Hodge conjecture for the moduli space of semi-stable sheaves over a nodal curve

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Abstract

In this article, we prove the Hodge conjecture for a desingularization of the moduli space of rank 2, semi-stable, torsion-free sheaves with fixed odd degree determinant over a very general irreducible nodal curve of genus at least 2. We also compute the algebraic Poincaré polynomial of the associated cohomology ring.

Keywords Hodge conjecture · Limit mixed Hodge structures · Cycle class · Algebraic Hodge polynomial · Nodal curves · Moduli space of semi-stable sheaves

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1 Introduction

The Hodge conjecture is one of the outstanding problems of present day mathematics. Although it has been known for over 70 years, the evidence for it has been rather limited (see [19, 34] for a survey). Recall, any smooth, projective variety X over $\mathbb{C}$ admits the Hodge decomposition $H^r(X, \mathbb{C}) = \bigoplus_i H^{i,r-i}(X, \mathbb{C})$. For $r = 2p$, elements of $H^{p,p}(X, \mathbb{Z}) := H^{2p}(X, \mathbb{Z}) \cap H^{p,p}(X, \mathbb{C})$ are called *Hodge classes*. Denote by $Z^p(X)$ the free abelian group of algebraic cycles of X of codimension p. Recall, cycle class map

$$c : Z^p(X) \to H^{2p}(X, \mathbb{Z})$$

which sends an algebraic cycle to its cohomology class (see [33, §11.1]). Denote by $H_A^{2p}(X, \mathbb{Z})$ the image of c, called the *algebraic cohomology group*. It is well-known that the cohomology class of any algebraic cycle is Hodge, i.e. $H_A^{2p}(X, \mathbb{Z}) \subset H^{p,p}(X, \mathbb{Z})$ (see [33,

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Proposition 11.20]). The *integral Hodge conjecture* predicts that every integral Hodge class comes from an algebraic cycle i.e. $H_A^{2p}(X, \mathbb{Z}) = H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Z})$. Although the conjecture is true for uniruled and Calabi Yau threefolds as well as cubic fourfolds (see [34]), the integral Hodge conjecture is false in general (see [3]). Therefore, we instead consider the rational Hodge conjecture i.e. $H_A^{2p}(X, \mathbb{Q}) = H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X, \mathbb{C})$, where

$$H_A^{2p}(X, \mathbb{Q}) := \operatorname{Im}(c : Z^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q})).$$

One interesting case where the conjecture holds true is that of the Jacobian, $\operatorname{Jac}(C)$ of a *very general*, smooth, projective curve C (see [8, §17.5]). Using this, Balaji-King-Newstead in [5] proved the conjecture for the moduli space $M_C(2, \mathcal{L})$ of rank 2 semi-stable, locally free sheaves with determinant $\mathcal{L}$ over C, for an odd degree invertible sheaf $\mathcal{L}$ on C. It has also been shown for the moduli space of stable pairs over a smooth, projective curve in [21]. However, nothing is known in the case the underlying curve is irreducible, nodal. The goal of this article is to prove the Hodge conjecture for the moduli space of stable rank 2 bundles with odd degree determinant in the case when the underlying curve is very general, irreducible, nodal.

Let X_0 be a very general, irreducible, nodal curve with exactly one node, say x_0 and $\mathcal{L}_0$ an invertible sheaf on X_0 of odd degree. Here a *very general nodal curve of genus g* means that the normalization, together with the two inverse images of the node, is a *very general* 2-pointed curve of genus $g - 1$. Equivalently, a very general nodal curve lies outside countably many, proper closed subsets of the image of the clutching map from $\mathcal{M}_{g-1,2}$ to $\overline{\mathcal{M}_g}$, where $\mathcal{M}_{g-1,2}$ denotes the moduli space of genus $g - 1$ curves with 2 marked points and $\overline{\mathcal{M}_g}$ is the moduli space of stable curves of genus g (see [2, Chapter XII, §10]). Given a torsion-free sheaf $\mathcal{E}$ on X_0, we say that $\mathcal{E}$ has *determinant* $\mathcal{L}_0$ if there is an $\mathcal{O}_{X_0}$-morphism $\wedge^2(\mathcal{E}) \to \mathcal{L}_0$ which is an isomorphism outside the node x_0. If $\mathcal{E}$ is locally free then this means $\wedge^2 \mathcal{E} \cong \mathcal{L}_0$. Using [31, Theorem 2], one can check that there exists a moduli space, denoted $U_{X_0}(2, \mathcal{L}_0)$, parameterizing rank 2, semi-stable sheaves on X_0 with determinant $\mathcal{L}_0$. However, the moduli space $U_{X_0}(2, \mathcal{L}_0)$ is singular. We show:

Theorem 1.1 *For X_0 a very general, irreducible nodal curve, there exists a desingularization $\mathcal{G}_0$ of $U_{X_0}(2, \mathcal{L}_0)$ (in the sense that $\mathcal{G}_0$ is non-singular and there is a proper birational morphism from $\mathcal{G}_0$ to $U_{X_0}(2, \mathcal{L}_0)$) such that the Hodge conjecture holds for $\mathcal{G}_0$.*

See Theorem 5.2 for a proof.

One obstacle to simply generalizing the techniques used in the smooth curve case is that an analogous description of the cohomology ring of $U_{X_0}(2, \mathcal{L}_0)$ is not available. More precisely, Balaji-King-Newstead in [5] prove that there are Hodge classes $\alpha \in H^2(M_C(2, \mathcal{L}), \mathbb{Z})$, $\beta \in H^4(M_C(2, \mathcal{L}), \mathbb{Z})$ and a surjective morphism

$$H^*(\operatorname{Jac}(C), \mathbb{Q}) \otimes \mathbb{Q}[\alpha, \beta] \to H^*(M_C(2, \mathcal{L}), \mathbb{Q})$$

inducing a surjective morphism $\nu : H_A^*(\operatorname{Jac}(C), \mathbb{Q}) \otimes \mathbb{Q}[\alpha, \beta] \to H_A^*(M_C(2, \mathcal{L}), \mathbb{Q})$. Since the Hodge conjecture holds for $\operatorname{Jac}(C)$ of a general smooth, projective curve C, they are able to conclude the Hodge conjecture for $M_C(2, \mathcal{L})$. Unfortunately such a morphism does not exist if we replace $M_C(2, \mathcal{L})$ by $U_{X_0}(2, \mathcal{L}_0)$. Moreover, since the Jacobian of a nodal curve is not projective, the statement of the Hodge conjecture does not apply to the Jacobian of X_0. As a result the classical tools fail in this setup.

Our strategy is as follows. We first embed the very general nodal curve X_0 as the central fiber of a regular, flat family of projective curves $\pi : \mathcal{X} \to \Delta$ (here Δ denotes the unit disc), smooth over $\Delta^* := \Delta \backslash \{0\}$ (see [4, Theorem B.2]) for the existence of such a family. Note that, the invertible sheaf $\mathcal{L}_0$ on X_0 lifts to a relative invertible sheaf $\mathcal{L}_\mathcal{X}$ over $\mathcal{X}$. By [31] and [32], there exists a relative Gieseker moduli space with fixed determinant over the family $\mathcal{X}$ given by a flat, projective morphism $\pi_2 : \mathcal{G}(2, \mathcal{L}) \to \Delta$ such that for all $s \in \Delta^*$, $\mathcal{G}(2, \mathcal{L})_s := \pi_2^{-1}(s) = M_{\mathcal{X}_s}(2, \mathcal{L}_s)$. The total space $\mathcal{G}(2, \mathcal{L})$ is regular and the central fiber $\pi_2^{-1}(0)$, denoted $\mathcal{G}_{X_0}(2, \mathcal{L}_0)$, is a reduced simple normal crossings divisor of $\mathcal{G}(2, \mathcal{L})$ with two smooth, irreducible components such that one of them is a desingularization of $U_{X_0}(2, \mathcal{L}_0)$. We denote this desingularization by $\mathcal{G}_0$. Since for all $s \neq 0$, $\mathcal{G}(2, \mathcal{L})_s = M_{\mathcal{X}_s}(2, \mathcal{L}_s)$ and we already know that the Hodge conjecture holds true for $M_{\mathcal{X}_s}(2, \mathcal{L}_s)$ by [5], it is natural to compare the Hodge classes and the algebraic classes on $\mathcal{G}_{X_0}(2, \mathcal{L}_0)$ using variation of mixed Hodge structures. We prove that the Hodge conjecture holds for both of the smooth components of the central fibre and therefore for a desingularization of $U_{X_0}(2, \mathcal{L}_0)$.

As a by-product we obtain the algebraic Poincaré polynomial of $\mathcal{G}_0$. Recall by [5, (5.1)], we have for any $s \in \Delta^*$, the algebraic Poincaré polynomial of $M_{\mathcal{X}_s}(2, \mathcal{L}_s)$, denoted

$$H(g, t) := \sum_i H_A^i \big(M_{\mathcal{X}_s}(2, \mathcal{L}_s) \big) t^i = \frac{(1 - t^g)(1 - t^{g+1})(1 - t^{g+2})}{(1 - t)(1 - t^2)(1 - t^3)},$$

where $\mathcal{X}_s := \pi^{-1}(s)$ and $\mathcal{L}_s := \mathcal{L}_\mathcal{X}|_{\mathcal{X}_s}$. Analogously, we prove (Theorems 5.2 and 5.3):

Theorem 1.2 *The algebraic Poincaré polynomial for $\mathcal{G}_0$ is given by*

$$P_A(\mathcal{G}_0) := \sum_i H_A^i(\mathcal{G}_0) t^i = H(g - 1, t)(t^2 + t^4) + H(g, t).$$

We note that this article is part of a series of articles in which we study related but different questions pertaining to the moduli space of stable, rank 2 sheaves on an irreducible nodal curve (see [6, 7, 9, 10]). The answers to these questions are well-known in the case when the underlying curve is smooth. Therefore we have often employed the theory of limit mixed Hodge structures to study the question for the nodal curve case using analogous results known for the case when the curve is smooth. However, the results in these articles are independent and overlap only in the background material.

Notation: Given any morphism $f : \mathcal{Y} \to S$ and a point $s \in S$, we denote by $\mathcal{Y}_s := f^{-1}(s)$. The open unit disc is denoted by Δ and $\Delta^* := \Delta \backslash \{0\}$ denotes the punctured disc. Unless mentioned otherwise, all cohomology groups are taken with $\mathbb{Q}$-coefficients.

2 Preliminaries: Limit mixed Hodge structures

We now briefly review the basics of limit-mixed Hodge structures. Since the theory of limit mixed Hodge structures are used just as a tool, we only state definitions and results (without proof) relevant to our setup. For a detailed treatment of the subject see [26].

Notation 2.1 Let $\rho : \mathcal{Y} \to \Delta$ be a flat, projective family of projective varieties, smooth over Δ^*, $\rho' : \mathcal{Y}_{\Delta^*} \to \Delta^*$ the restriction of ρ to Δ^*.

Definition 2.2 By Ehresmann's lemma (see [33, Theorem 9.3]), for all $i \geq 0$, $\mathbb{H}^i_{\mathcal{Y}_{\Delta^*}} := R^i \rho'_* \mathbb{Z}$ is a local system over Δ^* with fiber $H^i(\mathcal{Y}_t, \mathbb{Z})$, for $t \in \Delta^*$. One can associate to these local systems, the holomorphic vector bundles $\mathcal{H}^i_{\mathcal{Y}_{\Delta^*}} := \mathbb{H}^i_{\mathcal{Y}_{\Delta^*}} \otimes_{\mathbb{Z}} \mathcal{O}_{\Delta^*}$ called the *Hodge bundle*. There exist holomorphic sub-bundles $F^p \mathcal{H}^i_{\mathcal{Y}_{\Delta^*}} \subset \mathcal{H}^i_{\mathcal{Y}_{\Delta^*}}$ defined by the condition: for any $t \in \Delta^*$, the fibers $\left(F^p \mathcal{H}^i_{\mathcal{Y}_{\Delta^*}} \right)_t \subset \left(\mathcal{H}^i_{\mathcal{Y}_{\Delta^*}} \right)_t$ can be identified respectively with $F^p H^i(\mathcal{Y}_t, \mathbb{C}) \subset H^i(\mathcal{Y}_t, \mathbb{C})$. where F^p denotes the Hodge filtration (see [33, §10.2.1]).

In order to define a mixed Hodge structure on the family $\rho : \mathcal{Y} \to \Delta$, the Hodge bundles and their holomorphic sub-bundles need to be extended to the entire disc. By [26, Definition 11.4] there exists a canonical extension $\overline{\mathcal{H}}^i_{\mathcal{y}}$ of $\mathcal{H}^i_{\mathcal{Y}_{\Delta^*}}$ to Δ . Note that, $\overline{\mathcal{H}}^i_{\mathcal{y}}$ is locally-free over Δ. Denote by $j : \Delta^* \to \Delta$ the inclusion morphism, then the Hodge filtration F^p on Δ^* is extended to Δ by setting $F^p \overline{\mathcal{H}}^i_{\mathcal{y}} := j_* \left(F^p \mathcal{H}^i_{\mathcal{Y}_{\Delta^*}} \right) \cap \overline{\mathcal{H}}^i_{\mathcal{y}}$. Note that, $F^p \overline{\mathcal{H}}^i_{\mathcal{y}}$ is the locally-free sub-sheaf of $\overline{\mathcal{H}}^i_{\mathcal{y}}$ which extends $F^p \mathcal{H}^i_{\mathcal{Y}_{\Delta^*}}$.

Definition 2.3 Consider the universal cover $\mathfrak{h} \to \Delta^*$ of the punctured unit disc. Denote by $e : \mathfrak{h} \to \Delta^* \overset{j}{\to} \Delta$ the composed morphism and define by

$$\mathcal{Y}_\infty := \mathcal{Y} \times_\Delta \mathfrak{h}$$

the base change of the family $\mathcal{Y}$ over Δ to $\mathfrak{h}$, by the morphism e.

By [26, XI-8], for a choice of the parameter t on Δ, the central fiber of the canonical extension $\overline{\mathcal{H}}^i_{\mathcal{y}}$ can be identified with the cohomology group $H^i(\mathcal{Y}_\infty, \mathbb{C})$:

$$g^i_t : H^i(\mathcal{Y}_\infty, \mathbb{C}) \xrightarrow{\sim} \left(\overline{\mathcal{H}}^i_{\mathcal{y}} \right)_0. \tag{1}$$

As a consequence, there exist Hodge filtrations on $H^i(\mathcal{Y}_\infty, \mathbb{C})$ defined by

$$F^p H^i(\mathcal{Y}_\infty, \mathbb{C}) := (g^i_t)^{-1} \left(F^p \overline{\mathcal{H}}^i_{\mathcal{y}} \right)_0.$$

To define a weight filtration on $H^i(\mathcal{Y}_\infty, \mathbb{Z})$, we use the local monodromy transformations.

Definition 2.4 For any $s \in \Delta^*$ and $i \geq 0$, denote by

$$T_{s,i} : H^i(\mathcal{Y}_s, \mathbb{Z}) \to H^i(\mathcal{Y}_s, \mathbb{Z}) \text{ and } T^{\mathbb{Q}}_{s,i} : H^i(\mathcal{Y}_s) \to H^i(\mathcal{Y}_s)$$

the *local monodromy transformations* associated to the local system $\mathbb{H}^i_{\mathcal{Y}_{\Delta^*}}$, defined by parallel transport along a counterclockwise loop about $0 \in \Delta$ (see [26, §11.1.1]). By Ehresmann's lemma, the cohomology group $H^i(\mathcal{Y}_\infty)$ is (canonically) isomorphic to $H^i(\mathcal{Y}_s)$, depending only on the choice of $s' \in \mathfrak{h}$ with $e(s') = s$, induced by the inclusion of $\mathcal{Y}_{s'}$ into $\mathcal{Y}_\infty$. Then using the translation $s' \mapsto s' + 1$, the automorphism $T^{\mathbb{Q}}_{s,i}$ induces a $\mathbb{Q}$-automorphism

$$T_i : H^i(\mathcal{Y}_\infty) \to H^i(\mathcal{Y}_\infty). \tag{2}$$

See [17, §II.2.4] or [13, Theorem II.1.17] for more details.

We can now define a mixed Hodge structure on $H^i(\mathcal{Y}_\infty, \mathbb{Z})$.

Definition 2.5 Let N_i be the logarithm of the monodromy operator T_i. By [26, Lemma-Definition 11.9], there exists an unique increasing *monodromy weight filtration* $W_{\bullet}$ on $H^i(\mathcal{Y}_{\infty})$ such that,

(1) for $j \geq 2$, $N_i(W_j H^i(\mathcal{Y}_{\infty})) \subset W_{j-2} H^i(\mathcal{Y}_{\infty})$ and
(2) the map $N_i^l : \mathrm{Gr}_{i+l}^W H^i(\mathcal{Y}_{\infty}) \to \mathrm{Gr}_{i-l}^W H^i(\mathcal{Y}_{\infty})$ is an isomorphism for all $l \geq 0$.

By [28, Theorem 6.16] the induced filtrations on $H^i(\mathcal{Y}_{\infty}, \mathbb{C})$ define a mixed Hodge structure $(H^i(\mathcal{Y}_{\infty}, \mathbb{Z}), W_{\bullet}, F^{\bullet})$, called the *limit mixed Hodge structure* on $H^i(\mathcal{Y}_{\infty}, \mathbb{Z})$.

Definition 2.6 Recall, for any $t \in \Delta^*$, there exist natural specialization morphisms:

$$\mathrm{sp}_i : H^i(\mathcal{Y}_0) \to H^i(\mathcal{Y}_t)$$

obtained by composing the natural inclusion of the special fiber $\mathcal{Y}_t$ into $\mathcal{Y}$ with the retraction map to the central fiber $\mathcal{Y}_0$. Unfortunately, the resulting specialization maps are not morphism of mixed Hodge structures. However, if one identifies $H^i(\mathcal{Y}_t)$ with $H^i(\mathcal{Y}_{\infty})$ as mentioned above, then the resulting (modified) specialization morphisms are morphisms of mixed Hodge structures (see [26, Theorem 11.29]). Furthermore, by the local invariant cycle theorem [26, Theorem 11.43], we have the following exact sequence:

$$H^i(\mathcal{Y}_0) \xrightarrow{\mathrm{sp}_i} H^i(\mathcal{Y}_{\infty}) \xrightarrow{T_y - \mathrm{Id}} H^i(\mathcal{Y}_{\infty}) \tag{3}$$

where sp_i denotes the (modified) specialization morphism which is a morphism of mixed Hodge structures as discussed above.

Suppose that $\mathcal{Y}_0$ is a simple, normal crossings divisor and a union of two smooth, irreducible components, say Y_1, Y_2. Since $Y_1 \cap Y_2, Y_1, Y_2$ are non-singular (hence has pure Hodge structure), the Mayer-Vietoris sequence associated to $\mathcal{Y}_0$, gives rise to the exact sequence:

$$0 \to \mathrm{Gr}_i^W H^i(\mathcal{Y}_0) \to H^i(Y_1) \oplus H^i(Y_2) \xrightarrow{r_1 - r_2} H^i(Y_1 \cap Y_2), \tag{4}$$

where r_j is the restriction morphism from $H^i(Y_j)$ to $H^i(Y_1 \cap Y_2)$ for $j = 1, 2$. Note that the Gysin morphism from $H^{i-2}(Y_1 \cap Y_2)(-1)$ to $H^i(Y_1) \oplus H^i(Y_2)$ composed with $(r_1 - r_2)$ is the zero map (see [26, Proposition B.30]). Hence, $H^i(Y_1 \cap Y_2)$ factors through $\mathrm{Gr}_i^W H^i(\mathcal{Y}_0)$. Denote by

$$f_i : H^{i-2}(Y_1 \cap Y_2)(-1) \to \mathrm{Gr}_i^W H^i(\mathcal{Y}_0) \hookrightarrow H^i(\mathcal{Y}_0) \tag{5}$$

the composed morphism. We recall the following useful result which we use repeatedly.

Corollary 2.7 *Suppose that the central fiber $\mathcal{Y}_0$ is a simple, normal crossings divisor and an union of two smooth, irreducible components, say Y_1, Y_2. Then, $W_{i+1} H^i(\mathcal{Y}_{\infty}) = H^i(\mathcal{Y}_{\infty})$ and we have the following exact sequence of mixed Hodge structures*:

$$H^{i-2}(Y_1 \cap Y_2)(-1) \xrightarrow{f_i} H^i(\mathcal{Y}_0) \xrightarrow{\mathrm{sp}_i} H^i(\mathcal{Y}_{\infty}) \xrightarrow{g_i} \mathrm{Gr}_{i+1}^W H^i(\mathcal{Y}_{\infty}) \to 0, \tag{6}$$

where f_i is the morphism (5) and g_i is the natural quotient by $W_i H^i(\mathcal{Y}_{\infty})$.

Proof See [10, Corollary 2.4] for a proof. □

3 Limit mixed Hodge structures of the relative Gieseker moduli space

In this section, we recall the basic definitions and results on the relative Gieseker moduli space, the associated monodromy action and extend the exact sequence (6) in this setup.

3.1 Notation

Let X_0 be an irreducible nodal curve of genus $g \geq 2$, with exactly one node, say at x_0. Denote by $\pi_0 : \widetilde{X}_0 \to X_0$ the normalization map. There exists a flat, projective family of curves $\pi_1 : \mathcal{X} \to \Delta$ smooth over Δ^* and central fiber isomorphic to X_0 (see [4, Theorem B.2]). Moreover, $\mathcal{X}$ is a regular variety. Fix an invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ such that $\mathcal{L}|_{\mathcal{X}_s}$ is of odd degree, say d for all $s \in \Delta$. Set $\mathcal{L}_0 := \mathcal{L}|_{X_0}$, the restriction of $\mathcal{L}$ to the central fiber. Denote by $\widetilde{\mathcal{L}}_0 := \pi_0^*(\mathcal{L}_0)$.

3.2 Relative Gieseker moduli space

Recall, that a curve X_k *is semi-stably equivalent to* X_0 if it is the union of the normalization $\widetilde{X}_0$ and a chain of rational curves of length k. See [22, Definition-Notation 2] for the precise definition. We say that a family of curves $f : \mathcal{X}_S \to S$ *is semi-stably equivalent to the family* π_1 above, if

(1) there is a morphism from $\mathcal{X}_S$ to $\mathcal{X}$ inducing a morphism h from S to Δ such that the resulting diagram is commutative,
(2) for any point s mapping to $0 \in \Delta$, the fiber $f^{-1}(s)$ is semi-stably equivalent to X_0. For other points $s \in S$, the fiber $f^{-1}(s)$ is isomorphic to the fiber of π_1 over $h^{-1}(s)$.

There exists a relative moduli space, called the *relative Gieseker moduli space*, denoted $\mathcal{G}(2, \mathcal{L})$ which parametrizes certain rank 2, determinant $\mathcal{L}$ semi-stable sheaves defined over families of curves semi-stably equivalent to the family π_1. See [31, §3] or [32, §6] for the precise definitions. By [31, Theorem 2], there exists a flat, projective morphism $\pi_2 : \mathcal{G}(2, \mathcal{L}) \to \Delta$ such that for all $s \in \Delta^*$, $\mathcal{G}(2, \mathcal{L})_s := \pi_2^{-1}(s) = M_{\mathcal{X}_s}(2, \mathcal{L}_s)$ (see also [1, §5 and 6]). Moreover, $\mathcal{G}(2, \mathcal{L})$ is regular and the central fiber $\pi_2^{-1}(0)$, denoted $\mathcal{G}_{X_0}(2, \mathcal{L}_0)$, is a reduced simple normal crossings divisor of $\mathcal{G}(2, \mathcal{L})$ (see [32, §6]).

Denote by $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$ the fine moduli space of semi-stable sheaves of rank 2 and with determinant $\widetilde{\mathcal{L}}_0$ over $\widetilde{X}_0$ (see [14, Theorems 4.3.7 and 4.6.6]).

By [32, (6.2)], $\mathcal{G}_{X_0}(2, \mathcal{L}_0)$ can be written as the union of two irreducible components, say $\mathcal{G}_0$ and $\mathcal{G}_1$, where $\mathcal{G}_1$ (resp. $\mathcal{G}_0 \cap \mathcal{G}_1$) is isomorphic to a $\mathbb{P}^3$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$)-bundle over $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$. Note that, there is a proper morphism $\theta : \mathcal{G}_{X_0}(2, \mathcal{L}_0) \to U_{X_0}(2, \mathcal{L}_0)$ with the irreducible component $\mathcal{G}_0$ mapping surjectively to $U_{X_0}(2, \mathcal{L}_0)$ (use [31, Theorem 3.7] and [32, §6]). In fact, the restriction of θ to $\mathcal{G}_0$ is a birational morphism. Since $\mathcal{G}_0$ is non-singular, it is a desingularization of $U_{X_0}(2, \mathcal{L}_0)$ (see [31, 32]). Moreover, there exists an $\overline{\mathrm{SL}}_2$-bundle over $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$, denoted P_0, and closed subschemes $Z \subset P_0$, $Z' \subset \mathcal{G}_0$ such that $P_0 \backslash Z \cong \mathcal{G}_0 \backslash Z'$ and $\emptyset = Z' \cap \mathcal{G}_1$ (see [32, p. 27]), where $\overline{\mathrm{SL}}_2$ is the *wonderful compactification* of SL_2 defined as

$$\overline{\mathrm{SL}}_2 := \{[M, \lambda] \in \mathbb{P}(\mathrm{End}(\mathbb{C}^2) \oplus \mathbb{C})| \det(M) = \lambda^2\}$$

(see [27, Definition 3.3.1] for a general definition of wonderful compactification).

3.3 Monodromy action on the relative Gieseker moduli space

An important step in proving the Hodge conjecture for $\mathcal{G}_0$ is to study the monodromy action on $H^{2i}(\mathcal{G}(2, \mathcal{L})_s)$ for $s \in \Delta^*$. For this purpose, we need to consider the relative version of the construction of the Mumford-Newstead isomorphism [20]. Denote by

$$\mathcal{W} := \mathcal{X}_{\Delta^*} \times_{\Delta^*} \mathcal{G}(2, \mathcal{L})_{\Delta^*} \text{ and } \pi_3 : \mathcal{W} \to \Delta^*$$

the natural morphism. We set $\mathcal{W}_t := \pi_3^{-1}(t) \cong \mathcal{X}_t \times M_{\mathcal{X}_t}(2, \mathcal{L}_t)$ for all $t \in \Delta^*$. There exists a (relative) universal bundle $\mathcal{U}$ over $\mathcal{W}$ associated to the (relative) moduli space $\mathcal{G}(2, \mathcal{L})_{\Delta^*}$. Note that $\mathcal{U}$ is only well-defined up to pull-back by a line bundle on $\mathcal{G}(2, \mathcal{L})_{\Delta^*}$. For each $t \in \Delta^*$, $\mathcal{U}|_{\mathcal{W}_t}$ is a universal bundle over $\mathcal{X}_t \times M_{\mathcal{X}_t}(2, \mathcal{L}_t)$ associated to fine moduli space $M_{\mathcal{X}_t}(2, \mathcal{L}_t)$ (use [25, Theorem 9.1.1]). Let $\mathbb{H}^4_{\mathcal{W}} := R^4 \pi_{3_*} \mathbb{Z}_{\mathcal{W}}$ be the local system associated to $\mathcal{W}$. By Künneth decomposition, we have

$$\mathbb{H}^4_{\mathcal{W}} = \bigoplus_i \left(\mathbb{H}^i_{\mathcal{X}_{\Delta^*}} \otimes \mathbb{H}^{4-i}_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}} \right). \tag{7}$$

Denote by $c_2(\mathcal{U})^{1,3} \in \Gamma\left(\mathbb{H}^1_{\mathcal{X}_{\Delta^*}} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}} \right)$ the image of the second Chern class $c_2(\mathcal{U}) \in \Gamma(\mathbb{H}^4_{\mathcal{W}})$ under the natural projection from $\mathbb{H}^4_{\mathcal{W}}$ to $\mathbb{H}^1_{\mathcal{X}_{\Delta^*}} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}}$. Using Poincaré duality applied to the local system $\mathbb{H}^1_{\mathcal{X}_{\Delta^*}}$, we have

$$\mathbb{H}^1_{\mathcal{X}_{\Delta^*}} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}} \overset{PD}{\cong} \left(\mathbb{H}^1_{\mathcal{X}_{\Delta^*}} \right)^\vee \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}} \cong \mathcal{H}om \left(\mathbb{H}^1_{\mathcal{X}_{\Delta^*}}, \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}} \right). \tag{8}$$

Therefore, $c_2(\mathcal{U})^{1,3}$ induces a homomorphism $\Phi_{\Delta^*} : \mathbb{H}^1_{\mathcal{X}_{\Delta^*}} \to \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}}$. Note that, the restriction of the universal bundle $\mathcal{U}$ to the fiber over any point of the moduli space $\mathcal{G}(2, \mathcal{L})_{\Delta^*}$ is uniquely determined by the point (and not by the choice of the universal bundle). Hence, the value of the section $c_2(\mathcal{U})$ at each point of Δ^* is independent of the choice of $\mathcal{U}$. Since a (local) section of a local system is uniquely determined by its value at a point, the section $c_2(\mathcal{U})$ does not depend on the choice of the universal bundle. As a result, the homomorphism Φ_{Δ^*} is independent of the choice of $\mathcal{U}$. By [20, Lemma 1 and Proposition 1], we conclude that the homomorphism Φ_{Δ^*} is an isomorphism such that the induced isomorphism on the associated vector bundles:

$$\Phi_{\Delta^*} : \mathcal{H}^1_{\mathcal{X}_{\Delta^*}} \overset{\sim}{\longrightarrow} \mathcal{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}} \text{ satisfies } \Phi_{\Delta^*}(F^p \mathcal{H}^1_{\mathcal{X}_{\Delta^*}}) = F^{p+1} \mathcal{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}} \text{ for all } p \geq 0.$$

Denote by

$$\widetilde{\Phi}_s : H^1(\mathcal{X}_s, \mathbb{Z}) \overset{\sim}{\longrightarrow} H^3(\mathcal{G}(2, \mathcal{L})_s, \mathbb{Z}) \tag{9}$$

the restriction of Φ_{Δ^*} to the point $s \in \Delta^*$. Since $c_2(\mathcal{U})^{1,3}$ is a (single-valued) global section of $\mathbb{H}^1_{\mathcal{X}_{\Delta^*}} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^*}}$, we have

$$\widetilde{\Phi}_s \in \text{Hom}(H^1(\mathcal{X}_s, \mathbb{Z}), H^3(\mathcal{G}(2, \mathcal{L})_s, \mathbb{Z})) \cong H^1(\mathcal{X}_s, \mathbb{Z})^\vee \otimes H^3(\mathcal{G}(2, \mathcal{L})_s, \mathbb{Z})$$
$$\overset{\text{P.D.}}{\cong} H^1(\mathcal{X}_s, \mathbb{Z}) \otimes H^3(\mathcal{G}(2, \mathcal{L})_s, \mathbb{Z})$$

is monodromy invariant i.e., for all $s \in \Delta^*$, the following diagram is commutative:

$$H^1(\mathcal{X}_s, \mathbb{Z}) \xrightarrow[\sim]{\widetilde{\Phi}_s} H^3(\mathcal{G}(2,\mathcal{L})_s, \mathbb{Z})$$

$$T_{\mathcal{X}_s} \Big\downarrow \cong \quad \circlearrowleft \quad \cong \Big\downarrow T_{\mathcal{G}(2,\mathcal{L})_s} \qquad (10)$$

$$H^1(\mathcal{X}_s, \mathbb{Z}) \xrightarrow[\sim]{\widetilde{\Phi}_s} H^3(\mathcal{G}(2,\mathcal{L})_s, \mathbb{Z})$$

where $T_{\mathcal{X}_s}$ and $T_{\mathcal{G}(2,\mathcal{L})_s}$ are the monodromy transformations on $H^1(\mathcal{X}_s, \mathbb{Z})$ and $H^3(\mathcal{G}(2,\mathcal{L})_s, \mathbb{Z})$, respectively.

3.4 Limit Mumford-Newstead isomorphism

Let $\overline{\mathcal{H}}^1_{\mathcal{X}}$ and $\overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}$ be the canonical extensions of $\mathcal{H}^1_{\mathcal{X}_{\Delta^*}}$ and $\mathcal{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}$, respectively. Then, the morphism Φ_{Δ^*} extends to the entire disc (by canonicity of the extensions):

$$\widetilde{\Phi} : \overline{\mathcal{H}}^1_{\mathcal{X}} \xrightarrow{\sim} \overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}.$$

Using the identification (1) and restricting $\widetilde{\Phi}$ to the central fiber, we have an isomorphism:

$$\widetilde{\Phi}_0 : H^1(\mathcal{X}_\infty) \xrightarrow{\sim} H^3(\mathcal{G}(2,\mathcal{L})_\infty), \qquad (11)$$

where $\mathcal{X}_\infty$ (resp. $\mathcal{G}(2,\mathcal{L})_\infty$) is the base change of $\mathcal{X}$ (resp. $\mathcal{G}(2,\mathcal{L})$) to the upper half plane $\mathfrak{h}$ as described in Definition 2.3. Recall, $\widetilde{\Phi}_0$ is an isomorphism of mixed Hodge structures:

Theorem 3.1 *For the extended morphism $\widetilde{\Phi}$, we have $\widetilde{\Phi}(F^p \overline{\mathcal{H}}^1_{\mathcal{X}}) = F^{p+1}\overline{\mathcal{H}}^3_{\mathcal{G}(2,\mathcal{L})}$ for $p = 0, 1$ and $\widetilde{\Phi}(\overline{\mathbb{H}}^1_{\mathcal{X}}) = \overline{\mathbb{H}}^3_{\mathcal{G}(2,\mathcal{L})}$. Moreover, $\widetilde{\Phi}_0(W_i H^1(\mathcal{X}_\infty)) = W_{i+2}H^3(\mathcal{G}(2,\mathcal{L})_\infty)$ for all $i \geq 0$.*

Proof See [6, Proposition 4.1] for a proof of the statement. ☐

3.5 Leray-Hirsch cohomology decomposition

We now write the cohomology groups of $\mathcal{G}_1$ and $\mathcal{G}_0 \cap \mathcal{G}_1$ in terms of that of $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$, using the cohomology computations of the fibers of $\mathcal{G}_1$ and $\mathcal{G}_0 \cap \mathcal{G}_1$ over $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$.

Notation 3.2 Let ξ_0 be a generator of $H^2(\mathbb{P}^1)$, pr_i the natural projections from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ and $\xi_i := \mathrm{pr}_i^*(\xi_0)$. Using the Künneth decomposition, we have $H^{2i+1}(\mathbb{P}^1 \times \mathbb{P}^1) = 0$ for $i \geq 0$,

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q}, \ H^2(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q}\xi_1 \oplus \mathbb{Q}\xi_2 \text{ and } H^4(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q}\xi_1.\xi_2.$$

In other words, the cohomology ring $H^*(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q}[\xi_1, \xi_2]/(\xi_1^2, \xi_2^2)$. Recall, the cohomology ring $H^*(\mathbb{P}^3) = \mathbb{Q}[\xi'']/((\xi'')^4)$, where $\xi'' \in H^2(\mathbb{P}^3, \mathbb{Z})$ is a generator. Moreover, $\mathbb{P}^1 \times \mathbb{P}^1$ can be embedded (via Segre embedding) as a smooth quadric surface in $\mathbb{P}^3$ and under the induced pull-back morphism of (graded) cohomology rings:

$$\mathbb{Q}[\xi'']/((\xi'')^4) = H^*(\mathbb{P}^3) \to H^*(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q}[\xi_1, \xi_2]/(\xi_1^2, \xi_2^2),$$

ξ'' is mapped to $\xi_1 + \xi_2$. Denote by

$$i_0 : \mathcal{G}_0 \cap \mathcal{G}_1 \hookrightarrow \mathcal{G}_0 \text{ and } i_1 : \mathcal{G}_0 \cap \mathcal{G}_1 \hookrightarrow \mathcal{G}_1$$

the natural inclusions. Let $\rho_1 : \mathcal{G}_0 \cap \mathcal{G}_1 \longrightarrow M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$ and $\rho_2 : \mathcal{G}_1 \longrightarrow M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$ be the natural bundle morphisms. Note that the restriction morphism $r_j : H^{2j}(\mathbb{P}^3) \to H^{2j}(\mathbb{P}^1 \times \mathbb{P}^1)$ is an isomorphism (Lefschetz hyperplane section theorem), for any $j \neq \{1, 3\}$.

By the Deligne-Blanchard theorem [12] (the Leray spectral sequence degenerates at E_2 for smooth families), we have $H^i(\mathcal{G}_0 \cap \mathcal{G}_1) \cong \bigoplus_j H^{i-j}(R^j \rho_{1,*}\mathbb{Q})$ and $H^i(\mathcal{G}_1) \cong \bigoplus_j H^{i-j}(R^j \rho_{2,*}\mathbb{Q})$. Since $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$ is smooth and simply connected (as it is rationally connected [15, Proposition 2.3.7 and Remark 2.3.8] which implies simply connected [11, Corollary 4.29]), the local systems $R^j \rho_{1,*}\mathbb{Q}$ and $R^j \rho_{2,*}\mathbb{Q}$ are trivial. Therefore, for any $y \in M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$, the natural morphisms

$$H^i(\mathcal{G}_0 \cap \mathcal{G}_1) \twoheadrightarrow H^0(R^i \rho_{1,*}\mathbb{Q}) \to H^i((\mathcal{G}_0 \cap \mathcal{G}_1)_y) \text{ and } H^i(\mathcal{G}_1) \twoheadrightarrow H^0(R^i \rho_{2,*}\mathbb{Q}) \to H^i(\mathcal{G}_{1,y})$$

are surjective. By the Leray-Hirsch theorem (see [33, Theorem 7.33]), we then have:

$$
\begin{aligned}
H^i(\mathcal{G}_0 \cap \mathcal{G}_1) &\cong \bigoplus_{j \geq 0}(H^j((\mathcal{G}_0 \cap \mathcal{G}_1)_y) \otimes H^{i-j}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))) \\
&\cong H^i(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)) \oplus H^{i-2}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)) \otimes (\mathbb{Q}\xi_1 \oplus \mathbb{Q}\xi_2) \oplus H^{i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))\xi_1\xi_2,
\end{aligned}
\tag{12}
$$

$$H^i(\mathcal{G}_1) \cong \bigoplus_{j \geq 0}(H^j(\mathcal{G}_{1,y}) \otimes H^{i-j}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))) \cong \bigoplus_{j=0}^{3} H^{i-2j}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)) \otimes (\xi'')^j \tag{13}$$

Using this one can check that:

Proposition 3.3 *The following holds true*:

$$\ker((i_{0,*}, i_{1,*}) : H^{i-2}(\mathcal{G}_0 \cap \mathcal{G}_1) \to H^i(\mathcal{G}_0) \oplus H^i(\mathcal{G}_1)) \cong H^{i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi_1 \oplus -\xi_2).$$

Proof See [10, Theorem 4.2] for proof of the statement. □

Let $\mathcal{G}(2, \mathcal{L})_\infty$ denote the base change of the family $\mathcal{G}(2, \mathcal{L})$ (defined in §3.2) over Δ to the universal cover $\mathfrak{h}$ by the morphism $e : \mathfrak{h} \to \Delta^* \xrightarrow{j} \Delta$. Using the definitions and results from §2, we can equip $H^i(\mathcal{G}(2, \mathcal{L})_\infty)$ with a limit mixed Hodge structure and obtain a specialization morphism:

$$\mathrm{sp}_i : H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to H^i(\mathcal{G}(2, \mathcal{L})_\infty)$$

which is a morphism of mixed Hodge structures for all i. Replacing Y_1 (resp. Y_2) in §2 by $\mathcal{G}_0$ (resp. $\mathcal{G}_1$) from §3.2, we have by Proposition 3.3 that the kernel of the morphism f_i is isomorphic to $H^{i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(-2)$ i.e., we have an exact sequence of mixed Hodge structures (see Corollary 6):

$$
\begin{aligned}
0 &\to H^{i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(-2) \xrightarrow{h_i} H^{i-2}(\mathcal{G}_0 \cap \mathcal{G}_1)(-1) \xrightarrow{f_i} H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \xrightarrow{\mathrm{sp}_i} \\
&\xrightarrow{\mathrm{sp}_i} H^i(\mathcal{G}(2, \mathcal{L})_\infty) \xrightarrow{g_i} \mathrm{Gr}_{i+1}^W H^i(\mathcal{G}(2, \mathcal{L})_\infty) \to 0.
\end{aligned}
\tag{14}
$$

4 Limit Hodge conjecture for relative Gieseker moduli space

Recall, that for a very general smooth, projective curve Y, $H^{2p}_{\text{Hdg}}(\text{Jac}(Y)) \cong \mathbb{Q}$ for every p (see [8, Theorem 17.5.1]). It is natural to ask, given a degenerating family of smooth, projective curves, is the sub-space of p-th Hodge classes in the limit mixed Hodge structure associated with the corresponding family of Jacobians, one dimensional for each p? Apriori, this is false as can be seen from the theory of Noether-Lefschetz loci i.e., very generally, the rank of the Hodge lattice jumps. However, in this section, we prove that under the genericity assumption of the nodal curve X_0, we have a positive answer (see Theorem 4.3). This result will later play an important role in the proof of Theorem 5.2.

4.1 Hodge conjecture in the smooth case

We now recall from [5] the proof of the Hodge conjecture for the moduli space of rank 2 vector bundles with fixed determinants on a very general, smooth, projective curve. We give a brief sketch for the sake of completion.

Let C be a smooth, projective curve and $\mathcal{L}$ an invertible sheaf on C of odd degree. Denote by $M_C(2, \mathcal{L})$ the moduli of semi-stable, rank 2 sheaves on C with determinant $\mathcal{L}$. By [20, Proposition 1], there exists an isomorphism of pure Hodge structures:

$$\Phi : H^1(C, \mathbb{Z})(-1) \to H^3(M_C(2, \mathcal{L}), \mathbb{Z}).$$

By [24, Theorem 1], there exists $\alpha \in H^2(M_C(2, \mathcal{L}), \mathbb{Z})$ and $\beta \in H^4(M_C(2, \mathcal{L}), \mathbb{Z})$ such that the cohomology ring $H^*(M_C(2, \mathcal{L}))$ is generated by α, β and $H^3(M_C(2, \mathcal{L}))$. Now, there is a natural isomorphism of pure Hodge structures

$$H^1(\text{Jac}(C), \mathbb{Z}) \xrightarrow{\sim} H^1(C, \mathbb{Z})^\vee \xrightarrow{\sim} H^1(C, \mathbb{Z}) \tag{15}$$

where the first isomorphism comes from the cohomology of Jacobians (see [33, p. 169]) and the second isomorphism is the Poincaré duality. Combining with the isomorphism Φ above, we get an isomorphism of pure Hodge structures:

$$H^1(\text{Jac}(C), \mathbb{Z})(-1) \to H^1(C, \mathbb{Z})(-1) \xrightarrow{\Phi} H^3(M_C(2, \mathcal{L}), \mathbb{Z}).$$

Since cup-product is a morphism of pure Hodge structures, this isomorphism induces a surjective (graded) ring homomorphism which is a morphism of pure Hodge structures on each graded piece (α and β are given weights 2 and 4, respectively):

$$\delta_C : \mathbb{Q}[\alpha, \beta] \otimes (\oplus_i H^i(\text{Jac}(C))(-i)) \to H^*(M_C(2, \mathcal{L})) \tag{16}$$

(see [5, (1.2)]). As α and β are linear combination of Chern classes of the universal bundle associated to $M_C(2, \mathcal{L})$, they are algebraic cohomology classes (see [24, p. 338]). Then, by [5, Theorem 2] the isomorphism δ_C induces a surjective ring homomorphism:

$$\delta_C : \mathbb{Q}[\alpha, \beta] \otimes H^*_A(\text{Jac}(C)) \to H^*_A(M_C(2, \mathcal{L}))$$

where $H^{2i}_A(Y)$ denotes the vector sub-space of *algebraic* cohomology classes in $H^{2i}(Y)$ and $H^*_A(Y) = \oplus_i H^{2i}_A(Y)$. Recall by [8, Theorem 17.5.1], for a very general curve C, $H^*_{\text{Hdg}}(\text{Jac}(C))$ is generated (as a $\mathbb{Q}$-vector space) by the theta divisor. Hence, using δ_C we have

$$H^*_{\text{Hdg}}(\text{Jac}(C)) = H^*_A(\text{Jac}(C)) \text{ and } H^*_{\text{Hdg}}(M_C(2, \mathcal{L})) = H^*_A(M_C(2, \mathcal{L})). \tag{17}$$

4.2 Limit mixed Hodge structures for families of Jacobians

Keep notations §3.1. Recall, the degenerating family

$$\pi_1 : \mathcal{X} \to \Delta$$

of curves which is smooth over the punctured disc Δ^* and the central fiber is the irreducible nodal curve X_0. There exists a family of Jacobians of curves associated to the family π_1, i.e.

$$\pi_4 : \mathbf{J}_{\mathcal{X}_{\Delta^*}} \to \Delta^*$$

is a smooth, projective morphism such that for every $s \in \Delta^*$, the fiber $\pi_4^{-1}(s)$ is the Jacobian $\text{Jac}(\mathcal{X}_s)$ of the curve $\mathcal{X}_s := \pi_1^{-1}(s)$. By (15), there is a natural isomorphism of pure Hodge structures

$$\delta_s : H^1(\text{Jac}(\mathcal{X}_s), \mathbb{Z}) \to H^1(\mathcal{X}_s, \mathbb{Z}).$$

This induces an isomorphism of local systems

$$\mathbb{H}^1_{\mathbf{J}_{\Delta^*}} \xrightarrow{\sim} \mathbb{H}^1_{\mathcal{X}_{\Delta^*}} \tag{18}$$

which induces an isomorphism of Hodge bundles from $\mathcal{H}^1_{\mathbf{J}_{\Delta^*}}$ to $\mathcal{H}^1_{\mathcal{X}_{\Delta^*}}$ i.e., preserves the Hodge filtration F^p (see §2 for notations). Composing the above isomorphism with Φ_{Δ^*} from §3.3, we get an isomorphism of local systems

$$\delta_{\Delta^*} : \mathbb{H}^1_{\mathbf{J}_{\Delta^*}} \xrightarrow[\sim]{(4.4)} \mathbb{H}^1_{\mathcal{X}_{\Delta^*}} \xrightarrow[\sim]{\Phi_{\Delta^*}} \mathbb{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}.$$

which induces an isomorphism of Hodge bundles from $\mathcal{H}^1_{\mathbf{J}_{\Delta^*}}$ to $\mathcal{H}^3_{\mathcal{G}(2,\mathcal{L})_{\Delta^*}}$. Since the monodromy action on isomorphic local systems is the same, the weight filtrations associated to the two limit mixed Hodge structures are the same. In particular, we have

Theorem 4.1 *The morphism δ_{Δ^*} induces an isomorphism of mixed Hodge structures*:

$$\delta_\infty : H^1(\mathbf{J}_\infty)(-1) \xrightarrow{\sim} H^3(\mathcal{G}(2, \mathcal{L})_\infty)$$

where $\mathbf{J}_\infty$ (resp. $\mathcal{G}(2, \mathcal{L})_\infty$) is the base change of $\mathbf{J}_{\Delta^*}$ (resp. $\mathcal{G}(2, \mathcal{L})_{\Delta^*}$) to the upper half plane $\mathfrak{h}$ (see Definition 2.3 for notation) and the mixed Hodge structures on $H^1(\mathbf{J}_\infty)$ and $H^3(\mathcal{G}(2, \mathcal{L})_\infty)$ are as defined in §2.

Proof The proof is identical to the proof of Theorem 3.1 (or [6, Proposition 4.1]) after replacing $\mathcal{X}$ in the statement with $\mathbf{J}$. □

Remark 4.2 Since cup-product is a morphism of mixed Hodge structures (see [26, Corollary 5.45]), we have the following morphism of mixed Hodge structures for each $i \geq 0$:

$$\Phi_\infty^{(i)} : H^i(\mathbf{J}_\infty)(-i) = \bigwedge^i H^1(\mathbf{J}_\infty)(-1) \xrightarrow{\delta_\infty} \bigwedge^i H^3(\mathcal{G}(2, \mathcal{L})_\infty) \xrightarrow{\cup} H^{3i}(\mathcal{G}(2, \mathcal{L})_\infty),$$

where ∪ denotes the cup-product morphism. Choose a point $s' \in \mathfrak{h}$ such that $e(s') = s$. Since $\mathfrak{h}$ is contractible, by the Ehresmann's lemma, there is a natural isomorphism (induced by the inclusion of the fiber over s' into $\mathcal{G}(2, \mathcal{L})_\infty$):

$$\phi_s^i : H^i(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Z}) \to H^i(M_{\mathcal{X}_s}(2, \mathcal{L}_s), \mathbb{Z}) \tag{19}$$

commuting with the cup-product (cup-product commutes with pull-back by continuous maps)

i.e., $\phi_s^i(\xi_1) \cup \phi_s^j(\xi_2) = \phi_s^{i+j}(\xi_1 \cup \xi_2)$ for $\xi_1 \in H^i(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Z})$ and $\xi_2 \in H^j(\mathcal{G}(2, \mathcal{L})_\infty, \mathbb{Z})$.

Denote by $\alpha_\infty := (\phi_s^2)^{-1}(\alpha_s)$ and $\beta_\infty := (\phi_s^4)^{-1}(\beta_s)$, where $\alpha_s \in H^2(M_{\mathcal{X}_s}(2, \mathcal{L}_s), \mathbb{Z})$ and $\beta_s \in H^4(M_{\mathcal{X}_s}(2, \mathcal{L}_s), \mathbb{Z})$ arise as linear combination of the Chern classes of a universal bundle $\mathcal{U}_s$ for $M_{\mathcal{X}_s}(2, \mathcal{L}_s)$ (see [24, p. 337] for an explicit description). As $\mathcal{U}_s$ is the restriction to the fiber over s of the universal bundle $\mathcal{U}$ as in §3.3 for $\mathcal{G}(2, \mathcal{L})_{\Delta^*}$, the classes α_s and β_s are monodromy invariant. Hence, α_∞ and β_∞ does not depend on the choice of s. Then the morphism $\Phi_\infty^{(i)}$ induces a (graded) ring homomorphism

$$\Phi_\infty : \mathbb{Q}[\alpha_\infty, \beta_\infty] \otimes \left(\bigoplus_i H^i(\mathbf{J}_\infty)(-i) \right) \xrightarrow{\mathrm{id} \times \oplus \Phi_\infty^{(i)}} H^*(\mathcal{G}(2, \mathcal{L})_\infty)$$

which induces a morphism of mixed Hodge structures on each graded piece (α_∞ and β_∞ are considered as Hodge classes of weight 2 and 4, respectively). Since $H^*(M_{\mathcal{X}_s}(2, \mathcal{L}_s))$ is generated by α_s, β_s and $H^3(M_{\mathcal{X}_s}(2, \mathcal{L}_s))$ ([24, Theorem 1]) and ϕ_s^i commutes with cup-product, the cohomology ring $H^*(\mathcal{G}(2, \mathcal{L})_\infty)$ is generated by $\alpha_\infty, \beta_\infty$ and $H^3(\mathcal{G}(2, \mathcal{L})_\infty)$. Since the morphism δ_∞ is an isomorphism, this implies that the morphism Φ_∞ above is surjective.

4.3 Compactified Jacobians of curves

Let X_0 be an irreducible nodal curve. The *compactified Jacobian* $\overline{J}_{X_0}$ parametrizes rank 1, degree 0 torsion-free sheaves on X_0 (see [18, Theorem 3.3]). The compactified Jacobian is a semi-normal, projective variety. In particular, the normalization map

$$h : \widetilde{J}_{X_0} \to \overline{J}_{X_0}$$

is the desingularization of $\overline{J}_{X_0}$ i.e., $\widetilde{J}_{X_0}$ is non-singular and the pre-image under h of the singular locus of $\overline{J}_{X_0}$ consists of two disjoint non-singular divisors, say D_1 and D_2 in $\widetilde{J}_{X_0}$ which are both isomorphic to the Jacobian $\mathrm{Jac}(\widetilde{X}_0)$, of the normalization $\widetilde{X}_0$ of the curve X_0. Moreover, $\widetilde{J}_{X_0}$ is a $\mathbb{P}^1$-bundle over $\mathrm{Jac}(\widetilde{X}_0)$. See [23, p. 567] for a nice description.

One can also consider the compactified Jacobian $\overline{J}_{X_0}$ as a degeneration of Jacobians of smooth curves. In particular, given a family π_1 of smooth curves degenerating to X_0 as above, there exists a flat, projective family (see [18, Theorem 3.3])

$$\pi_5 : \overline{\mathbf{J}} \to \Delta$$

such that for any $s \in \Delta^*$, $\pi_5^{-1}(s) \cong \mathrm{Jac}(\mathcal{X}_s)$ and the central fiber $\pi_5^{-1}(0) = \overline{J}_{X_0}$. Blowing up the family $\overline{\mathbf{J}}$ along the singular locus of the central fiber $\overline{J}_{X_0}$ followed by an etale base change, we get a flat projective family (this is the *semi-stable reduction* of π_5, see [30, Example (2.15)])

$$\widetilde{\pi}_5 : \overline{\mathbf{J}}' \to \Delta$$

such that the fibers over Δ^* coincides with that of π_5 above and the central fiber $\widetilde{\pi}_5^{-1}(0)$ is a reduced simple normal crossings divisor consisting of two irreducible components $\widetilde{J}_{X_0}$ and E, where E is a $\mathbb{P}^1$-bundle over $\mathrm{Jac}(\widetilde{X}_0)$ and $E \cap \widetilde{J}_{X_0} = D_1 \cup D_2$.

4.4 Limit Hodge conjecture

Denote by $H_{\mathrm{Hdg}}^{2p}(\mathbf{J}_\infty) := H^{p,p}\mathrm{Gr}_{2p}^W H^{2p}(\mathbf{J}_\infty)$. Using the description of compactified Jacobians in families given above we prove:

Theorem 4.3 *For a very general nodal curve X_0 as defined in the Introduction, we have* $H_{\mathrm{Hdg}}^{2p}(\mathbf{J}_\infty) \cong \mathbb{Q}$ *for each p.*

Proof For the proof of this statement we use the exact sequence (6). Notations as in §4.3. Applying Corollary 2.7 to $\widetilde{\pi}_5$ above, we get the exact sequence of pure Hodge structures:

$$H^{2p-2}(D_1 \cup D_2)(-1) \xrightarrow{f_{2p}} \mathrm{Gr}_{2p}^W H^{2p}(\widetilde{J}_{X_0} \cup E) \xrightarrow{\mathrm{sp}_{2p}} \mathrm{Gr}_{2p}^W H^{2p}(\mathbf{J}_\infty) \to 0. \tag{20}$$

To compute the middle term in this exact sequence we use the Mayer-Vietoris sequence. In particular, we have an exact sequence of pure Hodge structures:

$$0 \to \mathrm{Gr}_{2p}^W H^{2p}(\widetilde{J}_{X_0} \cup E) \to H^{2p}(\widetilde{J}_{X_0}) \oplus H^{2p}(E) \to H^{2p}(D_1 \cup D_2). \tag{21}$$

Recall, E and $\widetilde{J}_{X_0}$ are $\mathbb{P}^1$-bundles over $\mathrm{Jac}(\widetilde{X}_0)$. Denote by

$$q_E : E \to \mathrm{Jac}(\widetilde{X}_0) \text{ and } q : \widetilde{J}_{X_0} \to \mathrm{Jac}(\widetilde{X}_0)$$

the natural bundle morphisms. Let $\xi_0 \in H^2(\mathbb{P}^1)$ be a generator. By the Leray-Hirsch theorem (see [33, Theorem 7.33]), we have for any closed point $y \in \mathrm{Jac}(\widetilde{X}_0)$ and $p \geq 0$:

$$H^{2p}(E) \cong \bigoplus_{j \geq 0} \left(H^j(E_y) \otimes q_E^* H^{2p-j}\left(\mathrm{Jac}(\widetilde{X}_0) \right) \right) \cong q_E^* H^{2p}\left(\mathrm{Jac}(\widetilde{X}_0) \right) \oplus q_E^* H^{2p-2}\left(\mathrm{Jac}(\widetilde{X}_0) \right) \xi_0 \tag{22}$$

$$H^{2p}(\widetilde{J}_{X_0}) \cong \bigoplus_{j \geq 0} \left(H^j(\widetilde{J}_{X_0,y}) \otimes q^* H^{2p-j}\left(\mathrm{Jac}(\widetilde{X}_0) \right) \right) \cong q^* H^{2p}\left(\mathrm{Jac}(\widetilde{X}_0) \right) \oplus q^* H^{2p-2}\left(\mathrm{Jac}(\widetilde{X}_0) \right) \xi_0 \tag{23}$$

Denote by $q_i : D_i \hookrightarrow \widetilde{J}_{X_0}$ and $q_{E,i} : D_i \hookrightarrow E$ the closed immersions. As $q \circ q_i$ and $q_E \circ q_{E,i}$ are isomorphisms, q_i^* (resp. $q_{E,i}^*$) maps $q^* H^{2p}\left(\mathrm{Jac}(\widetilde{X}_0) \right)$ (resp. $q_E^* H^{2p}\left(\mathrm{Jac}(\widetilde{X}_0) \right)$) isomorphically to $H^{2p}(D_i)$. Using the decompositions (22) and (23), we then conclude that the last (restriction) morphism of (21) is surjective. Since (by assumption) X_0 is very general, we have $H_{\mathrm{Hdg}}^{2p}\left(\mathrm{Jac}(\widetilde{X}_0) \right) \cong \mathbb{Q}$ for every $p \geq 0$. Hence, using the decompositions (22) and (23) we have

$$\dim H_{\mathrm{Hdg}}^{2p-2}(D_1 \cup D_2) = \dim H_{\mathrm{Hdg}}^{2p}(D_1 \cup D_2) = 2 = \dim H_{\mathrm{Hdg}}^{2p}(\widetilde{J}_{X_0}) = \dim H_{\mathrm{Hdg}}^{2p}(E).$$

Then the surjectivity on the right of (21) implies that $\dim H^{p,p}\mathrm{Gr}_{2p}^W H^{2p}(\widetilde{J}_{X_0} \cup E) = 2$. Using the exact sequence (20) it therefore suffices to check that the kernel of f_{2p} when restricted to $H_{\mathrm{Hdg}}^{2p-2}(D_1 \cup D_2)(-1)$ is one dimensional. Using the projection formula (see [26, Lemma B.26]) and the identification described in the Leray-Hirsch theorem (identifying certain cohomology classes with their restriction to y) we have for any $\gamma \in H^{2p-2}(\mathrm{Jac}(\widetilde{X}_0))$, that

$$q_{i,*} \circ q_i^* \circ q^*(\gamma) = q^*(\gamma)\xi_0 \text{ and } q_{E,i,*} \circ q_{E,i}^* \circ q_E^*(\gamma) = q_E^*(\gamma)\xi_0.$$

Taking $\gamma = \theta^{p-1}$, where $\theta \in H^2(\mathrm{Jac}(\widetilde{X}_0))$ is the class of the theta divisor, we then observe that the kernel of

$$f_{2p} : H_{\mathrm{Hdg}}^{2p-2}(D_1 \cup D_2)(-1) \to H^{p,p}\mathrm{Gr}_{2p}^W H^{2p}(\widetilde{J}_{X_0} \cup E)$$

is one dimensional, generated by (a scalar multiple of) the pull-back of θ^{p-1} (recall from §2 that f_{2p} is induced by the Gysin morphism). This proves the theorem. $\qquad\square$

Notation 4.4 Let ϕ_s^i as in (19). Denote by $H_{\mathrm{Hdg}}^{2p}(\mathcal{G}(2,\mathcal{L})_\infty) := H^{p,p}\mathrm{Gr}_{2p}^W H^{2p}(\mathcal{G}(2,\mathcal{L})_\infty)$ and

$$H_A^{2p}(\mathcal{G}(2,\mathcal{L})_\infty) = \left\{ \begin{array}{l} \sigma \in H^{2p}(\mathcal{G}(2,\mathcal{L})_\infty) \text{ such that for a very general } s \in \Delta^* \text{ and the} \\ \text{natural isomorphism } \phi_s^{2p} : H^{2p}(\mathcal{G}(2,\mathcal{L})_\infty) \to H^{2p}(\mathcal{G}(2,\mathcal{L})_s), \\ \phi_s^{2p}(\sigma) \in H_A^{2p}(\mathcal{G}(2,\mathcal{L})_s). \end{array} \right\}$$

Corollary 4.5 *Let X_0 be as in Theorem* 4.3. *Let θ_∞ be a generator of $H_{\mathrm{Hdg}}^2(\mathbf{J}_\infty)$ (see Theorem* 4.3). *Then, the graded ring $H_{\mathrm{Hdg}}^*(\mathcal{G}(2,\mathcal{L})_\infty) := \oplus_p H_{\mathrm{Hdg}}^{2p}(\mathcal{G}(2,\mathcal{L})_\infty)$ is generated (as a ring over $\mathbb{Q}$) by $\alpha_\infty, \beta_\infty$ and (the image of) θ_∞. In particular, $H_A^{2p}(\mathcal{G}(2,\mathcal{L})_\infty) = H_{\mathrm{Hdg}}^{2p}(\mathcal{G}(2,\mathcal{L})_\infty)$*
.

The above corollary is what we call the "limit" Hodge conjecture in the family of Gieseker moduli spaces. Also note that $H_{\mathrm{Hdg}}^*(\mathcal{G}(2,\mathcal{L})_\infty)$ is not a polynomial ring in 3 variables i.e., there are relations between $\alpha_\infty, \beta_\infty$ and θ_∞ (see [16, p. 408]).

Proof of Corollary 4.5 Recall from Remark 4.2 that $\Phi_\infty^{(i)}$ is a morphism of mixed Hodge structures and Φ_∞, which is the direct sum over all the graded pieces, is surjective. This implies that we have a surjective morphism induced by Φ_∞:

$$\Phi_\infty : \mathbb{Q}[\alpha_\infty, \beta_\infty] \otimes_{\mathbb{Q}} H_{\mathrm{Hdg}}^*(\mathbf{J}_\infty) \twoheadrightarrow H_{\mathrm{Hdg}}^*(\mathcal{G}(2,\mathcal{L})_\infty), \text{ where } H_{\mathrm{Hdg}}^*(\mathbf{J}_\infty) := \bigoplus_{p \geq 0} H_{\mathrm{Hdg}}^{2p}(\mathbf{J}_\infty).$$

By Theorem 4.3, $H_{\mathrm{Hdg}}^*(\mathbf{J}_\infty)$ is generated by θ_∞. Hence, $H_{\mathrm{Hdg}}^*(\mathcal{G}(2,\mathcal{L})_\infty)$ is generated by $\alpha_\infty, \beta_\infty$ and (the image of) θ_∞. This proves the first part of the corollary.

Observe that for a very general $s \in \Delta^*$, $H_A^*(\mathcal{G}(2,\mathcal{L})_s) = H_A^*(M_{\mathcal{X}_s}(2,\mathcal{L}_s))$ is generated as a ring over $\mathbb{Q}$, by α_s, β_s and (the image of) a generator θ_s of $H_A^2(\mathrm{Jac}(\mathcal{X}_s))$ (see §4.1). Since $\alpha_\infty, \beta_\infty$ and (the image of) θ_∞ maps to α_s, β_s and (the image of) θ_s, respectively (upto multiplication by a scalar) by the morphism ϕ_s^i as in (19) for $i = 2, 4, 6$, we conclude that $H_A^*(\mathcal{G}(2,\mathcal{L})_\infty)$ is generated by $\alpha_\infty, \beta_\infty$ and (the image of) θ_∞. Using the first part, we conclude $H_A^*(\mathcal{G}(2,\mathcal{L})_\infty) = H_{\mathrm{Hdg}}^*(\mathcal{G}(2,\mathcal{L})_\infty)$ as graded rings over $\mathbb{Q}$. This proves the corollary.

$\qquad\square$

5 Main results

Assume that X_0 is a very general, irreducible nodal curve as defined in the Introduction. Note that, by the genericity of X_0, we can choose the family of curves π_1 as in §3.1 such that for a very general $s \in \Delta^*$, we have $H^*_{\mathrm{Hdg}}(\mathrm{Jac}(\mathcal{X}_s))$ is generated by the cohomology class of the theta divisor $\theta_s \in H^2_{\mathrm{Hdg}}(M_{\mathcal{X}_s}(2, \mathcal{L}_s))$. We fix such a family π_1, for this section. Let $j_0 : \mathcal{G}_0 \hookrightarrow \mathcal{G}_{X_0}(2, \mathcal{L}_0)$ and $j_1 : \mathcal{G}_1 \hookrightarrow \mathcal{G}_{X_0}(2, \mathcal{L}_0)$ be the natural inclusions. Define the *algebraic cohomology groups* on $\mathcal{G}_{X_0}(2, \mathcal{L}_0)$ as:

$$H^{2i}_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) = \left\{ \sigma \in \mathrm{Gr}^W_{2i} H^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \text{ such that } j_t^*(\sigma) \in H^{2i}_A(\mathcal{G}_t), \ t = 0, 1 \right\}$$

We first obtain an exact sequence analogous to (14) for algebraic classes.

Lemma 5.1 *For any $i \geq 0$, the algebraic cohomology group $H^{2i}_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$ sits in the following exact sequence*:

$$0 \to H^{2i-4}_A\left(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)) \xrightarrow{h_{2i}} H^{2i-2}_A(\mathcal{G}_0 \cap \mathcal{G}_1) \xrightarrow{f_{2i}} H^{2i}_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0) \right) \xrightarrow{\mathrm{sp}_{2i}} H^{2i}_A(\mathcal{G}(2, \mathcal{L})_\infty) \to 0.$$

(24)

Proof The exact sequence (24) will be induced by the exact sequence (14). We first claim that the restriction of the specialization morphism sp_{2i} to the algebraic cohomology group $H^{2i}_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$ factors through $H^{2i}_A(\mathcal{G}(2, \mathcal{L})_\infty)$. Let $\gamma \in H^{2i}_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$. By definition, for $t = 0, 1$ we have $j_t^*(\gamma) \in H^{2i}_A(\mathcal{G}_t)$. In particular, $j_t^*(\gamma) \in H^{i,i}(\mathcal{G}_t)$ for $t = 0, 1$. By the Mayer-Vietoris sequence (4), this implies $\gamma \in H^{i,i}\mathrm{Gr}^W_{2i} H^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0), \mathbb{C})$. As sp_{2i} is a morphism of mixed Hodge structures, this implies $\mathrm{sp}_{2i}(\gamma) \in H^{2i}_{\mathrm{Hdg}}(\mathcal{G}(2, \mathcal{L})_\infty)$ (see Notation 4.4). By Corollary 4.5, $H^{2i}_A(\mathcal{G}(2, \mathcal{L})_\infty) = H^{2i}_{\mathrm{Hdg}}(\mathcal{G}(2, \mathcal{L})_\infty)$. Hence, $\mathrm{sp}_{2i}(\gamma) \in H^{2i}_A(\mathcal{G}(2, \mathcal{L})_\infty)$. This proves our claim.

Next, we see that restriction of the specialization morphism to algebraic classes is surjective. Let $\alpha_s \in H^2(M_{\mathcal{X}_s}(2, \mathcal{L}_s))$ and $\beta_s \in H^4(M_{\mathcal{X}_s}(2, \mathcal{L}_s))$ be as in Remark 4.2 and $\theta_s \in H^6(M_{\mathcal{X}_s}(2, \mathcal{L}_s))$ be as in the proof of Corollary 4.5 (same as $\phi_s^6(\theta_\infty)$). Note that, α_s, β_s and θ_s arise as restriction to the fiber over s, of linear combinations of the Chern classes of the universal bundle $\mathcal{U}$ on $\mathcal{G}(2, \mathcal{L})_{\Delta^*}$ (see Remark 4.2). Hence, α_s, β_s and θ_s are monodromy invariant (they are restrictions of global sections of local systems). By taking closure (over the unit disc) of the Chern classes of $\mathcal{U}$, the classes α_s, β_s and θ_s extend as algebraic classes to the central fiber $\mathcal{G}_{X_0}(2, \mathcal{L}_0)$. This implies that there exists $\alpha_0 \in H^2_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0)), \beta_0 \in H^4_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$ and $\theta_0 \in H^6_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$ such that the specialization morphism maps α_0 (resp. β_0, θ_0) to α_∞ (resp. $\beta_\infty, \theta_\infty$) in $H^*_A(\mathcal{G}(2, \mathcal{L})_\infty)$. By Corollary 4.5, $H^*_A(\mathcal{G}(2, \mathcal{L})_\infty)$ is generated by $\alpha_\infty, \beta_\infty$ and θ_∞. Hence,

$$\mathrm{sp}_{2i} : H^{2i}_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to H^{2i}_A(\mathcal{G}(2, \mathcal{L})_\infty)$$

is surjective. By the exactness of (14), $\ker(\mathrm{sp}_{2i}) \cap H^{2i}_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$ consists of classes

$$\sigma \in H^{2i}_{\mathrm{Hdg}}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \text{ such that } j_t^*\sigma \in H^{2i}_A(\mathcal{G}_t) \text{ and}$$

there exists $\gamma \in H^{2i-2}(\mathcal{G}_0 \cap \mathcal{G}_1)$ such that $i_{t,*}(\gamma) = j_t^*(\sigma)$, where $i_t : \mathcal{G}_0 \cap \mathcal{G}_1 \hookrightarrow \mathcal{G}_t$ for $t = 0, 1$ is the natural inclusion. Since $i_{t,*}$ is a Gysin morphism (of pure Hodge structures) from $H^{2i-2}(\mathcal{G}_0 \cap \mathcal{G}_1)(-1)$ to $H^{2i}(\mathcal{G}_t)$, γ is a Hodge class (as $j_t^*\sigma$ is a Hodge class). Moreover, by

assumption $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$ satisfies the Hodge conjecture and so does $\mathcal{G}_0 \cap \mathcal{G}_1$ as it is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$. In particular, γ is an algebraic class i.e., $\gamma \in H_A^{2i-2}(\mathcal{G}_0 \cap \mathcal{G}_1)$.

Similarly, as h_{2i} from $H^{2i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(-2)$ to $H^{2i-2}(\mathcal{G}_0 \cap \mathcal{G}_1)(-1)$ is a morphism of pure Hodge structures, the exactness of (14) implies that $\ker(f_{2i}) \cap H_A^{2i-2}(\mathcal{G}_0 \cap \mathcal{G}_1)$ equals the image of the morphism:

$$H_A^{2i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)) = H_{\mathrm{Hdg}}^{2i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(-2) \xrightarrow{h_{2i}} H_{\mathrm{Hdg}}^{2i-2}(\mathcal{G}_0 \cap \mathcal{G}_1)(-1) = H_A^{2i-2}(\mathcal{G}_0 \cap \mathcal{G}_1).$$

The injectivity of h_{2i} follows from the exactness of (14). This proves the lemma. $\qquad\square$

Theorem 5.2 *For the central fibre $\mathcal{G}_{X_0}(2, \mathcal{L}_0)$ of the relative Gieseker's moduli space we have,*

$$H_{\mathrm{Hdg}}^*(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) = H_A^*(\mathcal{G}_{X_0}(2, \mathcal{L}_0)).$$

Moreover, the restriction morphism $j_0^* : H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to H^i(\mathcal{G}_0)$ is surjective and the irreducible components $\mathcal{G}_0$ and $\mathcal{G}_1$ satisfy the Hodge conjecture.

In particular, the Hodge conjecture holds for a desingularization of $U_{X_0}(2, \mathcal{L}_0)$.

Proof Since $\mathrm{Gr}_{2i+1}^W H^{2i}(\mathcal{G}(2, \mathcal{L})_\infty)$ is pure of weight $2i + 1$ and the morphisms in (14) are morphisms of mixed Hodge structure, we have the following exact sequence:

$$0 \to H_{\mathrm{Hdg}}^{2i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)) \to H_{\mathrm{Hdg}}^{2i-2}(\mathcal{G}_0 \cap \mathcal{G}_1) \xrightarrow{f_{2i}} H_{\mathrm{Hdg}}^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \xrightarrow{\mathrm{sp}_{2i}} H_{\mathrm{Hdg}}^{2i}(\mathcal{G}(2, \mathcal{L})_\infty) \to 0.$$
$$(25)$$

We observed in the proof of Lemma 5.1 that using the Mayer-Vietoris sequence (4), there is a natural inclusion from $H_A^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$ to $H_{\mathrm{Hdg}}^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$. By Corollary 4.5, we have $H_A^{2i}(\mathcal{G}(2, \mathcal{L})_\infty) = H_{\mathrm{Hdg}}^{2i}(\mathcal{G}(2, \mathcal{L})_\infty)$. Then combining (25) with (24) we have the following diagram of short exact sequences:

To show that the middle arrow in the above diagram is an isomorphism, it suffices to prove that the first vertical arrow is an isomorphism. This follows from comparing exact sequences (24) and (25) and using that Hodge classes are algebraic both for $M_{\widetilde{X}}(2, \widetilde{\mathcal{L}}_0)$ by the discussion in §4.1 and for $\mathcal{G}_0 \cap \mathcal{G}_1$ which is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)$. Hence, $H_{\mathrm{Hdg}}^*(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) = H_A^*(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$. This completes the proof of the first part of the theorem.

We now show that the restriction morphism $j_0^* : H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to H^i(\mathcal{G}_0)$ is surjective. Consider the following exact sequences of pure Hodge structures ([26, Proposition 5.46]):

$$0 \to \mathrm{Gr}_i^W H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0), \mathcal{G}_0) \to \mathrm{Gr}_i^W H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to H^i(\mathcal{G}_0)$$
$$\to \mathrm{Gr}_i^W H^{i+1}(\mathcal{G}_{X_0}(2, \mathcal{L}_0), \mathcal{G}_0) \to \mathrm{Gr}_i^W H^{i+1}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to 0,$$
$$(26)$$

$$0 \to \mathrm{Gr}_i^W H^i(\mathcal{G}_1, \mathcal{G}_0 \cap \mathcal{G}_1) \to H^i(\mathcal{G}_1) \xrightarrow{i_1^*} H^i(\mathcal{G}_0 \cap \mathcal{G}_1) \to \mathrm{Gr}_i^W H^{i+1}(\mathcal{G}_1, \mathcal{G}_0 \cap \mathcal{G}_1) \to 0.$$
(27)

Note that r_1 (resp. r_3) is injective (resp. surjective) with cokernel (resp. kernel) isomorphic to $\mathbb{Q}(\xi_1 - \xi_2)$ (resp. $\mathbb{Q}(\xi'')^3$), where notations as in §3.5. Using (12) and (13), this implies that the restriction morphism

$$\ker(i_1^*) \cong H^{i-6}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi'')^3 \text{ and } \mathrm{coker}(i_1^*) \cong H^{i-2}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi_1 - \xi_2).$$

The exact sequence (27) then implies,

$$\mathrm{Gr}_i^W H^i(\mathcal{G}_1, \mathcal{G}_0 \cap \mathcal{G}_1) \cong H^{i-6}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi'')^3 \text{ and}$$
$$\mathrm{Gr}_i^W H^{i+1}(\mathcal{G}_1, \mathcal{G}_0 \cap \mathcal{G}_1) \cong H^{i-2}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi_1 - \xi_2).$$

By [26, Example B.5(2)], the pair $\mathcal{G}_0$ and $\mathcal{G}_1$ form an excisive couple i.e., the induced excision map $j_1^* : H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0), \mathcal{G}_0) \to H^i(\mathcal{G}_1, \mathcal{G}_0 \cap \mathcal{G}_1)$ is an isomorphism of mixed Hodge structures. Then, the exact sequence (26) becomes

$$0 \to H^{i-6}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi'')^3 \to \mathrm{Gr}_i^W H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to H^i(\mathcal{G}_0)$$
$$\to H^{i-2}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi_1 - \xi_2) \to \mathrm{Gr}_i^W H^{i+1}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to 0.$$
(28)

Let $N := \dim \mathcal{G}_{X_0}(2, \mathcal{L}_0)$. Using [29, Example 3.5] and Poincaré duality, we have that

$$\mathrm{Gr}_i^W H^{i+1}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \cong \mathrm{coker}((i_0^* - i_1^*) : H^i(\mathcal{G}_0) \oplus H^i(\mathcal{G}_1) \to H^i(\mathcal{G}_0 \cap \mathcal{G}_1))$$
$$\cong \ker((i_{0,*}, i_{1,*}) : H^{2N-i-2}(\mathcal{G}_0 \cap \mathcal{G}_1) \to H^{2N-i}(\mathcal{G}_0) \oplus H^{2N-i}(\mathcal{G}_1))^\vee.$$

Note that, $\dim M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)) = N - 3$. By Propositions 3.3 combined with Poincaré duality, we then conclude that

$$\mathrm{Gr}_i^W H^{i+1}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \cong (H^{2N-i-4}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi_1 \oplus -\xi_2))^\vee \cong H^{i-2}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)).$$

Therefore, the exact sequence (28) becomes the following short exact sequence (since a surjective morphism of vector spaces of same dimension is isomorphic):

$$0 \to H^{i-6}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi'')^3 \to \mathrm{Gr}_i^W H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to H^i(\mathcal{G}_0) \to 0.$$
(29)

The above sequence shows the surjectivity of j_0^*. Using this and the equality $H_A^*(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) = H_{\mathrm{Hdg}}^*(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$, the Hodge conjecture for $\mathcal{G}_0$ follows. Since $\mathcal{G}_1$ is $\mathbb{P}^3$-bundle over $M_{\widetilde{X}_0}(2, \mathcal{L}_0)$, it satisfies the Hodge conjecture. This proves the theorem. $\square$

As an easy consequence, we can compute the algebraic Poincaré polynomial of $\mathcal{G}_0$.

Theorem 5.3 *The algebraic Poincaré polynomial $P_A(\mathcal{G}_0)$, for $\mathcal{G}_0$ is given by*

$$P_A(\mathcal{G}_0) := \sum H_A^i(\mathcal{G}_0) t^i = \frac{(1-t^g)(1-t^{g+1})(t^2(1-t^{g-1})(1+t^2) + (1-t^{g+2}))}{(1-t)(1-t^2)(1-t^3)}.$$

Proof Let i_1 be as in Notation 3.2. Using the decompositions (12) and (13), we observe that

$$(i_1^*)^{-1}(H_A^{2i}(\mathcal{G}_0 \cap \mathcal{G}_1)) \cong H_A^{2i}(\mathcal{G}_1).$$

By Theorem 5.2, the restriction morphism $j_0^* : H^i(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \to H^i(\mathcal{G}_0)$ is surjective. As j_0^* is a morphism of mixed Hodge structures, it maps $H_A^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) = H_{\mathrm{Hdg}}^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$ surjectively to $H_{\mathrm{Hdg}}^{2i}(\mathcal{G}_0) = H_A^{2i}(\mathcal{G}_0)$, where the equalities follow from Theorem 5.2. Moreover, using the exactness of (4) and $H_A^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) = H_{\mathrm{Hdg}}^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$, $V := \ker(j_0^*) \cap H_A^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0))$ consists of pairs $(0, \gamma) \in H_{\mathrm{Hdg}}^{2i}(\mathcal{G}_0) \oplus H_{\mathrm{Hdg}}^{2i}(\mathcal{G}_1)$ such that $i_1^*(\gamma) = 0$ i.e., $V = \ker i_1^* \cap H_A^{2i}(\mathcal{G}_1)$ (as $H_A^{2i}(\mathcal{G}_1) = H_{\mathrm{Hdg}}^{2i}(\mathcal{G}_1)$). Using the decompositions (12) and (13), we observe that V is isomorphic to $H_A^{2i-6}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi'')^3$ i.e., we have the following short exact sequence:

$$0 \to H_A^{2i-6}(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))(\xi'')^3 \to H_A^{2i}(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) \xrightarrow{j_0^*} H_A^{2i}(\mathcal{G}_0) \to 0. \qquad (30)$$

Let $H(g, t) := (1 - t^g)(1 - t^{g+1})(1 - t^{g+2})/((1 - t)(1 - t^2)(1 - t^3))$. By [5, (5.1)],

$$P_A(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0)) = H(g - 1, t) \text{ and } P_A(M_{\mathcal{X}_s}(2, \mathcal{L}_s)) = H(g, t) \text{ for } s \in \Delta^* \text{ very general .}$$

By definition, $P_A(\mathcal{G}(2, \mathcal{L})_\infty) = P_A(M_{\mathcal{X}_s}(2, \mathcal{L}_s))$ for very general $s \in \Delta^*$. Using the identification (12), we have $P_A(\mathcal{G}_0 \cap \mathcal{G}_1) = H(g - 1, t)(1 + 2t^2 + t^4)$. The exact sequence (24) then implies

$$P_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) = P_A(\mathcal{G}_0 \cap \mathcal{G}_1)t^2 + P_A(\mathcal{G}(2, \mathcal{L})_\infty) - P_A(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))t^4. \qquad (31)$$

which equals $H(g - 1, t)(t^2 + t^4 + t^6) + H(g, t)$. Finally, using the short exact sequence (30), we conclude that

$$P_A(\mathcal{G}_0) = P_A(\mathcal{G}_{X_0}(2, \mathcal{L}_0)) - P_A(M_{\widetilde{X}_0}(2, \widetilde{\mathcal{L}}_0))t^6 = H(g - 1, t)(t^2 + t^4) + H(g, t).$$

Substituting for $H(g, t)$ one immediately gets the corollary. $\square$

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