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Consequences of minimal length discretization on line element, metric tensor, and geodesic equation

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When minimal length uncertainty emerging from a generalized uncertainty principle (GUP) is thoughtfully implemented, it is of great interest to consider its impacts on *gravitational* Einstein field equations (gEFEs) and to try to assess consequential modifications in metric manifesting properties of quantum geometry due to quantum gravity. GUP takes into account the gravitational impacts on the noncommutation relations of length (distance) and momentum operators or time and energy operators and so on. On the other hand, gEFE relates *classical geometry or general relativity gravity* to the energy–momentum tensors, that is, proposing quantum equations of state. Despite the technical difficulties, we intend to insert GUP into the metric tensor so that the line element and the geodesic equation in flat and curved space are accordingly modified. The latter apparently encompasses acceleration, jerk, and snap (jounce) of a particle in the *quasi-quantized* gravitational field. Finite higher orders of acceleration apparently manifest phenomena such as accelerating expansion and transitions between different radii of curvature and so on.

KEYWORDS

generalized uncertainty principle, geodesic equation, line element, metric tensor, noncommutative geometry, quantum gravity, relativity and gravitation

1 | INTRODUCTION

The formulation of a consistent theory for quantum gravity is the ultimate goal but, unfortunately, still one of the open questions in physics. There have been various attempts to reconcile principles of the theory of general relativity in an entire quantum framework (Donoghue 1994; Rovelli & Smolin 1990). Quantum gravity is conjectured to add new elements to theory of general relativity (GR) and quantum mechanics (QM). For the latter, we mention a modification of the Heisenberg uncertainty principle due to gravitational effects, that is, generalized uncertainty principle (GUP) (Tawfik & Diab 2014, 2015). The present work focuses on the former, namely, the possible modifications in line element, metric tensor, and geodesics. Concretely, it intends to tackle the long-standing fundamental problem where the Einstein field equations (EFEs) relate *nonquantized* semi-Riemannian geometry characterized by Ricci and Einstein tensors, which directly depend on the metric tensor, with the *full-quantized* energy–momentum tensor (Stephani et al. 2003). Our approach is based on implementing minimal length uncertainty obtained from the

GUP, which in turn is inspired by string theory, doubly special relativity, and black hole physics (Tawfik & Diab 2014, 2015) and seems to be comparable to the Planck length, where fluctuations in the *quasi-quantized* manifold likely emerge (Tawfik & Diab 2016). We propose that this basic approach helps in characterizing the potential impacts of *full* quantization on einstein field equations (EFE). We also believe that this likely unveils the quantum nature of the cosmic geometry. We elaborate the corresponding modification in the line element, metric tensor, and geodesics and then show how this helps in manifesting properties of quantum geometry due to quantum gravity. The present study introduces a fundamental approach to the observation that the universe likely expands faster than the GR expectation. Unless there are forces deriving this kind of expansion, factors such as dark energy and cosmological constant might remain overdue (Peebles & Ratra 2003).

2 | GENERALIZED UNCERTAINLY PRINCIPLE AND SPACETIME METRIC TENSOR

The Heisenberg uncertainty principle (HUP) dictates how to constrain the uncertainties, for instance, in the quantum noncommutation relations of length (position) and momentum operators or of time and energy operators. On the other hand, when the gravitational influences are thoughtfully taken into account, a GUP then emerges so that an alternative quantum gravity approach for string theory, doubly special relativity, and black hole physics has been provided (Tawfik & Diab 2014, 2015), for example, a finite minimal length (Kempf et al. 1995),

$$\Delta x \, \Delta p \geq \frac{\hbar}{2}\left[1+\beta(\Delta p)^2\right], \tag{1}$$

where Δx and Δp are length and momentum uncertainties, respectively. The GUP parameter $\beta = \beta_0(\ell_p/\hbar)^2 = \beta_0/(m_p c)^2$ with the Planck length $\ell_p = \sqrt{\hbar G/c^3} = 1.977 \times 10^{-16}$ GeV^{-1} and mass $m_p = \sqrt{\hbar c/G} = 1.22 \times 10^{19}$ GeV/c^2. Upper bounds on the dimensionless parameter β_0 should be obtained from astronomical observations, such as recent gravitational waves, $\beta_0 \lesssim 5.5 \times 10^{60}$. Equation (1) seems to demonstrate the existence of a minimum length uncertainty,

$$\Delta x_{\min} \approx \hbar\sqrt{\beta} = \ell_p\sqrt{\beta_0}. \tag{2}$$

GUP also exhibits features of the ultraviolet/infrared (UV/IR) correspondence, where Δx increases rapidly (IR) as the Δp extends beyond the order of the Planck scale (UV) (Gubser et al. 1998; Maldacena 1999; Witten 1998). The UV/IR correspondence could be applied to various subjects of short- versus Long-distance physics, for instance, the *deformed* commutation relations. We assume that the current problem of minimal length discretization would be solved by such a correspondence.

Analogous to Equation (1), the canonical noncommutation relation of quantum operators of length and momentum reads as follows:

$$\left[\hat{x}_i, \hat{p}_j\right] \geq \delta_{ij} i\hbar\left(1+\beta p^2\right), \tag{3}$$

where $p^2 = g_{ij} p^{0i} p^{0j}$, and g_{ij} is the Minkowski spacetime metric tensor, for example, $(-,+,+,+)$. The length and momentum operators, respectively, are defined as

$$\hat{x}_i = \hat{x}_{0i}\left(1+\beta p^2\right), \tag{4}$$

$$\hat{p}_j = \hat{p}_{0j}, \tag{5}$$

in which the operators $\hat{x}_{0i}$ and $\hat{p}_{0j}$ are to be derived from the corresponding noncommutation relation

$$\left[\hat{x}_{0i}, \hat{p}_{0j}\right] = \delta_{ij} i\hbar. \tag{6}$$

3 | SPACETIME GEOMETRY: LINE ELEMENT AND METRIC TENSOR

When including such quantum noncommutation operations, as given in Section 2, in the spacetime geometry, for instance, EFE, the Minkowskian manifold of the line element is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{7}$$

where μ, ν, and $\lambda = 0,1,2,3$, and it increases to an eight-dimensional spacetime tangent bundle similar to the one with the coordinates $x^A = (x^\mu(\zeta^a), \beta\dot{x}^\mu(\zeta^a))$, where $\dot{x}^\mu = dx^\mu/d\zeta^\mu$ (Brandt 2000). This leads to a modification in the line element, that is, a new metric related to quantum geometry,

$$d\tilde{s}^2 = g_{AB}\, dx^A\, dx^B, \tag{8}$$

where $g_{AB} = g_{\mu_\nu} \otimes g_{\mu\nu}$.

With some trivial approximations, the manifold can be reduced to the effective four-dimensional spacetime geometry, where $x^A = x^A(\zeta^\mu)$. Therefore, the modified four-dimensional metric tensor is as follows:

$$\begin{aligned}\tilde{g}_{\mu\nu} &= g_{AB}\frac{\partial x^A}{\partial \zeta^\mu}\frac{\partial x^B}{\partial \zeta^\nu} \simeq g_{ab}\left[\frac{\partial x^a}{\partial \zeta^\mu}\frac{\partial x^b}{\partial \zeta^\nu} + \beta\frac{\partial \dot{x}^a}{\partial \zeta^\mu}\frac{\partial \dot{x}^b}{\partial \zeta^\nu}\right] \\ &\simeq \left(1+\beta\ddot{x}^\lambda\ddot{x}_\lambda\right) g_{\mu\nu},\end{aligned} \tag{9}$$

where $\ddot{x}^\mu = \partial\dot{x}^\mu/\partial\zeta^\mu$ is the four-dimensional acceleration. The indices A, B, a, and b run over $0, 1, \cdots, 7$.

- For flat spacetime, where $g_{\mu\nu} = \eta_{\mu\nu}$, the modified metric tensor can be expressed as

$$\widetilde{g}_{\mu\nu} = \left(1 + \beta\ddot{x}^{\lambda}\ddot{x}_{\lambda}\right)\eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{10}$$

where $h_{\mu\nu} = \beta\ddot{x}^{\lambda}\ddot{x}_{\lambda}\eta_{\mu\nu}$ encompasses the quantum contributions to the spacetime geometry.

- In the limit where $h \to 0$, the added quantum corrections completely diminish, and the *classical* EFE, that is, GR EFE, can be restored.

Thus, the principle of general covariance is apparently also satisfied in the absence of the gravitational effects on the modified Minkowski metric.

The modified four-dimensional line element is then expressed as

$$\begin{aligned} d\tilde{s}^2 &= g_{\mu\nu}\left(dx^{\mu}dx^{\nu} + \beta^2 d\dot{x}^{\mu}\ \ d\dot{x}^{\nu}\right) \\ &= \left(1 + \beta^2\ddot{x}^{\lambda}\ddot{x}_{\lambda}\right)ds^2. \end{aligned} \tag{11}$$

4 | SPACETIME GEOMETRY: GEODESIC EQUATION

Accordingly, the properties of the manifold in special and general relativity can also be generalized. We limit the discussion to modified geodesics, where the notion of a *straight line* is generalized to curved spacetime. In this section, we propose a theory for the possible consequences of length discretization based on an approach to quantum gravity, GUP, on the world line of a free particle. In this regard, we recall that GR assumes gravity as a consequence of curved spacetime geometry, and the *quantized* energy–momentum tensor is the source of spacetime curvature. Our approach follows the same line with a major difference: The length is discretized and, accordingly, so is the rhs of EFE.

By using the variational principle and by extremizing the path s_{AB}, the geodesic equation can be formulated. Due to the proposed length quantization, we obtain

- for flat space:

$$\beta\mathcal{L}\frac{d^2\dot{x}^{\mu}}{d\tau^2} - \frac{dx^{\mu}}{d\tau} + c = 0, \tag{12}$$

- and for curved space:

$$\begin{aligned} &\frac{d^2x^2}{d\tau^2} - \beta\frac{d}{d\tau}\left(\mathcal{L}\frac{d^3x^2}{d\tau^3}\right) = \\ &-\Gamma^2_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{nu}}{d\tau} + \beta g^{2\alpha}g_{\mu\nu,\alpha}\frac{d^2x^{\mu}}{d^2\tau}\frac{d^2x^{nu}}{d\tau^2} \\ &+ \beta\mathcal{L}g^{2\alpha}g_{\mu\alpha,\gamma}\left[\frac{dx^{\gamma}}{d\tau}\frac{d^2\dot{x}^{\mu}}{d\tau^2} + \frac{d}{d\tau}\left(\frac{dx^{\gamma}}{d\tau}\frac{d\dot{x}^{\mu}}{d\tau}\right)\right] \\ &+ \beta\mathcal{L}g^{2\alpha}g_{\mu\alpha,\gamma,\delta}\frac{dx^{\delta}}{d\tau}\frac{dx^{\gamma}}{d\tau}\frac{d\dot{x}^{\mu}}{d\tau}, \end{aligned} \tag{13}$$

where

$$\begin{aligned} \tau &= \int \mathcal{L}\left(s, \dot{x}, \ddot{x}\right) ds, \\ \dot{x}^{\mu} &= \frac{dx^{\mu}}{ds}, \\ g_{\mu\nu,\alpha} &= \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}, \\ g_{\mu\alpha,\gamma,\delta} &= \frac{\partial}{\partial x^{\delta}}\left(\frac{\partial g_{\mu\alpha}}{\partial x^{\gamma}}\right), \\ \Gamma^2_{\mu\nu} &= \frac{1}{2}g^{2\alpha}\left[g_{\mu\alpha,\nu} - g_{\alpha\nu,\mu} + g_{\mu\nu,\alpha}\right], \\ \mathcal{L} &= \left[g_{\mu\nu}\left(\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds} + \beta\frac{d\dot{x}^{\mu}}{ds}\frac{d\dot{x}^{\nu}}{ds}\right)\right]^{1/2}. \end{aligned}$$

The β-terms in Equation (13) distinguish this expression from the GR geodesics. In the section that follows, we discuss these terms in which the consequences of the minimal length discretization are in and summarize our final conclusions.

5 | CONCLUSIONS

It is worth highlighting that the Christoffel connection and the equivalence principle are not affected by the minimal length uncertainty. Thus, we conclude that GUP reconciles with the equivalence principle, and simultaneously, the equivalence principle is not violated. In this way, this gives possibilities to recover the violation of the equivalence principle in the presence of quantum gravity through quantum geometry characterized by minimal length discretization. This essential result apparently contradicts those of references (Ghosh 2014; Scardigli & Casadio 2009) because we have implemented GUP for the basic metric tensor. Accordingly, we have obtained a modified metric tensor and line element. Both are fundamentals of the proposed geometry. By modifying both quantities, we could derive the corresponding geodesics. The appearance of vibration and/or sudden transition, as shall be discussed shortly, apparently stems from the *quasi-quantized* geometry or the proposed approach to quantum gravity, where the curvature demonstrated by the quantized energy–momentum tensor seems to gain corrections as well.

As outlined in Section 3, the corrections to the line element are combined in the term $\beta^2\ddot{x}^{\lambda}\ddot{x}_{\lambda}$, which manifest the essential contributions added by the acceleration of the expansion. In other words, it seems that the length discretization as guaranteed by GUP emphasizes that even the line element would not only expand but also accelerate. It is worth noticing that, even if the GUP parameter β squared would ensure that this factor remains small, the product $\ddot{x}^{\lambda}\ddot{x}_{\lambda}$ raises the values of this correction term. In the same way, the metric tensor obtains corrections, $\beta\ddot{x}^{\lambda}\ddot{x}_{\lambda}$, as well. Here, the factor to the product $\ddot{x}^{\lambda}\ddot{x}_{\lambda}$ is simply the GUP factor β.

Even if classical GR necessarily invokes metric tensor in four dimensions, we assume that the corresponding reference frames experience different expansions in time and space. The metric expansions proposed in the present work are related to changes in metric tensor with time. Despite the possible differences in temporal and spatial expansions, we straightforwardly impose the origin of an accelerated expansion. The spacetime seems to shrink or grow as the corresponding geodesics converge or diverge, where the length discretization plays a major role.

We also note that the geodesics, Equation (13), not only provide the acceleration of a particle in a gravitational field but higher-order derivatives as well (Eager et al. 2016), namely, the snap or jounce, $x^{(4)}$, which in turn is derived from the jerk, $x^{(3)}$. The jerk gives the change in the force acting on that particle, while the snap results from change in the jerk itself. That both quantities are finite means that vibration or sudden transitions would occur between different radii of the curvature. Acceleration, as in GR geodesics, without jerk is just a static load, that is, neither vibration nor transition is allowed.

Due to GUP and the corresponding minimal length discretization, the corrections added to the line element, metric tensor, and geodesics emphasize that the evolution of the universe as theorized by classical GR is also accelerated (Riess et al. 1998; Velten et al. 2018). While the line element and metric tensor obtain additional terms of acceleration products, $\ddot{x}^{\lambda}\ddot{x}_{\lambda}$, the geodesics is corrected with higher-order acceleration derivatives, such as snap $x^{(4)}$ and jerk $x^{(3)}$.

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