Arbitrage theory in models with transaction costs beyond efficient friction

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Chapter 1

Introduction

This introductory chapter aims to provide an overview of the arbitrage theory in models with proportional transaction costs relevant to this thesis and describes its main results.

1.1 Financial market models with transaction costs

In frictionless financial market models, there are no costs associated with trading. All assets can be bought and sold at the same price. Of course, these models are an idealization of the real world as transaction costs are an important feature of financial markets. Typical types of transaction costs are fixed costs and proportional costs. *Fixed costs* pertain to a constant fee, e.g., a brokerage fee, being charged for each transaction regardless of its size. Thus, fixed costs penalize the frequency of trading. The transaction costs that arise due to a positive bid-ask spread, i.e., the difference between the *ask price*, which is the lowest price an asset is sold for, and the *bid price*, which is the highest price offered for an asset, are proportional to the size of each trade. We refer to such costs as *proportional transaction costs*. In contrast to fixed costs, proportional costs penalize the size of the trades. In general, bid-ask spreads are caused by numerous factors (see, e.g., Harris [39, Chapter 14] and Madhavan [70]). For example, Glosten and Milgrom [33] show that a positive bid-ask spread can arise due to adverse selection, i.e., as liquidity providers may lose money to informed traders, they widen the spread to recover their losses from uninformed traders.

In this thesis, we only consider models with proportional transaction costs (for models with fixed costs see, e.g., [3, 9, 48]). In usual models of a stock market with proportional transaction costs and d risky assets (see, e.g., Jouini and Kallal [47]), there are two d-dimensional processes modeling the bid and ask price of each stock, and each transaction involves a *bank account* (or bond), i.e., a frictionless riskless asset with strictly positive price at any time. In particular, the actions of an investor are limited to buying and selling an asset in exchange for units of the bank account. After passing to relative prices, i.e., expressing all prices as multiples of the bank account, this means that each purchase of a self-financing strategy charges the bank account with the ask price, and each sale credits the bank account with the bid price.

As identified by Kabanov [50], the existence of a bank account is unrealistic for a currency market which, in contrast to a stock market, allows for the direct exchange of the different assets. Thus, Kabanov [50] introduces a "currency model" with finitely many assets (see, also, Kabanov and Safarian [55, Section 3.1]). In this general framework, which includes the models described above, a bank account need not exist and there is no one-dimensional wealth process. Portfolios are expressed in terms of *physical units* of the assets. The transaction costs are implicitly modeled by an adapted cone-valued process whose value at time t models the *cone of portfolios available at price zero at time t*. More specifically, in the discrete-time setup of Kabanov's model from Schachermayer [85], which we follow in the second chapter of this thesis, the cone of portfolios available at price zero at time t is induced by a random matrix whose entries specify the amount of physical units of an asset needed to purchase one unit of another asset at time t. In this case, a portfolio process is self-financing iff its increment at time t takes values within the cone of portfolios available at price zero at time t for each point in time t.

In the case of only two assets, Kabanov's model is equivalent to the usual model of a stock market, as described above, where all transfers are made via the bank account. In the third chapter of this thesis, we consider such a two-dimensional model in continuous time.

1.2 The discrete-time case

At the foundation of the modern mathematical theory of financial markets is the principle of *no-arbitrage*. Loosely speaking, a mathematical model of a financial market should not allow for any arbitrage opportunities, i.e., a chance to make a profit without any risk and any net investment. In discrete-time frictionless financial markets models, the absence of arbitrage opportunities is characterized by the existence of an equivalent martingale measure, i.e., an equivalent probability measure that turns discounted prices into martingales. This is known as the fundamental theorem of asset pricing (FTAP). In the case of finite probability spaces, it goes back to Harrison and Pliska [41] and its extension to general probability spaces is due to Dalang et al. [21]. The proof of the later uses finite dimensional separation arguments to obtain the existence of a martingale measure locally and, subsequently, uses measurable selection to extend the result to multiple periods. On the other hand, more modern versions of its proof, see [53, 83], use that the absence of arbitrage opportunities implies the closedness in probability of the set of hedgeable claims attainable from zero endowment. To show this, Schachermayer [83] passes to the set of trading strategies "orthogonal" to all trading strategies resulting in a terminal value of zero. Subsequently, he shows that any sequence of these strategies, whose associated sequence of terminal values converges, has to stay bounded by leading a possible explosion to a contradiction. Of course, this relies, just as the proof of Kabanov and Stricker [53], on the (obvious) fact that the set of strategies resulting in a terminal value of zero is a vector space in a frictionless financial market. To conclude, by no-arbitrage, the set of hedgeable claims attainable from zero endowment is closed *and* its intersection with the set of non-negative random variables reduces to zero. Thus, the Kreps-Yan theorem, i.e., a Hahn-Banach-type separation argument, can be applied to obtain an equivalent martingale measure.

For a finite probability space, Kabanov and Stricker [54] extend the FTAP to Kabanov's model of a currency market. They show that no-arbitrage (NA) is equivalent to the existence of a *consistent price system (CPS)*, which is a multidimensional martingale under the objective probability measure taking values within the polar of the cone of portfolios available at price zero at each point in time. The relationship of a CPS with the notion of an equivalent martingale measure is apparent in the special case of a stock market model with a bank account. In this case, a CPS corresponds one-to-one to a tuple of an equivalent measure and a price process whose components take values within the bid-ask spread of each asset and which is a martingale under the equivalent measure (see, e.g., Rohklin [82, Section 3]).

For infinite probability spaces, an analogue of the frictionless FTAP fails. Schachermayer [85] provides an example of an arbitrage-free market that allows for an approximate arbitrage, i.e., a non-zero and non-negative portfolio that is the limit in probability of a sequence of portfolios attainable from zero endowment. Consequently, a CPS cannot exist. This shows that under proportional transaction costs the set of hedgeable claims attainable from zero endowment may fail to be closed although (NA) is satisfied. A key observation is that there might be claims attainable from zero endowment by trading up to some time t that can subsequently be liquidated for sure which *cannot* be attained from zero endowment in the subsequent periods. In other words, although (NA) is satisfied, one may gain an advantage by trading up to time t since the same terminal position cannot be achieved by starting to trade at time t and there is no risk associated to the position as it can be liquidated for sure between t and the terminal time. This advantage may result in the set of hedgeable claims attainable from zero endowment not being closed and can, in some cases, be exploited to construct an approximate arbitrage (see, e.g., Schachermayer [85, Example 3.1]). Thus, there arises the question under which stronger absence of arbitrage conditions such effects can be excluded.

Schachermayer [85] introduces the *robust no-arbitrage* property (NA^{r}), i.e., there have to exist more favorable investment opportunities leading to smaller bid-ask spreads (for each pair of assets) such that the modified market still satisfies (NA). Schachermayer shows that, under (NA^{r}), the set of null-strategies, i.e., the increments of self-financing portfolio processes with vanishing terminal value, is a vector space. This ensures that the effects discussed above cannot occur. Thus, using the same idea as in [83], Schachermayer shows that the set of hedgeable claims attainable from zero endowment is closed in probability and, consequently, the existence of a CPS follows by the usual separation arguments. In addition, he shows that (NA^{r}) is actually equivalent to the existence of a *strictly consistent price system (SCPS)*, which is a martingale taking values within the *relative interior* of the polar of the cone of portfolios attainable at price zero at each point in time.

The alternative absence of arbitrage condition is *strict no-arbitrage* (NA^{s}) introduced earlier by Kabanov et al. [51]. A market model satisfies (NA^{s}) iff any claim which is attainable from zero endowment up to some intermediate time t and which can be liquidated in t for sure, can also be attained from zero endowment by trading at time t only. Together with *efficient friction*, i.e., nonvanishing transaction costs, Kabanov et al. [51] show that (NA^{s}) implies that the *only* null-strategy consists of not trading at all. Consequently, the set of hedgeable claims attainable from zero endowment is closed in probability. Combining the closedness with (NA^{s}), they show that a SCPS exists. The opposite implication also holds true, i.e., under efficient friction, (NA^{s}) and the existence of a SCPS are equivalent.

In the second chapter of this thesis, we introduce the novel *prospective strict no-arbitrage* ($\mathrm{NA}^{\mathrm{ps}}$) property which is a variant of (NA^{s}). Roughly speaking, the market model satisfies ($\mathrm{NA}^{\mathrm{ps}}$) iff any claim which is attainable from zero endowment by trading up to some time t and which can subsequently be liquidated for sure, can also be attained from zero endowment by trading between t and the terminal time. This means that either one cannot obtain an advantage by trading up to time t as the same terminal position can be achieved by starting to trade at time t, or there is some risk associated to the position as it cannot be liquidated for sure. This means that ($\mathrm{NA}^{\mathrm{ps}}$) is tailor-made to exclude the effect described above and, indeed, ($\mathrm{NA}^{\mathrm{ps}}$) is weaker as ($\mathrm{NA}^{\mathrm{r}}$) but already guarantees the closedness of the set of hedgeable claims attainable from zero endowment (see Theorem 2.2.6). In particular, ($\mathrm{NA}^{\mathrm{ps}}$) implies the existence of a CPS.

On the other hand, it is important to note that the closedness of the set of hedgeable claims attainable from zero endowment is *not* necessary for the existence of a CPS. In the case of only two assets, Grigoriev [34] shows that (NA) is equivalent to the existence of a CPS - although the set of hedgeable claims attainable from zero endowment need not be closed (see, e.g., [34, Example 1.3]). As already mentioned, this analogy of the frictionless FTAP fails in higher dimensions (see [65, Example 4.6] for a counterexample with three assets). This means that ($\mathrm{NA}^{\mathrm{ps}}$) cannot be equivalent to the existence of a CPS. Therefore, we introduce a weakening of ($\mathrm{NA}^{\mathrm{ps}}$) which turns out to be equivalent to the existence of a CPS (see Theorem 2.2.11).

While the closedness of the set of hedgeable claims attainable from zero endowment is not necessary for the existence of a CPS, it is still a desirable property of the market model. Similar to the frictionless theory, see, e.g., Föllmer and Schied [32, Chapter 7], the closedness is needed for superhedging. Thus, just as under (NA^{r}) (see Schachermayer [85, Theorem 4.1]), a similar superhedging results holds under ($\mathrm{NA}^{\mathrm{ps}}$) (see Remark 2.2.23). Even though superhedging may not be applicable from a practical point of view, it is of enormous theoretical importance, e.g., in the theory of optimal portfolio selection. Czichowsky et al. [16] use both the closedness of the set hedgeable claims attainable from zero endowment and the superhedging result to establish a duality result in analogy to the frictionless case [63, Theorem 2.1 and 2.2] under transaction costs. That means that they establish the existence of primal and dual optimizers and show how the solutions of the primal and the dual problem are connected. This result is central to answer the question whether a so-called shadow price, i.e., a fictitious frictionless price process that takes values within the bid-ask spread of the original market *and* that leads to the same *optimal* decisions and trading gains as under transaction costs, exists. In other words, the question arises whether the behavior of an agent under transaction costs can be explained by passing to a suitable *least favorable* frictionless market. For finite probability spaces Kallsen and Muhle-Karbe [57] show that the answer is affirmative, i.e., shadow prices always exist. On the other hand, Czichowsky et al. [16] show that shadow prices do not have to exist on general probability spaces. However, they also show that if the dual minimizer is a martingale it corresponds to a shadow price and, conversely, if a shadow price exists it is necessarily derived from a dual minimizer.

Coming back to the discussion of arbitrage under transaction costs, it was observed by Rásonyi [76] that under transaction costs an *arbitrage opportunity of a second kind*, i.e., a non-solvent portfolio which ends up solvent, can exist even though a strictly consistent price system exists. Thus, Rásonyi [76] introduces the condition of *no-arbitrage of a second kind* (NA2), which was originally called *no sure gain in liquidation value.* Assuming efficient friction, Rásonyi [76, Theorem 1] characterizes (NA2) by showing that it is equivalent to the condition of *prices consistently extendable* (PCE), i.e., the existence of strictly consistent price systems with arbitrary (inner) starting points (see also Denis and Kabanov [26, Section 4] for a complementary result). Interestingly, there is no analogue of this result for frictionless financial markets where an equivalent martingale measure may fail to exist although (NA2) is satisfied (see Rásonyi [76, Remark 4]). In contrast to the previously discussed absence of arbitrage conditions for market models with transaction costs, (NA2) is equivalent to its local version, that is: A position cannot surely be solvent in the next period, if it is not already solvent in the current period. Being able to pass from local to global is advantageous as, e.g., demonstrated by Bouchard and Nutz [8], who obtain an analogous result under model uncertainty, where the usual functional analytic approach does not work due to the lack of a reference measure.

In general, the previously discussed criteria try to establish a link between the absence of arbitrage, i.e., a property of the financial model, and the existence of a (strictly) consistent price system. Abstractly speaking, given an adapted sequence of random sets, the question is whether there exists a process taking values within the sequence of random sets which is a martingale under an equivalent measure. This more general point of view is introduced by Rokhlin [79, 80] as the *martingale selection problem (MSP).* Thus, the stronger absence of arbitrage criteria can be viewed as conditions under which the MSP is solvable. Rohklin [81] shows that the framework of the MSP is useful for models with portfolio constrains and, more recently, Burzoni and Šikić [10] use the relationship between no-arbitrage theory and the MSP to address the general theory of markets with frictions in a discrete-time setting with no probability measures.

1.3 The continuous-time case

To establish a fundamental theorem of asset pricing (FTAP) for continuous-time frictionless financial market models, several difficulties have to be overcome. As already noted by Harrison and Kreps [40, Section 6], arbitrages arising from doubling strategies need to be avoided by a suitable concept of admissibility, e.g., requiring the corresponding value process to be uniformly bounded from below. In addition, [22, Example 7.7] shows that, in arbitrage-free continuous-time models, there can still exist a sequence of strategies whose payoffs converge towards an arbitrage opportunity and, consequently, an equivalent local martingale measure does not exist. This is in contrast to finite discrete time where no-arbitrage already implies the closedness of the set of hedgeable claims. This need to complement the notion of no-arbitrage with a topological condition was already noted by Kreps [64]. He introduces the notion of a *free lunch*, i.e., a generalized sequence, also called a net, of payoffs converging towards an arbitrage in a weak sense. Then, *no free lunch*, which is the absence of free lunches, is equivalent to the existence of an equivalent local martingale measure (cf. Delbaen and Schachermayer [24, Theorem 5.2.2]).

The use of nets and a hard to interpret convergence is economically not satisfying (see Delbaen and Schachermayer [24, Chapter 5] for a detailed discussion). Thus, Delbaen and Schachermayer [22] introduce the economically meaningful condition of *no free lunch with vanishing risk* (NFLVR). Loosely speaking, there should not exist a sequence of admissible trading strategies whose final payoffs converge almost surely to an arbitrage opportunity while the associated losses converge to zero uniformly. For a locally bounded semimartingale price process Delbaen and Schachermayer [22] show that (NFLVR) is equivalent to the existence of an equivalent local martingale measure, i.e., they establish the FTAP in continuous time.

Later, Delbaen and Schachermayer [23], see also Kabanov [49], show that the boundedness assumption on the semimartingale price process is not really needed if one replaces the term "local martingale" with "sigma-martingale". It also turns out that the price process being a semimartingale is not really a restriction. Indeed, already under the weaker notion of *no unbounded profit with bounded risk* (NUPBR) for simple strategies the price process has to be a semimartingale (see Delbaen and Schachermayer [22, Section 7]). This connection between arbitrage and the semimartingale property of the price process has subsequently been studied in detail by various authors (see, e.g., [2, 6, 60]).

In the FTAP the (NFLVR) condition has to be formulated for general strategies as otherwise an equivalent martingale measure may fail to exist. Delbaen and Schachermayer [22, Section 7] provide an example of a bounded semimartingale satisfying no free lunch with vanishing risk for simple strategies, but there is a free lunch with vanishing risk if one allows strategies to sell before each rational number and buy back immediately after it. This problem arises as the jumps of the semimartingale do not occur at predictable stopping times (cf. Schachermayer [84]). Thus, the need for general strategies and a general theory of stochastic integration is a crucial finding of the frictionless investigation.

Jouini and Kallal [47] initiated the study of absence of arbitrage under proportional transaction costs in the same spirit as Harrison and Kreps [40] and Kreps [64] (cf. the first paragraph of the current subsection). Considering a stock market model with proportional transaction costs, they show that *no free lunch* formulated in L^2 is equivalent to the existence of a *consistent price system* (CPS), which, in their model, is a tuple of an equivalent measure and a price process whose components take values within the bid-ask spread of each asset and which is a martingale under the equivalent measure.

Later, it became more apparent how much the picture changes as soon as transaction costs are introduced. For example, in a frictionless setting the (geometric) fractional Brownian motion with Hurst parameter $H \neq 1/2$, which is not a semimartingale, allows for arbitrage (see Cheridito [12] and Rogers [78]). However, Guasoni [35] shows that the addition of arbitrary small transaction costs is sufficient to eliminate all arbitrage opportunities. Since the fractional Brownian motion has *conditional full support*, it is *sticky*, i.e., it stays within an arbitrary small interval around its current value with positive probability. Thus, already the transaction costs of an initial trade may not be recovered. Guasoni et al. [37] show that for continuous processes taken as the mid-price process *conditional full support* is sufficient for the existence of a CPS for arbitrary small transaction costs. Thus, *conditional full support* is a sufficient condition for the existence of a CPS. But, it is not necessary as shown, e.g., by Guasoni et al. [38, Appendix A]. For a continuous strictly positive mid-price, Guasoni et al. [38] provide a corresponding FTAP for arbitrary small transactions. A FTAP for locally bounded and càdlàg bid-ask processes is established by Guasoni et al. [36]. They show that *robust no free lunch with vanishing risk* (RNFLVR), which, in their formulation, implies efficient friction, i.e., nonvanishing transaction costs, is equivalent to the existence of a *strictly consistent price system* (SCPS). In their work, a simple strategy is admissible iff, after every transaction, the current position can be frozen and, subsequently, be liquidated for a bounded loss at a (possibly) later time. An important difference from the frictionless theory is that they formulate the condition of (RNFLVR) in terms of *simple admissible strategies* only. General admissible strategies are defined as predictable process of finite variation that can be approximated by simple admissible strategies. Then, under (RNFLV), Guasoni et al. [36] can show that the set of total variations of simple admissble trading strategies is bounded in probability. Using this, they show the Fatou-closedness of the set of claims dominated by outcomes of general admissible strategies. Finally, they use the usual separation method together with an argument from Jouini and Kallal [47] to deduce the existence of a SCPS. This improves previous work by Kabanov and Stricker [56] and Campi and Schachermayer [11] who show for continuous and càdlàg bid-ask processes, respectively, that efficient friction together with the existence of a SCPS implies the boundedness in probability of the set of total variations of trading strategies and deduce Fatou-closedness to establish superreplication theorems.

Efficient friction is a standing assumption in continuous-time financial market models with proportional transaction costs. The aim of the second part of this thesis is to go *beyond* efficient friction. Considering a model consisting of a bond and one risky asset whose bid and ask price processes *are not necessarily different*, we make a first step in this direction by introducing a reasonable set of general strategies for which the self-financing condition of the model can be defined. This set has to go beyond strategies of finite variation as without efficient friction strategies of infinite variation can make sense.

As an auxiliary result, which is of independent interest, we first show that (NUPBR) for simple long-only strategies implies the existence of a *semimartingale price system*, i.e., a semimartingale taking values between the bid and the ask price process (see Theorem 3.2.7). It is worth emphasizing that for this neither the bid nor the ask price process need to be semimartingales. Hence, assuming the existence of semimartingale price system is not really a restriction.

Since we deal with strategies that can be of infinite variation, we cannot directly use them as integrators. But we overcome this difficulty by using the semimartingale price system. Namely, to define the self-financing condition, see Section 3.3, we start out with bounded and predictable strategies specifying the amount of the risky asset. Using that these strategies are integrable with regard to any semimartingale, we take the semimartingale price system as an integrator to calculate the trading gains. Subsequently, as the semimartingale price system is more favorable than the original market with proportional transaction costs, we subtract a cost term to adjust for the difference. Roughly speaking, if the spread is away from zero the costs are a Riemann-Stieltjes integral similar to Guasoni et al. [36]. Then, we exhaust the costs when the spread is away from zero. These costs are always non-negative but can be infinite. As the trading gains charged in the semimartingale are finite, infinite costs cannot be compensated. After also subtracting the current stock position evaluated in the semimartingale price system, we end up with the corresponding self-financing position in the bond. Under a mild additional assumption on the behavior of the bid-ask spread at zero, our approach leads to a well-founded self-financing condition which we justify by suitable approximations with simple strategies (see Theorem 3.3.19). Especially, the self-financing condition does not depend on the semimartingale price system used in its construction.

Finally, we extend the self-financing condition from the set of bounded and predictable strategies to a maximal set of strategies for which it can be defined in a reasonable way. Roughly speaking, we show that this set consists of all predictable strategies with the following property: there exists an approximating sequence of bounded predictable strategies such that for some semimartingale price system the associate sequence of wealth processes is Cauchy w.r.t. uniform convergence in probability, *and* the approximation is better than all other pointwise approximations if the stock position is evaluated in the same semimartingale. The latter is needed as different approximations may lead to different costs. In the special case of a frictionless financial market, this maximal set equals the set of predictable processes which are integrable w.r.t. the semimartingale price process in the usual sense. Hence, we also obtain a new view on the frictionless case.

In a nutshell, we define a reasonable set of general strategies in a model beyond efficient friction by using a semimartingale price system. The existence of the latter is assumed, but not really a restriction as we show its existence under (NUPBR) for simple long-only strategies. We emphasize that we do *not* show a FTAP. However, as already discussed above, the need for general strategies is already proven in the special case of frictionless markets. Consequently, under transaction costs but beyond efficient friction general strategies can become an important tool to guarantee the existence of a CPS.

The idea to relate trading under transaction costs to a fictitious frictionless market is not new. In the theory of optimal portfolio selection, a *shadow price* is a frictionless pricing systems taking values within the bid-ask spread that leads to the same optimal decisions and trading gains as under transaction costs. This goes back to Cvitanić and Karatzas [15] who were the first to apply convex duality to optimization problems under transaction costs. In an Itô-process model, they show that if the dual optimizer exists and is a local martingal, then it is a shadow price. Kabanov [50, Section 4] shows a analogous duality result in a semimartingale currency model. To cite Cvitanić and Karatzas [15, Remark 6.1], the assumption that the dual optimizer is attained as a local martingale is a big one. Czichowsky and Schachermayer [17] develop a general duality theory under transaction costs. They show the existence of a dual optimizer and a shadow price in an appropiate general sense. In particular, they have to allow for double jumps. Building on these general results, Czichowsky et al. [19] show that the theory simplifies for continuous price processes. Specifically, they show that if the (ask) price process satisfies (NUPBR), a shadow price exists. Czichowsky and Schachermayer [18] provide a duality result for continuous and sticky processes, which includes, e.g., (geometric) fractional Brownian motions, for utility functions on the whole real line that are bounded from above. Subsequently, Czichowsky et al. [20] show that for utility functions defined on the positive real line the condition of *two-way crossing*, i.e., whenever the price process moves, it crosses its current level infinitely often over arbitrary small intervals, is sufficient for the existence of a shadow price. In particular, their result implies the existence of a shadow price in the fractional Black-Scholes model for all utility functions on the positive half-line.

1.4 Overview of the thesis

Each of the chapters is self-contained and introduced separately. Hereby, we knowingly allowed for redundancies. The thesis is structured as follows.

Chapter 2 corresponds to the article Kühn and Molitor [66]. The new *prospective strict-no arbitrage* condition is introduced in Section 2.2. Here, we also present the main results of the chapter (Theorem 2.2.6 and Theorem 2.2.11). The proofs of the main results are collected in Section 2.3. At last, in Section 2.4, we give various examples to illustrate the difference between the various no-arbitrage conditions in discrete time and highlight the effect of a cascade of *approximate hedges*.

Chapter 3 is the preprint Kühn and Molitor [67]. After an introduction to the chapter, we show, in Section 3.2, that under *no unbounded profit with bounded risk* for simple long-only strategies there exists a semimartingale price system. In Section 3.3, we show how to use a semimartingale price system to define the self-financing condition of the model. At first, this is done for bounded and predictable processes and we give a characterization (Theorem 3.3.19). Subsequently, in Section 3.4, we extend the self-financing condition to a maximal set for which it can be defined in a reasonable way. Finally, the technical proofs of the chapter are collected in Section 3.5 and Section 3.6

Chapter 4 provides an detailed German summary of the main results of this thesis.

Chapter 2

Prospective strict no-arbitrage and the fundamental theorem of asset pricing under transaction costs

2.1 Introduction

In frictionless finite discrete-time financial market models, the absence of arbitrage opportunities is equivalent to the existence of an equivalent probability measure under which the discounted price processes are martingales. This result is called the fundamental theorem of asset pricing (FTAP). In the case of a finite probability space, it goes back to the work of Harrison and Pliska [41]. The extension to arbitrary probability spaces is known as the Dalang-Morton-Willinger Theorem [21], whose original proof was subsequently refined by several authors, see, e.g., [53, 83]. In the later proofs, the implication that, in frictionless markets, the absence of arbitrage opportunities implies that the set of hedgeable claims attainable from zero endowment is closed in probability is identified as the key lemma.

For a finite probability space, Kabanov and Stricker [54] extend the FTAP of Harrison and Pliska to models with proportional transaction costs. They consider a general "currency model" with finitely many currencies (assets), which we also follow in the current chapter. It allows to buy any asset by paying with any other asset. In this general framework, there need not exist an asset which can play the role of a bank account, i.e., an asset which can be involved in every transaction at minimal costs. Kabanov and Stricker show that no-arbitrage (NA) is equivalent to the existence of a so-called consistent price system (CPS), which is a multidimensional martingale under the objective probability measure taking values within the dual of the cone of solvent portfolios at each point in time. For infinite probability spaces, this equivalence fails: Schachermayer [85] provides an example for an arbitrage-free market which allows for an approximate arbitrage, i.e., a non-zero and non-negative portfolio which is the limit in probability of a sequence of portfolios attainable from zero endowment, and consequently a CPS cannot exist (see Example 3.1 therein). There arises the obvious question under which stronger no-arbitrage conditions the existence of a CPS can be guaranteed. Schachermayer [85] introduces the concept of robust no-arbitrage (NA^{r}) – a no-arbitrage condition which is robust with respect to small changes in the bid-ask spreads. Loosely speaking, if the bid-ask spread (of a pair of assets) does not vanish, there have to exist more favorable bid-ask prices, leading to a smaller spread, such that the modified market still satisfies (NA). Schachermayer shows that (NA^{r}) implies that the set of hedgeable claims attainable from zero endowment, in the following denoted by $\mathcal{A}$, is closed in probability, and (NA^{r}) is equivalent to the existence of a strictly consistent price system (SCPS), that is a martingale taking values within the relative interior of the dual of the cone of solvent portfolios at each point in time.

An alternative condition is the strict no-arbitrage (NA^{s}) property introduced by Kabanov et al. [51]. Loosely speaking, a market model satisfies (NA^{s}) iff any claim which is attainable from zero endowment up to some intermediate time t and which can be liquidated in t for sure, can also be attained from zero endowment by trading at time t only. (NA^{s}) alone does not imply the existence of a CPS (see Example 3.3 in [85] for the existence of an approximate arbitrage under (NA^{s})), but together with the Penner-condition (2.4.1), due to I. Penner [73], this implication holds (see Theorem 2 of Kabanov et al. [52]). Loosely speaking, the Penner-condition postulates that any "free-round-trip" of exchanging assets that can be carried out in the next period for sure – given the information of the current period – can already be carried out in the current period. Together with (NA^{s}), it allows to show that the so-called null-strategies, i.e., the increments of self-financing portfolio processes with vanishing terminal value, form a linear space. This is also a crucial argument in [85] to show closedness of $\mathcal{A}$, which is the main step to show the existence of a CPS. Indeed, it is shown by Rokhlin [82] that the vector space property of null-strategies is equivalent to (NA^{r}).

A different approach to study the occurrence of an approximate arbitrage is followed in Jacka et al. [44]. They provide a necessary and sufficient condition for $\mathcal{A}$ to be closed in probability and construct adjusted trading prices such that the corresponding cone of hedgeable claims attainable from zero endowment either contains an arbitrage or equals the closure of $\mathcal{A}$. Put differently, they postulate the (weak) no-arbitrage condition for an adjusted trading model instead of postulating a stronger no-arbitrage condition for the original one (cf. also Remark 2.4.4 below).

Furthermore, it is important to note that the closedness of $\mathcal{A}$ is not necessary for the existence of a CPS. In the case of only two assets (e.g., a bank account and one risky stock), it is shown by Grigoriev [34] that (NA) already implies the existence of a CPS – although $\mathcal{A}$ need not be closed (see Example 1.3 in [34] and Proposition 3.5 in Lépinette and Zhao [69] for the non-closedness of the set of attainable liquidation values). This means that already in dimension two, additional conditions are required to guarantee that the set of attainable liquidation values is closed. For this, Lépinette and Zhao [69] provide an intuitive and easy to verify condition (Condition E) that takes the postponing of trades into consideration. Their proof uses the existence of a CPS that is guaranteed by Grigoriev [34] in the case of an arbitrage-free model with two assets. On the other hand, already for three assets, there is a counterexample showing that (NA) does not imply the existence of a CPS (see Example 4.6 in [65]). The goal of the current chapter is twofold:

- We want to provide an (easy to interpret) no-arbitrage condition which is as weak as possible and under which the set $\mathcal{A}$ of terminal portfolios attainable from zero endowment is closed.
- We want to establish a FTAP with CPSs which are not necessarily strict as in the FTAP of Schachermayer [85].

For this, we introduce a variant of (NA^{s}), that we call *prospective strict no-arbitrage* (NA^{ps}) and that turns out to be sufficient to guarantee that $\mathcal{A}$ is closed in probability (see Theorem 2.2.6). We say that the market model satisfies (NA^{ps}) iff any claim which is attainable from zero endowment by trading up to some time t and which can *subsequently* be liquidated for sure, can also be attained from zero endowment in the *subsequent* periods (here, "subsequent" is not understood in a strict sense). This means that in contrast to the (NA^{s}) criterion, we do not distinguish between a trade that can be realized at time t and a trade from which we know at time t for sure that it can be realized in the future. In the special case of efficient friction (i.e., positive bid-ask spreads, cf. Definition 2.2.5), (NA^{ps}) and (NA^{s}) are equivalent (see Proposition 2.2.21).

In our proofs, we cannot rely on the vector space property of the null-strategies, which was central in the arguments of Schachermayer and Kabanov et al. Indeed, it was shown by Rokhlin [82] that this property is equivalent to (NA^{r}), which is strictly stronger than (NA^{ps}). Our proof relies on a decomposition of the trading possibilities in "reversible" and "purely non-reversible" transactions at each point in time, where we call a transaction "reversible" if the resulting portfolio can be liquidated in the later periods for sure. This decomposition can be seen as a non-linear, only positively homogeneous generalization of the projection on the set of null-strategies that is used in the case that the null-strategies form a vector space. Given a trading strategy, we then consider only the "purely non-reversible" part at each point in time and postpone the "reversible" part to later points in time, where more information is available. This is possible by (NA^{ps}), and, as it turns out, sufficient to assert that $\mathcal{A}$ is closed in probability. Consequently, (NA^{ps}) implies the existence of a CPS.

On the other hand, as described above, a CPS can exist although the set $\mathcal{A}$ is not closed. Consequently, the existence of a CPS cannot be equivalent to (NA^{ps}). But, for a weak version of (NA^{ps}), called *weak prospective strict no-arbitrage* (NA^{wps}), we have equivalence to the existence of a CPS (see Theorem 2.2.11). A market satisfies (NA^{wps}) iff there exists an at least as favorable market which satisfies (NA^{ps}). Since the second market need not be strictly more favorable than the original one, (NA^{ps}) implies (NA^{wps}). Hence, we establish a FTAP, which complements those of Schachermayer [85] and Kabanov et al. [51, 52]. The main difference is that the resulting CPS may lie on the relative boundary of the bid-ask-spread. In Section 2.4, Figure 2.1 illustrates a very simple example for this. Alternatively, one may think of an actually frictionless market with one risky stock that is written as a model with efficient friction in the following way. Each point in time is split into two points. Under the same information, at the first point, the investor can only buy, and at the second point she can only sell arbitrary quantities of the stock. If the frictionless market satisfies (NA), the artificial market with friction has a CPS but not a SCPS. Thus, at least from a conceptual point of view, it is desirable to have a FTAP with arbitrary CPSs as well.

In the case of a finite probability space, ($\mathrm{NA}^{\mathrm{wps}}$) is equivalent to (NA), which means that our version of the FTAP can be seen as a generalization of the above mentioned FTAP by Kabanov and Stricker [54] (see part 2 of Theorem 1 therein) to the case of arbitrary probability spaces. Finally, we motivate the ($\mathrm{NA}^{\mathrm{wps}}$) condition by an example which shows that ($\mathrm{NA}^{\mathrm{wps}}$) cannot be replaced by a further weakening of the ($\mathrm{NA}^{\mathrm{ps}}$) condition (see Example 2.4.3).

The remainder of the chapter is organized as follows. In Section 2.2, we introduce the framework of financial modeling, the prospective strict no-arbitrage condition, and the weak prospective strict no-arbitrage condition. We relate these properties to the robust no-arbitrage and the strict no-arbitrage condition and state the main results of the chapter (Theorem 2.2.6 and Theorem 2.2.11). The proofs can be found in Section 2.3. In Section 2.4, there are two very simple examples that illustrate the differences between the above mentioned no-arbitrage conditions and a more sophisticated example (Example 2.4.3) that shows the effect of a possible "cascade" of approximate hedges.

2.2 Prospective strict no-arbitrage and consistent price systems

We now introduce the market model and the relevant notation. We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a discrete-time filtration $(\mathcal{F}_t)_{t=0}^T$, $T \in \mathbb{N}$, such that $\mathcal{F}_T = \mathcal{F}$. The space (of equivalence classes) of $\mathcal{F}_t$-measurable d-dimensional random vectors is denoted by $L^0(\mathbb{R}^d, \mathcal{F}_t)$. For a set-valued mapping $\omega \mapsto N(\omega) \subseteq \mathbb{R}^d$, we denote by $L^0(N, \mathcal{F}_t) := \{v \in L^0(\mathbb{R}^d, \mathcal{F}_t) : v(\omega) \in N(\omega) \text{ for a.e. } \omega \in \Omega\}$ the set of $\mathcal{F}_t$-measurable selectors of N. As usual, the spaces are equipped with the topology of the convergence in probability, and we write $L^0(N) := L^0(N, \mathcal{F}_T)$.

We work with the market model with proportional transaction costs from Schachermayer [85], where the reader may find a discussion about its economical meaning and its connection to the models of [51] and [54]. There are $d \in \mathbb{N}$ traded assets (one of which may, but need not, be a money market account), and a $d \times d$-matrix $\Pi = (\pi^{ij})_{1 \le i,j \le d}$ is called *bid-ask matrix* if

(i) $0 < \pi^{ij} < \infty$, for $1 \le i, j \le d$,

(ii) $\pi^{ii} = 1$, for $1 \le i \le d$,

(iii) $\pi^{ij} \leq \pi^{ik}\pi^{kj}$, for $1 \leq i, j, k \leq d$.

The terms of trade of the d assets are specified by a *bid-ask process* $(\Pi_t)_{t=0}^T$, i.e., an adapted $d \times d$–matrix-valued process such that for each $\omega \in \Omega$ and $t \in \{0, \dots, T\}$, $\Pi_t(\omega)$ is a bid-ask matrix. For each $t \in \{0, \dots, T\}$, the random matrix $\Pi_t = (\pi_t^{ij})_{1 \leq i,j \leq d}$ specifies the exchanges available to the investor at time t. More precisely, the entry π_t^{ij} denotes the number of units of asset i for which an agent can buy one unit of asset j at time t. Therefore, the *set of portfolios attainable at zero endowment at time* t, which, in this context, consists of $\mathcal{F}_t$-measurable $\mathbb{R}^d$-valued random variables, is modeled by the convex cone

$$\left\{ \sum_{1 \leq i,j \leq d} \lambda^{ij}(e^j - \pi_t^{ij} e^i) - r : (\lambda^{ij})_{1 \leq i,j \leq d} \in L^0(\mathbb{R}_+^{d \times d}, \mathcal{F}_t),\ r \in L^0(\mathbb{R}_+^d, \mathcal{F}_t) \right\}, \quad (2.2.1)$$

where e^i denotes the i-th unit vector of $\mathbb{R}^d$. This means that each portfolio is the result of an order $\lambda = (\lambda^{ij})_{1 \leq i,j \leq d} \in L^0(\mathbb{R}_+^{d \times d}, \mathcal{F}_t)$, where λ^{ij} denotes the units of assets j ordered in exchange for asset i, and some non-negative amount $r \in L^0(\mathbb{R}_+^d, \mathcal{F}_t)$, which corresponds to the decision of the investor to "throw away" some non-negative physical quantities of each asset. Next, we define for each $\omega \in \Omega$ the polyhedral cone

$$K\left(\Pi_t\left(\omega\right)\right) := \operatorname{cone}\left(\left\{\pi_t^{ij}(\omega)e^i - e^j\right\}_{1 \leq i,j \leq d}, \left\{e^i\right\}_{1 \leq i \leq d}\right),$$

which we abbreviate as $K_t(\omega) := K(\Pi_t(\omega))$. In Lemma 2.3.1 below, we briefly verify the intuitively obvious fact that the set given in (2.2.1) coincides with the set $L^0(-K_t, \mathcal{F}_t)$ of $\mathcal{F}_t$-measurable selectors of the set-valued mapping $\omega \mapsto -K_t(\omega)$. We use this equality **throughout the chapter** and refer to $L^0(-K_t, \mathcal{F}_t)$ as the set of portfolios attainable from zero endowment at time t.

Definition 2.2.1. An $\mathbb{R}^d$-valued adapted process $\vartheta = (\vartheta_t)_{t=0}^T$ is called *self-financing portfolio process* for the bid-ask process $(\Pi_t)_{t=0}^T$ if

$$\vartheta_t - \vartheta_{t-1} \in L^0(-K_t, \mathcal{F}_t) \quad \text{for all } t = 0, \dots, T, \quad (2.2.2)$$

where $\vartheta_{-1} := 0$. Consequently, for each pair (s, t) with $s, t \in \{0, \dots, T\}$ and $s \leq t$, the convex cone of *hedgeable claims attainable from zero endowment between* s *and* t is denoted by $\mathcal{A}_s^t$ and is defined to be

$$\mathcal{A}_s^t := \sum_{k=s}^{t} L^0(-K_k, \mathcal{F}_k).$$

For an alternative bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$, the corresponding set is denoted by $\widetilde{\mathcal{A}}_s^t$, where $-\widetilde{K}_t(\omega) := -K(\widetilde{\Pi}_t(\omega))$ for all $\omega \in \Omega$ and $t = 0, \dots, T$.

The primary object of interest in this chapter is the cone $\mathcal{A}_0^T$ of hedgeable claims attainable from zero endowment between 0 and T. However, we still need the following auxiliary notions. The convex cone $L^0(K_t, \mathcal{F}_t)$ is called the *set of solvent portfolios at time t* and the (polyhedral) cone $K_t(\omega)$ is called the *solvency cone* corresponding to the bid-ask matrix $\Pi_t(\omega)$. Indeed, for each portfolio $v \in L^0(K_t, \mathcal{F}_t)$ the portfolio $-v \in L^0(-K_t, \mathcal{F}_t)$ is attainable at price zero, thus the portfolio v can be liquidated to zero and, consequently, is solvent. Similarly, let $K_t^0(\omega) := K_t(\omega) \cap -K_t(\omega)$ for each $\omega \in \Omega$, then $L^0(K_t^0, \mathcal{F}_t)$ denotes the space of portfolios, which are attainable at zero endowment and are also solvent, i.e., can be liquidated to the zero portfolio.

Before we introduce our new no-arbitrage condition, we recall the concepts of no-arbitrage from the literature; compare to [52] and [85].

Definition 2.2.2.

(i) The bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the *no-arbitrage property (NA)* if

$$\mathcal{A}_0^T \cap L^0(\mathbb{R}_+^d) = \{0\}. \tag{2.2.3}$$

(ii) The bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the *strict no-arbitrage property (NA^s)* if

$$\mathcal{A}_0^t \cap L^0(K_t, \mathcal{F}_t) \subseteq L^0(K_t^0, \mathcal{F}_t) \quad \text{for all } t = 0, \dots, T. \tag{2.2.4}$$

(iii) The bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the *robust no-arbitrage condition (NA^r)* if there is a bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ with smaller bid-ask spreads in the sense that the spread $[1/\widetilde{\pi}_t^{ji}(\omega), \widetilde{\pi}_t^{ij}(\omega)]$ is contained in the relative interior of $[1/\pi_t^{ji}(\omega), \pi_t^{ij}(\omega)]$ for all $1 \leq i, j \leq d$, $t \in \{0, \dots, T\}$ and almost all $\omega \in \Omega$, such that $(\widetilde{\Pi}_t)_{t=0}^T$ satisfies the no-arbitrage condition (NA).

We just note that in the case of vanishing bid-ask spreads, the choice of $\widetilde{\Pi} = \Pi$ has smaller bid-ask spreads than Π, i.e., frictionless markets are not excluded in Definition 2.2.2 (iii).

It is well known that although each of the cones $L^0(-K_t, \mathcal{F}_t)$ is closed with regard to the convergence in probability, the cone $\mathcal{A}_0^T$ may fail to be closed. As already mentioned, neither (NA) nor (NA$^{\mathrm{s}}$) are strong enough to guarantee that $\mathcal{A}_0^T$ is closed (see Examples 3.1 and 3.3 in [85]). This is in contrast to the frictionless case, where (NA) is sufficient (see, e.g., Theorem 6.9.2 in [24]). In the present context Schachermayer [85] showed that the robust no-arbitrage condition (NA$^{\mathrm{r}}$) is strong enough to assure that $\mathcal{A}_0^T$ is closed. We now introduce a slight weakening of (NA$^{\mathrm{r}}$) called prospective strict no-arbitrage (NA$^{\mathrm{ps}}$), which is still sufficient to guarantee that $\mathcal{A}_0^T$ is closed.

Definition 2.2.3. The bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the *prospective strict no-arbitrage property (NA^{ps})* if

$$\mathcal{A}_0^t \cap (-\mathcal{A}_t^T) \subseteq \mathcal{A}_t^T \quad \text{for all } t = 0, \dots, T.$$

Remark 2.2.4. *The (NA^{ps}) property has the following interpretation: any claim $v \in \mathcal{A}_0^t$ attained by trading up to time t which can be reduced to the zero portfolio in t or in the subsequent periods, i.e., $-v \in \mathcal{A}_t^T$, has to be attainable by trading between t and T only, i.e., $v \in \mathcal{A}_t^T$. It is a variant of the (NA^s) condition, that postulates that any claim $v \in \mathcal{A}_0^t$ which can be liquidated at time t, i.e., $-v \in \mathcal{A}_t^t$, has to be attainable at time t as well, i.e., $v \in \mathcal{A}_t^t$. The only difference is that we do not distinguish between a trade at time t and a trade from which one knows for sure at time t that it can be realized in the future.*

Put differently, for every t, we review the trading up to time t. Either one does not gain advantage from the trading since the same terminal position can be achieved for sure by starting to trade at t. Or, one takes some risk by the trading up to time t since the position cannot be liquidated for sure in the future.

Definition 2.2.5. The bid-ask process $(\Pi_t)_{t=0}^T$ satisfies *efficient friction (EF)* if

$$K_t^0(\omega) := K_t(\omega) \cap \big(- K_t(\omega) \big) = \{0\} \text{ for all } t = 0, \dots, T \text{ and } \omega \in \Omega.$$

The bid-ask process $(\Pi_t)_{t=0}^T$ satisfies efficient friction if and only if $\pi_t^{ij}(\omega)\pi_t^{ji}(\omega) > 1$ for all $1 \le i \ne j \le d$, $t = 0, \dots, T$ and $\omega \in \Omega$ (see Proposition 2.2.20). Under efficient friction, the conditions (NA$^{\mathrm{ps}}$) and (NA$^{\mathrm{s}}$) coincide (see Proposition 2.2.21). We can already formulate the first main result of the chapter:

Theorem 2.2.6. *If the bid-ask process $(\Pi_t)_{t=0}^T$ has the prospective strict no-arbitrage property (NA^{ps}), then the convex cone $\mathcal{A}_0^T$ is closed with regard to the convergence in probability.*

The theorem above has obvious consequences for the existence of dual variables. For a given bid-ask matrix Π, the (positive) dual cone $K^\star$ of the solvency cone $K = K(\Pi)$ is defined by $K^\star := \{w \in \mathbb{R}^d : \langle v, w\rangle \ge 0 \text{ for all } v \in K\}$. For the bid-ask process $(\Pi_t)_{t=0}^T$, this induces the set-valued process $(K_t^\star)_{t=0}^T$ of dual cones. We can now define the notion of *consistent price systems*, which is dual to the notion of self-financing portfolio process and plays a similar role as the notion of an equivalent martingale measure in the frictionless theory. Once again, we refer to [85] for a detailed discussion of the economical interpretation.

Definition 2.2.7. An adapted $\mathbb{R}_+^d$-valued process $Z = (Z_t)_{t=0}^T$ is called a *consistent price system (CPS)* for the bid-ask process $(\Pi_t)_{t=0}^T$ if Z is a martingale under $\mathbb{P}$ and $Z_t \in L^0(K_t^\star \setminus \{0\}, \mathcal{F}_t)$, i.e., $Z_t(\omega) \in K_t^\star(\omega) \setminus \{0\}$ for a.e. $\omega \in \Omega$ and each $t \in \{0, \dots, T\}$.

We have the following consequence of Theorem 2.2.6.

Corollary 2.2.8. *If the bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the prospective strict no-arbitrage condition (NA^{ps}), then it admits a consistent price system (CPS). More generally, for any given strictly positive $\mathcal{F}_T$-measurable function $\varphi : \Omega \to (0,1]$, there is a CPS $Z = (Z_t)_{t=0}^T$ with $\|Z_T\| \le M\varphi$ a.s. for some $M \in \mathbb{R}_+ \setminus \{0\}$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.*

Remark 2.2.9. *An abstract version of (NA^{ps}) reads: If a strategy up to time t can be extented to a strategy without losses at T, then any other extension beyond t can be dominated at T by a strategy that does not trade before t.*

This scheme can be formalized in a quite canonical way in diverse market models including, e.g., capital gains taxes, uncertainty about the execution of limit orders, or dividend paying assets, where the basic problem from Example 3.1 in Schachermayer [85], can also occur (see, e.g., Example 4.5 in [65]). The arguments of our proofs may be adapted to these models to show that the set of attainable terminal portfolios is closed.

For example, in the context of optimal investment problems with utility functions on the positive real line, this means, roughly speaking, that the set $\mathcal{C}$ of non-negative random variables dominated by the liquidation value of an attainable portfolio (with a given initial endowment) is also closed in probability. Hence, defining the set of dual variables $\mathcal{D}$ as the polar set of $\mathcal{C}$, the abstract versions of the duality results in Kramkov and Schachermayer [63, Theorem 3.1 and 3.2] may also be applied to these models.

The converse of Corollary 2.2.8 fails to be true. More generally, by [55, Section 3.2.4, Example 1], there cannot exist a no-arbitrage criterion that both guarantees closedness of $\mathcal{A}_0^T$ and that is equivalent to the existence of a CPS, cf. also the discussion in Remark 2.2.14. We can, however, establish an equivalence if we pass from (NA$^{\mathrm{ps}}$) to a weaker notion of prospective strict no-arbitrage.

Definition 2.2.10. The bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the *weak prospective strict no-arbitrage property (NA^{wps})* if there is a bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ with $\widetilde{\Pi}_t \leq \Pi_t$ a.s. for all $t = 0, \dots, T$, such that $(\widetilde{\Pi}_t)_{t=0}^T$ satisfies the prospective strict no-arbitrage condition (NA$^{\mathrm{ps}}$).

The (NA$^{\mathrm{wps}}$) condition is obviously a weakening of the (NA$^{\mathrm{ps}}$) condition since the bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ in Definition 2.2.10 need not be strictly more favorable than $(\Pi_t)_{t=0}^T$. The difference between the two conditions is illustrated in Example 2.4.2 below; see also Remark 2.2.14. Our second main result is the following fundamental theorem of asset pricing.

Theorem 2.2.11. *A bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the weak prospective strict no-arbitrage condition (NA^{wps}) if and only if it admits a consistent price system (CPS).*

Remark 2.2.12. *Theorem 2.2.11 extends part 2 of Theorem 1 in Kabanov and Stricker [54] to the case of infinite probability spaces. Combining these two theorems, it can be seen that (NA^{wps}) possesses the nice property that it is equivalent to (NA) if $|\Omega| < \infty$.*

Remark 2.2.13. *In addition, (NA) and (NA^{wps}) coincide in the case of only two assets on arbitrary probability spaces, which follows from the equivalence of (NA) and the existence of a CPS, derived by Grigoriev [34].*

Remark 2.2.14. *In the following discussion, we identify an "absence of arbitrage" criterion $\mathcal{C}$ with the set of bid-ask processes which satisfy the criterion and call it monotone if for all bid-ask processes $\widetilde{\Pi} \leq \Pi$, $\widetilde{\Pi} \in \mathcal{C}$ implies that $\Pi \in \mathcal{C}$. Monotonicity is obviously satisfied by the simple (NA) condition. The more sophisticated criteria (NA^s), (NA^r), and (NA^{ps}) are in general only monotone if bid-ask matrices without efficient friction are excluded from the consideration, i.e., $\pi^{ij}\pi^{ji} \geq \widetilde{\pi}^{ij}\widetilde{\pi}^{ji} > 1$ for all $i \neq j$ (cf. Proposition 2.2.20). On the one hand, the equivalence to the existence of a CPS can only hold for a monotone criterion. It follows directly from Definition 2.2.7 that a CPS for $\widetilde{\Pi}$ is also a CPS for Π with $\Pi \geq \widetilde{\Pi}$. On the other hand, the closedness of the set of attainable portfolios does not transfer to a market with a less favorable bid-ask process (see, e.g., Examples [55, Section 3.2.4, Example 1] and [44, Example 2.1]). Thus, to guarantee closedness, e.g., in the context of optimal investment problems, the limitation to monotone criteria would be unnecessarily restrictive.*

The (NA^{wps}) criterion can be characterized as the "strongest monotone criterion which is weaker than (NA^{ps})", i.e., it follows directly from Definition 2.2.10 that

$$(NA^{wps}) = \bigcap_{(NA^{ps}) \subseteq \mathcal{C},\ \mathcal{C} \text{ is monotone}} \mathcal{C}. \tag{2.2.5}$$

In the special case of a frictionless market, the criteria (NA^{ps}) and (NA^{wps}) coincide (see Proposition 2.2.18).

We stress that the picture cannot be as clear-cut as in the frictionless case. In discrete-time frictionless markets, (NA) already implies closedness (see Lemma 2.1 in Schachermayer [83]). In continuous-time frictionless markets, Delbaen and Schachermayer [22] derived closedness in the appropriate topology under the economic meaningful assumption of "no free lunch with vanishing risk" (NFLVR), that is also necessary for the existence of an equivalent martingale measure. Under transaction costs, the FTAP of Delbaen and Schachermayer [22] cannot hold. Namely, Example 3.1 in Schachermayer [85] satisfies (NFLVR) defined for multivariate portfolio processes, i.e., we have $\overline{\mathcal{A}_0^T}^{\infty} \cap L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d_+) = \{0\}$, where $\overline{\mathcal{A}_0^T}^{\infty}$ denotes the closure of $\mathcal{A}_0^T \cap L^\infty(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^d)$ w.r.t. the topology of uniform convergence, but a CPS does nevertheless not exist.

Definition 2.2.15. An element $v \in L^0(\mathbb{R}^d_+)$ with $v \neq 0$ is called an *approximate arbitrage (in probability)* if there is a sequence $(v^n)_{n\in\mathbb{N}} \subseteq \mathcal{A}_0^T$ such that $v^n \to v$ in probability.

In general, (NA) does not guarantee the absence of an approximate arbitrage, i.e., $\overline{\mathcal{A}_0^T} \cap L^0(\mathbb{R}^d_+) = \{0\}$. On the other hand, even though the (NA$^{\mathrm{wps}}$) property is not sufficient to assure that $\mathcal{A}_0^T$ is closed in probability, we have the following easy consequence of Theorem 2.2.11.

Corollary 2.2.16. *If a bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the weak prospective strict no-arbitrage condition (NA^{wps}), then we have $\overline{\mathcal{A}_0^T} \cap L^0(\mathbb{R}^d_+) = \{0\}$.*

The short proof is also deferred to Section 2.3.

Remark 2.2.17. *(NA^{wps}) postulates the existence of a bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ such that $\widetilde{\Pi}_t \le \Pi_t$ a.s. for all $t = 0, \dots, T$ and*

$$\widetilde{\mathcal{A}}_0^t \cap \left(-\widetilde{\mathcal{A}}_t^T\right) \subseteq \widetilde{\mathcal{A}}_t^T \quad \text{for all } t = 0, \dots, T. \tag{2.2.6}$$

One may ask if one can replace this condition with the following slightly weaker condition: there exists a bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ satisfying (NA), such that $\widetilde{\Pi}_t \le \Pi_t$ a.s. for all $t = 0, \dots, T$ and

$$\mathcal{A}_0^t \cap \left(-\widetilde{\mathcal{A}}_t^T\right) \subseteq \widetilde{\mathcal{A}}_t^T \quad \text{for all } t = 0, \dots, T \tag{2.2.7}$$

Indeed, by Proposition 2.2.18, (2.2.6) implies that $\widetilde{\Pi}$ satisfies (NA), which means that the second condition is a weakening of the first one. In condition (2.2.7), the position at time t is achieved by trading in the original market, only its "evaluation" is made in the more favorable market model $\widetilde{\Pi}$. But, maybe surprisingly, it turns out that (2.2.7) does not exclude the existence of an approximate arbitrage and thus a CPS need not exist (see Example 2.4.3 below).

Proposition 2.2.18. *We have the following implications*

$$(NA^r) \Rightarrow (NA^{ps}) \Rightarrow (NA^{wps}) \Rightarrow (NA). \tag{2.2.8}$$

Remark 2.2.19. *All implications in (2.2.8) are strict (see Examples 2.4.1 and 2.4.2 below; for (NA) $\not\Rightarrow$ (NA^{wps}), consider an arbitrage-free model with an approximate arbitrage, Example 3.1 in [85], and apply Corollary 2.2.16).*

Proof of Proposition 2.2.18. Ad ($\mathrm{NA^r}$) $\Rightarrow$ ($\mathrm{NA^{ps}}$). Assume that the bid-ask process $(\Pi_t)_{t=0}^T$ satisfies ($\mathrm{NA^r}$) and let $v \in \mathcal{A}_0^t$ such that $-v \in \mathcal{A}_t^T$. We have to show $v \in \mathcal{A}_t^T$. According to our assumption, we have $v = \sum_{s=0}^t \widetilde{\xi}_s$ with $\widetilde{\xi}_s \in L^0(-K_s, \mathcal{F}_s)$ for $s = 0, \dots, t$ and $-v = \sum_{s=t}^T \widehat{\xi}_s$ with $\widehat{\xi}_s \in L^0(-K_s, \mathcal{F}_s)$ for $s = t, \dots, T$. Hence, we define $\xi_s \in L^0(-K_s, \mathcal{F}_s)$ by

$$\xi_s := \begin{cases} \widetilde{\xi}_s, & s < t, \\ \widetilde{\xi}_s + \widehat{\xi}_s, & s = t, \\ \widehat{\xi}_s, & s > t, \end{cases}$$

and notice that $\sum_{s=0}^T \xi_s = v - v = 0$. From Lemma 3.2.12 in [55], it follows that $\xi_s \in L^0(K_s^0, \mathcal{F}_s)$ for all $s = 0, \dots, T$. In particular, we have $\widehat{\xi}_s \in L^0(K_s, \mathcal{F}_s)$ for $s > t$. In addition, we have $\widehat{\xi}_t = -\widetilde{\xi}_t + \xi_t \in L^0(K_t, \mathcal{F}_t) + L^0(K_t, \mathcal{F}_t) = L^0(K_t, \mathcal{F}_t)$. This implies $v = \sum_{s=t}^T (-\widehat{\xi}_s) \in \mathcal{A}_t^T$, which concludes the proof of the first implication.

Ad ($\mathrm{NA^{ps}}$) $\Rightarrow$ ($\mathrm{NA^{wps}}$). Obvious.

Ad (NA$^{\text{wps}}$) $\Rightarrow$ (NA). Assume that the bid-ask process $(\Pi_t)_{t=0}^T$ satisfies (NA$^{\text{wps}}$), i.e., there exists a bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ with $\widetilde{\Pi}_t \leq \Pi_t$ a.s. for all $t = 0, \dots, T$ and $(\widetilde{\Pi}_t)_{t=0}^T$ satisfies (NA$^{\text{ps}}$). Let $v \in \mathcal{A}_0^T \cap L^0(\mathbb{R}_+^d) \subseteq \widetilde{\mathcal{A}}_0^T \cap L^0(\mathbb{R}_+^d)$. This obviously implies that $-v \in L^0(-\widetilde{K}_T, \mathcal{F}_T)$ and hence, by (NA$^{\text{ps}}$) of $\widetilde{\Pi}$, $v \in L^0(-\widetilde{K}_T, \mathcal{F}_T)$. Together with $(-\widetilde{K}_T(\omega)) \cap \mathbb{R}_+^d = \{0\}$ for each $\omega \in \Omega$, which holds by the properties of a bid-ask matrix, this implies $v = 0$ a.s. Thus, the bid-ask process $(\Pi_t)_{t=0}^T$ satisfies (NA). □

Proposition 2.2.20. *The bid-ask process* $(\Pi_t)_{t=0}^T$ *satisfies efficient friction (EF) if and only if*

$$\pi_t^{ij}(\omega)\pi_t^{ji}(\omega) > 1 \quad \textit{for all } 1 \leq i \neq j \leq d,\ t = 0, \dots, T \textit{ and } \omega \in \Omega.$$

Proof. If $\pi_t^{ij}(\omega)\pi_t^{ji}(\omega) = 1$ holds for some $1 \leq i \neq j \leq d$, $t \in \{0, \dots, T\}$ and $\omega \in \Omega$, we find

$$e^j - \pi_t^{ij}(\omega)e^i + \frac{1}{\pi_t^{ji}(\omega)}\left(e^i - \pi_t^{ji}(\omega)\, e^j\right) = \frac{1}{\pi_t^{ji}(\omega)}\left(1 - \pi_t^{ij}(\omega)\, \pi_t^{ji}(\omega)\right) e^i = 0.$$

This implies $e^j - \pi_t^{ij}(\omega)e^i = -\frac{1}{\pi_t^{ji}(\omega)}(e^i - \pi_t^{ji}(\omega)e^j) \in (-K_t(\omega)) \cap K_t(\omega)$ and thus efficient friction is not satisfied. This shows that efficient friction implies $\pi_t^{ij}(\omega)\pi_t^{ji}(\omega) > 1$ for all $1 \leq i \neq j \leq d$, $t = 0, \dots, T$ and $\omega \in \Omega$.

To show the reverse implication, we assume $\pi_t^{ij}(\omega)\pi_t^{ji}(\omega) > 1$ for all $1 \leq i \neq j \leq d$, $t = 0, \dots, T$, and $\omega \in \Omega$. Suppose that efficient friction does not hold, i.e., for some $t \in \{0, \dots, T\}$ and $\omega \in \Omega$ there is $v \in (-K_t(\omega)) \cap K_t(\omega)$ with $v \neq 0$. By definition of $-K_t(\omega)$, we have $v = \sum_{1\leq i\neq j\leq d} \lambda^{ij}(e^j - \pi_t^{ij}(\omega)e^i) - \sum_{i=1}^d \beta^i e^i$ with $\lambda^{ij}, \beta^i \geq 0$. Let $\pi_t^1(\omega) := (\pi_t^{11}(\omega), \dots, \pi_t^{1d}(\omega))^T$ and note that $\pi_t^1(\omega) \in K_t^\star(\omega)$ by property (iii) of the bid-ask matrix. We have $\langle v, \pi_t^1(\omega)\rangle = 0$ as $v \in (-K_t(\omega)) \cap K_t(\omega)$. Property (iii) of a bid-ask matrix implies $\lambda^{ij}(\pi_t^{1j}(\omega) - \pi_t^{1i}(\omega)\pi_t^{ij}(\omega)) \leq 0$ for all $1 \leq i \neq j \leq d$, which yields $0 = \langle v, \pi_t^1(\omega)\rangle = \sum_{1\leq i\neq j\leq d} \lambda^{ij}(\pi_t^{1j}(\omega) - \pi_t^{1i}(\omega)\pi_t^{ij}(\omega)) - \sum_{i=1}^d \beta^i\pi_t^{1i}(\omega) \leq -\sum_{i=1}^d \beta^i\pi_t^{1i}(\omega)$ and thus $\beta^i = 0$ for all $i = 1, \dots, d$. Therefore, $v \neq 0$ implies $\lambda^{kl} > 0$ for at least one pair $1 \leq k \neq l \leq d$. Applying the same arguments with $\pi_t^l(\omega) := (\pi_t^{l1}(\omega), \dots, \pi_t^{ld}(\omega))^T \in K_t^\star(\omega)$, we get

$$\begin{aligned} 0 = \langle v, \pi_t^l(\omega)\rangle &= \sum_{1\leq i\neq j\leq d} \lambda^{ij}(\pi_t^{lj}(\omega) - \pi_t^{li}(\omega)\pi_t^{ij}(\omega)) \\ &\leq \lambda^{kl}(\pi_t^{ll}(\omega) - \pi_t^{lk}(\omega)\pi_t^{kl}(\omega)) = \lambda^{kl}(1 - \pi_t^{lk}(\omega)\pi_t^{kl}(\omega)) < 0, \end{aligned}$$

which is a contradiction. □

Proposition 2.2.21. *Assume that the bid-ask process* $(\Pi_t)_{t=0}^T$ *satisfies efficient friction (EF). Then, we have the equivalence*

$$(NA^{ps}) \Leftrightarrow (NA^s). \tag{2.2.9}$$

Remark 2.2.22. *In general, (NA^s) is neither necessary nor sufficient for (NA^{ps}). Indeed, (NA^{ps}) $\not\Rightarrow$ (NA^s) is straightforward and (NA^s) $\not\Rightarrow$ (NA^{ps}) follows from Example 3.3 in [85].*

But, it is also well-known that under efficient friction (NA^r) and (NA^s) are equivalent (cf. Theorem 1 in [51] and Theorem 1.7 in [85]). Thus, in this case (NA^r), (NA^s), and (NA^{ps}) coincide.

Proof of Proposition 2.2.21. In view of Proposition 2.2.18 and the preceding remark, it is sufficient to show (NA$^{\mathrm{ps}}$)$\Rightarrow$(NA$^{\mathrm{s}}$). Hence, we assume that (NA$^{\mathrm{ps}}$) holds. Let us show by a backward induction on $t = T, T-1, \ldots, 0$ that $\mathcal{A}_0^t \cap L^0(K_t, \mathcal{F}_t) = \{0\}$. Let $t = T$ and $v \in \mathcal{A}_0^T \cap L^0(K_T, \mathcal{F}_T)$, then (NA$^{\mathrm{ps}}$) implies $v \in L^0(-K_T, \mathcal{F}_T)$, i.e., $v \in L^0(K_T \cap (-K_T), \mathcal{F}_T)$, which, under (EF), is tantamount to $v = 0$ a.s.

For the induction step $t+1 \rightsquigarrow t$, we let $t < T$ and assume $A_0^s \cap L^0(K_s, \mathcal{F}_s) = \{0\}$ for $s = t+1, \ldots, T$. Given $v \in \mathcal{A}_0^t \cap L^0(K_t, \mathcal{F}_t)$, we may write $v = \sum_{s=t}^T \xi_s$ for $\xi_s \in L^0(-K_s, \mathcal{F}_s)$ by (NA$^{\mathrm{ps}}$). Since $-v \in L^0(-K_t, \mathcal{F}_t)$ and $-v + \sum_{s=t}^{T-1} \xi_s = -\xi_T$, we obtain $-v + \sum_{s=t}^{T-1} \xi_s \in \mathcal{A}_0^T \cap L^0(K_T, \mathcal{F}_T)$. Thus, by the induction hypothesis, $-v + \sum_{s=t}^{T-2} \xi_s = -\xi_{T-1}$. Hence, $-v + \sum_{s=t}^{T-2} \xi_s \in \mathcal{A}_0^{T-1} \cap L^0(K_{T-1}, \mathcal{F}_{T-1})$ and, again by the induction hypothesis, $-v + \sum_{s=t}^{T-3} \xi_s = -\xi_{T-2}$. Continuing inductively, we get $-v + \xi_t = 0$, but this means $v \in L^0(K_t \cap (-K_t), \mathcal{F}_t)$ and thus $v = 0$ a.s. by (EF). $\square$

Remark 2.2.23. *With Theorem 2.2.6, the superhedging result in Schachermayer [85] (see Theorem 4.1 therein) and its proof hold one-to-one under the slightly weaker assumption that $(\Pi_t)_{t=0}^T$ satisfies (NA^{ps}) instead of (NA^r) – only without the statement with "strictly consistent price systems" in the brackets.*

2.3 Proofs of the main results

This section is devoted to the proof of Theorem 2.2.6. The other results of Section 2.2 are standard consequences of $\mathcal{A}_0^T$ being closed and thus we mainly refer to the known results in the literature and highlight the minor adjustments. The latter is postponed to the end of the section.

The main hurdle in the proof of Theorem 2.2.6 is that we do not have at hand that the null-strategies, i.e., the elements of $(\xi_0, \ldots, \xi_T) \in L^0(-K_0, \mathcal{F}_0) \times \cdots \times L^0(-K_T, \mathcal{F}_T)$ with $\sum_{t=0}^T \xi_t = 0$ a.s., form a linear space. Namely, it is shown by Rokhlin [82] that the implication

$$\sum_{t=0}^{T} \xi_t = 0 \ a.s. \text{ with } \xi_t \in L^0(-K_t, \mathcal{F}_t) \Rightarrow \xi_t \in L^0(K_t^0, \mathcal{F}_t) \text{ for all } t = 0, \ldots, T \quad (2.3.1)$$

is equivalent to (NA$^{\mathrm{r}}$), which is strictly stronger than (NA$^{\mathrm{ps}}$). Thus, in the following we propose a new proof method which overcomes this hurdle.

Before starting with the main proof, we show that $L^0(-K_t, \mathcal{F}_t)$ coincides with the set given in (2.2.1). This allows us to argue directly with orders $\lambda \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ and vectors $r \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$ instead of the resulting elements of $L^0(-K_t, \mathcal{F}_t)$. In order to ease notation, we define for all $t = 0, \ldots, T$ the mapping

$$L_t : L^0(\mathbb{R}^{d\times d}_+, \mathcal{F}_t) \to L^0(\mathbb{R}^d, \mathcal{F}_t)$$

by

$$L_t(\lambda_t) = \sum_{1\leq i,j\leq d} \lambda_t^{ij}(e^j - \pi_t^{ij} e^i) \quad \text{for all } \lambda_t = (\lambda_t^{ij})_{1\leq i,j\leq d} \in L^0(\mathbb{R}^{d\times d}_+, \mathcal{F}_t).$$

Lemma 2.3.1. *Let $\Pi = (\Pi_t)_{t=0}^T$ denote a bid-ask process. Then, we have*

$$L^0(-K_t, \mathcal{F}_t) = \left\{ L_t(\lambda_t) - r_t : \lambda_t \in L^0(\mathbb{R}^{d\times d}_+, \mathcal{F}_t), r_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \right\} \tag{2.3.2}$$

for all $t = 0, \ldots, T$ and, consequently, we have

$$\mathcal{A}_s^t = \left\{ \sum_{k=s}^{t} L_k(\lambda_k) - r : \lambda_k \in L^0(\mathbb{R}^{d\times d}_+, \mathcal{F}_k),\ k = s, \ldots, t,\ r \in L^0(\mathbb{R}^d_+, \mathcal{F}_t) \right\} \tag{2.3.3}$$

for all $0 \leq s \leq t \leq T$.

Proof. For each $\lambda_t \in L^0(\mathbb{R}^{d\times d}_+, \mathcal{F}_t)$ and $r_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$ the random vector $L_t(\lambda_t) - r_t$ is an element of $L^0(-K_t, \mathcal{F}_t)$. Hence, we only have to show that for each $v \in L^0(-K_t, \mathcal{F}_t)$, we can find $\lambda_t \in L^0(\mathbb{R}^{d\times d}_+, \mathcal{F}_t)$ and $r_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$ such that $v = L_t(\lambda_t) - r_t$ a.s.

For this, let $v \in L^0(-K_t, \mathcal{F}_t)$, i.e., $v(\omega) \in -K_t(\omega)$ for each $\omega \in \Omega \setminus N$, where $N \in \mathcal{F}_t$ is a set of measure zero. Then $\widetilde{v} := \mathbb{1}_{\Omega\setminus N} v \in L^0(-K_t, \mathcal{F}_t)$ satisfies $\widetilde{v} = v$ a.s. and $\widetilde{v}(\omega) \in -K_t(\omega)$ for all $\omega \in \Omega$. Next, we define the set-valued mapping $\omega \mapsto P(\omega) \subseteq \mathbb{R}^{d\times d} \times \mathbb{R}^d$ by

$$P(\omega) := \left\{ (\lambda, r) \in \mathbb{R}^{d\times d} \times \mathbb{R}^d : \lambda, r \geq 0, \sum_{1\leq i,j\leq d} \lambda^{ij} \left(e^j - \pi_t^{ij}(\omega) e^i \right) - r = \widetilde{v}(\omega) \right\}.$$

We have $P(\omega) \neq \emptyset$ for each $\omega \in \Omega$ by virtue of $\widetilde{v}(\omega) \in -K_t(\omega)$ for each $\omega \in \Omega$. In addition, the mapping $\omega \mapsto \sum_{1\leq i,j\leq d} \lambda^{ij}(e^j - \pi_t^{ij}(\omega)e^i) - r$ is $\mathcal{F}_t$-measurable for each $(\lambda, r) \in \mathbb{R}^{d\times d}_+ \times \mathbb{R}^d_+$, and the mapping $(\lambda, r) \mapsto \sum_{1\leq i,j\leq d} \lambda^{ij}(e^j - \pi_t^{ij}(\omega)e^i) - r$ is continuous for each $\omega \in \Omega$. Hence, we may apply Theorem 14.36 in [77] to find $\lambda_t \in L^0(\mathbb{R}^{d\times d}_+, \mathcal{F}_t)$ and $r_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_t)$ such that $(\lambda_t(\omega), r_t(\omega)) \in P(\omega)$ for all $\omega \in \Omega$. This yields $\widetilde{v}(\omega) = \sum_{1\leq i,j\leq d} \lambda_t^{ij}(\omega)(e^j - \pi_t^{ij}(\omega)e^i) - r(\omega)$ for each $\omega \in \Omega$ and, consequently, we have $v = L_t(\lambda_t) - r_t$ a.s. At last, (2.3.3) follows directly from (2.3.2). □

Definition 2.3.2. For any $t \in \{0, \ldots, T-1\}$, we define the (convex) cone of *reversible orders* at time t by

$$\mathcal{R}_t := \{\lambda \in L^0(\mathbb{R}^{d\times d}_+, \mathcal{F}_t) : -L_t(\lambda) \in \mathcal{A}_{t+1}^T\}.$$

The following lemma establishes a suitable decomposition of the elements of $L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ into reversible and "purely non-reversible" orders. For the decomposition, one needs that $\mathcal{R}_t$ is closed in probability. To achieve this, the lemma *assumes* that $\mathcal{A}_{t+1}^T$ is closed in probability, a property that is not yet shown at this place.

Lemma 2.3.3. *Let $t \in \{0, \ldots, T-1\}$ and assume that $\mathcal{A}_{t+1}^T$ is closed in probability. Then for any $\lambda \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ there is a unique pair (up to null sets) $\lambda_1 \in \mathcal{R}_t$ and $\lambda_2 \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ with $\lambda = \lambda_1 + \lambda_2$ such that for any decomposition $\lambda = \widetilde{\lambda}_1 + \widetilde{\lambda}_2$ with $\widetilde{\lambda}_1 \in \mathcal{R}_t$, $\widetilde{\lambda}_2 \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$, we have*

$$\|\lambda_2\| \leq \|\widetilde{\lambda}_2\| \quad \mathbb{P}\text{-}a.s., \tag{2.3.4}$$

where the inequality is strict on $\{\lambda_2 \neq \widetilde{\lambda}_2\}$ $\mathbb{P}$-a.s. and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{d\times d}$. In addition, the mappings

$$p_t : L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t) \to \mathcal{R}_t \quad \text{and} \quad q_t : L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t) \to L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$$

defined by $p_t(\lambda) = \lambda_1$ and $q_t(\lambda) = \lambda_2$ have the following properties:

(i) *For all $\lambda \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ and all non-negative $\mathcal{F}_t$-measurable scalars μ we have $p_t(\mu\lambda) = \mu p_t(\lambda)$,*

(ii) $\text{Image}(q_t) = \{\lambda \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t) : q_t(\lambda) = \lambda\}$,

(iii) $\text{Image}(p_t) \cap \text{Image}(q_t) = \{0\}$.

We refer to $p_t(\lambda)$ and $q_t(\lambda)$ as the reversible and the purely non-reversible part of the order $\lambda \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$, respectively. The following continuity of the decomposition is the last ingredient for the proof of Theorem 2.2.6.

Lemma 2.3.4. *Let $t \in \{0, \ldots, T-1\}$ and assume that $\mathcal{A}_{t+1}^T$ is closed in probability. Let $(\lambda_n)_{n\in\mathbb{N}} \subseteq L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ converge $\mathbb{P}$-a.s. to some $\lambda \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$. Then, $p_t(\lambda_n) \to p_t(\lambda)$ and $q_t(\lambda_n) \to q_t(\lambda)$ $\mathbb{P}$-a.s. for $n \to \infty$. Especially,* $\text{Image}(q_t)$ *is closed in probability.*

We postpone the proofs of the two lemmas to make some comments on their use. By the prospective strict no-arbitrage (NA^{ps}) property, reversible orders can be postponed to later periods $s \in \{t+1, \ldots, T\}$. Thus, any order at time t can be replaced by its purely non-reversible part at time t. On the other hand, if a sequence in $\mathcal{A}_t^T$ converges, we can show that the sequence of purely non-reversible orders at time t has to stay bounded by leading a possible explosion to a contradiction. The mapping p_t plays the role of the projection of an arbitrary self-financing strategy onto the set of null-strategies in [85]. There, the null-strategies form a linear subspace, which implies that the orthogonal part is automatically self-financing. This property is not available here, and thus we cannot argue with a projection, but with a more complicated decomposition.

Alternatively, the decomposition in Lemma 2.3.3 could also be defined on the level of portfolio changes $\vartheta_t - \vartheta_{t-1} \in L^0(\mathbb{R}^d, \mathcal{F}_t)$. But, in the proof of Lemma 2.3.4, we have to argue directly with the orders λ.

For the convenience of the reader, we recall a lemma on the existence of a measurable subsequence that is applied several times in the following proofs (see, e.g. [85] and [53]).

Lemma 2.3.5 (Lemma A.2 of [85]). *Let $t \in \{0, \dots, T\}$. For a sequence $(f_n)_{n\in\mathbb{N}} \subseteq L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$, there is a random subsequence $(\tau_k)_{k\in\mathbb{N}}$, i.e., a strictly increasing sequence of $\mathbb{N}$-valued $\mathcal{F}_t$-measurable random variables such that the sequence of random variables $(g_k)_{k\in\mathbb{N}}$ given by $g_k(\omega) := f_{\tau_k(\omega)}(\omega)$, $k \in \mathbb{N}$, converges a.s. in the one-point-compactification $\mathbb{R}_+^{d\times d} \cup \{\infty\}$ to a random variable in $f \in L^0(\mathbb{R}_+^{d\times d} \cup \{\infty\}, \mathcal{F}_t)$. In fact, we may find the subsequence such that*

$$\|f\| = \limsup_{n\to\infty} \|f_n\|, \ \mathbb{P}\text{-}a.s.$$

where $\|\infty\| = \infty$.

Proof of Lemma 2.3.3. First, we show the existence and uniqueness of the decomposition satisfying (2.3.4). Fix $\lambda \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ and define the non-empty set

$$X_\lambda := \left\{ \widetilde{\lambda} \in \mathcal{R}_t : \lambda - \widetilde{\lambda} \in L^0\big(\mathbb{R}_+^{d\times d}, \mathcal{F}_t\big) \right\},$$

which consists of the first components of the possible decompositions of λ. Under the assumptions made, the convex cone $\mathcal{R}_t$ is closed in probability and closed under multiplication with non-negative $\mathcal{F}_t$-measurable scalars. This implies that X_λ is closed in probability and closed under measurable convex combinations. We have to show that

$$x := \operatorname*{ess\,inf}_{\widetilde{\lambda}\in X_\lambda} \|\lambda - \widetilde{\lambda}\|$$

is attained and the minimizer is unique. Thus, notice that the set of random variables $\{\|\lambda - \widetilde{\lambda}\| : \widetilde{\lambda} \in X_\lambda\}$ is downward directed. Indeed, for each $\lambda_1, \lambda_2 \in X_\lambda$, one has $\|\lambda - \lambda_3\| = \|\lambda - \lambda_1\| \wedge \|\lambda - \lambda_2\|$, where

$$X_\lambda \ni \lambda_3 := \mathbb{1}_{\{\|\lambda-\lambda_1\|\le\|\lambda-\lambda_2\|\}}\lambda_1 + \mathbb{1}_{\{\|\lambda-\lambda_1\|>\|\lambda-\lambda_2\|\}}\lambda_2.$$

Hence, there is a sequence of random variables $(\lambda_n)_n \subseteq X_\lambda$ such that $\|\lambda - \lambda_n\| \to x$ $\mathbb{P}$-a.s. for $n \to \infty$. From the parallelogram law (see, e.g., Lemma 6.51 in [1]) of the Euclidean norm on $\mathbb{R}^{d\times d}$ and the convexity of X_λ, we obtain

$$\begin{aligned} \|\lambda_n - \lambda_m\|^2 &= 2\|\lambda - \lambda_n\|^2 + 2\|\lambda - \lambda_m\|^2 - 4\|\lambda - \frac{\lambda_n + \lambda_m}{2}\|^2 \\ &\le 2\|\lambda - \lambda_n\|^2 + 2\|\lambda - \lambda_m\|^2 - 4x \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.3.5}$$

(2.3.5) implies that $(\lambda_n)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.s. to some element of $L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$. By the closedness of X_λ, one derives the existence. Uniqueness in the postulated sense also follows from the estimate (2.3.5).

This means that the mappings p_t and q_t are well-defined and it remains to show that they satisfy the properties.

Ad (i): Let $\mu \geq 0$ be a $\mathcal{F}_t$-measurable random variable. As a consequence of $\mathcal{R}_t$ and $L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ being closed under multiplication with non-negative $\mathcal{F}_t$-measurable random variables, we have $X_{\mu\lambda} = \{\mu\widetilde{\lambda} : \widetilde{\lambda} \in X_\lambda\}$. Then, the assertion follows from the construction of p_t from above.

Ad (ii): Let $\lambda \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$. We have $p_t(\lambda) + p_t(q_t(\lambda)) \in \mathcal{R}_t + \mathcal{R}_t \subseteq \mathcal{R}_t$ and $\lambda - (p_t(\lambda) + p_t(q_t(\lambda))) = q_t(\lambda) - p_t(q_t(\lambda)) \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ by definition of p_t, thus in particular

$$p_t(\lambda) + p_t\big(q_t(\lambda)\big) \in X_\lambda. \tag{2.3.6}$$

On the other hand, one has

$$\begin{aligned} \|\lambda - \Big(p_t(\lambda) + p_t\big(q_t(\lambda)\big)\Big)\| &= \|q_t(\lambda) - p_t\big(q_t(\lambda)\big)\| \\ &\leq \|q_t(\lambda)\| = \|\lambda - p_t(\lambda)\| \quad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{2.3.7}$$

where the inequality holds since $p_t(q_t(\lambda))$ is the optimal reversible part of $q_t(\lambda)$. By (2.3.7), (2.3.6), and the uniqueness of the optimal reversible part in the decomposition of λ, it follows that $p_t(\lambda) + p_t(q_t(\lambda)) = p_t(\lambda)$ $\mathbb{P}$-a.s. and thus

$$\text{Image}(p_t \circ q_t) = \{0\}. \tag{2.3.8}$$

The assertion immediately follows from (2.3.8).

Ad (iii): Follows immediately from (ii). □

Proof of Lemma 2.3.4. We have to show that

$$\lambda_n \to \lambda \quad \mathbb{P}\text{-a.s.} \qquad \Longrightarrow \qquad p_t(\lambda_n) \to p_t(\lambda) \quad \mathbb{P}\text{-a.s.} \tag{2.3.9}$$

The property that Image(q_t) is closed in probability immediately follows from Lemma 2.3.3 (ii) and (2.3.9) by passing to an almost surely converging subsequence. To show (2.3.9), we define for each $n \in \mathbb{N}$ the $\mathcal{F}_t$-measurable real-valued random variable

$$\mu_n(\omega) := 1 \wedge \inf_{\substack{1\leq i,j\leq d \\ p_t(\lambda)^{ij}(\omega)>0}} \frac{\lambda_n^{ij}(\omega)}{p_t(\lambda)^{ij}(\omega)}. \tag{2.3.10}$$

One has that $\mu_n p_t(\lambda) \in \mathcal{R}_t$ and $\lambda_n - \mu_n p_t(\lambda) \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$, i.e., $\mu_n p_t(\lambda) \in X_{\lambda_n}$. This means that we compress the transfer matrix $p_t(\lambda)$ to use it for a (in general not optimal) decomposition of λ_n into a reversible and a non-reversible part. Note that in the trivial case that $p_t(\lambda)^{ij}(\omega) = 0$ for all (i,j), the compression is irrelevant, here one has $\mu_n(\omega) = 1$. As $p_t(\lambda_n)$ is the optimal reversible part of λ_n, it follows that

$$\|\lambda_n - p_t(\lambda_n)\| \leq \|\lambda_n - \mu_n p_t(\lambda)\| \quad \mathbb{P}\text{-a.s.} \quad \forall n \in \mathbb{N}. \tag{2.3.11}$$

In addition, by $\lambda^{ij} \geq p_t(\lambda)^{ij} \geq 0$ for all $i,j = 1,\dots,d$ and $\lambda_n \to \lambda$, we have that $\mu_n \to 1$ $\mathbb{P}$-a.s. Combining this with the triangle inequality of the Euclidean norm, we arrive at

$$\begin{aligned}\limsup_{n\to\infty} \|\lambda - p_t(\lambda_n)\| &= \limsup_{n\to\infty} \|\lambda_n - p_t(\lambda_n)\| \\ &\leq \limsup_{n\to\infty} \|\lambda_n - \mu_n p_t(\lambda)\| = \|\lambda - p_t(\lambda)\| \ \mathbb{P}\text{-a.s.},\end{aligned}$$

where the inequality follows from (2.3.11). Since $p_t(\lambda)$ is the optimal reversible part of λ, this just means that

$$\|\lambda - p_t(\lambda_n)\| \to \|\lambda - p_t(\lambda)\|, \quad n \to \infty, \quad \mathbb{P}\text{-a.s.} \tag{2.3.12}$$

To complete the proof, we define the $\mathcal{F}_t$-measurable random variable

$$\varepsilon(\omega) := \sup\{1/k : k \in \mathbb{N},\ \|p_t(\lambda_n) - p_t(\lambda)\|(\omega) \geq 1/k \text{ for infinitely many } n\}$$

that is strictly positive on the set $A := \{p_t(\lambda_n) \not\to p_t(\lambda)\} \in \mathcal{F}_t$. Then, we construct the random subsequence $(\tau_k)_{k\in\mathbb{N}}$ recursively by $\tau_0 := 0$ and $\tau_k := \inf\{n \in \mathbb{N} : n > \tau_{k-1},\ \|p_t(\lambda_n) - p_t(\lambda)\| \geq \varepsilon\}$ on A and $\tau_k := k$ on $\Omega \setminus A$. By construction, we have that

$$\mathbb{P}\left[\|p_t(\lambda_{\tau_k}) - p_t(\lambda)\| \geq \varepsilon,\ \forall k \in \mathbb{N} \;\middle|\; A\right] = 1. \tag{2.3.13}$$

By $\lambda_n \to \lambda$ and $0 \leq p_t(\lambda_n)^{ij} \leq \lambda_n^{ij}$, one has $\sup_{n\in\mathbb{N}} \|p_t(\lambda_n)\| \leq \sup_{n\in\mathbb{N}} \|\lambda_n\| < \infty$ $\mathbb{P}$-a.s. Thus, by Lemma 2.3.5, there exists a random subsequence $(\widetilde{\tau}_k)_{k\in\mathbb{N}}$ of $(\tau_k)_{k\in\mathbb{N}}$ and an $f \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ s.t. $p_t(\lambda_{\widetilde{\tau}_k}) \to f$ $\mathbb{P}$-a.s. Together with (2.3.12), this implies that $\|\lambda - f\| = \|\lambda - p_t(\lambda)\|$ $\mathbb{P}$-a.s.. In addition, we have $f \in X_\lambda$. On the other hand, by (2.3.13), $f \neq p_t(\lambda)$ on A $\mathbb{P}$-a.s. Since $p_t(\lambda)$ is the unique optimal reversible part of λ in the sense of Lemma 2.3.3, these two properties can only hold simultaneously if $\mathbb{P}[A] = 0$ and we are done. □

Remark 2.3.6. *We note that for the proof of Theorem 2.2.6, we only need the weaker assertion that* Image(q_t) *is closed in probability. To show this assertion, one can restrict oneself to sequences with $\lambda_n = q_t(\lambda_n)$, i.e., $p_t(\lambda_n) = 0$, for all $n \in \mathbb{N}$, and the above proof would already be completed with (2.3.12).*

We are now in the position to prove Theorem 2.2.6. As in Kabanov et al. [52] we argue by induction on the periods. The key difference is that reversible orders are postponed to later periods, instead of being executed and compensated in the same period. The later is not possible since the null-strategies do not form a linear space.

Proof of Theorem 2.2.6. Assume that the bid-ask-process $(\Pi_t)_{t=0}^T$ satisfies $(\mathrm{NA}^{\mathrm{ps}})$. Let us prove by a backward induction on $t = T, T-1, \dots, 0$ that $\mathcal{A}_t^T$ is closed in probability. The induction basis $t = T$ is trivial since $\mathcal{A}_T^T$ coincides with $L^0(-K_T, \mathcal{F}_T)$ which is closed in probability.

Induction step $t+1 \rightsquigarrow t$: We assume that $\mathcal{A}_{t+1}^T$ is closed in probability for some $t \le T-1$ and have to show that $\mathcal{A}_t^T$ is closed too. Therefore, let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_t^T$ which converges to some $\xi \in L^0(\mathbb{R}^d)$ in probability. Obviously, we may assume that $\xi_n \to \xi$ almost surely by passing to a subsequence. We have to show that $\xi \in \mathcal{A}_t^T$.

Step 1. According to Lemma 2.3.1, we may write

$$\xi_n = \sum_{s=t}^{T} L_s(\lambda_s^n) - r^n, \quad n \in \mathbb{N}, \tag{2.3.14}$$

where $(\lambda_s^n)_{n\in\mathbb{N}} \subseteq L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_s)$ for each $s = t, \dots, T$ and $(r^n)_{n\in\mathbb{N}} \subseteq L^0(\mathbb{R}_+^d)$. Recall that $L^0(\mathbb{R}_+^d) = L^0(\mathbb{R}_+^d, \mathcal{F}_T)$ by convention. Under the induction hypothesis that $\mathcal{A}_{t+1}^T$ is closed in probability, we apply Lemma 2.3.3 in order to decompose λ_t^n into $p_t(\lambda_t^n) + q_t(\lambda_t^n)$ and thus

$$L_t(\lambda_t^n) = L_t\big(p_t(\lambda_t^n)\big) + L_t\big(q_t(\lambda_t^n)\big),$$

where $p_t(\lambda_t^n)$ is reversible and $q_t(\lambda_t^n)$ is purely non-reversible. This means that

$$L_t\big(p_t(\lambda_t^n)\big) \in \mathcal{A}_0^t \cap \big(-\mathcal{A}_{t+1}^T\big)\,.$$

The prospective strict no-arbitrage $(\mathrm{NA}^{\mathrm{ps}})$ property implies that $\mathcal{A}_0^t \cap (-\mathcal{A}_{t+1}^T) \subseteq \mathcal{A}_0^{t+1} \cap (-\mathcal{A}_{t+1}^T) \subseteq \mathcal{A}_{t+1}^T$ and thus $L_t(p_t(\lambda_t^n)) \in \mathcal{A}_{t+1}^T$. This allows us to rewrite (2.3.14) as

$$\xi_n = L_t\big(q_t(\lambda_t^n)\big) + L_t\big(p_t(\lambda_t^n)\big) + \sum_{s=t+1}^{T} L_s(\lambda_s^n) - r^n =: L_t\big(q_t(\lambda_t^n)\big) + x_n$$

with $x_n \in \mathcal{A}_{t+1}^T$. Hence, from now on we can assume w.l.o.g. that $(\lambda_t^n)_{n\in\mathbb{N}} \subseteq \mathrm{Image}(q_t)$.

Step 2. Our next goal is to show that

$$\mathbb{P}[A] = 0, \quad \text{where} \quad A := \left\{ \limsup_{n\to\infty} \|\lambda_t^n\| = \infty \right\}. \tag{2.3.15}$$

By Lemma 2.3.5, we may pass to a measurable subsequence $(\tau_k)_{k\in\mathbb{N}}$ such that for a.e. $\omega \in A$ we have $\lambda_t^{\tau_k(\omega)}(\omega) \neq 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \|\lambda_t^{\tau_k(\omega)}(\omega)\| = \infty$. Then, by the stability of $\mathrm{Image}(q_t)$ under multiplication with non-negative $\mathcal{F}_t$-measurable scalars (see Lemma 2.3.3 i), we find that $\widetilde{\lambda}_t^n := \frac{\lambda_t^{\tau^n}}{\|\lambda_t^{\tau_n}\|} \mathbb{1}_A$ belongs to $\mathrm{Image}(q_t)$ and, in addition, we define

$$\widetilde{\lambda}_s^n := \frac{\lambda_s^{\tau_n}}{\|\lambda_t^{\tau_n}\|} \mathbb{1}_A \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_s) \text{ for } s = t+1, \dots, T \text{ and } \widetilde{r}^n := \frac{r^n}{\|\lambda_t^{\tau_n}\|} \mathbb{1}_A \in L^0(\mathbb{R}_+^d).$$

We have $\sum_{s=t}^{T} L_s(\widetilde{\lambda}_s^n) - \widetilde{r}^n = \mathbb{1}_A \xi_{\tau_n}/\|\lambda_t^{\tau_n}\| \to 0$ a.s. Now, we may apply once again Lemma 2.3.5 to find a measurable subsequence $(\sigma_k)_{k\in\mathbb{N}}$ such that

$$\widetilde{\lambda}_t := \lim_{k\to\infty} \widetilde{\lambda}_t^{\sigma_k} \tag{2.3.16}$$

exists and $\|\widetilde{\lambda}_t\| = \lim_{k\to\infty} \|\widetilde{\lambda}_t^{\sigma_k}\|$. Consequently, $L_t(\widetilde{\lambda}_t^{\sigma_k}) \to L_t(\widetilde{\lambda}_t)$ and thus the sequence

$$\left(\sum_{s=t+1}^{T} L_s(\widetilde{\lambda}_s^{\sigma_k}) - \widetilde{r}^{\sigma_k} \right)_{k\in\mathbb{N}} \subseteq \mathcal{A}_{t+1}^T$$

converges to $-L_t(\widetilde{\lambda}_t)$. Since $\mathcal{A}_{t+1}^T$ is closed and due to Lemma 2.3.1, the limit can be written as $\sum_{s=t+1}^{T} L_s(\widetilde{\lambda}_s) - \widetilde{r}$, i.e., we have

$$L_t(\widetilde{\lambda}_t) + \sum_{s=t+1}^{T} L_s(\widetilde{\lambda}_s) - \widetilde{r} = 0 \quad \mathbb{P}\text{-a.s.}$$

with $\widetilde{\lambda}_s \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_s)$ and $\widetilde{r} \in L^0(\mathbb{R}_+^d, \mathcal{F}_s)$. Thus we have that $\widetilde{\lambda}_t$ is reversible, i.e., $\widetilde{\lambda}_t \in \mathcal{R}_t = \mathrm{Image}(p_t)$. However, on the other hand, the sequence $(\widetilde{\lambda}_t^{\sigma_k})_{k\in\mathbb{N}}$ belonged to $\mathrm{Image}(q_t)$, thus, by Lemma 2.3.4, $\widetilde{\lambda}_t \in \mathrm{Image}(q_t)$. Therefore $\widetilde{\lambda}_t \in \mathrm{Image}(p_t) \cap \mathrm{Image}(q_t)$, hence $\widetilde{\lambda}_t = 0$ a.s. according to Lemma 2.3.3 (iii). Since $\mathbb{P}[A] = \mathbb{P}[\widetilde{\lambda}_t \neq 0]$, this is only possible if $\mathbb{P}[A] = 0$, i.e., (2.3.15) holds true.

Step 3. According to step 2, we can apply Lemma 2.3.5 to find a measurable subsequence $(\tau_k)_{k\in\mathbb{N}}$ such that $\lambda_t^{\tau_k} \to \lambda_t \in L^0(\mathbb{R}_+^{d\times d}, \mathcal{F}_t)$ $\mathbb{P}$-a.s. for $k \to \infty$ and, consequently, $L_t(\lambda_t^{\tau_k}) \to L_t(\lambda_t)$ a.s. Hence, $\sum_{s=t+1}^{T} L_s(\lambda_s^{\tau_k}) - r^{\tau_k}$ converges a.s. to $\xi - L_t(\lambda_t)$, which, by the induction hypothesis, belongs to $\mathcal{A}_{t+1}^T$. This implies that $\xi \in \mathcal{A}_t^T$. □

Finally, we finish up the remaining proofs. Notice that every result is a standard consequence of the set $\mathcal{A}_0^T$ being closed in probability under the (NA$^{\mathrm{ps}}$) condition, hence we only give the respective references and point out where some changes are needed.

Proof of Corollary 2.2.8. It suffices to repeat the arguments on page 29 between lines 5-33 of the proof of Theorem 2.1 in [85] with $\mathcal{A}_0^T$ (instead of $\widetilde{\mathcal{A}}_T$), which is closed by Theorem 2.2.6. □

Proof of Theorem 2.2.11. (NA$^{\mathrm{wps}}$)$\Rightarrow$ $\exists$ CPS: According to the (NA$^{\mathrm{wps}}$) condition there is a bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ with $\widetilde{\Pi}_t \le \Pi_t$ a.s. for all $t = 0, \dots, T$ satisfying (NA$^{\mathrm{ps}}$). Corollary 2.2.8 implies that $(\widetilde{\Pi}_t)_{t=0}^T$ admits a CPS, which is obviously a CPS for $(\Pi_t)_{t=0}^T$ as well.

$\exists$ CPS $\Rightarrow$ (NA$^{\mathrm{wps}}$): It is again sufficient to repeat the arguments on page 30 between lines 1-12 of the proof of Theorem 2.1 in [85] to define a frictionless bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$, i.e., $\widetilde{\pi}_t^{ij} = 1/\widetilde{\pi}_t^{ji}$, with $\widetilde{\Pi}_t \le \Pi_t$ a.s. for all $t = 0, \dots, T$ satisfying

(NA), which in the frictionless case coincides with (NA$^{\text{ps}}$) by Proposition 2.2.18. Thus $(\Pi_t)_{t=0}^T$ satisfies (NA$^{\text{wps}}$). □

Proof of Corollary 2.2.16. This is a well known consequence of the existence of a consistent price system. Indeed, we may use Proposition 3.2.6 in [55] to see that the existence of a consistent price system implies $\overline{\mathcal{A}_0^T} \cap L^0(K_T, \mathcal{F}_T) \subseteq L^0(\partial K_T, \mathcal{F}_T)$. To complete the proof, we observe that $\partial K_T \cap \mathbb{R}^d_+ = \{0\}$. Indeed, by $\pi^{ij} < \infty$, the existence of a $v \in \mathbb{R}^d_+ \setminus \{0\}$ and a sequence $(v_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^d \setminus K_T(\omega)$ with $v_n \to v$ can easily be led to a contradiction. □

Remark 2.3.7. *Our results can be extended to the Kabanov model as defined in Subsection 3.2 of [55], which, in addition to the barter market considered here, also covers a wider range of models, e.g., models of a barter market where a bank account is charged the transaction costs and models where baskets of assets are exchanged. To see this, we briefly highlight the minor adjustments. On the other hand, the proofs of Lemmas 2.3.3 and 2.3.4 are heavily based on the polyhedral structure of the solvency cones. The key argument that μ_n defined in (2.3.10) converges to 1 does not work for general closed solvency cones.*

The Kabanov model is defined as follows. Let $((X^i_t)_{t=0}^T)_{i\in\mathbb{N}}$ be a sequence of adapted $\mathbb{R}^d$-valued processes, such that for all t and ω the set $\{i \in \mathbb{N} : X^i_t(\omega) \neq 0\}$ is non-empty and finite, and set

$$K_t(\omega) := \operatorname{cone}\left(-X^i_t(\omega) : i \in \mathbb{N}\right).$$

In this case, $K = (K_t)_{t=0}^T$ is called a cone-valued process. In addition, we assume $\mathbb{R}^d_+ \setminus \{0\} \subseteq \operatorname{int} K_T(\omega)$ for all ω (which corresponds to the possibility to freely dispose of assets and $\pi^{ij}_T < \infty$ for all i, j in the base model) and $-K_T(\omega) \cap \mathbb{R}^d_+ = \{0\}$ (which corresponds to $\pi^{ij}_T \leq \pi^{ik}_T \pi^{kj}_T$ and $\pi^{ii}_T = 1$ for all i, j, k). The cone of hedgeable claims attainable from zero endowment by trading between s and t is given by $\mathcal{A}^t_s = \sum_{k=s}^t L^0(-K_k, \mathcal{F}_k)$, $s \leq t$. The (NA) and (NA^{ps}) conditions are defined accordingly. We now sketch how the arguments of the previous proofs can be applied in this more general setting. Let $I_t(\omega) := \sup\{n \in \mathbb{N} : X^n_t(\omega) \neq 0\}$ for $\omega \in \Omega$ and $t = 0, \dots, T$. The assumptions above guarantee that I_t is a $\mathbb{N}$-valued $\mathcal{F}_t$-measurable random variable. By $\Omega = \bigcup_{I\in\mathbb{N}}\{I_t = I\}$, the arguments from Lemmas 2.3.1, 2.3.3, and 2.3.4 can be separately applied on the sets $\{I_t = I\}$ for $I \in \mathbb{N}$. In particular, portfolio changes can be represented by $L_t(\lambda_t) := \sum_{i=1}^{I_t} \lambda^i_t X^i_t$, where $\lambda^i_t \in L^0(\mathbb{R}_+, \mathcal{F}_t)$. The Euclidean norm on $\mathbb{R}^{d\times d}$ that is used for the decomposition of an order into the reversible and the purely non-reversible part is replaced by

$$\|\lambda\| := \sqrt{\sum_{i=1}^{I} (\lambda^i)^2} \quad \text{on } \{I_t = I\}.$$

With these adjustments, Theorem 2.2.6 extends to the Kabanov model by arguing along the lines of the original proofs. We note again that it is crucial that for fixed ω, only linear combinations from finitely many $X^i_t(\omega)$ have to be considered.

At last, we say that $K = (K_t)_{t=0}^T$ satisfies the (NA^{wps}) property if there is a cone-valued process $\widetilde{K} = (\widetilde{K}_t)_{t=0}^T$ with the (NA^{ps}) property such that $K_t(\omega) \subseteq \widetilde{K}_t(\omega)$ for all ω and t. Then, Theorem 2.2.11 holds true in the Kabanov model as well. Indeed, (NA^{wps}) implies the existence of a consistent price system for $K = (K_t)_{t=0}^T$ and, on the other hand, given a CPS $Z = (Z_t)_{t=0}^T$, the cone-valued process $\widetilde{K} = (\widetilde{K}_t)_{t=0}^T$ defined by $\widetilde{K}_t(\omega) := (\mathbb{R}_+ Z_t(\omega))^\star$ satisfies (NA^{ps}) and $K_t(\omega) \subseteq \widetilde{K}_t(\omega)$ for all t and ω, i.e., K satisfies (NA^{wps}).

In addition, our reasoning to show the closedness of $\mathcal{A}_0^T$ can also be applied to models with incomplete information such as those considered in [7, 27], where arguing on the level of orders is quite natural.

2.4 (Counter-)Examples

We start with two very simple examples that illustrate the difference between (NA^{r}), ($\mathrm{NA}^{\mathrm{ps}}$), and ($\mathrm{NA}^{\mathrm{wps}}$) and the need to consider CPSs which do not lie in the relative interior of the bid-ask spread.

Example 2.4.1 (($\mathrm{NA}^{\mathrm{ps}}$) $\not\Rightarrow$ (NA^{r}))**.** *We consider a deterministic two-asset one-period model with the bid-ask process given by*

$$\Pi_t = \begin{pmatrix} 1 & \overline{S}_t \\ 1/\underline{S}_t & 1 \end{pmatrix}, \quad t = 0, 1,$$

where the deterministic processes $(\underline{S}_t)_{t=0,1}$ and $(\overline{S}_t)_{t=0,1}$ are illustrated in Figure 2.1 below. This is just a two-asset model with a bank account that does not pay interest and one stock with bid-price $(\underline{S}_t)_{t=0,1}$ and ask-price $(\overline{S}_t)_{t=0,1}$.

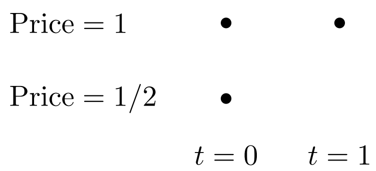

Figure 2.1: Deterministic model satisfying ($\mathrm{NA}^{\mathrm{ps}}$) and ($\mathrm{NA}^{\mathrm{s}}$), but not ($\mathrm{NA}^{\mathrm{r}}$). At $t = 0$ the bid-price $\underline{S}_0$ equals $1/2$ and the ask price $\overline{S}_0$ equals 1; at $t = 1$ the market is frictionless with price $\underline{S}_1 = \overline{S}_1 = 1$.

Here, we have $K_t = \mathrm{cone}(\overline{S}_t e^1 - e^2, e^2 - \underline{S}_t e^1, e^1, e^2)$, and its dual is given by $K_t^\star = \{(Y^1, Y^2) \in \mathbb{R}_+^2 : \underline{S}_t Y^1 \le Y^2 \le \overline{S}_t Y^1\}$ for $t = 0, 1$. Since Ω is a singleton, a CPS has to be constant in time. Thus, the market admits the unique (up to scalar multiples) CPS $Z = (Z_t^1, Z_t^2)_{t=0,1}$ given by $Z_t = (1, 1)$ for $t = 0, 1$.

Recall that a strictly consistent price system (SCPS) is a CPS with $Z_t \in L^0(\mathrm{ri} K_t^\star, \mathcal{F}_t)$ for all t, where $\mathrm{ri} K_t^\star$ denotes the relative interior of $K_t^\star$, and that (NA^r) is equivalent to the existence of a SCPS (see Theorem 1.7 in Schachermayer [85]). We have

$\text{ri}K_t^\star = \{(Y^1, Y^2) \in \text{int}\mathbb{R}_+^2 : Y^2/Y^1 \in \text{ri}[\underline{S}_t, \overline{S}_t]\}$ *for* $t = 0, 1$ *(see, e.g., equation (3.2) in Roklin [82]). Thus,* Z *is not a SCPS since* $Z_0^2/Z_0^1 = 1 \notin (1/2, 1)$*. Hence, the model cannot satisfy (NA*^{r}*). Also the Penner-condition*

$$L^0(K_t^0, \mathcal{F}_{t-1}) \subseteq L^0(K_{t-1}^0, \mathcal{F}_{t-1}) \quad \textit{for all } t = 1, \ldots, T. \tag{2.4.1}$$

is not satisfied. On the other hand, we have that $\mathcal{A}_0^0 \cap (-\mathcal{A}_1^1) = \text{cone}(e^2 - e^1) \subseteq \text{cone}(e^2 - e^1, e^1 - e^2, -e^1, -e^2) = \mathcal{A}_1^1$*, i.e., only a long stock position built up at time* 0 *can be liquidated without losses at time* 1*, but the purchase of the stock (asset* 2*) can also be postponed to time* 1*. Thus the model satisfies (NA*^{ps}*).*

Example 2.4.2 ((NA$^{\text{wps}}$) $\not\Rightarrow$ (NA$^{\text{ps}}$))**.** *We consider a variant of Example 2.4.1, where the processes* $(\underline{S}_t)_{t=0,1}$ *and* $(\overline{S}_t)_{t=0,1}$ *are given in Figure 2.2 below.*

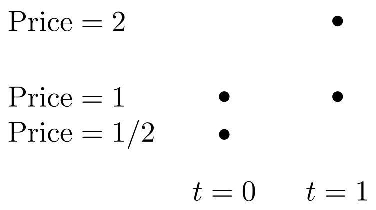

Figure 2.2: Deterministic model satisfying (NA$^{\text{wps}}$) but not (NA$^{\text{ps}}$). One has $\underline{S}_0 = 1/2$, $\overline{S}_0 = 1$, $\underline{S}_1 = 1$, and $\overline{S}_1 = 2$.

The market still admits the unique (up to scalar multiples) CPS $Z_t = (1, 1)$ *for* $t = 0, 1$*, but now fails (NA*^{ps}*) since* $\mathcal{A}_0^0 \cap (-\mathcal{A}_1^1) = \text{cone}(e^2 - e^1) \not\subseteq \text{cone}(e^2 - 2e^1, e^1 - e^2, -e^1, -e^2) = \mathcal{A}_1^1$*. On the other hand, the model satisfies (NA*^{wps}*) since the more favorable bid-ask process in Figure 2.1 satisfies (NA*^{ps}*).*

Finally, we provide an example showing that (NA$^{\text{wps}}$) cannot be replaced by the "next weaker" condition that there exists a more favorable market, i.e., a bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ with $\widetilde{\Pi}_t \leq \Pi_t$ for each $t = 0, \ldots, T$, such that $(\widetilde{\Pi}_t)_{t=0}^T$ satisfies (NA) and

$$\mathcal{A}_0^t \cap \left(-\widetilde{\mathcal{A}}_t^T\right) \subseteq \widetilde{\mathcal{A}}_t^T \quad \text{for all } t = 0, \ldots, T \tag{2.4.2}$$

(cf. Remark 2.2.17). We show that there is a bid-ask process $(\Pi_t)_{t=0}^3$ with four assets satisfying condition (2.4.2) which allows for an approximate arbitrage (see Definition 2.2.15). Hereby, in the spirit of the basic Example 3.1 of Schachermayer [85], which can be used to achieve an approximate arbitrage, the example is based on the idea of two consecutive approximate hedges. An approximate hedge is a sequence of trading strategies that hedge a given portfolio position in the limit. There exists a more favorable bid-ask process $(\widetilde{\Pi}_t)_{t=0}^3$ s.t. (2.4.2) holds, but which only turns the first approximate hedge into a perfect hedge and thus the model still satisfies (NA).

The example highlights the importance of a possible "cascade" of approximate hedges, which is, to the best of our knowledge, a phenomenon not discussed in the previous literature. It is also of interest for the discussion of adjusted bid-ask processes as introduced in Jacka et al. [44](see Remark 2.4.4).

Example 2.4.3 (A cascade of approximate hedges). *Let $T = 3$, $\Omega = \mathbb{N}^2\times\{-1/2, 1/2\}^2$, $\mathcal{F} = 2^\Omega$ and all states have positive probability. In addition, the information structure is given by $\mathcal{F}_0 = \{\emptyset, \Omega\}$,*

$$\mathcal{F}_1 = \sigma\Big(\big\{\{(n,m,i,j) : (m,i,j) \in \mathbb{N}\times\{-1/2,1/2\}^2\} : n \in \mathbb{N}\big\}\Big),$$
$$\mathcal{F}_2 = \sigma\Bigg(\Big\{\big\{(n,m,i,-1/2),(n,m,i,1/2)\big\} : (n,m,i) \in \mathbb{N}^2\times\{-1/2,1/2\}\Big\}\Bigg),$$

and $\mathcal{F}_3 = 2^\Omega = \mathcal{F}$. This means n is revealed at time 1, m and i are revealed at time 2 and, at last, j is revealed at time 3. Next, we define a bid-ask process $(\Pi_t)_{t=0}^3$ depending on parameter $a > 0$ for $t = 0, 1$ as follows

$$\Pi_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & 1 & \cdot & \cdot \\ a & \cdot & 1 & \cdot \\ a & \cdot & \cdot & 1 \end{pmatrix}, \quad \Pi_1 \equiv \begin{pmatrix} 1 & a & a & a \\ a & 1 & \cdot & \cdot \\ a & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}$$

and for $t = 2, 3$ depending on the state $(n,m,i,j) \in \mathbb{N}^2\times\{-1/2,1/2\}^2$ as

$$\Pi_2(n,m,i,j) = \begin{pmatrix} 1 & a & a & a \\ a & 1 & \cdot & \cdot \\ \frac{1}{1+i} & \cdot & 1 & \cdot \\ \frac{1}{1-\frac{i}{n}} & \cdot & \cdot & 1 \end{pmatrix}, \quad \Pi_3(n,m,i,j) = \begin{pmatrix} 1 & a & a & a \\ \frac{1}{1+\frac{1}{4}+j} & 1 & \cdot & \cdot \\ \frac{1}{1+i}\frac{1}{1-\frac{j}{m}} & \cdot & 1 & \cdot \\ a & \cdot & \cdot & 1 \end{pmatrix}.$$

The missing entries are specified via the direct transfer over the first asset, i.e., $\pi_t^{ij} := \pi_t^{i1}\pi_t^{1j}$ for $2 \le i \ne j \le 4$. This means that the first asset plays the role of a money market account and the assets 2,3, and 4 represent risky stocks. Finally, we choose the parameter a prohibitively high such that the corresponding transfers are unattractive, more precisely, we set $a := 5 > 4$.

The market is actually frictionless with special short- and long-selling constraints. Asset 2 yields the random return $1/4 + j$, $j \in \{-1/2, 1/2\}$. It can be approximately hedged, yielding an extra profit, by the return $-j/m$ of asset 3 between time 2 and time 3. On the other hand, asset 3 has to be bought already at time 0 which leads to the prior random return i, $i \in \{-1/2, 1/2\}$. The latter return can be approximately hedged by asset 4. This means that there is a cascade of approximate hedges – hedge asset 2 by asset 3 and asset 3 by asset 4 – leading to an approximate arbitrage.

To exclude an arbitrage, it is crucial that assets 2, 3, and 4 have to be bought already at time 0, without the knowledge of n and m. If n and m were known at time 0, the return of asset 3 between time 0 and time 2 could be perfectly hedged by asset 4 with hedging ratio n, and then, asset 2 could be perfectly hedged by the return of asset 3 between time 2 and time 3 with a hedging ratio exploding with m.

On the other hand, asset 4 can be sold at its initial purchasing price at time 1, after n is revealed, and the same with the portfolio of one asset 3 and n assets 4 at time 2, after m is revealed. Thus, one can buy large quantities of assets 4 and 3 at time 0 and sell the units which are not needed later on at their initial prices. But, since there is no a priori upper bound for n and m, there remains the risk that one does not have enough quantities of assets 4 and 3. Thus, buying more and more units at time 0 only leads to an approximate arbitrage.

The extension from Π *to* $\widetilde{\Pi}$ *in* (2.4.3) *allows the investor to postpone the purchase of assets 3 and 4 by one period to time 1. This means that she can now use her knowledge of n to perfectly hedge the return of asset 3 up to time 2 by asset 4. On the other hand, m is still not known at the time assets 2 and 3 have to be purchased. Thus, the extension only turns the first of the two consecutive approximate hedges into a perfect hedge. This is the reason why* (2.4.2) *holds and* $\widetilde{\Pi}$ *satisfies (NA). In the following, these ideas are worked out in detail.*

Step 1. *Let us show that* $(\Pi_t)_{t=0}^3$ *allows for an approximate arbitrage, i.e.,* $\overline{\mathcal{A}_0^3} \cap L^0(\mathbb{R}^4_+) \supsetneq \{0\}$. *Therefore, we define for fixed* $k \in \mathbb{N}$ *the following strategy. For* $t = 0$, *we set*

$$\xi_0^k = e^2 - e^1 + k(e^3 - e^1) + k^2(e^4 - e^1) \in -K_0.$$

For $t = 1$, *we define*

$$\xi_1^k(n,m,i,j) = \big(k^2 - (k \wedge n)\,k\big)\,\big(e^1 - e^4\big) \in -K_1(n,m,i,j).$$

For $t = 2$, *we define*

$$\begin{aligned}\xi_2^k(n,m,i,j) &= (k \wedge n)\,k\left(1 - \frac{i}{n}\right)\left(e^1 - \frac{1}{1-\frac{i}{n}}e^4\right)\\ &\quad + \left(k - \frac{m}{1+i} \wedge k\right)(1+i)\left(e^1 - \frac{1}{1+i}e^3\right) \in -K_2(n,m,i,j).\end{aligned}$$

Finally, at $t = 3$ *we liquidate the remaining positions in the assets* 2 *and* 3. *Thus, we define*

$$\begin{aligned}\xi_3^k(n,m,i,j) &= \left(1 + \frac{1}{4} + j\right)\left(e^1 - \frac{1}{1+\frac{1}{4}+j}e^2\right)\\ &\quad + \left(\frac{m}{1+i} \wedge k\right)(1+i)\left(1 - \frac{j}{m}\right)\left(e_1 - \frac{1}{1+i}\frac{1}{1-\frac{j}{m}}e^3\right),\end{aligned}$$

which belongs to $-K_3(n,m,i,j)$. *Thus* $v^k = \xi_0^k + \xi_1^k + \xi_2^k + \xi_3^k$ *belongs to* $\mathcal{A}_0^T$ *and we have*

$$v^k(n,m,i,j) = \left(\frac{1}{4} + ik\Big(1 - \frac{n \wedge k}{n}\Big) + j\Big(1 - \frac{1+i}{m}\Big(\frac{m}{1+i} \wedge k\Big)\Big)\right)e^1.$$

Finally, letting $k \to \infty$*, we obtain* $v \in \overline{\mathcal{A}_0^T}$ *given by*

$$v(n,m,i,j) = \lim_{k\to\infty} v^k(n,m,i,j) = \frac{e^1}{4},$$

which is the desired asymptotic arbitrage. Hence, the model cannot admit a CPS (see Proposition 3.2.6. in [55]).

Next, we introduce the bid-ask process $(\widetilde{\Pi}_t)_{t=0}^3$ *given by* $\widetilde{\Pi}_0 = \Pi_0$*,* $\widetilde{\Pi}_2 = \Pi_2$*,* $\widetilde{\Pi}_3 = \Pi_3$*, and*

$$\widetilde{\Pi}_1 \equiv \begin{pmatrix} 1 & a & 1 & 1 \\ a & 1 & \cdot & \cdot \\ a & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}, \tag{2.4.3}$$

which satisfies $\widetilde{\Pi}_t \leq \Pi_t$ *for all* $t = 0,1,2,3$*. We want to show that* $(\widetilde{\Pi}_t)_{t=0}^3$ *has the (NA) property and satisfies* $\mathcal{A}_0^t \cap -\widetilde{\mathcal{A}}_t^T \subseteq \widetilde{\mathcal{A}}_t^T$ *for* $t = 0,1,2,3$*.*

Step 2. *We start with the (NA) property for* $(\widetilde{\Pi}_t)_{t=0}^T$*. Let* $\widetilde{v} \in \widetilde{\mathcal{A}}_0^3$ *with* $\widetilde{v}^i = 0$ *for* $i = 2,3,4$ *and* $\widetilde{v}^1 \geq 0$ *a.s. We have to show that this already implies* $\widetilde{v}^1 = 0$ *a.s. We may pass to a* $v \in \widetilde{\mathcal{A}}_0^3$ *with* $v^i = 0$ *for* $i = 2,3,4$ *a.s. and* $v^1 \geq \widetilde{v}^1$ *s.t.* v *can be represented solely by transfers*

$$\lambda_t^{1j} \text{ with } \pi_t^{1j} < a \quad \text{and} \quad \lambda_t^{j1} \text{ with } \pi_t^{j1} < a. \tag{2.4.4}$$

Indeed, purchasing an asset $i \in \{2,3,4\}$ *at price* $a = 5$ *or short-selling it at price* $1/a = 1/5$ *(in terms of the asset* 1*) leads to a sure loss after liquidating this position afterwards. Hence, we only need to consider* $v = \xi_0 + \xi_1 + \xi_2 + \xi_3 \in \widetilde{\mathcal{A}}_0^3$*, where* $\xi_0 \in \mathrm{cone}(e^2 - e^1)$*,* $\xi_1 \in L^0(\mathrm{cone}(e^3 - e^1, e^4 - e^1), \mathcal{F}_1)$*,* $\xi_2 \in L^0(\mathrm{cone}(e^1 - \widetilde{\pi}_2^{3,1} e^3, e^1 - \widetilde{\pi}_2^{4,1} e^4), \mathcal{F}_2)$*, and* $\xi_3 \in L^0(\mathrm{cone}(e^1 - \widetilde{\pi}_3^{2,1} e^2, e^1 - \widetilde{\pi}_3^{3,1} e^3), \mathcal{F}_3)$ *with the additional restrictions* $\xi_1^3 + \xi_2^3 \geq 0$ *a.s. and* $\xi_1^4 + \xi_2^4 = 0$ *a.s. Under the assumptions above, we get*

$$\begin{aligned} 0 \leq v^1(n,m,i,j) \leq \xi_0^2 \cdot (j + 1/4) + i \cdot \left(\xi_1^3(n) - \frac{\xi_1^4(n)}{n} \right) \\ + \left(\xi_1^3(n) + \xi_2^3(n,m,i) \right) \cdot \frac{-j}{m}(1+i) \end{aligned} \tag{2.4.5}$$

for each state $(n,m,i,j) \in \Omega$ *with* $\xi_0^2 \geq 0$*,* $\xi_1^3(n) \geq 0$*,* $\xi_2^3(n,m,i) \leq 0$ *and* $\xi_1^4(n) \geq 0$*. Hereby the notation highlights the required measurability of the random variables; for instance, we write* $n \mapsto \xi_1^3(n)$ *since the value of the* $\mathcal{F}_1$*-measurable random-variable* ξ_1^3 *can only depend on* n*. We have* $\xi_1^3(n) + \xi_2^3(n,m,i) \leq \xi_1^3(n)$*, i.e., the investment in asset 3 between* $t = 2$ *and* $t = 3$ *is bounded from above by the* $\mathcal{F}_1$*-measurable random variable* ξ_1^3*. Consequently, the third summand in equation* (2.4.5) *becomes arbitrarily small for large* $m \in \mathbb{N}$*. But, this implies that the sum of the first two terms, i.e.,*

$$\xi_0^2 \cdot (j + 1/4) + i \cdot \left(\xi_1^3(n) - \frac{\xi_1^4(n)}{n} \right),$$

that does not depend on m*, has to be almost surely non-negative, which is only possible if*

$$\xi_0^2 = 0 \quad \text{and} \quad \xi_1^3(n) = \frac{\xi_1^4(n)}{n} \text{ for each } n \in \mathbb{N}. \tag{2.4.6}$$

But then (2.4.5) *reduces to*

$$0 \le v^1(n,m,i,j) \le \left(\xi_1^3(n) + \xi_2^3(n,m,i)\right) \cdot \frac{-j}{m}(1+i).$$

Taking $j = 1/2$*, this implies* $\xi_1^3(n) = -\xi_2^3(n,m,i)$ *and, consequently,* $v^1 \equiv 0$*. Hence* $(\widetilde{\Pi}_t)_{t=0}^3$ *satisfies (NA).*

Step 3. *Let us now show* $\mathcal{A}_0^t \cap -\widetilde{\mathcal{A}}_t^T \subseteq \widetilde{\mathcal{A}}_t^T$ *for* $t = 1,2,3,4$ *(for* $t = 0$*, there is nothing to show). This is akin to the proof in step 2. As in (2.4.4), we can restrict to portfolios which can be represented by transfers that do not trade at price* $a = 5$*. Indeed, since we only consider positions that can be liquidated for sure, the cancellation of a transfer at price a (with re-transfer at a later time to asset 1), would lead to a strict improvement and thus an arbitrage. Since this would contradict to the (NA) property of* $(\widetilde{\Pi}_t)_{t=0}^3$ *shown in Step 2, we can exclude such silly trades in the following considerations.*

By the arguments leading to (2.4.6), it follows that $e^2 - e^1 \notin -\widetilde{\mathcal{A}}_1^3$*. Consequently, we have* $\mathcal{A}_0^1 \cap -\widetilde{\mathcal{A}}_1^3 = \operatorname{cone}(e^3 - e^1, e^4 - e^1) + L^0(\operatorname{cone}(e^1 - e^4), \mathcal{F}_1) \subseteq \widetilde{\mathcal{A}}_1^3$.

Now consider the case $t = 2$*. Let* $w \in \mathcal{A}_0^2 \cap -\widetilde{\mathcal{A}}_2^3$*, i.e. we may write* $w = \xi_0 + \xi_1 + \xi_2 = -\widetilde{\xi}_2 - \widetilde{\xi}_3$ *for* $\xi_0 \in \operatorname{cone}(e^2 - e^1, e^3 - e^1, e^4 - e^1)$*,* $\xi_1 \in L^0(\operatorname{cone}(e^4 - e^1), \mathcal{F}_1)$*,* $\xi_2, \widetilde{\xi}_2 \in L^0(\operatorname{cone}(e^1 - \pi_2^{31} e^3, e^1 - \pi_2^{41} e^4), \mathcal{F}_2)$ *and* $\widetilde{\xi}_3 \in L^0(\operatorname{cone}(e^1 - \pi^{21} e^2, e^1 - \pi^{31} e^3), \mathcal{F}_3)$ *with the restrictions* $\xi_0^3 + \xi_2^3 + \widetilde{\xi}_2^3 \ge 0$*,* $\xi_0^4 + \xi_1^4 \ge 0$ *and* $\xi_0^4 + \xi_1^4 + \xi_2^4 + \widetilde{\xi}_2^4 = 0$*. Indeed, this is a consequence of (NA) and the avoidance of silly trades. But, then we may consider* $v := w - w = \xi_0 + \xi_1 + \xi_2 - (-\widetilde{\xi}_2 - \widetilde{\xi}_3)$*. Note that we have* $v^i = 0$ *for* $i = 2,3,4$*, which uniquely determines* v^1 *as*

$$\begin{aligned} v^1(n,m,i,j) = {} & \xi_0^2 \cdot (j + 1/4) + i \cdot \left(\xi_0^3 - \frac{\xi_0^4 + \xi_1^4(n)}{n}\right) \\ & + \left(\xi_0^3 + \xi_2^3(n,m,i) + \widetilde{\xi}_2^3(n,m,i)\right) \frac{-j}{m}(1+i). \end{aligned}$$

On the other hand, we also have $v^1 = 0$*. Since the first two terms do not depend on* $m \in \mathbb{N}$*, we must have that*

$$\xi_0^2 \cdot (j + 1/4) + i \cdot \left(\xi_0^3 - \frac{\xi_0^4 + \xi_1^4(n)}{n}\right) = 0.$$

Considering $j = -1/2$ *and* $i = \pm 1/2$*,* $\xi_0^2 \ge 0$ *implies that* $\xi_0^2 = 0$ *and*

$$\xi_0^3 - \left(\xi_0^4 + \xi_1^4(n)\right)/n = 0 \textit{ for all } n \in \mathbb{N}.$$

Hence, $\xi_0^4 = -\xi_1^4(n)$ *for all* $n \in \mathbb{N}$ *and* $\xi_0^3 = 0$*. Consequently, we also have* $\xi_2^3(n,m,i) = \widetilde{\xi}_2^3(n,m,i) = 0$*. But, then we have shown* $w = 0$*, which is tantamount to* $\mathcal{A}_0^2 \cap -\widetilde{\mathcal{A}}_2^3 = \{0\} \subseteq \widetilde{\mathcal{A}}_2^3$*.*

The same arguments apply for $t = 3$ *and thus* $\mathcal{A}_0^3 \cap -\widetilde{\mathcal{A}}_3^3 = \{0\} \subseteq \widetilde{\mathcal{A}}_3^3$*.*

Remark 2.4.4. *Example 2.4.3 also allows us to discuss the following related question: Does the existence of a bid-ask process* $(\widehat{\Pi}_t)_{t=0}^T$ *with* $\widehat{\Pi}_t \leq \Pi_t$ *a.s. for all* $t = 0,\dots,T$ *such that* $(\widehat{\Pi}_t)_{t=0}^T$ *satisfies (NA) and*

$$\sum_{t=0}^{T} \xi_t = 0 \text{ a.s. with } \xi_t \in L^0(-K_t,\mathcal{F}_t) \Rightarrow \xi_t \in L^0(\widehat{K}_t^0,\mathcal{F}_t) \textit{ for all } t = 0,\dots,T \tag{2.4.7}$$

already imply the absence of an approximate arbitrage, i.e. $\overline{\mathcal{A}_0^T} \cap L^0(\mathbb{R}_+^d) = \{0\}$*? Hereby* (2.4.7) *means that each transaction which is involved in a null-strategy in the original model is carried out at frictionless prices in the adjusted market.*

It turns out that the answer to the question is negative. Indeed, in the setting of Example 2.4.3, we define the adjusted bid-ask process $(\widehat{\Pi}_t)_{t=0}^3$ *by*

$$\widehat{\Pi}_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & 1 & \cdot & \cdot \\ a & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix}, \quad \widehat{\Pi}_1 = \begin{pmatrix} 1 & a & a & 1 \\ a & 1 & \cdot & \cdot \\ a & \cdot & 1 & \cdot \\ 1 & \cdot & \cdot & 1 \end{pmatrix},$$

$\widehat{\Pi}_2 = \Pi_2$ *and* $\widehat{\Pi}_3 = \Pi_3$*. Then, we have* $\widehat{\Pi}_t \leq \Pi_t$ *a.s. for all* $t = 0,1,2,3$*. In addition, the previously considered adjusted bid-ask process* $(\widetilde{\Pi}_t)_{t=0}^3$ *satisfies (NA) and yields better terms of trade than* $(\widehat{\Pi}_t)_{t=0}^3$*, i.e.,* $\widehat{\mathcal{A}}_0^T \subseteq \widetilde{\mathcal{A}}_0^T$*. Consequently,* $(\widehat{\Pi}_t)_{t=0}^3$ *inherits the (NA) property. In addition, the only (up to multiplication with non-negative scalars) null strategy in the original market is* $(\vartheta_t - \vartheta_{t-1})_{t=0}^3 = (e^4 - e^1, e^1 - e^4, 0, 0)$*. Thus, condition* (2.4.7) *is satisfied, but, as we have shown,* $\overline{\mathcal{A}_0^T} \cap L^0(\mathbb{R}_+^d) \supsetneq \{0\}$*.*

The key observation is that the bid-ask process $(\widehat{\Pi}_t)_{t=0}^3$ *satisfies (2.4.7) but not (2.3.1). This means, a transfer* $\xi_s \in L^0(-K_s,\mathcal{F}_s)$*,* $s \in \{0,\dots,3\}$*, which can be extended to a null-strategy* $(\xi_0,\dots,\xi_3)$ *in the original market* $(\Pi_t)_{t=0}^3$ *is frictionless in the better market* $(\widehat{\Pi}_t)_{t=0}^3$*, i.e.,* $\xi_s \in L^0((-\widehat{K}_s) \cap \widehat{K}_s, \mathcal{F}_s)$*. On the other hand, by the extension of the market from* $(\Pi_t)_{t=0}^3$ *to* $(\widehat{\Pi}_t)_{t=0}^3$*, additional null-strategies occur, and their transfers need not be frictionless.*

The bid-ask process $\widehat{\Pi}$ *is obtained by the following adjustment of the trading prices:*

$$\widehat{\pi}_t^{ij} = \frac{1}{\pi_t^{ji}} \mathbb{1}_{B_t^{ji}} + \pi_t^{ij} \mathbb{1}_{\Omega \setminus B_t^{ji}}, \tag{2.4.8}$$

where B^{ji} *is given by*

$$\mathbb{1}_{B_t^{ji}} = \operatorname{esssup}\{\mathbb{1}_B \ : \ B \in \mathcal{F}_t \textit{ with } -(e^i - \pi_t^{ji} e^j)\mathbb{1}_B \in \mathcal{A}_0^T\}, \tag{2.4.9}$$

with the essential supremum w.r.t. the σ-algebra $\mathcal{F}_t$. This means, if the transfer $(e^i - \pi_t^{ji} e^j)\mathbb{1}_B$, $B \in \mathcal{F}_t$ can be compensated by trades during $\{0, \ldots, T\}$, the adjusted price $\widehat{\pi}_t^{ij}$ allows to compensate it by trading at time t only. In this special example, one has $\mathbb{1}_{B_0^{14}} = 1$, $\mathbb{1}_{B_1^{41}} = 1$ and $\mathbb{1}_{B_t^{ji}} = 0$ for all other triplets (j, i, t).

The adjustment (2.4.8)/(2.4.9) *ensures that the null-strategies in the original market are frictionless in the extended market (see the arguments on pages 592 and 593 of Jacka et al. [44]). But, it was a finding by Jacka et al. [44] (see the paragraph before Definition 3.2 therein) that the latter is not sufficient to obtain closedness of the set $\widehat{\mathcal{A}}_0^T$ of attainable portfolio values in the extended market. Thus, they define their "adjusted trading prices" slightly different from (2.4.8)/(2.4.9) with $\mathcal{A}_0^T$ replaced by its closure in probability $\overline{\mathcal{A}_0^T}$. Example 2.4.3 shows that the price adjustment* (2.4.8)/(2.4.9) *with $\mathcal{A}_0^T$ instead of $\overline{\mathcal{A}_0^T}$ does not imply that the set $\widehat{\mathcal{A}}_0^T$ of attainable portfolio values is closed. Thus the example justifies the approach by Jacka et al. [44].*

On the other hand, in Example 3.3 of [44] with $T = 1$, the adjustment (2.4.8)/(2.4.9) *with $\mathcal{A}_0^T$ already yields that the set $\widehat{\mathcal{A}}_0^T$ of attainable portfolio values is closed. Indeed, in Example 3.3 of [44], the price adjustment* (2.4.8)/(2.4.9) *consists of changing π_0^{12} to $\widehat{\pi}_0^{12} = 1 = 1/\pi_0^{21}$ and π_1^{21} to $\widehat{\pi}_1^{21} = 1 = 1/\pi_1^{12}$ on Ω, which makes the null-strategy $(e^1 - e^2, e^2 - e^1)$ of the original market frictionless in the extended market. Considering the scenario $\omega = 2$, it can easily be seen that $\widehat{\mathcal{A}}_0^0 \cap (-\widehat{\mathcal{A}}_1^1) = \mathrm{cone}(e^1 - e^2, e^2 - e^1, e^3 - e^4)$, which is contained in $\widehat{\mathcal{A}}_1^1$. From this one can easily deduce that the extended market $\widehat{\Pi}$ satisfies (NA^{ps}). Consequently, $\widehat{\mathcal{A}}_0^1$ is closed by Theorem 2.2.6.*

Chapter 3

Semimartingale price systems in models with transaction costs beyond efficient friction

3.1 Introduction

In frictionless markets, asset price processes have to be semimartingales unless they allow for an "unbounded profit with bounded risk" (UPBR) with simple strategies (see Delbaen and Schachermayer [22]). With semimartingale price processes, the powerful tools of stochastic calculus can be used to construct the gains from dynamic trading. A trading strategy specifying the amounts of shares an investor holds in her portfolio is a predictable process that is integrable w.r.t. the vector-valued price process. Strategies can be of infinite variation since in the underlying limiting procedure, one directly considers the (book) profits made rather than the portfolio rebalancings.

On the other hand, under arbitrary small transaction costs also non-semimartingales can lead to markets without "approximate arbitrage opportunities". Guasoni [35] and Guasoni, Rásonyi, and Schachermayer [37] derive the sufficient condition of "conditional full support" of the mid-price process, that is satisfied, e.g., by a fractional Brownian motion, and arbitrary small constant proportional costs. Guasoni, Rásonyi, and Schachermayer [38] derive a fundamental theorem of asset pricing for a family of transaction costs models.

Under the assumptions of efficient friction, i.e., nonvanishing bid-ask spreads, and the existence of a strictly consistent price system, Kabanov and Stricker [56] and Campi and Schachermayer [11] show for continuous and càdlàg processes, respectively, that a finite credit line implies that the variation of the trading strategies is bounded in probability. A similar assertion is shown in Guasoni, Lépinette, and Rásonyi [36] under the condition of "robust no free lunch with vanishing risk". An important consequence for hedging and portfolio optimization is that the set of portfolios that are attainable with strategies of finite variation is Fatou-closed. For a detailed discussion, we refer to the monograph of Kabanov and Safarian [55].

In this chapter, we consider càdlàg bid and ask price processes that are not necessarily different. The ask price is bigger or equal to the bid price. The spread, which models the transaction costs, can vary in time and can even vanish. The contribution of this chapter is twofold. First, we show that if the bid-ask model satisfies “no unbounded profit with bounded risk” (NUPBR) for simple long-only strategies, then there exists a semimartingale lying between the bid and the ask price process. This generalizes Theorem 7.2 of Delbaen and Schachermayer [22] for the frictionless case. The proof in [22] is very intuitive. Roughly speaking, it first shows that an explosion of the quadratic increments of the price process along stopping times would lead to an (UPBR). Then, it considers a discrete-time Doob decomposition of the asset price process and shows that an explosion of the drift part as the mesh of the grid tends to zero would lead to an (UPBR). This already yields that under (NUPBR), the asset price process has to be a good integrator and thus a semimartingale by the Bichteler-Dellacherie theorem. More recently, Beiglböck, Schachermayer, and Veliyev [6] provide an alternative proof of the Bichteler-Dellacherie theorem combining these no-arbitrage arguments with Komlós type arguments. Kardaras and Platen [60] follow a quite different approach that only requires long investments. They construct supermartingale deflators as dual variables in suitable utility maximization problems under a variation of (NUPBR) for simple long-only strategies. Bálint and Schweizer [2] assume that asset prices are expressed in a possibly nontradable accounting unit. In their setting there need not exist an asset with a strictly positive price process that can be used as a numéraire. They show that if there exists a portfolio with strictly positive value process then, under a discounting invariant form of absence of arbitrage, which generalizes the condition used in [60], the asset prices discounted by the portfolio value are semimartingales. Since in transaction costs models it is natural to start with the relative prices of the tradable assets, there is no obvious analogy of discounting by a portfolio value. In our model, we implicitly assume the existence of an asset with strictly positive price process that serves as a reference asset.

In the bid-ask model, we consider a Dynkin zero-sum stopping game in which the lower payoff process is the bid price and the upper payoff process the ask price. The Doob decomposition of the dynamic value of the discrete-time game along arbitrarily fine grids is used to identify smart investment opportunities. The crucial point is that the drift of the Dynkin value can be earned by trading in the bid-ask market. This we combine with the brilliant idea in Lemma 4.7 of [22] to control the martingale part. We complete the proof by showing that under the assumptions above, the continuous-time Dynkin value has to be a local quasimartingale.

In the second part of the chapter, we show how a semimartingale between the bid and the ask process can be used to define the self-financing condition of the model beyond efficient friction. Without efficient friction, strategies of infinite variation can make sense since they do not produce infinite trading costs. This of course means that we cannot use them as integrators without major hesitation. In the first step, we only consider bounded amounts of risky assets. Thus, the trading gains charged in the semimartingale are finite. Then, we add the costs caused by the fact that the trades are carried out at the less favorable bid-ask prices. Roughly speaking, if the spread is away from zero the costs are a Riemann-Stieltjes integral similar to Guasoni, Lépinette, and Rásonyi [36]. Then, we exhaust the costs when the spread is away from zero. The crucial point is that these costs are always nonnegative, and the semimartingale gains are finite. Especially, infinite costs cannot be compensated and lead to ruin. Under a rather mild additional assumption on the behavior of the spread at zero (see Assumption 3.3.18), that goes at least far beyond the frictionless case and the case of efficient friction, this approach leads to a well-founded self-financing condition. Especially, the self-financing risk-less position does not depend on the choice of the semimartingale we use in the construction (see Corollary 3.3.22).

A self-financing condition for general strategies has to be justified by *suitable* approximations with simple strategies. With transaction costs, this is a delicate issue. Namely, under pointwise convergence of the strategies alone, one should not expect that portfolio processes converge. By the strict Fatou-type inequality (see Theorem A.9(iv) of [36]), some variation/costs can disappear in the limit. Thus, roughly speaking, we postulate the following: first, the limit strategy is better than all (almost) pointwise converging simple strategies and second, for each strategy there exists a special sequence of approximating simple strategies s.t. the wealth processes converge (see Theorem 3.3.19).

In the second step, we extend the self-financing condition from the bounded strategies to the maximal set of strategies for which it can be defined in a "reasonable" way. In the special case of a frictionless market, this maximal set coincides with the set of predictable processes which are integrable w.r.t. the semimartingale price process in the classic sense (see, e.g., [46]). Thus, we also provide a further characterization of this ubiquitous set.

In the no-arbitrage theory, the need for general strategies is already proven in the special case of frictionless markets. Indeed, Delbaen and Schachermayer [22, Lemma 7.9 and Lemma 7.10] provide an example with a bounded asset price process showing that no free lunch with vanishing risk (NFLVR) for simple strategies does not imply the existence of an equivalent martingale measure (EMM). Consequently, under transaction costs general strategies can become an important tool to guarantee the existence of a consistent price system (CPS), which plays a similar role as an EMM in the frictionless theory, under an appropriate no-arbitrage condition. On the other hand, in general a CPS does not exist even though (NFLVR) for multivariate portfolio processes is satisfied. This can already be seen in discrete time (see Schachermayer [85, Example 3.1]) with the observation that general strategies as described in Definition 3.4.1 coincide with simple strategies if the time is discrete.

In a nutshell, we provide a well-founded self-financing condition for models beyond efficient friction by relating the original trading gains under transaction costs with the gains in a fictitious frictionless market defined by a semimartingale and subtracting the appropriate costs. The idea to relate markets under transaction costs with fictitious frictionless markets is not new. It is already widely used in the theory of portfolio optimization. Here, shadow price processes, i.e., fictitious frictionless pricing systems that lead to the same *optimal* decisions and trading gains as under transaction costs, are utilized to determine optimal trading strategies. The existence of shadow prices and their relationship with a suitable dual problem goes back to Cvitanić and Karatzas [15]. In discrete time, Kallsen and Muhle-Karbe [57] show that on finite probability spaces shadow price processes always exist as long as the original problem has a solution, and Czichowsky et al. [16] provide counterexamples on infinite probability spaces. Conditions for the existence of a shadow price process in a semimartingale model are established by Czichowsky et al. [19] and starting with Kallsen and Muhle-Karbe [58] various explicit constructions of shadow prices processes have been given in Black-Scholes type models. Even in non-semimartingale models this *dual* approach is successfully applied (see, e.g., [17, 18, 20]) under efficient friction. In the proof of Theorem 3.4.5, we provide a direct connection between our work and shadow price processes for particular optimization problems.

The chapter is organized as follows. In Section 3.2, we show the existence of a semimartingale price system (Theorem 3.2.7). In Section 3.3, we construct the cost process which allows us to introduce the self-financing condition for bounded strategies, which is justified by Theorem 3.3.19 and Corollary 3.3.22. In Section 3.4, the extension to unbounded strategies is established (Proposition 3.4.2). In addition, the special case of a frictionless market is considered (Proposition 3.4.3) and the separate convergence of trading gains and cost terms of the approximating bounded strategies is discussed (Theorem 3.4.5). Technical proofs are postponed to Section 3.5 and Section 3.6.

3.2 Existence of a semimartingale price system

Throughout the chapter, we fix a terminal time $T \in \mathbb{R}_+$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions. The predictable σ-algebra on $\Omega \times [0,T]$ is denoted by $\mathcal{P}$, the set of bounded predictable processes starting at zero by $\mathbf{b}\mathcal{P}$. To simplify the notation, a stopping time τ is allowed to take the value ∞, but $[\![\tau]\!] := \{(\omega, t) \in \Omega \times [0,T] : t = \tau(\omega)\}$. Especially, we use the notation τ_A, $A \in \mathcal{F}_\tau$, for the stopping time that coincides with τ on A and is infinite otherwise. $\mathrm{Var}_a^b(X)$ denotes the pathwise variation of a process X on the interval $[a, b]$. A process X is called làglàd iff all paths possess finite left and right limits (but they can have double jumps). We set $\Delta^+X := X_+ - X$ and $\Delta X := \Delta^- X := X - X_-$, where $X_{t+} := \lim_{s\downarrow t} X_s$ and $X_{t-} := \lim_{s\uparrow t} X_s$. For a random variable Y, we set $Y^+ := \max(Y, 0)$ and $Y^- := \max(-Y, 0)$.

The financial market consists of one risk-free bond with price 1 and one risky asset with bid price $\underline{S}$ and ask price $\overline{S}$. Throughout the chapter, we make the following assumption.

Assumption 3.2.1. *$(\underline{S}_t)_{t\in[0,T]}$ and $(\overline{S}_t)_{t\in[0,T]}$ are adapted processes with càdlàg paths. In addition, $\underline{S}_t \le \overline{S}_t$ for all $t \in [0,T]$ and $\underline{S}$ is locally bounded from below.*

In this section, we only consider simple trading strategies in the following sense.

Definition 3.2.2. A *simple trading strategy* is a stochastic process $(\varphi_t)_{t\in[0,T]}$ of the form

$$\varphi = \sum_{i=1}^{n} Z_{i-1} \mathbb{1}_{]\!]T_{i-1},T_i]\!]}, \tag{3.2.1}$$

where $n \in \mathbb{N}$ is a finite number, $0 = T_0 \leq T_1 \leq \cdots \leq T_n = T$ is an increasing sequence of stopping times and Z_i is $\mathcal{F}_{T_i}$-measurable for all $i = 0, \ldots, n-1$.

The strategy φ specifies the amount of risky assets in the portfolio. The next definition corresponds to the self-financing condition of the model. It specifies the holdings in the risk-free bond given a simple trading strategy.

Definition 3.2.3. Let $(\varphi_t)_{t\in[0,T]}$ be a simple trading strategy. The *corresponding position in the risk-free bond* $(\varphi_t^0)_{t\in[0,T]}$ is given by

$$\varphi_t^0 := \sum_{0\leq s<t} \left(\underline{S}_s(\Delta^+\varphi_s)^- - \overline{S}_s(\Delta^+\varphi_s)^+ \right), \quad t \in [0,T]. \tag{3.2.2}$$

Definition 3.2.4. Let $(\varphi_t)_{t\in[0,T]}$ be a simple trading strategy. The *liquidation value process* $(V_t^{\mathrm{liq}}(\varphi))_{t\in[0,T]}$ is given by

$$V_t^{\mathrm{liq}}(\varphi) := \varphi_t^0 + (\varphi_t)^+\underline{S}_t - (\varphi_t)^-\overline{S}_t, \quad t \in [0,T]. \tag{3.2.3}$$

If it is clear from the context, we write $(V_t^{\mathrm{liq}})_{t\in[0,T]}$ instead of $(V_t^{\mathrm{liq}}(\varphi))_{t\in[0,T]}$.

We adapt the notion of an unbounded profit with bounded risk (UPBR) from Bayraktar and Yu [4] to the present setting of simple long-only trading strategies.

Definition 3.2.5. We say that $(\underline{S}_t, \overline{S}_t)_{t\in[0,T]}$ admits an *unbounded profit with bounded risk (UPBR) for simple long-only strategies* if there exists a sequence of simple trading strategies $(\varphi^n)_{n\in\mathbb{N}}$ with $\varphi^n \geq 0$ s.t.

(i) $V_t^{\mathrm{liq}}(\varphi^n) \geq -1$ for all $t \in [0,T]$ and $n \in \mathbb{N}$,

(ii) The sequence $(V_T^{\mathrm{liq}}(\varphi^n))_{n\in\mathbb{N}}$ is unbounded in probability, i.e.,

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left(V_T^{\mathrm{liq}}(\varphi^n) \geq m\right) > 0. \tag{3.2.4}$$

If no such sequence exists, we say that the bid-ask process $(\underline{S}, \overline{S})$ satisfies the *no unbounded profit with bounded risk (NUPBR) condition for simple long-only strategies.*

Remark 3.2.6. *The admissibility condition (i) is rather restrictive, e.g., compared to [36], see Definition 4.4. therein, which means that the present version of (NUPBR) is a weak condition. But, for the following first main result of the chapter, it is already sufficient.*

Theorem 3.2.7. *Let $(\underline{S}_t, \overline{S}_t)_{t\in[0,T]}$ satisfy Assumption 3.2.1 and the (NUPBR) condition for simple long-only strategies. Then, there exists a semimartingale $S = (S_t)_{t\in[0,T]}$ s.t.*

$$\underline{S}_t \leq S_t \leq \overline{S}_t \quad \textit{for all } t \in [0,T]. \tag{3.2.5}$$

A semimartingale S satisfying (3.2.5) we call a *semimartingale price system*. The remaining part of the section is devoted to the proof of Theorem 3.2.7. As a first step, we will show that it is actually sufficient to prove the following seemingly weaker version of the theorem.

Theorem 3.2.8. *Suppose that $0 \leq \underline{S} \leq \overline{S} \leq 1$, and that (NUPBR) for simple long-only strategies holds. Then there exists a semimartingale $S = (S_t)_{t\in[0,T]}$ s.t.*

$$\underline{S}_t \leq S_t \leq \overline{S}_t \quad \textit{for all } t \in [0,T].$$

Proposition 3.2.9. *Theorem 3.2.8 implies Theorem 3.2.7.*

Proof. We assume that Theorem 3.2.8 holds true.

Step 1: Let $\underline{S}$ be locally bounded from below, $\overline{S} \leq 1$, and $(\underline{S}, \overline{S})$ satisfies (NUPBR). Thus, there is an increasing sequence $(\sigma^n)_{n\in\mathbb{N}}$ of stopping times with $\mathbb{P}(\sigma^n = \infty) \to 1$ s.t. $\underline{S} \geq -n$ on $[\![0, \sigma^n]\!]$ for all $n \in \mathbb{N}$. With $(\underline{S}, \overline{S})$, a fortiori $((\underline{S}^{\sigma^n} + n)/(n+1), (\overline{S}^{\sigma^n} + n)/(n+1))$ satisfies (NUPBR). By Theorem 3.2.8, there is a semimartingale S^n for each $n \in \mathbb{N}$ s.t. $(\underline{S}^{\sigma^n} + n)/(n+1) \leq S^n \leq (\overline{S}^{\sigma^n} + n)/(n+1)$. Therefore, the process $S := \sum_{n=1}^{\infty} \mathbb{1}_{[\![\sigma^{n-1},\sigma^n[\![}((n+1)S^n - n)$, where $\sigma^0 := 0$, lies between $\underline{S}$ and $\overline{S}$. S is a local semimartingale and, thus, a semimartingale. Consequently, Theorem 3.2.8 holds true under the milder condition that $\underline{S}$ is only locally bounded from below instead of nonnegative.

Step 2: Let $\underline{S}$ be locally bounded from below and $(\underline{S}, \overline{S})$ satisfies (NUPBR) for simple long-only strategies. Consider the stopping times $\tau^n := \inf\{t \geq 0 : \overline{S}_t > n\}$, $n \in \mathbb{N}$. One has that $\mathbb{P}(\tau^n = \infty) = \mathbb{P}(\overline{S}_t \leq n \ \forall t \in [0,T]) \to 1$ as $n \to \infty$. With short-selling constraints, liquidation value processes that are attainable in the market $((\underline{S}^{\tau^n}/n) \wedge 1, (\overline{S}^{\tau^n}/n) \wedge 1)$ can be dominated by those in $(\underline{S}, \overline{S})$. Indeed, for $t < \tau^n$, one has $(\overline{S}_t^{\tau^n}/n) \wedge 1 = \overline{S}_t/n$, and a purchase at time τ^n cannot generate a profit in the market $((\underline{S}^{\tau^n}/n) \wedge 1, (\overline{S}^{\tau^n}/n) \wedge 1)$. Thus, $((\underline{S}^{\tau^n}/n) \wedge 1, (\overline{S}^{\tau^n}/n) \wedge 1)$ satisfies (NUPBR) with simple *long-only* strategies and by Step 1 there exist semimartingales S^n with $(\underline{S}^{\tau^n}/n) \wedge 1 \leq S^n \leq (\overline{S}^{\tau^n}/n) \wedge 1$ for all $n \in \mathbb{N}$. Then, $S := \sum_{n=1}^{\infty} \mathbb{1}_{[\![\tau^{n-1},\tau^n[\![} nS^n$, where $\tau^0 := 0$, shows the assertion. □

For the remainder of the section, we work under the assumptions of Theorem 3.2.8. More specifically we assume the following.

Assumption 3.2.10. *We assume $0 \leq \underline{S} \leq \overline{S} \leq 1$ and that $(\underline{S}, \overline{S})$ satisfies (NUPBR) for simple long-only strategies for the remainder of the section.*

In addition, we set w.l.o.g. $T = 1$. We now proceed with the proof of Theorem 3.2.8. The candidate for the semimartingale will be the value process of a Dynkin zero-sum stopping game played on the bid and ask price, i.e., let $(S_t)_{t\in[0,1]}$ be the right-continuous version of

$$\begin{aligned} S_t &:= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,1}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_{t,1}} \mathbb{E}\left[\underline{S}_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + \overline{S}_\sigma \mathbb{1}_{\{\tau>\sigma\}} \mid \mathcal{F}_t\right] \\ &= \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}_{t,1}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{t,1}} \mathbb{E}\left[\underline{S}_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + \overline{S}_\sigma \mathbb{1}_{\{\tau>\sigma\}} \mid \mathcal{F}_t\right], \end{aligned} \tag{3.2.6}$$

where $\mathcal{T}_{t,1}$ is the set of $[t, 1]$-valued stopping times for $t \in [0, 1]$. The existence of such a process and the non-trivial equality in (3.2.6) is guaranteed by *Théorème 7 & 9 and Corollaire 12* in [68]. Obviously, $S = (S_t)_{t\in[0,1]}$ satisfies $\underline{S} \leq S \leq \overline{S}$. Thus, we only have to show that (NUPBR) for simple long-only trading strategies implies that S is a semimartingale. We note that all arguments remain valid for a different terminal value of the game between $\underline{S}_1$ and $\overline{S}_1$.

The arguments below also provide a financial interpretation of the value process S of this Dynkin game. In the special case that the terminal bid- and ask price coincide, a discrete-time approximation of S can be interpreted as a shadow price for a utility maximization problem with a risk-neutral investor and the constraint that her dynamic stock position has to take values in $[-1, 1]$. Put differently, in the bid-ask market, an investor can earn the same expected profit as via an optimal strategy in the frictionless market with price process S (besides a finite deviation caused by different liquidation values).

Next, we recall the notion of a quasimartingale and Rao's Theorem (see, e.g., Theorem 17 in [74, Chapter 3] or Theorem 3.1 in [5]).

Definition 3.2.11. Let $X = (X_t)_{t\in[0,1]}$ be an adapted process s.t. $\mathbb{E}(|X_t|) < \infty$ for all $t \in [0, 1]$. Given a deterministic partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0, 1]$ the *mean-variation of X along π* is defined as

$$MV(X, \pi) := \mathbb{E}\left[\sum_{t_i\in\pi} \left|\mathbb{E}\left[X_{t_i} - X_{t_{i+1}} \mid \mathcal{F}_{t_i}\right]\right|\right]$$

and the *mean variation of X* is defined as

$$MV(X) := \sup_\pi MV(X, \pi).$$

Finally, X is called a *quasimartingale* if $MV(X) < \infty$.

Theorem 3.2.12 (Rao). *Let X be an adapted right-continuous process. Then, X is a quasimartingale if and only if X has a decomposition $X = Y - Z$ where Y and Z are each positive right-continuous supermartingales. In this case, the paths of X are a.s. càdlàg.*

Remark 3.2.13. *Usually, Rao's theorem is formulated for an adapted* càdlàg *process X. However, to show that X can be written as the difference of two right-continuous supermartingales, the existence of the finite left limits of X is not needed (see the proofs of Theorem 8.13 in [42] or Theorem 14 in [74, Chapter 3]). On the other hand, right-continuous supermartingales possess a.s. finite left limits (see Theorem VI.3 in [25]). This means that the theorem can be formulated for an a priori only right-continuous quasimartingale that turns out to be càdlàg.*

If we can show that the right-continuous process S is a local quasimartingale, Rao's theorem (in the version of Theorem 3.2.12) yields that S can locally be written as the difference of two supermartingales, and it admits a càdlàg modification. Thus, S is a semimartingale by the Doob-Meyer-Theorem (Case without Class D) [74, Chapter 3, Theorem 16]. Hence, we now want to show that S is a local quasimartingale.

For this, we consider a discrete-time approximation $S^n = (S^n_t)_{t\in D_n}$ of S on the set $D_n := \{0, 1/2^n, \dots (2^n-1)/2^n, 1\}$ of dyadic numbers defined by $S^n_1 = \underline{S}_1$ and

$$S^n_t := \min\left(\overline{S}_t, \max\left(\underline{S}_t, \mathbb{E}\left[S^n_{t+1/2^n} \mid \mathcal{F}_t\right]\right)\right), \quad t \in D_n,\ t < 1. \tag{3.2.7}$$

Indeed, it is well-known (see, e.g., [72, Proposition VI-6-9]) that

$$\begin{aligned} S^n_t &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}^n_{t,1}} \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}^n_{t,1}} \mathbb{E}\left[\underline{S}_\tau \mathbb{1}_{\{\tau\le\sigma\}} + \overline{S}_\sigma \mathbb{1}_{\{\tau>\sigma\}} \mid \mathcal{F}_t\right] \\ &= \operatorname*{ess\,inf}_{\sigma\in\mathcal{T}^n_{t,1}} \operatorname*{ess\,sup}_{\tau\in\mathcal{T}^n_{t,1}} \mathbb{E}\left[\underline{S}_\tau \mathbb{1}_{\{\tau\le\sigma\}} + \overline{S}_\sigma \mathbb{1}_{\{\tau>\sigma\}} \mid \mathcal{F}_t\right], \quad t \in D_n, \end{aligned} \tag{3.2.8}$$

where $\mathcal{T}^n_{t,1}$ denotes the set of all $\{t, t+1/2^n, \dots, 1\}$-valued stopping times. The following proposition generalizes Kifer [61, Proposition 3.2] from continuous processes to right-continuous processes.

Proposition 3.2.14. *Let $m \in \mathbb{N}$ and $t \in D_m$. Then, we have*

$$\lim_{\substack{n\to\infty \\ n\ge m}} S^n_t = S_t \quad \mathbb{P}\text{-a.s.}$$

Proof. Let $n \in \mathbb{N}$ with $n \ge m$ and $t \in D_m$. The pair of $\{t, t+1/2^n, \dots, 1\}$-valued stopping times

$$\begin{aligned} \tau^n_t &:= \inf\{s \ge t : s \in D_n, S^n_s = \underline{S}_s\}, \\ \sigma^n_t &:= \inf\{s \ge t : s \in D_n, S^n_s = \overline{S}_s\} \end{aligned}$$

is a Nash equilibrium of the discrete-time game started at time t, i.e.,

$$\mathbb{E}\left[R(\tau, \sigma^n_t) \mid \mathcal{F}_t\right] \le S^n_t \le \mathbb{E}\left[R(\tau^n_t, \sigma) \mid \mathcal{F}_t\right] \quad \text{for all } \tau, \sigma \in \mathcal{T}^n_{t,T}, \tag{3.2.9}$$

where $R(\tau, \sigma) := \underline{S}_\tau \mathbb{1}_{\{\tau\le\sigma\}} + \overline{S}_\sigma \mathbb{1}_{\{\tau>\sigma\}}$. This follows from [72, Proposition VI-6-9] and its proof with the observation that in *finite* discrete time the assertion also holds for

$\varepsilon = 0$ by dominated convergence. For any $\tau \in \mathcal{T}_{t,T}$, we let $D_n(\tau) := \inf\{t \geq \tau : t \in D_n\}$ and

$$\eta_n(\tau)(\omega) := \sup_{s \in (\tau(\omega), \tau(\omega)+1/2^n)} \max\left(\left|\underline{S}_s(\omega) - \underline{S}_\tau(\omega)\right|, \left|\overline{S}_s(\omega) - \overline{S}_\tau(\omega)\right|\right), \quad \omega \in \Omega.$$

This yields the estimates

$$R(\tau, D_n(\sigma)) - \eta_n(\tau) \leq R(D_n(\tau), D_n(\sigma)) \leq R(D_n(\tau), \sigma) + \eta_n(\sigma) \tag{3.2.10}$$

for all $\tau, \sigma \in \mathcal{T}_{0,T}$. Let $\varepsilon > 0$. For the continuous-time game, the pair of stopping times

$$\tau_t^* := \inf\{s \geq t : S_s \leq \underline{S}_s + \varepsilon\},$$
$$\sigma_t^* := \inf\{s \geq t : S_s \geq \overline{S}_s - \varepsilon\}$$

is an ε-Nash equilibrium, i.e.,

$$\mathbb{E}\left[R(\tau, \sigma_t^*) \mid \mathcal{F}_t\right] - \varepsilon \leq S_t \leq \mathbb{E}\left[R(\tau_t^*, \sigma) \mid \mathcal{F}_t\right] + \varepsilon, \quad \text{for all } \tau, \sigma \in \mathcal{T}_{t,T}. \tag{3.2.11}$$

This is shown in Corollaire 12 and its proof in [68]. Combining the first inequality in (3.2.9) with $\tau = D_n(\tau_t^*)$, the first inequality in (3.2.10) and the second inequality in (3.2.11) yields

$$\begin{aligned} S_t^n &\geq \mathbb{E}\left[R\left(D_n(\tau_t^*), \sigma_t^n\right) \mid \mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[R(\tau_t^*, \sigma_t^n) \mid \mathcal{F}_t\right] - \mathbb{E}\left[\eta_n(\tau_t^*) \mid \mathcal{F}_t\right] \\ &\geq S_t - \varepsilon - \mathbb{E}\left[\eta_n(\tau_t^*) \mid \mathcal{F}_t\right]. \end{aligned}$$

Similar, applying the second inequality (3.2.9) with $\sigma = D_n(\sigma_t^*)$, the second inequality in (3.2.10) and the first inequality in (3.2.11), yields the corresponding upper estimate on S_t^n. Putting together, we get

$$S_t + \varepsilon + \mathbb{E}\left[\eta_n(\sigma_t^*) \mid \mathcal{F}_t\right] \geq S_t^n \geq S_t - \varepsilon - \mathbb{E}\left[\eta_n(\tau_t^*) \mid \mathcal{F}_t\right].$$

Finally, as $\eta_n(\tau_t^*) \to 0$ and $\eta_n(\sigma_t^*) \to 0$ a.s. by the right-continuity of $\overline{S}$ and $\underline{S}$, the dominated convergence theorem for conditional expectations implies

$$S_t + \varepsilon \geq \limsup_{\substack{n\to\infty \\ n \geq m}} S_t^n \geq \liminf_{\substack{n\to\infty \\ n \geq m}} S_t^n \geq S_t - \varepsilon \quad \mathbb{P}\text{-a.s.},$$

which is the assertion as $\varepsilon > 0$ is arbitrary. □

In the following, we will consider the discrete-time Doob-decomposition of the processes $(S^n)_{n\in\mathbb{N}}$, i.e., we write $S_t^n = S_0^n + M_t^n + A_t^n$ with

$$A_t^n := \sum_{t_i \in D_n, 0 < t_i \leq t} \mathbb{E}\left[S_{t_i}^n - S_{t_{i-1}}^n \mid \mathcal{F}_{t_{i-1}}\right], \tag{3.2.12}$$

$$M_t^n := \sum_{t_i \in D_n, 0 < t_i \leq t} \left(S_{t_i}^n - S_{t_{i-1}}^n - \mathbb{E}\left[S_{t_i}^n - S_{t_{i-1}}^n \mid \mathcal{F}_{t_{i-1}}\right]\right) \tag{3.2.13}$$

for $t \in D_n$. In particular, we have (with a slight abuse of notation)

$$MV(S^n, D_n) := \mathbb{E}\left[\sum_{t_i \in D_n} \left|\mathbb{E}\left[S^n_{t_{i+1}} - S^n_{t_i} \mid \mathcal{F}_{t_i}\right]\right|\right] = \mathbb{E}\left[\sum_{t_i \in D_n} \left|A^n_{t_{i+1}} - A^n_{t_i}\right|\right]. \quad (3.2.14)$$

The following observation is at the core of why our approach works.

Lemma 3.2.15. *Let $n \in \mathbb{N}$ and $t = 0, 1/2^n, \ldots, (2^n - 1)/2^n$. Then, we have*

$$\{A^n_{t+1/2^n} - A^n_t > 0\} \subseteq \{S^n_t = \overline{S}_t\},$$
$$\{A^n_{t+1/2^n} - A^n_t < 0\} \subseteq \{S^n_t = \underline{S}_t\}.$$

Proof. From definition (3.2.12) we get $\mathbb{E}\left[S^n_{t+1/2^n} \mid \mathcal{F}_t\right] - S^n_t = A^n_{t+1/2^n} - A^n_t$, which together with $S^n_t = \min(\overline{S}_t, \max(\underline{S}_t, \mathbb{E}[S^n_{t+1/2^n} \mid \mathcal{F}_t]))$ yields the assertion. □

We now start to establish a uniform bound on (3.2.14) (after some stopping).

Lemma 3.2.16. *Let Assumption 3.2.10 hold. Then, the set*

$$\left\{\sup_{t \in D_n} |M^n_t| : n \in \mathbb{N}\right\}$$

is bounded in probability.

Proof. Before we begin, we roughly sketch the idea of the proof. If $\{\sup_{t \in D_n} |M^n_t| : n \in \mathbb{N}\}$ failed to be bounded in probability, the same would hold in some sense for the sequence $(A^n)_{n \in \mathbb{N}}$. Indeed, this is a consequence of $S^n = S^n_0 + M^n + A^n$ and the fact that $|S^n| \leq 1$. Keeping Lemma 3.2.15 in mind, we show that by suitable long-only investments in the bid-ask market, one can earn the increasing parts of A^n without suffering from the decreasing parts. In doing so, we would achieve an (UPBR) since the gains from A^n are of a higher order than the potential losses from the martingale part M^n. The proof of the latter relies on the brilliant ideas of Delbaen and Schachermayer [22, Lemma 4.7], which we adapt to the present setting. The present setting is easier than in [22, Lemma 4.7] since the jumps of S^n are uniformly bounded.

Step 1: Assume that the claim does not hold true, i.e., there is a subsequence $(\sup_{t \in D_{m_n}} |M^{m_n}_t|)_{n \in \mathbb{N}}$ and $\alpha \in (0, 1/10)$ s.t.

$$\mathbb{P}(\sup_{t \in D_{m_n}} |M^{m_n}_t| \geq n^3) > 10\alpha, \quad n \in \mathbb{N}.$$

In the following, we write $(\sup_{t \in D_n} |M^n_t|)_{n \in \mathbb{N}}$ instead of $(\sup_{t \in D_{m_n}} |M^{m_n}_t|)_{n \in \mathbb{N}}$ in order to simplify the notation. For this, it is important to note that from now on, we do not use properties of M^n that do not hold for M^{m_n}. Let $T_n := \inf\{t \in D_n : |M^n_t| \geq n^3\}$

and define the process $(\widetilde{S}^n_t)_{t\in D_n}$ by $\widetilde{S}^n_t := \frac{1}{n^2} S^n_{t\wedge T_n}$. Note that the (discrete-time) Doob decomposition of $\widetilde{S}^n$ is given by

$$\widetilde{S}^n_t = \widetilde{S}^n_0 + \widetilde{M}^n_t + \widetilde{A}^n_t = \frac{1}{n^2} S^n_0 + \frac{1}{n^2} M^n_{t\wedge T_n} + \frac{1}{n^2} A^n_{t\wedge T_n}, \quad t \in D_n,$$

where $(\widetilde{M}^n_t)_{t\in D_n} = (\frac{1}{n^2} M^n_{t\wedge T_n})_{t\in D_n}$ is the martingale part and $(\widetilde{A}^n_t)_{t\in D_n} = (\frac{1}{n^2} A^n_{t\wedge T_n})_{t\in D_n}$ the predictable part. In addition, we have

$$\mathbb{P}(\sup_{t\in D_n} |\widetilde{M}^n_t| \geq n) > 10\alpha, \quad |\widetilde{S}^n_t - \widetilde{S}^n_{t-1/2^n}| \leq \frac{1}{n^2},\ t \in D_n. \tag{3.2.15}$$

Next, we define $T_{n,0} := 0$ and, recursively,

$$T_{n,i} := \inf\{t \geq T_{n,i-1} : t \in D_n,\ |\widetilde{M}^n_t - \widetilde{M}^n_{T_{n,i-1}}| \geq 1\}, \quad i \in \mathbb{N}.$$

Since $|A^n_t - A^n_{t-1/2^n}| \leq 1$ and thus

$$|M^n_t - M^n_{t-1/2^n}| \leq |S^n_t - S^n_{t-1/2^n}| + |A^n_t - A^n_{t-1/2^n}| \leq 2 \tag{3.2.16}$$

for all $t \in D_n \setminus \{0\}$, we get

$$|\widetilde{M}^n_{T_{n,i}\wedge 1} - \widetilde{M}^n_{T_{n,i-1}\wedge 1}| \leq 1 + |\widetilde{M}^n_{T_{n,i}\wedge 1} - \widetilde{M}^n_{(T_{n,i}-1/2^n)\wedge 1}| \leq 1 + 2/n^2 \leq 3, \tag{3.2.17}$$

for all $n, i \in \mathbb{N}$. (3.2.17) implies

$$\mathbb{P}(T_{n,i} < \infty) > 10\alpha \quad \text{for } n \in \mathbb{N} \quad \text{and} \quad i = 0, \dots, k_n, \tag{3.2.18}$$

where $k_n := \lfloor (n-1)/3 \rfloor$ denotes the integer part of $(n-1)/3$.

Next, we establish a lower bound in $L^0(\mathbb{P})$ on $(\widetilde{M}^n_{T_{n,i}\wedge 1} - \widetilde{M}^n_{T_{n,i-1}\wedge 1})^-$ for $i = 1, \dots, k_n$. The martingale property of $\widetilde{M}^n$ together with (3.2.18) implies

$$\mathbb{E}\left[(\widetilde{M}^n_{T_{n,i}\wedge 1} - \widetilde{M}^n_{T_{n,i-1}\wedge 1})^-\right] = \frac{1}{2}\mathbb{E}\left[|\widetilde{M}^n_{T_{n,i}\wedge 1} - \widetilde{M}^n_{T_{n,i-1}\wedge 1}|\right] \geq \frac{1}{2}\mathbb{P}(T_{n,i} < \infty) > 5\alpha.$$

For $B_{n,i} := \{(\widetilde{M}^n_{T_{n,i}\wedge 1} - \widetilde{M}^n_{T_{n,i-1}\wedge 1})^- \geq 2\alpha\}$, we get

$$\mathbb{E}\left[(\widetilde{M}^n_{T_{n,i}\wedge 1} - \widetilde{M}^n_{T_{n,i-1}\wedge 1})^- \mathbb{1}_{B_{n,i}}\right] \geq \mathbb{E}\left[(\widetilde{M}^n_{T_{n,i}\wedge 1} - \widetilde{M}^n_{T_{n,i-1}\wedge 1})^-\right] - 2\alpha > 3\alpha$$

and thus by (3.2.17)

$$\mathbb{P}\left(B_{n,i}\right) > \alpha \quad \text{for } n \in \mathbb{N} \quad \text{and} \quad i = 0, \dots, k_n. \tag{3.2.19}$$

We now turn our attention to the increments $(\widetilde{A}^n_{T_{n,i}\wedge 1} - \widetilde{A}^n_{T_{n,i-1}\wedge 1})_{i=1,\dots,k_n}$ for $n \in \mathbb{N}$. Since $|\widetilde{S}^n_{T_{n,i}\wedge 1} - \widetilde{S}^n_{T_{n,i-1}\wedge 1}| \leq 1/n^2$, (3.2.19) implies

$$\mathbb{P}\left(\widetilde{A}^n_{T_{n,i}\wedge 1} - \widetilde{A}^n_{T_{n,i-1}\wedge 1} \geq \alpha\right) \geq \mathbb{P}\left(\widetilde{A}^n_{T_{n,i}\wedge 1} - \widetilde{A}^n_{T_{n,i-1}\wedge 1} \geq 2\alpha - \frac{1}{n^2}\right) \geq \mathbb{P}\left(B_{n,i}\right) > \alpha$$

for all $n \geq \sqrt{\alpha}$ and $i = 1, \dots, k_n$. In particular, if we define $(\widetilde{A}_t^{n,\uparrow})_{t \in D_n}$ by

$$\widetilde{A}_t^{n,\uparrow} := \sum_{t_i \in D_n, 0 < t_i \leq t} (\widetilde{A}_{t_i}^n - \widetilde{A}_{t_{i-1}}^n)^+, \quad t \in D_n,$$

we also get

$$\mathbb{P}\left(\widetilde{A}^{n,\uparrow}_{T_{n,i} \wedge 1} - \widetilde{A}^{n,\uparrow}_{T_{n,i-1} \wedge 1} \geq \alpha\right) > \alpha \tag{3.2.20}$$

for all all $n \geq 1/\sqrt{\alpha}$ and $i = 1, \dots, k_n$.

Step 2: In the second part of the proof, we construct an (UPBR) by placing smart bets on the process $(\widetilde{A}_t^{n,\uparrow})_{t \in D_n}$. This is similar to the second part of [22, Lemma 4.7] with the major difference that we cannot invest directly into S^n. We define two sequences of $D_n \cup \{\infty\}$-valued stopping times $(\sigma_k^n)_{k=1}^{2^n}$ and $(\tau_k^n)_{k=1}^{2^n}$ by

$$\sigma_1^n := \inf\{t \in D_n \mid A^n_{t+1/2^n} - A^n_t > 0\}, \quad \tau_1^n := \inf\{t > \sigma_1^n \mid t \in D_n,\ A^n_{t+1/2^n} - A^n_t < 0\},$$

and, recursively,

$$\begin{aligned} \sigma_k^n &:= \inf\{t > \tau_{k-1}^n \mid t \in D_n,\ A^n_{t+1/2^n} - A^n_t > 0\}, \\ \tau_k^n &:= \inf\{t > \sigma_k^n \mid t \in D_n,\ A^n_{t+1/2^n} - A^n_t < 0\} \end{aligned}$$

for $k = 2, 3, \dots, 2^n$. Next, define a sequence of simple trading strategies $(\varphi^n)_{n \in \mathbb{N}}$ by

$$\varphi^n := \left(\sum_{k=1}^{2^n} \frac{1}{n^2} \mathbb{1}_{]\!]\sigma_k^n, \tau_k^n]\!]}\right) \mathbb{1}_{]\!]0, T_{n,k_n}]\!]}.$$

By Lemma 3.2.15, the strategies φ^n only buy if $S_t^n = \overline{S}_t$ and sell if $S_t^n = \underline{S}_t$, despite of a possible liquidation at T_{n,k_n} Together with $S^n_{t_i} - \underline{S}_t \leq 1$ for all $t_i \in D_n$, $t \in [0,1]$, this implies that $V^{\text{liq}}(\varphi^n)$ can be bounded from below by

$$\begin{aligned} V_t^{\text{liq}}(\varphi^n) &\geq \sum_{t_i \in D_n, 0 < t_i \leq t} \varphi^n_{t_i}(S^n_{t_i} - S^n_{t_{i-1}}) - \frac{1}{n^2} \\ &= \widetilde{A}^{n,\uparrow}_{\lfloor 2^n t \rfloor / 2^n \wedge T_{n,k_n}} + \sum_{t_i \in D_n, 0 < t_i \leq t} \varphi^n_{t_i}(M^n_{t_i} - M^n_{t_{i-1}}) - \frac{1}{n^2} \\ &\geq \sum_{t_i \in D_n, 0 < t_i \leq t} \varphi^n_{t_i}(M^n_{t_i} - M^n_{t_{i-1}}) - \frac{1}{n^2} \\ &= \sum_{t_i \in D_n, 0 < t_i \leq t} (n^2 \varphi^n_{t_i})(\widetilde{M}^n_{t_i} - \widetilde{M}^n_{t_{i-1}}) - \frac{1}{n^2}, \quad t \in [0,1]. \end{aligned} \tag{3.2.21}$$

This means that the strategy allows us to invest in $\widetilde{A}^{n,\uparrow}$, but we still do not know if it actually allows for an (UPBR) as we need to get some control on the martingale part in (3.2.21). Therefore notice that

$$\left\| \sum_{t_i \in D_n, 0<t_i \leq T_{n,k_n}} (n^2 \varphi^n_{t_i})(\widetilde{M}^n_{t_i} - \widetilde{M}^n_{t_{i-1}}) \right\|_{L^2(\mathbb{P})} \leq \left\| \widetilde{M}_{T_{n,k_n} \wedge 1} \right\|_{L^2(\mathbb{P})}$$
$$\leq \sqrt{\sum_{i=1}^{k_n} \left\| \widetilde{M}^n_{T_{n,i} \wedge 1} - \widetilde{M}^n_{T_{n,i-1} \wedge 1} \right\|^2_{L^2(\mathbb{P})}} \leq 3\sqrt{k_n}. \tag{3.2.22}$$

Thus, Doob's maximal inequality yields

$$\left\| \sup_{t \in D_n,\ t \leq T_{n,k_n}} \left| \sum_{t_i \in D_n, 0<t_i \leq t} (n^2 \varphi^n_{t_i})(\widetilde{M}^n_{t_i} - \widetilde{M}^n_{t_{i-1}}) \right| \right\|_{L^2(\mathbb{P})} \leq 6\sqrt{k_n}. \tag{3.2.23}$$

Consequently, we get the estimate

$$\mathbb{P}\left(\inf_{t \in [0, T_{n,k_n} \wedge 1]} V^{\mathrm{liq}}_t(\varphi^n) \leq -k_n^{3/4} n^{-1/8} - n^{-2} \right)$$
$$\leq \mathbb{P}\left(\sup_{t \in D_n,\ t \leq T_{n,k_n}} \left| \sum_{t_i \in D_n, 0<t_i \leq t} (n^2 \varphi^n_{t_i})(\widetilde{M}^n_{t_i} - \widetilde{M}^n_{t_{i-1}}) \right| \geq k_n^{3/4} n^{-1/8} \right) \leq \frac{36 n^{1/4}}{\sqrt{k_n}} \tag{3.2.24}$$

by Tschebyscheff's inequality. Thus, let us define the stopping times

$$U_n := \inf\{t \geq 0 \ : \ V^{\mathrm{liq}}_t(\varphi^n) \leq -k_n^{3/4} n^{-1/8} - n^{-2}\} \wedge T_{n,k_n},$$

which satisfy $\mathbb{P}(U_n < T_{n,k_n}) \leq 36 n^{1/4}/\sqrt{k_n}$. We now pass to the strategy

$$\widetilde{\varphi}^n := (k_n)^{-3/4} \varphi^n \mathbb{1}_{]\!]0,U_n]\!]}.$$

The left and right jumps of $V^{\mathrm{liq}}(\widetilde{\varphi}^n)$ are bounded from below by $-k_n^{-3/4} n^{-2}$, which is a direct consequence of $0 \leq \underline{S} \leq \overline{S} \leq 1$. We obtain

$$\inf_{t \in [0, T_{n,k_n} \wedge 1]} V^{\mathrm{liq}}_t(\widetilde{\varphi}^n) \geq -n^{-1/8} - 2k_n^{-3/4} n^{-2} \to 0, \quad \text{for } n \to \infty. \tag{3.2.25}$$

It remains to show (3.2.4). First notice that using (3.2.20) in conjunction with [22, Corollary A1.3], yields

$$\mathbb{P}\left(\widetilde{A}^{n,\uparrow}_{T_{n,k_n} \wedge 1} \geq \frac{\alpha^2}{2} \right) > \frac{\alpha}{2}.$$

It follows that

$$\mathbb{P}\left((k_n)^{-3/4} \widetilde{A}^{n,\uparrow}_{T_{n,k_n} \wedge \frac{\lfloor 2^n U_n \rfloor}{2^n} \wedge 1} \geq k_n^{1/4} \frac{\alpha^2}{2} \right) > \frac{\alpha}{2} - \mathbb{P}(U_n < T_{n,k_n})$$
$$\geq \frac{\alpha}{2} - \frac{36 n^{1/4}}{\sqrt{k_n}}. \tag{3.2.26}$$

Putting (3.2.21), (3.2.24), (3.2.25), and (3.2.26) together yields that $(\widetilde{\varphi}^n)_{n\in\mathbb{N}}$ is an (UPBR). □

Lemma 3.2.17. *Let Assumption 3.2.10 hold. For each $\varepsilon > 0$, there exists a constant $C > 0$ and a sequence of $D_n \cup \{\infty\}$-valued stopping times $(\tau_n)_{n\in\mathbb{N}}$ s.t. $\mathbb{P}(\tau_n < \infty) < \varepsilon$ and the stopped processes $S^{n,\tau_n} = (S^n_{t\wedge\tau_n})_{t\in D_n}$, $A^{n,\tau_n} = (A_{t\wedge\tau_n})_{t\in D_n}$ satisfy*

$$\sum_{t_i\in D_n} \left| A^{n,\tau_n}_{t_{i+1}} - A^{n,\tau_n}_{t_i} \right| \le C \tag{3.2.27}$$

$$\text{and, consequently,} \quad MV(S^{n,\tau_n}, D_n) = \mathbb{E}\left[\sum_{t_i\in D_n} \left| A^{n,\tau_n}_{t_{i+1}} - A^{n,\tau_n}_{t_i} \right| \right] \le C. \tag{3.2.28}$$

Proof. The idea of the proof is akin to the proofs of Proposition 3.1 and Lemma 3.4 in Beiglböck et al. [6]. Thus, we only give a sketch of the proof and leave the details to the reader. We first claim that

$$\left\{ \sum_{t_i\in D_n} \left(A^n_{t_{i+1}} - A^n_{t_i} \right)^+ : n \in \mathbb{N} \right\} \tag{3.2.29}$$

is bounded in probability. We proceed by contraposition, i.e., we suppose otherwise and want to show that this leads to an (UPBR). Using Lemma 3.2.15, we can analogously to the previous proof construct a sequence of simple trading strategies $(\varphi^n)_{n\in\mathbb{N}}$ with $0 \le \varphi^n \le 1$ s.t. φ^n invests in $\sum_{t_i\in D_n} \left(A^n_{t_{i+1}} - A^n_{t_i} \right)^+$ while only making potential losses in the martingale part M^n and at liquidation. Indeed, similar as in step 2 of the proof of Lemma 3.2.16, it can be shown that the associated liquidation values can be bounded from below by

$$V^{\mathrm{liq}}_t(\varphi^n) \ge \sum_{t_i\in D_n, 0<t_i\le t} \left(A^n_{t_{i+1}} - A^n_{t_i} \right)^+ + \sum_{t_i\in D_n, 0<t_i\le t} \varphi^n_{t_i} \left(M^n_{t_i} - M^n_{t_{i-1}} \right) - 1. \tag{3.2.30}$$

By the previous Lemma 3.2.16 and some stopping, there is no loss of generality by assuming that $(M^n)_{n\in\mathbb{N}}$ is uniformly bounded. Hence, by Doob's maximal inequality, the pathwise maxima of the martingale parts in (3.2.30) are bounded in L^2. Thus, by further stopping (cf. the arguments used in Beiglböck et al. [6, page 2433, lines 11-15]), we may assume that (3.2.30) is uniformly bounded from below. On the other hand, by assumption, the RHS of (3.2.30) is unbounded in probability from above. Thus, the (adjusted) strategies yield an (UPBR) with long-only strategies (after rescaling), and we arrive at a contradiction. Consequently, (3.2.29) has to be bounded in probability. Since the martingale parts are also bounded in probability by Lemma 3.2.16, the same holds for $\left\{ \sum_{t_i\in D_n} \left(A^n_{t_{i+1}} - A^n_{t_i} \right)^- : n \in \mathbb{N} \right\}$, and we are done. □

In order to finish the proof of Theorem 3.2.8 we still need a couple of auxiliary results, which give us some more information about $MV(S^n, D_n)$ in comparison to $MV(S^m, D_m)$. Given a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ of $[0,1]$ and a stopping time τ, we have the following notation $\pi(\tau) := \inf\{t \in \pi : t \geq \tau\}$. Recall the following useful result from [5].

Lemma 3.2.18 (Lemma 3.2 of [5]). *Let Assumption 3.2.10 hold. Then*

$$\mathrm{MV}(S^{\pi(\tau)}, \pi) = \mathbb{E}\left[\sum_{t_i \in \pi} \mathbb{1}_{\{t_i < \tau\}} \left|\mathbb{E}\left[S_{t_{i+1}} - S_{t_i} \mid \mathcal{F}_{t_i}\right]\right|\right]$$

and $\left|\mathrm{MV}(S^{\pi(\tau)}, \pi) - \mathrm{MV}(S^{\tau}, \pi)\right| \leq 1$.

Compared to the frictionless case with $S^n = \underline{S} = \overline{S}$, the analysis is complicated by the fact that in general $S^m_t \neq S^n_t$ for $t \in D_n$. We have nevertheless the following monotonicity result.

Lemma 3.2.19. *Let Assumption 3.2.10 hold. In addition, let $n, m \in \mathbb{N}$ with $m > n$ and let τ_m be a $D_m \cup \{\infty\}$-valued stopping time. For any $s \in D_n$, we have*

$$\begin{aligned}&\mathbb{E}\left[\sum_{t_i \in D_n, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} \left|\mathbb{E}\left[S^n_{t_{i+1}} - S^n_{t_i} \mid \mathcal{F}_{t_i}\right]\right| \mid \mathcal{F}_s\right] \\ &\leq \mathbb{E}\left[\sum_{t_i \in D_m, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} \left|\mathbb{E}\left[S^m_{t_{i+1}} - S^m_{t_i} \mid \mathcal{F}_{t_i}\right]\right| \mid \mathcal{F}_s\right] + \left(2 - |S^n_s - S^m_s|\right) \mathbb{1}_{\{s < \tau_m\}}.\end{aligned}$$

In particular, for $s = 0$ this yields

$$MV(S^{n, D_n(\tau_m)}, D_n) \leq MV(S^{m, \tau_m}, D_m) + 2.$$

In addition, we have

$$\mathrm{MV}(S^{m, D_n(\tau)}, D_n) \leq \mathrm{MV}(S^{m, D_m(\tau)}, D_m) + 1. \tag{3.2.31}$$

for all $[0,1] \cup \{\infty\}$-valued stopping times τ.

Proof. Step 1: In a first step, we keep the grid D_n but replace S^n with S^m. Thus, we want to show

$$\begin{aligned}&\mathbb{E}\left[\sum_{t_i \in D_n, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} \left|\mathbb{E}\left[S^n_{t_{i+1}} - S^n_{t_i} \mid \mathcal{F}_{t_i}\right]\right| \mid \mathcal{F}_s\right] \\ &\leq \mathbb{E}\left[\sum_{t_i \in D_n, t_i \geq s} \mathbb{1}_{\{t_i < \tau_m\}} \left|\mathbb{E}\left[S^m_{t_{i+1}} - S^m_{t_i} \mid \mathcal{F}_{t_i}\right]\right| \mid \mathcal{F}_s\right] + \left(1 - |S^n_s - S^m_s|\right) \mathbb{1}_{\{s < \tau_m\}}.\end{aligned} \tag{3.2.32}$$

We start by showing the one-step estimate

$$\begin{aligned}
&\left|\mathbb{E}\left[S^n_{s+1/2^n} - S^n_s \mid \mathcal{F}_s\right]\right| \\
= &\left|\mathbb{E}\left[S^n_{s+1/2^n} - S^m_s \mid \mathcal{F}_s\right]\right| - |S^n_s - S^m_s| \\
\leq &\left|\mathbb{E}\left[S^m_{s+1/2^n} - S^m_s \mid \mathcal{F}_s\right]\right| + \mathbb{E}\left[\left|S^m_{s+1/2^n} - S^n_{s+1/2^n}\right| \mid \mathcal{F}_s\right] \\
&-|S^n_s - S^m_s| \qquad (3.2.33)
\end{aligned}$$

for all $s = 1-1/2^n, 1-2/2^n, \dots, 0$. The equality in (3.2.33) can be checked separately on the $\mathcal{F}_s$-measurable sets $B_1 := \{\mathbb{E}\left[S^n_{s+2^{-n}} \mid \mathcal{F}_s\right] > \overline{S}_s\}$, $B_2 := \{\mathbb{E}\left[S^n_{s+2^{-n}} \mid \mathcal{F}_s\right] < \underline{S}_s\}$, and $B_3 := \{\underline{S}_s \leq \mathbb{E}\left[S^n_{s+2^{-n}} \mid \mathcal{F}_s\right] \leq \overline{S}_s\}$. By the definition of S^n, $B_1 \subseteq \{S^n_s = \overline{S}_s\}$. On the other hand, $S^m_s \leq \overline{S}_s$, which implies the equality on B_1. On the set $B_2 \subseteq \{S^n_s = \underline{S}_s\}$, the situation is completely symmetric. Finally, on $B_3 = \{S^n_s = \mathbb{E}\left[S^n_{s+2^{-n}} \mid \mathcal{F}_s\right]\}$, the equality is obvious. The inequality in (3.2.33) follows from Jensen's inequality for conditional expectations and the triangle inequality.

Now, we show (3.2.32) by a backward-induction on $s = 1-1/2^n, 1-2/2^n, \dots, 0$. For the initial step $s = 1-1/2^n$, we only have to multiply (3.2.33) for $s = 1-1/2^n$ by $\mathbb{1}_{\{1-2^{-n}<\tau_m\}}$ and use that $|S^m_1 - S^n_1| \leq 1$.

Induction step $s + 1/2^n \rightsquigarrow s$: By the induction hypothesis, one has

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t_i \in D_t, t_i \geq s+1/2^n} \mathbb{1}_{\{t_i<\tau^m\}} \left|\mathbb{E}\left[S^n_{t_{i+1}} - S^n_{t_i} \mid \mathcal{F}_{t_i}\right]\right| \middle| \mathcal{F}_s\right] \\
\leq &\,\mathbb{E}\left[\sum_{t_i \in D_n, t_i \geq s+1/2^n} \mathbb{1}_{\{t_i<\tau_m\}} \left|\mathbb{E}\left[S^m_{t_{i+1}} - S^m_{t_i} \mid \mathcal{F}_{t_i}\right]\right| \middle| \mathcal{F}_s\right] \\
&+\mathbb{1}_{\{s<\tau_m\}}\mathbb{E}\left[1 - |S^n_{s+1/2^n} - S^m_{s+1/2^n}| | \mathcal{F}_s\right], \qquad (3.2.34)
\end{aligned}$$

where we take on both sides of (3.2.32) for $s+1/2^n$ the conditional expectation under $\mathcal{F}_s$ and use that $\{s + 1/2^n < \tau_m\} \subseteq \{s < \tau_m\}$. Multiplying (3.2.33) by $\mathbb{1}_{\{s<\tau_m\}}$ and adding (3.2.34) yields (3.2.32).

Step 2: We still need to pass from D_n to D_m for the process S^m, i.e., we now want to show that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t_i \in D_n, t_i \geq s} \mathbb{1}_{\{t_i<\tau_m\}} |\mathbb{E}\left[S^m_{t_{i+1}} - S^m_{t_i} \mid \mathcal{F}_{t_i}\right]| \mid \mathcal{F}_s\right] \\
\leq &\,\mathbb{E}\left[\sum_{t_i \in D_m, t_i \geq s} \mathbb{1}_{\{t_i<\tau_m\}} |\mathbb{E}\left[S^m_{t_{i+1}} - S^m_{t_i} \mid \mathcal{F}_{t_i}\right]| \mid \mathcal{F}_s\right] + \mathbb{1}_{\{s<\tau_m\}}. \qquad (3.2.35)
\end{aligned}$$

This is less tricky: for $\tau_m = 1$, it directly follows from the triangle inequality together with Jensen's inequality for conditional expectations and the second summand on the

RHS is not needed. However, in the general case there is the problem that τ_m can stop in $D_m \setminus D_n$. Thus, for every $i \in \{s2^n, s2^n+1, \dots, 2^n-1\}$, we have to make the following calculations

$$\begin{aligned}
&\mathbb{1}_{\{i/2^n<\tau_m\}} \left|\mathbb{E}\left[S^m_{(i+1)/2^n} - S^m_{i/2^n} \mid \mathcal{F}_{i/2^n}\right]\right| \\
&= \mathbb{1}_{\{i/2^n<\tau_m\}} \left|\mathbb{E}\left[\sum_{j=i2^{m-n}}^{(i+1)2^{m-n}-1} \left(S^m_{(j+1)/2^m} - S^m_{j/2^m}\right) \mid \mathcal{F}_{i/2^n}\right]\right| \\
&\leq \mathbb{E}\left[\sum_{j=i2^{m-n}}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{j/2^m<\tau_m\}} \left|\mathbb{E}\left[S^m_{(j+1)/2^m} - S^m_{j/2^m} \mid \mathcal{F}_{j/2^m}\right]\right| \mid \mathcal{F}_{i/2^n}\right] \\
&\quad + \left|\mathbb{E}\left[\mathbb{1}_{\{i/2^n<\tau_m\}} \sum_{j=i2^{m-n}}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{j/2^m\geq\tau_m\}} \left(S^m_{(j+1)/2^m} - S^m_{j/2^m}\right) \mid \mathcal{F}_{i/2^n}\right]\right|.
\end{aligned} \tag{3.2.36}$$

For the second summand, we can use the estimate

$$\begin{aligned}
&\left|\mathbb{1}_{\{i/2^n<\tau_m\}} \sum_{j=i2^{m-n}}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{j/2^m\geq\tau_m\}} \left(S^m_{(j+1)/2^m} - S^m_{j/2^m}\right)\right| \\
&= \left|\sum_{j=i2^{m-n}+1}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{(j-1)/2^m<\tau_m\leq j/2^m\}} \left(S^m_{(i+1)/2^n} - S^m_{j/2^m}\right)\right| \\
&\leq \sum_{j=i2^{m-n}+1}^{(i+1)2^{m-n}-1} \mathbb{1}_{\{(j-1)/2^m<\tau_m\leq j/2^m\}} \leq \mathbb{1}_{\{i/2^n<\tau_m\leq(i+1)/2^n\}},
\end{aligned} \tag{3.2.37}$$

where we use $0 \leq S^m_{t_i} \leq 1$ for all $t_i \in D_m$. Putting (3.2.36) and (3.2.37) together and summing up over all i, we arrive at (3.2.35). Together with (3.2.32), this yields the main assertion. (3.2.31) is just (3.2.35). □

For the convenience of the reader, we recall the following result from [5].

Lemma 3.2.20 (Lemma 4.2 in [5]). *Assume that* $(\tau_n)_{n\in\mathbb{N}}$ *is a sequence of* $[0,1]\cup\{\infty\}$*-valued stopping times s.t.* $\mathbb{P}(\tau_n = \infty) \geq 1-\varepsilon$ *for some* $\varepsilon > 0$ *and all* $n \in \mathbb{N}$*. Then, there exists a stopping time* τ *and for each* $n \in \mathbb{N}$ *convex weights* $\mu^n_n, \dots, \mu^n_{N_n}$*, i.e.,* $\mu^n_k \geq 0$*,* $k = n, \dots, N_n$ *and* $\sum_{k=n}^{N_n} \mu^n_k = 1$*, s.t.* $\mathbb{P}(\tau = \infty) \geq 1-3\varepsilon$ *and*

$$\mathbb{1}_{[\![0,\tau]\!]} \leq 2\sum_{k=n}^{N_n} \mu^n_k \mathbb{1}_{[\![0,\tau_k]\!]}, \quad n \in \mathbb{N}. \tag{3.2.38}$$

We are now in the position to prove Theorem 3.2.8.

Proof of Theorem 3.2.8. Let Assumption 3.2.10 hold. Let $\varepsilon > 0$, $(\tau_n)_{n\in\mathbb{N}}$ and $C > 0$ as in Lemma 3.2.17. In addition, let τ as in Lemma 3.2.20. We have

$$\begin{aligned} MV(S^{n,D_n(\tau)}, D_n) &= \mathbb{E}\left[\sum_{t_i\in D_n} \mathbb{1}_{\{t_i<\tau\}} \left|\mathbb{E}\left[S^n_{t_{i+1}} - S^n_{t_i} \mid \mathcal{F}_{t_i}\right]\right|\right] \\ &\leq 2\mathbb{E}\left[\sum_{t_i\in D_n}\sum_{k=n}^{N_n} \mu^n_k \mathbb{1}_{\{t_i<\tau_k\}} \left|\mathbb{E}\left[S^n_{t_{i+1}} - S^n_{t_i} \mid \mathcal{F}_{t_i}\right]\right|\right] \\ &= 2\sum_{k=n}^{N_n} \mu^n_k MV(S^{n,D_n(\tau_k)}, D_n) \\ &\leq 2\sum_{k=n}^{N_n} \mu^n_k (MV(S^{k,\tau_k}, D_k) + 2) \leq 2C + 4, \quad n \in \mathbb{N}. \end{aligned} \tag{3.2.39}$$

Indeed, both equalities hold by Lemma 3.2.18. The first inequality is due to Lemma 3.2.20 and the second inequality follows from Lemma 3.2.19. The third inequality holds by Lemma 3.2.17. Next, let us show that

$$\mathrm{MV}(S^{D_n(\tau)}, D_n) = \lim_{\substack{m\to\infty\\ m\geq n}} \mathrm{MV}(S^{m,D_n(\tau)}, D_n) \leq \limsup_{\substack{m\to\infty\\ m\geq n}} \mathrm{MV}(S^{m,D_m(\tau)}, D_m) + 1 \leq 2C + 5,$$

$n \in \mathbb{N}$, where S is the value process of the continuous-time game. Indeed, the equality follows from Proposition 3.2.14 and the dominated convergence theorem. The first inequality is (3.2.31) and the second follows from (3.2.39). Together with Lemma 3.2.18, we arrive at

$$\mathrm{MV}(S^\tau, D_n) \leq 2C + 6, \quad n \in \mathbb{N}. \tag{3.2.40}$$

Finally, by the right-continuity of S^τ and (3.2.40), we get

$$\mathrm{MV}(S^\tau) = \lim_{n\to\infty} \mathrm{MV}(S^\tau, D_n) \leq 2C + 6.$$

Together with $\mathbb{P}(\tau < \infty) \leq 3\varepsilon$, this establishes that the right-continuous process S is a local quasimartingale and, thus, a semimartingale by Rao's theorem (in the version of Theorem 3.2.12) and the Doob-Meyer-Decomposition [74, Chapter 3, Theorem 16]. □

Proof of Theorem 3.2.7. Having shown that Theorem 3.2.8 holds the assertion follows directly by Proposition 3.2.9. □

Remark 3.2.21. *The arguments presented here rely heavily on the two-dimensional setting. However, Theorem 2.7 can be directly applied to a model with a bank account and finitely many risky assets since in this case it is sufficient to have a semimartingale price system for each risky asset separately (cf. also [22, Theorem 7.2]). On the other hand, it seems that the approach cannot be adapted to the general Kabanov model (cf. Kabanov and Safarian [55, Section 3.6]), in which there need not exist a bank account that is involved in every transaction.*

3.3 The self-financing condition

As already discussed in the introduction, we use the semimartingale to define the self-financing condition in the bid-ask model for general strategies. A self-financing condition can be identified with an operator $\varphi \mapsto \Pi(\varphi)$ that maps each amount of risky assets to the corresponding position in the risk-less bank account (if the later exists). Here, we assume that the initial position and the risk-less interest are zero. In addition, **for the rest of the chapter, we assume that there exists a semimartingale price system** S, i.e., S is a semimartingale s.t. $\underline{S} \le S \le \overline{S}$ (cf. Theorem 3.2.7). The aim is to define $\Pi(\varphi)$ as $\varphi \bullet S - \varphi S -$ "costs", where the process $\varphi \bullet S$ denotes the stochastic integral. At this stage, the process φ is bounded (see Proposition 3.4.2 for the extension to general strategies). The costs are caused by the fact that the trades are carried out at the less favorable bid-ask prices. Since the gains in the semimartingale are finite, they cannot compensate infinite costs and the latter lead to ruin.

3.3.1 Construction of the cost term

We construct the cost associated to a strategy $\varphi \in \mathbf{b}\mathcal{P}$ path-by-path, i.e., in the following, $\omega \in \Omega$ is fixed and $\varphi, \underline{S}, \overline{S}, S$ are identified with functions in time.

We follow a two-step procedure. First, we calculate the costs on intervals in which the left limit of the spread is bounded away from zero by means of a modified Riemann-Stieltjes integral. The integral turns out to always exist (but it can take the value ∞). In the second step, we exhaust the set of points with positive spread by finite unions of such intervals and define the total costs as the supremum of the costs along these unions. One may see a vague analogy between the second step and the way a Lebesgue integral is constructed.

This approach leads to a well-founded self-financing condition under the additional Assumption 3.3.18 on the behavior of the spread at zero. Very roughly speaking, there should not occur costs if the investor builds up positions at times the spread is zero and the positions are already closed *before* the spread reaches any positive value (cf. Example 3.3.23 for a counterexample). Since for the construction of our cost process itself, the assumption is not needed, we introduce it later on.

In order to introduce the integral, we need the following notation.

Definition 3.3.1. Let $I = [a, b] \subseteq [0, T]$ with $a < b$.

(i) A collection $P = \{t_0, \dots t_n\}$ of points $t_i \in [a, b]$ for $n \in \mathbb{N}$ and $i = 0, \dots, n$ with $a = t_0 < t_1 < \dots < t_n = b$ is called a *partition of* I.

(ii) A partition $P' = \{t'_0, \dots, t'_m\}$ with $P' \supseteq P$ is called a *refinement of* P.

(iii) If P, P' are two partitions of I, the *common refinement* $P \cup P'$ is the partition obtained by ordering the points of $\{t_0, \dots t_n\} \cup \{t'_0, \dots, t'_m\}$ in increasing order.

(iv) Given a partition $P = \{t_0, \dots, t_n\}$ of I a collection $\lambda = \{s_1, \dots, s_n\}$ with $s_i \in [t_{i-1}, t_i)$ for $i = 1, \dots, n$ is called a *modified intermediate subdivision of* P.

(v) Let $\varphi \in \mathbf{b}\mathcal{P}$, $P = \{t_0, \dots, t_n\}$ be a partition of I and $\lambda = \{s_1, \dots, s_n\}$ be an modified intermediate subdivision of P, the *modified Riemann-Stieltjes sum* is defined by

$$R(\varphi, P, \lambda) := \sum_{i=1}^{n} (\overline{S}_{s_i} - S_{s_i})(\varphi_{t_i} - \varphi_{t_{i-1}})^+ + \sum_{i=1}^{n} (S_{s_i} - \underline{S}_{s_i})(\varphi_{t_i} - \varphi_{t_{i-1}})^-.$$

Definition 3.3.2. Let $\varphi \in \mathbf{b}\mathcal{P}$ and $I = [a, b] \subseteq [0, T]$ with $a < b$. The *cost term of φ on I* exists and equals $C(\varphi, I) \in \mathbb{R}_+ \cup \{\infty\}$ if for all $\varepsilon > 0$ there is a partition P_ε of I s.t. for all refinements P of P_ε and all modified intermediate subdivisions λ of P the following is satisfied:

(i) In the case of $C(\varphi, I) < \infty$, we have $|C(\varphi, I) - R(\varphi, P, \lambda)| < \varepsilon$,

(ii) In the case of $C(\varphi, I) = \infty$, we have $|R(\varphi, P, \lambda)| > \frac{1}{\varepsilon}$.

In addition, we set $C(\varphi, \{a\}) := 0$ for all $a \in [0, T]$ and $C(\varphi, \emptyset) := 0$.

The next proposition establishes the existence of the cost term on an interval I where the spread is bounded away from zero.

Proposition 3.3.3. *Let $\varphi \in \mathbf{b}\mathcal{P}$ and $I = [a, b] \subseteq [0, T]$ with $a < b$ s.t. $\inf_{t \in [a,b)}(\overline{S}_t - \underline{S}_t) > 0$. Then, the cost term $C(\varphi, I)$ in Definition 3.3.2 exists and is unique. In addition, we have*

$$\begin{cases} C(\varphi, I) < \infty, & \textit{if } \operatorname{Var}_a^b(\varphi) < \infty \\ C(\varphi, I) = \infty, & \textit{if } \operatorname{Var}_a^b(\varphi) = \infty, \end{cases}$$

where $\operatorname{Var}_a^b(\varphi)$ denotes the pathwise variation of φ on the interval $[a, b]$.

We postpone the technical proof of Proposition 3.3.3 to Section 3.6.

Remark 3.3.4. *First note that a priori, φ need not be of finite variation. Thus, we cannot decompose it into its increasing part $\varphi^\uparrow$ and decreasing part $\varphi^\downarrow$ to define $\int_a^b (\overline{S}_s - S_s) d\varphi_s^\uparrow + \int_a^b (S_s - \underline{S}_s) d\varphi_s^\downarrow := C(\varphi^\uparrow, [a, b]) + C(\varphi^\downarrow, [a, b])$. Instead, we consider the increasing and decreasing parts of φ along grids and weight them with the corresponding prices before passing to the limit.*

However, if $\operatorname{Var}_a^b(\varphi) < \infty$, it can be shown that $C(\varphi^\uparrow, [a, b]) + C(\varphi^\downarrow, [a, b]) = C(\varphi, [a, b])$. This can be seen by an inspection of the proof of Proposition 3.3.3, in which the condition $\inf_{t \in [a,b)}(\overline{S}_t - \underline{S}_t) > 0$ can be dropped if $\operatorname{Var}_a^b(\varphi) < \infty$.

Remark 3.3.5. *Definition 3.3.2(i) only requires that the cost term exists in the Moore-Pollard-Stieltjes-sense (see, e.g., Hildebrandt [43, Section 4] and Mikosch and Norvaiša [71, Section 2.3]), i.e., as the limit of the net $R(\varphi, \cdot, \cdot)$ indexed by the directed set of tuples (P, λ) with the partial order $(P, \lambda) \geq (P', \lambda')$ iff P is a refinement of P'. This is weaker than the existence in the norm-sense, i.e., as the limit of* *the net* $R(\varphi, \cdot, \cdot)$ *indexed by the tuples* (P, λ) *with the partial order* $(P, \lambda) \geq (P', \lambda')$ *iff* $\max_{i=1,\ldots,n}(t_i - t_{i-1}) \leq \max_{i=1,\ldots,m}(t'_i - t'_{i-1})$*, that is guaranteed for the usual Riemann-Stieltjes integral with a continuous integrator of finite variation. A straightforward adaptation of the existence in the norm-sense of the usual Riemann-Stieltjes integral to the present context would read:*

The cost term is said to exist and equal to $C(\varphi, I) \in \mathbb{R}_+$ *if for each* $\varepsilon > 0$ *there is* $\delta > 0$ *s.t.* $|C(\varphi, I) - R(\varphi, P, \lambda)| < \varepsilon$ *for all partitions* $P = \{t_0, \ldots, t_n\}$ *with* $\max_{i=1,\ldots,n}(t_i - t_{i-1}) < \delta$ *and all intermediates subdivision* $\lambda = \{s_1, \ldots, s_n\}$ *with* $s_i \in [t_{i-1}, t_i)$*.*

But, the following example, similar to Guasoni et al. [36, Example A.3] shows that $C(\varphi, I)$ *does in general not exist in the norm sense: let* $T = 2$*,* $\overline{S} - \underline{S} = \mathbb{1}_{[1,2]}$ *and* $\varphi = \mathbb{1}_{(1,2]}$*. Namely, if* $t_i = 1$ *is not included in the partition* P*,* $R(\varphi, P, \lambda)$ *can oscillate between* 0 *and* 1*.*

The example shows that the points of common discontinuities of integrator and integrand are critical to calculate the costs. Thus, they have to be included in the partition, which is guaranteed by the Moore-Pollard-Stieltjes approach.

Remark 3.3.6. *The restriction that the point* s_i *of the intermediate subdivision* λ *has to lie in the interval* $[t_{i-1}, t_i)$*, and not only in* $[t_{i-1}, t_i]$*, has a clear financial interpretation.*

If an investor buys $\varphi_s - \varphi_{s-}$ *shares at time* s*, she pays* $(\varphi_s - \varphi_{s-})\overline{S}_{s-}$ *monetary units. Consequently, if she updates her position between* t_{i-1} *and* t_i*, only the stock prices on the time interval* $[t_{i-1}, t_i)$ *have to be considered. In the limit, the choice of the price in* $[t_{i-1}, t_i)$ *does not matter. Indeed, a well-known way to guarantee the existence of Riemann-Stieltjes integrals in the case of simultaneous jump discontinuities on the same side of integrator and integrand is to exclude the boundary points (see Hildebrandt [43, Section 6]).*

Finally, we mention that in the case of $\mathrm{Var}_a^b(\varphi) < \infty$*, the integrals are the same as in Guasoni et al. [36, Section A.2]. But, besides considering different processes, we introduce the integrals in a different way.*

The next proposition states that the cost term is additive with regard to the underlying interval. Its proof is obvious.

Proposition 3.3.7. *Let* $\varphi \in \mathbf{b}\mathcal{P}$*,* $I = [a, b] \subseteq [0, T]$ *s.t.* $\inf_{t \in [a,b)}(\overline{S}_t - \underline{S}_t) > 0$ *and* $c \in [a, b]$*. Then, we have*

$$C(\varphi, [a, b]) = C(\varphi, [a, c]) + C(\varphi, [c, b]).$$

Having defined the costs for all subintervals $I = [a, b] \subseteq [0, T]$ with $\inf_{t \in [a,b)}(\overline{S}_t - \underline{S}_t) > 0$, we now proceed to define the accumulated costs as a process. Therefore, we let

$$\mathcal{I} := \left\{ \cup_{i=1}^n [a_i, b_i] : \begin{array}{l} n \in \mathbb{N},\ 0 \leq a_1 \leq b_1 \leq a_2 \leq \cdots \leq a_n \leq b_n \leq T, \\ \inf_{t \in [a_i, b_i)}(\overline{S}_t - \underline{S}_t) > 0,\ i = 1, \ldots, n \end{array} \right\} \cup \{\emptyset\}. \quad (3.3.1)$$

We now extend the cost term to $\mathcal{I}$. Given $\varphi \in \mathbf{b}\mathcal{P}$ and $J = \cup_{i=1}^n [a_i, b_i]$ with $\inf_{t\in[a_i,b_i)}(\overline{S}_t - \underline{S}_t) > 0$ for all $i = 1, \dots, n$, we define *the costs along* J by

$$C(\varphi, J) := \sum_{i=1}^{n} C(\varphi, [a_i, b_i]), \tag{3.3.2}$$

where the cost terms $C(\varphi, [a_i, b_i])$ for $i = 1, \dots, n$ are defined in Definition 3.3.2. By Proposition 3.3.7, the RHS of (3.3.2) does not depend on the representation of J. Thus, the cost term $C(\varphi, J)$ is well-defined for all $J \in \mathcal{I}$.

Definition 3.3.8. (Cost process) Let $\varphi \in \mathbf{b}\mathcal{P}$. Then, the *cost process* $(C_t(\varphi))_{t\in[0,T]}$ is defined by

$$C_t(\varphi) := \sup_{J\in\mathcal{I}} C(\varphi, J \cap [0, t]) \in [0, \infty], \quad t \in [0, T]$$

(Note that $\{0\} \in \mathcal{I}$ with $C(\varphi, \{0\}) = 0$ and thus the supremum is nonnegative). If it is clear from the context, we also write $(C_t)_{t\in[0,T]}$ for the cost process associated to φ.

Proposition 3.3.9. *Let $\varphi \in \mathbf{b}\mathcal{P}$. The cost process $(C_t(\varphi))_{t\in[0,T]}$ is $[0,\infty]$-valued, increasing and, consequently, làglàd (if finite). In addition, the following assertions hold.*

(i) For any $0 \le s \le t \le T$, we have $C_t(\varphi) = C_s(\varphi) + \sup_{J\in\mathcal{I}} C(\varphi, J \cap [s, t])$,

(ii) For any $0 \le s \le t \le T$ with $\inf_{\tau\in[s,t)}(\overline{S}_\tau - \underline{S}_\tau) > 0$, we have $C_t(\varphi) = C_s(\varphi) + C(\varphi, [s, t])$,

(iii) For any $0 \le s \le t \le T$, we have $C_t(\varphi) \le C_s(\varphi) + \sup_{\tau\in[s,t)}(\overline{S}_\tau - \underline{S}_\tau)\mathrm{Var}_s^t(\varphi)$.

The assertions above follow directly from Definitions 3.3.2 and 3.3.8. Thus, we leave the easy proof to the reader.

The next proposition determines sequences of partitions whose corresponding Riemann-Stieltjes sums converge to the cost term on an interval where the spread is bounded away from zero. This will be crucial to show that the cost term is predictable. For this purpose, recall that the oscillation $\mathrm{osc}(f, I)$ of a function $f : [0, T] \to \mathbb{R}$ on an interval $I \subseteq [0, T]$ is defined by $\mathrm{osc}(f, I) := \sup\{|f(t) - f(s)| : s, t \in I\}$.

Proposition 3.3.10. *Let $\varphi \in \mathbf{b}\mathcal{P}$ and $I = [a, b] \subseteq [0, T]$ with $a < b$ and $\inf_{t\in[a,b)}(\overline{S}_t - \underline{S}_t) > 0$. In addition, let $(P_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions of I, i.e., $P_n = \{t_0^n, \dots, t_{m_n}^n\}$ with $a = t_0^n < t_1^n < \dots < t_{m_n}^n = b$ and $P_{n+1} \supseteq P_n$, s.t.*

(i) $\lim_{n\to\infty} \max\big(\sup_{i=1,\dots,m_n} \mathrm{osc}(\overline{S} - S, [t_{i-1}^n, t_i^n)), \sup_{i=1,\dots,m_n} \mathrm{osc}(S - \underline{S}, [t_{i-1}^n, t_i^n))\big) = 0$

(ii) $\lim_{n\to\infty} \sum_{i=1}^{m_n} |\varphi_{t_i^n} - \varphi_{t_{i-1}^n}| = \mathrm{Var}_a^b(\varphi)$.

Then, for any sequence $\lambda_n = \{s_1^n, \ldots, s_{m_n}^n\}$ *of modified intermediate subdivision, we have*

$$R(\varphi, P_n, \lambda_n) \to C(\varphi, [a,b]) \quad \text{as } n \to \infty.$$

In addition, such a sequence $(P_n)_{n\in\mathbb{N}}$ *always exists.*

The proof of Proposition 3.3.10 is closely related to the proof of Proposition 3.3.3. Thus, we also postpone it to Section 3.6. We now conclude the subsection with a first approximation result.

Proposition 3.3.11. *Let* $\varphi, \varphi^n \in \mathbf{b}\mathcal{P}$, $n \in \mathbb{N}$, $t \in [0,T]$ *and* $J \in \mathcal{I}$. *Then, we have the implication*

$$\varphi^n \to \varphi \quad \text{pointwise} \quad \Rightarrow \quad \liminf_{n\to\infty} C(\varphi^n, J \cap [0,t]) \geq C(\varphi, J \cap [0,t]). \tag{3.3.3}$$

Proof. Let $\varphi^n \to \varphi$ pointwise and $t \in [0,T]$. We start by noting that the claim is trivial if $J = \{a\}$ for some $a \in [0,T]$ or if $J = \emptyset$.

Step 1. We now treat the special case $J = [a,b] \in \mathcal{I}$ with $a < b$. In this case, we have $C(\varphi, J \cap [0,t]) = C(\varphi, [a, b \wedge t])$ and $C(\varphi^n, J \cap [0,t]) = C(\varphi^n, [a, b \wedge t])$ for all $n \in \mathbb{N}$, where we use the convention $[c,d] = \emptyset$ if $d < c$. In addition, by the preceding observation, we may assume $t > a$.

We only consider the case $C(\varphi, [a, b \wedge t]) < \infty$ since the opposite case $C(\varphi, [a, b \wedge t]) = \infty$ is analogous. Let $\varepsilon > 0$. There is a partition $P_\varepsilon = \{t_0, \ldots, t_m\}$ of $[a, b \wedge t]$ s.t.

$$\begin{aligned}&\sum_{i=1}^m \inf_{s\in[t_{i-1},t_i)} (\overline{S}_s - S_s)(\varphi_{t_i} - \varphi_{t_{i-1}})^+ + \sum_{i=1}^m \inf_{s\in[t_{i-1},t_i)} (S_s - \underline{S}_s)(\varphi_{t_i} - \varphi_{t_{i-1}})^- \\ &\geq C(\varphi, [a, b \wedge t]) - \varepsilon.\end{aligned} \tag{3.3.4}$$

Using the pointwise convergence of $(\varphi^n)_{n\in\mathbb{N}}$, we can find $N \in \mathbb{N}$ s.t. for all $n \geq N$, we have

$$\begin{aligned}&\sum_{i=1}^m \inf_{s\in[t_{i-1},t_i)} (\overline{S}_s - S_s)(\varphi^n_{t_i} - \varphi^n_{t_{i-1}})^+ + \sum_{i=1}^m \inf_{s\in[t_{i-1},t_i)} (S_s - \underline{S}_s)(\varphi^n_{t_i} - \varphi^n_{t_{i-1}})^- \\ &\geq C(\varphi, [a, b \wedge t]) - 2\varepsilon.\end{aligned} \tag{3.3.5}$$

Keeping this in mind, for each n, we choose a partition $\overline{P}_n$ s.t. for all refinements P of $\overline{P}_n$ and intermediate subdivisions λ of P, we have $C(\varphi^n, [a, b\wedge t]) \geq R(\varphi^n, P, \lambda) - \varepsilon$. Now, we let $P_n := P_\varepsilon \cup \overline{P}_n$ and write $P_n = \{t_0^n, \ldots, t_{m_n}^n\}$. Denoting by $t_{i-1} = t^n_{i_1} < t^n_{i_2} < \cdots < t^n_{i_j} = t_i$ the points of P_n in between t_{i-1} and t_i, we have

$$\sum_{k=2}^j (\varphi^n_{t^n_{i_k}} - \varphi^n_{t^n_{i_{k-1}}})^+ \geq (\varphi^n_{t_i} - \varphi^n_{t_{i-1}})^+ \quad \text{and} \quad \sum_{k=2}^j (\varphi^n_{t^n_{i_k}} - \varphi^n_{t^n_{i_{k-1}}})^- \geq (\varphi^n_{t_i} - \varphi^n_{t_{i-1}})^-.$$

Together with (3.3.5) this yields

$$C(\varphi^n, [a, b \wedge t]) \geq R(\varphi^n, P_n, \lambda_n) - \varepsilon \geq C(\varphi, [a, b \wedge t]) - 3\varepsilon$$

for all $n \geq N$ and intermediate subdivision λ_n of P_n. Hence, we have

$$\liminf_{n\to\infty} C(\varphi^n, [a, b \wedge t]) \geq C(\varphi, [a, b \wedge t]) - 3\varepsilon,$$

which tantamount to the claim as $\varepsilon \downarrow 0$.

Step 2. Finally, let $J = \cup_{i=1}^m [a_i, b_i] \in \mathcal{I}$. Then, using the non-negativity of the sequences $(C(\varphi^n, [a_i, b_i \wedge t]))_{n\in\mathbb{N}}$ for $i = 1, \ldots, m$, we have

$$\liminf_{n\to\infty} C(\varphi^n, J \cap [0, t]) = \liminf_{n\to\infty} \sum_{i=1}^m C(\varphi^n, [a_i, b_i \wedge t]) \geq \sum_{i=1}^m \liminf_{n\to\infty} C(\varphi^n, [a_i, b_i \wedge t]).$$

Thus, (3.3.3) follows directly from step 1 and the observation at the start of the proof. □

3.3.2 The cost term as a stochastic process

Until now we kept $\omega \in \Omega$ fixed, i.e., the construction is path-by-path. To show some measurability properties of the cost term, we now consider it as a stochastic process.

Proposition 3.3.12. *Let $\varphi \in \mathbf{b}\mathcal{P}$. The cost process $C(\varphi) = (C_t(\varphi))_{t\in[0,T]}$ coincides with a predictable process up to evanescence.*

In order to prove Proposition 3.3.12, we need the following lemma, whose proof relies on some deep results of Doob [28] and thus is postponed to Section 3.6.

Lemma 3.3.13. *Let $\varphi \in \mathbf{b}\mathcal{P}$ and $\sigma \leq \tau$ two stopping times s.t. $\inf_{\sigma(\omega)\leq t<\tau(\omega)}(\overline{S}_t(\omega) - \underline{S}_t(\omega) > 0$ for all $\omega \in \Omega$. Then, the process $C(\varphi, [\sigma\wedge\cdot, \tau\wedge\cdot])$ coincides with a predictable process up to evanescence.*

In order to establish Proposition 3.3.12, we still need to approximate the supremum in Definition 3.3.8 in a measurable way. Therefore, we define for each $n \in \mathbb{N}$ a sequence of stopping times by $\tau_0^n := 0$ and

$$\tau_k^n := \begin{cases} \inf\{t \geq \tau_{k-1}^n : \overline{S}_t - \underline{S}_t \leq 2^{-(n+1)}\}, & k \text{ odd} \\ \inf\{t > \tau_{k-1}^n : \overline{S}_t - \underline{S}_t \geq 2^{-n}\}, & k \text{ even} \end{cases}, \quad \text{for } k \in \mathbb{N}. \tag{3.3.6}$$

Note that only a finite number of $\{\tau_k^n(\omega)\}_{k\in\mathbb{N}}$ is less than infinity as the process $\overline{S} - \underline{S}$ has càdlàg sample paths, $\tau_{2k}^n < \tau_{2k+1}^n$ on $\{\tau_{2k} < \infty\}$, and

$$\inf_{\tau_{2k}^n(\omega)\leq t<\tau_{2k+1}^n(\omega)} \left(\overline{S}_t(\omega) - \underline{S}_t(\omega)\right) \geq 2^{-(n+1)} \quad \text{for} \quad k \in \mathbb{N}_0$$

for all $\omega \in \Omega$. In particular, this means that the process $C^n(\varphi) = (C^n(\varphi)_t)_{t\in[0,T]}$

$$C_t^n(\varphi) := \sum_{k=0}^{\infty} C(\varphi, [\tau_{2k}^n \wedge t, \tau_{2k+1}^n \wedge t]) \tag{3.3.7}$$

is well-defined and coincides with a predictable process up to evanescence for each $n \in \mathbb{N}$ by Lemma 3.3.13.

Lemma 3.3.14. *Let $\varphi \in \mathbf{b}\mathcal{P}$ and $(C^n(\varphi))_{n\in\mathbb{N}}$ as above. Then, $C^n(\varphi) \to C(\varphi)$ pointwise.*

Proof. We write C^n instead of $C^n(\varphi)$ to not overburden the notation. Let $(\omega, t) \in \Omega \times [0,T]$. For $C_t(\omega) < \infty$, we claim: for each $\varepsilon > 0$ there is $N = N(\omega) \in \mathbb{N}$ s.t.

$$C_t(\omega) - \varepsilon \leq C_t^n(\omega) \leq C_t(\omega) \quad \text{for all} \quad n \geq N. \tag{3.3.8}$$

Thus, let us prove (3.3.8). It is obvious from Definitions 3.3.8 that we have $C_t(\omega) \geq C_t^n(\omega)$ for all $n \in \mathbb{N}$. To prove the other inequality, let $\varepsilon > 0$ and choose $0 \leq a_1 < b_1 \leq a_2 < \cdots \leq a_n < b_n \leq t$ s.t. $\inf_{t\in[a_i,b_i)}(\overline{S}_t(\omega) - \underline{S}_t(\omega)) > 0$ for $i = 1, \ldots, n$ and

$$C_t(\omega) - \varepsilon \leq \sum_{i=1}^{n} C(\varphi(\omega), [a_i, b_i]) \tag{3.3.9}$$

Let $\delta := \min_{i=1,\ldots,n} \inf_{t\in[a_i,b_i)}(\overline{S}_t(\omega) - \underline{S}_t(\omega)) > 0$ and choose $N \in \mathbb{N}$ s.t. $2^{-N} < \delta$. Then, it follows from the definition of the stopping times (3.3.6) that

$$\bigcup_{i=1}^{n} [a_i, b_i] \subseteq \bigcup_{k=0}^{\infty} [\tau_{2k}^n(\omega) \wedge t, \tau_{2k+1}^n(\omega) \wedge t] \quad \text{for all} \quad n \geq N.$$

Combining (3.3.9) this with Proposition 3.3.7, we find $C_t(\omega) - \varepsilon \leq C_t^n(\omega)$ for all $n \geq N$, which proves (3.3.8). Of course, for $C_t(\omega) = \infty$ the arguments are completely analogous. □

Proof of Proposition 3.3.12. Applying Lemma 3.3.13, we find that C^n coincides with a predictable process up to evanescence. Together with Lemma 3.3.14 this yields that C does the same. □

Next, we want to calculate the cost of an "almost simple" trading strategy (cf. Guasoni et al. [36] for a detailed discussion).

Definition 3.3.15. A predictable stochastic process φ of finite variation is called an *almost simple strategy* if there is a sequence of stopping times $(\tau_n)_{n\geq 0}$ with $\tau_n < \tau_{n+1}$ on $\{\tau_n < \infty\}$ and $\#\{n : \tau_n(\omega) < \infty\} < \infty$ for all $\omega \in \Omega$, s.t.

$$\varphi = \sum_{n=0}^{\infty} (\varphi_{\tau_n} \mathbb{1}_{[\![\tau_n]\!]} + \varphi_{\tau_n+} \mathbb{1}_{]\!]\tau_n,\tau_{n+1}[\![}).$$

Proposition 3.3.16. *Let φ be an almost simple strategy. We have*

$$\begin{aligned} C_t(\varphi) = &\sum_{n=0}^{\infty} \mathbb{1}_{\{\tau_n \le t\}} \left((\overline{S}_{\tau_n-} - S_{\tau_n-})(\varphi_{\tau_n} - \varphi_{\tau_n-})^+ + (S_{\tau_n-} - \underline{S}_{\tau_n-})(\varphi_{\tau_n} - \varphi_{\tau_n-})^- \right) \\ &+ \sum_{n=0}^{\infty} \mathbb{1}_{\{\tau_n < t\}} \left((\overline{S}_{\tau_n} - S_{\tau_n})(\varphi_{\tau_n+} - \varphi_{\tau_n})^+ + (S_{\tau_n} - \underline{S}_{\tau_n})(\varphi_{\tau_n+} - \varphi_{\tau_n})^- \right) \end{aligned}$$

for all $t \in [0, T]$.

Proof. For $\omega \in \Omega$ fixed, there is some $n \in \mathbb{N}_0$ with $\tau_0(\omega) < \ldots < \tau_{n-1}(\omega) \le T$ and $\tau_n(\omega) = \infty$. Now, it is sufficient to consider partitions containing $\tau_i(\omega) - \delta, \tau_i(\omega)$ if $(\overline{S}_{\tau_i-}(\omega) - S_{\tau_i-}(\omega)) \wedge (S_{\tau_i-}(\omega) - \underline{S}_{\tau_i-}(\omega)) > 0$ and $\tau_i(\omega), \tau_i(\omega) + \delta$ if $(\overline{S}_{\tau_i}(\omega) - S_{\tau_i}(\omega)) \wedge (S_{\tau_i}(\omega) - \underline{S}_{\tau_i}(\omega)) > 0$ for $i = 0, \ldots, n-1$ and $\delta > 0$ small. We leave the details to the reader. □

At last, we show how a $\varphi \in \mathbf{b}\mathcal{P}$, which incurs finite cost on a stochastic interval where the spread is bounded away from zero, can be approximated by almost simple strategies on this interval s.t. the cost terms converges as well.

Proposition 3.3.17. *Let $\varphi \in \mathbf{b}\mathcal{P}$ and $\sigma \le \tau$ two stopping times s.t.*

$$\inf_{\sigma(\omega) \le t < \tau(\omega)} (\overline{S}_t(\omega) - \underline{S}_t(\omega) > 0$$

for all $\omega \in \Omega$ and $C(\varphi, [\sigma \wedge T, \tau \wedge T]) < \infty$ a.s. Then, there exists a uniformly bounded sequence $(\varphi^n)_{n \in \mathbb{N}}$ s.t. $\varphi^n \mathbb{1}_{]\!]\sigma, \tau]\!]}$ is almost simple with $\varphi^n_\sigma = \varphi_\sigma$ on $\{\sigma < \infty\}$ and $|\varphi - \varphi^n| \le 1/n$ on $[\![\sigma, \tau]\!]$ (up to evanescence) for all $n \in \mathbb{N}$, and s.t.

$$\sup_{t \in [0,T]} |C(\varphi^n, [\sigma \wedge t, \tau \wedge t]) - C(\varphi, [\sigma \wedge t, \tau \wedge t])| \to 0, \quad \mathbb{P}\text{-}a.s. \tag{3.3.10}$$

The proof is postponed to Section 3.6.

3.3.3 Definition and characterization

For the remainder of the chapter, we make the following assumption on the bid-ask spread.

Assumption 3.3.18. *For every $(\omega, t) \in \Omega \times [0, T)$ with $\overline{S}_t(\omega) = \underline{S}_t(\omega)$ there exists an $\varepsilon > 0$ s.t. $\overline{S}_s(\omega) = \underline{S}_s(\omega)$ for all $s \in (t, (t+\varepsilon) \wedge T)$ or $\overline{S}_s(\omega) > \underline{S}_s(\omega)$ for all $s \in (t, (t+\varepsilon) \wedge T)$.*

This means that each zero of the path $t \mapsto \overline{S}_t(\omega) - \underline{S}_t(\omega)$ is either an inner point from the right of the zero set or a starting point of an excursion away from zero. This excludes, e.g., Brownian behavior of the spread, which is exploited in Example 3.3.23, where we show what can go wrong without this assumption.

For the rest of the chapter, we work with the predictable versions of the cost processes (cf. Proposition 3.3.12), and identify processes that coincide up to evanescence. Given a semimartingale S, we define the operator Π that maps a bounded, predictable strategy φ starting at zero, i.e., $\varphi \in \mathbf{b}\mathcal{P}$, to the associated $[-\infty,\infty)$-valued risk-less position (also starting at zero) by

$$\Pi_t(\varphi) := \varphi \bullet S_t - \varphi_t S_t - C_t(\varphi), \quad t \in [0,T], \tag{3.3.11}$$

which coincides with $\varphi \bullet S_{t-} - \varphi_t S_{t-} - C_t(\varphi)$. Throughout the chapter, $\varphi \bullet S$ denotes the *standard* stochastic integral as defined by Definition III.6.17 in [46]. If stock positions are evaluated by the semimartingale S, the wealth process is given by $V_t(\varphi) := \varphi \bullet S_t - C_t(\varphi) = \Pi_t(\varphi) + \varphi_t S_t$. If there is ambiguity about the semimartingale S used in the construction, we write $C^S(\varphi), \Pi^S(\varphi), V^S(\varphi)$ instead of $C(\varphi), \Pi(\varphi), V(\varphi)$.

We still have to introduce a measure that gives some information about the convergence of integrals w.r.t. S. There exists a probability measure $Q \sim \mathbb{P}$ s.t. the semimartingale S possesses a decomposition $S = M + A$, where M is a Q-square integrable martingale and A is a process of Q-integrable variation (Theorem 58 in Chapter VII of Dellacherie and Meyer [25]). We introduce the finite measure

$$\mu^S(B) := \mathbb{E}_Q\left(\mathbb{1}_B \bullet \langle M, M\rangle_T\right) + \mathbb{E}_Q\left(\mathbb{1}_B \bullet \mathrm{Var}_T(A)\right), \quad B \in \mathcal{P}, \tag{3.3.12}$$

where $\langle M, M\rangle$ denotes the predictable quadratic variation of M (see, e.g., [46, Chapter 1, Theorem 4.2]).

The following theorem characterizes the process $V(\varphi)$ as the limit of wealth processes associated with suitable almost simple strategies. Note that for almost simple strategies, V coincides with the intuitive wealth process that can be written down without any limiting procedure.

Theorem 3.3.19. *Let $\varphi \in \mathbf{b}\mathcal{P}$ and let μ be a σ-finite measure on the predictable σ-algebra with $\mu^S \ll \mu$.*

(i) For all $\{0,1\}$-valued decreasing predictable processes A and all uniformly bounded sequences of predictable processes $(\varphi^n)_{n\in\mathbb{N}}$, the following implication holds:

$$\begin{aligned} \varphi^n \to \varphi \quad & \text{pointwise on } \{\overline{S}_- > \underline{S}_-\} \cap \{A=1\}, \\ & \mu^S\text{-a.e. on } \{\overline{S}_- = \underline{S}_-\} \cap \{A=1\} \\ \Longrightarrow \quad & \liminf_{n\to\infty} V(\varphi^n) \le V(\varphi) \quad \text{on } \{A=1\} \text{ up to evanescence.} \end{aligned}$$

(ii) There exists a uniformly bounded sequence of almost simple strategies $(\varphi^n)_{n\in\mathbb{N}}$ s.t.

$$\begin{aligned} \varphi^n \to \varphi \quad & \text{pointwise on } \{\overline{S}_- > \underline{S}_-\} \cap \{C(\varphi) < \infty\}, \\ & \mu\text{-a.e. on } \{\overline{S}_- = \underline{S}_-\} \cap \{C(\varphi) < \infty\}, \end{aligned}$$

and

$$\sup_{t\in[0,T]} |V_t(\varphi^n) - V_t(\varphi)| \mathbb{1}_{\{C_t(\varphi)\le K\}} \to 0$$

in probability for $n \to \infty$ and all $K \in \mathbb{N}$.

Remark 3.3.20. *In the special case $C(\varphi) < \infty$, which is equivalent to $V(\varphi) > -\infty$, setting $A = 1$ yields the following characterization of the wealth process of a bounded strategy: (i) The wealth of the strategy exceeds the limiting wealth of (almost) pointwise converging simple strategies and (ii) there exists a special approximating sequence s.t. the wealth processes converge.*

On the set $\{V(\varphi) = -\infty\} = \{C(\varphi) = \infty\}$, one cannot expect the existence of a sequence of simple strategies that converge pointwise to φ on $\{\overline{S}_- > \underline{S}_-\}$. Nevertheless, Theorem 3.3.19(i) provides a motivation for $V(\varphi) = -\infty$.

For the proof of Corollary 3.3.22, we need the theorem in this general form, covering the case of infinite costs, since a priori *it is not clear that the latter property does not depend on the choice of S.*

Remark 3.3.21. *In Theorem 3.3.19(i), one cannot expect convergence "uniformly in probability" as in the frictionless case. Indeed, consider $\underline{S} = 1$, $\overline{S} = 2$, and $\varphi^n = \mathbb{1}_{]\!]1/n,1]\!]}$ which converges pointwise to $\varphi = \mathbb{1}_{]\!]0,1]\!]}$ but $V(\varphi^n) - V(\varphi) = \mathbb{1}_{]\!]0,1/n]\!]}$.*

Corollary 3.3.22. *Let $\varphi \in \mathbf{b}\mathcal{P}$. The self-financing condition, i.e., the risk-less position $\Pi(\varphi)$, does not depend on the choice of the semimartingale price system up to evanescence.*

Proof. Let $\varphi \in \mathbf{b}\mathcal{P}$ and $S, \widetilde{S}$ be semimartingale price systems. Of course, the measure Q in (3.3.12) can be chosen jointly for S and $\widetilde{S}$ and w.l.o.g. $Q = \mathbb{P}$. We set $\mu := \mu^S + \mu^{\widetilde{S}}$. Let us fix $K \in \mathbb{N}$ and show that

$$\Pi^{\widetilde{S}}(\varphi) \geq \Pi^S(\varphi) \quad \text{on } \{C^S(\varphi) \leq K\} \text{ up to evanescence.} \tag{3.3.13}$$

Observe that (3.3.13) for all $K \in \mathbb{N}$ implies that $\Pi^{\widetilde{S}}(\varphi) \geq \Pi^S(\varphi)$ up to evanescence since $\Pi^S(\varphi) = -\infty$ on $\{C^S(\varphi) = \infty\} = (\Omega \times [0,T]) \setminus \cup_{K\in\mathbb{N}}\{C^S(\varphi) \leq K\}$. Then, the assertion of the corollary follows by symmetry. Thus, it is sufficient to show (3.3.13).

For this, let $(\varphi^n)_{n\in\mathbb{N}}$ be a sequence of almost simple strategies satisfying the properties in Theorem 3.3.19(ii) for the semimartingale S and μ given above. According to Theorem 3.3.19(ii), we may suppose that

$$\sup_{t\in[0,T]} |V_t^S(\varphi^n) - V_t^S(\varphi)| \mathbb{1}_{\{C_t^S(\varphi)\leq K\}} \to 0 \quad \mathbb{P}\text{-a.s.} \tag{3.3.14}$$

by passing to a subsequence. On the other hand, by applying Theorem 3.3.19(i) with regard to the semimartingale $\widetilde{S}$ and $A := \mathbb{1}_{\{C^S(\varphi)\leq K\}}$, we get

$$\liminf_{n\to\infty} V^{\widetilde{S}}(\varphi^n) \leq V^{\widetilde{S}}(\varphi) \quad \text{on } \{C^S(\varphi) \leq K\} \text{ up to evanescence.} \tag{3.3.15}$$

In addition, Proposition 3.3.16 and elementary calculations yield the assertion of the corollary for almost simple strategies, i.e.,

$$V^{\widetilde{S}}(\varphi^n) = V^S(\varphi^n) + \varphi^n(\widetilde{S} - S), \quad n \in \mathbb{N}. \tag{3.3.16}$$

It remains to analyze $(\varphi^n - \varphi)(\widetilde{S} - S)$, especially on $\{\overline{S}_- = \underline{S}_-\} \cap \{\overline{S} > \underline{S}\}$. If a sequence of càdlàg processes converges to zero uniformly in probability, the same holds for the associated jump processes. Thus, the choice of μ and the same arguments as in the proof of Theorem 3.3.19(i) yield

$$\begin{aligned} &\sup_{t \in [0,T]} |\varphi^n_t \Delta S_t - \varphi_t \Delta S_t| \mathbb{1}_{\{\overline{S}_{t-} - \underline{S}_{t-} = 0,\ C^S_t(\varphi) < \infty\}} \to 0 \text{ in probability for } n \to \infty \\ &\sup_{t \in [0,T]} |\varphi^n_t \Delta \widetilde{S}_t - \varphi_t \Delta \widetilde{S}_t| \mathbb{1}_{\{\overline{S}_{t-} - \underline{S}_{t-} = 0,\ C^S_t(\varphi) < \infty\}} \to 0 \text{ in probability for } n \to \infty. \end{aligned} \tag{3.3.17}$$

By passing to a further subsequence (again denoted by $(\varphi^n)_{n \in \mathbb{N}}$), we can and do assume that the convergence in (3.3.17) holds for $\mathbb{P}$-a.e. $\omega \in \Omega$. Thus, on $\{\overline{S}_- = \underline{S}_-, C^S(\varphi) < \infty\}$ we have $\varphi^n(\widetilde{S} - S) = \varphi^n(\widetilde{S}_- - S_-) + \varphi^n(\Delta\widetilde{S} - \Delta S) = \varphi^n(\Delta\widetilde{S} - \Delta S) \to \varphi(\Delta\widetilde{S} - \Delta S) = \varphi(\widetilde{S} - S)$ up to evanescence. In addition, Theorem 3.3.19(ii) yields $\varphi^n(\widetilde{S} - S) \to \varphi(\widetilde{S} - S)$ on $\{\overline{S}_- > \underline{S}_-, C^S(\varphi) < \infty\}$, i.e., we have $\varphi^n(\widetilde{S} - S) \to \varphi(\widetilde{S} - S)$ on $\{C^S(\varphi) < \infty\}$ up to evanescence. Combining this with (3.3.14), (3.3.15), and (3.3.16) yields

$$\begin{aligned} \Pi^{\widetilde{S}}(\varphi) - \Pi^S(\varphi) &= V^{\widetilde{S}}(\varphi) - \varphi\widetilde{S} - \left(V^S(\varphi) - \varphi S\right) \\ &\geq \liminf_{n \to \infty} \left(V^{\widetilde{S}}(\varphi^n) - V^S(\varphi^n) - \varphi^n \left(\widetilde{S} - S\right)\right) \\ &= 0 \quad \text{on} \quad \{C^S(\varphi) \leq K\} \text{ up to an evanescence,} \end{aligned}$$

and we are done. We note that the differences above are well-defined since $\Pi^S(\varphi)$ and $V^S(\varphi)$ are finite on $\{C^S(\varphi) \leq K\}$. □

The following example shows that our approach does not work without Assumption 3.3.18.

Example 3.3.23. *Let $\underline{S} = -|B| + L^B$ and $\overline{S} = |B| + L^B$, where B is a standard Brownian motion and L^B its local time at zero in the sense of [74, page 212]. Consider the strategy $\varphi := \mathbb{1}_{\{\underline{S} = \overline{S}\} \cap (\Omega \times (0,T])} = \mathbb{1}_{\{B = 0\} \cap (\Omega \times (0,T])}$ and different semimartingale price systems $S = \alpha|B| + L^B$ for $\alpha \in [-1, 1]$. By Definition 3.3.8, we get $C(\varphi) = 0$. By [74, Theorem IV.69 and Corollary 3 of Theorem IV.70], we have $\varphi \bullet S = (\alpha + 1)L^B$. Together this implies $\Pi(\varphi) = (\alpha + 1)L^B - \mathbb{1}_{\{B=0\}} L^B$. Since L^B does not vanish, the self-financing condition would depend on the choice of α.*

Corollary 3.3.24. *Let $\varphi \in \mathbf{b}\mathcal{P}$ and $(\varphi^n)_{n \in \mathbb{N}}$ be uniformly bounded. If $\varphi^n \to \varphi$ pointwise on $\{\overline{S}_- > \underline{S}_-\}$ and μ^S-a.s. on $\{\overline{S}_- = \underline{S}_-\}$, then there exists a deterministic subsequence $(n_k)_{k \in \mathbb{N}}$ s.t.*

$$\lim_{k \to \infty} \left(V(\varphi^{n_k}) - V(\varphi)\right)^+ = 0 \textit{ up to evanescence.}$$

Proof. The proof of Theorem 3.19 (i) shows that we have $\varphi^n \bullet S \to \varphi \bullet S$ uniformly in probability. Hence, we can choose a subsequence $(n_k)_{k \in \mathbb{N}}$ s.t. $\varphi^{n_k} \bullet S \to \varphi \bullet S$ up to evanescence. Finally, together with $\liminf_{k \to \infty} C(\varphi^{n_k}) \geq \liminf_{n \to \infty} C(\varphi^n) \geq C(\varphi)$ the assertion follows. □

3.4 Extension to unbounded strategies

Let $(\mathbf{b}\mathcal{P})^\Pi := \{\varphi \in \mathbf{b}\mathcal{P} \ : \ \Pi(\varphi) > -\infty$ up to evanescence$\}$. Note that by Corollary 3.3.22 this set does not depend on the semimartingale price system. In this section, we want to extend the self-financing condition, i.e., the operator Π from $(\mathbf{b}\mathcal{P})^\Pi$ to an as large as possible set of predictable strategies. Therefore, recall that the space of adapted làdlàg processes $\mathcal{L}$ endowed with the topology of uniform convergence in probability, which is defined by the quasinorm $\|X\|_{up} = \mathbb{E}\left[\sup_{t\in[0,T]} |X_t| \wedge 1\right],\ X \in \mathcal{L}$, is a complete metric space with metric $d_{up}(X,Y) := \|X - Y\|_{up}$ for $X, Y \in \mathcal{L}$. Indeed, this is a consequence of the completeness of the space of làdlàg functions (also called regulated functions) equipped with the supremum norm (see, e.g., Fraňková [31, Point 1.8]). In addition, if $(X^n)_{n\in\mathbb{N}} \subseteq \mathcal{L}$ converges to $X \in \mathcal{L}$ with regard to d_{up}, we write $\text{up-lim}_{n\to\infty} X^n = X$. At this step, the restriction from $\mathbf{b}\mathcal{P}$ to $(\mathbf{b}\mathcal{P})^\Pi$ is not critical since the latter is sufficiently large to approximate finite portfolio processes, in which we are finally interested, in a reasonable way.

Definition 3.4.1. Let L denote the subset of real-valued, predictable strategies φ s.t. there exists a sequence $(\varphi^n)_{n\in\mathbb{N}} \subset (\mathbf{b}\mathcal{P})^\Pi$ with

(i) $\varphi^n \to \varphi$ pointwise on $\Omega \times [0,T]$ and $(\varphi^n)^+ \leq \varphi^+$, $(\varphi^n)^- \leq \varphi^-$ for all $n \in \mathbb{N}$,

(ii) there exists a semimartingale S with $\underline{S} \leq S \leq \overline{S}$ s.t.

$$(V^S(\varphi^n))_{n\in\mathbb{N}} = (\varphi^n \bullet S - C^S(\varphi^n))_{n\in\mathbb{N}}$$

is Cauchy in $(\mathcal{L}, d_{up})$ and s.t. for all sequences $(\widetilde{\varphi}^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ satisfying (i), there exists a deterministic subsequence $(n_k)_{k\in\mathbb{N}}$ s.t.

$$\left(V^S(\widetilde{\varphi}^{n_k}) - V^S(\varphi^{n_k})\right)^+ \to 0, \quad k \to \infty, \text{ up to evanescence.} \tag{3.4.1}$$

The requirement (ii) means that in the limit, the approximation with $(\varphi^n)_{n\in\mathbb{N}}$ is better than all other pointwise approximations $(\widetilde{\varphi}^n)_{n\in\mathbb{N}}$ if the stock position is evaluated by the same semimartingale. In (3.4.1), we cannot expect uniform convergence in time, but exceptional $\mathbb{P}$-null sets can be chosen independently of time. By Corollary 3.3.24, we have $(\mathbf{b}\mathcal{P})^\Pi \subseteq L$.

Proposition 3.4.2. *Let $\varphi \in L$. If $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ is a sequence satisfying the assertions of Definition 3.4.1 for φ with regard to a semimartingales S and $(\widetilde{\varphi}^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ is another sequence satisfying the same assertions for φ with regard to a semimartingale $\widetilde{S}$, then we have*

$$\underset{n\to\infty}{\text{up-lim}}\, V^S(\varphi^n) - \varphi S = \underset{n\to\infty}{\text{up-lim}}\, V^{\widetilde{S}}(\widetilde{\varphi}^n) - \varphi \widetilde{S}$$

up to evanescence.

We now can extend the operator Π to L by setting

$$\Pi(\varphi) := \underset{n\to\infty}{\text{up-lim}}\, V^S(\varphi^n) - \varphi S, \quad \varphi \in L,$$

where $(\varphi^n)_{n\in\mathbb{N}}$ is a sequence satisfying the assertions of Definition 3.4.1 with regard to the semimartingale S. By Proposition 3.4.2, Π is well-defined on L, i.e., it does not depend on the choice of the approximating sequence and the semimartingale.

Proof of Proposition 3.4.2. Let $(\varphi^n)_{n\in\mathbb{N}}$ and $(\widetilde{\varphi}^n)_{n\in\mathbb{N}}$ be sequences that satisfy the assumptions of the proposition. Corollary 3.3.22 states that the process $\Pi(\widetilde{\varphi}^n)$ does not depend on the semimartingale, i.e., we have

$$V^S(\widetilde{\varphi}^n) - \widetilde{\varphi}^n S = V^{\widetilde{S}}(\widetilde{\varphi}^n) - \widetilde{\varphi}^n \widetilde{S} \quad \text{up to evanescence for all } n \in \mathbb{N}, \tag{3.4.2}$$

and thus

$$\begin{aligned}&\Big(V^{\widetilde{S}}(\widetilde{\varphi}^n) - \widetilde{\varphi}^n\widetilde{S} - \big(V^S(\varphi^n) - \varphi^n S\big)\Big)^+ = \big(V^S(\widetilde{\varphi}^n) - V^S(\varphi^n) + (\varphi^n - \widetilde{\varphi}^n)S\big)^+\\ &\leq \big(V^S(\widetilde{\varphi}^n) - V^S(\varphi^n)\big)^+ + ((\varphi^n - \widetilde{\varphi}^n)S)^+ \end{aligned} \tag{3.4.3}$$

up to evanescence for all $n \in \mathbb{N}$. We have that $\varphi^n \to \varphi$ and $\widetilde{\varphi}^n \to \varphi$ pointwise as $n \to \infty$. We may pass to a subsequence s.t. $((V^S(\widetilde{\varphi}^n) - V^S(\varphi^n))^+)_{n\in\mathbb{N}}$ converges to zero pointwise up to evanescence by (3.4.1). In addition, we may further pass to subsequences, s.t. $(V^{\widetilde{S}}(\widetilde{\varphi}^n))_{n\in\mathbb{N}}$, $(V^S(\varphi^n))_{n\in\mathbb{N}}$ converge pointwise up to evanescence. Thus, by symmetry, (3.4.3) yields the assertion. ☐

3.4.1 Frictionless markets

We now turn towards the frictionless case, i.e., $\overline{S} = \underline{S} = S$, and show that L equals the set $L(S)$ of S-integrable processes:

Proposition 3.4.3. *Let $\underline{S} = \overline{S} = S$ be a semimartingale. Then, we have $L = L(S)$ and $\Pi(\varphi) = \varphi \bullet S - \varphi S$ for all $\varphi \in L$.*

The set $L(S)$ was introduced as given in Definition III.6.17 of [46] by Jacob [45], but there are equivalent definitions that may look a bit smarter and that are based on $\mathbf{b}\mathcal{P} \subseteq L(S)$. For this, recall that the space of semimartingales $\mathbb{S}$ endowed with the semimartingale topology defined by the metric

$$d_{\mathbb{S}}(X,Y) := \sup_{H\in\mathbf{b}\mathcal{P},\ \|H\|_\infty\leq 1} \|H \bullet (X-Y)\|_{up}, \quad X, Y \in \mathbb{S} \tag{3.4.4}$$

is a complete metric space by Émery [30, Theorem 1]. The following characterization of S-integrability is effectively due to Chou et al. [13].

Note 3.4.4. *Let S be a semimartingale and φ be a predictable process. The following assertions are equivalent* *(i)* $\varphi \in L(S)$.

(ii) *There exists a sequence* $(\varphi^n)_{n\in\mathbb{N}} \subseteq \mathbf{b}\mathcal{P}$ *s.t.* $\varphi^n \to \varphi$ *pointwise,* $(\varphi^n)^+ \leq \varphi^+$, $(\varphi^n)^- \leq \varphi^-$ *for all* $n \in \mathbb{N}$*, and* $(\varphi^n \bullet S)_{n\in\mathbb{N}}$ *is Cauchy in* $(\mathbb{S}, d_{\mathbb{S}})$.

(iii) *For all sequences* $(\varphi^n)_{n\in\mathbb{N}} \subseteq \mathbf{b}\mathcal{P}$ *with* $\varphi^n \to \varphi$ *pointwise and* $|\varphi^n| \leq |\varphi|$ *for all* $n \in \mathbb{N}$*, the sequence* $(\varphi^n \bullet S)_{n\in\mathbb{N}}$ *is Cauchy in* $(\mathbb{S}, d_{\mathbb{S}})$.

In this case, the integral $\varphi \bullet S$ *is given by the* $d_{\mathbb{S}}$*-limit of any such sequence* $(\varphi^n \bullet S)_{n\in\mathbb{N}}$.

Proof of Note 3.4.4. In the definition on page 130, Chou et al. [13] (see also [25, Chapter VIII, 75]) introduce the special approximating sequence $\varphi^n := \varphi \mathbb{1}_{\{|\varphi|\leq n\}}$ for some predictable process φ. Later on, the only properties of $(\varphi^n)_{n\in\mathbb{N}}$ they use is that $\varphi^n \in \mathbf{b}\mathcal{P}$ for $n \in \mathbb{N}$, $|\varphi^n| \leq |\varphi|$ for $n \in \mathbb{N}$, and $\varphi^n \to \varphi$ pointwise. Thus, the note is just a reformulation of their results [13, Properties b), c), d) on page 130 and Theoreme 1] (see also [25, Chapter VIII, 74-77]) □

A similar characterization is provided in Eberlein and Kallsen [29], page 193 by

$$L(S) = \{\varphi \text{ predictable} : \exists \text{ semimartingale } Z \text{ s.t. } (\varphi \mathbb{1}_{\{|\varphi|\leq n\}}) \bullet S = \mathbb{1}_{\{|\varphi|\leq n\}} \bullet Z,\ n \in \mathbb{N}\}.$$

It emphasizes the maximality of $L(S)$ if one *requires* that the integral $\varphi \bullet S := Z$ itself is a semimartingale. By contrast, in our characterization from Definition 3.4.1, the semimartingale property can be seen more as a result since it is stated with the up-metric and not with the semimartingale metric.

Proof of Proposition 3.4.3. Ad $L(S) \subseteq L$*:* This follows from (i)⇒ (ii)⇒(iii) in Note 3.4.4.

Ad $L \subseteq L(S)$*:* Let $\varphi \in L$. Thus, there exists $(\varphi^n)_{n\in\mathbb{N}} \subseteq \mathbf{b}\mathcal{P}$ satisfying Definition 3.4.1(i) and (ii). In particular, the sequence $(V^S(\varphi^n))_{n\in\mathbb{N}} = (\varphi^n \bullet S)_{n\in\mathbb{N}}$ is Cauchy with regard to d_{up}. Let us demonstrate that the sequence is also Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$ by contradiction, i.e., we assume that there exists $\varepsilon > 0$, a sequence $(H^n)_{n\in\mathbb{N}}$ of predictable processes with $0 \leq H^n \leq 1$ for all $n \in \mathbb{N}$ and a subsequence $(m_n)_{n\in\mathbb{N}}$ with $m_n \geq n$ s.t.

$$\mathbb{P}\left(((H^n(\varphi^n - \varphi^{m_n}) \bullet S))^*_T > \varepsilon\right) > \varepsilon, \quad \forall n \in \mathbb{N} \tag{3.4.5}$$

We note that in (3.4.5), it can be assumed that H^n is $[0,1]$-valued and not only $[-1,1]$-valued, since otherwise it can be decomposed into its positive and its negative part. Next, we define the strategies $\psi^n := H^n\varphi^n + (1 - H^n)\varphi^{m_n} \in \mathbf{b}\mathcal{P}$ and $\theta^n := (1 - H^n)\varphi^n + H^n\varphi^{m_n} \in \mathbf{b}\mathcal{P}$ for $n \in \mathbb{N}$. The strategies satisfy $\psi^n \to \varphi$, $\theta^n \to \varphi$ pointwise and $(\psi^n)^+ \vee (\theta^n)^+ \leq \varphi^+, (\psi^n)^- \vee (\theta^n)^- \leq \varphi^-$, i.e., they satisfy Definition 3.4.1 (i).

Let $\sigma^n := \inf\{t \geq 0 : \psi^n \bullet S_t - \varphi^n \bullet S_t > \varepsilon/2\}$ and $\tau^n := \inf\{t \geq 0 : \theta^n \bullet S_t - \varphi^n \bullet S_t > \varepsilon/2\}$. As $(\varphi^n - \varphi^{m_n}) \bullet S \to 0$ uniformly in probability by Definition 3.4.1 (ii), there is $N \in \mathbb{N}$ s.t. $\mathbb{P}(((\varphi^n - \varphi^{m_n}) \bullet S)^*_T > \varepsilon/2) < \varepsilon/2$ for all $n \geq N$. Thus, we have

$$\begin{aligned}\mathbb{P}(\sigma^n \wedge \tau^n \leq T) &\geq \mathbb{P}\left(((H^n(\varphi^n - \varphi^{m_n}) \bullet S))^*_T > \varepsilon\right) - \mathbb{P}\left(((\varphi^n - \varphi^{m_n}) \bullet S)^*_T > \varepsilon/2\right) \\ &> \varepsilon/2 \quad \forall n \geq N.\end{aligned}$$

Next, we define the strategies $\widetilde{\psi}^n := \psi^n \mathbb{1}_{[\![0,\sigma^n]\!]} + \varphi^n \mathbb{1}_{]\!]\sigma^n,T]\!]}$ and $\widetilde{\theta}^n := \theta^n \mathbb{1}_{[\![0,\tau^n]\!]} + \varphi^n \mathbb{1}_{]\!]\tau^n,T]\!]}$ that still satisfy Definition 3.4.1 (i). Thus, together with

$$\mathbb{P}\left(\left\{\widetilde{\psi}^n \bullet S_T - \varphi^n \bullet S_T > \varepsilon/2\right\} \cup \left\{\widetilde{\theta}^n \bullet S_T - \varphi^n \bullet S_T > \varepsilon/2\right\}\right) \geq \mathbb{P}(\sigma^n \wedge \tau^n \leq T) > \varepsilon/2$$

for all $n \geq N$, we have arrived at a contradiction to (3.4.1). Thus $(\varphi^n \bullet S)_{n\in\mathbb{N}}$ is Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$ and the assertion follows by $(ii) \Rightarrow (i)$ in Note 3.4.4. □

One of the referees in the review process for the article Kühn and Molitor [67] raised the following interesting question that can be considered as a generalization of Proposition 3.4.3 to markets with friction. Does $\varphi \in L$ imply that there exists a semimartingale price system S s.t. $\varphi \in L(S)$? This would mean, if stock positions are evaluated by S, the trading gains and the cost term of the approximating bounded strategies converge separately (and not only the sum).

Under additional assumptions, the following theorem gives a positive answer to this question. Especially, the considered model is deterministic, see Remark 3.4.6 below for a discussion.

Theorem 3.4.5. *Let $\Omega = \{\omega\}$ and $\overline{S}, \underline{S}$ be continuous. If $\varphi \in L$, $\varphi > 0$ on $(0,T]$, and φ is lower semi-continuous at all $t \in [0,T]$ with $\overline{S}_t > \underline{S}_t$, then there exists a semimartingale price system S with $\varphi \in L(S)$.*

Proof. We fix a semimartingale price system $\widetilde{S}$ (whose existence is assumed in this section).

Step 1: Let us show that

$$\sup_{\psi \text{ bounded},\ 0\leq\psi\leq\varphi} V_T^{\widetilde{S}}(\psi) < \infty. \tag{3.4.6}$$

Assume by contradiction that there exist bounded strategies ψ^n, $n \in \mathbb{N}$ s.t. $0 \leq \psi^n \leq \varphi$ and $V_T^{\widetilde{S}}(\psi^n) \to \infty$. On the other hand, since $\varphi \in L$ and by (3.4.2), there exist bounded φ^n, $n \in \mathbb{N}$ with $0 \leq \varphi^n \leq \varphi$, $\varphi^n \to \varphi$, and $V_T^{\widetilde{S}}(\varphi^n) \to V_T^{\widetilde{S}}(\varphi) \in \mathbb{R}$. Thus, there is a null sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subset (0,1)$ s.t.

$$V_T^{\widetilde{S}}(\varepsilon_n\psi^n + (1-\varepsilon_n)\varphi^n) \geq \varepsilon_n V_T^{\widetilde{S}}(\psi^n) + (1-\varepsilon_n)V_T^{\widetilde{S}}(\varphi^n) \to \infty,$$

which is a contradiction to $\varphi \in L$.

Step 2: Next, we show that for each nonnegative bounded function $\widetilde{\psi}$,

$$\sup_{0\leq\psi\leq\widetilde{\psi}} V_T^{\widetilde{S}}(\psi) \tag{3.4.7}$$

is attained by a maximizer ψ^*. To see this, let $(\psi^n)_{n\in\mathbb{N}}$ be a maximizing sequence, i.e., $V_T^{\widetilde{S}}(\psi^n) \to \sup_{0\leq\psi\leq\widetilde{\psi}} V_T^{\widetilde{S}}(\psi)$. Since $\psi^n \bullet \widetilde{S}_T \leq \sup_{t\in[0,T]} \widetilde{\psi}_t \cdot \mathrm{Var}_T(\widetilde{S})$ for all $n \in \mathbb{N}$, the sequence of cost terms $(C_T^{\widetilde{S}}(\psi^n))_{n\in\mathbb{N}}$ is bounded. In addition, the set $\{\overline{S} > \underline{S}\}$

can be written as a countable union of closed intervals on which either $\widetilde{S} \geq \underline{S} + 1/3(\overline{S} - \underline{S})$ or $\widetilde{S} \leq \underline{S} + 2/3(\overline{S} - \underline{S})$. In the first case, sells lead to essential costs on such an interval $[a, b]$. Consequently, one must have $\sup_{n\in\mathbb{N}} \operatorname{Var}_a^b(\psi^n) < \infty$. Then, by the same arguments as in Campi and Schachermayer [11], proof of Proposition 3.4, after passing to convex combinations, we obtain a pointwise limit $\lim_{n\to\infty} \psi^n =: \psi^*$ everywhere on $\{\overline{S} > \underline{S}\}$ and $\operatorname{Var}(\widetilde{S})$-a.e. on $\{\overline{S} = \underline{S}\}$, which has to be a maximizer by Theorem 3.3.19(i).

Step 3: We now construct a sequence $(\widehat{\varphi}^n)_{n\in\mathbb{N}}$ s.t. $\widehat{\varphi}^n$ is a solution of (3.4.7) with $\widetilde{\psi} = \varphi \wedge n$ for all $n \in \mathbb{N}$ and for $n < m$ the strategy $\widehat{\varphi}^m$ has to "buy/sell" if $\widehat{\varphi}^n$ "buys/sells".

Starting with solutions $\widehat{\eta}^k$ of (3.4.7) with $\widetilde{\psi} = (\varphi - (k-1))^+ \wedge 1$ for each $k \in \mathbb{N}$, we define the strategies $\eta^{n,k} := \left(\sum_{l=1}^n \widehat{\eta}^l - (k-1)\right)^+ \wedge 1$ for $n \in \mathbb{N}$ and $k \leq n$. We have

$$\sum_{k=1}^n V_T^{\widetilde{S}}(\eta^{n,k}) = V_T^{\widetilde{S}}\left(\sum_{k=1}^n \eta^{n,k}\right) = V_T^{\widetilde{S}}\left(\sum_{k=1}^n \widehat{\eta}^k\right) \geq \sum_{k=1}^n V_T^{\widetilde{S}}(\widehat{\eta}^k).$$

Indeed, $V_T^{\widetilde{S}}(\cdot)$ is superadditiv and additive for $\eta^{n,k}$, $k = 1, \ldots, n$. The later can be seen by the additivity of the cost term for approximating simple strategies. Together with $V_T^{\widetilde{S}}(\widehat{\eta}^k) \geq V_T^{\widetilde{S}}(\eta^{n,k})$ for all $k \leq n$, this implies $V_T^{\widetilde{S}}(\widehat{\eta}^k) = V_T^{\widetilde{S}}(\eta^{n,k})$ for all $n \in \mathbb{N}$ and $k \leq n$. Defining $\eta^k := \lim_{n\to\infty} \eta^{n,k} = (\sum_{l=1}^\infty \widehat{\eta}^l - (k-1))^+ \wedge 1$, $k \in \mathbb{N}$, we observe $\eta^k = 0$ on $\{\eta^{k-1} < 1\}$ and $\eta^k \leq (\varphi - (k-1)) \wedge 1$. In addition, we have $V_T^{\widetilde{S}}(\eta^k) \geq \lim_{n\to\infty} V_T^{\widetilde{S}}(\eta^{n,k}) = V_T^{\widetilde{S}}(\widehat{\eta}^k)$ by Theorem 3.3.19 (i) and, thus, η^k solves (3.4.7) with $\widetilde{\psi} = (\varphi - (k-1)) \wedge 1$. Finally, we set $\widehat{\varphi}^n := \sum_{k=1}^n \eta^k$, $n \in \mathbb{N}$. Then, for an arbitrary strategy ψ with $\psi \leq \varphi \wedge n$, the optimality of η^k yields $V_T^{\widetilde{S}}(\psi) = \sum_{k=1}^n V_T^{\widetilde{S}}((\psi - (k-1))^+ \wedge 1) \leq \sum_{k=1}^n V_T^{\widetilde{S}}(\eta^k) = V_T^{\widetilde{S}}(\widehat{\varphi}^n)$, i.e., $\widehat{\varphi}^n$ solves (3.4.7) with $\widetilde{\psi} = \varphi \wedge n$.

Step 4: Let $(\widehat{\varphi}^n)_{n\in\mathbb{N}}$ be the sequence of maximizers from the previous step. Since short positions are forbidden, we can replace $\underline{S}_T$ by $\widetilde{S}_T$ and assume that positions are sold at T. The aim is to construct a finite variation process S s.t. $V_T^{\widetilde{S}}(\widehat{\varphi}^n) = \widehat{\varphi}^n \bullet S_T$ and $\psi \bullet S_T \leq \widehat{\varphi}^n \bullet S_T$ for all strategies $0 \leq \psi \leq \varphi \wedge n$, i.e., S is a shadow price simultaneously for all problems (3.4.7) with $\widetilde{\psi} = \varphi \wedge n$, $n \in \mathbb{N}$. Under Assumption 3.3.18 and by an exhaustion argument, it is possible to construct S in the following way. On the frictionless intervals, cf. Lemma 3.5.2, S is defined as $S = \overline{S} = \underline{S}$. Now, let a be a "buying time" with $\overline{S}_a > \underline{S}_a$, i.e., there exists $n \in \mathbb{N}$ s.t. in any neighborhood of a there are $t_1 < t_2$ with $\widehat{\varphi}^n_{t_2} > \widehat{\varphi}^n_{t_1}$. Let b be the next selling time (defined as infimum over $n \in \mathbb{N}$), and d the next buying time after b. In addition, c is the last selling time before d. We have that $a < b \leq c \leq d$. The strict inequality is crucial for the exhaustion argument. It holds since, by $\overline{S}_a > \underline{S}_a$ and the continuity of the bid-ask prices, any investment needs some time to amortize, and by Step 3, for any pair of buying and selling time, there is a joint strategy $\widehat{\varphi}^n$ that realizes this investment. Summing up, all $\widehat{\varphi}^n$, $n \in \mathbb{N}$, are nondecreasing on (a, b), nonincreasing on (b, c), and constant on (c, d).

For $t \in [a, b)$, we define

$$\tau_t := \inf\{s \in [a,t] : \exists \varepsilon > 0 \inf_{u\in(s,t+\varepsilon)} \varphi_u > \inf_{u\in(t,b)} \varphi_u\} \wedge t \tag{3.4.8}$$

and

$$S_t := \inf_{u\in[\tau_t,b)} \overline{S}_u \wedge \underline{S}_b.$$

Roughly speaking, S can only increase at a "bottleneck" on the way to b, where the constraint is binding. For $t \in [b, c)$, we define

$$\sigma_t := \sup\{s \in [t,c) : \forall \varepsilon > 0 \inf_{u\in(t+\varepsilon,s)} \varphi_u > \inf_{u\in[b,t]} \varphi_u\} \vee t$$

and

$$S_t := \sup_{u\in[b,\sigma_t]} \underline{S}_u. \tag{3.4.9}$$

For $[c, d)$, $c < d$, we make a case differentiation. For $\widehat{\varphi}^1 = 0$ on (c, d), we define S on $[c, d)$ as the Snell envelope of the process $L_t := \underline{S}_t \mathbb{1}_{\{t<d\}} + \overline{S}_d \mathbb{1}_{\{t=d\}}$, $t \in [c, d]$, i.e., $S_t := \sup_{u\in[t,d]} L_u$, $t \in [c, d)$]. Otherwise, we define $S_t := \underline{S}_c \mathbb{1}_{\{t<\widetilde{\tau}_d\}} + \overline{S}_d \mathbb{1}_{\{t\geq\widetilde{\tau}_d\}}$, where $\widetilde{\tau}_d := \inf\{s \in [c,d] : \inf_{u\in(s,d)} \varphi_u > \inf_{u\in(d,\widetilde{b})} \varphi_u\} \wedge d$ with $\widetilde{b}$ being the next selling time after d. By using the maximality and the monotonicity of all $\widehat{\varphi}^n$, $n \in \mathbb{N}$, it is easy to check that S has to lie in the bid-ask spread.

Now, any excursion of the spread away from zero, cf. Lemma 3.5.1, can be exhausted by intervals of the form $[a, b)$, $[b, c)$, and $[c, d)$. In the special case that there is no further buying time, (3.4.9) is applied to the closed interval from b to the end of the excursion of the spread away from zero or to T. The resulting process S is càdlàg and does not depend on the choice of the intervals. Note that $\overline{S}_a > \underline{S}_a$ is only needed to guarantee that $b > a$.

Step 5: Let us show that S is of finite variation and

$$\widehat{\varphi}^n \bullet S_T = V_T^S(\widehat{\varphi}^n) = V_T^{\widetilde{S}}(\widehat{\varphi}^n), \quad n \in \mathbb{N}. \tag{3.4.10}$$

Let a be a buying time and $\widetilde{a}$ be the time $\inf\{t > a : \widehat{\varphi}_t^1 = 0\}$ truncated at the end of the excursion. We have that $S_a = \overline{S}_a \geq \widetilde{S}_a$ and $S_{\widetilde{a}} = \underline{S}_{\widetilde{a}} \leq \widetilde{S}_{\widetilde{a}}$, and S is nondecreasing on $[a, \widetilde{a}]$. From $\widetilde{a}$ up to (and including) the next buying time, S is nonincreasing. This yields $\mathrm{Var}_T(S) \leq \mathrm{Var}_T(\widetilde{S}) < \infty$. Finally, by construction of S, the cost terms $C^S(\widehat{\varphi}^n)$ vanish for all $n \in \mathbb{N}$ and thus (3.4.10) holds. E.g., on $[a, b)$, the process $\widehat{\varphi}^n$ is nondecreasing and has to be constant on $\{S < \overline{S}\}$ by optimality.

Step 6: Next, we show that

$$\psi \bullet S_T \leq \widehat{\varphi}^n \bullet S_T \text{ for all } n \in \mathbb{N} \text{ and all strategies } \psi \text{ with } 0 \leq \psi \leq \varphi \wedge n. \tag{3.4.11}$$

Of course, it is sufficient to show this assertion for excursions of the spread away from zero (cf., again, Lemma 3.5.1).

From now on, we need the assumed lower semi-continuity, i.e.,

$$\varphi_t = \lim_{\varepsilon\to 0} \inf_{u\in[t-\varepsilon,t+\varepsilon]} \varphi_u \quad \text{for all } t \in (0,T) \text{ with } \overline{S}_t > \underline{S}_t. \tag{3.4.12}$$

We start with the buying period, i.e., the interval $[a,b)$ (cf. Step 4). Setting $\xi_t := \inf_{u\in[t,b)} \varphi_u$, we claim that

$$\int_{[a,b)} \psi_t \, dS_t \le \int_{[a,b)} (\varphi_t \wedge n) \, dS_t \le \int_{[a,b)} (\xi_t \wedge n) \, dS_t \le \int_{[a,b)} \widehat{\varphi}^n_t \, dS_t \tag{3.4.13}$$

for every strategy ψ with $\psi \le \varphi \wedge n$.

The first inequality is obvious as S is nondecreasing on $[a,b)$. We start by showing the second inequality in (3.4.13). It follows from (3.4.12) that $(\xi_t)_{t\in[a,b)}$ is left-continuous and the set $\{t \in [a,b) : \xi_t < \varphi_t\}$ is open. Hence, we find a sequence of open intervals (u^k_1, u^k_2), $u^k_1 \le u^k_2$, $k \in \mathbb{N}$ s.t.

$$\{t \in [a,b) : \xi_t < \varphi_t\} = \bigcup_{k\in\mathbb{N}} (u^1_k, u^2_k). \tag{3.4.14}$$

For all t_1, t_2 with $u^k_1 < t_1 < t_2 < u^k_2$, we have that $\inf_{t\in[t_1,t_2]}(\varphi_t - \xi_t) > 0$ and, thus, $S_{t_2} = S_{t_1}$. This yields $S_{u^k_2-} = S_{u^k_1}$ if $u^k_1 < u^k_2$ and, hence, $\int_{[a,b)}(\varphi_t \wedge n) \, dS_t = \int_{[a,b)}(\xi_t \wedge n) \, dS_t$ due to (3.4.14).

Moving towards the last inequality in (3.4.13), we exclude the trivial case that $\overline{S}_a = \underline{S}_b$. For a given $\varepsilon > 0$, there is a partition $a = t_0 < t_1 < \cdots < t_m = b$ s.t.

$$\int_{[a,b)} (\xi_t \wedge n) \, dS_t \le \sum_{i=1}^{m-1} (\xi_{t_{i-1}} \wedge n)(S_{t_i} - S_{t_{i-1}}) + (\xi_{t_{m-1}} \wedge n)(S_{b-} - S_{t_{m-1}}) + \varepsilon \tag{3.4.15}$$

by [74, Theorem II.21] and the left-continuity of ξ. Let $s := \sup\{u > a : \overline{S}_u < \underline{S}_b\} \le b$. Next, we define a perturbation $\widehat{\varphi}^{n,p}$ of the optimal strategy $\widehat{\varphi}^n$ in the bid-ask model, which approximately realizes the gains on the RHS of (3.4.15) on $[a,b)$. We set $\widehat{\varphi}^{n,p} = \widehat{\varphi}^n$ on $[0,a) \cup [s,T]$ and construct $\widehat{\varphi}^{n,p}$ on $[a,s)$ by iteratively specifying possible purchases. At time $t_0 = a$, we buy until we reach $\widehat{\varphi}^{n,p}_a := \xi_{t_0} \wedge n \ge \widehat{\varphi}^n_a$, paying price $\overline{S}_a = S_a$ per share (time t_0 has the special property that it is a "buying time" in the sense of Step 4). We proceed as follows: if $S_{t_1} < S_{t_2}$ (which is equivalent to $\inf_{u\in[\tau_{t_1},\tau_{t_2})} \overline{S}_u < S_{t_2}$ and, in this case, $S_{t_1} = \inf_{u\in[\tau_{t_1},\tau_{t_2})} \overline{S}_u$), we buy until we reach $\xi_{t_1} \wedge n$ shares at time $t^*_1 := \arg\min_{u\in[\tau_{t_1},\tau_{t_2})} \overline{S}_u$. Hereby, we have $\overline{S}_{t^*_1} < S_{t_2} \le \underline{S}_b$, i.e., $t^*_1 < s$, and, since $t^*_1 \ge \tau_{t_1}$, the constraint $\varphi \wedge n$ is also satisfied. This is repeated for the intervals $[\tau_{t_{i-1}}, \tau_{t_i})$ for $i = 3, \dots, m$. Since purchasing prices are strictly below $\underline{S}_b$, in the bid-ask market, purchases take place on $[a,s)$. For $s < b$, we have $\widehat{\varphi}^{n,p}_{s-} \le \xi_s \wedge n = \widehat{\varphi}^n_s$, where the equality follows from the optimality of $\widehat{\varphi}^n$ and (3.4.12). Finally, the missing position $\widehat{\varphi}^n_s - \widehat{\varphi}^{n,p}_{s-} \ge 0$ is purchased at price $\overline{S}_s = \underline{S}_b$ if $s < b$. In the case $s = b$, we must have $\underline{S}_b = \overline{S}_b$ and need not care about the sign of the missing position.

Hence, the optimality of $\widehat{\varphi}^n$, together with $V_T^{\widetilde{S}}(\widehat{\varphi}^n) - V_T^{\widetilde{S}}(\widehat{\varphi}^{n,p}) = V_T^S(\widehat{\varphi}^n) - V_T^S(\widehat{\varphi}^{n,p})$, yields

$$0 \le V_T^S(\widehat{\varphi}^n) - V_T^S(\widehat{\varphi}^{n,p}) \le \int_{[a,b)} \widehat{\varphi}_t^n dS_t - \int_{[a,b)} (\xi_t \wedge n) dS_t + \varepsilon, \tag{3.4.16}$$

where for the second inequality we use (3.4.15) and the fact that $\widehat{\varphi}^{n,p}$ does not produce any costs w.r.t. S. (3.4.16) implies the last inequality in (3.4.13) as the $\varepsilon > 0$ is arbitrary.

It remains to show $\psi_t \, dS_t \le \widehat{\varphi}_t^n \, dS_t$ on sets other than $[a,b)$. After a time reversal, the proof for a selling interval $[b,c)$ is the same as for a buying interval $[a,b)$. Namely, w.l.o.g. we assume that $\underline{S}_c > \underline{S}_b$ and consider an approximation similar to (3.4.15) "backward in time" (the last point is $b-$ with $S_{b-} = \underline{S}_b$). Time s from above is replaced by $\widetilde{s} := \inf\{u > b \ : \ \underline{S}_u > \underline{S}_b\} \le c$. From the optimality of $\widehat{\varphi}^n$, the assumption that b is a selling time in the sense of Step 4, and (3.4.12), it follows that $\widehat{\varphi}_{b-}^n \ge \inf_{u \in [b,\widetilde{s}]} \varphi_u \wedge n$. We leave the details as an exercise for the reader. On intervals with $\widehat{\varphi}^1 = 0$, we use that the Snell envelope is nonincreasing.

Step 7: By $\varphi \in L$ and (3.4.2), we can find a sequence of strategies $(\varphi^n)_{n\in\mathbb{N}}$ with $\varphi^n \to \varphi$ and $0 \le \varphi^n \le \varphi \wedge n$ s.t. for all other strategies $(\widetilde{\varphi}^n)_{n\in\mathbb{N}}$ with $\widetilde{\varphi}^n \to \varphi$ and $0 \le \widetilde{\varphi}^n \le \varphi \wedge n$, one has $(V_T^S(\widetilde{\varphi}^n) - V_T^S(\varphi^n))^+ \to 0$. Let us show that $(\varphi^n \bullet S)_{n\in\mathbb{N}}$ has to be Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$. We first show that

$$\forall \varepsilon > 0 \ \exists K \in \mathbb{R}_+ \ \forall n \in \mathbb{N}, B \in \mathcal{B}([0,T]) \qquad (\mathbb{1}_{\{\varphi > K\} \cap B} \varphi^n) \bullet S_T \le \varepsilon. \tag{3.4.17}$$

Indeed, since S is a shadow price, see (3.4.11), and by (3.4.6), we have

$$(\mathbb{1}_{\{\varphi > K\} \cap B} \varphi^n) \bullet S_T \le (\mathbb{1}_{\{\varphi > K\}} \widehat{\varphi}^n) \bullet S_T \le \sum_{k=1}^{\infty} ((\mathbb{1}_{\{\varphi > K\}} \eta^k) \bullet S_T) < \infty \tag{3.4.18}$$

for all $K \in \mathbb{R}_+$ and $B \in \mathcal{B}([0,T])$. By (3.4.18), $(\mathbb{1}_{\{\varphi > K\}} \eta^k) \bullet S_T \le \eta^k \bullet S_T$ (which follows from (3.4.11)), and dominated convergence, we obtain (3.4.17). Let us show that

$$\forall \varepsilon > 0 \ \exists K \in \mathbb{R}_+, N \in \mathbb{N} \ \forall n \ge N, B \in \mathcal{B}([0,T]) \ (\mathbb{1}_{\{\varphi > K\} \cap B} \varphi^n) \bullet S_T \ge -\varepsilon. \tag{3.4.19}$$

Assume by contradiction that there exists $\varepsilon > 0$, a subsequence $(n_k)_{k\in\mathbb{N}}$, and a sequence $(B_k)_{k\in\mathbb{N}} \subset \mathcal{B}([0,T])$ s.t. $(\mathbb{1}_{\{\varphi > k\} \cap B_k} \varphi^{n_k}) \bullet S_T < -\varepsilon$ for all $k \in \mathbb{N}$. On the other hand, since $d_{\mathbb{S}}(\mathbb{1}_{\{\varphi > k\}} \bullet S, 0) \to 0$ for $k \to \infty$, there must exist a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ with $\lambda_k \to \infty$ slowly enough s.t. $\mathbb{1}_{\{\varphi > k\} \cap B_k} (\varphi^{n_k} \wedge \lambda_k) \bullet S_T \to 0$ for $k \to \infty$. Thus, we have $(\mathbb{1}_{\{\varphi > k\} \cap B_k} (\varphi^{n_k} - \lambda_k)^+) \bullet S_T < -\varepsilon/2$ for k large enough. As in (3.4.18), we can estimate $(\mathbb{1}_{[0,T] \setminus (\{\varphi > k\} \cap B_k)} (\varphi^{n_k} - \lambda_k)^+) \bullet S_T = (\mathbb{1}_{\{\varphi > \lambda_k\} \setminus (\{\varphi > k\} \cap B_k)} (\varphi^{n_k} - \lambda_k)^+) \bullet S_T \le \sum_{l=1}^{\infty} ((\mathbb{1}_{\{\varphi > \lambda_k\}} \eta^l) \bullet S_T)$, which converge to 0 as $\lambda_k \to \infty$ for $k \to \infty$. This yields that $((\varphi^{n_k} - \lambda_k)^+) \bullet S_T < -\varepsilon/4$ for k large enough. Since the cost term of φ^{n_k} exceeds those of $\varphi^{n_k} \wedge \lambda_k$, we arrive at $V_T^S(\varphi^{n_k}) < V_T^S(\varphi^{n_k} \wedge \lambda_k) - \varepsilon/4$ for k large enough. This is a contradiction to the maximality of $(\varphi^n)_{n\in\mathbb{N}}$ stated at the beginning of this step. Thus, (3.4.19) holds.

Putting (3.4.17), (3.4.19), and $\varphi^n \to \varphi$ with $\varphi^n \le \varphi$ for all $n \in \mathbb{N}$ together, we obtain that $(\varphi^n \bullet S)_{n\in\mathbb{N}}$ is Cauchy in $(\mathbb{S}, d_{\mathbb{S}})$. This implies that $\varphi \in L(S)$ (cf. Note 3.4.4). □

Remark 3.4.6. *The proof demonstrates how the maximality condition in the definition of L works. For $\varphi \in L$, problem (3.4.6) has to be finite, but its maximizer $\widehat{\varphi} := \lim_{n\to\infty} \widehat{\varphi}^n$ can be different from $\varphi = \lim_{n\to\infty} \varphi^n$. Also, in the frictionless shadow price market, $\widehat{\varphi}^n$ dominates all other strategies that are bounded by $\varphi \wedge n$. This upper bound is key to show that $\varphi^n \bullet S$ is Cauchy w.r.t. the semimartingale topology.*

It is an open (but possibly insolvable) problem whether the theorem also holds in the general stochastic case. The construction of the shadow price S is essentially based on the assumptions that the model is deterministic and φ is lower semi-continuous. The latter is needed since on the intervals with friction, S has its upward movements at the "bottlenecks" of the constraint $\varphi \wedge n$.

Nevertheless, we think that the proof already provides the basic intuition for the relation between L and $L(S)$ in the general stochastic case. In addition, the sequence of strategies constructed in Step 3 and the ideas from Step 7 should also be of general use to solve related problems in the stochastic model. By contrast, the other assumptions are less essential. They are made to focus on the main ideas and to avoid further case differentiations and technicalities.

3.5 Proof of Theorem 3.3.19

We start with two lemmas that prepare the proof of Theorem 3.3.19. In the following, we set $X := \overline{S} - \underline{S}$ with the convention that $X_{0-} := 0$. Let M be the set of starting points of excursions of the spread away from zero, i.e.,

$$M := (\{X = 0\} \cup \{X_- = 0\}) \\ \cap \{(\omega, t) \in \Omega \times [0,T) : \exists \varepsilon > 0\ \forall s \in (t, (t+\varepsilon)\wedge T)\ X_s(\omega) > 0\}.$$

Here, we follow the convention that an excursion also ends (and thus a new excursion can start) if only the left limit of the spread process is zero. Under the usual conditions and Assumption 3.3.18, the process $Y := \mathbb{1}_{\{(\omega,t)\in\Omega\times[0,T):\exists\varepsilon>0\ \forall s\in(t,(t+\varepsilon)\wedge T)\ X_s(\omega)>0\}}$ is right-continuous on $\Omega\times[0,T)$ and adapted (for the latter one uses that for all $t \in [0,T)$ and $\widetilde{\varepsilon} \in (0, T-t)$, one has $\{\omega \in \Omega : \exists\varepsilon > 0\ \forall s \in (t,(t+\varepsilon)\wedge T)\ X_s(\omega) > 0\} = \Omega \setminus \{\omega \in \Omega : \exists \varepsilon \in (0,\widetilde{\varepsilon})\cap\mathbb{Q}\ \forall s \in (t, t+\varepsilon)\cap\mathbb{Q}\ X_s(\omega) = 0\}$). Thus, Y is a progressive process (see, e.g., Theorem 3.11 in [42]), which implies that M is a progressive set. Consequently, $\{\omega \in \Omega\ :\ \tau(\omega) < \infty,\ (\omega, \tau(\omega)) \notin M\} \in \mathcal{F}$ if τ is a stopping time.

For a stopping time τ, we define the associated stopping time $\Gamma_2(\tau)$ by

$$\Gamma_2(\tau) := \inf\{t > \tau\ :\ X_t = 0 \text{ or } X_{t-} = 0\}.$$

Lemma 3.5.1. *There exists a sequence of stopping times $(\tau_1^n)_{n\in\mathbb{N}}$ with $\mathbb{P}(\{\omega \in \Omega : \tau_1^n(\omega) < \infty,\ (\omega, \tau_1^n(\omega)) \notin M\}) = 0$ for all $n \in \mathbb{N}$, $\mathbb{P}(\tau_1^{n_1} = \tau_1^{n_2} < \infty) = 0$ for all*

$n_1 \neq n_2$, and

$$\{X_- > 0\} \subset \cup_{n\in\mathbb{N}} \rrbracket \tau_1^n, \Gamma_2(\tau_1^n) \rrbracket \quad \textit{up to evanescence.} \tag{3.5.1}$$

Proof. We define a finite measure μ on the predictable σ-algebra by

$$\mu(A) := \sum_{k=1}^{\infty} 2^{-k} \mathbb{P}(\{\omega \in \Omega : (\omega, q_k) \in A\}), \quad A \in \mathcal{P},$$

where $(q_k)_{k\in\mathbb{N}}$ is a counting of the rational numbers. Let $\mathcal{M}$ be the set of predictable processes of the form $\mathbb{1}_{\rrbracket \tau, \Gamma_2(\tau)\rrbracket}$, where τ runs through all stopping times satisfying $\mathbb{P}(\{\omega \in \Omega : \tau(\omega) < \infty,\ (\omega, \tau(\omega)) \notin M\}) = 0$. The essential supremum of $\mathcal{M}$ w.r.t. μ can be written as

$$\operatorname{esssup} \mathcal{M} = \sup_{n\in\mathbb{N}} \mathbb{1}_{\rrbracket \tau_1^n, \tau_2^n \rrbracket} = \mathbb{1}_{\cup_{n\in\mathbb{N}} \rrbracket \tau_1^n, \tau_2^n \rrbracket} \quad \mu\text{-a.e.},$$

where $\tau_2^n = \Gamma_2(\tau_1^n)$. Obviously, the sequence $(\tau_1^n)_{n\in\mathbb{N}}$ can be chosen s.t. $\mathbb{P}(\tau_1^{n_1} = \tau_1^{n_2} < \infty) = 0$ holds for all $n_1 \neq n_2$. Then, by the definition of M and Γ_2, one has that $\rrbracket \tau_1^{n_1}, \tau_2^{n_1} \rrbracket \cap \rrbracket \tau_1^{n_2}, \tau_2^{n_2} \rrbracket = \emptyset$ up to evanescence for all $n_1 \neq n_2$.

Now consider the random time $\sigma := \inf\{t \in (0,T] : X_{t-} > 0 \text{ and } t \notin \cup_{n\in\mathbb{N}} (\tau_1^n, \tau_2^n]\}$. Since σ can be written as the debut $\inf\{t \in (0,T] : Z_t > 0\}$, where $Z := X_-(1 - \sum_{n=1}^{\infty} \mathbb{1}_{\rrbracket \tau_1^n, \tau_2^n \rrbracket})$ is a finite predictable process, it is a stopping time (see Theorem 7.3.4 in [14]). By the definition of the infimum and Γ_2, we must have that $X_\sigma = 0$ or $X_{\sigma-} = 0$ on the set $\{\sigma < \infty\}$. Together with Assumption 3.3.18, this means that in σ there starts an excursion, and it is not yet overlapped. By the definition of the essential supremum, one has $\mu(\rrbracket \sigma, \Gamma_2(\sigma) \rrbracket) = 0$. Since $\Gamma_2(\sigma) > \sigma$ on $\{\sigma < \infty\}$, this is only possible if $\mathbb{P}(\sigma < \infty) = 0$ and thus $\mathbb{P}(\{\omega \in \Omega : \exists t \in (0,T]\ X_{t-}(\omega) > 0 \text{ and } t \notin \cup_{n\in\mathbb{N}} (\tau_1^n(\omega), \tau_2^n(\omega)]\}) = 0$. □

Next, we analyze the time the spread spends at zero. Define

$$\begin{aligned} M_1 := &\{(\omega, t) \in \Omega \times [0,T] : t = 0 \text{ or } \forall \varepsilon > 0\ \exists s \in ((t-\varepsilon) \vee 0, t)\ X_s(\omega) > 0\} \\ &\cap \{X_- = 0\} \\ \text{and } M_2 := &\{X_- > 0\} \cap \{X = 0\}. \end{aligned}$$

The optional set $M_1 \cup M_2$ consists of the ending points of an excursion and of their accumulation points. For a stopping time τ, we define the starting point of the next excursion after τ by $(\Gamma_1(\tau))(\omega) := \inf\{t \geq \tau(\omega) : (\omega, t) \in M\}$ for $\omega \in \Omega$, which is the debut of a progressive set and thus a stopping time by [14, Theorem 7.3.4].

Lemma 3.5.2. *There exists a sequence of stopping times $(\sigma_1^n)_{n\in\mathbb{N}}$ with $\mathbb{P}(\{\omega \in \Omega : \sigma_1^n(\omega) < \infty,\ (\omega, \sigma_1^n(\omega)) \notin M_1 \cup M_2\}) = 0$ s.t. $(\sigma_1^n)_{\{X_{\sigma_1^n -} = 0\}}$ are predictable stopping times for all $n \in \mathbb{N}$, $\mathbb{P}(\sigma_1^{n_1} = \sigma_1^{n_2} < \infty) = 0$ for all $n_1 \neq n_2$, and*

$$\{X_- = 0\} \subset \cup_{n\in\mathbb{N}} \left(\llbracket (\sigma_1^n)_{\{X_{\sigma_1^n -} = 0\}} \rrbracket \cup \rrbracket \sigma_1^n, \Gamma_1(\sigma_1^n) \rrbracket \right) \quad \textit{up to evanescence.} \tag{3.5.2}$$

for Γ_1 from above.

(3.5.2) can be interpreted as follows. If the spread approaches zero continuously at some time t, the investment between $t-$ and t already falls into the "frictionless regime". On the other hand, if the spread jumps to zero at time t, the frictionless regime only starts immediately after t (if at all).

Proof of Lemma 3.5.2. We take the starting points τ_1^n of the excursions from Lemma 3.5.1 and define the measure $\mu(A) := \sum_{n=1}^\infty 2^{-n}\mathbb{P}(\{\omega \in \Omega : (\omega, \tau_1^n(\omega)) \in A\}) + \mathbb{P}(\{\omega \in \Omega : (\omega, T) \in A\})$ for all $A \in \mathcal{P}$. Consider the essential supremum w.r.t. μ of the set of predictable processes $\mathbb{1}_{[\![\sigma_{\{X_{\sigma-}=0\}}]\!]\cup]\!]\sigma,\Gamma_1(\sigma)]\!]}$, where σ runs through the set of stopping times satisfying $\mathbb{P}(\{\omega \in \Omega : \sigma(\omega) < \infty,\ (\omega, \sigma(\omega)) \notin M_1 \cup M_2\}) = 0$ with the further constraint that $\sigma_{\{X_{\sigma-}=0\}}$ is a predictable stopping time. Again, the supremum can be written as

$$\mathbb{1}_{\cup_{n\in\mathbb{N}}\left([\![(\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}}]\!]\cup]\!]\sigma_1^n,\Gamma_1(\sigma_1^n)]\!]\right)} \quad \mu\text{-a.e.}$$

Consider the random time

$$\sigma := \inf\{t \geq 0 : X_{t-} = 0 \text{ and } t \notin \cup_{n\in\mathbb{N}}\left([(\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}}] \cup (\sigma_1^n, \sigma_2^n]\right)\}, \tag{3.5.3}$$

where $\sigma_2^n := \Gamma_1(\sigma_1^n)$. Since $\sigma = \inf\{t \geq 0 : Z_t = 0\}$, where

$$Z := X_- + \sum_{n=1}^{\infty} \mathbb{1}_{[\![(\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}}]\!]\cup]\!]\sigma_1^n,\sigma_2^n]\!]}$$

is predictable, σ is a stopping time (see Theorem 7.3.4 in [14]). In addition, one has

$$\begin{aligned}[\![\sigma_{\{X_{\sigma-}=0\}}]\!] &= [\![\sigma]\!] \cap \{X_- = 0\} \\ &= \left([\![0,\sigma]\!] \setminus \cup_{n\in\mathbb{N}}[\![(\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}}]\!]\cup]\!]\sigma_1^n,\sigma_2^n]\!]\right) \cap \{X_- = 0\} \in \mathcal{P},\end{aligned}$$

where we use that the infimum in (3.5.3) must be attained if $X_{\sigma-} = 0$. Thus, $\sigma_{\{X_{\sigma-}=0\}}$ is a predictable stopping time. Finally, we have that $\mathbb{P}(\{\omega \in \Omega : \sigma(\omega) < \infty,\ (\omega, \sigma(\omega)) \notin M_1 \cup M_2) = 0$. By the maximality of the supremum, one has

$$\mu([\![\sigma_{\{X_{\sigma-}=0\}}]\!]\cup]\!]\sigma,\Gamma_1(\sigma)]\!]) = 0.$$

Since the intervals overlap T or some $\tau_1^n(\omega)$ if they are nonempty, we arrive at $\mathbb{P}(\sigma < \infty) = 0$, and thus (3.5.2) holds. □

Note 3.5.3. *For any $\varphi \in \mathbf{b}\mathcal{P}$ and any σ-finite measure μ on $\mathcal{P}$ with $\mu^S \ll \mu$, there exists a uniformly bounded sequence of simple strategies $(\varphi^n)_{n\in\mathbb{N}}$ with $\varphi^n \to \varphi$, μ-a.e., and for any such sequence $(\varphi^n)_{n\in\mathbb{N}}$ one has $\varphi^n \bullet S \to \varphi \bullet S$ uniformly in probability.*

Proof. The existence of such a sequence with $\varphi^n \to \varphi$, μ-a.e. follows from the approximation theorem for measures (see, e.g., Theorem 1.65(ii) in [62]). Then, the convergence of the integrals follows for the martingale parts by (3) on page 49 of [46] and for the finite variation parts by dominated convergence. □

Proof of Theorem 3.3.19. Obviously, it is sufficient to show the theorem under an equivalent measure $Q \sim \mathbb{P}$. Hence, we assume w.l.o.g. that $\mathbb{P} = Q$, where Q is the measure introduced above (3.3.12).

Ad (i): Let $(\varphi^n)_{n\in\mathbb{N}} \subset \mathbf{b}\mathcal{P}$ satisfy $\varphi^n \to \varphi$ pointwise on $\{\overline{S}_- > \underline{S}_-,\ A = 1\}$. For any $J \in \mathcal{I}$ from (3.3.1), Proposition 3.3.11 yields that $\liminf_{n\to\infty} C(\varphi^n, J \cap [0,t])(\omega) \geq C(\varphi, J \cap [0,t])(\omega)$ for all $(\omega, t) \in \{A = 1\}$. It follows that $\liminf_{n\to\infty} C_t(\varphi^n)(\omega) \geq \sup_{J\in\mathcal{I}} C(\varphi, J \cap [0,t])(\omega) = C_t(\varphi)(\omega)$ for all $(\omega,t) \in \{A = 1\}$. If in addition $(\varphi^n)_{n\in\mathbb{N}}$ is uniformly bounded and $\varphi^n \to \varphi$ μ^S-a.e. on $\{\overline{S}_- = \underline{S}_-,\ A = 1\}$, we have that

$$(\varphi^n \mathbb{1}_{\{A=1\}}) \bullet S \to (\varphi \mathbb{1}_{\{A=1\}}) \bullet S \quad \text{uniformly in probability} \tag{3.5.4}$$

(see Note 3.5.3). Since $\{A = 1\}$ is a predictable set of interval type, there is an increasing sequence of stopping times $(T^m)_{m\in\mathbb{N}}$ s.t. $\{A = 1\} \cup (\Omega \times \{0\}) = \cup_{m\in\mathbb{N}} [\![0, T^m]\!]$ (see, e.g., [42, Theorem 8.18]). For each $m \in \mathbb{N}$, we obviously have

$$\left(\left(\mathbb{1}_{[\![0,T^m]\!]}\varphi\right) \bullet S\right) \mathbb{1}_{[\![0,T^m]\!]} = (\varphi \bullet S)^{T^m} \mathbb{1}_{[\![0,T^m]\!]} = (\varphi \bullet S)\, \mathbb{1}_{[\![0,T^m]\!]}.$$

Letting $m \to \infty$ this yields

$$(\varphi \mathbb{1}_{\{A=1\}} \bullet S)\mathbb{1}_{\{A=1\}} = (\varphi \bullet S)\mathbb{1}_{\{A=1\}} \tag{3.5.5}$$

up to evanescence by Note 3.5.3 and, analogously, $(\varphi^n \mathbb{1}_{\{A=1\}} \bullet S)\mathbb{1}_{\{A=1\}} = (\varphi^n \bullet S)\mathbb{1}_{\{A=1\}}$ up to evanescence for $n \in \mathbb{N}$. Thus, together with (3.5.4), we have

$$\liminf_{n\to\infty} (\varphi^n \bullet S - \varphi \bullet S)^+ \mathbb{1}_{\{A=1\}} = 0 \quad \text{up to evanescence.}$$

Putting the cost terms and the trading gains w.r.t. S together, we arrive at (i).

Ad (ii): The following analysis is based on the stopping times $(\tau_1^n)_{n\in\mathbb{N}}$ and $(\sigma_1^n)_{n\in\mathbb{N}}$ from Lemma 3.5.1 and Lemma 3.5.2, respectively. We can and do choose $(\sigma_1^n)_{n\in\mathbb{N}}$ s.t.

$$\mathbb{P}(\sigma_1^n = \tau_1^m < \infty,\ X_{\sigma_n^1-} > 0) = 0, \quad \forall n, m \in \mathbb{N}. \tag{3.5.6}$$

This means that if the spread X only touches zero at a single point and its left limit is non-zero, there directly starts the next excursion without a one point frictionless regime in between.

For the rest of the proof, we write $\{X_{\tau-} \in B\}$ for the set $\{\omega \in \Omega : \exists t \in [0,T]\ \tau(\omega) = t,\ X_{t-}(\omega) \in B\}$, where τ is a $[0,T] \cup \{\infty\}$-valued stopping time and $B \subset \mathbb{R}$. Let

$$\begin{aligned}
A^n &:=]\!](\tau_1^n)_{\{X_{\tau_1^n-}>0\}}, \Gamma_2(\tau_1^n)[\![\cup [\![(\Gamma_2(\tau_1^n))_{\{X_{\tau_1^n-}>0\}\cap\{X_{\Gamma_2(\tau_1^n)-}>0\}}]\!] \in \mathcal{P},\ n \in \mathbb{N},\\
B^n &:= [\![(\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}}]\!] \cup]\!]\sigma_1^n, \Gamma_1(\sigma_1^n)]\!] \in \mathcal{P},\ n \in \mathbb{N},\\
\widetilde{B}^n &:=]\!]\Gamma_1(\sigma_1^n), \Gamma_2(\Gamma_1(\sigma_1^n))[\![\cup [\![(\Gamma_2(\Gamma_1(\sigma_1^n)))_{X_{\{\Gamma_2(\Gamma_1(\sigma_1^n)))->0\}}}]\!] \in \mathcal{P},\ n \in \mathbb{N},
\end{aligned}$$

and

$$\varphi^N := \varphi \mathbb{1}_{\cup_{n=1,\dots,N}(A^n \cup B^n \cup \widetilde{B}^n)}, \quad N \in \mathbb{N}. \tag{3.5.7}$$

Excursions away from zero are either included by A^n or by $\widetilde{B}^n$ with the frictionless forerunner B^n. In the first case, the spread cannot jump away from zero since $X_{\tau_1^n} = 0$ on $\{X_{\tau_1^n -} > 0\}$. In the latter case, the frictionless forerunner avoids that φ^N produces costs when the spread jumps away from zero, which do not occur with the strategy φ. Namely, at a time the spread jumps away from zero, φ^N either remains zero or it already coincides with φ. Note that the frictionless forerunner may consist of a single point only. For example, this is the case if the jump time is an accumulation point of starting/ending points of excursions shortly before.

First, we approximate φ by the strategies φ^N.

Step 1: Let $E \in \mathcal{F}_T$ be a set with $\mathbb{P}(E) = 1$ s.t. the properties from Lemma 3.5.1 and Lemma 3.5.2 hold for all $\omega \in E$. Let us show that $\varphi_t^N(\omega) \to \varphi_t(\omega)$ for all $t \in [0,T]$ and $\omega \in E$. By construction of φ^N, we only have to show that for each $n \in \mathbb{N}$, the excursion starting in $\tau_1^n(\omega)$ is overlapped by $A_\omega^n := \{t \in [0,T] : (\omega,t) \in A^n\}$, the ω-intersection of A^n, or by some $\widetilde{B}_\omega^m$, $m \in \mathbb{N}$. In the case that $X_{\tau_1^n(\omega)-}(\omega) > 0$, the excursion is overlapped by A_ω^n. In the case that $X_{\tau_1^n(\omega)-}(\omega) = 0$, we have by Lemma 3.5.2 that $\tau_1^n(\omega) \in [\sigma_1^m(\omega), \Gamma_1(\sigma_1^m(\omega))]$ for some $m \in \mathbb{N}$ and thus the excursion starting in $\tau_n^1(\omega)$ is overlapped by $\widetilde{B}_\omega^m$. By Note 3.5.3, it follows that $\varphi^N \bullet S$ to $\varphi \bullet S$ uniformly in probability for $N \to \infty$.

Step 2: W.l.o.g we assume that the bounded process φ takes values in $[-1/2, 1/2]$ to get rid of a further constant. Let us show that

$$\sup_{t\in[0,T]} |C_t(\varphi^N) - C_t(\varphi)| \mathbb{1}_{\{C_t(\varphi)\le K\}} \to 0, \ N \to \infty, \text{ pointwise on } E \ \forall K \in \mathbb{N}. \tag{3.5.8}$$

From $X_{\tau_1^n} = 0$ on $\{X_{\tau_1^n -} > 0\}$ and $X_{\sigma_1^n} = 0$ on $\{X_{\sigma_1^n -} > 0\}$, we conclude: for fixed $\omega \in E$ and $0 \le a \le b \le T$ with $\inf_{u\in[a,b)} X_u(\omega) > 0$, we either have that $\varphi_u^N(\omega) = \varphi_u(\omega)$ for all $u \in [a,b]$ or $\varphi_u^N(\omega) = 0$ for all $u \in [a,b]$. By the definition of the cost term in (3.3.2), this yields $C(\varphi^N, I \cap [0,t]) \le C(\varphi, I \cap [0,t])$ for all $I \in \mathcal{I}$, $(\omega,t) \in E \times [0,T]$ and thus $C_t(\varphi^N) \le C_t(\varphi)$ for all $(\omega,t) \in E \times [0,T]$. We define

$$\theta^m := \inf\{t \ge 0 \ : \ C_t(\varphi) > m\} \wedge T \quad \text{for } m \in \mathbb{N}. \tag{3.5.9}$$

By $\Delta^- C_{\theta^m}(\varphi) \le \sup_{u\in[0,T]} X_u$, the paths of the stopped process $C^{\theta^m}(\varphi)$ are bounded. Fix $\omega \in E$ and $\varepsilon > 0$. For $K \in \mathbb{N}$ we set $u := \theta^K$. Proposition 3.3.7 yields that $C(\varphi, I \cap [0,u]) = C(\varphi, I \cap [0,t]) + C(\varphi, I \cap [t,u])$ for all $I \in \mathcal{I}$ and $t \le u$. Therefore, together with Proposition 3.3.9(i), there exists $I \in \mathcal{I}$ s.t.

$$\sup_{t\in[0,T]} \left(C_t(\varphi) - C(\varphi, I \cap [0,t])\right) \mathbb{1}_{\{C_t(\varphi)\le K\}} \le \varepsilon.$$

The set I is overlapped by finitely many ω-intersections of A^n and $B^n \cup \widetilde{B}^n$, i.e., for N large enough, one has $I \subset \cup_{n\le N}(A^n \cup B^n \cup \widetilde{B}^n)_\omega$, i.e., $C(\varphi^N, I \cap [0,t]) =$

$C(\varphi, I \cap [0,t])$ and, consequently, $(C_t(\varphi) - C_t(\varphi^N))\mathbb{1}_{\{C_t(\varphi)\le K\}} \le (C(\varphi, I\cap[0,t]) - C(\varphi^N, I\cap[0,t]))\mathbb{1}_{\{C_t(\varphi)\le K\}} + \varepsilon = \varepsilon$ for all $t \in [0,T]$. This implies (3.5.8). Together with Step 1, we have that

$$\varphi^N \to \varphi \text{ pointwise up to evanescence} \tag{3.5.10}$$

$$\text{and } \sup_{t\in[0,T]} |V_t(\varphi^N) - V_t(\varphi)|\mathbb{1}_{\{C_t(\varphi)\le K\}} \to 0 \text{ in probability} \tag{3.5.11}$$

for $N \to \infty$ and each $K \in \mathbb{N}$.

Step 3: It remains to approximate the strategies φ^N, $N \in \mathbb{N}$, by almost simple strategies. Since the pointwise convergence that we need on $\{X_- > 0\} \cap \{C(\varphi) < \infty\}$ is not metrizable, it is not sufficient to approximate each φ^N separately by a sequence of almost simple strategies. Let μ be a σ-finite measure on $\mathcal{P}$ with $\mu^S \ll \mu$. We fix some $N \in \mathbb{N}$ and let $\varepsilon := 2^{-N}$. In the following, we construct an almost simple strategy step by step on disjoint stochastic intervals. The main idea is to approximate the cost term on subintervals of excursions where the spread is bounded away from zero while controlling the error at the beginning and the end of the excursions. We start with the construction of an almost simple strategy on A^n with $n \le N$. We recall that $\tau_2^n := \Gamma_2(\tau_1^n)$. There exists a stopping time $\tau_1^{n,N}$ with $\theta^N \wedge \tau_2^n \ge \tau_1^{n,N} > \tau_1^n$ on $\{\tau_1^n < \theta^N\} \cap \{X_{\tau_1^n-} > 0\}$, $\tau_1^{n,N} = \theta^N$ on $\{\theta^N \le \tau_1^n\}$ and, for notational convenience, $\tau_1^{n,N} = \tau_1^n$ elsewhere s.t.

$$\mathbb{P}(\tau_1^n \wedge \theta^N \le \tau_1^{n,N} \le \tau_1^n + \varepsilon) = 1, \tag{3.5.12}$$

$\mathbb{P}((\varphi^N \mathbb{1}_{]\!]\tau_1^n,\tau_1^{n,N}]\!]} \bullet S)^\star > \varepsilon) \le \varepsilon$, $\mathbb{P}(\tau_1^n < \infty,\ |X_{\tau_1^{n,N}} - X_{\tau_1^n\wedge\theta^N}| > \varepsilon) \le \varepsilon$, and $\mathbb{P}(\tau_1^n < \infty,\ C_{\tau_1^{n,N}}(\varphi^N) - C_{\tau_1^n\wedge\theta^N}(\varphi^N) > \varepsilon) \le \varepsilon$, where we use the notation $Y^\star := \sup_{t\in[0,T]}|Y_t|$ and θ^N was defined in (3.5.9). This follows from the right-continuity of the processes $\varphi^N\mathbb{1}_{]\!]\tau_1^n,T]\!]} \bullet S$, X and from the definition of the cost process together with $X_{\tau_1^n} = 0$ on $\{X_{\tau_1^n-} > 0\}$. In addition, since $[\![(\tau_2^n)_{\{X_{\tau_2^n-}=0\}}]\!] =]\!]\tau_1^n, \tau_2^n]\!] \cap \{X_- = 0\} \in \mathcal{P}$, the stopping time $(\tau_2^n)_{\{X_{\tau_2^n-}=0\}}$ is predictable. Thus, by the existence of an announcing sequence (see, e.g., [42, Theorem 4.34]), there is a stopping time $\tau_2^{n,N}$ with $\tau_1^{n,N} \le \tau_2^{n,N} \le \tau_2^n \wedge \theta^N$ and $\tau_2^{n,N} < \tau_2^n$ on $\{X_{\tau_2^n-} = 0, \tau_1^{n,N} < \tau_2^n\}$ s.t.

$$\mathbb{P}(\tau_2^{n,N} < \tau_2^n \wedge \theta^N - \varepsilon) \le \varepsilon, \qquad \mathbb{P}(X_{\tau_2^n-} > 0,\ \tau_2^{n,N} < \tau_2^n \wedge \theta^N) \le \varepsilon, \tag{3.5.13}$$

$$\mathbb{P}((\varphi^N \mathbb{1}_{]\!]\tau_2^{n,N},\tau_2^n\wedge\theta^N[\![\cup[\![(\tau_2^n)_{\{X_{\tau_2^n-}>0,\ \tau_2^{n,N}<\tau_2^n\wedge\theta^N\}}]\!]} \bullet S)^\star > \varepsilon) \le \varepsilon,$$

$$\mathbb{P}(X_{\tau_2^{n,N}} > \varepsilon, \tau_2^{n,N} < \tau_2^n \wedge \theta^N) \le \varepsilon, \text{ and } \mathbb{P}(\tau_2^n < \infty,\ C_{\tau_2^n\wedge\theta^N}(\varphi^N) - C_{\tau_2^{n,N}}(\varphi^N) > \varepsilon) \le \varepsilon.$$

By Proposition 3.3.17 applied to the stopping times $\tau_1^{n,N} \le \tau_2^{n,N}$, there exists an almost simple strategy $\widetilde{\psi}^N$ with $\widetilde{\psi}^N_{\tau_1^{n,N}} = \varphi^N_{\tau_1^{n,N}}$,

$$\sup_{t\in[\tau_1^{n,N},\tau_2^{n,N}]} |\widetilde{\psi}^N_t - \varphi^N_t| \le \varepsilon, \tag{3.5.14}$$

$$\mathbb{P}\left(\sup_{t\in[\tau_1^{n,N},\tau_2^{n,N}]} |C_t(\widetilde{\psi}^N) - C_{\tau_1^{n,N}}(\widetilde{\psi}^N) - (C_t(\varphi^N) - C_{\tau_1^{n,N}}(\varphi^N))| > \varepsilon\right) \leq \varepsilon,$$

and $\mathbb{P}(((\widetilde{\psi}^N - \varphi^N)\mathbb{1}_{]\!]\tau_1^{n,N},\tau_2^{n,N}]\!]} \bullet S)^\star > \varepsilon) \leq \varepsilon$ (the later also uses Note 3.5.3). We define the almost simple strategy by

$$\psi_t^N := \widetilde{\psi}_t^N \mathbb{1}_{(\tau_1^{n,N} < t \leq \tau_2^{n,N})} \quad \text{on} \quad A^n. \tag{3.5.15}$$

Since ψ^N can be updated for free at the left endpoint of A^n, for the increments of the process $V(\psi^N) - V(\varphi^N) = (\psi^N - \varphi^N) \bullet S - (C(\psi^N) - C(\varphi^N))$ we get the estimate

$$\mathbb{P}\Bigg(\sup_{\substack{t\in(\tau_1^n,\tau_2^n)\\ \cup[(\tau_2^n)_{\{X_{\tau_2^n-}>0\}}]}} |V_t(\psi^N) - V_{\tau_1^n}(\psi^N) - (V_t(\varphi^N) - V_{\tau_1^n}(\varphi^N))|\mathbb{1}_{\{C_t(\varphi)\leq K\}} > 8\varepsilon$$
$$\tau_1^n < \infty,\ X_{\tau_1^n-} > 0,\Bigg) \leq 8\varepsilon \qquad \text{for all } n = 1,\dots,N,\ K \leq N, \tag{3.5.16}$$

regardless of how ψ^N is defined outside A^n, especially at time τ_1^n. Indeed, in the worst case, there are 2 error terms on $(\tau_1^n, \tau_1^{n,N}]$, 3 error terms on $(\tau_1^{n,N}, \tau_2^{n,N}]$, and 3 error terms between $(\tau_2^{n,N}, \tau_2^n) \cup [(\tau_2^n)_{\{X_{\tau_2^n-}>0\}}]$.

We proceed with the construction of the almost simple strategy on $B^n \cup \widetilde{B}^n$ with $n \leq N$. A strategy with support B^n has zero costs, and by Note 3.5.3, we find an (almost) simple strategy $\widehat{\psi}^N$ with

$$\mu(|\widehat{\psi}^N - \varphi^N|\mathbb{1}_{B^n} > \varepsilon) \leq \varepsilon, \tag{3.5.17}$$

$$\mathbb{P}(\Gamma_1(\sigma_1^n) < \infty,\ |\widehat{\psi}^N_{\Gamma_1(\sigma_1^n)} - \varphi^N_{\Gamma_1(\sigma_1^n)}|X_{\Gamma_1(\sigma_1^n)} > \varepsilon) \leq \varepsilon, \tag{3.5.18}$$

and $\mathbb{P}(((\widehat{\psi}^N - \varphi^N)\mathbb{1}_{B^n} \bullet S)^\star > \varepsilon) \leq \varepsilon$. After $\Gamma_1(\sigma_1^n)$, we proceed similar to (3.5.15). Setting $\widetilde{\tau}_2^n := \Gamma_2(\Gamma_1(\sigma_1^n))$, there exists a stopping time $\widetilde{\tau}_1^{n,N}$ with $\widetilde{\tau}_1^{n,N} = \theta^N$ on $\{\theta^N \leq \Gamma_1(\sigma_1^n)\}$, $\widetilde{\tau}_1^{n,N} = \Gamma_1(\sigma_1^n)$ on $\{\Gamma_1(\sigma_1^n) < \theta^N,\ X_{\Gamma_1(\sigma_1^n)} > 0\}$ and $\theta^N \wedge \widetilde{\tau}_2^n \geq \widetilde{\tau}_1^{n,N} > \Gamma_1(\sigma_1^n)$ on $\{\Gamma_1(\sigma_1^n) < \theta^N,\ X_{\Gamma_1(\sigma_1^n)} = 0\}$ s.t. $\mathbb{P}(\Gamma_1(\sigma_1^n) \wedge \theta^N \leq \widetilde{\tau}_1^{n,N} \leq \Gamma_1(\sigma_1^n) + \varepsilon) = 1$, $\mathbb{P}(((\varphi^N - \varphi^N_{\Gamma_1(\sigma_1^n)})\mathbb{1}_{]\!]\Gamma_1(\sigma_1^n),\widetilde{\tau}_1^{n,N}]\!]} \bullet S)^\star > \varepsilon) \leq \varepsilon$, $\mathbb{P}(\Gamma_1(\sigma_1^n) < \infty,\ |X_{\widetilde{\tau}_1^{n,N}} - X_{\Gamma_1(\sigma_1^n)\wedge\theta^N}| > \varepsilon) \leq \varepsilon$, and $\mathbb{P}(\Gamma_1(\sigma_1^n) < \infty,\ C_{\widetilde{\tau}_1^{n,N}}(\varphi^N) - C_{\Gamma_1(\sigma_1^n)\wedge\theta^N}(\varphi^N) > \varepsilon) \leq \varepsilon$. $\widetilde{\tau}_2^{n,N}$ is defined completely analogous to $\tau_2^{n,N}$ from above. We set

$$\psi_t^N := \widehat{\psi}_t^N \mathbb{1}_{(t\leq\Gamma_1(\sigma_1^n)\wedge\theta^N)} + \overline{\psi}_t^N \mathbb{1}_{(\widetilde{\tau}_1^{n,N} < t \leq \widetilde{\tau}_2^{n,N})} \text{ on } B^n \cup \widetilde{B}^n \tag{3.5.19}$$

for some almost simple strategy $\overline{\psi}^N$ with $\overline{\psi}^N_{\widetilde{\tau}_1^{n,N}} = \varphi^N_{\widetilde{\tau}_1^{n,N}}$ and $\sup_{t\in[\widetilde{\tau}_1^{n,N},\widetilde{\tau}_2^{n,N}]} |\overline{\psi}_t^N - \varphi_t^N| \leq \varepsilon$. As in (3.5.16), but with the additional error terms on B^n and (3.5.18) for the case that the spread jumps away from zero, we get that

$$\mathbb{P}\left(\sup_{\substack{t\in[(\sigma_1^n)_{\{X_{\sigma_1^n-}=0\}}\cup(\sigma_1^n,\Gamma_2(\Gamma_1(\sigma_1^n)))\\ \cup[(\Gamma_2(\Gamma_1(\sigma_1^n)))_{\{X_{\Gamma_2(\Gamma_1(\sigma_1^n))-}>0\}}]}} |V_t(\psi^N) - V^1 - (V_t(\varphi^N) - V^2)|\mathbb{1}_{\{C_t(\varphi)\leq K\}} > 10\varepsilon\right)$$
$$\leq 10\varepsilon \qquad \text{for all } n = 1,\dots,N,\ K \leq N, \tag{3.5.20}$$

where $V^1 := V_{\sigma_1^n-}(\psi^N)$, $V^2 := V_{\sigma_1^n-}(\varphi^N)$ on $\{X_{\sigma_1^n-} = 0\}$ and $V^1 := V_{\sigma_1^n}(\psi^N)$, $V^2 := V_{\sigma_1^n}(\varphi^N)$ on $\{X_{\sigma_1^n-} > 0\}$. By (3.5.6), A^n and $B^m \cup \widetilde{B}^m$ are disjoint. Thus, (3.5.15) and (3.5.19) can be used to define an almost simple strategy on $\Omega \times [0,T]$: for $n \leq N$, define ψ^N on $\cup_{n\leq N}(A^n \cup B^n \cup \widetilde{B}^n)$ as above and set $\psi^N := 0$ on $(\Omega \times [0,T]) \setminus \cup_{n\leq N}(A^n \cup B^n \cup \widetilde{B}^n)$. By $V_0(\psi^N) = V_0(\varphi^N) = 0$ and the construction of A^n and $B^n \cup \widetilde{B}^n$, for each (ω,t), $(V_t(\psi_t^N)(\omega) - V_t(\varphi^N)(\omega))\mathbb{1}_{\{C_t(\varphi)\leq K\}}(\omega)$ can be written as a finite sum of increments from (3.5.16) and (3.5.20). For this, we again use that at the right endpoint of A^n and $\widetilde{B}^n$, the position can be liquidated without any costs. Summing up the error terms and recalling that $\varepsilon = 2^{-N}$, this yields $\mathbb{P}(\sup_{t\in[0,T]} |V_t(\psi^N) - V_t(\varphi^N)|\mathbb{1}_{\{C_t(\varphi)\leq K\}} > 18N2^{-N}) \leq 18N2^{-N}$ for all $N \geq K$. Together with (3.5.11), we obtain $\sup_{t\in[0,T]} |V_t(\psi^N) - V_t(\varphi)|\mathbb{1}_{\{C_t(\varphi)\leq K\}} \to 0$ in probability for $N \to \infty$ and all $K \in \mathbb{N}$.

By (3.5.17) and (3.5.19), we have that $(\psi^N)_{N\in\mathbb{N}}$ converges to φ μ-a.e. on $\{X_- = 0\}\cap\{C(\varphi) < \infty\}$. It remains to show that $(\psi^N)_{N\in\mathbb{N}}$ converges pointwise up to evanescence to φ on the set $\{X_- > 0\} \cap \{C(\varphi) < \infty\}$. Let $(\omega,t) \in \Omega \times [0,T]$ with $X_{t-}(\omega) > 0$ and $C_t(\varphi)(\omega) < \infty$. By the arguments in Step 1, there exists an $n \in \mathbb{N}$ with $(\omega,t) \in A^n \cup \widetilde{B}^n$. W.l.o.g. $(\omega,t) \in A^n$. By (3.5.12), one has $\tau_1^{n,N}(\omega) \leq \tau_1^n(\omega) + 2^{-N} < t$ and, as the costs at t are finite, $\theta^N(\omega) \geq t$ for N large enough.

Case 1: $t < \tau_2^n(\omega)$. By (3.5.13) and the lemma of Borel-Cantelli, we have that $\mathbb{P}(E^n) = 0$, where $E^n := \cap_{\widetilde{N}\in\mathbb{N}} \cup_{N\geq\widetilde{N}} \{\tau_2^{n,N} < \tau_2^n - 2^{-N}\}$. If $\omega \notin E^n$, this implies that $t < \tau_2^n(\omega) - 2^{-N} \leq \tau_2^{n,N}(\omega)$ for N large enough and thus by (3.5.14), $|\psi_t^N(\omega) - \varphi_t^N(\omega)| \leq 2^{-N}$ for N large enough.

Case 2: $t = \tau_2^n(\omega)$ and thus $X_{\tau_2^n(\omega)-} > 0$. By (3.5.13) and the lemma of Borel-Cantelli, we have that $\mathbb{P}(\widetilde{E}^n) = 0$, where $\widetilde{E}^n := \cap_{\widetilde{N}\in\mathbb{N}} \cup_{N\geq\widetilde{N}} \{X_{\tau_2^n-} > 0,\ \tau_2^{n,N} < \tau_2^n\}$. If $\omega \notin \widetilde{E}^n$, this implies that $t = \tau_2^{n,N}(\omega)$ for N large enough and thus by (3.5.14), $|\psi_t^N(\omega) - \varphi_t^N(\omega)| \leq 2^{-N}$ for N large enough.

Since $\varphi_t^N(\omega) = \varphi_t(\omega)$ for all $N \geq n$, we conclude that the sequence $(\psi^N)_{N\in\mathbb{N}}$ converges pointwise up to evanescence to φ on the set $\{X_- > 0\} \cap \{C(\varphi) < \infty\}$. □

3.6 Technical results: Construction of the cost term

Proof of Proposition 3.3.3 and Proposition 3.3.10. As the two propositions are interrelated, we give their proofs together. Recall that the arguments below are path-by-path, i.e., $\omega \in \Omega$ is fixed.

Step 1: We begin by establishing the uniqueness of the cost term. Therefore, assume that there are exist $C_1, C_2 \in [0,\infty]$ satisfying the condition in Definition 3.3.2. This means that for each $i \in \{1,2\}$, $\varepsilon > 0$, we find a partition P^i_ε of $I = [a,b]$ s.t. for every refinement P of P^i_ε and every modified intermediate subdivision λ of P, we have $d(C_i, R(\varphi, P, \lambda)) < \varepsilon$, where $d(x,y) := |\arctan(x) - \arctan(y)|$ with $\arctan(\infty) := \pi/2$, which defines a metric on $[0,\infty]$. But, letting λ denote an arbitrary modified intermediate subdivision of $P^1_\varepsilon \cup P^2_\varepsilon$, this means

$$d(C_1, C_2) \le d(C_1, R(\varphi, P^1_\varepsilon \cup P^2_\varepsilon, \lambda)) + d(C_2, R(\varphi, P^1_\varepsilon \cup P^2_\varepsilon, \lambda)) < 2\varepsilon,$$

which means $C_1 = C_2$ as the above holds for all $\varepsilon > 0$.

Step 2: We now turn towards existence. Let $(\delta_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}} \subseteq (0,\infty)$ be sequences with $\delta_n \downarrow 0$ and $\eta_n \downarrow 0$. It follows from a minor adjustment of [71, Lemma 2.1] that for each $n \in \mathbb{N}$ there is a partition $P_n = \{t^n_0, \ldots, t^n_{k_n}\}$ of I s.t.

$$\operatorname{osc}(\overline{S} - S, [t^n_{i-1}, t^n_i)) < \delta_n \quad \text{and} \quad \operatorname{osc}(S - \underline{S}, [t^n_{i-1}, t^n_i)) < \delta_n \tag{3.6.1}$$

for $i = 1, \ldots, k_n$. By the definition of the oscillation of a function, (3.6.1) also holds for every refinement of P_n. Hence, P_n can be chosen s.t. we also have

$$\begin{cases} \sum_{i=1}^{k_n} |\varphi_{t^n_i} - \varphi_{t^n_{i-1}}| + \eta_n \ge \operatorname{Var}^b_a(\varphi), & \text{if } \operatorname{Var}^b_a(\varphi) < \infty \\ \sum_{i=1}^{k_n} |\varphi_{t^n_i} - \varphi_{t^n_{i-1}}| > 1/\eta_n, & \text{if } \operatorname{Var}^b_a(\varphi) = \infty \end{cases} \quad \text{for all } n \in \mathbb{N}. \tag{3.6.2}$$

In addition, we can obviously choose the sequence $(P_n)_{n\in\mathbb{N}}$ s.t. it is refining. This shows that there exists a refining sequence of partitions satisfying assertions (i) and (ii) of Proposition 3.3.10.

Step 3: Next, let $(P_n)_{n\in\mathbb{N}}$ be a refining sequence of partitions from step 2, i.e., $P_n = \{t^n_0, \ldots, t^n_{k_n}\}$ satisfies (3.6.1) and (3.6.2).

Case 1: Let us first assume $\operatorname{Var}^b_a(\varphi) < \infty$. Let $M := \sup_{t\in I}(\overline{S}_t - \underline{S}_t)$. We claim that for all subdivisions $\lambda = \{s_1, \ldots, s_{k_n}\}$ of P_n, all refinements $P' = \{t'_0, \ldots, t'_m\}$ of P_n, and all subdivisions $\lambda' = \{s'_1, \ldots, s'_m\}$ of P', we have

$$|R(\varphi, P_n, \lambda) - R(\varphi, P', \lambda')| \le \eta_n M + \delta_n \operatorname{Var}^b_a(\varphi). \tag{3.6.3}$$

The key estimate to derive (3.6.3) is

$$\begin{aligned}
&\left|(\overline{S}_{s_i} - S_{s_i})\left(\varphi_{t_i^n} - \varphi_{t_{i-1}^n}\right)^+ - \sum_{k=1}^{n_i}\left(\overline{S}_{s'_{i_k}} - S_{s'_{i_k}}\right)\left(\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}}\right)^+\right| \\
&\le \left|(\overline{S}_{s_i} - S_{s_i})\left(\left(\varphi_{t_i^n} - \varphi_{t_{i-1}^n}\right)^+ - \sum_{k=1}^{n_i}\left(\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}}\right)^+\right)\right| \\
&\quad + \left|\sum_{k=1}^{n_i}\left(\left(\overline{S}_{s'_{i_k}} - S_{s'_{i_k}}\right) - \left(\overline{S}_{s_i} - S_{s_i}\right)\right)\left(\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}}\right)^+\right| \\
&\le M\left(\sum_{k=1}^{n_i}\left(\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}}\right)^+ - \left(\varphi_{t_i^n} - \varphi_{t_{i-1}^n}\right)^+\right) + \delta_n \sum_{k=1}^{n_i}\left(\varphi_{t'_{i_k}} - \varphi_{t'_{i_{k-1}}}\right)^+,
\end{aligned}$$

where $i \in \{1, \dots, k_n\}$ and $t'_{i_1}, \dots, t'_{i_{n_i}}$ denote the elements of P' with $t_{i-1}^n = t'_{i_1} < \dots < t'_{i_{n_i}} = t_i^n$.

Now, let $(\lambda_n)_{n\in\mathbb{N}}$ be arbitrary modified intermediate subdivisions of $(P_n)_{n\in\mathbb{N}}$. Then, as the sequence $(P_n)_{n\in\mathbb{N}}$ is refining, (3.6.3) yields

$$\sup_{m\ge n} |R(\varphi, P_m, \lambda_m) - R(\varphi, P_n, \lambda_n)| \le \eta_n M + \delta_n \mathrm{Var}_a^b(\varphi).$$

Thus, the sequence $(R(\varphi, P_n, \lambda_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}_+$ and $C := \lim_{n\to\infty} R(\varphi, P_n, \lambda_n) \in \mathbb{R}_+$ exists. It remains to show that C satisfies Definition 3.3.2(i). Therefore, let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ s.t. $\eta_n M + \delta_n \mathrm{Var}_a^b(\varphi) < \varepsilon/2$ and $|C - R(\varphi, P_n, \lambda_n)| < \varepsilon/2$. Together with (3.6.3), this implies that for all refinements P' of P_n and subdivisions λ' of P', we have

$$|C - R(\varphi, P', \lambda')| \le |C - R(\varphi, P_n, \lambda_n)| + |R(\varphi, P_n, \lambda_n) - R(\varphi, P', \lambda')| < \varepsilon.$$

Thus, C satisfies Definition 3.3.2(i).

Case 2: We now treat the case $\mathrm{Var}_a^b(\varphi) = \infty$. In this case, we will show that the cost term exists and $C(\varphi, I) = \infty$. Recall that we assumed $\delta := \inf_{t\in[a,b)}(\overline{S}_t - \underline{S}_t) > 0$. We define a sequence $(\sigma_k)_{k\ge 0}$ by $\sigma_0 = a$ and

$$\sigma_k := \begin{cases} \inf\{t \ge \sigma_{k-1} : S_t \le \underline{S}_t + \delta/3\} \wedge b, & k \text{ odd} \\ \inf\{t \ge \sigma_{k-1} : S_t \le \overline{S}_t - \delta/3\} \wedge b, & k \text{ even.} \end{cases}$$

As $\underline{S}$, S, and $\overline{S}$ are càdlàg, we have $\sigma_k = b$ for k large enough. Hence, let $K \in \mathbb{N}$ denote the smallest number s.t. $\sigma_K = b$. In addition, note that we also have $\sigma_0 \le \sigma_1 < \sigma_2 < \dots < \sigma_K = b$ and, per construction,

$$\inf_{t\in[\sigma_{2k}, \sigma_{2k+1})} S_t - \underline{S}_t > \delta/3, \quad \text{and} \quad \inf_{t\in[\sigma_{2k+1}, \sigma_{2(k+1)})} \overline{S}_t - S_t > \delta/3. \tag{3.6.4}$$

Recall that $\mathrm{Var}_a^b(\varphi) = \infty$ implies that $\sum_{i=1}^{k_n} |\varphi_{t_i^n} - \varphi_{t_{i-1}^n}| \to \infty$ as $n \to \infty$ by (3.6.2). Since $K < \infty$ and φ is bounded, this implies that for at least one $k \in \{0, 1, \ldots, K-1\}$, we have

$$\sum_{\substack{t_i^n, t_{i-1}^n \in P_n \\ t_i^n, t_{i-1}^n \in [\sigma_k, \sigma_{k+1}]}} |\varphi_{t_i^n} - \varphi_{t_{i-1}^n}| \to \infty, \quad n \to \infty,$$

which, again by the boundedness of φ, implies that

$$\begin{aligned} &\sum_{\substack{t_i^n, t_{i-1}^n \in P_n \\ t_i^n, t_{i-1}^n \in [\sigma_k, \sigma_{k+1}]}} (\varphi_{t_i^n} - \varphi_{t_{i-1}^n})^+ \to \infty, \ n \to \infty \\ \text{and} \quad &\sum_{\substack{t_i^n, t_{i-1}^n \in P_n \\ t_i^n, t_{i-1}^n \in [\sigma_k, \sigma_{k+1}]}} (\varphi_{t_i^n} - \varphi_{t_{i-1}^n})^- \to \infty, \ n \to \infty. \end{aligned} \tag{3.6.5}$$

By (3.6.4), this implies that $R(\varphi, P_n, \lambda_n) \to \infty$ as $n \to \infty$ for arbitrary subdivisions λ_n of P_n. Since the sums in (3.6.5) get even bigger if P_n are replaced by refining partitions G_n, the cost term $C(\varphi, I)$ exists and is ∞.

This finishes the proof of Propositions 3.3.3 and 3.3.10. Indeed, in step 2 above, we showed that there exists a sequence of partitions satisfying the assumptions (i) and (ii) of Proposition 3.3.10. Subsequently, in step 3 we showed that for every refining sequence of partitions with these properties the corresponding Riemann-Stieltjes sums converge and their limits satisfy Definition 3.3.2. Thus, by the uniqueness shown in step 1, their limits coincide and we are done. □

We now turn to the proof of Lemma 3.3.13. This will rely on the following concept and result of Doob [28].

Definition 3.6.1. Let φ be a stochastic process. A sequence $(T_n)_{n\in\mathbb{N}}$ of predictable stopping times is called a *predictable separability set for φ* if for each $\omega \in \Omega$ the set $\{T_n(\omega) : n \in \mathbb{N}\}$ contains 0 and is dense in $[0, T]$ and

$$\{(t, \varphi_t(\omega)) : t \in [0, T]\} = \overline{\{(T_n(\omega), \varphi_{T_n(\omega)}(\omega)) : n \in \mathbb{N}\}}, \tag{3.6.6}$$

i.e., the graph of the sample function $t \mapsto \varphi_t(\omega)$ is the closure of the graph restricted to the set $\{T_n(\omega) : n \in \mathbb{N}\}$. A stochastic process φ having a predictable separability set is called *predictably separable.*

Theorem 3.6.2 (Doob [28], Theorem 5.2)**.** *A predictable process coincides with some predictably separable predictable process up to evanescence.*

Proof of Lemma 3.3.13. By Theorem 3.6.2, we have to show that for a predictably separable predictable process φ, the process $C(\varphi, [\sigma \wedge \cdot, \tau \wedge \cdot])$ is predictable.

Let $\{T_n : n \in \mathbb{N}\}$ denote the predictable separability set for φ. By (3.6.6), we can find a sequence of finite sequences of (not necessarily predictable) stopping times

$\sigma = T_0^n \leq T_1^n \leq \cdots \leq T_{m_n}^n = \tau$ s.t.

$$\mathrm{Var}_{\sigma\wedge t}^{\tau\wedge t}(\varphi) = \lim_{n\to\infty} \sum_{i=1}^{m_n} |\varphi_{T_i^n \wedge t} - \varphi_{T_{i-1}^n \wedge t}|, \quad \text{pointwise}, \quad t \in [0,T].$$

Next, we define for each $n \in \mathbb{N}$ and $i \in \{1, \ldots, m_n\}$ a sequence $(V_l^{n,i})_{l\in\mathbb{N}}$ of stopping times by $V_0^{n,i} = T_{i-1}^n$ and recursively

$$V_l^{n,i} := \inf\{t > V_{l-1}^{n,i} : |\overline{S}_t - S_t - (\overline{S}_{V_{l-1}^{n,i}} - S_{V_{l-1}^{n,i}})| > \frac{1}{2n}$$
$$\text{or } |S_t - \underline{S}_t - (S_{V_{l-1}^{n,i}} - \underline{S}_{V_{l-1}^{n,i}})| > \frac{1}{2n}\} \wedge T_i^n.$$

This leads to the sequence of random partitions $\overline{P}_n := \bigcup_{k\leq n} \bigcup_{i=1,\ldots,m_k} \bigcup_{l\in\mathbb{N}_0} \{V_l^{i,k}\}$, $n \in \mathbb{N}$, which is for each ω refining. Note that for ω and n fixed, P_n is finite. Rearranging the resulting stopping times in increasing order yields a refining sequence of increasing sequences of stopping times $(\nu_k^n)_{k\in\mathbb{N}}$, $n \in \mathbb{N}$, s.t. $\#\{k : \nu_k^n(\omega) < \infty\} < \infty$ for all $n \in \mathbb{N}$, $\mathrm{Var}_{\sigma\wedge t}^{\tau\wedge t}(\varphi) = \lim_{n\to\infty} \sum_{k=0}^{\infty} |\varphi_{\nu_k^n \wedge t} - \varphi_{\nu_{k-1}^n \wedge t}|$ for all $t \in [0,T]$, and $\max(\mathrm{osc}(\overline{S} - S, [\nu_k^n, \nu_{k+1}^n)), \mathrm{osc}(S - \underline{S}, [\nu_k^n, \nu_{k+1}^n))) \leq 1/n$ for all $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$. In particular, this means that for each $\omega \in \{\sigma < \tau\}$ and $t \in [0,T]$ the sequence of partitions $(P_n(\omega))_{n\in\mathbb{N}}$ defined by $P_n(\omega) := \{\nu_k^n(\omega) \wedge t : k \in \mathbb{N}\}$ satisfies the assumptions of Proposition 3.3.10. Hence, Proposition 3.3.10 together with $C(\varphi, [\sigma \wedge \cdot, \tau \wedge \cdot]) = 0$ on $\{\sigma = \tau\}$ implies that the sequence of predictable processes

$$\sum_{k=1}^{\infty} (\overline{S}_{\nu_{k-1}^n} - S_{\nu_{k-1}^n})(\varphi_{\nu_k^n \wedge \cdot} - \varphi_{\nu_{k-1}^n \wedge \cdot})^+ + \sum_{k=1}^{\infty} (S_{\nu_{k-1}^n} - \underline{S}_{\nu_{k-1}^n})(\varphi_{\nu_k^n \wedge \cdot} - \varphi_{\nu_{k-1}^n \wedge \cdot})^-, \quad n \in \mathbb{N}$$

converges pointwise to $C(\varphi, [\sigma \wedge \cdot, \tau \wedge \cdot])$, which yields the assertion. □

Proof of Proposition 3.3.17. In the following, we can and do assume with no loss of generality that σ and τ are $[0,T]$-valued stopping times. In addition, by Proposition 3.3.3, we have $\mathrm{Var}_\sigma^\tau(\varphi) < \infty$ a.s. and thus w.l.o.g. also for all paths. This implies that the paths of φ are làglàd on $[\![\sigma, \tau]\!]$.

Step 1. We start by constructing the sequence $(\varphi^n)_{n\in\mathbb{N}}$. Therefore, we define

$$T_0^n := \sigma, \quad T_k^n := \inf\{t \in (T_{k-1}^n, \tau] : |\varphi_t - \varphi_{T_{k-1}^n +}| \geq 1/n\}, \quad k \in \mathbb{N}, \tag{3.6.7}$$

which are obviously stopping times. In addition, we have $T_{k-1}^n < T_k^n$ on $\{T_{k-1}^n < \infty\}$ and $\#\{k : T_k^n(\omega) \leq \tau\} < \infty$ for all $\omega \in \Omega$ as $\mathrm{Var}_\sigma^\tau(\varphi) < \infty$. We have to distinguish between a portfolio adjustment at T_k^n and at T_k^n+. For this, we define further stopping times:

$$\pi_0^n := \sigma, \quad \pi_k^n := (T_k^n)_{\{|\varphi_{T_k^n} - \varphi_{T_{k-1}^n +}| \geq 1/n\}}, \quad k \in \mathbb{N}$$

and note that π_k^n is a predictable stopping time for all $k \in \mathbb{N}$. Indeed, for $k \geq 1$ we have

$$[\![\pi_k^n]\!] = [\![0, T_k^n]\!] \cap \{(\omega, t) : Y_t(\omega) \geq 1/n\} \in \mathcal{P}$$

since the process $Y_t := |\varphi_t - \varphi_{T_{k-1}^n+}| \mathbb{1}_{]\!]T_{k-1}^n, \tau]\!]}$ is a predictable. Hence, we may define $(\varphi^n)_{n \in \mathbb{N}}$ by

$$\varphi^n := \sum_{k=0}^{\infty} \left(\varphi_{\pi_k^n} \mathbb{1}_{[\![\pi_k^n]\!]} + \varphi_{T_k^n+} \mathbb{1}_{]\!]T_k^n, T_{k+1}^n]\!] \setminus [\![\pi_{k+1}^n]\!]} \right)$$

which satisfies $\varphi_\sigma^n = \varphi_\sigma$ and $\varphi^n \mathbb{1}_{]\!]\sigma, \tau]\!]}$ is predictable and, consequently, almost simple. In addition, the definition ensures $|\varphi - \varphi^n| \leq 1/n$ on $[\![\sigma, \tau]\!]$.

Step 2: Let us show that $\sup_{t \in [\sigma, \tau]} |\mathrm{Var}_\sigma^t(\varphi) - \mathrm{Var}_\sigma^t(\varphi^n)| \to 0$ pointwise. Let $\omega \in \Omega$ and $\varepsilon > 0$ be fixed. We take a partition $P = \{t_0, \ldots, t_m\}$ s.t. $\mathrm{Var}_\sigma^\tau(\varphi(\omega)) \leq \sum_{i=1}^m |\varphi_{t_i}(\omega) - \varphi_{t_{i-1}}(\omega)| + \varepsilon$. This yields

$$\mathrm{Var}_\sigma^t(\varphi(\omega)) \leq \sum_{i=1}^m |\varphi_{t_i \wedge t}(\omega) - \varphi_{t_{i-1} \wedge t}(\omega)| + \varepsilon, \quad \forall t \in [\sigma(\omega), \tau(\omega)]. \tag{3.6.8}$$

Now, recall from Step 1 that $\varphi^n(\omega) \to \varphi(\omega)$ uniformly on $[\sigma(\omega), \tau(\omega)]$. Thus, we may choose $N \in \mathbb{N}$ large enough s.t. for all $n \geq N$ we have $|\varphi_t(\omega) - \varphi_t^n(\omega)| \leq \varepsilon/(2m)$ for all $t \in [\sigma(\omega), \tau(\omega)]$. Therefore, we get

$$\begin{aligned} \mathrm{Var}_\sigma^t(\varphi(\omega)) - \mathrm{Var}_\sigma^t(\varphi^n(\omega)) &\leq \sum_{i=1}^m |\varphi_{t_i \wedge t}(\omega) - \varphi_{t_{i-1} \wedge t}(\omega)| + \varepsilon - \mathrm{Var}_\sigma^t(\varphi^n(\omega)) \\ &\leq \sum_{i=1}^m |\varphi_{t_i \wedge t}^n(\omega) - \varphi_{t_{i-1} \wedge t}^n(\omega)| + 2\varepsilon - \mathrm{Var}_\sigma^t(\varphi^n(\omega)) \leq 2\varepsilon \end{aligned}$$

for all $t \in [\sigma(\omega), \tau(\omega)]$. Hence, we have proven the claim as we have $\mathrm{Var}_\sigma^t(\varphi(\omega)) \geq \mathrm{Var}_\sigma^t(\varphi^n(\omega))$ by construction.

Step 3: We now show that (3.3.10) holds. We again argue path-by-path, i.e., $\omega \in \Omega$ is fixed without explicitly mentioning it. Therefore, note that the jumps of the cost term on $[\sigma, \tau]$ are given by

$$\begin{aligned} \Delta C_t(\varphi) &= \lim_{s \uparrow t} C(\varphi, [s, t]) = (\overline{S}_{t-} - S_{t-})(\Delta\varphi_t)^+ + (S_{t-} - \underline{S}_{t-})(\Delta\varphi_t)^-, \quad t \in (\sigma, \tau], \\ \Delta^+ C_t(\varphi) &= \lim_{s \downarrow t} C(\varphi, [t, s]) = (\overline{S}_t - S_t)(\Delta^+\varphi_t)^+ + (S_t - \underline{S}_t)(\Delta^+\varphi_t)^-, \quad t \in [\sigma, \tau). \end{aligned}$$

In the following, given $k \in \mathbb{N}$, we use the notation $C(\varphi, (T_{k-1}^n, T_k^n]) := C(\varphi, [T_{k-1}^n, T_k^n]) - \Delta^+ C_{T_{k-1}^n}(\varphi)$ and $C(\varphi, (T_{k-1}^n, T_k^n)) := C(\varphi, (T_{k-1}^n, T_k^n]) - \Delta C_{T_k^n}(\varphi)$, where it is tacitly assumed that $T_k^n \leq \tau$. In particular, this means that for φ^n, we have $C(\varphi^n, (T_{k-1}^n, T_k^n]) = (\overline{S}_{T_k^n-} - S_{T_k^n-})(\varphi_{T_k^n}^n - \varphi_{T_k^n-}^n)^+ + (S_{T_k^n-} - \underline{S}_{T_k^n-})(\varphi_{T_k^n}^n - \varphi_{T_k^n-}^n)^-$ as $C(\varphi^n, (T_{k-1}^n, T_k^n)) = 0$ according to Proposition 3.3.16. We now want to get an estimate on

$$|C(\varphi, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi) - (C(\varphi^n, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi^n))| \tag{3.6.9}$$

(this means that we move forward from $T^n_{k-1}+$ to T^n_k+ and tacitly assume $T^n_k < \tau$).

Step 3.1: We start by establishing a strong bound on the difference (3.6.9), which only holds if the prices do not vary too much between T^n_{k-1} and T^n_k. To formalize this, we take $\delta > 0$, which will be specified later, and define $(\rho_m)_{m\geq 0}$ by $\rho_0 := \sigma$ and

$$\rho_m := \inf\{t \in (\rho_{m-1}, \tau] : |\overline{S}_t - S_t - (\overline{S}_{\rho_{m-1}} - S_{\rho_{m-1}})| > \delta \text{ or } |S_t - \underline{S}_t - (S_{\rho_{m-1}} - \underline{S}_{\rho_{m-1}})| > \delta\}.$$

We now claim that on $\{\rho_{m-1} \leq T^n_{k-1} < T^n_k < \rho_m\}$ for some $m \geq 1$, we have

$$\begin{aligned}
&|C(\varphi, (T^n_{k-1}, T^n_k]) + \Delta^+ C_{T^n_k}(\varphi) - (C(\varphi^n, (T^n_{k-1}, T^n_k]) + \Delta^+ C_{T^n_k}(\varphi^n))| \\
&\leq \delta \mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi) + \sup_{t\in[0,T]} (\overline{S}_t - \underline{S}_t) \left(\mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi) - \mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi^n) \right), \quad k \geq 1.
\end{aligned} \tag{3.6.10}$$

In order to prove this, we distinguish between two cases.

Case 1: We start by considering the event $\{T^n_k = \pi^n_k\}$, i.e., the infimum in (3.6.7) is attained and $\Delta^+ C_{T^n_k}(\varphi) = \Delta^+ C_{T^n_k}(\varphi^n)$. First, we assume that $\varphi_{T^n_k} - \varphi_{T^n_{k-1}+} \geq 0$, i.e., the strategy φ buys (after netting buying and selling)

$$a := \varphi_{T^n_k} - \varphi_{T^n_{k-1}+} = \mathrm{Var}^{T^n_k}_{T^n_{k-1}+}(\varphi^n) \geq 0$$

stocks on $(T^n_{k-1}, T^n_k]$. Now observe that φ^n buys a stocks at a cost of $\overline{S}_{T^n_k-} - S_{T^n_k-}$ and φ buys at least a stocks at different cost, which differs from $\overline{S}_{T^n_k-} - S_{T^n_k-}$ by at most δ. In addition, the continuous strategy purchases $\varphi^\uparrow_{T^n_k} - \varphi^\uparrow_{T^n_{k-1}+} - a$ additional stocks and sells $\varphi^\downarrow_{T^n_k} - \varphi^\downarrow_{T^n_{k-1}+}$ stocks on the same interval. But the cost of those trades can be estimated above by $\sup_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)$. Putting these arguments together, we get

$$\begin{aligned}
&|C(\varphi, (T^n_{k-1}, T^n_k]) + \Delta^+ C_{T^n_k}(\varphi) - (C(\varphi^n, (T^n_{k-1}, T^n_k]) + \Delta^+ C_{T^n_k}(\varphi^n))| \\
&\leq \delta a + \sup_{t\in[0,T]} (\overline{S}_t - \underline{S}_t)(\varphi^\uparrow_{T^n_k} - \varphi^\uparrow_{T^n_{k-1}+} - a + \varphi^\downarrow_{T^n_k} - \varphi^\downarrow_{T^n_{k-1}+}) \\
&= \delta \mathrm{Var}^{T^n_k}_{T^n_{k-1}+}(\varphi^n) + \sup_{t\in[0,T]} (\overline{S}_t - \underline{S}_t) \left(\mathrm{Var}^{T^n_k}_{T^n_{k-1}+}(\varphi) - \mathrm{Var}^{T^n_k}_{T^n_{k-1}+}(\varphi^n) \right) \\
&\leq \delta \mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi) + \sup_{t\in[0,T]} (\overline{S}_t - \underline{S}_t) \left(\mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi) - \mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi^n) \right),
\end{aligned}$$

where we used

$$\mathrm{Var}^{T^n_k}_{T^n_{k-1}+}(\varphi^n) \leq \mathrm{Var}^{T^n_k}_{T^n_{k-1}+}(\varphi) \leq \mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi)$$

and $\Delta^+ C_{T^n_k}(\varphi) = \Delta^+ C_{T^n_k}(\varphi^n)$ on $\{T^n_k = \pi^n_k\}$. For $\varphi_{T^n_k} - \varphi_{T^n_{k-1}+} < 0$, the argument is analogue.

Case 2: We still have to prove the claim on $\{T^n_k \neq \pi^n_k\}$. Here, we have $\Delta C_{T^n_k}(\varphi^n) = 0$ and, therefore, the argument is similar to the previous case but this time with $a := \varphi_{T^n_k+} - \varphi_{T^n_{k-1}+}$. Thus, we skip the details.

Step 3.2: We still need a bound on (3.6.9) if the costs vary by more than δ between T^n_{k-1} and T^n_k. Fortunately, a weaker bound will be sufficient here. We now claim that, in general, we have

$$\begin{aligned} &|C(\varphi,(T^n_{k-1},T^n_k]) + \Delta^+ C_{T^n_k}(\varphi) - (C(\varphi^n,(T^n_{k-1},T^n_k]) + \Delta^+ C_{T^n_k}(\varphi^n))| \\ &\leq \sup_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)\left[\frac{2}{n} + (\mathrm{Var}^{T^n_k+}_{T^n_{k-1+}}(\varphi) - \mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi^n))\right]. \end{aligned} \tag{3.6.11}$$

We distinguish between the same cases as above.

Case 1: We first consider the event $\{T^n_k = \pi^n_k\}$. Recall that in this case we have $\Delta^+ C_{T^n_k}(\varphi) = \Delta^+ C_{T^n_k}(\varphi^n)$. In addition, let us assume that $\varphi_{T^n_k} - \varphi_{T^n_{k-1}+} \geq 0$. In this case, we have $\varphi_{T^n_k} - \varphi_{T^n_{k-1}+} \geq 1/n$ and $\varphi_{T^n_k-} - \varphi_{T^n_{k-1}+} \leq 1/n$ by the Definition of T^n_k. This implies

$$\varphi_{T^n_k} - \varphi_{T^n_k-} = \varphi_{T^n_k} - \varphi_{T^n_{k-1}+} - \left(\varphi_{T^n_k-} - \varphi_{T^n_{k-1}+}\right) \geq 0,$$

i.e., both strategies buy at T^n_k, but possibly different amounts. Thus, we have $\Delta C_{T^n_k}(\varphi) = (\overline{S}_{T^n_k-} - S_{T^n_k-})(\varphi_{T^n_k} - \varphi_{T^n_k-})$ and can write

$$\begin{aligned} &|C(\varphi,(T^n_{k-1},T^n_k]) + \Delta^+ C_{T^n_k}(\varphi) - (C(\varphi^n,(T^n_{k-1},T^n_k]) + \Delta^+ C_{T^n_k}(\varphi^n))| \\ &= |C(\varphi,(T^n_{k-1},T^n_k)) + (\overline{S}_{T^n_k-} - S_{T^n_k-})(\varphi_{T^n_k} - \varphi_{T^n_k-}) - (\overline{S}_{T^n_k-} - S_{T^n_k-})(\varphi_{T^n_k} - \varphi_{T^n_{k-1+}})| \\ &= |C(\varphi,(T^n_{k-1},T^n_k)) - (\overline{S}_{T^n_k-} - S_{T^n_k-})(\varphi_{T^n_k-} - \varphi_{T^n_{k-1}+})|. \end{aligned}$$

Since the costs per share are bounded by $\sup_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)$, this yields

$$\begin{aligned} &|C(\varphi,(T^n_{k-1},T^n_k)) - (\overline{S}_{T^n_k-} - S_{T^n_k-})(\varphi_{T^n_k-} - \varphi_{T^n_{k-1}+})| \\ &\leq \sup_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)\left[\mathrm{Var}^{T^n_k-}_{T^n_{k-1+}}(\varphi) + |\varphi_{T^n_k-} - \varphi_{T^n_{k-1}+}|\right] \\ &\leq \sup_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)\left[\frac{2}{n} + \mathrm{Var}^{T^n_k+}_{T^n_{k-1+}}(\varphi) - \mathrm{Var}^{T^n_k+}_{T^n_{k-1}+}(\varphi^n)\right], \end{aligned}$$

where we use $|\varphi_{T^n_k-} - \varphi_{T^n_{k-1}+}| \leq 1/n$ per construction of T^n_k, $\mathrm{Var}^{T^n_k-}_{T^n_{k-1+}}(\varphi) - |\varphi_{T^n_k-} - \varphi_{T^n_{k-1}+}| \leq \mathrm{Var}^{T^n_k}_{T^n_{k-1+}}(\varphi) - \mathrm{Var}^{T^n_k}_{T^n_{k-1+}}(\varphi^n)$, and $\Delta^+ C_{T^n_k}(\varphi) = \Delta^+ C_{T^n_k}(\varphi)$ on $\{T^n_k = \pi^n_k\}$.

The case $\varphi_{T^n_k} - \varphi_{T^n_{k-1}+} \leq 0$ is analogous.

Case 2: We still need to consider the event $\{T^n_k \neq \pi^n_k\}$, i.e., $\Delta C_{T^n_k}(\varphi^n) = 0$. However, as this is analogous to Case 1, we leave it to the reader.

In addition, note that on $\{T^n_{k-1} \leq t < T^n_k\}$, we have $\mathrm{Var}^t_{T^n_{k-1}+}(\varphi^n) = 0$ and, thus, the trivial estimate

$$\begin{aligned} &|C(\varphi,(T^n_{k-1},t]) - C(\varphi^n,(T^n_{k-1},t])| \\ &\leq \sup_{t\in[0,T]}(\overline{S}_t - \underline{S}_t)(\mathrm{Var}^t_{T^n_{k-1}+}(\varphi) - \mathrm{Var}^t_{T^n_{k-1}+}(\varphi^n)). \end{aligned} \tag{3.6.12}$$

Step 4: We can now finish the proof by putting the different estimates together. Therefore, let $a(\delta) := \#\{m : \rho_m \leq \tau\}$ and note that $a(\delta) < \infty$ (recall that $\omega \in \Omega$ is fixed). Next, note that we have $\Delta^+ C_\sigma(\varphi) = \Delta^+ C_\sigma(\varphi^n)$ by construction of φ^n. For $t \in [\sigma, \tau]$ let $K_n := \#\{k : T_k^n \leq t\}$. We get

$$\begin{aligned}
&|C(\varphi, [\sigma, t]) - C(\varphi^n, [\sigma, t])| \\
&\leq \sum_{k=1}^{K_n} |C(\varphi, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi) \mathbb{1}_{\{T_k^n < t\}} - (C(\varphi^n, (T_{k-1}^n, T_k^n]) + \Delta^+ C_{T_k^n}(\varphi^n) \mathbb{1}_{\{T_k^n < t\}})| \\
&\quad + |C(\varphi, (T_{K_n}^n, t]) - C(\varphi^n, (T_{K_n}^n, t])| \qquad (3.6.13)
\end{aligned}$$

On $\{T_{K_n}^n < t\}$ we apply the estimate (3.6.11) to all pairs T_{k-1}^n, T_k^n with $k = 1, \dots, K_n$ s.t. there is at least one $m = 1, \dots, a(\delta)$ with $T_{k-1}^n < p_m \leq T_k^n$, the estimate (3.6.12) to the last interval $(T_{K_n}^n, t]$ and for all other pairs we use the stronger estimate (3.6.10). On $\{T_{K_n}^n = t\}$ we apply the same estimates to all pairs T_{k-1}^n, T_k^n with $k = 1, \dots, K_n - 1$. In addition, on $\{T_{K_n}^n = \pi_{K_n}^n = t\}$ the arguments in Step 3.1, Case 1 resp. Step 3.2, Case 1 show that $|C(\varphi, (T_{K_n-1}^n, T_{K_n}^n]) - C(\varphi^n, (T_{K_n-1}^n, T_{K_n}^n])|$ is bounded from above by the RHS of (3.6.10) if there is no $m \in \{1, \dots, a(\delta)\}$ with $T_{K_n-1}^n < p_m \leq T_{K_n}^n$ or by the RHS of (3.6.11) if there is. Finally, on $\{T_{K_n}^n = t, \pi_{K_n}^n = \infty\}$, we have $\mathrm{Var}^t_{T_{K_n-1}^n +}(\varphi^n) = 0$ and thus $|C(\varphi, (T_{K_n-1}^n, T_{K_n}^n]) - C(\varphi^n, (T_{K_n-1}^n, T_{K_n}^n])|$ is bounded from above by the RHS of (3.6.12). Plugging all this into (3.6.13), we get

$$\begin{aligned}
&|C(\varphi, [\sigma, t]) - C(\varphi^n, [\sigma, t])| \\
&\leq \delta \mathrm{Var}^t_\sigma(\varphi) + \sup_{t \in [0,T]} (\overline{S}_t - \underline{S}_t) \left(\mathrm{Var}^t_\sigma(\varphi) - \mathrm{Var}^t_\sigma(\varphi^n) + \frac{2a(\delta)}{n} \right) \\
&\leq \delta \mathrm{Var}^\tau_\sigma(\varphi) + \sup_{t \in [0,T]} (\overline{S}_t - \underline{S}_t) \left(\sup_{t \in [\sigma, \tau]} (\mathrm{Var}^t_\sigma(\varphi) - \mathrm{Var}^t_\sigma(\varphi^n)) + \frac{2a(\delta)}{n} \right) \qquad (3.6.14)
\end{aligned}$$

for all $t \in [\sigma, \tau]$. Given an $\varepsilon > 0$, we first choose $\delta < \varepsilon/(2\mathrm{Var}^\tau_\sigma(\varphi))$ and, subsequently, applying Step 2 together with the fact that $a(\delta) < \infty$ and $\sup_{t \in [0,T]}(\overline{S}_t - \underline{S}_t) < \infty$ (for fixed $\omega \in \Omega$). We can choose $N \in \mathbb{N}$ s.t. for $n \geq N$ the second term in (3.6.14) is smaller than $\varepsilon/2$. At last this yields $\sup_{t \in [\sigma, \tau]} |C(\varphi, [\sigma, t]) - C(\varphi^n, [\sigma, t])| < \varepsilon$ for $n \geq N$. Thus, we have established the assertion. □

Chapter 4

Deutsche Zusammenfassung

Eine *Arbitragemöglichkeit* ist die Chance auf einen risikolosen Gewinn ohne Startkapital. Das Grundprinzip der *Arbitragefreiheit* besagt, dass es in einem finanzmathematischem Marktmodell keine Arbitragemöglichkeiten geben sollte. In friktionslosen zeitdiskreten Finanzmarktmodellen ist die Arbitragefreiheit des Marktmodells äquivalent zur Existenz eines äquivalenten Martingalmaßes. Diese Aussage ist als *Fundamentalsatz der Preistheorie* bekannt (siehe, z.B., [21, 41, 53, 83]).

Unter proportionalen Transaktionskosten spielen *(strikt) konsistente Preissysteme* eine ähnliche Rolle wie äquivalente Martingalemaße in der friktionslosen Theorie. Jedoch ist die Arbitragefreiheit des Marktmodells im Allgemeinen nicht mehr äquivalent zur Existenz eines konsistenten Preissystems. Im Gegensatz zu friktionslosen Modellen, in denen die Arbitragefreiheit bereits die Abgeschlossenheit der Menge der superreplizierbaren Claims bzgl. der Konvergenz in Wahrscheinlichkeit impliziert, existieren arbitragefreie Modelle mit Transaktionskosten in denen eine Arbitrage beliebig gut approximiert werden kann. Im ersten Teil der Arbeit, siehe Kapitel 2, führen wir die *prospektive strict no-arbitrage*-Bedingung ein. *Prospective strict no-arbitrage* ist schwächer als *robust no-arbitrage*, aber impliziert bereits die Abgeschlossenheit der Menge der superreplizierbaren Claims und schließt somit approximative Arbitragemöglichkeiten aus. Dementsprechend impliziert *prospective strict no-arbitrage* die Existenz eines konsistenten Preissystems. Obwohl die Abgeschlossenheit für die theoretisch bedeutsamen *Superhedging*-Resultate und für Anwendungen in der Portfoliooptimierung zentral ist, ist sie nicht notwendig für die Existenz eines konsistenten Preissystems. Jedoch stellt sich heraus, dass eine abgeschwächte Form der *prospective strict no-arbitrage*-Bedingung äquivalent zur Existenz eines konsistenten Preissystems ist. Anders als bei *strikt* konsistenten Preissystemen, können die Preissysteme hierbei Werte auf dem Rand des Bid-Ask-Spreads annehmen.

Die Hauptresultate des ersten Teils der Arbeit sind im Folgenden zusammengefasst.

Prospective strict no-arbitrage

In *Kapitel 2* arbeiten wir mit dem zeitdiskreten Modell mit proportionalen Transaktionskosten aus Schachermayer [85]. Es gibt $d \in \mathbb{N}$ Wertpapiere. Wir nennen eine $d \times d$-Matrix $\Pi = (\pi^{ij})_{1 \le i,j \le d}$ eine *Bid-Ask Matrix*, falls

(i) $0 < \pi^{ij} < \infty$, für $1 \le i, j \le d$,

(ii) $\pi^{ii} = 1$, für $1 \le i \le d$,

(iii) $\pi^{ij} \le \pi^{ik}\pi^{kj}$, für $1 \le i, j, k \le d$.

Die Handelsbedingungen der d Wertpapiere werden durch einen adaptierten $d \times d$–Matrix-wertigen Prozess $(\Pi_t)_{t=0}^T$, wobei die Matrix $\Pi_t(\omega)$ für jedes $\omega \in \Omega$ und $t \in \{0, \dots, T\}$ eine Bid-Ask Matrix ist, modelliert. Der Prozess $(\Pi_t)_{t=0}^T$ wird *Bid-Ask Prozess* genannt. Die zufällige Matrix $\Pi_t = (\pi_t^{ij})_{1 \le i,j \le d}$ spezifiziert die Wechselkurse zwischen den einzelnen Wertpapieren zum Zeitpunkt t. Der Eintrag π_t^{ij} gibt die Anzahl der Einheiten von Wertpapier i, welche ein Investor zum Erwerb einer Einheit des Wertpapiers j zum Zeitpunkt t benötigt, an. Die Menge, der zum Zeitpunkt t ohne Anfangsausstattung erwerbbaren Portfolios wird dementsprechend durch den konvexen Kegel

$$\left\{ \sum_{1 \le i,j \le d} \lambda^{ij}(e^j - \pi_t^{ij} e^i) - r : (\lambda^{ij})_{1 \le i,j \le d} \in L^0(\mathbb{R}_+^{d \times d}, \mathcal{F}_t),\ r \in L^0(\mathbb{R}_+^d, \mathcal{F}_t) \right\}, \tag{1}$$

modelliert, wobei e^i den i-ten Einheitsvektor des $\mathbb{R}^d$ bezeichnet. Jedes Portfolio ist eine $\mathcal{F}_t$-messbare $\mathbb{R}^d$-wertige Zufallsvariable, welche sich als Differenz des Ergebnisses eines Handelsauftrages $\lambda = (\lambda^{ij})_{1 \le i,j \le d} \in L^0(\mathbb{R}_+^{d \times d}, \mathcal{F}_t)$ und einer nicht-negativen Zufallsvariable $r \in L^0(\mathbb{R}_+^d, \mathcal{F}_t)$ ergibt. Hierbei beschreibt λ^{ij} die Anzahl, der bestellten Einheiten von Wertpapier j im Austausch mit Wertpapier i und r die Möglichkeit eines Investors eine nicht-negative Menge jedes Wertpapiers „weg zuschmeißen".

Sei $K(\Pi_t(\omega)) := \text{cone}(\{\pi_t^{ij}(\omega)e^i - e^j\}_{1 \le i,j \le d}, \{e^i\}_{1 \le i \le d})$ für $\omega \in \Omega$ und $t \in \{0, \dots, T\}$. Im Folgenden schreiben wir kurz $K_t = K(\Pi_t)$. Die Menge in (1) stimmt mit der Menge $L^0(-K_t, \mathcal{F}_t)$ der $\mathcal{F}_t$-messbaren Selektoren des zufälligen polyhedrischen Kegels $-K_t$ überein (siehe Lemma 2.3.1). Dementsprechend bezeichnen wir $L^0(-K_t, \mathcal{F}_t)$ als die *Menge der zum Zeitpunkt t ohne Anfangsausstattung erwerbbaren Portfolios.*

Definition 1. Ein $\mathbb{R}^d$-wertiger adaptierter Prozess $\vartheta = (\vartheta_t)_{t=0}^T$ mit

$$\vartheta_t - \vartheta_{t-1} \in L^0(-K_t, \mathcal{F}_t) \quad \text{für alle } t = 0, \dots, T, \tag{2}$$

wobei $\vartheta_{-1} := 0$, heißt *selbstfinanzierender Protfolioprozess* für den Bid-Ask Prozess $(\Pi_t)_{t=0}^T$. Für jedes Paar (s, t) mit $s, t \in \{0, \dots, T\}$ und $s \le t$ definieren wir den konvexen Kegel $\mathcal{A}_s^t$, der *ohne Anfangsausstattung zwischen s und t superreplizierbaren*

Claims, durch

$$\mathcal{A}_s^t := \sum_{k=s}^{t} L^0(-K_k, \mathcal{F}_k).$$

Für einen weiteren Bid-Ask Prozess $(\widetilde{\Pi}_t)_{t=0}^T$ bezeichnen wir die entsprechende Menge mit $\widetilde{\mathcal{A}}_s^t$, wobei $\widetilde{K}_t = K(\widetilde{\Pi}_t)$ für alle $t = 0, \ldots, T$.

Definition 2. Der Bid-Ask Prozess $(\Pi_t)_{t=0}^T$ erfüllt die *prospective strict no-arbitrage Bedingung (NA^{ps})*, falls

$$\mathcal{A}_0^t \cap (-\mathcal{A}_t^T) \subseteq \mathcal{A}_t^T \quad \text{für alle } t = 0, \ldots, T$$

gilt.

Bemerkung 3. *Die (NA^{ps}) Eigenschaft kann wie folgt interpretiert werden: Jedes Portfolio $v \in \mathcal{A}_0^t$, welches bis zum Zeitpunkt t aufgebaut wurde und welches in t bzw. in den darauf folgenden Perioden liquidiert werden kann, d.h. $-v \in \mathcal{A}_t^T$, muss auch durch Handel nur zwischen t und T erreichbar seien, d.h. $v \in \mathcal{A}_t^T$. (NA^{ps}) ist eine Variante der* strict no-arbitrage (NA^s) *Bedingung aus Kabanov et al. [51], welche besagt, dass jedes Portfolio $v \in \mathcal{A}_0^t$, welches zum Zeitpunkt t liquidiert werden kann, d.h. $-v \in \mathcal{A}_t^t$, ebenfalls zum Zeitpunkt t ohne Startausstattung aufgebaut werden kann, d.h. $v \in \mathcal{A}_t^t$. Der einzige Unterschied zwischen den Bedingungen ist, dass (NA^{ps}) nicht zwischen Transaktionen zum Zeitpunkt t und Transaktionen von denen bereits zum Zeitpunkt t bekannt ist, dass sie sicher in Zukunft realisiert werden können, unterscheidet.*

In anderen Worten überprüfen wir für jedes t die bis zum Zeitpunkt t erzielbaren Portfolios. Entweder ist das erzielte Portfolio nicht vorteilhaft, da dasselbe Portfolio auch durch Handel ab t erzielt werden kann, oder das Portfolio ist risikobehaftet, d.h. es kann in den folgenden Perioden nicht sicher liquidiert werden.

Wir können nun bereits das erste Hauptresultat der Arbeit präsentieren.

Theorem 4. *Erfüllt der Bid-Ask Prozess $(\Pi_t)_{t=0}^T$ die* prospective strict no-arbitrage *Bedingung (NA^{ps}), so ist der konvexe Kegel $\mathcal{A}_0^T$ abgeschlossen bzgl. der Konvergenz in Wahrscheinlichkeit.*

Für eine Bid-Ask Matrix Π ist der duale Kegel $K^\star$ von $K = K(\Pi)$ durch $K^\star := \{w \in \mathbb{R}^d : \langle v, w \rangle \geq 0 \text{ für alle } v \in K\}$ definiert. Dementsprechend induziert der Bid-Ask Prozess $(\Pi_t)_{t=0}^T$ einen zugehörigen mengenwertigen Prozess $(K_t^\star)_{t=0}^T$ von dualen Kegeln. Wir führen nun *konsistente Preissysteme*, welche das Analogon zu Dichten äquivalenter Martingalmaße aus der friktionslosen Theorie sind, ein. Für eine detaillierte ökonomische Interpretation von konsistenten Preissystemen verweisen wir auf Schachermayer [85].

Definition 5. Ein $\mathbb{R}_+^d$-wertiges $\mathbb{P}$-Martingal $Z = (Z_t)_{t=0}^T$ mit $Z_t \in L^0(K_t^\star \setminus \{0\}, \mathcal{F}_t)$, d.h. $Z_t(\omega) \in K_t^\star(\omega) \setminus \{0\}$ für fast alle $\omega \in \Omega$ und jedes $t \in \{0, \ldots, T\}$, heißt *konsistentes Preissystem, engl. consistent price system (CPS),* für den Bid-Ask Prozess $(\Pi_t)_{t=0}^T$.

Wir haben nun die folgende Konsequenz von Theorem 4.

Korollar 6. *Erfüllt der Bid-Ask Prozess* $(\Pi_t)_{t=0}^T$ *die prospective strict no-arbitrage Bedingung* (NA^{ps})*, so existiert ein konsistentes Preissystem (CPS). Allgemeiner, existiert für jede* $\mathcal{F}_T$*-messbare Funktion* $\varphi : \Omega \to (0, 1]$ *ein CPS* $Z = (Z_t)_{t=0}^T$ *mit* $\|Z_T\| \leq M\varphi$ *f.s. für ein* $M \in \mathbb{R}_+ \setminus \{0\}$*, wobei* $\|\cdot\|$ *die euklidische Norm auf dem* $\mathbb{R}^d$ *bezeichnet.*

Die umgekehrte Implikation von Korollar 6 gilt nicht. Im Allgemeinen folgt aus [55, Section 3.2.4, Example 1], dass keine no-arbitrage Bedingung existiert, welche die Abgeschlossenheit von $\mathcal{A}_0^T$ garantiert und gleichzeitig äquivalent zur Existenz eines CPS ist. Wir können jedoch eine Äquivalenz zeigen, wenn wir zu einer schwächeren Form von $(\mathrm{NA^{ps}})$ übergehen.

Definition 7. Der Bid-Ask Prozess $(\Pi_t)_{t=0}^T$ erfüllt die *schwache prospective strict no-arbitrage Bedingung* (NA^{wps}), wenn ein Bid-Ask Prozess $(\widetilde{\Pi}_t)_{t=0}^T$ mit $\widetilde{\Pi}_t \leq \Pi_t$ f.s. für alle $t = 0, \dots, T$ existiert, sodass $(\widetilde{\Pi}_t)_{t=0}^T$ die prospective strict no-arbitrage Bedingung $(\mathrm{NA^{ps}})$ erfüllt.

Der Bid-Ask Prozess $(\widetilde{\Pi}_t)_{t=0}^T$ in Definition 7 muss nicht *strikt* vorteilhafter als $(\Pi_t)_{t=0}^T$ sein, somit ist $(\mathrm{NA^{wps}})$ schwächer als $(\mathrm{NA^{ps}})$. Das zweite Hauptresultat des ersten Teils dieser Arbeit ist der folgende Fundamentalsatz der Preistheorie.

Theorem 8. *Ein Bid-Ask Prozess* $(\Pi_t)_{t=0}^T$ *erfüllt die schwache prospective strict no-arbitrage Bedingung* (NA^{wps}) *genau dann, wenn ein konsistentes Preissystem (CPS) existiert.*

Bemerkung 9. *Theorem 8 ist eine Verallgemeinerung des zweiten Teils von Theorem 1 aus Kabanov and Stricker [54] auf allgemeine Wahrscheinlichkeitsräume. Im Fall* $|\Omega| < \infty$ *ergibt sich aus der Kombination beider Theoreme die* nette *Eigenschaft, dass* (NA^{wps}) *äquivalent zur üblichen no-arbitrage Bedingung, d.h.* $\mathcal{A}_0^T \cap L^0(\mathbb{R}_+^d, \mathcal{F}_T) = \{0\}$*, ist.*

Bemerkung 10. *Aus dem Satz von Grigoriev [34], d.h. der Äquivalenz zwischen der Existenz eines CPS und (NA) im Falle von nur zwei Wertpapieren, ergibt sich, dass (NA) und* (NA^{wps}) *in diesem Fall auch auf beliebigen Wahrscheinlichkeitsräumen übereinstimmen.*

Semimartingalpreissysteme in Modellen mit proportionalen Transaktionskosten

In zeitstetigen Finanzmarktmodellen mit Transaktionskosten ist *efficient friction*, d.h. nichtverschwindende Transaktionskosten, eine Standardannahme. Zusammen mit *robust no free lunch with vanishing risk* schließt *efficient friction* Handelsstrategien von unendlicher Variation, welche in friktionslosen Finanzmarktmodellen üblich sind, aus (siehe Guasoni et al. [36]). Im zweiten Teil dieser Arbeit, siehe Kapitel 3, zeigen wir, wie Modelle mit und ohne Transaktionskosten vereinheitlicht werden können. Wir betrachten *eindimensionale* Bid- und Askpreisprozesse mit càdlàg Pfaden, die adaptiert und lokal nach unten beschränkt sind. Insbesondere können sie an einigen Stellen übereinstimmen. Wir zeigen zunächst, dass die Bedingung *no unbounded profit with bounded risk* für nichtnegative elementar vorhersehbare Strategien die Existenz eines Semimartingals mit Werten zwischen Bid-und Askpreisprozess impliziert.

Unter der zusätzlichen Annahme, dass jede Nullstelle des Bid-Ask Spreads entweder ein innerer Punkt von rechts der Nullstellenmenge oder der Startpunkt einer Exkursion weg von der Null ist, zeigen wir in einem zweiten Schritt, wie das Semimartingal verwendet werden kann, um für jede beschränkte und vorhersehbare Strategie die zugehörige selbstfinanzierende Position im Bankkonto zu definieren. Abschließend setzen wir die Selbstfinanzierungsbedingung auf eine *größtmögliche* Menge von Strategien fort. Im friktionslosen Spezialfall stimmt diese Menge mit der Menge der vorhersehbaren Prozesse, welche bezüglich des Semimartingals integrierbar sind, überein. Wir erhalten somit eine neue Charakterisierung und damit auch einen neuen Blick auf diese Menge.

Die Hauptresultate des zweiten Teils sind im Folgenden zusammengefasst.

Existenz von Semimartingalpreissystemen

Das Finanzmarktmodell besteht aus einem risikolosen Bond mit Preis 1 und einer risikobehafteten Aktie mit Bid-Preis $\underline{S}$ und Ask-Preis $\overline{S}$. Wir nehmen an, dass $(\underline{S}_t)_{t\in[0,T]}$ und $(\overline{S}_t)_{t\in[0,T]}$ adaptierte Prozesse mit càdlàg Pfaden sind. Zusätzlich sei $\underline{S}_t \leq \overline{S}_t$ für alle $t \in [0,T]$ und $\underline{S}$ sei lokal von unten beschränkt.

Definition 11. Eine *elementar vorhersehbare Handelsstrategie* ist ein stochastischer Prozess $(\varphi_t)_{t\in[0,T]}$ von der Form

$$\varphi = \sum_{i=1}^{n} Z_{i-1} \mathbb{1}_{]\!]T_{i-1},T_i]\!]}, \tag{3}$$

wobei $n \in \mathbb{N}$ eine endliche Zahl ist, $0 = T_0 \leq T_1 \leq \cdots \leq T_n = T$ eine aufsteigende Folge von Stoppzeiten ist, und für alle $i = 0,\dots,n-1$ ist Z_i messbar bzgl. $\mathcal{F}_{T_i}$.

Die Strategie φ spezifiziert die Menge der risikobehafteten Aktie im Portfolio. Ist eine elementar vorhersehbare Handelsstrategie φ gegeben, so lässt sich die entsprechende selbstfinanzierende Position im Bond direkt aufschreiben. Diese Selbstfinanzierungbedingung ist Inhalt der folgenden Definition.

Definition 12. Sei $(\varphi_t)_{t\in[0,T]}$ eine elementar vorhersehbare Handelsstrategie. Die *zugehörige Position im risikolosen Bond* $(\varphi_t^0)_{t\in[0,T]}$ ist gegeben durch

$$\varphi_t^0 := \sum_{0\leq s<t} \left(\underline{S}_s(\Delta^+\varphi_s)^- - \overline{S}_s(\Delta^+\varphi_s)^+ \right), \quad t \in [0,T]. \tag{4}$$

Definition 13. Sei $(\varphi_t)_{t\in[0,T]}$ eine elementar vorhersehbare Handelsstrategie. Der *Liquidierungswertprozess* $(V_t^{\mathrm{liq}}(\varphi))_{t\in[0,T]}$ ist gegeben durch

$$V_t^{\mathrm{liq}}(\varphi) := \varphi_t^0 + (\varphi_t)^+ \underline{S}_t - (\varphi_t)^- \overline{S}_t, \quad t \in [0,T]. \tag{5}$$

Definition 14. Wir sagen, dass $(\underline{S}_t, \overline{S}_t)_{t\in[0,T]}$ einen *unbeschränkten Profit mit beschränkten Risiko, engl. unbounded profit with bounded risk (UPBR), für elementar vorhersehbare nichtnegative Strategien* erlaubt, wenn eine Folge elementar vorhersehbarer Handelsstrategien $(\varphi^n)_{n\in\mathbb{N}}$ mit $\varphi^n \geq 0$ existiert, sodass

(i) $V_t^{\mathrm{liq}}(\varphi^n) \geq -1$ für alle $t \in [0,T]$ und $n \in \mathbb{N}$, und

(ii) die Folge $(V_T^{\mathrm{liq}}(\varphi^n))_{n\in\mathbb{N}}$ unbeschränkt in Wahrscheinlichkeit ist, d.h.

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left(V_T^{\mathrm{liq}}(\varphi^n) \geq m\right) > 0. \tag{6}$$

Falls keine solche Folge existiert, sagen wir, dass der Bid-Ask Prozess $(\underline{S}, \overline{S})$ die *no unbounded profit with bounded risk (NUPBR) Bedingung für elementar vorhersehbare nichtnegative Strategien* erfüllt.

Das erste Hauptresultat des zweiten Teils der Arbeit ist das folgende Theorem.

Theorem 15. *Sei angenommen, dass* $(\underline{S}_t, \overline{S}_t)_{t\in[0,T]}$ *die (NUPBR) Bedingung für elementar vorhersehbare nichtnegative Handelsstrategien erfüllt. Dann existiert ein Semimartingal* $S = (S_t)_{t\in[0,T]}$ *mit*

$$\underline{S}_t \leq S_t \leq \overline{S}_t \quad \textit{für alle } t \in [0,T]. \tag{7}$$

Ein Semimartingale S, welches (7) erfüllt, nennen wir *Semimartingalpreissystem.*

Die Selbstfinanzierungsbedingung

Wir bereits angedeutet, verwenden wir das Semimartingalpreissystem S, um eine Selbstfinanzierungsbedingung für das Modell zu definieren. Eine Selbstfinanzierungsbedingung kann hierbei mit einem Operator $\varphi \mapsto \Pi(\varphi)$, welcher die Position im risikobehafteten Wertpapier auf die zugehörige Position im risikolosen Bond abbildet, identifiziert werden. Wir nehmen stets an, dass die Startposition und die risikolose Zinsrate Null sind. Ferner sei für den Rest dieses Kapitels angenommen, dass ein Semimartingalpreissystem S existiert. Ziel ist es $\Pi(\varphi)$ als $\varphi \bullet S - \varphi S -$ „Kosten$''$ zu definieren, wobei der Prozess $\varphi \bullet S$ das herkömmliche stochastische Integral bezeichnet. Die Kosten entstehen dadurch, dass die einzelnen Handelsgeschäfte im Transaktionskostenmarkt nicht zum Preis S, sondern zum jeweiligen Bid- bzw. Ask Preis durchgeführt werden. Wir starten zunächst mit beschränkten und vorhersehbaren Handelsstrategien. Die Handelsgewinne $\varphi \bullet S$ sind hierdurch stets wohldefiniert und endlich. Dementsprechend können „unendliche" Verluste nicht durch Handelsgewinne kompensiert werden.

Konstruktion des Kostenterms

Im Folgenden bezeichne $\mathbf{b}\mathcal{P}$ die Menge der beschränkten und vorhersehbaren Prozesse φ mit $\varphi_0 = 0$. Die Konstruktion des Kostenterms für eine Strategie $\varphi \in \mathbf{b}\mathcal{P}$ erfolgt pfadweise, d.h. $\omega \in \Omega$ ist fix und $\varphi, \underline{S}, \overline{S}$ und S werden mit Funktionen in der Zeit identifiziert.

Unsere Konstruktion besteht aus zwei Schritten. Zunächst berechnen wir die Kosten auf Intervallen, in denen die linksstetige Version des Spreads von der Null entfernt ist, durch ein modifiziertes Riemann-Stieltjes Integral. Dieses Integral existiert immer, kann jedoch den Wert ∞ annehmen. Anschließend schöpfen wir die Menge der Punkte mit positiven Spread durch endliche Vereinigungen obiger Intervalle aus und definieren die Gesamtkosten als Supremum über solche Vereinigungen.

Definition 16. Sei $I = [a, b] \subseteq [0, T]$ ein Intervall mit $a < b$.

(i) Eine Menge $P = \{t_0, \dots t_n\}$ von Punkten $t_i \in [a, b]$ mit $n \in \mathbb{N}$ und $i = 0, \dots, n$ sowie $a = t_0 < t_1 < \dots < t_n = b$ heißt *Partition von I.*

(ii) Eine Partition $P' = \{t'_0, \dots, t'_m\}$ mit $P' \supseteq P$ heißt *Verfeinerung von P.*

(iii) Die *gemeinsame Verfeinerung* $P \cup P'$ zweier Partitionen P, P' von I ist die Anordnung der Punkte in $\{t_0, \dots t_n\} \cup \{t'_0, \dots, t'_m\}$ in aufsteigender Reihenfolge.

(iv) Ist $P = \{t_0, \dots, t_n\}$ eine Partition von I, so nennen wir eine Menge $\lambda = \{s_1, \dots, s_n\}$ mit $s_i \in [t_{i-1}, t_i)$ für $i = 1, \dots, n$ eine *modifizierte Zwischenunterteilung von P.*

(v) Sei $\varphi \in \mathbf{b}\mathcal{P}$, $P = \{t_0, \dots, t_n\}$ eine Partition von I und $\lambda = \{s_1, \dots, s_n\}$ eine modifizierte Zwischenunterteilung von P, die *modifizierte Riemann-Stieltjes Summe zu φ, P und λ* ist

$$R(\varphi, P, \lambda) := \sum_{i=1}^{n} (\overline{S}_{s_i} - S_{s_i})(\varphi_{t_i} - \varphi_{t_{i-1}})^+ + \sum_{i=1}^{n} (S_{s_i} - \underline{S}_{s_i})(\varphi_{t_i} - \varphi_{t_{i-1}})^-.$$

Definition 17. Sei $\varphi \in \mathbf{b}\mathcal{P}$ und $I = [a, b] \subseteq [0, T]$ ein Intervall mit $a < b$. Der *Kostenterm von φ auf I* existiert und ist gleich $C(\varphi, I) \in \mathbb{R}_+ \cup \{\infty\}$, wenn für alle $\varepsilon > 0$ eine Partition P_ε von I existiert, sodass für alle Verfeinerungen P von P_ε und alle modifizierten Zwischeneinteilungen λ von P das Folgende gilt:

(i) Im Falle $C(\varphi, I) < \infty$, haben wir $|C(\varphi, I) - R(\varphi, P, \lambda)| < \varepsilon$,

(ii) Im Falle $C(\varphi, I) = \infty$, haben wir $|R(\varphi, P, \lambda)| > \frac{1}{\varepsilon}$.

Zusätzlich, setzen wir $C(\varphi, \{a\}) := 0$ für alle $a \in [0, T]$ und $C(\varphi, \emptyset) := 0$.

Die nächste Proposition stellt die Existenz des Kostenterms auf einem Intervall I, in dem der Spread von der Null entfernt ist, fest.

Proposition 18. *Sei $\varphi \in \mathbf{b}\mathcal{P}$ und $I = [a,b] \subseteq [0,T]$ ein Intervall mit $a < b$ und $\inf_{t\in[a,b)}(\overline{S}_t - \underline{S}_t) > 0$. Dann existiert der Kostenterm $C(\varphi, I)$ aus Definition 3.3.2 und ist eindeutig. Zusätzlich gilt*

$$\begin{cases} C(\varphi, I) < \infty, & \textit{if } \operatorname{Var}_a^b(\varphi) < \infty \\ C(\varphi, I) = \infty, & \textit{if } \operatorname{Var}_a^b(\varphi) = \infty, \end{cases}$$

wobei $\operatorname{Var}_a^b(\varphi)$ die pfadweise Variation von φ auf dem Intervall $[a,b]$ bezeichnet.

Nachdem wir die Kosten für alle Teilintervalle $I = [a,b] \subseteq [0,T]$ mit $\inf_{t\in[a,b)}(\overline{S}_t - \underline{S}_t) > 0$ definiert haben, gehen wir nun dazu über die kumulierten Kosten als Prozess zu definieren. Dafür sei

$$\mathcal{I} := \left\{ \cup_{i=1}^n [a_i, b_i] : \begin{array}{l} n \in \mathbb{N},\ 0 \le a_1 \le b_1 \le a_2 \le \cdots \le a_n \le b_n \le T, \\ \inf_{t\in[a_i,b_i)}(\overline{S}_t - \underline{S}_t) > 0,\ i = 1, \ldots, n \end{array} \right\} \cup \{\emptyset\}. \quad (8)$$

Wir setzen den Kostenterm nun auf $\mathcal{I}$ fort. Gegeben $\varphi \in \mathbf{b}\mathcal{P}$ und $J = \cup_{i=1}^n [a_i, b_i] \in \mathcal{I}$, definieren wir *die Kosten entlang* J durch

$$C(\varphi, J) := \sum_{i=1}^n C(\varphi, [a_i, b_i]), \quad (9)$$

wobei die Kostenterme $C(\varphi, [a_i, b_i])$ für $i = 1, \ldots, n$ in Definition 17 definiert sind.

Definition 19. (Kostenprozess) Sei $\varphi \in \mathbf{b}\mathcal{P}$. Der *Kostenprozess* $(C_t(\varphi))_{t\in[0,T]}$ ist gegeben durch

$$C_t(\varphi) := \sup_{J\in\mathcal{I}} C(\varphi, J \cap [0,t]) \in [0,\infty], \quad t \in [0,T].$$

Bisher haben wir $\omega \in \Omega$ fest gehalten, d.h. die Konstruktion erfolgte Pfad für Pfad. Betrachtet als stochastischen Prozess haben wir die folgende Messbarkeitseigenschaft:

Proposition 20. *Sie $\varphi \in \mathbf{b}\mathcal{P}$. Der Kostenprozess $C(\varphi) = (C_t(\varphi))_{t\in[0,T]}$ stimmt mit einem vorhersehbaren Prozess bis auf Ununterscheidbarkeit überein.*

Definition und Charakterisierung

Für den Rest des Abschnittes brauchen wir die folgende Voraussetzung an den Bid-Ask Spread.

Annahme 21. Für jedes $(\omega, t) \in \Omega \times [0,T)$ mit $\overline{S}_t(\omega) = \underline{S}_t(\omega)$ existiert ein $\varepsilon > 0$ sodass $\overline{S}_s(\omega) = \underline{S}_s(\omega)$ für alle $s \in (t, (t+\varepsilon) \wedge T)$ oder $\overline{S}_s(\omega) > \underline{S}_s(\omega)$ für alle $s \in (t, (t+\varepsilon) \wedge T)$.

Dementsprechend ist jede Nullstelle des Pfades $t \mapsto \overline{S}_t(\omega) - \underline{S}_t(\omega)$ entweder ein innerer Punkt der Nullstellenmenge von rechts oder ein Startpunkt einer Exkursion weg von der Null.

Zusätzlich verwenden wir im Folgenden stets die vorhersehbare Version des Kostenprozesses (cf. Proposition 20) und identifizieren ununterscheidbare Prozesse miteinander. Sei nun S ein Semimartingalpreissystem. Dann definieren wir den Operator Π, welcher eine beschränkte vorhersehbare Strategie φ mit Startwert null, d.h., $\varphi \in \mathbf{b}\mathcal{P}$, auf die zugehörige $[-\infty, \infty)$-wertige risikolose Position abbildet, durch

$$\Pi_t(\varphi) := \varphi \bullet S_t - \varphi_t S_t - C_t(\varphi), \quad t \in [0, T]. \tag{10}$$

Hierbei ist $\varphi \bullet S$ das übliche stochastische Integral. Wird die Aktienpostion im Semimartingal S bewertet, so ist der zugehörige Vermögensprozess durch $V_t(\varphi) := \varphi \bullet S_t - C_t(\varphi) = \Pi_t(\varphi) + \varphi_t S_t$ gegeben.

Wir benötigen im Folgenden ein Maß, welches Informationen über die Konvergenz von Integralen bzgl. S liefert. Dazu bemerken wir, dass ein Wahrscheinlichkeitsmaß $Q \sim \mathbb{P}$ existiert, sodass das Semimartingal S eine Zerlegung $S = M + A$ besitzt, wobei M ein Q-quadratintegrierbares Martingal und A ein Prozess mit Q-integrierbarer Variation ist (Theorem 58 in Kapitel VII von Dellacherie und Meyer [25]). Wir führen nun das endliche Maß

$$\mu^S(B) := \mathbb{E}_Q\left(\mathbb{1}_B \bullet \langle M, M\rangle_T\right) + \mathbb{E}_Q\left(\mathbb{1}_B \bullet \mathrm{Var}_T(A)\right), \quad B \in \mathcal{P}, \tag{11}$$

ein, wobei $\langle M, M\rangle$ die vorhersehbare quadratische Variation von M (siehe, z.B., [46, Kapitel 1, Theorem 4.2]) bezeichnet und $\mathcal{P}$ die vorhersehbare σ-Algebra ist.

Definition 22. Ein vorhersehbarer stochastischer Prozess φ von endlicher Variation heißt *fast elementar vorhersehbare Strategie*, falls eine Folge von Stoppzeiten $(\tau_n)_{n\geq 0}$ mit $\tau_n < \tau_{n+1}$ auf $\{\tau_n < \infty\}$ und $\#\{n : \tau_n(\omega) < \infty\} < \infty$ für alle $\omega \in \Omega$ existiert, sodass

$$\varphi = \sum_{n=0}^{\infty} (\varphi_{\tau_n} \mathbb{1}_{[\![\tau_n]\!]} + \varphi_{\tau_n +} \mathbb{1}_{]\!]\tau_n, \tau_{n+1}[\![}).$$

Für eine Strategie $\varphi \in \mathbf{b}\mathcal{P}$ charakterisiert das folgende Theorem den Prozess $V(\varphi)$ als Grenzwert von Vermögensprozessen, welche zu einer passenden Folge fast elementar vorhersehbarer Strategien gehören. Hierbei sei bemerkt, dass der Vermögensprozess V für fast elementar vorhersehbare Strategien mit dem intuitiven Vermögensprozess, welcher ohne jede Grenzwertprozedur aufgeschrieben werden kann, übereinstimmt.

Theorem 23. *Sei $\varphi \in \mathbf{b}\mathcal{P}$ und μ ein σ-endliches Maß auf der vorhersehbaren σ-Algebra mit $\mu^S \ll \mu$.*

(i) Für alle $\{0,1\}$-wertigen monoton fallenden vorhersehbaren Prozesse A und gleichmäßig beschränkten Folgen vorhersehbarer Prozesse $(\varphi^n)_{n\in\mathbb{N}}$ gilt die Implikation:

$$\begin{aligned} \varphi^n \to \varphi \quad & \textit{punktweise auf } \{\overline{S}_- > \underline{S}_-\} \cap \{A = 1\}, \\ & \mu^S\textit{-f.ü. auf } \{\overline{S}_- = \underline{S}_-\} \cap \{A = 1\} \\ \Longrightarrow \quad & \liminf_{n\to\infty} V(\varphi^n) \leq V(\varphi) \quad \textit{auf } \{A = 1\} \textit{ bis auf Ununterscheidbarkeit.} \end{aligned}$$

(ii) Es existiert eine gleichmäßig beschränkte Folge fast elementar vorhersehbarer Strategien $(\varphi^n)_{n\in\mathbb{N}}$, s.d.

$$\varphi^n \to \varphi \quad \textit{punktweise auf } \{\overline{S}_- > \underline{S}_-\} \cap \{C(\varphi) < \infty\},$$
$$\mu\textit{-f.ü. auf } \{\overline{S}_- = \underline{S}_-\} \cap \{C(\varphi) < \infty\},$$

und

$$\sup_{t\in[0,T]} |V_t(\varphi^n) - V_t(\varphi)| \mathbb{1}_{\{C_t(\varphi)\leq K\}} \to 0$$

in Wahrscheinlichkeit für $n \to \infty$ und alle $K \in \mathbb{N}$.

Bemerkung 24. *Im Spezialfall $C(\varphi) < \infty$, welcher äquivalent zu $V(\varphi) > -\infty$ ist, ergibt sich mit $A = 1$ die folgende Charakterisierung des Vermögensprozesses für eine beschränkte Strategie: (i) Das Vermögen der Strategie dominiert das Grenzvermögen aller (fast) punktweise konvergierender Strategien und (ii) es existiert eine ausgezeichnete Folge approximierender Strategien, sodass die Vermögensprozesse konvergieren.*

Auf der Menge $\{V(\varphi) = -\infty\} = \{C(\varphi) = \infty\}$ kann die Existenz einer punktweise auf $\{\overline{S}_- > \underline{S}_-\}$ gegen φ konvergierenden Folge fast elementar vorhersehbarer Strategien jedoch nicht erwartet werden. Trotzdem liefert Theorem 23(i) eine Motivation für $V(\varphi) = -\infty$.

Korollar 25. *Sei $\varphi \in \mathbf{b}\mathcal{P}$. Die Selbstfinanzierungsbedingung, d.h. die risikolose Position $\Pi(\varphi)$, hängt bis auf Ununterscheidbarkeit nicht von der Wahl des Semimartingalpreissystems ab.*

Erweiterung auf unbeschränkte Strategien

Sei $(\mathbf{b}\mathcal{P})^{\Pi} := \{\varphi \in \mathbf{b}\mathcal{P} : \Pi(\varphi) > -\infty$ bis auf Ununterscheidbarkeit$\}$. Nach dem vorangegangen Korollar hängt diese Menge nicht vom Semimartingalpreissystem ab. In diesem Abschnitt wollen wir die Selbstfinanzierungsbedingung, d.h. den Operator Π von $(\mathbf{b}\mathcal{P})^{\Pi}$ auf eine größtmögliche Menge von vorhersehbaren Strategien fortsetzen. Hierzu sei daran erinnert, dass der Raum der adaptierten làdlàg Prozesse $\mathcal{L}$ ausgestattet mit der Topologie der gleichmäßigen Konvergenz in Wahrscheinlichkeit ein vollständiger metrischer Raum mit Metrik $d_{up}(X,Y) := \|X - Y\|_{up} = \mathbb{E}\left[\sup_{t\in[0,T]} |X_t - Y_t| \wedge 1\right]$ für $X, Y \in \mathcal{L}$ ist. Im Folgenden schreiben wir

$$\operatorname*{up-lim}_{n\to\infty} X^n = X,$$

falls eine Folge $(X^n)_{n\in\mathbb{N}} \subseteq \mathcal{L}$ gegen $X \in \mathcal{L}$ bzgl. d_{up} konvergiert.

Definition 26. Sei L die Teilmenge der reellwertigen, vorhersehbaren Strategien φ, sodass eine Folge $(\varphi^n)_{n\in\mathbb{N}} \subset (\mathbf{b}\mathcal{P})^{\Pi}$ mit den folgenden Eigenschaften existiert:

(i) $\varphi^n \to \varphi$ punktweise auf $\Omega \times [0,T]$ und $(\varphi^n)^+ \leq \varphi^+$, $(\varphi^n)^- \leq \varphi^-$ für alle $n \in \mathbb{N}$.

(ii) Es existiert ein Semimartingal S mit $\underline{S} \leq S \leq \overline{S}$, sodass

$$(V^S(\varphi^n))_{n\in\mathbb{N}} = (\varphi^n \bullet S - C^S(\varphi^n))_{n\in\mathbb{N}}$$

Cauchy in $(\mathcal{L}, d_{up})$ ist, und sodass für alle Folgen $(\widetilde{\varphi}^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$, die (i) erfüllen, eine deterministische Teilfolge $(n_k)_{k\in\mathbb{N}}$ existiert mit

$$\left(V^S(\widetilde{\varphi}^{n_k}) - V^S(\varphi^{n_k})\right)^+ \to 0, \quad k \to \infty, \text{ bis auf Ununterscheidbarkeit} \tag{12}$$

Die Bedingung (ii) bedeutet, dass die Approximation mit $(\varphi^n)_{n\in\mathbb{N}}$ im Grenzwert besser ist als alle anderen punktweise Approximationen $(\widetilde{\varphi}^n)_{n\in\mathbb{N}}$, wenn die Aktienposition im selben Semimartingal bewertet wird. In (12) können wir keine gleichmäßige Konvergenz in der Zeit erwarten. Die Ausnahmenullmenge kann jedoch unabhängig von der Zeit gewählt werden. Insbesondere gilt $(\mathbf{b}\mathcal{P})^\Pi \subseteq L$.

Proposition 27. *Sei $\varphi \in L$. Falls $(\varphi^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ eine Folge von Strategien ist, welche Definition 26 für φ im Bezug auf ein Semimartingal S erfüllt, und $(\widetilde{\varphi}^n)_{n\in\mathbb{N}} \subseteq (\mathbf{b}\mathcal{P})^\Pi$ eine weitere Folge von Strategien ist, welche dieselben Bedingung im Bezug auf φ und ein Semimartingal $\widetilde{S}$ erfüllt, dann gilt*

$$\operatorname*{up-lim}_{n\to\infty} V^S(\varphi^n) - \varphi S = \operatorname*{up-lim}_{n\to\infty} V^{\widetilde{S}}(\widetilde{\varphi}^n) - \varphi \widetilde{S}$$

bis auf Ununterscheidbarkeit.

Wir können den Operator Π nun durch

$$\Pi(\varphi) := \operatorname*{up-lim}_{n\to\infty} V^S(\varphi^n) - \varphi S, \quad \varphi \in L$$

auf L fortsetzen, wobei $(\varphi^n)_{n\in\mathbb{N}}$ eine Folge von Strategien, welche Definition 26 bzgl. des Semimartingals S erfüllt, ist. Nach Propostion 27 ist Π auf L wohldefiniert, d.h. der Operator hängt weder von der Wahl der approximierende Folge noch der Wahl des Semimartingals ab. Wir schließen nun die Zusammenfassung der Hauptresultate dieser Arbeit mit einer Proposition ab, welche zeigt, dass im friktionslosen Fall, d.h. $\overline{S} = \underline{S} = S$, die Menge L mit der Menge $L(S)$, der S-integrierbaren Prozesse, übereinstimmt.

Proposition 28. *Sei $\underline{S} = \overline{S} = S$ ein Semimartingal. Dann gilt $L = L(S)$ und $\Pi(\varphi) = \varphi \bullet S - \varphi S$ für alle $\varphi \in L$.*

Chapter 5

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