\mathrm{fi}} &\equiv \mathbf{1}. \end{aligned} \tag{5.7.46}$$

In full analogy with $\boldsymbol{l}$, we denote by $\hat{\boldsymbol{l}}$ the list (vector) of $\hat{l}$-lengths.

Let us now go back to (5.7.40): with $f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}})$ standing for the number of $\hat{\pi}$-paths which share k_i edges with π between the hyperplanes H_{i-1} and H_i with prescribed lengths $\boldsymbol{l}$ (for π), $\hat{\boldsymbol{l}}$ (for $\hat{\pi}$) and with $\boldsymbol{\sigma}$ directionality/order, it holds

$$f_\pi(n, k) = \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\hat{\boldsymbol{l}}} f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}). \tag{5.7.47}$$

We will now derive a formula for the f_π-summands on the r.h.s. above in terms of the number of paths satisfying the prescriptions *locally*: this requires discriminating between the cases where first and last common edge both lie within the same slab (i.e. between successive H-planes), or in two different slabs. Let $h(i) \equiv \min\{a, a \geq i, k_a > 0\}$ and $h(i) = K + 1$ if $\{a, a \geq i, k_a > 0\}$ is empty or $i = K + 1$. Finally, $h(0) \equiv 0$.

- Same slab.
  - For $k_{h(i)} \geq 1$, we denote by $\mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i)$ the number of stretched subpaths sharing $k_{h(i)}$ edges with π between $v_{h(i)}^{\mathrm{fi}}$ and $v_{h(i)}^{\mathrm{la}}$, knowing that first and last edge are in common.
- Different slabs.
  - We denote by $\overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i)$ the number of paths connecting $\hat{v}_{h(i)}^{\mathrm{la}}$ to $\hat{v}_{h(i+1)}^{\mathrm{fi}}$.

See below for a graphical rendition:

With these definitions, denoting by $b \equiv \#\{i : k_i > 0\}$, it clearly holds that

$$f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}) = \prod_{i=1}^{b} \mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \prod_{i=0}^{b} \overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i)\,. \tag{5.7.48}$$

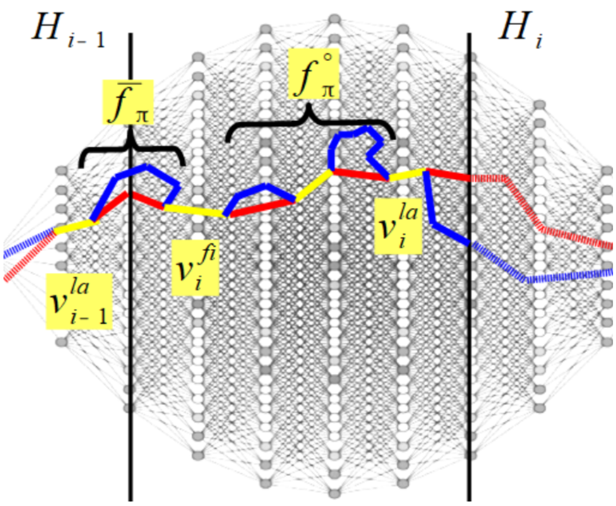

Figure 5.18: The blue and the red paths are admissible paths, which cross the different common edges in yellow.

The new goal is to get a handle on the $\mathring{f}_\pi$ and $\overline{f}_\pi$-terms. As for the former, we claim that for n big enough, for i, $k_{h(i)} > 0$ and with $\alpha \equiv \frac{5}{6}$,

$$\mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \leq \tanh\left(\mathsf{E}\frac{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}} \times \cosh\left(\mathsf{E}\frac{\left(d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}\right) - k_{h(i)})}{\mathsf{L}_{opt}n}\right)^{n} \times \left(\frac{\mathsf{L}_{opt}n}{e\mathsf{E}}\right)^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}} n^{n^\alpha} n^{\frac{1}{2}}. \tag{5.7.49}$$

In order to see this, we first observe that substrands are stretched between successive H-planes: the number of subpaths which share $k_{h(i)} \geq 2$ edges with π between $v^{\text{fi}}_{h(i)}$ and $v^{\text{la}}_{h(i)}$ therefore equals the number of directed subpaths that share $k_{h(i)} - 2$ edges with the subpath of π between $v^{\text{fi}}_{h(i)}$ and $v^{\text{la}}_{h(i)}$ on a hypercube of dimension $d\left(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}\right) - 2$. Hence

$$\mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \leq F\left(d\left(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}\right) - 2, k_{h(i)} - 2\right). \tag{5.7.50}$$

Next we note that for n large enough,

$$n^6 \leq \binom{n}{\mathfrak{n}_\mathfrak{e}}, \tag{5.7.51}$$

and therefore, by Lemma 49, the following rough bound holds *for all* $k \leq n$:

$$F(n,k) \leq (n-k)! \binom{n}{\mathfrak{n}_\mathfrak{e}} . \tag{5.7.52}$$

Using this in (5.7.50) yields

$$F\left(d\left(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}\right) - 2, k_{h(i)} - 2\right) \leq \left(d\left(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}\right) - k_{h(i)}\right)! \binom{n}{\mathfrak{n}_\mathfrak{e}} . \tag{5.7.53}$$

Furthermore,

$$\binom{n}{\mathfrak{n}_\mathfrak{e}} \leq \frac{n!}{\mathfrak{n}_\mathfrak{e}!} = \frac{n!}{(n-5e(n+3)^{2/3})!} \leq n^{5e(n+3)^{2/3}} \leq n^{n^\alpha} , \tag{5.7.54}$$

for n big enough, where $\alpha \equiv \frac{5}{6}$. Using this in (5.7.53), and plugging the ensuing estimates in (5.7.50) we obtain

$$\mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \leq (d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)})! n^{n^\alpha} . \tag{5.7.55}$$

By elementary Stirling approximation,

$$\begin{aligned}(d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)})! &\lesssim \left(d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}\right)^{1/2} \left[\frac{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}}{e}\right]^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}} \\ &\lesssim n^{1/2} \left[\frac{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}}{e}\right]^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}} ,\end{aligned} \tag{5.7.56}$$

the last inequality using that the dimension of an hypercube embedded between two hyperplanes is bounded above by their distance, i.e $d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) \leq \frac{n}{K} < n$.

Plugging (5.7.56) in (5.7.55) yields

$$\mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim n^{n^\alpha + 1/2} \left[\frac{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}}{e}\right]^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}} . \tag{5.7.57}$$

The above bound strongly depends on local specifications, which turn out to be rather untractable especially when it comes to the full product (5.7.48). We will circumvent this problem by means of a series of tricks: in a first step we recognize the term involving the $d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)})$ in (5.7.57) as a constituent part of a Stanley's bound, which we thus introduce artificially. In a second step, we will perform a rather elementary asymptotic analysis of the product (5.7.48) which is enabled by some monotonicity properties of the hyperbolic functions. To see how the first step comes about, we note that $\sinh(x) \geq x$ and $\cosh(x) \geq 1$ for all $x > 0$, hence the following holds

$$1 \leq \frac{\sinh(y)^d}{y^d} \cosh(y)^{n-d} = \tanh(y)^d \cosh(y)^n \frac{1}{y^d}, \tag{5.7.58}$$

for any $y > 0$ and $d \leq n$. We use this inequality with

$$y := \mathsf{E}\frac{(d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)})}{\mathsf{L}_{opt} n}, \quad d := d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}, \tag{5.7.59}$$

in which case we see that

$$\begin{aligned} 1 \leq \tanh & \left(\mathsf{E}\frac{(d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)})}{\mathsf{L}_{opt} n} \right) \\ & \times \cosh \left(\mathsf{E}\frac{(d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)})}{\mathsf{L}_{opt} n} \right)^n \\ & \times \left[\frac{\mathsf{L}_{opt} n}{\mathsf{E}(d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)})} \right]^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}} . \end{aligned} \tag{5.7.60}$$

Artificially upperbounding with the help of this estimate the r.h.s. of (5.7.57), and factoring out the $(d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)})^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}}$-terms then yields

$$\begin{aligned} \mathring{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim n^{n^\alpha + 1/2} \tanh & \left(\mathsf{E}\frac{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt} n} \right)^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}} \times \\ & \times \cosh \left(\mathsf{E}\frac{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt} n} \right)^n \left(\frac{\mathsf{L}_{opt} n}{e\mathsf{E}} \right)^{d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)}} . \end{aligned} \tag{5.7.61}$$

Claim (5.7.49) is therefore settled for $k_{h(i)} \geq 2$ and easely holds for $k_{h(i)} = 1$.

We now move to estimating the $\overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i)$-terms. Note that $\boldsymbol{l}$ fixes the vertices $v^{\text{fi}}_i, v^{\text{la}}_i$, and in particular the Hamming distance between two successive commons edges, which are not between the same H-planes, $\boldsymbol{\sigma}$ fixes $\hat{v}^{\text{fi}}_i, \hat{v}^{\text{la}}_i$, while $\hat{\boldsymbol{l}}$ gives the length of the subpaths $\hat{\pi}$ between these common edges.
For all $i \in \{0 \dots K\}$, we set

$$\hat{l}_i \equiv l_{\hat{\pi}}(\hat{v}^{\text{la}}_{h(i)}, \hat{v}^{\text{fi}}_{h(i+1)}). \tag{5.7.62}$$

We claim that

$$\overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim \tanh \left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt} n} \right)^{d\left(\hat{v}^{\text{la}}_{h(i)}, \hat{v}^{\text{fi}}_{h(i+1)}\right)} \cosh \left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt} n} \right)^n \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e} \right)^{\hat{l}_i} n^{\frac{1}{2}} . \tag{5.7.63}$$

Indeed, it clearly holds that

$$\overline{f}_{\pi}(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \leq M_{n,\hat{l}_i,d\left(\hat{v}^{\mathrm{la}}_{h(i)},\hat{v}^{\mathrm{fi}}_{h(i+1)}\right)}. \tag{5.7.64}$$

To get a handle on the r.h.s. above we make use of the following estimate, the derivation of which follows the by now classical route[6], and is thus omitted:

$$M_{n,l,nd} \lesssim n^{\frac{1}{2}} \tanh\left(\frac{\mathsf{E}l}{\mathsf{L}_{opt}n}\right)^{nd} \cosh\left(\frac{\mathsf{E}l}{\mathsf{L}_{opt}n}\right)^{n}\left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{l}. \tag{5.7.65}$$

Using (5.7.65) with $l := \hat{l}_i$, $nd := d(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)})$ in (5.7.64) steadily yields the claim (5.7.63).

Having obtained explicit estimates for the $\mathring{f}_{\pi}$ and $\overline{f}_{\pi}$-terms, we need bounds to their products as appearing in (5.7.48). This will be done exploiting the aforementioned monotonicity properties of hyperbolic functions: for any $y_i, d_i \geq 0$, and $k \in \mathbb{N}$ it holds

$$\prod_{i=1}^{k} \tanh\left(y_i\right)^{d_i} \leq \prod_{i=1}^{k} \tanh\left(\sum_{i=1}^{k} y_i\right)^{d_i} = \tanh\left(\sum_{i=1}^{k} y_i\right)^{\sum_{i=1}^{k} d_i}, \tag{5.7.66}$$

since tanh is increasing, and

$$\prod_{i=1}^{k} \cosh\left(y_i\right) \leq \cosh\left(\sum_{i=1}^{k} y_i\right), \tag{5.7.67}$$

which can be steadily checked iterating $\cosh(a+c) = \cosh(a)\cosh(c) + \sinh(a)\sinh(c) \geq \cosh(a)\cosh(c)$, for $a, c > 0$.

These bounds allow to remove most of the local dependencies appearing in the products (5.7.48): shortening

$$\mathcal{D}_b \equiv \sum_{i=1}^{b}\left[d\left(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}\right) - k_{h(i)}\right], \tag{5.7.68}$$

and combining (5.7.66), (5.7.67) and (5.7.49) we get

$$\prod_{i=1}^{b} \mathring{f}_{\pi}(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim n^{Kn^{\alpha}+\frac{K}{2}} \tanh\left(\frac{\mathsf{E}\mathcal{D}_b}{\mathsf{L}_{opt}n}\right)^{\mathcal{D}_b} \cosh\left(\frac{\mathsf{E}\mathcal{D}_b}{\mathsf{L}_{opt}n}\right)^{n}\left(\frac{\mathsf{L}_{opt}n}{e\mathsf{E}}\right)^{\mathcal{D}_b}. \tag{5.7.69}$$

On the other hand, shortening

$$\widehat{\mathcal{D}}_b \equiv \sum_{i=0}^{b} d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right), \quad \widehat{L}_b \equiv \sum_{i=0}^{b} \hat{l}_i, \tag{5.7.70}$$

[6] Stanley's bound (5.2.14) with $x := \frac{l\mathsf{E}}{\mathsf{L}_{opt}n}$ / Stirling approximation / some elementary rearrangements.

and combining (5.7.66), (5.7.67) with (5.7.63) we obtain

$$\prod_{i=0}^{b} \overline{f}_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}, i) \lesssim n^{\frac{K+1}{2}} \tanh\left(\frac{\mathsf{E}\widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{\widehat{\mathcal{D}}_b} \cosh\left(\frac{\mathsf{E}\widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\widehat{L}_b}. \tag{5.7.71}$$

Plugging (5.7.69) and (5.7.71) in (5.7.48) thus leads to

$$\begin{aligned} f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}) \lesssim n^{\frac{2K+1}{2}+Kn^\alpha} \tanh\left(\frac{\mathsf{E}\mathcal{D}_b}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b} \cosh\left(\frac{\mathsf{E}\mathcal{D}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\mathcal{D}_b} \times \\ \times \tanh\left(\frac{\mathsf{E}\widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{\widehat{\mathcal{D}}_b} \cosh\left(\frac{\mathsf{E}\widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\widehat{L}_b}. \end{aligned} \tag{5.7.72}$$

The above estimate still involves the product of two tanh-, and two cosh-terms: using once more the monotonicity tricks (5.7.66) and (5.7.67) we get

$$\begin{aligned} & f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}) \lesssim \\ & n^{\frac{2K+1}{2}+Kn^\alpha} \tanh\left(\mathsf{E}\frac{\mathcal{D}_b + \widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b+\widehat{\mathcal{D}}_b} \cosh\left(\mathsf{E}\frac{\mathcal{D}_b + \widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\mathcal{D}_b+\widehat{L}_b}. \end{aligned} \tag{5.7.73}$$

But paths in $\mathcal{J}$ have the same, prescribed length, and it holds that

$$\mathcal{D}_b + \widehat{L}_b = \mathsf{L}_{opt} n - k. \tag{5.7.74}$$

Using this, (5.7.73) simplifies to

$$\begin{aligned} & f_\pi(n, \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{\sigma}, \hat{\boldsymbol{l}}) \lesssim \\ & n^{\frac{2K+1}{2}+Kn^\alpha} \tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b+\widehat{\mathcal{D}}_b} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\mathsf{L}_{opt} n-k} \end{aligned} \tag{5.7.75}$$

Remark, in particular, that the r.h.s. above depends on the local prescriptions *only through the* tanh-*exponent*. It will come hardly as a surprise that this feature leads to a dramatic simplification of the computations. As a matter of fact, even the exponent depends only very mildly on the local prescriptions: indeed, we claim that

Lemma 51.

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \geq \max\left(n - k, \frac{\mathsf{L}_{opt} n - k}{4}\right). \tag{5.7.76}$$

Proving this claim will unfortunately require a fair amount of work, so we assume its validity for the time being.

By monotonicity,

$$\tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right) \leq \tanh\left(\mathsf{E}\right) = \frac{1}{\sqrt{2}} < 1, \tag{5.7.77}$$

hence Lemma 51 applied to (5.7.75) yields the upperbound

$$f_\pi(n,\boldsymbol{k},\boldsymbol{l},\boldsymbol{\sigma},\hat{\boldsymbol{l}}) \lesssim \\ n^{\frac{2K+1}{2}+Kn^\alpha}\tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^{\max\left(n-k,\frac{\mathsf{L}_{opt}n-k}{4}\right)}\cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^n\left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}, \tag{5.7.78}$$

no longer depends on $\boldsymbol{l},\boldsymbol{\sigma},\hat{\boldsymbol{l}},\boldsymbol{k}$; plugging this in (5.7.48), and the ensuing estimate in (5.7.47) therefore leads to

$$f_\pi(n,k) \lesssim n^{\frac{2K+1}{2}+Kn^\alpha}\sum_{\boldsymbol{l}}\sum_{\boldsymbol{\sigma}}\sum_{\hat{\boldsymbol{l}}}\sum_{\boldsymbol{k}}\mathfrak{T}(n,k)\,, \tag{5.7.79}$$

where

$$\mathfrak{T}(n,k) \equiv \tanh\left(\frac{\mathsf{E}(\mathsf{L}_{opt}n-k)}{\mathsf{L}_{opt}n}\right)^{\max\left(n-k,\frac{\mathsf{L}_{opt}n-k}{4}\right)}\cosh\left(\frac{\mathsf{E}(\mathsf{L}_{opt}n-k)}{\mathsf{L}_{opt}n}\right)^n\left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}. \tag{5.7.80}$$

Since $\mathfrak{T}(n,k)$ depends on the number of common edges, but not on the local prescriptions, we thus only need estimates on the cardinalities of the sums appearing in (5.7.79). As for the first sum, since $v_\cdot^{\text{fi}}$ can only move along the path π between two successive hyperplanes, the number of ways to place such $v_\cdot^{\text{fi}}$'s is *at most* n (the same of course holds true for $v_\cdot^{\text{la}}$), hence

$$\sum_{\boldsymbol{l}} \leq n^{2K}, \tag{5.7.81}$$

and by analogous reasoning

$$\sum_{\boldsymbol{l}'} \leq n^{2K}. \tag{5.7.82}$$

Moreover, it clearly holds that

$$\sum_{\boldsymbol{\sigma}} \leq 2^K. \tag{5.7.83}$$

Finally,

$$\sum_{\boldsymbol{k}} = \sum_{\substack{k_i,\\ k_1+k_2+\ldots+k_K=k}} = \binom{k+K-1}{K-1} \lesssim \frac{(k+K-1)^{k+K-1}}{(K-1)^{K-1}k^k}, \tag{5.7.84}$$

by Stirling approximation. Since $(K-1)^{K-1} \geq 1$, and $\log(1+x) \leq x$, we see that

$$(5.7.84) \leq k^{K-1}\left(1+\frac{K-1}{k}\right)^{k+K-1} = k^{K-1}\exp\left[(k+K+1)\log\left(1+\frac{K-1}{k}\right)\right]$$
$$\leq k^{K-1}\exp\left[(k+K-1)\,\frac{K-1}{k}\right] \leq k^{K-1}\exp K\,(K-1). \tag{5.7.85}$$

Combining (5.7.79), (5.7.81), (5.7.82), (5.7.83) and (5.7.85), we obtain

$$f_\pi(n,k) \leq P_n n^{Kn^\alpha} \tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^{\max\left(n-k,\frac{\mathsf{L}_{opt}n-k}{4}\right)} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^{n}\left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}, \tag{5.7.86}$$

where P_n is a finite degree polynomial, which is indeed the claim of Lemma 46. □

Proof of Lemma 51. . Recall that the claim reads

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \geq \max\left(n-k, \frac{\mathsf{L}_{opt}n-k}{4}\right). \tag{5.7.87}$$

The validity of the first inequality, to wit

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \geq n-k, \tag{5.7.88}$$

relies on a self-evident fact, namely that the total distance of shared edges in the directed case is a lower bound for the undirected case. More precisely, since common edges contribute to the number of steps performed while connecting **0** to **1**, as soon as a backstep acts on a shared edge, the total distance between shared edges is bound to increase: the path has eventually to make up for the "lost ground". Another way to put it: the contribution $\mathcal{D}_b + \widehat{\mathcal{D}}_b$ is smallest when *all* shared edges are steps forward, in which case the total distance between these edges must be *at least* the minimal number of steps required to connect **0** to **1**. Since this minimal number is clearly the dimension minus the number of shared (prescribed) edges, i.e. $n-k$, (5.7.88) is settled.

The second inequality

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \geq \frac{\mathsf{L}_{opt}n-k}{4}, \tag{5.7.89}$$

requires more work and depends on some key properties of paths in $\mathcal{J}$. We begin with a couple of observations:

i) First we note that
$$d\left(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}\right) = d\left(\hat{v}^{\text{fi}}_{h(i)}, \hat{v}^{\text{la}}_{h(i)}\right),$$
since inverting directionality clearly has no impact on the distance.

ii) Furthemore, in a (fully) stretched phase distance and length do, in fact, coincide:

$$d\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) = l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) .$$

iii) Finally, and by definition,

$$\sum_{i=1}^{b} k_{h(i)} = k .$$

Plugging items i-iii) above in the $\mathcal{D}_b$-definition (5.7.68) yields

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b = \sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k + \sum_{i=0}^{b} d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right) . \tag{5.7.90}$$

We now claim that for $i \in \{0, 1, \dots b\}$, it holds:

$$d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right) \geq \frac{1}{4} l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) . \tag{5.7.91}$$

This is, in fact, our key technical claim, but since its proof requires some involved analysis, we assume its validity for the time being, and first show how it implies (5.7.89): plugging (5.7.91) in (5.7.90) we obtain

$$\mathcal{D}_b + \widehat{\mathcal{D}}_b \geq \sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k + \frac{1}{4}\sum_{i=0}^{b} l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) . \tag{5.7.92}$$

But by construction,

$$l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) \geq k_{h(i)}, \tag{5.7.93}$$

hence

$$\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k \geq \sum_{i=1}^{b} k_{h(i)} - k \geq 0, \tag{5.7.94}$$

the last inequality by item iii) above. This positivity implies, in particular, that

$$\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k \geq \frac{1}{4}\left(\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k\right) , \tag{5.7.95}$$

and using this lower bound in (5.7.92) then yields

$$\begin{aligned} \mathcal{D}_b + \widehat{\mathcal{D}}_b &\geq \frac{1}{4}\left(\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) - k\right) + \frac{1}{4}\sum_{i=0}^{b} l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) \\ &= \frac{1}{4}\left(\underbrace{\sum_{i=1}^{b} l_{\hat{\pi}}\left(\hat{v}^{\mathrm{fi}}_{h(i)}, \hat{v}^{\mathrm{la}}_{h(i)}\right) + \sum_{i=0}^{b} l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)})}_{=\mathsf{L}_{opt} n} - k\right) , \end{aligned} \tag{5.7.96}$$

which settles our key claim (5.7.89).

It thus remains to prove (5.7.91). Recall that we are considering the situation where shared edges are separated by (at least) one H-plane[7]. Since by definition an H-plane is also an H'-plane, prescribing the number of separating H'-planes allows to discriminate among different scenarios. Indeed, introducing, for $i = 0 \ldots b$,

$$c_{\hat{\pi}}(i) \equiv \quad \text{number of } H'\text{-planes which lie between } \hat{v}^{\mathrm{la}}_{h(i)} \text{ and } \hat{v}^{\mathrm{fi}}_{h(i+1)}\,, \tag{5.7.97}$$

a minute's thought suggests that there are three scenarios which are "structurally" manifestly different:

- $c_{\hat{\pi}}(i) > 2$: the common edges are separated by at least one H-plane, and multiple H'-planes. We will refer to this as the H'HH'-case.
- $c_{\hat{\pi}}(i) = 2$: in this case the common edges are separated by one H-plane, and one H'-plane (which is however not an H-plane). We will refer to this as the HH'-case.
- $c_{\hat{\pi}}(i) = 1$: the separating hyperplane must be an H-plane: we will refer to this as the H-case.

We will establish the validity of (5.7.91) in all three possible scenarios. We anticipate that (5.7.91) becomes more delicate the less hyperplanes are separating the common edges: this is due to the fact that the larger the number of separating hyperplanes the further apart (in terms of Hamming distance d) the common edges must lie, a feature which renders (5.7.91) all the more likely. In line with this observation, the $c_{\hat{\pi}}(i) = 1$ will turn out to be the most delicate. We emphasize that the index i is given and fixed. To lighten notation we will thus omit it in the expressions, whenever no confusion can possibly arise.

A number of insights are common to the treatment of all three scenarios. Given the nature of the inequality we are aiming to prove, it will not come as a surprise that we will need a good control - in the form of *lower bounds* - on the distance of two common edges, as well as a good control - this time around in the form of *upper bounds* - on the length of the substrands connecting the shared edges.

A reasonably tight, but what's more: valid for any of the three $c_{\hat{\pi}}$-scenarios, lower bound for the distance is provided by technical input **(T1)** below. Let H'_{fi} be the first hyperplane on the right of $\hat{v}^{\mathrm{la}}_{h(i)}$ and H'_{la} be the last hyperplane on the left of $\hat{v}^{\mathrm{fi}}_{h(i+1)}$, and shorten $d^{\mathrm{fi}}_{\hat{\pi}} \equiv d(\hat{v}^{\mathrm{la}}_{h(i)}, H'_{\mathrm{fi}})$, and $d^{\mathrm{la}}_{\hat{\pi}} \equiv d(H'_{\mathrm{la}}, \hat{v}^{\mathrm{fi}}_{h(i+1)})$. A graphical depiction of this is given in Figure 5.19 below. The following estimate holds by definition/construction[8]

[7] as otherwise the claim would be trivial anyhow: if the shared edges lie within two successive H-planes, the polymer is in a stretched phase in which case distance (d) and length (l) coincide, with the inequality (5.7.91) thus trivially holding.

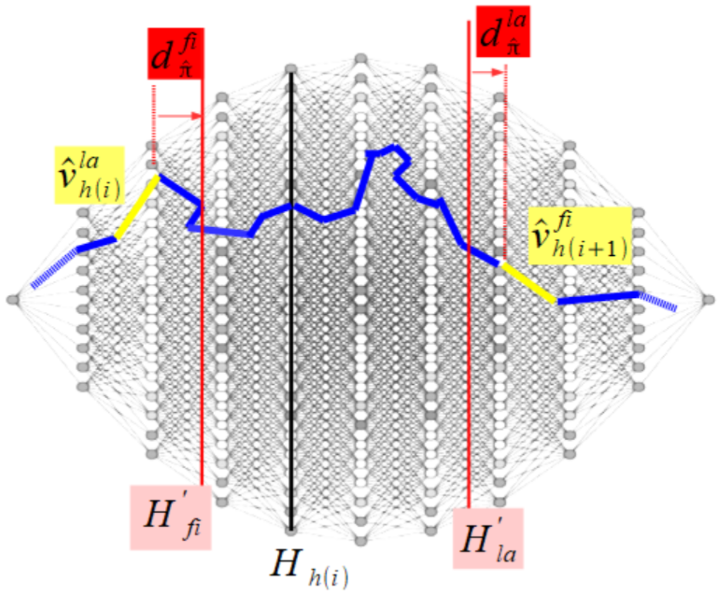

Figure 5.19: $\hat{v}^{\text{la}}_{h(i)}$ and $\hat{v}^{\text{fi}}_{h(i+1)}$ separated by three hyperplanes.

$$d\left(\hat{v}^{\text{la}}_{h(i)}, \hat{v}^{\text{fi}}_{h(i+1)}\right) \geq d^{\text{fi}}_{\hat{\pi}} + \frac{c_{\hat{\pi}}(i)-1}{KK'}n + d^{\text{la}}_{\hat{\pi}} \qquad \textbf{(T1)}\,.$$

(We note in passing that equality holds if and only if $\hat{v}^{\text{la}}_{h(i)}, \hat{v}^{\text{fi}}_{h(i+1)}$ are connected by a directed substrand; since a stretched substrand may have to perform backsteps while connecting these two vertices, **(T1)** is in general only a lower bound).

As mentioned, the second technical input, **(T2)** below, concerns *upperbounds* on the length of a substrand connecting H'-planes. To see how these come about, let us denote by $\boldsymbol{v} \in H'_{i,j}, \boldsymbol{w} \in H'_{i,j+1}$ *the* vertices by which the $\hat{\pi}$-substrand connects the finer mesh. It is important to observe that in virtue of (5.4.2), there is no absolutely no ambiguity in the way we identify these vertices: in fact,

$$\text{these vertices are unequivocally identified through the } \textit{length} \text{ of the substrand connecting the successive } H'\text{-planes.} \tag{5.7.98}$$

We now claim that

$$l_{\hat{\pi}}(\boldsymbol{v}, \boldsymbol{w}) \leq \frac{1.46}{KK'}n \qquad \textbf{(T2)}$$

The proof of **(T2)** is rather immediate: first recall that in virtue of (5.4.2),

$$l_{\hat{\pi}}(\boldsymbol{v}, \boldsymbol{w}) = (\mathsf{ef}_i + \mathsf{eb}_i)\,\frac{n}{K'} = \left(\frac{1}{K} + 2\mathsf{eb}_i\right)\frac{n}{K'}, \tag{5.7.99}$$

[8]it can also immediately evinced from Figure 5.19.

the second equality by (5.2.36). But by (5.3.19) (and again (5.2.36)), the number of effective backsteps between H-planes in the stretched phase satisfies

$$\mathsf{eb}_i = \sinh(\overline{\mathsf{a}}_{i-1}\mathsf{E})\sinh(\mathsf{a}_i\mathsf{E})\sinh(\underline{\mathsf{a}}_i\mathsf{E}), \tag{5.7.100}$$

and by (5.4.12),

$$\sinh(\mathsf{a}_i\mathsf{E}) \leq \frac{1}{K} + \frac{1}{6K^3}, \tag{5.7.101}$$

which combined with (5.7.100) yields

$$\begin{aligned} \mathsf{eb}_i &\leq \sinh(\overline{\mathsf{a}}_i\mathsf{E})\sinh(\underline{\mathsf{a}}_i\mathsf{E})\left(\frac{1}{K} + \frac{1}{6K^3}\right) \\ &\leq \sinh\left(\frac{\mathsf{E}}{2}\right)^2\left(\frac{1}{K} + \frac{1}{6K^3}\right). \end{aligned} \tag{5.7.102}$$

the second inequality by (5.4.10). Since

$$\sinh\left(\frac{\mathsf{E}}{2}\right)^2 \overset{(5.4.14)}{=} \frac{\sqrt{2}-1}{2} \leq 0.22, \tag{5.7.103}$$

and using that $K > 10^7$, one plainly checks that

$$\mathsf{eb}_i \leq 0.23 \times \frac{1}{K}. \tag{5.7.104}$$

Plugging (5.7.104) in (5.7.99) settles **(T2)**.

If it's true that there is no ambiguity in the way *vertices* on the H'-plane are identified (recall remark (5.7.98) above), it is nonetheless true there there is a certain amount of uncertainty in the way the polymer *connects* these planes. This is due to the fact that (contrary to the H-planes) the H'-planes are not repulsive, hence a polymer might cross them multiple times. Such excursions increase of course the length of the substrand, and introduce some "fuzziness" into the picture. Notwithstanding, we claim that

during one such excursion a polymer can overshoot,
in terms of Hamming distance, an H'-plane by at most
$\frac{0.23}{KK'}n$ units. **(T3)**

Figure 5.20 below provides an elementary proof of this fact.

The above insight, captured by **(T3)**, suggests to introduce the following set

$$\mathfrak{F}_{i,j} \equiv \left\{ \boldsymbol{v} \in V_n,\, d(v, H'_{i,j}) \leq \frac{0.23}{KK'}n \right\}. \tag{5.7.105}$$

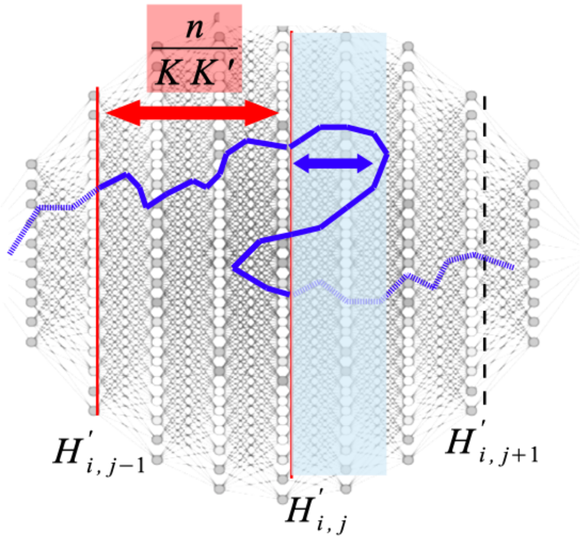

Figure 5.20: The proof of **(T3)** relies on two observations: *i)* By **(T2)**, the length of the path connecting first and second H'-planes (the continuous blue strand) is at most $\frac{1.46}{KK'}n$. *ii)* By construction, the Hamming distance of these planes is $\frac{n}{KK'}$. Taking into account that the polymer must return to the second H'-plane, we see that the blue arrow is at most half the difference of these quantities, indeed $\frac{0.23}{KK'}n$, as claimed by **(T3)**. (Remark that this case corresponds to a worst-case scenario: the polymer performs first all available forward steps, and only then all availble backsteps).

We emphasize that whenever a common edge lies in this set, it can be crossed by a sub-strand which *either* connects $H'_{i,j-1}$ with $H'_{i,j}$ *or* $H'_{i,j}$ with $H'_{i,j+1}$: for this reason, we refer to $\mathfrak{F}_{i,j}$ (which is nothing but "twice" the blue-shaded region in Figure 5.20) as the fuzzy zone.

We now record two useful consequences of **(T2)** and **(T3)** on the lengths of substrand which will play a role in the proof of (5.7.91). For reasons which will become clear, we will only need to consider the case where the first common edge lies in the fuzzy zone of the $H'-$plane which is on the left of H'_{fi}, and/or the other common edge lies on the right of H'_{la}. There are two cases: either shared edges lie outside the fuzzy zone, OuF for short, or inside, InF.

(InF) Remark that $\hat{v}^{\text{la}}_{h(i)}$ being in a fuzzy zone is equivalent to $d^{\text{fi}}_{\hat{\pi}} \geq \frac{0.77}{KK'}n$. Analogously, $\hat{v}^{\text{fi}}_{h(i+1)}$ is in a fuzzy zone if and only if $d^{\text{la}}_{\hat{\pi}} \geq \frac{0.77}{KK'}n$. Furthermore, a path crossing $\hat{v}^{\text{la}}_{h(i)}$ (or $\hat{v}^{\text{fi}}_{h(i+1)}$) can cross multiple H'-planes besides that to which this vertex belongs: by **(T3)**, this phenomenon can contribute to the length of the substrand at most

$\frac{0.46}{KK'}n$ units.

(OuF) If neither $\hat{v}^{\text{la}}_{h(i)}$ nor $v^{\text{fi}}_{h(i+1)}$ are in a fuzzy zone, by **(T2)**, the connecting substrands satisfy

$$l_{\hat{\pi}}(\hat{v}^{\text{la}}_{h(i)}, \hat{v}^{\text{fi}}_{h(i+1)}) \leq \frac{\left(c_{\hat{\pi}(i)} + 1\right) 1.46}{KK'} n\,.$$

We can finally move to the proof of (5.7.91): this will be done via case-by-case analysis of the three possible $c_{\hat{\pi}}$-scenarios.

The H'HH'-case.

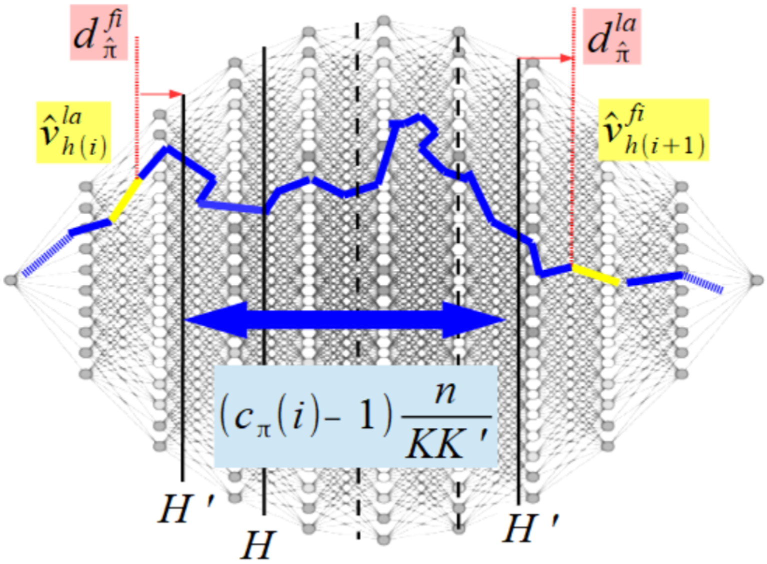

Figure 5.21: $c(i) \geq 3$: at least three hyperplanes, i.e. at least two H' and one H, separating the common edges.

This case is graphically summarized in Figure 5.21 below: combining (OuF) and (InF), we immediately evince from this picture that

$$l_{\hat{\pi}}(\hat{v}^{\text{la}}_{h(i)}, \hat{v}^{\text{fi}}_{h(i+1)}) \leq \frac{(c_{\hat{\pi}}(i) + 1)\, 1.46}{KK'} n + \frac{0.46}{KK'} n \left(1_{d^{\text{fi}}_{\hat{\pi}} \geq \frac{0.77n}{KK'}} + 1_{d^{\text{la}}_{\hat{\pi}} \geq \frac{0.77n}{KK'}} \right) . \tag{5.7.106}$$

The H'HH'-scenario at hand is characterized by $c_{\hat{\pi}}(i) > 2$, in which case the following inequality is immediate:

$$\frac{(c_{\hat{\pi}}(i) + 1)\, 1.46}{KK'} \leq \frac{4(c_{\hat{\pi}}(i) - 1)}{KK'}. \tag{5.7.107}$$

Using this in (5.7.106) we obtain

$$l_{\hat\pi}(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}) \leq \frac{4(c_{\hat\pi}(i)-1)}{KK'} + \frac{0.46}{KK'} n \left(1_{d^{\text{fi}}_{\hat\pi} \geq \frac{0.77n}{KK'}} + 1_{d^{\text{la}}_{\hat\pi} \geq \frac{0.77n}{KK'}} \right) . \tag{5.7.108}$$

Concerning the last two terms on the r.h.s. above, we first observe that obviously

$$d \geq \frac{0.77}{KK'} n \Longrightarrow 4d \geq \frac{0.46}{KK'} n \,, \tag{5.7.109}$$

hence

$$\frac{0.46}{KK'} n 1_{d^{\text{fi}}_{\hat\pi} \geq \frac{0.77}{KK'} n} \leq 4 d^{\text{fi}}_{\hat\pi}(\hat\pi), \qquad \frac{0.46}{KK'} n 1_{d^{\text{la}}_{\hat\pi} \geq \frac{0.77}{KK'} n} \leq 4 d^{\text{la}}_{\hat\pi}(\hat\pi). \tag{5.7.110}$$

Plugging this in (5.7.108) yields

$$\begin{aligned} l_{\hat\pi}(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}) &\leq \frac{4(c_{\hat\pi(i)}-1)}{KK'} n + 4 d^{\text{fi}}_{\hat\pi}(\hat\pi) + 4 d^{\text{la}}_{\hat\pi}(\hat\pi) \\ &\leq 4d \left(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)} \right) , \end{aligned} \tag{5.7.111}$$

the last step by **(T1)**. Claim (5.7.91) is therefore settled for the H'HH'-case.

<u>The HH'-case.</u>

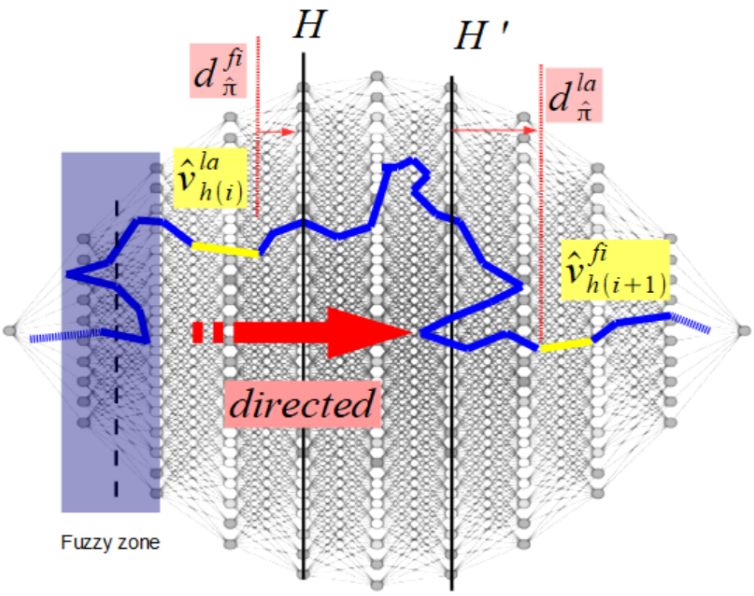

Figure 5.22: The common edges are separated by $c_{\hat\pi}(i) = 2$.

In this case, see Figure 5.22 below for a graphical rendition, a subpath connecting $\hat v^{\text{la}}_{h(i)}$ and $\hat v^{\text{fi}}_{h(i+1)}$, crosses $c_{\hat\pi(i)} = 2$ many H'-planes, one of which is also an H-plane. Without loss of generality, we assume that H'_{fi} is the H-plane. We will here distinguish two subcases: $d^{\text{fi}}_{\hat\pi} \geq \frac{0.77}{KK'}n$, and its complement. It holds:

- If $d^{\text{fi}}_{\hat\pi} \geq \frac{0.77}{KK'}n$, i.e. the vertex $\hat v^{\text{la}}_{h(i)}$ is in the fuzzy zone, it follows from (OuF) and (InF) (cfr. also with Figure 5.22) that

$$
\begin{aligned}
l_{\hat\pi}(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}) &\leq \frac{3\times 1.46}{KK'}n + \frac{0.46}{KK'}n\left(1_{d^{\text{fi}}_{\hat\pi}\geq\frac{0.77n}{KK'}} + 1_{d^{\text{la}}_{\hat\pi}\geq\frac{0.77n}{KK'}}\right),\\
&= \frac{4.38}{KK'}n + \frac{0.46}{KK'}n1 + \frac{0.46}{KK'}n1_{d^{\text{la}}_{\hat\pi}\geq\frac{0.77n}{KK'}}\\
&\overset{(5.7.109)}{\leq} \frac{4.84}{KK'}n + 4d^{\text{la}}_{\hat\pi}\\
&\leq \frac{4}{KK'}n + 4\frac{0.77}{KK'}n + 4d^{\text{la}}_{\hat\pi}\\
&\overset{(\mathbf{T1})}{\leq} 4d(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)})\,.
\end{aligned}
\tag{5.7.112}
$$

- If $d^{\text{fi}}_{\hat\pi} < \frac{0.77}{KK'}n$, the vertex $\hat v^{\text{la}}_{h(i)}$ is no longer in the fuzzy zone. However, and crucially, the "complement" of the fuzzy zone is necessarily the repulsive phase, cfr. Figure 5.22 below. This in particular implies that the substrand will connect $\hat v^{\text{la}}_{h(i)}$ with the H-plane in a directed fashion, and therefore

$$
\begin{aligned}
l_{\hat\pi}(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}) &= l_{\hat\pi}(\hat v^{\text{la}}_{h(i)}, H_{h(i)}\cap\hat\pi) + l_{\hat\pi}(H_{h(i)}\cap\hat\pi, \hat v^{\text{fi}}_{h(i+1)})\\
&= d^{\text{fi}}_{\hat\pi} + l_{\hat\pi}(H_{h(i)}\cap\hat\pi, \hat v^{\text{fi}}_{h(i+1)}),
\end{aligned}
\tag{5.7.113}
$$

  As before, we estimate the last term on the r.h.s. above by OuF and InF. Here is the upshot:

$$
\begin{aligned}
l_{\hat\pi}(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}) &\leq d^{\text{fi}}_{\hat\pi} + \frac{2\times 1.46}{KK'}n + \frac{0.46}{KK'}n1_{d^{\text{la}}_{\hat\pi}\geq\frac{0.77}{KK'}n}\\
&\overset{(5.7.109)}{\leq} d^{\text{fi}}_{\hat\pi} + \frac{2.92}{KK'}n + 4d^{\text{la}}_{\hat\pi}\\
&\leq 4\frac{0.77}{KK'}n + \frac{4}{KK'}n + 4d^{\text{la}}_{\pi}\\
&\overset{(\mathbf{T_1})}{\leq} 4d(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}).
\end{aligned}
\tag{5.7.114}
$$

The claim (5.7.91) is thus settled for the HH'-case.

The H-case.

In this case, see Figure 5.23 above, a subpath connecting $\hat v^{\text{la}}_{h(i)}$ and $\hat v^{\text{fi}}_{h(i+1)}$, crosses $c_{\hat\pi(i)} = 1$ many H'-planes which is also an H-plane. Four subcases are possible:

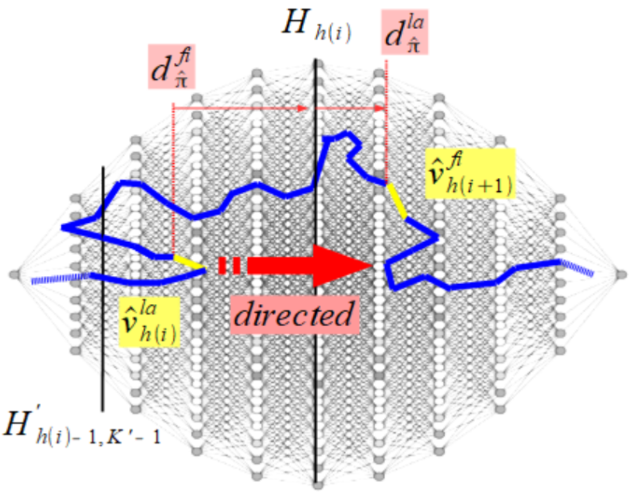

Figure 5.23: The common edges are separated by $c_{\hat\pi}(i) = 1$.

- $d^{\text{fi}}_{\hat\pi} < \frac{0.77}{KK'}n$ and $d^{\text{la}}_{\hat\pi} < \frac{0.77}{KK'}n$, i.e. both vertices $\hat v^{\text{la}}_{h(i)}$ and $\hat v^{\text{fi}}_{h(i+1)}$ are in the (same) *repulsive phase*: the substrand thus connects them in directed fashion, in which case length and distance coincide, and

$$l_{\hat\pi}(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}) = d(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}) \leq 4d(\hat v^{\text{la}}_{h(i)}, \hat v^{\text{fi}}_{h(i+1)}). \tag{5.7.115}$$

- $d^{\text{fi}}_{\hat\pi} < \frac{0.77}{KK'}n$ and $d^{\text{la}}_{\hat\pi} \geq \frac{0.77}{KK'}n$. In this case:

  - the vertex $\hat v^{\text{la}}_{h(i)}$ is in the repulsive phase (cfr. with the second subcase in the HH'-regime above): in this first part of the journey, the substrand thus connects it with the H-plane in directed fashion, where again, and crucially, length and distance coincide.
  - as for the "rest of the journey", i.e. in order to deal with the length of the strand connecting H-plane and target vertex $\hat v^{\text{fi}}_{h(i+1)}$, we proceed exactly as in (5.7.113).

Splitting the substrand in first/second part of the journey, and then by these observations, we get

$$
\begin{aligned}
l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) &= l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, H_{h(i)} \cap \hat{\pi}) + l_{\hat{\pi}}(H_{h(i)} \cap \hat{\pi}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) \\
&\leq d^{\mathrm{fi}}_{\hat{\pi}} + \frac{1.46}{KK'}n + \frac{0.46}{KK'}n \\
&\leq 4d^{\mathrm{fi}}_{\hat{\pi}} + 4\frac{0.77}{KK'}n \\
&\overset{\textbf{(T1)}}{\leq} 4d(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}).
\end{aligned}
\tag{5.7.116}
$$

- $d^{\mathrm{fi}}_{\hat{\pi}} \geq \frac{0.77}{KK'}n$ and $d^{\mathrm{la}}_{\hat{\pi}} < \frac{0.77}{KK'}n$: this case is, by symmetry, equivalent to the previous.

- $d^{\mathrm{fi}}_{\hat{\pi}} \geq \frac{0.77}{KK'}n$ and $d^{\mathrm{la}}_{\hat{\pi}} \geq \frac{0.77}{KK'}n$: both vertices being in the fuzzy zone, we proceed exactly as in (5.7.106) to obtain

$$
\begin{aligned}
l_{\hat{\pi}}(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}) &\leq 2\frac{1.46}{KK'}n + 2\frac{0.46}{KK'}n \\
&\leq 4\frac{0.77}{KK'}n + 4\frac{0.77}{KK'}n \\
&\overset{\textbf{(T1)}}{\leq} 4d(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)},
\end{aligned}
\tag{5.7.117}
$$

Claim (5.7.91) thus holds true for all possible sub-scenarios of the third (and last) H-case: this finishes the proof of Lemma 51. □

Proof of Lemma 48. We want now to estimate $f^{(s)}_{\pi}(n,k)$: Let $f^{(s)}_{l,\pi}(n,k)$ (respectively $f^{(s)}_{r,\pi}(n,k)$) the number of paths which are sharing k edges with π with at least one common edge betweeen H_m and the middle of the hypercube (respectively between the middle of the hypercube and H_{K-m}) but without considering first and last edge. It holds

$$
f^{(s)}_{\pi}(n,k) = f^{(s)}_{l,\pi}(n,k) + f^{(s)}_{r,\pi}(n,k) = 2f^{(s)}_{l,\pi}(n,k), \tag{5.7.118}
$$

the last equality by symmetry (see (5.7.11)). Using (5.7.48), (5.7.49) and(5.7.63), it clearly holds

$$
\begin{aligned}
f^{(s)}_{l,\pi}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^{\alpha}} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\boldsymbol{l'}} \prod_{i=1}^{b} \tanh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt} n}\right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}} \\
\times \cosh\left(\mathsf{E}\frac{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{e\mathsf{E}}\right)^{d(v^{\mathrm{fi}}_{h(i)}, v^{\mathrm{la}}_{h(i)}) - k_{h(i)}} \\
\times \prod_{i=0}^{b} \tanh\left(\frac{\hat{l}_i \mathsf{E}}{\mathsf{L}_{opt} n}\right)^{d\left(\hat{v}^{\mathrm{la}}_{h(i)}, \hat{v}^{\mathrm{fi}}_{h(i+1)}\right)} \cosh\left(\frac{\hat{l}_i \mathsf{E}}{\mathsf{L}_{opt} n}\right)^{n} \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\hat{l}_i}.
\end{aligned}
\tag{5.7.119}
$$

Using the monotonicity of the cosh-function (5.7.67), and the fact that all paths in $\mathcal{J}$ have the same length[9], in (5.7.119) yields

$$f_{l,\pi}^{(s)}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^\alpha} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\boldsymbol{l}'} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}$$
$$\prod_{i=0}^{b} \tanh\left(\frac{\hat{l}_i\mathsf{E}}{\mathsf{L}_{opt}n}\right)^{d\left(\hat{v}^{\text{la}}_{h(i)},\hat{v}^{\text{fi}}_{h(i+1)}\right)} \prod_{i=1}^{b} \tanh\left(\mathsf{E}\frac{d(v^{\text{fi}}_{h(i)},v^{\text{la}}_{h(i)})-k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\text{fi}}_{h(i)},v^{\text{la}}_{h(i)})-k_{h(i)}}. \tag{5.7.120}$$

Let $q \equiv \min\{h(i) > m, k_{h(i)} > 0\}$, splitting the product of the tanh-terms according to q, we obtain

$$\begin{aligned}
&\prod_{i=0}^{b} \tanh\left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt}n}\right)^{d\left(\hat{v}^{\text{la}}_{h(i)},\hat{v}^{\text{fi}}_{h(i+1)}\right)} \prod_{i=1}^{b} \tanh\left(\mathsf{E}\frac{d(v^{\text{fi}}_{h(i)},v^{\text{la}}_{h(i)})-k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\text{fi}}_{h(i)},v^{\text{la}}_{h(i)})-k_{h(i)}} \\
&= \prod_{i=0}^{q-1} \tanh\left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt}n}\right)^{d\left(\hat{v}^{\text{la}}_{h(i)},\hat{v}^{\text{fi}}_{h(i+1)}\right)} \prod_{i=1}^{q} \tanh\left(\mathsf{E}\frac{d(v^{\text{fi}}_{h(i)},v^{\text{la}}_{h(i)})-k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\text{fi}}_{h(i)},v^{\text{la}}_{h(i)})-k_{h(i)}} \\
&\times \prod_{i=q}^{b} \tanh\left(\frac{\mathsf{E}\hat{l}_i}{\mathsf{L}_{opt}n}\right)^{d\left(\hat{v}^{\text{la}}_{h(i)},\hat{v}^{\text{fi}}_{h(i+1)}\right)} \prod_{i=q+1}^{b} \tanh\left(\mathsf{E}\frac{d(v^{\text{fi}}_{h(i)},v^{\text{la}}_{h(i)})-k_{h(i)}}{\mathsf{L}_{opt}n}\right)^{d(v^{\text{fi}}_{h(i)},v^{\text{la}}_{h(i)})-k_{h(i)}} \\
&\le \tanh\left(\mathsf{E}\frac{\widehat{L}_{q-1}+\mathcal{D}_q}{\mathsf{L}_{opt}n}\right)^{\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_q} \times \tanh\left(\mathsf{E}\frac{\widehat{L}_b-\widehat{L}_{q-1}+\mathcal{D}_b-\mathcal{D}_q}{\mathsf{L}_{opt}n}\right)^{\widehat{\mathcal{D}}_b-\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_b-\mathcal{D}_q},
\end{aligned} \tag{5.7.121}$$

the last r.h.s using the monotonicity of the tanh-terms (5.7.66) two times: one time for the first line and a second time for the second line of the second equality. Putting (5.7.121) into (5.7.120) yields

$$f_{l,\pi}^{(s)}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^\alpha} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\boldsymbol{l}'} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt}n-k}{\mathsf{L}_{opt}n}\right)^n \left(\frac{\mathsf{L}_{opt}n}{\mathsf{E}e}\right)^{\mathsf{L}_{opt}n-k}$$
$$\tanh\left(\mathsf{E}\frac{\widehat{L}_{q-1}+\mathcal{D}_q}{\mathsf{L}_{opt}n}\right)^{\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_q} \times \tanh\left(\mathsf{E}\frac{\widehat{L}_b-\widehat{L}_{q-1}+\mathcal{D}_b-\mathcal{D}_q}{\mathsf{L}_{opt}n}\right)^{\widehat{\mathcal{D}}_b-\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_b-\mathcal{D}_q}. \tag{5.7.122}$$

We now claim that for $0 < x \le y \le \mathsf{E}$,

$$\tanh(x) \le \frac{3}{4}\tanh(x+y). \tag{5.7.123}$$

[9] Recall from (5.7.74) that $\sum_{i=0}^{b} \hat{l}_i + \sum_{i=1}^{b} d(v^{\text{fi}}_{h(i)}, v^{\text{la}}_{h(i)}) - k_{h(i)} = \mathsf{L}_{\mathsf{opt}}\mathsf{n} - \mathsf{k},$.

Indeed, using the addition formula for the tanh function, it holds

$$\frac{\tanh(x)}{\tanh(x+y)} = \frac{\tanh(x)\,(1+\tanh(x)\tanh(y))}{\tanh(x)+\tanh(y)} = \frac{1+\tanh(x)\tanh(y)}{1+\frac{\tanh(y)}{\tanh(x)}} \leq \frac{1+\tanh(\mathsf{E})^2}{2} = \frac{3}{4}, \tag{5.7.124}$$

the last inequality because the function tanh is increasing and the claim (5.7.123) is settled.

Again using that tanh is increasing we also have that

$$\tanh(y) \leq \tanh(x+y). \tag{5.7.125}$$

Using in (5.7.122) the estimates (5.7.123) and (5.7.125) with

$$x \equiv \min\{\widehat{L}_{q-1} + \mathcal{D}_q, \widehat{L}_b - \widehat{L}_{q-1} + \mathcal{D}_b - \mathcal{D}_q\}, \tag{5.7.126}$$

and

$$y \equiv \max\{\widehat{L}_{q-1} + \mathcal{D}_q, \widehat{L}_b - \widehat{L}_{q-1} + \mathcal{D}_b - \mathcal{D}_q\}, \tag{5.7.127}$$

we obtain

$$f_{l,\pi}^{(s)}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^\alpha} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\boldsymbol{l}'} \left(\frac{3}{4}\right)^{\min\{\widehat{\mathcal{D}}_{q-1}+\mathcal{D}_q, \widehat{\mathcal{D}}_b - \widehat{\mathcal{D}}_{q-1} + \mathcal{D}_b - \mathcal{D}_q\}}$$
$$\tanh\left(\mathsf{E}\frac{\mathcal{D}_b + \widehat{L}_b}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b + \widehat{\mathcal{D}}_b} \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^n \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\mathsf{L}_{opt} n - k}. \tag{5.7.128}$$

With the same line of reasoning as in (5.7.88), we clearly have that

$$\widehat{\mathcal{D}}_{q-1} + \mathcal{D}_q \geq m\hat{n}_K - k, \tag{5.7.129}$$

and

$$\widehat{\mathcal{D}}_b - \widehat{\mathcal{D}}_{q-1} + \mathcal{D}_b - \mathcal{D}_q \geq \frac{n}{2} - k. \tag{5.7.130}$$

Thus, it follows from (5.7.129) and (5.7.130) that

$$\min\{\widehat{\mathcal{D}}_{q-1} + \mathcal{D}_q, \widehat{\mathcal{D}}_b - \widehat{\mathcal{D}}_{q-1} + \mathcal{D}_b - \mathcal{D}_q\} \geq m\hat{n}_K - k. \tag{5.7.131}$$

Plugging (5.7.131) into (5.7.128) and recalling that paths in $\mathcal{J}$ have the same, prescribed length (recall once more (5.7.74) or, which is the same, footnote 9), it holds

$$f_{l,\pi}^{(s)}(n,k) \lesssim n^{\frac{2K+1}{2}} n^{Kn^\alpha} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{l}} \sum_{\boldsymbol{\sigma}} \sum_{\boldsymbol{l}'} \left[\left(\frac{3}{4}\right)^{m\hat{n}_K - k} \tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^{\mathcal{D}_b + \widehat{\mathcal{D}}_b} \right.$$
$$\left. \times \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^n \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\mathsf{L}_{opt} n - k} \right]. \tag{5.7.132}$$

We follow *exactly* the same steps which from (5.7.73) lead to (5.7.86), this time of course with the factor $\left(\frac{3}{4}\right)^{m\hat{n}_K - k}$. Omitting the details, we obtain

$$f^{(s)}_{l,\pi}(n,k) \leq P_n n^{Kn^\alpha} \left(\frac{3}{4}\right)^{m\hat{n}_K - k} \tanh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^{\max\left(n-k, \frac{\mathsf{L}_{opt} n - k}{4}\right)} \times \cosh\left(\mathsf{E}\frac{\mathsf{L}_{opt} n - k}{\mathsf{L}_{opt} n}\right)^n \left(\frac{\mathsf{L}_{opt} n}{\mathsf{E} e}\right)^{\mathsf{L}_{opt} n - k}, \tag{5.7.133}$$

where P_n is a finite degree polynomial. Combining (5.7.118) and (5.7.133) and the fact that for $k \leq 200\hat{n}_K$, $\left(\frac{3}{4}\right)^{m\hat{n}_K - k} \leq \left(\frac{3}{4}\right)^{(m-200)\hat{n}_K}$ finishes the proof of Lemma 48.

□

5.8 Concentration of the optimal length: proof of Theorem 3

Recall that claim (5.2.60) reads

$$\lim_{n\to\infty} \mathbb{P}\left(\#\left\{\pi \in \Pi_n : X_\pi \leq E + \epsilon^2, \frac{|l_\pi - \mathsf{L}n|}{n} \geq a\epsilon\right\} > 0\right) = 0\,, \tag{5.8.1}$$

for $a > 0$ large enough. The proof, which is (vaguely) inspired by the *saddle point method* [30], exploits the strong concentration of the expansion of the sinh-function on specific Taylor-terms. To see how this goes, in virtue of the by now "classical" route (union bounds and Markov's inequality / independence / tail estimates) it holds

$$\mathbb{P}\left(\#\left\{\pi \in \Pi_n : X_\pi \leq E + \epsilon^2, \frac{|l_\pi - \mathsf{L}n|}{n} \geq a\epsilon\right\}\right) \lesssim \sum_{\frac{|l - \mathsf{L}n|}{n} \geq a\epsilon} M_{n,l} \frac{(E+\epsilon^2)^l}{l!}\,. \tag{5.8.2}$$

Splitting the above sum

$$\sum_{\frac{|l_\pi - \mathsf{L}n|}{n} \geq a\epsilon} M_{n,l} \frac{(E+\epsilon^2)^l}{l!} = \sum_{l=0}^{(\mathsf{L}-a\epsilon)n} M_{n,l} \frac{(E+\epsilon^2)^l}{l!} + \sum_{l=(\mathsf{L}+a\epsilon)n}^{\infty} M_{n,l} \frac{(E+\epsilon^2)^l}{l!}\,, \tag{5.8.3}$$

we claim that both contributions vanish in the large-n limit.

Concerning the first sum, by Stanley's M-bound (5.2.14), and for *any* $x > 0$, we have that

$$\sum_{l=0}^{(\mathsf{L}-a\epsilon)n} M_{n,l} \frac{(E+\epsilon^2)^l}{l!} \leq \sinh(x)^n \sum_{l=0}^{(\mathsf{L}-a\epsilon)n} \left(\frac{E+\epsilon^2}{x}\right)^l, \tag{5.8.4}$$

We choose $x \equiv E + \epsilon^2 - \epsilon$, in which case the largest term in the above sum is given by $l = \mathsf{L} - a\epsilon$, and therefore

$$\begin{aligned}(5.8.4) &\lesssim \sinh\left(E + \epsilon^2 - \epsilon\right)^n \left(\frac{E+\epsilon^2}{E+\epsilon^2-\epsilon}\right)^{(\mathsf{L}-a\epsilon)n} \times n \\ &= n \exp\left\{ n \log\sinh(E+\epsilon^2-\epsilon) - (\mathsf{L}-a\epsilon)\log\left(1 - \frac{\epsilon}{E+\epsilon^2}\right)\right\} .\end{aligned} \tag{5.8.5}$$

To get a handle on the above exponent we proceed by Taylor expansions around E:

$$\begin{aligned}\sinh(E+\epsilon^2-\epsilon) &= \sinh(E) + (\epsilon^2-\epsilon)\cosh(E) + (\epsilon^2-\epsilon)^2\frac{\sinh(E)}{2} + o(\epsilon^2) \\ &= 1 + (\epsilon^2-\epsilon)\sqrt{2} + \frac{\epsilon^2}{2} + o(\epsilon^2) \qquad (\epsilon \downarrow 0).\end{aligned} \tag{5.8.6}$$

Further using that $\log(1-x) = 1 - x - \frac{x^2}{2} + o(x^2)$ for $x \downarrow 0$, we thus get

$$\begin{aligned}&\log\sinh(E+\epsilon^2-\epsilon) - (\mathsf{L}-a\epsilon)\log\left(1-\frac{\epsilon}{E+\epsilon^2}\right) \\ &\quad = (\epsilon^2-\epsilon)\sqrt{2} + \frac{\epsilon^2}{2} - (\mathsf{L}-a\epsilon)\left(-\frac{\epsilon}{E} - \frac{\epsilon^2}{2E^2}\right) + o(\epsilon^2) \\ &\quad = \epsilon^2\left(\frac{1}{2} + \sqrt{2} + \frac{1}{\sqrt{2}E} - \frac{a}{E}\right) + o(\epsilon^2),\end{aligned} \tag{5.8.7}$$

for $\epsilon \downarrow 0$. But the r.h.s. (5.8.7) is clearly negative as soon as $a > \frac{E}{2} + \sqrt{2}E + \frac{1}{\sqrt{2}}$, implying that the first sum in (5.8.3) yields no contribution in the large-n limit, as claimed.

We proceed in full analogy for the second sum, but this time around via Stanley's M-bound with $x \equiv E + \epsilon^2 + \epsilon$: an elementary estimate of the ensuing geometric series yields

$$\begin{aligned}\sum_{l=(\mathsf{L}+a\epsilon)n}^{\infty} M_{n,l}\frac{(E+\epsilon^2)^l}{l!} &\lesssim \sinh\left(E+\epsilon^2+\epsilon\right)^n\left(\frac{E+\epsilon^2}{E+\epsilon^2+\epsilon}\right)^{(\mathsf{L}+a\epsilon)n}\frac{E+\epsilon^2+\epsilon}{\epsilon} \\ &\lesssim \exp n\left\{\log\sinh(E+\epsilon^2+\epsilon) - (\mathsf{L}+a\epsilon)\log\left(1+\frac{\epsilon}{E+\epsilon^2}\right)\right\},\end{aligned} \tag{5.8.8}$$

recalling in the last step the definition of $l_{\epsilon,n} = \mathsf{L} + a\epsilon$. Once again Taylor-expanding the exponent (around E) we get

$$\log\sinh(\mathsf{E}+\epsilon^2+\epsilon) - (\mathsf{L}+a\epsilon)\log\left(1+\frac{\epsilon}{E+\epsilon^2}\right) = \epsilon^2\left(\frac{1}{2}+\sqrt{2}+\frac{1}{\sqrt{2}E} - \frac{a}{E}\right) + o(\epsilon^2), \tag{5.8.9}$$

for $\epsilon \downarrow 0$: as this is also negative for $a > \frac{E}{2} + \sqrt{2}E + \frac{1}{\sqrt{2}}$, the second claim is also settled, and the proof of the Theorem 3 follows.

□

5.9 Appendix

We give for completeness the short proof of Stanley's formula (5.1.3), which states that

$$\sinh(x)^d \cosh(x)^{n-d} = \sum_{l=0}^{\infty} M_{n,l,d} \frac{x^l}{l!} \,. \tag{5.9.1}$$

Indeed, by the Binomial Theorem, it holds

$$\begin{aligned}\sinh(x)^d \cosh(x)^{n-d} &= \frac{1}{2^n}\left(e^x - e^{-x}\right)^d \left(e^x + e^{-x}\right)^{n-d} \\ &= \frac{1}{2^n}\left(\sum_{j=0}^{d}\binom{d}{j}(-1)^j e^{(d-2j)x}\right)\left(\sum_{i=0}^{n-d}\binom{n-d}{i} e^{(n-d-2i)x}\right) \\ &= \frac{1}{2^n}\sum_{j=0}^{d}\sum_{i=0}^{n-d}\binom{n-d}{i}\binom{d}{j}(-1)^j \exp\left(n - 2(i+j)x\right) \,.\end{aligned} \tag{5.9.2}$$

Taylor expanding the exponential function, we get that the r.h.s. above equals

$$\begin{aligned}&\sum_{l=0}^{\infty}\frac{1}{2^n}\sum_{i=0}^{n-d}\sum_{j=0}^{d}\binom{d}{j}\binom{n-d}{i}(-1)^j (n-2(i+j))^l \frac{x^l}{l!} \\ &\qquad = \sum_{l=0}^{\infty}\left\{\frac{1}{2^n}\sum_{i'=j}^{n-d+j}\sum_{j=0}^{d}\binom{d}{j}\binom{n-d}{i'-j}(-1)^j (n-2i')^l \mathbb{1}_{j\le i'}\right\}\frac{x^l}{l!} \,,\end{aligned} \tag{5.9.3}$$

the last step by the substitution $i' \hookrightarrow i+j$. By definition of the $M's$, Stanley's formula thus follows . □

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Curriculum Vitae

Adrien Schertzer

Date of birth:	4th May 1991
Place of birth:	Paris
Phone:	++49 - 15 - 90 - 3765910
Email:	`schertzer@hotmail.fr`
Homepage:	`http://www.math.uni-frankfurt.de/~schertze`

J.W. Goethe-Universität Frankfurt
Robert-Mayer-Str. 10, Office 6
D - 60325 Frankfurt

Education

Ph.D. in Mathematics, supervisor: Prof. Nicola Kistler	Goethe-University, 04/2018-present
Master studies in Mathematics	Goethe-University, 11/2016-02/2018
AMF (L'Autorité des marchés financiers) Certification	Paris, 01/2016
Master in Mathematics	University of Lorraine, 09/2015
Engineering Degree	Ecole des Mines, Nancy, 09/2015
Bac (high school leaving diploma) passed with distinction	Claude Monet, 07/2009

Employment

Goethe-University, Teaching Assistant and Ph.D. candidate	04/2018-present
Junior Trader at Rivage Investment in Paris, Asset: Equity	10/2015-11/2016
Internship at Societe Generale, Paris, Risk Analyst	03/2015-09/2015
Internship at Goethe-University, Research in biostatistics	06/2014-09/2014
Internship, Worker on a construction site	winter 2013

Publications

with N. Kistler and M. Schmidt, *Oriented first passage percolation in the mean field limit, 2. The extremal process*, Ann. Appl. Prob., Vol. **30** no.2, 788-811 (2020)

with N. Kistler and M. Schmidt, *Oriented first passage percolation in the mean field limit*, Brazilian Jour. Prob. Stat., Vol. **34** no. 2, 414-425 (2020)

with G. Kersting, N. Kistler and M. Schmidt, *From Parisi to Boltzmann*, In Statistical Mechanics of Classical and Disordered Systems, Springer PROMS **293** (2019)

Preprints

with N. Kistler, *Undirected polymer in random environment* submitted

Teaching experience

Current tutorials. *Extremwerttheorie* (Master), Seminar *Ausgewählte Kapitel der Stochastik* (Bachelor), Seminar *Statistische Mechanik* (Master)

Past tutorials. *Stochastische Prozesse* (Bachelor), *Elementare Stochastik* (Bachelor), *Höhere Stochastik* (Master), 04/2018-07/2020

Advising. 1 Bachelor student 2019

Seminars

Mainz University (2019)

Frankfurt University (2019)

Other

Computer skills. VBA, Bloomberg, MS Office (Word, Excel, Power Point), SAS, R, SQL, Jmp, Matlab, C++.

Language. French: Native, English: Fluent (IELTS: 6.5/9, 2014), German: Fluent (C1), Russian: Beginner

Leisure activities. Reading, movies, music, playing piano, travel. Sports: Soccer, Tennis, Basketball (High school League Champion-Paris 2007) , Athletics (second place in Athletics Team Championship, Paris 2005)

References

Prof. Dr. Nicola Kistler
Institut für Mathematik
J.W. Goethe-Universität Frankfurt
Robert-Mayer-Str. 10
D - 60325 Frankfurt
Email: `kistler@math.uni-frankfurt.de`
Phone: ++49 - 69 - 798 - 22644

Prof. Dr. Anton Wakolbinger
Institut für Mathematik
J.W. Goethe-Universität Frankfurt
Robert-Mayer-Str. 10
D - 60325 Frankfurt
Email: `wakolbinger@math.uni-frankfurt.de`
Phone: ++49 - 69 - 798 - 28651

Prof. Dr. Lisa Hartung
Institut für Mathematik
Johannes Gutenberg-Universität Mainz
Staudingerweg 9
D-55099 Mainz
Email: lhartung@uni-mainz.de
Phone: ++49 - 6131-392-7998