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FEM–BEM coupling for the thermoelastic wave equation with transparent boundary conditions in 3D

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Abstract. We consider the thermoelastic wave equation in three dimensions with transparent boundary conditions on a bounded, not necessarily convex domain. In order to solve this problem numerically, we introduce a coupling of the thermoelastic wave equation in the interior domain with time-dependent boundary integral equations. Here, we want to highlight that this type of problem differs from other wave-type problems that dealt with FEM–BEM coupling so far, e.g., the acoustic as well as the elastic wave equation, since our problem consists of coupled partial differential equations involving a vector-valued displacement field and a scalar-valued temperature field. This constitutes a nontrivial challenge which is solved in this paper. Our main focus is on a coercivity property of a Calderón operator for the thermoelastic wave equation in the Laplace domain, which is valid for all complex frequencies in a half-plane. Combining Laplace transform and energy techniques, this coercivity in the frequency domain is used to prove the stability of a fully discrete numerical method in the time domain. The considered numerical method couples finite elements and the leapfrog time-stepping in the interior with boundary elements and convolution quadrature on the boundary. Finally, we present error estimates for the semi- and full discretization.

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1. Introduction and outline

In this paper, we study the thermoelastic wave equation in $\mathbb{R}^3$, which describes the interaction of the elastic behavior of a material and its temperature. Initial conditions and inhomogeneities are assumed to have support in a bounded, possibly non-convex domain, such that waves may re-enter the domain. Instead of considering this domain explicitly, we equip the wave equation with transparent boundary conditions, which are nonlocal in space and time. We achieve a stable numerical coupling of the interior and exterior problems by use of a so-called Calderón operator. Thus, it suffices to solve a problem which couples the thermoelastic wave equation to a boundary integral equation which represents the transmission conditions between the interior and exterior domain.

The kind of coupling we intend to use has been applied successfully for other wave-type equations, e.g., in [4] for the acoustic wave equation, in [8] for the elastodynamic wave equation, and in [15] for Maxwell's equations. Further, we want to mention the comparison of the acoustic and elastic FEM–BEM coupling in [10] as well as the implementation of the elastodynamic problem in [9]. However, the functional analytic setting is different as thermoelasticity has as variables a vector-valued[1] displacement field $\vec{U}$ and a scalar-valued temperature field T. One of the challenges is to include the equation involving the time

[1]Scalar-valued quantities are denoted by capital Latin letters, vector-valued quantities are denoted by capital Latin letters with an arrow above $\vec{\cdot}$, matrices are denoted by capital Latin letters with two arrows above $\vec{\vec{\cdot}}$.

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derivative of T as well as the time derivative of $\vec{\nabla} \cdot \vec{U}$. All in all, we have a coupled set of a partial differential equation of second order in time and space (w.r.t. $\vec{U}$) as well as a second-order PDE which is first order in time (w.r.t. T). This is one of the main differences between the thermoelastic wave equation and the wave-type equations which have been considered in the above literature and thus an extension to the existing literature.

So far, most articles considering thermoelastic wave propagation used either a finite element method, such as, e.g., [14,20] or more recently [23], or a boundary element method, e.g., [6,7,21]. We also want to mention [1] which deals with a CQM-based BEM formulation for uncoupled transient quasistatic thermoelasticity analysis. However, methods which couple finite element methods in the interior of a domain with boundary elements, as we do in this work, have not been considered so far, although they build an essential step in the framework of FEM–BEM coupling.

Our main result is the derivation of a stable and convergent fully discrete numerical method that couples discretizations in the interior and on the boundary, without requiring convexity of the domain. The key component for this is an analysis of the underlying equations in the Laplace domain by means of a Calderón operator involving boundary integral operators which cannot be directly adopted from the elastodynamic case as shown in [8] and has to be modified and extended due to the coupling of partial differential equations for the displacement vector $\vec{U}$ and the temperature T. We want to mention that we have to transform the equation involving the time derivative of the temperature T, since the original formulation would lead to problems in several proofs later on.

This paper is organized as follows: We start with the introduction of the time-dependent thermoelastic wave equation as it can be found, e.g., in [5], [13] Equation (3.1.13), or [16] (I,12.13), and equip it with transparent boundary conditions. This set of equations is then transformed with respect to time into the Laplace domain. There, we construct the aforementioned Calderón operator, for which we need the fundamental solutions to the Laplace transformed thermoelastic wave equation. This is done in Sect. 3. Having constructed the Calderón operator, we show its coercivity in Sect. 4. We discretize our system in Sect. 5 by transforming the thermoelastic wave equation into a system which is first order with respect to time and derive a variational formulation for this system. Next, we take a look at the spatial discretization, where we apply finite elements in the interior and boundary elements on the boundary. The time discretization is performed by a leapfrog scheme and convolution quadrature. Sections 6–9 deal with stability and error bounds of our semi-discrete and fully discrete schemes. For this purpose, we derive several energies for our system and consider their behavior in time. Finally, we show an asymptotically optimal $\mathcal{O}\left(h + \Delta t^2\right)$ convergence under the corresponding CFL condition. The paper is completed by a conclusion and an outlook on further research directions.

2. The thermoelastic wave equation

It is well known that elastic solids expand if their temperature rises. This interaction also works the other way around, i.e., elastic deformations influence the temperature. Such interactions are summarized in the term thermoelasticity. In a simple, linear, but non-stationary setting, an adequate description in three-dimensional Euclidean space is given by the system (see, e.g., [13], Eq. (3.1.13))

$$\rho\partial_t^2\vec{U}(\vec{x},t) = \mu\Delta\vec{U}(\vec{x},t) + (\lambda+\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{U}(\vec{x},t)) - \beta\vec{\nabla}T(\vec{x},t) + \rho\partial_t\vec{F}(\vec{x},t), \quad (2.1)$$

$$\partial_t(\rho c_p T(\vec{x},t) + \beta\, T_{\mathrm{ref}}\vec{\nabla}\cdot\vec{U}(\vec{x},t)) = \kappa\Delta T(\vec{x},t) + \varpi\, T_{\mathrm{ref}}\partial_t G(\vec{x},t), \quad (2.2)$$

$$\vec{U}(\vec{x},0) = \vec{U}_0 \quad \text{in } \mathbb{R}^3, \quad (2.3)$$

$$\partial_t\vec{U}(\vec{x},0) = \vec{V}_0 \quad \text{in } \mathbb{R}^3, \quad (2.4)$$

$$T(\vec{x},0) = T_0 \quad \text{in } \mathbb{R}^3, \quad (2.5)$$

where $\vec{U}$ is the displacement vector, T the temperature, ρ is the density, μ and λ the Lamé parameters, which we assume to be constant, $\beta = 3K\alpha$ with the compression modulus $K = \lambda + \frac{2}{3}\mu$ and the thermal expansion coefficient α, c_p the specific heat capacity, $T_{\text{ref}} > 0$ a reference temperature, κ the thermal diffusivity coefficient, the abbreviation $\varpi = \frac{\rho c_p}{T_{\text{ref}}}$, $\partial_t \vec{F}$ the inhomogeneity w.r.t. $\vec{U}$, $\varpi\, T_{\text{ref}} \partial_t G$ the inhomogeneity w.r.t. T, and the first two equations are supposed to hold for all $(\vec{x}, t) \in \mathbb{R}^3 \times (0, t_{\text{end}}]$. The fact that we consider time derivatives of functions as inhomogeneities is due to technical reasons which will become clear once we derive a first-order formulation. For short, we refer to Eqs. (2.1) and (2.2) as thermoelastic wave equation. They are complemented by the radiation conditions (see, e.g., [13], Eq. (3.3.43))

$$\lim_{\|\vec{x}\|_2 \to \infty} \|\vec{x}\|_2\, \vec{U}_{\text{c.f.}}(\vec{x}, t) = 0, \qquad \lim_{\|\vec{x}\|_2 \to \infty} \|\vec{x}\|_2^2\, \frac{\partial}{\partial x_j} \vec{U}_{\text{c.f.}}(\vec{x}, t) < \infty \text{ for } j = 1, 2, 3, \tag{2.6}$$

$$\lim_{\|\vec{x}\|_2 \to \infty} \|\vec{x}\|_2\, \vec{U}_{\text{d.f.}}(\vec{x}, t) < \infty, \qquad \lim_{\|\vec{x}\|_2 \to \infty} \|\vec{x}\|_2 \left(\frac{\partial}{\partial r} \vec{U}_{\text{d.f.}}(\vec{x}, t) + \sqrt{\frac{\rho}{\mu}}\, \frac{\partial}{\partial t} \vec{U}_{\text{d.f.}}(\vec{x}, t) \right) = 0, \tag{2.7}$$

$$\lim_{\|\vec{x}\|_2 \to \infty} \|\vec{x}\|_2\, T(\vec{x}, t) = 0, \qquad \lim_{\|\vec{x}\|_2 \to \infty} \|\vec{x}\|_2^2\, \frac{\partial}{\partial x_j} T(\vec{x}, t) < \infty \text{ for } j = 1, 2, 3. \tag{2.8}$$

Here, $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^3$, the index c.f. means the curl-free part of $\vec{U}$, the index d.f. means the divergence-free part of $\vec{U}$, and $\frac{\partial}{\partial r}$ denotes a derivative in radial direction.

In order to obtain a system in which each PDE is of second order with respect to time, we apply the variable transformation $T = \partial_t Q$ such that Eqs. (2.1)–(2.5) become

$$\rho \partial_t^2 \vec{U}(\vec{x}, t) = \mu \Delta \vec{U}(\vec{x}, t) + (\lambda + \mu) \vec{\nabla}(\vec{\nabla} \cdot \vec{U}(\vec{x}, t)) - \beta \vec{\nabla} \partial_t Q(\vec{x}, t) + \rho \partial_t \vec{F}(\vec{x}, t), \tag{2.9}$$

$$\varpi \partial_t^2 Q(\vec{x}, t) = \frac{\kappa}{T_{\text{ref}}} \Delta \partial_t Q(\vec{x}, t) - \beta \vec{\nabla} \cdot \partial_t \vec{U}(\vec{x}, t) + \varpi \partial_t G(\vec{x}, t), \quad (\vec{x}, t) \in \mathbb{R}^3 \times (0, t_{\text{end}}], \tag{2.10}$$

$$\vec{U}(\vec{x}, 0) = \vec{U}_0 \quad \text{in } \mathbb{R}^3, \tag{2.11}$$

$$\partial_t \vec{U}(\vec{x}, 0) = \vec{V}_0 \quad \text{in } \mathbb{R}^3, \tag{2.12}$$

$$Q(\vec{x}, 0) = Q_0 \quad \text{in } \mathbb{R}^3, \tag{2.13}$$

$$\partial_t Q(\vec{x}, 0) = T_0 \quad \text{in } \mathbb{R}^3. \tag{2.14}$$

This transformation is essential for successfully applying several techniques in the proofs later on in this paper. We also want to highlight that this choice is not trivial and required a deep examination of the underlying problem. In contrast to other equations for which the numerical method we have in mind has been considered, we now have to deal with a vector-valued and a scalar-valued quantity. Moreover, our system involves terms which mix spatial and temporal derivatives. Note that due to the mixed derivative term $\frac{\kappa}{T_{\text{ref}}} \Delta \partial_t Q(\vec{x}, t)$ in Eq. (2.10), we have a third-order system here.

2.1. Transmission conditions between interior and exterior space

To account for bounded support of initial conditions and inhomogeneities, we split our problem: For the *interior part* of a bounded Lipschitz-domain $\Omega \subset \mathbb{R}^3$ which contains the supports of initial conditions and right-hand sides, find the solution $\vec{U}^-$ for the system (2.9)–(2.14), whereas for the *exterior part* $\Omega^+ = \mathbb{R}^3 \setminus \overline{\Omega}$, where $\vec{F} = \vec{0}$, $G = 0$ and initial conditions vanish, determine the solution $\vec{U}^+$ of the homogenous version of (2.9)–(2.14). Both problems are coupled by *transmission conditions* on the boundary $\partial\Omega = \Gamma$. Thus, we introduce Dirichlet as well as Neumann traces. We remind the reader that Neumann traces are related to co-normal derivatives, which may differ from normal derivatives. In our case, they even correspond to different entities for $\vec{U}$ and Q as can be seen in the boundary conditions

$$\vec{\gamma}_D \vec{U}(\vec{x},t) = \vec{U}(\vec{x},t)|_\Gamma, \qquad (\vec{x},t) \in \Gamma \times [0, t_{\text{end}}], \tag{2.15}$$

$$\vec{\gamma}_N \vec{U}(\vec{x},t) = \left(\overset{\rightrightarrows}{S}_{\text{el}}(\vec{U}(\vec{x},t)) - \beta \overset{\rightrightarrows}{I} \partial_t Q(\vec{x},t) \right) \vec{n}(\vec{x}), \qquad (\vec{x},t) \in \Gamma \times [0, t_{\text{end}}], \tag{2.16}$$

$$\gamma_D Q(\vec{x},t) = Q(\vec{x},t)|_\Gamma, \qquad (\vec{x},t) \in \Gamma \times [0, t_{\text{end}}], \tag{2.17}$$

$$\gamma_N Q(\vec{x},t) = \frac{\kappa}{T_{\text{ref}}} \vec{\nabla} \partial_t Q(\vec{x},t) \cdot \vec{n}(\vec{x}), \qquad (\vec{x},t) \in \Gamma \times [0, t_{\text{end}}], \tag{2.18}$$

where $\vec{n}$ is the outer unit normal to Ω, $\overset{\rightrightarrows}{I}$ is the unit matrix in $\mathbb{R}^{3\times 3}$ and

$$\overset{\rightrightarrows}{S}_{\text{el}}(\vec{U}(\vec{x},t)) := \mu \left(\vec{\nabla}\vec{U} + \left(\vec{\nabla}\vec{U}\right)^T \right) + \lambda \left(\vec{\nabla} \cdot \vec{U} \right) \overset{\rightrightarrows}{I} = 2\mu \overset{\rightrightarrows}{E}(\vec{U}) + \lambda \left(\vec{\nabla} \cdot \vec{U} \right) \overset{\rightrightarrows}{I} \tag{2.19}$$

is the elastic stress tensor. Here, we also introduced the elastic strain tensor

$$\overset{\rightrightarrows}{E}(\vec{U}) := \frac{1}{2} \left(\vec{\nabla}\vec{U} + \left(\vec{\nabla}\vec{U}\right)^T \right). \tag{2.20}$$

The thermoelastic stress tensor is

$$\overset{\rightrightarrows}{S}((\vec{U},Q)(\vec{x},t)) := \overset{\rightrightarrows}{S}_{\text{el}}(\vec{U}(\vec{x},t)) - \beta \overset{\rightrightarrows}{I} \partial_t Q(\vec{x},t). \tag{2.21}$$

Based on this, we finally state the transmission conditions

$$\vec{\gamma}_D^- \vec{U}^- = \vec{\gamma}_D^+ \vec{U}^+, \quad \vec{\gamma}_N^- \vec{U}^- = \vec{\gamma}_N^+ \vec{U}^+, \quad \gamma_D^- Q^- = \gamma_D^+ Q^+, \quad \gamma_N^- Q^- = \gamma_N^+ Q^+. \tag{2.22}$$

We also extend the trace operators to Sobolev spaces in the usual way, such that

$$\vec{\gamma}_D \colon (\mathrm{H}^1(\Omega))^3 \to (\mathrm{H}^{\frac{1}{2}}(\Gamma))^3, \qquad \vec{\gamma}_N \colon \left\{ \vec{U} \in (\mathrm{H}^1(\Omega))^3 : \nabla \cdot \overset{\rightrightarrows}{S}_{\text{el}}(\vec{U}) \in (\mathrm{L}^2(\Omega))^3 \right\} \to (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3 \tag{2.23}$$

$$\gamma_D \colon \mathrm{H}^1(\Omega) \to \mathrm{H}^{\frac{1}{2}}(\Gamma), \qquad \gamma_N \colon \{ Q \in \mathrm{H}^1(\Omega) : \Delta Q \in \mathrm{L}^2(\Omega) \} \to \mathrm{H}^{-\frac{1}{2}}(\Gamma). \tag{2.24}$$

2.2. Laplace transform of the thermoelastic wave equation

We introduce the Laplace transforms[2]

$$\vec{\mathcal{U}}(\vec{x},s) := \mathcal{L}\vec{U}(\vec{x},t), \qquad \mathcal{Q}(\vec{x},s) := \mathcal{L}Q(\vec{x},t)$$

with the parameter $s \in \mathbb{C}$. If we assume vanishing initial conditions for the moment, transferring the modified thermoelastic wave equations (2.9)–(2.10) to the Laplace domain yields

$$\rho s^2 \vec{\mathcal{U}}(\vec{x},s) = \mu \Delta \vec{\mathcal{U}}(\vec{x},s) + (\lambda + \mu) \vec{\nabla}(\vec{\nabla} \cdot \vec{\mathcal{U}}(\vec{x},s)) - \beta s \vec{\nabla} \mathcal{Q}(\vec{x},s) + \rho s \vec{\mathcal{F}}(\vec{x},s) \qquad \text{in } \Omega, \tag{2.25}$$

$$\varpi s^2 \mathcal{Q}(\vec{x},s) = \frac{\kappa}{T_{\text{ref}}} s\, \Delta \mathcal{Q}(\vec{x},s) - \beta\, s\, \vec{\nabla} \cdot \vec{\mathcal{U}}(\vec{x},s) + \varpi\, s\, \mathcal{G}(\vec{x},s) \qquad \text{in } \Omega, \tag{2.26}$$

$$\vec{\gamma}_D \vec{\mathcal{U}}(\vec{x},s) = \vec{\mathcal{U}}(\vec{x},s)|_\Gamma \qquad \text{on } \Gamma, \tag{2.27}$$

$$\gamma_D \mathcal{Q}(\vec{x},s) = \mathcal{Q}(\vec{x},s)|_\Gamma \qquad \text{on } \Gamma, \tag{2.28}$$

$$\vec{\gamma}_N \vec{\mathcal{U}}(\vec{x},s) = \left(\overset{\rightrightarrows}{S}_{\text{el}}(\vec{\mathcal{U}}(\vec{x},s)) - s\beta \overset{\rightrightarrows}{I} \mathcal{Q}(\vec{x},s) \right) \vec{n}(\vec{x}) \qquad \text{on } \Gamma, \tag{2.29}$$

$$\gamma_N \mathcal{Q}(\vec{x},s) = \frac{\kappa}{T_{\text{ref}}} s \left(\vec{\nabla} \mathcal{Q}(\vec{x},s) \right) \cdot \vec{n}(\vec{x}) \qquad \text{on } \Gamma. \tag{2.30}$$

[2]For the Laplace domain, we use calligraphic Latin letters and differentiate between scalars, vectors and matrices in the same way as in the time domain (see [1]).

This leads to the relation

$$\begin{pmatrix} \rho s^2 \vec{u}(\vec{x},s) \\ \varpi s^2 Q(\vec{x},s) + \beta s \vec{\nabla}\cdot\vec{u}(\vec{x},s)\end{pmatrix} = \begin{pmatrix} \vec{\nabla}\cdot\left(\overset{\Rightarrow}{S}_{\mathrm{el}}(\vec{u}(\vec{x},s)) - s\beta \overset{\Rightarrow}{I} Q(\vec{x},s)\right) \\ \vec{\nabla}\cdot \frac{\kappa}{T_{\mathrm{ref}}} s \vec{\nabla} Q(\vec{x},s)\end{pmatrix} + \begin{pmatrix} \rho\, s\, \vec{\mathcal{F}}(\vec{x},s) \\ \varpi\, s\, \mathcal{G}(\vec{x},s)\end{pmatrix}. \tag{2.31}$$

Further on, we drop the dependence on $(\vec{x},s)$ as it is clear from context whether functions are considered in the time or the Laplace domain.

3. Potentials and Calderón operator

The material of this section is taken from [19], Chapters 6 and 7, and included for the convenience of the reader. We introduce the trace operators

$$\overset{\Rightarrow}{\gamma}_D\colon (\mathrm{H}^1(\Omega))^4 \to (\mathrm{H}^{\frac12}(\Gamma))^4, \tag{3.1}$$

$$\overset{\Rightarrow}{\gamma}_N\colon \left\{\left(\vec{U},Q\right)^T \in (\mathrm{H}^1(\Omega))^3\times \mathrm{H}^1(\Omega) : \left(\nabla\cdot \overset{\Rightarrow}{S}_{\mathrm{el}}(\vec{U}), \Delta Q\right)^T \in (\mathrm{L}^2(\Omega))^4\right\} \to (\mathrm{H}^{-\frac12}(\Gamma))^4, \tag{3.2}$$

such that

$$\overset{\Rightarrow}{\gamma}_D \left(\vec{u}, Q\right)^T := \left(\vec{\gamma}_D \vec{u}, \gamma_D Q\right)^T, \qquad \overset{\Rightarrow}{\gamma}_N \left(\vec{u}, Q\right)^T := \left(\vec{\gamma}_N \vec{u}, \gamma_N Q\right)^T. \tag{3.3}$$

For any linear partial differential equation, the single layer potential is given by

$$\mathcal{S}(s)[\overset{\Rightarrow}{\phi}](\vec{x}) := \int_\Gamma \overset{\Rightarrow}{G}_s(\vec{x},\vec{y})\overset{\Rightarrow}{\phi}(\vec{y})\,\mathrm{d}\vec{s}_{\vec{y}} \tag{3.4}$$

and the double layer potential by

$$\mathcal{D}(s)[\overset{\Rightarrow}{\psi}](\vec{x}) := \int_\Gamma [\overset{\Rightarrow}{\gamma}{}^*_N \overset{\Rightarrow}{G}{}^*_s(\vec{x},\vec{y})]^* \overset{\Rightarrow}{\psi}(\vec{y})\,\mathrm{d}\vec{s}_{\vec{y}} \tag{3.5}$$

for $\vec{x}\in\mathbb{R}^3\setminus\Gamma$, where $\overset{\Rightarrow}{G}{}^*_s(\vec{x},\vec{y})$ is the adjoint of $\overset{\Rightarrow}{G}_s(\vec{x},\vec{y})$. The fundamental solution tensor $\overset{\Rightarrow}{G}_s(\vec{x},\vec{y})$ can be adapted from [13], Section 3.2(b), or [16], Chapter II, §3. We include it in Appendix A. The layer potentials and related traces fulfill the following mapping properties:

$$\mathcal{S}(s)\colon (\mathrm{H}^{-\frac12}(\Gamma))^4 \to (\mathrm{H}^1(\Omega))^4, \qquad \mathcal{D}(s)\colon (\mathrm{H}^{\frac12}(\Gamma))^4 \to (\mathrm{H}^1(\Omega))^4, \tag{3.6}$$

$$\overset{\Rightarrow}{\gamma}_D\mathcal{S}(s) : (\mathrm{H}^{-\frac12}(\Gamma))^4 \to (\mathrm{H}^{\frac12}(\Gamma))^4, \qquad \overset{\Rightarrow}{\gamma}_D\mathcal{D}(s)\colon (\mathrm{H}^{\frac12}(\Gamma))^4 \to (\mathrm{H}^{\frac12}(\Gamma))^4, \tag{3.7}$$

$$\overset{\Rightarrow}{\gamma}_N\mathcal{S}(s)\colon (\mathrm{H}^{-\frac12}(\Gamma))^4 \to (\mathrm{H}^{-\frac12}(\Gamma))^4, \qquad \overset{\Rightarrow}{\gamma}_N\mathcal{D}(s)\colon (\mathrm{H}^{\frac12}(\Gamma))^4 \to (\mathrm{H}^{-\frac12}(\Gamma))^4. \tag{3.8}$$

The jumps of the traces over Γ are defined as

$$\left[(\vec{u},Q)^T\right] = \overset{\Rightarrow}{\gamma}{}^-_D(\vec{u},Q)^T - \overset{\Rightarrow}{\gamma}{}^+_D(\vec{u},Q)^T, \tag{3.9}$$

$$\left[\overset{\Rightarrow}{\gamma}_N(\vec{u},Q)^T\right] = \overset{\Rightarrow}{\gamma}{}^-_N(\vec{u},Q)^T - \overset{\Rightarrow}{\gamma}{}^+_N(\vec{u},Q)^T, \tag{3.10}$$

which allows us to state the corresponding jump relations

$$[\mathcal{S}(s)\overset{\Rightarrow}{\phi}] = \overset{\Rightarrow}{0}, \qquad [\overset{\Rightarrow}{\gamma}_N\mathcal{S}(s)\overset{\Rightarrow}{\phi}] = \overset{\Rightarrow}{\phi} \qquad [\mathcal{D}(s)\overset{\Rightarrow}{\psi}] = -\overset{\Rightarrow}{\psi}, \qquad [\overset{\Rightarrow}{\gamma}_N\mathcal{D}(s)\overset{\Rightarrow}{\psi}] = \overset{\Rightarrow}{0}, \tag{3.11}$$

such that with the definitions

$$\overset{\Rightarrow}{\psi} = \begin{pmatrix}\psi_{\vec{u}} \\ \psi_Q\end{pmatrix} = -\begin{pmatrix}[\vec{\gamma}_D\vec{u}] \\ [\gamma_D Q]\end{pmatrix}, \qquad \overset{\Rightarrow}{\phi} = \begin{pmatrix}\phi_{\vec{u}} \\ \phi_Q\end{pmatrix} = \frac{1}{s}\begin{pmatrix}[\vec{\gamma}_N\vec{u}] \\ [\gamma_N Q]\end{pmatrix}, \tag{3.12}$$

it holds that

$$\begin{pmatrix} \vec{\mathcal{U}} \\ \mathcal{Q} \end{pmatrix} = s\mathcal{S}(s) \begin{pmatrix} \vec{\phi}_{\vec{u}} \\ \phi_{\mathcal{Q}} \end{pmatrix} + \mathcal{D}(s) \begin{pmatrix} \vec{\psi}_{\vec{u}} \\ \psi_{\mathcal{Q}} \end{pmatrix}. \tag{3.13}$$

Moreover, we define the operators

$$\mathcal{J}(s)\vec{\vec{\phi}} := \vec{\vec{\gamma}}_D \mathcal{S}(s)\vec{\vec{\phi}} \colon (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4 \to (\mathrm{H}^{\frac{1}{2}}(\Gamma))^4, \tag{3.14}$$

$$\mathcal{W}(s)\vec{\vec{\psi}} := -\vec{\vec{\gamma}}_N \mathcal{D}(s)\vec{\vec{\psi}} \colon (\mathrm{H}^{\frac{1}{2}}(\Gamma))^4 \to (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4, \tag{3.15}$$

$$\mathcal{K}(s)\vec{\vec{\psi}} := \frac{1}{2}\left(\vec{\vec{\gamma}}_D^{\,+}\mathcal{D}(s)\vec{\vec{\psi}} + \vec{\vec{\gamma}}_D^{\,-}\mathcal{D}(s)\vec{\vec{\psi}}\right) = \{\{\vec{\vec{\gamma}}_D \mathcal{D}(s)\vec{\vec{\psi}}\}\} \colon (\mathrm{H}^{\frac{1}{2}}(\Gamma))^4 \to (\mathrm{H}^{\frac{1}{2}}(\Gamma))^4, \tag{3.16}$$

$$\widetilde{\mathcal{K}}^*(s)\vec{\vec{\phi}} := \frac{1}{2}\left(\vec{\vec{\gamma}}_N^{\,+}\mathcal{S}(s)\vec{\vec{\phi}} + \vec{\vec{\gamma}}_N^{\,-}\mathcal{S}(s)\vec{\vec{\phi}}\right) = \{\{\vec{\vec{\gamma}}_N \mathcal{S}(s)\vec{\vec{\phi}}\}\} \colon (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4 \to (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4, \tag{3.17}$$

where we also introduced the notation $\{\{\cdot\}\}$ for averages of inner and outer limits at the boundary. This allows us to compactly summarize the limit and jump relations

$$\begin{pmatrix} \left[\vec{\vec{\gamma}}_D \cdot\right] \\ \frac{1}{s}\left[\vec{\vec{\gamma}}_N \cdot\right] \end{pmatrix} \begin{pmatrix} s\mathcal{S}(s)\vec{\vec{\phi}} & \mathcal{D}(s)\vec{\vec{\psi}} \end{pmatrix} = \begin{pmatrix} \vec{\vec{0}} & -\vec{\vec{\psi}} \\ \vec{\vec{\phi}} & \vec{\vec{0}} \end{pmatrix}, \tag{3.18}$$

$$\begin{pmatrix} \left\{\left\{\vec{\vec{\gamma}}_D \cdot\right\}\right\} \\ -\frac{1}{s}\left\{\left\{\vec{\vec{\gamma}}_N \cdot\right\}\right\} \end{pmatrix} \begin{pmatrix} s\mathcal{S}(s)\vec{\vec{\phi}} & \mathcal{D}(s)\vec{\vec{\psi}} \end{pmatrix} = \begin{pmatrix} s\mathcal{J}(s)\vec{\vec{\phi}} & \mathcal{K}(s)\vec{\vec{\psi}} \\ -\widetilde{\mathcal{K}}^*(s)\vec{\vec{\phi}} & \frac{1}{s}\mathcal{W}(s)\vec{\vec{\psi}} \end{pmatrix}. \tag{3.19}$$

Moreover, we introduce the Calderón operator

$$\mathcal{B}(s) = \begin{pmatrix} s\mathcal{J}(s) & \mathcal{K}(s) \\ -\widetilde{\mathcal{K}}^*(s) & \frac{1}{s}\mathcal{W}(s) \end{pmatrix} : (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4 \times (\mathrm{H}^{\frac{1}{2}}(\Gamma))^4 \to (\mathrm{H}^{\frac{1}{2}}(\Gamma))^4 \times (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4. \tag{3.20}$$

4. Coercivity results for the Calderón operator

In order to show stability of the suggested numerical model, we need to show that the Calderón operator satisfies a coercivity estimate, both in the Laplace as well as in the time domain. Our considerations here extend the results for the elastic wave equations in [8] as we consider a vector-valued partial differential equation (second order in time and space) which is coupled with a scalar-valued third-order PDE (second order in time), which stems from the transformation of the original problem as well as the corresponding boundary conditions.

4.1. Coercivity of the Laplace transformed Calderón operator $\mathcal{B}(s)$

We need the following estimates which can be found in [11], p. 28, and [19], Theorem 10.2, respectively.

Lemma 1. *Let Ω be a domain with a Lipschitz boundary Γ. Then, it holds the trace inequality*

$$\|\vec{\vec{S}}_{\mathrm{el}}(\vec{\mathcal{U}})\vec{n}\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3} \leq \|\vec{\vec{S}}_{\mathrm{el}}(\vec{\mathcal{U}})\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times 3}} + \|\vec{\nabla}\cdot\vec{\vec{S}}_{\mathrm{el}}(\vec{\mathcal{U}})\|_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} \tag{4.1}$$

and Korn's second inequality

$$\|\vec{\vec{E}}(\vec{\mathcal{U}})\|^2_{(\mathrm{L}^2(\Omega))^{3\times 3}} \geq \mathsf{a}\|\vec{\nabla}\vec{\mathcal{U}}\|_{(\mathrm{L}^2(\Omega))^{3\times 3}} - \mathsf{b}\|\vec{\mathcal{U}}\|_{(\mathrm{L}^2(\Omega))^3}. \tag{4.2}$$

With these estimates, we can show coercivity of the Calderón operator. In the following, $\langle\cdot,\cdot\rangle_\Gamma$ denotes the anti-duality in $(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4 \times (\mathrm{H}^{\frac{1}{2}}(\Gamma))^4$ and $(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4 \times (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4$. Furthermore, we define for $\vec{\vec{\phi}}$, $\vec{\vec{\psi}}$ as in Eq. (3.12)

$$\|\vec{\vec{\phi}}\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4} := \|\vec{\phi}_{\vec{u}}\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3} + \|\phi_{\mathcal{Q}}\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)}, \qquad \|\vec{\vec{\psi}}\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4} := \|\vec{\psi}_{\vec{u}}\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^3} + \|\psi_{\mathcal{Q}}\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma)}. \tag{4.3}$$

Lemma 2. *There exists a constant $\widetilde{\mathsf{c}} > 0$ such that the Calderón operator $\mathcal{B}(s)$ as defined in Eq. (3.20) satisfies*

$$\operatorname{Re}\left\langle \begin{pmatrix}\vec{\vec{\phi}}\\ \vec{\vec{\psi}}\end{pmatrix}, \mathcal{B}(s)\begin{pmatrix}\vec{\vec{\phi}}\\ \vec{\vec{\psi}}\end{pmatrix}\right\rangle_\Gamma \geq \widetilde{\mathsf{c}}\,\frac{\operatorname{Re}(s)}{|s|^3}\min\left(1,|s|^3\right)\left(\|\vec{\vec{\phi}}\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4} + \|\vec{\vec{\psi}}\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma)^4}\right) \tag{4.4}$$

for $\operatorname{Re}(s) > 0$ *and for all* $\vec{\vec{\phi}} \in (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4$ *and* $\vec{\vec{\psi}} \in (\mathrm{H}^{\frac{1}{2}}(\Gamma))^4$.

Proof. We start with the consideration of

$$\begin{aligned}\left\langle \begin{pmatrix}\vec{\vec{\phi}}\\ \vec{\vec{\psi}}\end{pmatrix}, \mathcal{B}(s)\begin{pmatrix}\vec{\vec{\phi}}\\ \vec{\vec{\psi}}\end{pmatrix}\right\rangle_\Gamma &= \left\langle \begin{pmatrix}\frac{1}{s}\left[\vec{\gamma}_N\vec{u}\right]\\ \frac{1}{s}\left[\gamma_N\mathcal{Q}\right]\\ -\left[\vec{\gamma}_D\vec{u}\right]\\ -\left[\gamma_D\mathcal{Q}\right]\end{pmatrix}, \begin{pmatrix}\{\{\vec{\gamma}_D\vec{u}\}\}\\ \{\{\gamma_D\mathcal{Q}\}\}\\ -\frac{1}{s}\{\{\vec{\gamma}_N\vec{u}\}\}\\ -\frac{1}{s}\{\{\gamma_N\mathcal{Q}\}\}\end{pmatrix}\right\rangle_\Gamma \\ &= \frac{1}{s}\left(\langle\vec{\gamma}_N^-\vec{u},\vec{\gamma}_D^-\vec{u}\rangle_\Gamma + \langle\gamma_N^-\mathcal{Q},\gamma_D^-\mathcal{Q}\rangle_\Gamma - \langle\vec{\gamma}_N^+\vec{u},\vec{\gamma}_D^+\vec{u}\rangle_\Gamma - \langle\gamma_N^+\mathcal{Q},\gamma_D^+\mathcal{Q}\rangle_\Gamma\right).\end{aligned} \tag{4.5}$$

If we now substitute the definitions of $\vec{\gamma}_N\vec{u}$ and $\gamma_N\mathcal{Q}$ from Eqs. (2.28) and (2.30), we can apply a modified version of Green's second identity for poroelasticity from [2], Theorem 4.3., to obtain

$$\begin{aligned}&\operatorname{Re}\Bigg(\frac{1}{s}\left(\langle\vec{\gamma}_N^-\vec{u},\vec{\gamma}_D^-\vec{u}\rangle_\Gamma + \langle\gamma_N^-\mathcal{Q},\gamma_D^-\mathcal{Q}\rangle_\Gamma - \langle\vec{\gamma}_N^+\vec{u},\vec{\gamma}_D^+\vec{u}\rangle_\Gamma - \langle\gamma_N^+\mathcal{Q},\gamma_D^+\mathcal{Q}\rangle_\Gamma\right)\Bigg)\\ &= \operatorname{Re}\Bigg(2\mu\frac{1}{s}\left(\vec{\vec{E}}(\vec{u}),\vec{\vec{E}}(\vec{u})\right)_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times3}} + \lambda\frac{1}{s}\left(\vec{\nabla}\cdot\vec{u},\vec{\nabla}\cdot\vec{u}\right)_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \rho s\left(\vec{u},\vec{u}\right)_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}\\ &\qquad + \frac{\kappa}{T_{\mathrm{ref}}}\left(\vec{\nabla}\mathcal{Q},\vec{\nabla}\mathcal{Q}\right)_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \varpi s\left(\mathcal{Q},\mathcal{Q}\right)_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}\Bigg)\\ &\geq \mathsf{d}_{\min}\operatorname{Re}(s)\min\left(1,\frac{1}{|s|}\right)\Bigg(\left\|\frac{1}{s}\vec{\vec{E}}(\vec{u})\right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times3}} + \left\|\frac{1}{s}\vec{\nabla}\cdot\vec{u}\right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \|\vec{u}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}\\ &\qquad + \|\vec{\nabla}\mathcal{Q}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \|\mathcal{Q}\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}\Bigg),\end{aligned} \tag{4.6}$$

where $\mathsf{d}_{\min} = \min\left\{2\mu,\lambda,\rho,\frac{\kappa}{T_{\mathrm{ref}}},\varpi\right\}$.

Next, we take a look at the norms of the boundary densities. It is

$$\begin{aligned}&\|\vec{\phi}_{\vec{u}}\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3}\\ &= \left\|\frac{1}{s}\left[\left(\vec{\vec{S}}_{\mathrm{el}}(\vec{u}) - \beta s\mathcal{Q}\vec{\vec{I}}\right)\vec{n}\right]\right\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3}\\ &\leq \mathsf{d}_1\left(\left\|\frac{1}{s}\left(\vec{\vec{S}}_{\mathrm{el}}(\vec{u}) - \beta s\mathcal{Q}\vec{\vec{I}}\right)\right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times3}} + \|\rho s\vec{u}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}\right)\\ &\leq \mathsf{d}_{1,\max}\left(\left\|\frac{1}{s}\vec{\vec{E}}(\vec{u})\right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times3}} + \left\|\frac{1}{s}\vec{\nabla}\cdot\vec{u}\right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \|\mathcal{Q}\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + |s|^2\|\vec{u}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}\right),\end{aligned} \tag{4.7}$$

where we have also used Eqs. (4.1) and (2.9), the triangle inequality, and defined $\mathsf{d}_{1,\max} := \mathsf{d}_1 \max(8\mu^2, 2\lambda^2, \beta^2, \rho^2)$.

Further, we have

$$\begin{aligned}
\|\phi_Q\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)} &= \left\| \frac{1}{s} \left[\frac{\kappa}{T_{\mathrm{ref}}} s \left(\vec{\nabla} Q \right) \cdot \vec{n} \right] \right\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)} \\
&\leq \mathsf{d}_2 \left(\left\| \frac{\kappa}{T_{\mathrm{ref}}} \vec{\nabla} Q \right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \left\| \frac{\kappa}{T_{\mathrm{ref}}} \Delta Q \right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} \right) \\
&= \mathsf{d}_2 \left(\left\| \frac{\kappa}{T_{\mathrm{ref}}} \vec{\nabla} Q \right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \| \varpi s Q + \beta \vec{\nabla} \cdot \vec{u} \|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} \right) \\
&\leq \mathsf{d}_{2,\max} \left(\| \vec{\nabla} Q \|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + |s|^2 \| Q \|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \| \vec{\nabla} \cdot \vec{u} \|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} \right),
\end{aligned} \tag{4.8}$$

where we used Eqs. (4.1) and (2.10), the triangle inequality, the relation

$$\begin{aligned}
\|\vec{\psi}_{\vec{u}}\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^3} &\leq \mathsf{d} \left(\|\vec{u}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \|\vec{\nabla}\vec{u}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times 3}} \right) \\
&\leq \mathsf{d}_3 \left(\|\vec{u}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \|\vec{\vec{E}}(\vec{u})\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times 3}} \right)
\end{aligned} \tag{4.9}$$

which is obtained by using Korn's inequality (4.2), and

$$\|\psi_Q\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma)} \leq \mathsf{d}_4 \left(\|Q\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \|\vec{\nabla} Q\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} \right), \tag{4.10}$$

as well as defined $\mathsf{d}_{2,\max} := \mathsf{d}_2 \max\left(\left(\frac{\kappa}{T_{\mathrm{ref}}} \right)^2, \varpi^2, \beta^2 \right)$. By adding the estimates for the norms of the four boundary densities, we find

$$\begin{aligned}
&\|\vec{\phi}_{\vec{u}}\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3} + \|\phi_Q\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)} + \|\vec{\psi}_{\vec{u}}\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^3} + \|\psi_Q\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma)} \\
&\leq \mathsf{d}_{\max} \max\left(1, |s|^2\right) \left(\left\| \frac{1}{s} \vec{\vec{E}}(\vec{u}) \right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times 3}} + \left\| \frac{1}{s} \vec{\nabla} \cdot \vec{u} \right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \|\vec{u}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} \right. \\
&\quad \left. + \|Q\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \|\vec{\nabla} Q\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} \right),
\end{aligned} \tag{4.11}$$

where $\mathsf{d}_{\max} := 3 \max\{\mathsf{d}_{1,\max}, \mathsf{d}_{2,\max}, \mathsf{d}_3, \mathsf{d}_4\}$. By combining the final estimates in Eqs. (4.6) and (4.11), we achieve the desired result

$$\begin{aligned}
&\mathrm{Re} \left\langle \begin{pmatrix} \vec{\vec{\phi}} \\ \vec{\vec{\psi}} \end{pmatrix}, \mathcal{B}(s) \begin{pmatrix} \vec{\vec{\phi}} \\ \vec{\vec{\psi}} \end{pmatrix} \right\rangle_\Gamma \\
&\geq \tilde{\mathsf{c}} \frac{\mathrm{Re}(s)}{|s|^3} \min\left(1, |s|^3\right) \left(\|\vec{\phi}_{\vec{u}}\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3} + \|\phi_Q\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)} + \|\vec{\psi}_{\vec{u}}\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^3} + \|\psi_Q\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma)} \right)
\end{aligned} \tag{4.12}$$

with $\tilde{\mathsf{c}} := \frac{\mathsf{d}_{\min}}{\mathsf{d}_{\max}}$. □

4.2. Coercivity of the time-dependent Calderón operator $\mathcal{B}(\partial_t)$

In order to analyze the stability of our numerical scheme and find error estimates, we have to transfer the coercivity result for the Calderón operator from the Laplace domain to the time domain. This is directly possible due to an operator-valued variant of the classical Herglotz theorem and the convolution quadrature as shown in [4]. The following lemmas follow from this Herglotz theorem in a way similar to the corresponding results for the elastic wave equation derived in [8] and from the estimates

$$\|s\mathcal{J}(s)\| \leq M(\sigma)|s|^3, \qquad \|\mathcal{K}(s)\| \leq M(\sigma)|s|^2, \qquad \left\|s^{-1}\mathcal{W}(s)\right\| \leq M(\sigma)|s|^2, \tag{4.13}$$

where M depends only on $\sigma = \mathrm{Re}(s)$. These estimates can be derived in a similar way as in [3,17]. Exemplary for $\|s\mathcal{J}(s)\|$, we define $\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T := s\mathcal{S}(s)\vec{\vec{\phi}}$ for $\vec{\vec{\phi}} \in (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4$, so that $\vec{\vec{\phi}} = \frac{1}{s}\left[\vec{\vec{\gamma}}_N\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T\right]$ and $\vec{\vec{\psi}} = \vec{\vec{\gamma}}_D\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T = s\mathcal{J}(s)\vec{\vec{\phi}}$. Further, we have

$$\left[\vec{\vec{\gamma}}_D\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T\right] = \left[\vec{\vec{\gamma}}_D s\mathcal{S}(s)\vec{\vec{\phi}}\right] = \vec{\vec{0}}. \tag{4.14}$$

Hence,

$$\vec{\vec{\psi}} = \vec{\vec{\gamma}}_D^{\,-}\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T = \vec{\vec{\gamma}}_D^{\,+}\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T = \left\{\left\{\vec{\vec{\gamma}}_D\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T\right\}\right\}. \tag{4.15}$$

Then

$$\mathrm{Re}\left\langle\vec{\vec{\phi}}, s\mathcal{J}(s)\vec{\vec{\phi}}\right\rangle = \mathrm{Re}\left\langle\vec{\vec{\phi}}, \vec{\vec{\psi}}\right\rangle = \mathrm{Re}\left\langle\frac{1}{s}\left[\vec{\vec{\gamma}}_N\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T\right], \left\{\left\{\vec{\vec{\gamma}}_D\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T\right\}\right\}\right\rangle \tag{4.16}$$

$$= \mathrm{Re}\left(\frac{1}{s}\left(\langle\vec{\gamma}_N^-\vec{\mathcal{U}}, \vec{\gamma}_D^-\vec{\mathcal{U}}\rangle_\Gamma + \langle\gamma_N^-\mathcal{Q}, \gamma_D^-\mathcal{Q}\rangle_\Gamma - \langle\vec{\gamma}_N^+\vec{\mathcal{U}}, \vec{\gamma}_D^+\vec{\mathcal{U}}\rangle_\Gamma - \langle\gamma_N^+\mathcal{Q}, \gamma_D^+\mathcal{Q}\rangle_\Gamma\right)\right). \tag{4.17}$$

As in the proof of Lemma 2, we get

$$\mathrm{Re}\left\langle\vec{\vec{\phi}}, s\mathcal{J}(s)\vec{\vec{\phi}}\right\rangle \geq \tilde{\mathsf{c}}\frac{\mathrm{Re}(s)}{|s|^3}\min\left(1, |s|^3\right)\|\vec{\vec{\phi}}\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4}. \tag{4.18}$$

The Lax–Milgram lemma in the form of [22] (Lemma 2.1.51 with the definition of ellipticity as in (2.43)) then gives

$$\left\|(s\mathcal{J}(s))^{-1}\right\| \leq \frac{1}{\tilde{\mathsf{c}}}\frac{|s|^3}{\mathrm{Re}(s)}\max\left(1, \frac{1}{|s|^3}\right). \tag{4.19}$$

Now

$$\left|\left\langle\frac{1}{s}\mathcal{J}^{-1}(s)\vec{\vec{\psi}}, \vec{\vec{\psi}}\right\rangle\right| \tag{4.20}$$

$$\geq \mathrm{Re}\left\langle\frac{1}{s}\mathcal{J}^{-1}(s)\vec{\vec{\psi}}, \vec{\vec{\psi}}\right\rangle = \mathrm{Re}\left\langle\vec{\vec{\phi}}, \vec{\vec{\psi}}\right\rangle = \mathrm{Re}\left\langle\frac{1}{s}\left[\vec{\vec{\gamma}}_N\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T\right], \left\{\left\{\vec{\vec{\gamma}}_D\left(\vec{\mathcal{U}}, \mathcal{Q}\right)^T\right\}\right\}\right\rangle$$

$$= \mathrm{Re}\left(\frac{1}{s}\left(\langle\vec{\gamma}_N^-\vec{\mathcal{U}}, \vec{\gamma}_D^-\vec{\mathcal{U}}\rangle_\Gamma + \langle\gamma_N^-\mathcal{Q}, \gamma_D^-\mathcal{Q}\rangle_\Gamma - \langle\vec{\gamma}_N^+\vec{\mathcal{U}}, \vec{\gamma}_D^+\vec{\mathcal{U}}\rangle_\Gamma - \langle\gamma_N^+\mathcal{Q}, \gamma_D^+\mathcal{Q}\rangle_\Gamma\right)\right). \tag{4.21}$$

With Eq. (4.6), we get

$$\left|\left\langle\frac{1}{s}\mathcal{J}^{-1}(s)\vec{\vec{\psi}}, \vec{\vec{\psi}}\right\rangle\right|$$

$$\geq \mathsf{d}_{\min}\mathrm{Re}(s)\min\left(1, \frac{1}{|s|}\right)\left(\left\|\frac{1}{s}\vec{\vec{E}}(\vec{\mathcal{U}})\right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times3}} + \left\|\frac{1}{s}\vec{\nabla}\cdot\vec{\mathcal{U}}\right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \|\vec{\mathcal{U}}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}\right.$$

$$\left. + \|\vec{\nabla}\mathcal{Q}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \|\mathcal{Q}\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}\right), \tag{4.22}$$

Using the trace inequality and Korn's lemma as in Eq. (4.2), we obtain

$$\left\| \vec{\vec{\psi}} \right\|^2 \tag{4.23}$$

$$\leq \mathsf{d}_{\max} \max(1,|s|^2) \left(\|\vec{\mathcal{U}}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \|\mathcal{Q}\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \left\| \frac{1}{s} \vec{\vec{E}}(\vec{\mathcal{U}}) \right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times 3}} + \|\vec{\nabla}\mathcal{Q}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} \right)$$

$$\leq \mathsf{d}_{\max} \max(1,|s|^2) \left(\|\vec{\mathcal{U}}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} + \|\mathcal{Q}\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} + \left\| \frac{1}{s} \vec{\vec{E}}(\vec{\mathcal{U}}) \right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times 3}} + \|\vec{\nabla}\mathcal{Q}\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3} \right.$$
$$\left. + \left\| \frac{1}{s} \vec{\nabla} \cdot \vec{\mathcal{U}} \right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)} \right) \tag{4.24}$$

so that

$$\left| \left\langle \frac{1}{s} \mathcal{J}^{-1}(s) \vec{\vec{\psi}}, \vec{\vec{\psi}} \right\rangle \right| \geq \frac{\mathsf{d}_{\min}}{\mathsf{d}_{\max}} \min\left(1, \frac{1}{|s|^3}\right) \mathrm{Re}(s) \left\| \vec{\vec{\psi}} \right\|^2 \tag{4.25}$$

and

$$\left\| \frac{1}{s} \mathcal{J}^{-1}(s) \right\| \geq \frac{\mathsf{d}_{\min}}{\mathsf{d}_{\max}} \min\left(1, \frac{1}{|s|^3}\right) \mathrm{Re}(s). \tag{4.26}$$

The Lax–Milgram lemma in the form of [22] (Lemma 2.1.51 with the definition of ellipticity as in (2.43)) then gives

$$\|s\mathcal{J}(s)\| \leq \frac{\mathsf{d}_{\max}}{\mathsf{d}_{\min}} \frac{1}{\mathrm{Re}(s)} \max\left(1, |s|^3\right) \tag{4.27}$$

$$= \frac{\mathsf{d}_{\max}}{\mathsf{d}_{\min}} \frac{1}{\mathrm{Re}(s)} \max\left(1, \frac{1}{|s|^3}\right) |s|^3 \tag{4.28}$$

$$\leq \frac{\mathsf{d}_{\max}}{\mathsf{d}_{\min}} \left(\frac{1}{\sigma} + \frac{1}{\sigma^4}\right) |s|^3. \tag{4.29}$$

Lemma 3. *From Lemma 2, we get*

$$\int_0^{t_{\mathrm{end}}} e^{-\frac{2t}{t_{\mathrm{end}}}} \left\langle \begin{pmatrix} \vec{\vec{\phi}}(\cdot,t) \\ \vec{\vec{\psi}}(\cdot,t) \end{pmatrix}, \mathcal{B}(\partial_t) \begin{pmatrix} \vec{\vec{\phi}}(\cdot,t) \\ \vec{\vec{\psi}}(\cdot,t) \end{pmatrix} (\cdot,t) \right\rangle_\Gamma dt$$
$$\geq \tilde{\mathsf{c}}\, \mathsf{d}_{\mathrm{end}} \int_0^{t_{\mathrm{end}}} e^{-\frac{2t}{t_{\mathrm{end}}}} \left(\|\partial_t^{-2} \vec{\vec{\phi}}(\cdot,t)\|^2_{(\mathrm{H}^{-\frac12}(\Gamma))^4} + \|\partial_t^{-2} \vec{\vec{\psi}}(\cdot,t)\|^2_{(\mathrm{H}^{\frac12}(\Gamma))^4} \right) dt \tag{4.30}$$

for any $t_{\mathrm{end}} > 0$ and for all $\vec{\vec{\phi}} \in C^5([0,t_{\mathrm{end}}], (\mathrm{H}^{-\frac12}(\Gamma))^4)$, $\vec{\vec{\psi}} \in C^4([0,t_{\mathrm{end}}], (\mathrm{H}^{\frac12}(\Gamma))^4)$, with $\vec{\vec{\phi}}(\cdot,0) = \vec{\vec{0}} = \partial_t^k \vec{\vec{\phi}}(\cdot,0)$, for $k \in \{1,2,3,4\}$ and $\vec{\vec{\psi}}(\cdot,0) = \vec{\vec{0}} = \partial_t^j \vec{\vec{\psi}}(\cdot,0)$, for $j \in \{1,2,3\}$. Here, $\mathsf{d}_{\mathrm{end}} = \min(t_{\mathrm{end}}^{-2}, t_{\mathrm{end}}^{-5})$, and ∂_t^{-1} is a short-hand notation for an integration with respect to time from 0 to t which corresponds to the factor $\frac{1}{s}$ in the Laplace domain.

Lemma 3 can be extended to bound the time behavior of an energy.

Lemma 4. *Let $\mathbb{E} : [0,\infty) \to [0,\infty)$, $\mathbb{S} : \mathbb{R} \to \mathbb{R}$, $\vec{\vec{\phi}} \in C^5([0,t_{\mathrm{end}}], (\mathrm{H}^{-\frac12}(\Gamma))^4)$ and $\vec{\vec{\psi}} \in C^4([0,t_{\mathrm{end}}], (\mathrm{H}^{\frac12}(\Gamma))^4)$ with $\vec{\vec{\phi}}(\cdot,0) = \vec{\vec{0}} = \partial_t^k \vec{\vec{\phi}}(\cdot,0)$ for $k \in \{1,2,3,4\}$ and $\vec{\vec{\psi}}(\cdot,0) = \vec{\vec{0}} = \partial_t^j \vec{\vec{\psi}}(\cdot,0)$ for $j \in \{1,2,3\}$. If*

$$\dot{\mathbb{E}} + \left\langle \begin{pmatrix} \vec{\vec{\phi}}(\cdot,t) \\ \vec{\vec{\psi}}(\cdot,t) \end{pmatrix}, \mathcal{B}(\partial_t) \begin{pmatrix} \vec{\vec{\phi}}(\cdot,t) \\ \vec{\vec{\psi}}(\cdot,t) \end{pmatrix} \right\rangle_\Gamma = \mathbb{S} \qquad \text{in } [0,t_{\mathrm{end}}], \tag{4.31}$$

then

$$\mathbb{E}(t_{\text{end}}) + \tilde{\mathsf{c}}\,\mathsf{d}_{\text{end}} \int_0^{t_{\text{end}}} \left(\|\partial_t^{-2} \vec{\vec{\phi}}(\cdot,t)\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4} + \|\partial_t^{-2} \vec{\vec{\psi}}(\cdot,t)\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4} \right) dt \tag{4.32}$$
$$\leq e^2\,\mathbb{E}(0) + \int_0^{t_{\text{end}}} e^{2\left(1-\frac{t}{t_{\text{end}}}\right)} \mathbb{S}(t)\,dt,$$

where $\mathsf{d}_{\text{end}} = \min(t_{\text{end}}^{-2}, t_{\text{end}}^{-5})$ *and* $\dot{\mathbb{E}}$ *denotes the derivative of* $\mathbb{E}$ *with respect to* t*.*

For both, Lemma 3 as well as Lemma 4, time discrete analogs hold (see [4]).

Remark 1. Here, we want to highlight one of the major differences to the acoustic [4], elastic [8] and Maxwell's [15] cases. In both, Lemmas 3 and 4, we have integrals involving $\|\partial_t^{-2} \vec{\vec{\phi}}(t,\cdot)\|_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4}$ and $\|\partial_t^{-2} \vec{\vec{\psi}}(t,\cdot)\|_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4}$ instead of $\|\partial_t^{-1} \vec{\vec{\phi}}(t,\cdot)\|_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4}$ and $\|\partial_t^{-1} \vec{\vec{\psi}}(t,\cdot)\|_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4}$. This leads to nontrivial challenges in the energy estimates later on, which are solved in this paper.

Lemma 5. *In the situation of Lemma 3, we have for* $t_{\text{end}} = m\Delta t$ *and* $0 < \Delta t \leq \Delta t_0$ *that*

$$\sum_{n=0}^{m} e^{-\frac{2n\Delta t}{t_{\text{end}}}} \left\langle \begin{pmatrix} \vec{\vec{\phi}}^n \\ \vec{\vec{\psi}}^n \end{pmatrix}, \mathcal{B}(\partial_t^{\Delta t}) \begin{pmatrix} \vec{\vec{\phi}}^n \\ \vec{\vec{\psi}}^n \end{pmatrix} \right\rangle_\Gamma \tag{4.33}$$
$$\geq \tilde{\mathsf{c}}\,\mathsf{d}_{\text{end}} \sum_{n=0}^{m} e^{-\frac{2n\Delta t}{t_{\text{end}}}} \left(\|(\partial_t^{\Delta t})^{-2} \vec{\vec{\phi}}^n\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4} + \|(\partial_t^{\Delta t})^{-2} \vec{\vec{\psi}}^n\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4} \right),$$

for all sequences $\vec{\vec{\phi}}^n = \vec{\vec{\phi}}(n\Delta t,\cdot)$ *in* $(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4$ *and* $\vec{\vec{\psi}}^n = \vec{\vec{\psi}}(n\Delta t,\cdot)$ *in* $(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4$, $n = 0,\ldots,m$*. Here,* $\mathsf{d}_{\text{end}} = \mathsf{c}\min(t_{\text{end}}^{-2}, t_{\text{end}}^{-5})$*, where* $\mathsf{c} > 0$ *depends only on* Δt_0 *and tends to* 1 *as* Δt_0 *tends to zero.*

Lemma 6. *Let* $\mathbb{E} : [0,\infty) \to [0,\infty)$, $\mathbb{S} : \mathbb{R} \to \mathbb{R}$, *and* $\vec{\vec{\phi}}^n$ *and* $\vec{\vec{\psi}}^n$ *as in Lemma 5. Further, let* $t_{\text{end}} = m\Delta t$ *and* $0 < \Delta t \leq \Delta t_0$*. If*

$$\frac{\mathbb{E}^{n+1} - \mathbb{E}^n}{\Delta t} + \left\langle \begin{pmatrix} \vec{\vec{\phi}}^n \\ \vec{\vec{\psi}}^n \end{pmatrix}, \mathcal{B}(\partial_t^{\Delta t}) \begin{pmatrix} \vec{\vec{\phi}}^n \\ \vec{\vec{\psi}}^n \end{pmatrix} \right\rangle_\Gamma = \mathbb{S}^n \qquad \text{for } n = 0,\ldots,m, \tag{4.34}$$

then

$$\mathbb{E}^{m+1} + \tilde{\mathsf{c}}\,\mathsf{d}_{\text{end}} \Delta t \sum_{n=0}^{m} \left(\|(\partial_t^{\Delta t})^{-2} \vec{\vec{\phi}}^n\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4} + \|(\partial_t^{\Delta t})^{-2} \vec{\vec{\psi}}^n\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4} \right) \tag{4.35}$$
$$\leq \mathsf{c} \left(e^2\,\mathbb{E}^0 + \Delta t \sum_{n=0}^{m} e^{2(1-\frac{n}{m})} \mathbb{S}^n \right),$$

where $\mathsf{d}_{\text{end}} = \min(t_{\text{end}}^{-2}, t_{\text{end}}^{-5})$ *and* $\mathsf{c} > 0$ *depends only on* Δt_0 *and tends to* 1 *as* Δt_0 *tends to zero.*

5. Discretization

As we now have all the necessary theoretical results, let us discuss the discretization of our problem. We do this in two steps, first a semi-discretization in space and then a full discretization including space and time.

5.1. Variational formulation

Returning to the time domain, the second relation in Eq. (3.12) translates to the more explicit formulation

$$\vec{\phi}_{\vec{U}} = -\partial_t^{-1}\vec{\vec{\gamma}}_D(2\mu\vec{\vec{E}}(\vec{U}) + \lambda(\vec{\nabla}\cdot\vec{U})\vec{\vec{I}} - \beta\partial_t Q\vec{\vec{I}})\vec{n}, \tag{5.1}$$

$$\phi_Q = -\partial_t^{-1}\vec{\gamma}_D\left(\frac{\kappa}{T_{\text{ref}}}\vec{\nabla}\partial_t Q\right)\cdot\vec{n}, \tag{5.2}$$

which yields for the Calderón operator

$$\mathcal{B}(\partial_t)\begin{pmatrix}\vec{\phi}_{\vec{U}}\\ \phi_Q\\ \vec{\psi}_{\vec{U}}\\ \psi_Q\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\vec{\gamma}_D\vec{U}\\ \gamma_D Q\\ -\partial_t^{-1}\vec{\vec{\gamma}}_D(2\mu\vec{\vec{E}}(\vec{U}) + \lambda(\vec{\nabla}\cdot\vec{U})\vec{\vec{I}} - \beta\partial_t Q\vec{\vec{I}})\vec{n}\\ -\partial_t^{-1}\vec{\gamma}_D\left(\frac{\kappa}{T_{\text{ref}}}\vec{\nabla}\partial_t Q\right)\cdot\vec{n}\end{pmatrix}. \tag{5.3}$$

Together with Eqs. (2.9) and (2.10), this leads to the following formulation which is of first order in time:

$$\rho\partial_t\vec{U} = \mu\vec{\nabla}\cdot\vec{\vec{V}} + \lambda\vec{\nabla}W - \beta\vec{\nabla}Q + \rho\vec{F}, \tag{5.4}$$

$$\partial_t\vec{\vec{V}} = 2\vec{\vec{E}}(\vec{U}) = \vec{\nabla}\vec{U} + (\vec{\nabla}\vec{U})^T, \tag{5.5}$$

$$\partial_t W = \vec{\nabla}\cdot\vec{U}, \tag{5.6}$$

$$\varpi\partial_t Q = \frac{\kappa}{T_{\text{ref}}}\Delta Q - \beta\vec{\nabla}\cdot\vec{U} + \varpi G, \tag{5.7}$$

$$\mathcal{B}(\partial_t)\begin{pmatrix}\vec{\phi}_{\vec{U}}\\ \phi_Q\\ \vec{\psi}_{\vec{U}}\\ \psi_Q\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\vec{\gamma}_D\vec{U}\\ \gamma_D Q\\ -\vec{\vec{\gamma}}_D(\mu\vec{\vec{V}} + \lambda W\vec{\vec{I}} - \beta Q\vec{\vec{I}})\vec{n}\\ -\frac{\kappa}{T_{\text{ref}}}\left(\vec{\gamma}_D\vec{\nabla}Q\right)\cdot\vec{n}\end{pmatrix}. \tag{5.8}$$

With the test functions $X, R \in \mathrm{H}^1(\Omega)$, $\vec{Z} \in (\mathrm{H}^1(\Omega))^3$, $\frac{1}{2}\vec{\vec{Y}} \in (\mathrm{H}^1(\Omega))^{3\times 3}$, $\xi_Q \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$, $\chi_Q \in \mathrm{H}^{\frac{1}{2}}(\Gamma)$, $\vec{\xi}_U \in (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3$, and $\vec{\chi}_U \in (\mathrm{H}^{\frac{1}{2}}(\Gamma))^3$, we perform integration by parts to obtain a variational formulation. However, we split up some of the terms before performing integration by parts in order to end up with antisymmetries. As it should be clear from context, we only write $(\cdot,\cdot)$ to denote the L^2-inner product in each of the spaces $\mathrm{H}^1(\Omega)$, $(\mathrm{H}^1(\Omega))^3$, and $(\mathrm{H}^1(\Omega))^{3\times 3}$. By observing that $\vec{\vec{V}} = \vec{\vec{V}}^T$, we arrive at

$$\begin{aligned}\rho(\partial_t\vec{U},\vec{Z}) = & -\frac{1}{2}\mu(\vec{\vec{V}},\vec{\nabla}\vec{Z}) + \frac{1}{2}\mu(\vec{\nabla}\cdot\vec{\vec{V}},\vec{Z}) + \frac{1}{2}\mu\langle\vec{\vec{\gamma}}_D(\vec{\vec{V}})\vec{n},\vec{\gamma}_D\vec{Z}\rangle_\Gamma \\ & -\frac{1}{2}\lambda(W,\vec{\nabla}\cdot\vec{Z}) + \frac{1}{2}\lambda(\vec{\nabla}W,\vec{Z}) + \frac{1}{2}\lambda\langle\vec{\vec{\gamma}}_D(W\vec{\vec{I}})\vec{n},\vec{\gamma}_D\vec{Z}\rangle_\Gamma \\ & +\frac{1}{2}\beta(Q,\vec{\nabla}\cdot\vec{Z}) - \frac{1}{2}\beta(\vec{\nabla}Q,\vec{Z}) - \frac{1}{2}\beta\langle\vec{\vec{\gamma}}_D(Q\vec{\vec{I}})\vec{n},\vec{\gamma}_D\vec{Z}\rangle_\Gamma + \rho(\vec{F},\vec{Z}),\end{aligned} \tag{5.9}$$

$$(\partial_t\vec{\vec{V}},\frac{1}{2}\vec{\vec{Y}}) = \frac{1}{2}(\vec{\nabla}\vec{U},\vec{\vec{Y}}) - \frac{1}{2}(\vec{U},\vec{\nabla}\cdot\vec{\vec{Y}}) + \frac{1}{2}\langle\vec{\gamma}_D\vec{U},\vec{\vec{\gamma}}_D(\vec{\vec{Y}})\vec{n}\rangle_\Gamma, \tag{5.10}$$

$$(\partial_t W, X) = \frac{1}{2}(\vec{\nabla}\cdot\vec{U}, X) - \frac{1}{2}(\vec{U},\vec{\nabla}X) + \frac{1}{2}\langle\vec{\gamma}_D\vec{U},\vec{\vec{\gamma}}_D(X\vec{\vec{I}})\vec{n}\rangle_\Gamma, \tag{5.11}$$

$$\varpi\,(\partial_t Q, R) = -\frac{\kappa}{T_{\text{ref}}}(\vec{\nabla}Q,\vec{\nabla}R) + \frac{\kappa}{T_{\text{ref}}}\langle(\vec{\gamma}_D\vec{\nabla}Q)\cdot\vec{n},\gamma_D R\rangle_\Gamma \tag{5.12}$$

$$-\frac{1}{2}\beta(\vec{\nabla}\cdot\vec{U},R)+\frac{1}{2}\beta(\vec{U},\vec{\nabla}R)-\frac{1}{2}\beta\langle\vec{\gamma}_D\vec{U},\vec{\vec{\gamma}}_D(R\vec{\vec{I}})\vec{n}\rangle_\Gamma+\varpi(G,R),$$

$$\left\langle\begin{pmatrix}\vec{\xi}_{\vec{U}}\\ \xi_Q\\ \vec{\chi}_{\vec{U}}\\ \chi_Q\end{pmatrix},\mathcal{B}(\partial_t)\begin{pmatrix}\vec{\phi}_{\vec{U}}\\ \phi_Q\\ \vec{\psi}_{\vec{U}}\\ \psi_Q\end{pmatrix}\right\rangle_\Gamma=\frac{1}{2}\langle\vec{\xi}_{\vec{U}},\vec{\gamma}_D\vec{U}\rangle_\Gamma-\frac{1}{2}\langle\vec{\chi}_{\vec{U}},\vec{\vec{\gamma}}_D(\mu\vec{\vec{V}}+\lambda W\vec{\vec{I}}-\beta Q\vec{\vec{I}})\vec{n}\rangle_\Gamma$$
$$+\frac{1}{2}\langle\xi_Q,\gamma_D Q\rangle_\Gamma-\frac{1}{2}\left\langle\chi_Q,\vec{\gamma}_D\left(\frac{\kappa}{T_{\mathrm{ref}}}\vec{\nabla}Q\right)\cdot\vec{n}\right\rangle_\Gamma. \tag{5.13}$$

These equations are completed by the relations

$$\vec{\phi}_{\vec{U}}=-\vec{\vec{\gamma}}_D(\mu\vec{\vec{V}}+\lambda W\vec{\vec{I}}-\beta Q\vec{\vec{I}})\vec{n},\quad \phi_Q=-\frac{\kappa}{T_{\mathrm{ref}}}\left(\vec{\gamma}_D\vec{\nabla}Q\right)\cdot\vec{n},\quad \vec{\psi}_{\vec{U}}=\vec{\gamma}_D\vec{U},\quad \psi_Q=\gamma_D Q, \tag{5.14}$$

which are valid due to the transmission conditions (2.22).

Choosing as test functions $\vec{Z}=\vec{U}$, $\vec{\vec{Y}}=\mu\vec{\vec{V}}$, $X=\lambda W$, $R=Q$, $\vec{\xi}_{\vec{U}}=\vec{\phi}_{\vec{U}}$, $\xi_Q=\phi_Q$, $\vec{\chi}_{\vec{U}}=\vec{\psi}_{\vec{U}}$, as well as $\chi_Q=\psi_Q$, and rearranging terms slightly yields

$$\begin{aligned}\rho(\partial_t\vec{U},\vec{U})=&-\mu\frac{1}{2}(\vec{\vec{V}},\vec{\nabla}\vec{U})+\mu\frac{1}{2}(\vec{\nabla}\cdot\vec{\vec{V}},\vec{U})-\lambda\frac{1}{2}(W,\vec{\nabla}\cdot\vec{U})+\lambda\frac{1}{2}(\vec{\nabla}W,\vec{U})\\&+\frac{1}{2}\beta(Q,\vec{\nabla}\cdot\vec{U})-\frac{1}{2}\beta(\vec{\nabla}Q,\vec{U})\\&+\frac{1}{2}\langle\vec{\vec{\gamma}}_D\left(\mu\vec{\vec{V}}+\lambda W\vec{\vec{I}}-\beta Q\vec{\vec{I}}\right)\vec{n},\vec{\gamma}_D\vec{U}\rangle_\Gamma\\&+\rho(\vec{F},\vec{U}),\end{aligned} \tag{5.15}$$

$$(\partial_t\vec{\vec{V}},\frac{\mu}{2}\vec{\vec{V}})=+\mu\frac{1}{2}(\vec{\nabla}\vec{U},\vec{\vec{V}})-\mu\frac{1}{2}(\vec{U},\vec{\nabla}\cdot\vec{\vec{V}})+\mu\frac{1}{2}\langle\vec{\gamma}_D\vec{U},\vec{\vec{\gamma}}_D(\vec{\vec{V}})\vec{n}\rangle_\Gamma, \tag{5.16}$$

$$(\partial_t W,\lambda W)=+\lambda\frac{1}{2}(\vec{\nabla}\cdot\vec{U},W)-\lambda\frac{1}{2}(\vec{U},\vec{\nabla}W)+\lambda\frac{1}{2}\langle\vec{\gamma}_D\vec{U},\vec{\vec{\gamma}}_D(W\vec{\vec{I}})\vec{n}\rangle_\Gamma, \tag{5.17}$$

$$\begin{aligned}\varpi(\partial_t Q,Q)=&-\frac{\kappa}{T_{\mathrm{ref}}}(\vec{\nabla}Q,\vec{\nabla}Q)+\frac{\kappa}{T_{\mathrm{ref}}}\langle(\vec{\gamma}_D\vec{\nabla}Q)\cdot\vec{n},\gamma_D Q\rangle_\Gamma\\&-\frac{1}{2}\beta(\vec{\nabla}\cdot\vec{U},Q)+\frac{1}{2}\beta(\vec{U},\vec{\nabla}Q)-\frac{1}{2}\beta\langle\vec{\gamma}_D\vec{U},\vec{\vec{\gamma}}_D(Q\vec{\vec{I}})\vec{n}\rangle_\Gamma\\&+\varpi(G,Q),\end{aligned} \tag{5.18}$$

$$\left\langle\begin{pmatrix}\vec{\phi}_{\vec{U}}\\ \phi_Q\\ \vec{\psi}_{\vec{U}}\\ \psi_Q\end{pmatrix},\mathcal{B}(\partial_t)\begin{pmatrix}\vec{\phi}_{\vec{U}}\\ \phi_Q\\ \vec{\psi}_{\vec{U}}\\ \psi_Q\end{pmatrix}\right\rangle_\Gamma=\frac{1}{2}\langle\vec{\phi}_{\vec{U}},\vec{\gamma}_D\vec{U}\rangle_\Gamma-\frac{1}{2}\langle\vec{\psi}_{\vec{U}},\vec{\vec{\gamma}}_D(\mu\vec{\vec{V}}+\lambda W\vec{\vec{I}}-\beta Q\vec{\vec{I}})\vec{n}\rangle_\Gamma$$
$$+\frac{1}{2}\langle\phi_Q,\gamma_D Q\rangle_\Gamma-\frac{\kappa}{2T_{\mathrm{ref}}}\langle\psi_Q,\left(\vec{\gamma}_D\vec{\nabla}Q\right)\cdot\vec{n}\rangle_\Gamma. \tag{5.19}$$

Taking Eq. (5.14) into account, terms given the same color cancel out when we sum up these equations to obtain

$$\rho(\partial_t\vec{U},\vec{U})+\frac{1}{2}\mu(\partial_t\vec{\vec{V}},\vec{\vec{V}})+\lambda(\partial_t W,W)+\varpi(\partial_t Q,Q)+\frac{\kappa}{T_{\mathrm{ref}}}(\vec{\nabla}Q,\vec{\nabla}Q)+\left\langle\begin{pmatrix}\vec{\vec{\phi}}\\ \vec{\vec{\psi}}\end{pmatrix},\mathcal{B}(\partial_t)\begin{pmatrix}\vec{\vec{\phi}}\\ \vec{\vec{\psi}}\end{pmatrix}\right\rangle_\Gamma$$
$$=\rho(\vec{F},\vec{U})+\varpi(G,Q) \tag{5.20}$$

$$\begin{aligned}\iff \frac{\mathrm{d}}{\mathrm{d}t}\Bigg(&\frac{1}{2}\rho\left\|\vec{U}\right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}+\frac{1}{4}\mu\left\|\vec{\vec{V}}\right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times3}}+\frac{1}{2}\lambda\|W\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}+\frac{1}{2}\varpi\|Q\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}\\&+\frac{\kappa}{T_{\mathrm{ref}}}\int_0^t\|\vec{\nabla}Q(\cdot,t')\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}\,\mathrm{d}t'\Bigg)+\left\langle\begin{pmatrix}\vec{\vec{\phi}}\\\vec{\vec{\psi}}\end{pmatrix},B(\partial_t)\begin{pmatrix}\vec{\vec{\phi}}\\\vec{\vec{\psi}}\end{pmatrix}\right\rangle_\Gamma\\&=\rho(\vec{F},\vec{U})_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}+\varpi(G,Q)_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}.\end{aligned}\tag{5.21}$$

Hence, we end up with an energy given by

$$\begin{aligned}\mathbb{E}(t)=&\frac{1}{2}\rho\left\|\vec{U}\right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^3}+\frac{1}{4}\mu\left\|\vec{\vec{V}}\right\|^2_{(\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma))^{3\times3}}+\frac{1}{2}\lambda\left\|W\right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}+\frac{1}{2}\varpi\left\|Q\right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}\\&+\frac{\kappa}{T_{\mathrm{ref}}}\int_0^t\left\|\vec{\nabla}Q(\cdot,t')\right\|^2_{\mathrm{L}^2(\mathbb{R}^3\setminus\Gamma)}\,\mathrm{d}t'.\end{aligned}\tag{5.22}$$

Here, we see the first consequence of the presence of a Laplace operator in Eq. (5.7). This is actually similar to the dependence on $\vec{U}$ as $\vec{\vec{V}}$ is the time-integrated symmetrized Jacobi matrix of $\vec{U}$ and W its time-integrated curl. However, we want to remark that this energy is a mathematical construct and should not be interpreted as a quantity with a physical meaning.

Remark 2. We want to mention that this special choice of testing (see (5.15)–(5.19)) is required for the construction of the energy (5.22) and is only possible due to our variable substitution $T \leftarrow \partial_t Q$, we introduced in (2.10).

If $\rho > 0$, $\mu > 0$, $\lambda > 0$, $\kappa > 0$ which is valid for most common materials, and $T_{\mathrm{ref}} > 0$, Lemma 4 implies for $\vec{F} = \vec{0}$ and $G = 0$ (i.e., $\mathbb{S} = 0$) that the field energy is bounded for $t > 0$:

$$\mathbb{E}(t_{\mathrm{end}})+\tilde{\mathsf{c}}\,\mathsf{d}_{\mathrm{end}}\int\limits_0^{t_{\mathrm{end}}}\left(\|\partial_t^{-2}\vec{\vec{\phi}}(\cdot,t)\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4}+\|\partial_t^{-2}\vec{\vec{\psi}}(\cdot,t)\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4}\right)\mathrm{d}t\le e^2\,\mathbb{E}(0).\tag{5.23}$$

5.1.1. FEM–BEM spatial semi-discretization. To discretize in space[3], let $\mathfrak{W}_h = \mathfrak{Q}_h \subset \mathrm{H}^1(\Omega)$, $\mathfrak{U}_h = \mathfrak{Q}_h^3 \subset (\mathrm{H}^1(\Omega))^3$, $\mathfrak{V}_h = \mathfrak{Q}_h^{3\times3} \subset (\mathrm{H}^1(\Omega))^{3\times3}$, $\mathfrak{Z}_{Q,h} \subset \mathrm{H}^{\frac{1}{2}}(\Gamma)$, $\mathfrak{Z}_{\vec{U},h} = \mathfrak{Z}_{Q,h}^3 \subset (\mathrm{H}^{\frac{1}{2}}(\Gamma))^3$, $\mathfrak{P}_{Q,h} \subset \mathrm{H}^{-\frac{1}{2}}(\Gamma)$, and $\mathfrak{P}_{\vec{U},h} = \mathfrak{P}_{Q,h}^3 \subset (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3$ be finite dimensional subspaces of the given Sobolev spaces. $\mathfrak{Q}_h$ is a finite element space of continuous piecewise linear function, $\mathfrak{Z}_{Q,h}$ a boundary element space of continuous piecewise linear functions, and $\mathfrak{P}_{Q,h}$ a boundary element space of piecewise constant functions. We denote the chosen bases of these spaces by $(b_i^Q) = (b_i^W)$, $(\vec{b}_i^U)$, $(\vec{\vec{b}}_i^V)$, $(b_i^{\Psi_Q})$, $(\vec{b}_i^{\Psi_U})$, $(b_i^{\Phi_Q})$, and $(\vec{b}_i^{\Phi_U})$. Furthermore, $\mathfrak{Z}_{Q,h}$ and $\mathfrak{P}_{Q,h}$ contain the traces of $\mathfrak{Q}_h$, i.e., $\gamma_D\mathfrak{Q}_h \subseteq \mathfrak{Z}_{Q,h}$ and $\gamma_N\mathfrak{Q}_h \subseteq \mathfrak{P}_{Q,h}$. This is not problematic since we assume that the boundary of Ω is approximated in a piecewise polygonal manner. The inclusion also implies $\vec{\gamma}_D\mathfrak{U}_h \subseteq \mathfrak{Z}_{\vec{U},h}$ and $\vec{\gamma}_N\mathfrak{U}_h \subseteq \mathfrak{P}_{\vec{U},h}$. The semi-discretized problem then reads:

Find $\vec{U}_h(\cdot,t) \in \mathfrak{U}_h$, $\vec{\vec{V}}_h(\cdot,t) \in \mathfrak{V}_h$, $W_h(\cdot,t) \in \mathfrak{Q}_h$, $Q_h(\cdot,t) \in \mathfrak{Q}_h$, $\phi_{Q,h}(\cdot,t) \in \mathfrak{P}_{Q,h}$, $\psi_{Q,h}(\cdot,t) \in \mathfrak{Z}_{Q,h}$, $\vec{\phi}_{\vec{U},h}(\cdot,t) \in \mathfrak{P}_{\vec{U},h}$, and $\vec{\psi}_{\vec{U},h}(\cdot,t) \in \mathfrak{Z}_{\vec{U},h}$ such that for all $\vec{Z}_h \in \mathfrak{U}_h$, $\vec{\vec{Y}}_h \in \mathfrak{V}_h$, $X_h \in \mathfrak{Q}_h$, $R_h \in \mathfrak{Q}_h$, $\xi_{Q,h} \in \mathfrak{P}_{Q,h}$, $\chi_{Q,h} \in \mathfrak{Z}_{Q,h}$, $\vec{\xi}_{\vec{U},h} \in \mathfrak{P}_{\vec{U},h}$, and $\vec{\chi}_{\vec{U},h} \in \mathfrak{Z}_{\vec{U},h}$, Eqs. (5.9)–(5.14) hold with the respective substitutions. We can turn the result into matrix-vector-form by testing with the according basis functions $(b_i^Q) = (b_i^W)$, $(\vec{b}_i^U)$, $(\vec{\vec{b}}_i^V)$, $(b_i^{\Psi_Q})$, $(\vec{b}_i^{\Psi_U})$, $(b_i^{\Phi_Q})$, and $(\vec{b}_i^{\Phi_U})$ (provided the occurring functions are evaluated

[3]The discrete subspaces of the corresponding Sobolev spaces are denoted with bold fractional letters.

in the nodes of the discretization)[4]:

$$\rho\underline{\underline{\boldsymbol{M}}}_0\partial_t\boldsymbol{U} = \mu\underline{\underline{\boldsymbol{D}}}_0\boldsymbol{V} + \lambda\underline{\underline{\boldsymbol{D}}}_1\boldsymbol{W} - \beta\underline{\underline{\boldsymbol{D}}}_1\boldsymbol{Q} - \underline{\underline{\boldsymbol{C}}}_0\boldsymbol{\phi}_U + \rho\underline{\underline{\boldsymbol{M}}}_0\boldsymbol{F}, \tag{5.24}$$

$$\underline{\underline{\boldsymbol{M}}}_1\partial_t\boldsymbol{V} = -\underline{\underline{\boldsymbol{D}}}_0^T\boldsymbol{U} + \underline{\underline{\boldsymbol{C}}}_1\boldsymbol{\psi}_U\left\{+\underline{\underline{\boldsymbol{M}}}_1\boldsymbol{I}\right\}, \tag{5.25}$$

$$\underline{\underline{\boldsymbol{M}}}_2\partial_t\boldsymbol{W} = -\underline{\underline{\boldsymbol{D}}}_1^T\boldsymbol{U} + \underline{\underline{\boldsymbol{C}}}_2\boldsymbol{\psi}_U\left\{+\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{K}\right\}, \tag{5.26}$$

$$\varpi\underline{\underline{\boldsymbol{M}}}_2\partial_t\boldsymbol{Q} = -\frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{D}}}_2\boldsymbol{Q} - \underline{\underline{\boldsymbol{C}}}_3\boldsymbol{\phi}_Q + \frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{C}}}_4\boldsymbol{\psi}_Q + \beta\underline{\underline{\boldsymbol{D}}}_1^T\boldsymbol{U} - \beta\underline{\underline{\boldsymbol{C}}}_2\boldsymbol{\psi}_U + \varpi\,\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{G}, \tag{5.27}$$

$$\boldsymbol{B}(\partial_t)\begin{pmatrix}\boldsymbol{\phi}_U\\ \boldsymbol{\phi}_Q\\ \boldsymbol{\psi}_U\\ \boldsymbol{\psi}_Q\end{pmatrix} = \begin{pmatrix}\underline{\underline{\boldsymbol{C}}}_0^T\boldsymbol{U}\\ \underline{\underline{\boldsymbol{C}}}_3^T\boldsymbol{Q}\\ -\mu\underline{\underline{\boldsymbol{C}}}_1^T\boldsymbol{V} - \lambda\underline{\underline{\boldsymbol{C}}}_2^T\boldsymbol{W} + \beta\underline{\underline{\boldsymbol{C}}}_2^T\boldsymbol{Q}\\ -\frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{C}}}_4^T\boldsymbol{Q}\end{pmatrix}\left\{+\begin{pmatrix}\nu\\ \varsigma\\ \eta\\ \xi\end{pmatrix}\right\}. \tag{5.28}$$

Here, we have

$$\underline{\underline{\boldsymbol{M}}}_0|_{ij} := (\vec{b}_i^U, \vec{b}_j^U), \qquad \underline{\underline{\boldsymbol{M}}}_1|_{ij} := \frac{1}{2}\left(\overset{\rightrightarrows}{b}_i^V, \overset{\rightrightarrows}{b}_j^V\right), \qquad \underline{\underline{\boldsymbol{M}}}_2|_{ij} := (b_i^Q, b_j^Q), \tag{5.29}$$

$$\begin{aligned}\underline{\underline{\boldsymbol{D}}}_0|_{ij} &:= \frac{1}{2}\left(\vec{b}_i^U, \vec{\nabla}\cdot\overset{\rightrightarrows}{b}_j^V\right) - \frac{1}{2}\left(\vec{\nabla}\vec{b}_i^U, \overset{\rightrightarrows}{b}_j^V\right), \qquad \underline{\underline{\boldsymbol{D}}}_1|_{ij} := \frac{1}{2}\left(\vec{b}_i^U, \vec{\nabla}b_j^Q\right) - \frac{1}{2}\left(\vec{\nabla}\cdot\vec{b}_i^U, b_j^Q\right),\\ \underline{\underline{\boldsymbol{D}}}_2|_{ij} &:= \left(\vec{\nabla}b_i^Q, \vec{\nabla}b_j^Q\right),\end{aligned} \tag{5.30}$$

and for the boundary

$$\begin{aligned}\underline{\underline{\boldsymbol{C}}}_0|_{ij} &:= \frac{1}{2}\left\langle\vec{b}_i^{\Psi_U}, \vec{b}_j^{\Phi_U}\right\rangle_\Gamma, \qquad \underline{\underline{\boldsymbol{C}}}_1|_{ij} := \frac{1}{2}\langle\overset{\rightrightarrows}{\gamma}_D(\overset{\rightrightarrows}{b}_i^V)\vec{n}, \vec{b}_j^{\Psi_U}\rangle_\Gamma, \qquad \underline{\underline{\boldsymbol{C}}}_2|_{ij} := \frac{1}{2}\langle b_i^{\Psi_Q}\vec{n}, \vec{b}_j^{\Psi_U}\rangle_\Gamma,\\ \underline{\underline{\boldsymbol{C}}}_3|_{ij} &:= \frac{1}{2}\langle b_i^{\Psi_Q}, b_j^{\Phi_Q}\rangle_\Gamma, \qquad \underline{\underline{\boldsymbol{C}}}_4|_{ij} := \frac{1}{2}\langle\left(\vec{\gamma}_D\vec{\nabla}b_i^Q\right)\cdot\vec{n}, b_j^{\Psi_Q}\rangle_\Gamma.\end{aligned} \tag{5.31}$$

The terms in gray are meant to be perturbations. They do not have any physical meaning, but need to be considered in our stability analysis later on. For Eqs. (5.24) and (5.27), perturbations are considered as parts of the actual right-hand sides $\boldsymbol{F}$ and $\boldsymbol{G}$, respectively. Further, we divided the boundary term from Eq. (5.12) into two terms and plugged in $\phi_{Q,h}$ for one, but $\psi_{Q,h}$ for the other. Thus, terms involving these boundary contributions and the related terms from Eq. (5.28) also cancel out in the semi-discrete and discrete settings.

With $\overset{\rightrightarrows}{b}_i^\Psi$ and $\overset{\rightrightarrows}{b}_i^\Phi$ suitable basis functions of $\mathfrak{Z}_h := \left(\mathfrak{Z}_{Q,h}\right)^4$ and $\mathfrak{P}_h := \left(\mathfrak{P}_{Q,h}\right)^4$, respectively, we can define the blocks

$$\underline{\underline{\boldsymbol{J}}}(s)|_{ij} := \left\langle\overset{\rightrightarrows}{b}_i^\Phi, \mathcal{J}(s)\overset{\rightrightarrows}{b}_j^\Phi\right\rangle_\Gamma, \qquad \underline{\underline{\boldsymbol{K}}}(s)|_{ij} := \left\langle\overset{\rightrightarrows}{b}_i^\Phi, \mathcal{K}(s)\overset{\rightrightarrows}{b}_j^\Psi\right\rangle_\Gamma, \qquad \underline{\underline{\boldsymbol{W}}}(s)|_{ij} := \left\langle\overset{\rightrightarrows}{b}_i^\Psi, \mathcal{W}(s)\overset{\rightrightarrows}{b}_j^\Psi\right\rangle_\Gamma \tag{5.32}$$

and, thus, represent the matrix $\boldsymbol{B}(s)$ by

$$\boldsymbol{B}(s) = \begin{pmatrix} s\underline{\underline{\boldsymbol{J}}}(s) & \underline{\underline{\boldsymbol{K}}}(s)\\ -\underline{\underline{\widetilde{\boldsymbol{K}}}}^*(s) & \frac{1}{s}\underline{\underline{\boldsymbol{W}}}(s)\end{pmatrix}. \tag{5.33}$$

[4]Discrete vector-valued quantities are written in bold and matrices in addition with $\underline{\underline{\cdot}}$.

5.2. Time discretization via leapfrog and convolution quadrature

In order to apply the leapfrog method, see, e.g., [12], and convolution quadrature, we use the same ideas as in [8], but have to account for the additional terms involving $\underline{\underline{\boldsymbol{D}_1}}\boldsymbol{Q}$ as well as the equation for $\boldsymbol{Q}$ and related boundary conditions. As a consequence, we get a time-stepping scheme which is implicit for $\boldsymbol{Q}$, in accordance with [12], Section 1.8. Thus, we obtain

$$\begin{aligned}
\underline{\underline{\boldsymbol{M}}}_1\boldsymbol{V}^{n+\frac{1}{2}} &= \underline{\underline{\boldsymbol{M}}}_1\boldsymbol{V}^n - \frac{\Delta t}{2}\underline{\underline{\boldsymbol{D}_0^T}}\boldsymbol{U}^n + \frac{\Delta t}{2}\underline{\underline{\boldsymbol{C}}}_1\psi_U^n \left\{+\frac{\Delta t}{2}\underline{\underline{\boldsymbol{M}}}_1\boldsymbol{I}^n\right\}, && (5.34)\\
\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{W}^{n+\frac{1}{2}} &= \underline{\underline{\boldsymbol{M}}}_2\boldsymbol{W}^n - \frac{\Delta t}{2}\underline{\underline{\boldsymbol{D}_1^T}}\boldsymbol{U}^n + \frac{\Delta t}{2}\underline{\underline{\boldsymbol{C}}}_2\psi_U^n \left\{+\frac{\Delta t}{2}\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{K}^n\right\}, && (5.35)\\
\varpi\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{Q}^{n+1} &= \varpi\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{Q}^n - \Delta t\,\frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{D}}}_2\overline{\boldsymbol{Q}}^{n+\frac{1}{2}} - \Delta t\underline{\underline{\boldsymbol{C}}}_3\phi_Q^{n+\frac{1}{2}} + \Delta t\,\frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{C}}}_4\overline{\boldsymbol{\psi}}_Q^{n+\frac{1}{2}}\\
&\quad + \Delta t\beta\underline{\underline{\boldsymbol{D}_1^T}}\overline{\boldsymbol{U}}^{n+\frac{1}{2}} - \Delta t\beta\underline{\underline{\boldsymbol{C}}}_2\overline{\boldsymbol{\psi}}_U^{n+\frac{1}{2}} + \Delta t\varpi\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{G}^{n+\frac{1}{2}}, && (5.36)\\
\rho\underline{\underline{\boldsymbol{M}}}_0\boldsymbol{U}^{n+1} &= \rho\underline{\underline{\boldsymbol{M}}}_0\boldsymbol{U}^n + \Delta t\mu\underline{\underline{\boldsymbol{D}}}_0\boldsymbol{V}^{n+\frac{1}{2}} + \Delta t\lambda\underline{\underline{\boldsymbol{D}}}_1\boldsymbol{W}^{n+\frac{1}{2}}\\
&\quad - \Delta t\beta\underline{\underline{\boldsymbol{D}}}_1\overline{\boldsymbol{Q}}^{n+\frac{1}{2}} - \Delta t\underline{\underline{\boldsymbol{C}}}_0\phi_U^{n+\frac{1}{2}} + \Delta t\rho\underline{\underline{\boldsymbol{M}}}_0\boldsymbol{F}^{n+\frac{1}{2}}, && (5.37)\\
\underline{\underline{\boldsymbol{M}}}_1\boldsymbol{V}^{n+1} &= \underline{\underline{\boldsymbol{M}}}_1\boldsymbol{V}^{n+\frac{1}{2}} - \frac{\Delta t}{2}\underline{\underline{\boldsymbol{D}_0^T}}\boldsymbol{U}^{n+1} + \frac{\Delta t}{2}\underline{\underline{\boldsymbol{C}}}_1\psi_U^{n+1}\left\{+\frac{\Delta t}{2}\underline{\underline{\boldsymbol{M}}}_1\boldsymbol{I}^{n+1}\right\}, && (5.38)\\
\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{W}^{n+1} &= \underline{\underline{\boldsymbol{M}}}_2\boldsymbol{W}^{n+\frac{1}{2}} - \frac{\Delta t}{2}\underline{\underline{\boldsymbol{D}_1^T}}\boldsymbol{U}^{n+1} + \frac{\Delta t}{2}\underline{\underline{\boldsymbol{C}}}_2\psi_U^{n+1}\left\{+\frac{\Delta t}{2}\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{K}^{n+1}\right\}, && (5.39)
\end{aligned}$$

$$\left[\boldsymbol{B}(\partial_t^{\Delta t})\begin{pmatrix}\phi_U\\ \phi_Q\\ \overline{\psi}_U\\ \overline{\psi}_Q\end{pmatrix}\right]^{n+\frac{1}{2}} = \begin{pmatrix}\underline{\underline{\boldsymbol{C}_0^T}}\overline{\boldsymbol{U}}^{n+\frac{1}{2}}\\ \underline{\underline{\boldsymbol{C}_3^T}}\overline{\boldsymbol{Q}}^{n+\frac{1}{2}}\\ -\mu\underline{\underline{\boldsymbol{C}_1^T}}\widetilde{\boldsymbol{V}}^{n+\frac{1}{2}} - \lambda\underline{\underline{\boldsymbol{C}_2^T}}\widetilde{\boldsymbol{W}}^{n+\frac{1}{2}} + \beta\underline{\underline{\boldsymbol{C}_2^T}}\overline{\boldsymbol{Q}}^{n+\frac{1}{2}}\\ -\frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{C}_4^T}}\overline{\boldsymbol{Q}}^{n+\frac{1}{2}}\end{pmatrix}\left\{+\begin{pmatrix}\boldsymbol{\nu}^{n+\frac{1}{2}}\\ \boldsymbol{\varsigma}^{n+\frac{1}{2}}\\ \boldsymbol{\eta}^{n+\frac{1}{2}}\\ \boldsymbol{\xi}^{n+\frac{1}{2}}\end{pmatrix}\right\}, \tag{5.40}$$

where the bar above a quantity denotes an average, e.g.,

$$\overline{\boldsymbol{Q}}^{n+\frac{1}{2}} = \frac{1}{2}\left(\boldsymbol{Q}^{n+1} + \boldsymbol{Q}^n\right) \tag{5.41}$$

and analogously for all others. Further, we introduced

$$\begin{aligned}
\widetilde{\boldsymbol{V}}^{n+\frac{1}{2}} &= \boldsymbol{V}^{n+\frac{1}{2}} + \delta\,(\Delta t)^2\,\underline{\underline{\boldsymbol{M}_1^{-1}}}\underline{\underline{\boldsymbol{C}}}_1\dot{\psi}_U^{n+\frac{1}{2}}, && (5.42)\\
\widetilde{\boldsymbol{W}}^{n+\frac{1}{2}} &= \boldsymbol{W}^{n+\frac{1}{2}} + \delta\,(\Delta t)^2\,\underline{\underline{\boldsymbol{M}_2^{-1}}}\underline{\underline{\boldsymbol{C}}}_2\dot{\psi}_U^{n+\frac{1}{2}} && (5.43)
\end{aligned}$$

where $\delta > 0$ is a stabilization parameter which is discussed in more detail in Section 8 and the dot above a quantity denotes a central difference with respect to time, i.e.,

$$\dot{\boldsymbol{\psi}}_U^{n+\frac{1}{2}} = \frac{1}{\Delta t}\left(\dot{\boldsymbol{\psi}}_U^{n+1} - \dot{\boldsymbol{\psi}}_U^{n}\right) \tag{5.44}$$

and similarly for other quantities. Please note that there is another discretization for the first-order derivative with respect to time in play here, which is denoted by $\partial_t^{\Delta t}$. It is used for the discretization of the Calderón operator and given by the backward differentiation formula BDF-2 according to [4], Section 2.3.

Let us compare the above system of equations to the known ones for the acoustic and the elastic wave equations. For both of those, only the system of equations incorporating the Calderón operator is implicit. Thus, it is possible to express the analog to $\overline{\boldsymbol{U}}^{n+\frac{1}{2}}$ explicitly without referring to $\boldsymbol{U}^{n+1}$. This allows for a decoupling such that a more concrete expression can be given for the (still implicit) equations involving the Calderón operator, for the acoustic wave equation in [4], Section 5.3, and for the elastic wave equation in [8], Section 5.3. Due to the implicit nature of Eqs. (5.36) and (5.37), we cannot do something like this here. Only Eqs. (5.34), (5.35), (5.38), and (5.39) can be treated as explicit time-stepping schemes, whereas all other equations must be considered together as a coupled system of implicit equations.

Further, we provide the equations for $\boldsymbol{V}$ and $\boldsymbol{W}$ using a staggered grid, cf. [12], Eq. (1.7), as this is the version used to prove stability results and is obtained by substituting (5.38) and (5.39) into (5.34) and (5.35), respectively, so that

$$\underline{\underline{\boldsymbol{M}}}_1\boldsymbol{V}^{n+\frac{1}{2}} = \underline{\underline{\boldsymbol{M}}}_1\boldsymbol{V}^{n-\frac{1}{2}} - \Delta t\underline{\underline{\boldsymbol{D}}}_0^T\boldsymbol{U}^n + \Delta t\underline{\underline{\boldsymbol{C}}}_1\boldsymbol{\psi}_U^n \left\{+\Delta t\,\underline{\underline{\boldsymbol{M}}}_1\boldsymbol{I}^n\right\}, \tag{5.45}$$

$$\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{W}^{n+\frac{1}{2}} = \underline{\underline{\boldsymbol{M}}}_2\boldsymbol{W}^{n-\frac{1}{2}} - \Delta t\underline{\underline{\boldsymbol{D}}}_1^T\boldsymbol{U}^n + \Delta t\underline{\underline{\boldsymbol{C}}}_2\boldsymbol{\psi}_U^n \left\{+\Delta t\,\underline{\underline{\boldsymbol{M}}}_2\boldsymbol{K}^n\right\}. \tag{5.46}$$

We will also need a set of semi-discrete equations which is of third order in time. These can be derived by taking derivatives with respect to time of Eqs. (5.24), (5.27), and (5.28) as well as using Eqs. (5.25) and (5.26) to replace the time derivatives of $\boldsymbol{V}$ and $\boldsymbol{W}$ that appear on the right-hand when time derivatives are applied to Eq. (5.24).

For our stability considerations, we also need a formulation which is closer to a time discretized version of that system which reads

$$\begin{aligned}
\rho\underline{\underline{\boldsymbol{M}}}_0\left(\dot{\boldsymbol{U}}^{n+1} - 2\dot{\boldsymbol{U}}^n + \dot{\boldsymbol{U}}^{n-1}\right) = &-(\Delta t)^2\,\mu\underline{\underline{\boldsymbol{D}}}_0\underline{\underline{\boldsymbol{M}}}_1^{-1}\left(\underline{\underline{\boldsymbol{D}}}_0^T\dot{\boldsymbol{U}}^n - \underline{\underline{\boldsymbol{C}}}_1\dot{\boldsymbol{\psi}}_U^n\right)\\
&-(\Delta t)^2\,\lambda\underline{\underline{\boldsymbol{D}}}_1\underline{\underline{\boldsymbol{M}}}_2^{-1}\left(\underline{\underline{\boldsymbol{D}}}_1^T\dot{\boldsymbol{U}}^n - \underline{\underline{\boldsymbol{C}}}_2\dot{\boldsymbol{\psi}}_U^n\right)\\
&-(\Delta t)^2\,\beta\underline{\underline{\boldsymbol{D}}}_1\overline{\ddot{\boldsymbol{Q}}}^n - (\Delta t)^2\,\underline{\underline{\boldsymbol{C}}}_0\ddot{\boldsymbol{\phi}}_U^n\\
&+(\Delta t)^2\,\rho\underline{\underline{\boldsymbol{M}}}_0\ddot{\boldsymbol{F}}^n\left\{+(\Delta t)^2\,\mu\underline{\underline{\boldsymbol{D}}}_0\dot{\boldsymbol{I}}^n + (\Delta t)^2\,\lambda\underline{\underline{\boldsymbol{D}}}_1\dot{\boldsymbol{K}}^n\right\},
\end{aligned} \tag{5.47}$$

$$\begin{aligned}
\varpi\underline{\underline{\boldsymbol{M}}}_2\left(\dot{\boldsymbol{Q}}^{n+1} - 2\dot{\boldsymbol{Q}}^n + \dot{\boldsymbol{Q}}^{n-1}\right) = &-(\Delta t)^2\,\frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{D}}}_2\overline{\ddot{\boldsymbol{Q}}}^n - (\Delta t)^2\,\underline{\underline{\boldsymbol{C}}}_3\ddot{\boldsymbol{\phi}}_Q^n + (\Delta t)^2\,\frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{C}}}_4\overline{\ddot{\boldsymbol{\psi}}}_Q^n\\
&+(\Delta t)^2\,\beta\underline{\underline{\boldsymbol{D}}}_1^T\overline{\ddot{\boldsymbol{U}}}^n - (\Delta t)^2\,\beta\underline{\underline{\boldsymbol{C}}}_2\overline{\ddot{\boldsymbol{\psi}}}_U^n + (\Delta t)^2\,\varpi\,\underline{\underline{\boldsymbol{M}}}_2\ddot{\boldsymbol{G}}^n,
\end{aligned} \tag{5.48}$$

where the dot above a quantity is to be interpreted as a central difference in time as in Eq. (5.44) and a bar denotes averaging, see Eq. (5.41). Hence we have

$$\overline{\ddot{\boldsymbol{Q}}}^n = \frac{1}{2\,\Delta t}\left(\dot{\boldsymbol{Q}}^{n+1} - \dot{\boldsymbol{Q}}^{n-1}\right), \qquad \ddot{\boldsymbol{Q}}^n = \frac{1}{(\Delta t)^2}\left(\dot{\boldsymbol{Q}}^{n+1} - 2\dot{\boldsymbol{Q}}^n + \dot{\boldsymbol{Q}}^{n-1}\right). \tag{5.49}$$

The accompanying equations for the boundary densities are

$$\left[\boldsymbol{B}(\partial_t)\begin{pmatrix}\ddot{\boldsymbol{\phi}}_U\\ \ddot{\boldsymbol{\phi}}_Q\\ \overline{\ddot{\boldsymbol{\psi}}}_U\\ \overline{\ddot{\boldsymbol{\psi}}}_Q\end{pmatrix}\right]^n = \begin{pmatrix}\underline{\underline{\boldsymbol{C}}}_0^T\overline{\ddot{\boldsymbol{U}}}^n\\ \underline{\underline{\boldsymbol{C}}}_3^T\overline{\ddot{\boldsymbol{Q}}}^n\\ -\mu\underline{\underline{\boldsymbol{C}}}_1^T\ddot{\tilde{\boldsymbol{V}}}^n - \lambda\underline{\underline{\boldsymbol{C}}}_2^T\ddot{\tilde{\boldsymbol{W}}}^n + \beta\underline{\underline{\boldsymbol{C}}}_2^T\overline{\ddot{\boldsymbol{Q}}}^n\\ -\frac{\kappa}{T_{\mathrm{ref}}}\underline{\underline{\boldsymbol{C}}}_4^T\overline{\ddot{\boldsymbol{Q}}}^n\end{pmatrix}\left\{+\begin{pmatrix}\ddot{\boldsymbol{\nu}}^{n+\frac{1}{2}}\\ \ddot{\varsigma}^{n+\frac{1}{2}}\\ \ddot{\boldsymbol{\eta}}^{n+\frac{1}{2}}\\ \ddot{\xi}^{n+\frac{1}{2}}\end{pmatrix}\right\}, \tag{5.50}$$

where $\ddot{\tilde{\boldsymbol{V}}}^n$ and $\ddot{\tilde{\boldsymbol{W}}}^n$ are to be understood as placeholders such that

$$\ddot{\tilde{\boldsymbol{V}}}^n = \underline{\underline{\boldsymbol{M}}}_1^{-1}\left(-\underline{\underline{\boldsymbol{D}}}_0^T\dot{\boldsymbol{U}}^n + \underline{\underline{\boldsymbol{C}}}_1\dot{\boldsymbol{\psi}}_U^n\right) + \delta\,(\Delta t)^2\,\underline{\underline{\boldsymbol{M}}}_1^{-1}\underline{\underline{\boldsymbol{C}}}_1\dddot{\boldsymbol{\psi}}_U^n\left\{+\dot{\boldsymbol{I}}^n\right\}, \tag{5.51}$$

$$\ddot{\tilde{\boldsymbol{W}}}^n = \underline{\underline{\boldsymbol{M}}}_2^{-1}\left(-\underline{\underline{\boldsymbol{D}}}_1^T\dot{\boldsymbol{U}}^n + \underline{\underline{\boldsymbol{C}}}_2\dot{\boldsymbol{\psi}}_U^n\right) + \delta\,(\Delta t)^2\,\underline{\underline{\boldsymbol{M}}}_2^{-1}\underline{\underline{\boldsymbol{C}}}_2\dddot{\boldsymbol{\psi}}_U^n\left\{+\dot{\boldsymbol{K}}^n\right\}. \tag{5.52}$$

6. Stability of the spatial semi-discretization

Let us assume that our bases of $\mathfrak{U}_h$, $\mathfrak{V}_h$, $\mathfrak{Q}_h$, $\mathfrak{Z}_h$, and $\mathfrak{P}_h$ are orthonormal in $\mathrm{L}^2(\Omega)^3$, $(\mathrm{L}^2(\Omega))^{3\times 3}$, $\mathrm{L}^2(\Omega)$, $(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4$, and $(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4$, respectively. Then the matrices $\underline{\underline{\boldsymbol{M}}}_0$ and $\underline{\underline{\boldsymbol{M}}}_2$ become unit matrices. The matrix $\underline{\underline{\boldsymbol{M}}}_1$, however, becomes a diagonal matrix whose diagonal elements are all $\frac{1}{2}$. Consequently, $\underline{\underline{\boldsymbol{M}}}_1^{-1}$ is a diagonal matrix with a 2 for each entry on the diagonal.

To analyze the propagation of errors in the spatial discretization, we consider bounds of the Euclidean norms of the solution of these equations, where we take the norm of the perturbations $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{I}$, $\boldsymbol{K}$, $\boldsymbol{\nu}$, $\boldsymbol{\varsigma}$, $\boldsymbol{\eta}$, and $\boldsymbol{\xi}$ into account. Similar to the continuous field energy we introduce a semi-discrete field energy and show its boundedness.

Lemma 7. *The semi-discrete field energy*

$$\mathbb{E}(t) = \frac{1}{2}\rho\,\|\boldsymbol{U}(t)\|_2^2 + \frac{1}{4}\mu\,\|\boldsymbol{V}(t)\|_2^2 + \frac{1}{2}\lambda\,\|\boldsymbol{W}(t)\|_2^2 + \frac{1}{2}\varpi\,\|\boldsymbol{Q}(t)\|_2^2 + \frac{\kappa}{T_{\mathrm{ref}}}\int_0^t \left\|\underline{\underline{\boldsymbol{D}}}_2^{\frac{1}{2}}\boldsymbol{Q}(\tau)\right\|_2^2 d\tau \tag{6.1}$$

is bounded along the solution by

$$\begin{aligned}\mathbb{E}(t) \leq\ & \mathsf{f}(\widetilde{\mathsf{c}})\Bigg(\mathbb{E}(0) + \frac{t}{2}\int_0^t \left(\rho\|\boldsymbol{F}(\tau)\|_2^2 + \frac{\mu}{2}\|\boldsymbol{I}(\tau)\|_2^2 + \lambda\|\boldsymbol{K}(\tau)\|_2^2 + \varpi\|\boldsymbol{G}(\tau)\|_2^2\right) d\tau \\ &+ \max\left(t^2, t^8\right)\int_0^t \left(\|\partial_t^3\boldsymbol{\nu}(\tau)\|_2^2 + \|\partial_t^3\boldsymbol{\varsigma}(\tau)\|_2^2 + \|\partial_t^3\boldsymbol{\eta}(\tau)\|_2^2 + \|\partial_t^3\boldsymbol{\xi}(\tau)\|_2^2\right) d\tau\Bigg)\end{aligned} \tag{6.2}$$

for $t > 0$. This estimate holds provided $\boldsymbol{\nu}(0) = \partial_t\boldsymbol{\nu}(0) = \partial_t^2\boldsymbol{\nu}(0) = 0$, $\boldsymbol{\varsigma}(0) = \partial_t\boldsymbol{\varsigma}(0) = \partial_t^2\boldsymbol{\varsigma}(0) = 0$, $\boldsymbol{\eta}(0) = \partial_t\boldsymbol{\eta}(0) = \partial_t^2\boldsymbol{\eta}(0) = 0$, and $\boldsymbol{\xi}(0) = \partial_t\boldsymbol{\xi}(0) = \partial_t^2\boldsymbol{\xi}(0) = 0$.

To prove this lemma, we start by taking inner products of Eq. (5.24) with $\boldsymbol{U}$, Eq. (5.25) with $\mu\boldsymbol{V}$, Eq. (5.26) with $\lambda\boldsymbol{W}$, Eq. (5.27) with $\boldsymbol{Q}$, and Eq. (5.28) with $\left(\boldsymbol{\phi}_U, \boldsymbol{\phi}_Q, \boldsymbol{\psi}_U, \boldsymbol{\psi}_Q\right)^T$. Summing up the results of this procedure gives us an ordinary differential equation for the semi-discrete field energy $\mathbb{E}$ which allows us to apply Lemma 4. From there, the proof follows along the lines of the proof for Lemma 6.1 in [4]. For this purpose, observe that the coercivity result

$$\begin{aligned}&\int_0^{t_{\mathrm{end}}} e^{-\frac{2t}{t_{\mathrm{end}}}} \left\langle \begin{pmatrix}\boldsymbol{\phi}_U(t)\\ \boldsymbol{\phi}_Q(t)\\ \boldsymbol{\psi}_U(t)\\ \boldsymbol{\psi}_Q(t)\end{pmatrix}, \boldsymbol{B}(\partial_t)\begin{pmatrix}\boldsymbol{\phi}_U(t)\\ \boldsymbol{\phi}_Q(t)\\ \boldsymbol{\psi}_U(t)\\ \boldsymbol{\psi}_Q(t)\end{pmatrix}\right\rangle_\Gamma \mathrm{d}t \\ &\geq \widetilde{\mathsf{c}}\,\mathsf{d}_{\mathrm{end}}\int_0^{t_{\mathrm{end}}} e^{-\frac{2t}{t_{\mathrm{end}}}}\left(\|\partial_t^{-2}\boldsymbol{\phi}_U(t)\|_2^2 + \|\partial_t^{-2}\boldsymbol{\phi}_Q(t)\|_2^2 + \|\partial_t^{-2}\boldsymbol{\psi}_U(t)\|_2^2 + \|\partial_t^{-2}\boldsymbol{\psi}_Q(t)\|_2^2\right)\mathrm{d}t\end{aligned} \tag{6.3}$$

with $\widetilde{\mathsf{c}} > 0$ independent of the grid size is inherited by the Calderón operator for the semi-discrete case from the continuous result in Lemma 3. However, due to the fact that our coercivity result in Lemma 2 gives an estimate which involves $|s|^3$ instead of $|s|^2$, our result involves third-order derivatives of $\boldsymbol{\nu}$, $\boldsymbol{\varsigma}$, $\boldsymbol{\eta}$, and $\boldsymbol{\xi}$, necessitating that the values and derivatives of these functions up to order 2 vanish at $t = 0$. For the same reason, our estimate involves t^8 instead of t^6. Further, our result is similar to Lemma 7 in [8]. However, it was assumed there that $\underline{\underline{\boldsymbol{M}}}_1$ is a unit matrix, resulting in some minor differences.

We can estimate the norms of the boundary densities. For that we define

$$\begin{aligned}\mathbb{H}(t) = &\ \frac{1}{2}\rho\left\|\partial_t^2 \boldsymbol{U}(t)\right\|_2^2 + \mu\left\|\underline{\underline{\boldsymbol{D}_0^T}}\partial_t\boldsymbol{U}(t) - \underline{\underline{\boldsymbol{C}_1}}\partial_t\boldsymbol{\psi}_U(t)\right\|_2^2 + \frac{1}{2}\lambda\left\|\underline{\underline{\boldsymbol{D}_1^T}}\partial_t\boldsymbol{U}(t) - \underline{\underline{\boldsymbol{C}_2}}\partial_t\boldsymbol{\psi}_U(t)\right\|_2^2 \\ &+ \frac{1}{2}\varpi\|\partial_t^2\boldsymbol{Q}\|_2^2 + \frac{\kappa}{T_{\mathrm{ref}}}\int_0^t \|\underline{\underline{\boldsymbol{D}_2^{\frac{1}{2}}}}\partial_t^2\boldsymbol{Q}(\tau)\|_2^2 d\tau\end{aligned} \tag{6.4}$$

Lemma 8. *The semi-discrete boundary functions $\boldsymbol{\phi}_U$, $\boldsymbol{\phi}_Q$, $\boldsymbol{\psi}_U$, and $\boldsymbol{\psi}_Q$ of Eqs. (5.24)–(5.28) are bounded in the form*

$$\begin{aligned}&\int_0^t \left(\|\boldsymbol{\phi}_U(\tau)\|_2^2 + \left\|\boldsymbol{\phi}_Q(\tau)\right\|_2^2 + \|\boldsymbol{\psi}_U(\tau)\|_2^2 + \left\|\boldsymbol{\psi}_Q(\tau)\right\|_2^2\right) d\tau \\ &\leq \mathsf{h}(\widetilde{\mathsf{c}})\Bigg(\max\left(t^2,t^5\right)\mathbb{H}(0) \\ &\quad + \max\left(t^3,t^6\right)\int_0^t \left(\rho\left\|\partial_t^2\boldsymbol{F}(\tau)\right\|_2^2 + \mu\left\|\partial_t^2\boldsymbol{I}(\tau)\right\|_2^2 + \lambda\left\|\partial_t^2\boldsymbol{K}(\tau)\right\|_2^2 + \varpi\left\|\partial_t^2\boldsymbol{G}(\tau)\right\|_2^2\right) d\tau \\ &\quad + \max\left(t^2,t^8\right)\int_0^t \left(\left\|\partial_t^3\boldsymbol{\nu}(\tau)\right\|_2^2 + \left\|\partial_t^3\boldsymbol{\varsigma}(\tau)\right\|_2^2 + \left\|\partial_t^3\boldsymbol{\eta}(\tau)\right\|_2^2 + \left\|\partial_t^3\boldsymbol{\xi}(\tau)\right\|_2^2\right) d\tau\Bigg)\end{aligned} \tag{6.5}$$

for $t > 0$. This estimate holds provided $\boldsymbol{F}(0) = \partial_t\boldsymbol{F}(0) = 0$, $\boldsymbol{G}(0) = \partial_t\boldsymbol{G}(0) = 0$, $\boldsymbol{I}(0) = \partial_t\boldsymbol{I}(0) = 0$, $\boldsymbol{K}(0) = \partial_t\boldsymbol{K}(0) = 0$, $\boldsymbol{\nu}(0) = \partial_t\boldsymbol{\nu}(0) = \partial_t^2\boldsymbol{\nu}(0) = 0$, $\boldsymbol{\varsigma}(0) = \partial_t\boldsymbol{\varsigma}(0) = \partial_t^2\boldsymbol{\varsigma}(0) = 0$, $\boldsymbol{\eta}(0) = \partial_t\boldsymbol{\eta}(0) = \partial_t^2\boldsymbol{\eta}(0) = 0$, and $\boldsymbol{\xi}(0) = \partial_t\boldsymbol{\xi}(0) = \partial_t^2\boldsymbol{\xi}(0) = 0$.

For the proof, we have to provide estimates for the three cases i) $\mathbb{E}(0) = \mathbb{H}(0) = \boldsymbol{F} = \boldsymbol{G} = \boldsymbol{I} = \boldsymbol{K} = 0$, ii) $\boldsymbol{I} = \boldsymbol{K} = \partial_t\boldsymbol{\nu} = \partial_t\boldsymbol{\varsigma} = \partial_t\boldsymbol{\eta} = \partial_t\boldsymbol{\xi} = 0$, and iii) $\mathbb{E}(0) = \mathbb{H}(0) = \boldsymbol{F} = \boldsymbol{G} = \partial_t\boldsymbol{\nu} = \partial_t\boldsymbol{\varsigma} = \partial_t\boldsymbol{\eta} = \partial_t\boldsymbol{\xi} = 0$, and follow the same ideas as Lemma 6.3 in [4] in that we use intermediate results from and modifications of the proofs of both previous lemmas to prove the cases i) and iii). However, in order to get rid of the ∂_t^{-2} in Lemma 4, we have to use the third-order system obtained by differentiating the second-order system and then follow the case ii) of the proof of Lemma 6.3 in [4]. This results in the energy $\mathbb{H}(0)$.

7. Error bound for the spatial semi-discretization

Let us first take a look at the consistency error for the spatial semi-discretization.

Lemma 9. *Let $\mathcal{P}_h$ denote the $\mathrm{L}^2(\Omega)$-orthogonal projection from $\left(\mathrm{H}^1(\Omega)\right)^4$ onto $(\mathfrak{Q}_h)^4$. In the case of a quasi-uniform triangulation of Ω, there exists a positive constant k such that*

$$\left\|\vec{\vec{Z}} - \mathcal{P}_h\vec{\vec{Z}}\right\|_{(\mathrm{H}^1(\Omega))^4} \leq \mathsf{k}h\left\|\vec{\vec{Z}}\right\|_{(\mathrm{H}^2(\Omega))^4} \qquad \textit{for all } \vec{\vec{Z}} \in (\mathrm{H}^2(\Omega))^4. \tag{7.1}$$

Proof. We take a look at

$$\vec{\vec{Z}} - \mathcal{P}_h\vec{\vec{Z}} = \left(\vec{\vec{Z}} - \mathcal{I}_h\vec{\vec{Z}}\right) + \left(\mathcal{I}_h\vec{\vec{Z}} - \mathcal{P}_h\vec{\vec{Z}}\right), \tag{7.2}$$

where $\mathcal{I}_h$ is the finite element interpolation operator. The H^1-norm for the first term is of order $\mathcal{O}(h)$, whereas the L^2-norm for the second term is of order $\mathcal{O}(h^2)$. The desired estimate then follows from a standard inverse inequality. □

Lemma 10. *There exists a constant* $\mathsf{m}(t)$*, growing at most polynomially with* t*, such that*

$$\int_0^t \left\| B(\partial_t) \begin{pmatrix} (I - \mathcal{P}_{\mathfrak{P}_{\vec{U},h}})\vec{\phi}_{\vec{U}}(\cdot,\tau) \\ (I - \mathcal{P}_{\mathfrak{P}_{Q,h}})\phi_Q(\cdot,\tau) \\ (I - \mathcal{P}_{\mathfrak{Z}_{\vec{U},h}})\vec{\psi}_{\vec{U}}(\cdot,\tau) \\ (I - \mathcal{P}_{\mathfrak{Z}_{Q,h}})\psi_Q(\cdot,\tau) \end{pmatrix} \right\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^4 \times (\mathrm{H}^{-\frac{1}{2}}(\Gamma))^4} d\tau$$
$$\leq \mathsf{m}(t) h^2 \int_0^t \Big(\|\partial_t^3 \vec{\phi}_{\vec{U}}(\cdot,\tau)\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^3} + \|\partial_t^3 \phi_Q(\cdot,\tau)\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma)}$$
$$+ \|\partial_t^3 \vec{\psi}_{\vec{U}}(\cdot,\tau)\|^2_{(\mathrm{H}^{\frac{3}{2}}(\Gamma))^3} + \|\partial_t^3 \psi_Q(\cdot,\tau)\|^2_{\mathrm{H}^{\frac{3}{2}}(\Gamma)} \Big) \, d\tau \tag{7.3}$$

for any $t > 0$ *and for all* $\vec{\phi}_{\vec{U}} \in C^3([0,t], (\mathrm{H}^{\frac{1}{2}}(\Gamma))^3)$, $\phi_Q \in C^3([0,t], \mathrm{H}^{\frac{1}{2}}(\Gamma))$, $\vec{\psi}_{\vec{U}} \in C^3([0,t], (\mathrm{H}^{\frac{3}{2}}(\Gamma))^3)$, $\psi_Q \in C^3([0,t], \mathrm{H}^{\frac{3}{2}}(\Gamma))$, *with* $\vec{\phi}_{\vec{U}}(\cdot,0) = \partial_t \vec{\phi}_{\vec{U}}(\cdot,0) = \partial_t^2 \vec{\phi}_{\vec{U}}(\cdot,0) = 0$, $\phi_Q(\cdot,0) = \partial_t \phi_Q(\cdot,0) = \partial_t^2 \phi_Q(\cdot,0) = 0$, $\vec{\psi}_{\vec{U}}(\cdot,0) = \partial_t \vec{\psi}_{\vec{U}}(\cdot,0) = \partial_t^2 \vec{\psi}_{\vec{U}}(\cdot,0) = 0$, $\psi_Q(\cdot,0) = \partial_t \psi_Q(\cdot,0) = \partial_t^2 \psi_Q(\cdot,0) = 0$. *Here* $\mathcal{P}_{-}$ *denotes the* $\mathrm{L}^2(\Gamma)$*-orthogonal projection onto the corresponding boundary element spaces given as indices.*

Proof. This lemma is proven in a similar way as the corresponding results for the acoustic wave equation [4] or the elastodynamic wave equation [8] by applying the bounds of the boundary integral operators. □

By combining the previous results, we obtain the following theorem.

Theorem 11. *Assume that the initial values* $\vec{U}(\cdot,0)$, $\vec{\vec{V}}(\cdot,0)$, $W(\cdot,0)$, *and* $Q(\cdot,0)$ *have their support in* Ω. *Let the initial values for the semi-discretization be chosen as* $\partial_t^j \vec{U}_h(0) = \mathcal{P}_{\mathfrak{U}_h} \partial_t^j \vec{U}(0)$, $j = 0,1,2$, $\vec{\vec{V}}_h(0) = \mathcal{P}_{\mathfrak{V}_h} \vec{\vec{V}}(0)$, $W_h(0) = \mathcal{P}_{\mathfrak{Q}_h} W(0)$, *and* $\partial_t^k Q_h(0) = \mathcal{P}_{\mathfrak{Q}_h} \partial_t^k Q(0)$, $k = 0,1,2$. *Here* $\mathcal{P}_{-}$ *denotes the* $\mathrm{L}^2(\Omega)$*-orthogonal projection onto the corresponding finite element spaces. If we assume that the solution of the wave equation is sufficiently smooth, then the error of the FEM–BEM semi-discretization is bounded by*

$$\rho \|\vec{U}_h(t) - \vec{U}(t)\|_{(\mathrm{L}^2(\Omega))^3} + \frac{1}{2}\mu \|\vec{\vec{V}}_h(t) - \vec{\vec{V}}(t)\|_{(\mathrm{L}^2(\Omega))^{3\times 3}} + \lambda \|W_h(t) - W(t)\|_{\mathrm{L}^2(\Omega)}$$
$$+ \varpi \|Q_h(t) - Q(t)\|_{\mathrm{L}^2(\Omega)} + \frac{\kappa}{T_{\mathrm{ref}}} \int_0^t \left\| \vec{\nabla} Q_h(\tau) - \vec{\nabla} Q(\tau)) \right\|_{(\mathrm{L}^2(\Omega)^3)} d\tau$$
$$+ \Bigg(\int_0^t \|\vec{\phi}_{\vec{U},h}(\tau) - \vec{\phi}_{\vec{U}}(\tau)\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3} + \|\phi_{Q,h}(\tau) - \phi_Q(\tau)\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)} \tag{7.4}$$
$$+ \|\vec{\psi}_{\vec{U},h}(\tau) - \vec{\psi}_{\vec{U}}(\tau)\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^3} + \|\psi_{Q,h}(\tau) - \psi_Q(\tau)\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma)} \, d\tau \Bigg)^{\frac{1}{2}}$$
$$\leq \mathsf{m}(t) h,$$

where the constant $\mathsf{m}(t)$ *grows at most polynomially with* t.

Proof. For the first term on the left-hand side, the triangle inequality yields

$$\|\vec{U}_h(t) - \vec{U}(t)\|_{(\mathrm{L}^2(\Omega))^3} \leq \|\vec{U}_h(t) - \mathcal{P}_{\mathfrak{U}_h} \vec{U}(t)\|_{(\mathrm{L}^2(\Omega))^3} + \|\mathcal{P}_{\mathfrak{U}_h} \vec{U}(t) - \vec{U}(t)\|_{(\mathrm{L}^2(\Omega))^3} \tag{7.5}$$

and similar estimates hold for all other quantities. The second term on the right can be estimated according to Lemma 9. For the sum of the first terms, observe that the differences $\left(\vec{U}_h(t) - \mathcal{P}_{\mathfrak{U}_h}\vec{U}(t)\right)$, $\left(\vec{\vec{V}}_h(t) - \mathcal{P}_{\mathfrak{V}_h}\vec{\vec{V}}(t)\right)$, $(W_h(t) - \mathcal{P}_{\mathfrak{Q}_h}W(t))$, $(Q_h(t) - \mathcal{P}_{\mathfrak{Q}_h}Q(t))$, $\left(\vec{\phi}_{\vec{U},h}(t) - \mathcal{P}_{\mathfrak{P}_{\vec{U},h}}\vec{\phi}_{\vec{U}}(t)\right)$, $\left(\phi_{Q,h}(t) - \mathcal{P}_{\mathfrak{P}_{Q,h}}\phi_Q(t)\right)$, $\left(\vec{\psi}_{\vec{U},h}(t) - \mathcal{P}_{\mathfrak{Z}_{\vec{U},h}}\vec{\psi}_{\vec{U}}(t)\right)$, and $\left(\psi_{Q,h}(t) - \mathcal{P}_{\mathfrak{Z}_{Q,h}}\psi_Q(t)\right)$ satisfy Eqs. (5.9)–(5.13) for test functions from the respective FEM and BEM spaces if the right-hand sides are defined such that

$$
\begin{aligned}
\rho(\vec{F}_h, \vec{Z}_h) = {} & -\frac{1}{2}\mu(\vec{\vec{V}} - \mathcal{P}_{\mathfrak{V}_h}\vec{\vec{V}}, \vec{\nabla}\vec{Z}_h) + \frac{1}{2}\mu(\vec{\nabla}\cdot(\vec{\vec{V}} - \mathcal{P}_{\mathfrak{V}_h}\vec{\vec{V}}), \vec{Z}_h) \\
& - \frac{1}{2}\lambda(W - \mathcal{P}_{\mathfrak{Q}_h}W, \vec{\nabla}\cdot\vec{Z}_h) + \frac{1}{2}\lambda(\vec{\nabla}(W - \mathcal{P}_{\mathfrak{Q}_h}W), \vec{Z}_h) \\
& + \frac{1}{2}\beta(Q - \mathcal{P}_{\mathfrak{Q}_h}Q, \vec{\nabla}\cdot\vec{Z}_h) - \frac{1}{2}\beta(\vec{\nabla}(Q - \mathcal{P}_{\mathfrak{Q}_h}Q), \vec{Z}_h) \\
(\vec{\vec{I}}_h, \vec{\vec{Y}}_h) = {} & \frac{1}{2}(\vec{\nabla}(\vec{U} - \mathcal{P}_{\mathfrak{U}_h}\vec{U}), \vec{\vec{Y}}_h) - \frac{1}{2}(\vec{U} - \mathcal{P}_{\mathfrak{U}_h}\vec{U}, \vec{\nabla}\cdot\vec{\vec{Y}}_h) \\
(K_h, X_h) = {} & \frac{1}{2}(\vec{\nabla}\cdot(\vec{U} - \mathcal{P}_{\mathfrak{U}_h}\vec{U}), X_h) - \frac{1}{2}(\vec{U} - \mathcal{P}_{\mathfrak{U}_h}\vec{U}, \vec{\nabla}X_h) \\
\varpi(G_h, R_h) = {} & -\frac{\kappa}{T_{\mathrm{ref}}}(\vec{\nabla}(Q - \mathcal{P}_{\mathfrak{Q}_h}Q), \vec{\nabla}R_h) \\
& - \frac{1}{2}\beta(\vec{\nabla}\cdot(\vec{U} - \mathcal{P}_{\mathfrak{U}_h}\vec{U}), R_h) + \frac{1}{2}\beta(\vec{U} - \mathcal{P}_{\mathfrak{U}_h}\vec{U}, \vec{\nabla}R_h)
\end{aligned}
\tag{7.6}
$$

and for the boundary densities

$$
\begin{pmatrix} \vec{\nu}_h \\ \varsigma_h \\ \vec{\eta}_h \\ \xi_h \end{pmatrix} = \mathcal{B}(\partial_t)\begin{pmatrix} \vec{\phi}_{\vec{U}} - \mathcal{P}_{\mathfrak{P}_{\vec{U},h}}\vec{\phi}_{\vec{U}} \\ \phi_Q - \mathcal{P}_{\mathfrak{P}_{Q,h}}\phi_Q \\ \vec{\psi}_{\vec{U}} - \mathcal{P}_{\mathfrak{Z}_{\vec{U},h}}\vec{\psi}_{\vec{U}} \\ \psi_Q - \mathcal{P}_{\mathfrak{Z}_{Q,h}}\psi_Q \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \vec{\gamma}_D(\vec{U} - \mathcal{P}_{\mathfrak{U}_h}\vec{U}) \\ \gamma_D(Q - \mathcal{P}_{\mathfrak{Q}_h}Q) \\ -\vec{\vec{\gamma}}_D(\mu(\vec{\vec{V}} - \mathcal{P}_{\mathfrak{V}_h}\vec{\vec{V}}) + \lambda(W - \mathcal{P}_{\mathfrak{Q}_h}W)\vec{\vec{I}} - \beta(Q - \mathcal{P}_{\mathfrak{Q}_h}Q)\vec{\vec{I}})\vec{n} \\ -\vec{\gamma}_D\left(\frac{\kappa}{T_{\mathrm{ref}}}\vec{\nabla}(Q - \mathcal{P}_{\mathfrak{Q}_h}Q)\right)\cdot\vec{n} \end{pmatrix}
\tag{7.7}
$$

All that is left to do is apply the stability results from the previous section together with Lemmas 9 and 10 to get the desired result, similar to the proof of Theorem 7.1 in [4] and Theorem 6.1 in [15]. ☐

8. Stability of the full discretization

We want to study the stability of the full discrete scheme under the corresponding CFL condition. Inspecting Eqs. (5.36), (5.37), (5.40), (5.45), and (5.46), we observe that Eq. (5.36) is similar in spirit to the Crank–Nicolson scheme for parabolic PDEs. Thus, there is no CFL- or other stability condition with regard to $\boldsymbol{Q}$ and we only need the CFL condition from elasticity, which reads

$$
\Delta t \le \frac{\sqrt{\rho}}{\sqrt{2\mu\left\|\underline{\underline{\boldsymbol{D}_0}}\right\|^2 + \lambda\left\|\underline{\underline{\boldsymbol{D}_1}}\right\|^2}}.
\tag{8.1}
$$

As a further stability condition, it is sufficient to assume

$$
\delta \ge 2
\tag{8.2}
$$

resulting from the considerations of the proof of Lemma 12. This guarantees that the discrete energy $\mathbb{E}^n \ge 0$ for all $n = 0, 1, 2, \ldots$.

Similar to the continuous and semi-discrete case, the discrete field energy for Eqs. (5.34)–(5.40) is given by

$$\begin{aligned}\mathbb{E}^n = \; & \frac{1}{2}\rho\|\boldsymbol{U}^n\|_2^2 + \frac{1}{8}\mu\left(\|\boldsymbol{V}^{n+\frac{1}{2}}\|_2^2 + \|\boldsymbol{V}^{n-\frac{1}{2}}\|_2^2\right) + \frac{1}{4}\lambda\left(\|\boldsymbol{W}^{n+\frac{1}{2}}\|_2^2 + \|\boldsymbol{W}^{n-\frac{1}{2}}\|_2^2\right) \\ & + \frac{1}{2}\varpi\|\boldsymbol{Q}^n\|_2^2 + \Delta t\,\frac{\kappa}{T_{\text{ref}}}\sum_{j=0}^{n-1}\left\|\underline{\underline{\boldsymbol{D}_2}}^{\frac{1}{2}}\overline{\boldsymbol{Q}}^{j+\frac{1}{2}}\right\|_2^2\end{aligned} \tag{8.3}$$

under the assumption that all needed bases are orthonormal.

Lemma 12. *For $0 < \Delta t \le \Delta t_0$, the discrete field energy (8.3) is bounded at $t = n\Delta t$ by*

$$\begin{aligned}\mathbb{E}^n \; \le \mathsf{p}\Bigg(& \mathbb{E}^0 + \frac{t}{2}\Delta t\sum_{j=0}^{n}\rho\|\boldsymbol{F}^{j+\frac{1}{2}}\|_2^2 + \frac{\mu}{2}\|\boldsymbol{I}^j\|_2^2 + \lambda\|\boldsymbol{K}^j\|_2^2 + \varpi\|\boldsymbol{G}^{j+\frac{1}{2}}\|_2^2 \\ & + \max(t^2,t^8)\Delta t\sum_{j=0}^{n}\left(\|(\partial_t^{\Delta t})^3\boldsymbol{\nu}^{j+\frac{1}{2}}\|_2^2 + \|(\partial_t^{\Delta t})^3\boldsymbol{\varsigma}^{j+\frac{1}{2}}\|_2^2 + \|(\partial_t^{\Delta t})^3\boldsymbol{\eta}^{j+\frac{1}{2}}\|_2^2 + \|(\partial_t^{\Delta t})^3\boldsymbol{\xi}^{j+\frac{1}{2}}\|_2^2\right)\Bigg),\end{aligned} \tag{8.4}$$

where p is independent of h, Δt, and n.

As mentioned, the proof of this lemma follows essentially the steps sketched in the proof of Lemma 8.1 in [4] by using the staggered equations (5.45) and (5.46).

Similar to the semi-discrete case, we obtain an estimate for the boundary densities. For that, observe that

$$\begin{aligned}\mathbb{H}^{n+\frac{1}{2}} = \; & \frac{1}{2}\,\rho\left|\ddot{\boldsymbol{U}}^{n+\frac{1}{2}}\right|^2 + \frac{1}{2}\,\varpi\left|\ddot{\boldsymbol{Q}}^{n+\frac{1}{2}}\right|^2 + \Delta t\,\frac{\kappa}{T_{\text{ref}}}\sum_{j=1}^{n}\left|\underline{\underline{\boldsymbol{D}_2}}^{\frac{1}{2}}\ddot{\overline{\boldsymbol{Q}}}^{j}\right|^2 \\ & + \mu\left|\underline{\underline{\boldsymbol{D}_0}}^T\dot{\overline{\boldsymbol{U}}}^{n+\frac{1}{2}} - \underline{\underline{\boldsymbol{C}_1}}\dot{\overline{\boldsymbol{\psi}}}_U^{n+\frac{1}{2}}\right|^2 + \frac{1}{2}\,\lambda\left|\underline{\underline{\boldsymbol{D}_1}}^T\dot{\overline{\boldsymbol{U}}}^{n+\frac{1}{2}} - \underline{\underline{\boldsymbol{C}_2}}\dot{\overline{\boldsymbol{\psi}}}_U^{n+\frac{1}{2}}\right|^2.\end{aligned} \tag{8.5}$$

Lemma 13. *For $0 < \Delta t \le \Delta t_0$, the discrete boundary densities $\boldsymbol{\phi}_U^{n+\frac{1}{2}}$, $\boldsymbol{\phi}_Q^{n+\frac{1}{2}}$, $\overline{\boldsymbol{\psi}}_U^{n+\frac{1}{2}}$, $\overline{\boldsymbol{\psi}}_Q^{n+\frac{1}{2}}$ are bounded at $t = n\Delta t$ by*

$$\begin{aligned}&\sum_{j=0}^{n}\left(\left|\boldsymbol{\phi}_U^{j+\frac{1}{2}}\right|^2 + \left|\boldsymbol{\phi}_Q^{j+\frac{1}{2}}\right|^2 + \left|\overline{\boldsymbol{\psi}}_U^{j+\frac{1}{2}}\right|^2 + \left|\overline{\boldsymbol{\psi}}_Q^{j+\frac{1}{2}}\right|^2\right) \\ &\quad \le \mathsf{r}\Bigg(\max(t^2,t^5)\mathbb{H}^{-\frac{1}{2}} \\ &\qquad + \max(t^3,t^6)\sum_{j=0}^{n}\left(\rho\left\|\ddot{\boldsymbol{F}}^j\right\|_2^2 + \varpi\left\|\ddot{\boldsymbol{G}}^j\right\|_2^2 + \frac{\mu}{2}\left\|(\partial_t^{\Delta t})^2\boldsymbol{I}^j\right\|_2^2 + \lambda\left\|(\partial_t^{\Delta t})^2\boldsymbol{K}^j\right\|_2^2\right) \\ &\qquad + \max(t^2,t^8)\sum_{j=0}^{n}\left(\|(\partial_t^{\Delta t})^3\boldsymbol{\nu}^{j+\frac{1}{2}}\|_2^2 + \|(\partial_t^{\Delta t})^3\boldsymbol{\varsigma}^{j+\frac{1}{2}}\|_2^2 + \|(\partial_t^{\Delta t})^3\boldsymbol{\eta}^{j+\frac{1}{2}}\|_2^2 + \|(\partial_t^{\Delta t})^3\boldsymbol{\xi}^{j+\frac{1}{2}}\|_2^2\right)\Bigg)\end{aligned} \tag{8.6}$$

where r is independent of h, Δt, and n.

Similar to Lemma 8, we have a contribution from $\mathbb{H}^{-\frac{1}{2}}$ here. This time however, this contribution does not vanish in the setting relevant for the next section. Instead, this particular term yields a condition on the implementation of initial conditions, cf. [18], Theorem 13.2.

9. Error bound for the full discretization

As our final result, we present an error bound for the full discretization.

Theorem 14. *Assume that the initial values and the inhomogeneities of the thermoelastic wave equation have their support in* Ω. *Let the initial values for the semi-discretization be chosen as* $\vec{U}_h(0) = \mathcal{P}_{\mathfrak{U}_h}\vec{U}(0)$, $\vec{\vec{V}}_h(0) = \mathcal{P}_{\mathfrak{V}_h}\vec{\vec{V}}(0)$, $W_h(0) = \mathcal{P}_{\mathfrak{Q}_h}W(0)$, $Q_h(0) = \mathcal{P}_{\mathfrak{Q}_h}Q(0)$, *where* $\mathcal{P}_{-}$ *denotes the* L^2*-orthogonal projection onto the corresponding finite element spaces. Further, we set the initial values of the time discretization to*

$$\vec{U}_h^1 = \mathcal{P}_{\mathfrak{U}_h}\left(\vec{U}(0) + \Delta t\partial_t\vec{U}(0) + \frac{(\Delta t)^2}{2}\partial_t^2\vec{U}(0)\right), \tag{9.1}$$

$$\vec{U}_h^0 = \mathcal{P}_{\mathfrak{U}_h}\vec{U}(0), \tag{9.2}$$

$$\vec{U}_h^{-1} = \mathcal{P}_{\mathfrak{U}_h}\left(\vec{U}(0) - \Delta t\partial_t\vec{U}(0) + \frac{(\Delta t)^2}{2}\partial_t^2\vec{U}(0)\right), \tag{9.3}$$

$$\vec{U}_h^{-2} = \mathcal{P}_{\mathfrak{U}_h}\left(\vec{U}(0) - 2\Delta t\partial_t\vec{U}(0) + 2(\Delta t)^2\partial_t^2\vec{U}(0) - \Delta t^3\partial_t^3\vec{U}(0)\right), \tag{9.4}$$

$$Q_h^1 = \mathcal{P}_{\mathfrak{Q}_h}\left(Q(0) + \Delta t\partial_t Q(0) + \frac{(\Delta t)^2}{2}\partial_t^2 Q(0)\right), \tag{9.5}$$

$$Q_h^0 = \mathcal{P}_{\mathfrak{Q}_h}Q(0), \tag{9.6}$$

$$Q_h^{-1} = \mathcal{P}_{\mathfrak{Q}_h}\left(Q(0) - \Delta t\partial_t Q(0) + \frac{(\Delta t)^2}{2}\partial_t^2 Q(0)\right), \tag{9.7}$$

$$Q_h^{-2} = \mathcal{P}_{\mathfrak{Q}_h}\left(Q(0) - 2\Delta t\partial_t Q(0) + 2(\Delta t)^2\partial_t^2 Q(0) - \Delta t^3\partial_t^3 Q(0)\right). \tag{9.8}$$

If the solution of the wave equation is sufficiently smooth, then, under the CFL condition and if the stability parameter $\delta \geq 2$, *the error of the FEM and BEM and leapfrog and convolution quadrature full discretization is bounded at* $t = n\Delta t$ *by*

$$\begin{aligned}
&\rho\|\vec{U}_h^n - \vec{U}(t)\|_{(\mathrm{L}^2(\Omega))^3} + \frac{1}{2}\mu\|\vec{\vec{V}}_h^n - \vec{\vec{V}}(t)\|_{(\mathrm{L}^2(\Omega))^{3\times 3}} + \lambda\|W_h^n - W(t)\|_{\mathrm{L}^2(\Omega)} \\
&+ \varpi\|Q_h^n - Q(t)\|_{\mathrm{L}^2(\Omega)} + \Delta t\frac{\kappa}{T_{ref}}\sum_{j=0}^{n-1}\|\vec{\nabla}Q_h^j - \vec{\nabla}Q(t_j)\|_{(\mathrm{L}^2(\Omega))^3} \\
&+ \Bigg(\Delta t\sum_{j=0}^{n-1} \|\vec{\phi}_{\vec{U},h}^{j+\frac{1}{2}} - \vec{\phi}_{\vec{U}}(t_{j+\frac{1}{2}})\|^2_{(\mathrm{H}^{-\frac{1}{2}}(\Gamma))^3} + \|\phi_{Q,h}^{j+\frac{1}{2}} - \phi_Q(t_{j+\frac{1}{2}})\|^2_{\mathrm{H}^{-\frac{1}{2}}(\Gamma)} \\
&+ \|\vec{\psi}_{\vec{U},h}^{j+\frac{1}{2}} - \vec{\psi}_{\vec{U}}(t_{j+\frac{1}{2}})\|^2_{(\mathrm{H}^{\frac{1}{2}}(\Gamma))^3} + \|\overline{\psi}_{Q,h}^{j+\frac{1}{2}} - \psi_Q(t_{j+\frac{1}{2}})\|^2_{\mathrm{H}^{\frac{1}{2}}(\Gamma)}\Bigg)^{\frac{1}{2}} \\
&\leq \mathsf{q}(t)(h + (\Delta t)^2)
\end{aligned} \tag{9.9}$$

where the constant $\mathsf{q}(t)$ *grows at most polynomially with* t.

Proof. We are searching for $\vec{U}_h^n \in \mathfrak{U}_h$, $\vec{\vec{V}}_h^n$, $\vec{\vec{V}}_h^{n+\frac{1}{2}} \in \mathfrak{V}_h$, W_h^n, $W_h^{n+\frac{1}{2}}$, Q_h^n, $Q_h^{n+\frac{1}{2}} \in \mathfrak{Q}_h$, $\vec{\phi}_{\vec{U},h}^{n+\frac{1}{2}} \in \mathfrak{P}_{\vec{U},h}$, $\phi_{Q,h}^{n+\frac{1}{2}} \in \mathfrak{P}_{Q,h}$, $\vec{\psi}_{\vec{U},h}^n \in \mathfrak{Z}_{\vec{U},h}$, and $\psi_{Q,h}^{n+\frac{1}{2}} \in \mathfrak{Z}_{Q,h}$ such that

$$\frac{2}{\Delta t}(\vec{\vec{V}}_h^{n+\frac{1}{2}} - \vec{\vec{V}}_h^n, \frac{1}{2}\vec{\vec{Y}}_h) = \frac{1}{2}(\vec{\nabla}\vec{U}_h^n, \vec{\vec{Y}}_h) - \frac{1}{2}(\vec{U}_h^n, \vec{\nabla}\cdot\vec{\vec{Y}}_h) + \frac{1}{2}\langle\vec{\psi}_{\vec{U},h}^n, \vec{\vec{\gamma}}_D(\vec{\vec{Y}}_h)\vec{n}\rangle_\Gamma, \tag{9.10}$$

$$\frac{2}{\Delta t}(W_h^{n+\frac{1}{2}} - W_h^n, X_h) = -\frac{1}{2}(\vec{U}_h^n, \vec{\nabla} X_h) + \frac{1}{2}(\vec{\nabla}\cdot\vec{U}_h^n, X_h) + \frac{1}{2}\langle \vec{\psi}_{\vec{U},h}^n, \vec{\vec{\gamma}}_D(X_h \vec{\vec{I}})\vec{n}\rangle_\Gamma, \tag{9.11}$$

$$\begin{aligned}\frac{\varpi}{\Delta t}\left(Q_h^{n+1} - Q_h^n, R_h\right) =\; & -\frac{\kappa}{T_{\mathrm{ref}}}(\vec{\nabla} Q_h^{n+\frac{1}{2}}, \vec{\nabla} R_h) - \frac{1}{2}\langle \phi_{Q,h}^{n+\frac{1}{2}}, \gamma_D R_h\rangle_\Gamma + \frac{1}{2}\frac{\kappa}{T_{\mathrm{ref}}}\langle \overline{\psi}_{Q,h}^{n+\frac{1}{2}}, \vec{\gamma}_D(\vec{\nabla} R_h)\cdot\vec{n}\rangle_\Gamma \\ & - \frac{1}{2}\beta(\vec{\nabla}\cdot\vec{U}_h^{n+\frac{1}{2}}, R_h) + \frac{1}{2}\beta(\vec{U}_h^{n+\frac{1}{2}}, \vec{\nabla} R_h) - \frac{1}{2}\beta\langle \overline{\vec{\psi}}_{\vec{U},h}^{n+\frac{1}{2}}, \vec{\vec{\gamma}}_D(R_h\vec{\vec{I}})\vec{n}\rangle_\Gamma \\ & + \varpi(G(t_n), R_h),\end{aligned} \tag{9.12}$$

$$\begin{aligned}\frac{\rho}{\Delta t}(\vec{U}_h^{n+1} - \vec{U}_h^n, \vec{Z}_h) =\; & -\frac{\mu}{2}(\vec{\vec{V}}_h^{n+\frac{1}{2}}, \vec{\nabla}\vec{Z}_h) + \frac{\mu}{2}(\vec{\nabla}\cdot\vec{\vec{V}}_h^{n+\frac{1}{2}}, \vec{Z}_h) \\ & - \frac{\lambda}{2}(W_h^{n+\frac{1}{2}}, \vec{\nabla}\cdot\vec{Z}_h) + \frac{\lambda}{2}(\vec{\nabla} W_h^{n+\frac{1}{2}}, \vec{Z}_h) \\ & + \frac{\beta}{2}(Q_h^{n+\frac{1}{2}}, \vec{\nabla}\cdot\vec{Z}_h) - \frac{\beta}{2}(\vec{\nabla} Q_h^{n+\frac{1}{2}}, \vec{Z}_h) \\ & - \frac{1}{2}\langle \vec{\phi}_{\vec{U},h}^{n+\frac{1}{2}}, \vec{\gamma}_D\vec{Z}_h\rangle_\Gamma + \rho(\vec{F}(t_{n+\frac{1}{2}}, \vec{Z}_h),\end{aligned} \tag{9.13}$$

$$\frac{2}{\Delta t}(\vec{\vec{V}}_h^{n+1} - \vec{\vec{V}}_h^{n+\frac{1}{2}}, \frac{1}{2}\vec{\vec{Y}}_h) = \frac{1}{2}(\vec{\nabla}\vec{U}_h^{n+1}, \vec{\vec{Y}}_h) - \frac{1}{2}(\vec{U}_h^{n+1}, \vec{\nabla}\cdot\vec{\vec{Y}}_h) + \frac{1}{2}\langle \vec{\psi}_{\vec{U},h}^{n+1}, \vec{\vec{\gamma}}_D(\vec{\vec{Y}}_h)\vec{n}\rangle_\Gamma, \tag{9.14}$$

$$\frac{2}{\Delta t}(W_h^{n+1} - W_h^{n+\frac{1}{2}}, X_h) = -\frac{1}{2}(\vec{U}_h^{n+1}, \vec{\nabla} X_h) + \frac{1}{2}(\vec{\nabla}\cdot\vec{U}_h^{n+1}, X_h) + \frac{1}{2}\langle \vec{\psi}_{\vec{U},h}^{n+1}, \vec{\vec{\gamma}}_D(X_h\vec{\vec{I}})\vec{n}\rangle_\Gamma, \tag{9.15}$$

for $\vec{Z}_h, \vec{N}_h \in \mathfrak{U}_h$, $\vec{\vec{Y}}_h \in \mathfrak{V}_h$, $X_h, R_h \in \mathfrak{Q}_h$, and

$$\begin{aligned}\left\langle \begin{pmatrix}\vec{\xi}_{\vec{U},h}\\ \xi_{Q,h}\\ \vec{\chi}_{\vec{U},h}\\ \chi_{Q,h}\end{pmatrix}, \left[\mathcal{B}(\partial_t^{\Delta t})\begin{pmatrix}\vec{\phi}_{\vec{U},h}\\ \phi_{Q,h}\\ \vec{\psi}_{\vec{U},h}\\ \psi_{Q,h}\end{pmatrix}\right]^{n+\frac{1}{2}}\right\rangle_\Gamma =\; & \frac{1}{2}\langle \vec{\xi}_{\vec{U},h}, \vec{\gamma}_D\vec{U}_h^{n+\frac{1}{2}}\rangle_\Gamma + \frac{1}{2}\langle \xi_{Q,h}, \gamma_D Q_h^{n+\frac{1}{2}}\rangle_\Gamma \\ & -\frac{1}{2}\langle \vec{\chi}_{\vec{U},h}, \vec{\vec{\gamma}}_D(\mu\vec{\vec{V}}_h^{n+\frac{1}{2}} + \lambda W_h^{n+\frac{1}{2}}\vec{\vec{I}} - \beta Q_h^{n+\frac{1}{2}}\vec{\vec{I}})\vec{n}\rangle_\Gamma \\ & -\delta\,(\Delta t)^2\,\langle \vec{\chi}_{\vec{U},h}, \dot{\vec{\psi}}_{\vec{U}}^{n+\frac{1}{2}}\rangle_\Gamma - \delta\,(\Delta t)^2\left\langle \chi_{Q,h}, \dot{\psi}_Q^{n+\frac{1}{2}}\right\rangle_\Gamma \\ & -\frac{1}{2}\left\langle \chi_{Q,h}, \vec{\gamma}_D\left(\frac{\kappa}{T_{\mathrm{ref}}}\vec{\nabla} Q_h^{n+\frac{1}{2}}\vec{\vec{I}}\right)\vec{n}\right\rangle_\Gamma\end{aligned} \tag{9.16}$$

for $\vec{\xi}_{\vec{U},h} \in \mathfrak{P}_{\vec{U},h}$, $\xi_{Q,h} \in \mathfrak{P}_{Q,h}$, $\vec{\chi}_{\vec{U},h} \in \mathfrak{Z}_{\vec{U},h}$, and $\chi_{Q,h} \in \mathfrak{Z}_{Q,h}$, where

$$\overline{\vec{U}}_h^{n+\frac{1}{2}} = \frac{1}{2}\left(\vec{U}_h^{n+1} + \vec{U}_h^n\right), \qquad \overline{\vec{\psi}}_{\vec{U},h}^{n+\frac{1}{2}} = \frac{1}{2}\left(\vec{\psi}_{\vec{U},h}^{n+1} + \vec{\psi}_{\vec{U},h}^n\right), \tag{9.17}$$

$$\dot{\vec{\psi}}_{\vec{U},h}^{n+\frac{1}{2}} = \frac{1}{\Delta t}\left(\vec{\psi}_{\vec{U},h}^{n+1} - \vec{\psi}_{\vec{U},h}^n\right), \quad \dot{\psi}_{Q,h}^{n+\frac{1}{2}} = \frac{1}{\Delta t}\left(\psi_{Q,h}^{n+\frac{1}{2}} - \psi_{Q,h}^{n-\frac{1}{2}}\right). \tag{9.18}$$

Again, we take a look at the defects under the projection of the exact solution to the variational formulation of the fully discretized scheme. However, as in [4,8], we replace $\mathcal{P}_{\mathfrak{V}_h}\vec{\vec{V}}_h^{n+\frac{1}{2}}$ by a slightly different term. We define $\vec{\vec{V}}_{\mathrm{mid}}^{n+\frac{1}{2}} := \vec{\vec{V}}(t_{n+\frac{1}{2}}) - \frac{1}{8}\Delta t^2\partial_t^2\vec{\vec{V}}(t_{n+\frac{1}{2}})$, such that $\vec{\vec{V}}_{\mathrm{mid}}^{n+\frac{1}{2}} = \vec{\vec{V}}(t_n) + \frac{1}{2}\Delta t\partial_t\vec{\vec{V}}(t_{n+1}) + \mathcal{O}(\Delta t^3)$ and $\vec{\vec{V}}(t_{n+1}) = \vec{\vec{V}}_{\mathrm{mid}}^{n+\frac{1}{2}} + \frac{1}{2}\Delta t\partial_t\vec{\vec{V}}(t_{n+1}) + \mathcal{O}(\Delta t^3)$. If the solution is smooth enough, it can be shown by Taylor expansion that then

$$\frac{2}{\Delta t}(\mathcal{P}_{\mathfrak{V}_h}\vec{\vec{V}}^{n+\frac{1}{2}}_{\mathrm{mid}} - \mathcal{P}_{\mathfrak{V}_h}\vec{\vec{V}}(t_n), \frac{1}{2}\vec{\vec{Y}}_h) = \frac{1}{2}(\vec{\nabla}(\mathcal{P}_{\mathfrak{U}_h}(\vec{U}(t_n))), \vec{\vec{Y}}_h) - \frac{1}{2}(\mathcal{P}_{\mathfrak{U}_h}\vec{U}(t_n), \vec{\nabla}\cdot\vec{\vec{Y}}_h)$$
$$+ \frac{1}{2}\langle \mathcal{P}_{\mathfrak{Z}_{\vec{U},h}}\vec{\psi}_{\vec{U}}(t_n), \vec{\vec{\gamma}}_D(\vec{\vec{Y}}_h)\vec{n}\rangle_\Gamma + (\vec{\vec{d}}^{\,h}_{V,n}, \vec{\vec{Y}}_h)$$

holds for all $\vec{\vec{Y}}_h \in \mathfrak{V}_h$. Here, the defect is

$$\vec{\vec{d}}^{\,h}_{V,n} = \vec{\vec{I}}_h(t_n) + \frac{2}{\Delta t}\mathcal{P}_{\mathfrak{V}_h}\left(\vec{\vec{V}}^{n+\frac{1}{2}}_{\mathrm{mid}} - \vec{\vec{V}}(t_n) - \frac{\Delta t}{2}\partial_t\vec{\vec{V}}(t_n)\right), \tag{9.19}$$

where the first term represents the consistency error and the second one is of order $\mathcal{O}(\Delta t^2)$. For the further defects of the full discrete equations, we refer to the equations of the spatial semi-discretization. In addition, we obtain for a temporally smooth solution terms of order $\mathcal{O}(\Delta t^2)$. Further, the initial values of the time discretization are chosen as described in [18], so that $\mathbb{H}^{-\frac{1}{2}}$ is well defined and is of order $\mathcal{O}(\Delta t^2)$ as well. Summarizing the discrete stability results and the error bounds of the spatial semi-discretization, we finally obtain the error bound for the full discretization. □

10. Summary and conclusion

In this paper, we presented a numerically stable FEM–BEM coupling for the thermoelastic wave equation with transparent boundary conditions. We started with the introduction of the involved equations and stated the corresponding fundamental solution as well as the required layer potentials. Based on these, we took a look at the boundary integral operators which are needed for the construction of the Calderón operator. The key issue for the stability analysis was the coercivity result of the Calderón operator together with energy techniques. Here, we first considered the semi-discretization followed by the full discretization including time, where we applied a leapfrog scheme in the domain's interior and convolution quadrature on the boundary. Finally, we ended up with an asymptotically optimal convergence result.

In order to manage the coupled system of a vector-valued PDE (second order in time and space) and a third-order scalar-valued PDE (first order in time and second order in space), we introduced a time-integrated temperature deviation. In particular, this allowed us to handle the coupling term without having to use a factor s in the test functions for the displacement vector, see, e.g., [24], Theorem 3.9. All in all, we understand this contribution as a step-stone to generalize the methods from [4,8] to more general sets of partial differential equations, as, e.g., those which describe wave propagation in poroelastic media [24].

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Appendix A. Fundamental solution tensor

For the thermoelastic wave equation in the Laplace domain as given by Eq. (2.31), a fundamental solution tensor can be computed explicitly [13]. In order to state it, we need the following abbreviations:

$$\mathsf{c}_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad \mathsf{c}_2 = \sqrt{\frac{\mu}{\rho}}, \quad \mathsf{e} = \frac{\beta^2}{\varpi(\lambda + 2\mu)}, \quad \mathsf{d}_1^2 = -\frac{s^2}{\mathsf{c}_1^2}, \quad \mathsf{d}_2^2 = -\frac{s^2}{\mathsf{c}_2^2}, \quad \mathsf{q} = \frac{-\varpi T_{\mathrm{ref}}}{\frac{\kappa}{T_{\mathrm{ref}}}} s, \tag{A.1}$$

$$\mathsf{k}_{1,2}^2 = \frac{1}{2}\left\{\mathsf{d}_1^2 + \mathsf{q} + \mathsf{eq} \pm \left[\left(\mathsf{d}_1^2 + \mathsf{q} + \mathsf{eq}\right)^2 - 4\mathsf{q}\mathsf{d}_1^2\right]^{\frac{1}{2}}\right\}, \tag{A.2}$$

$$\mathcal{H}(\mathsf{r}, s) = \frac{\mathsf{d}_1^2\left(\mathsf{k}_1^2 - \mathsf{q}\right)}{\mathsf{k}_1^2\left(\mathsf{k}_1^2 - \mathsf{k}_2^2\right)}\frac{\exp(\mathsf{k}_1\mathsf{r})}{\mathsf{r}} - \frac{\mathsf{d}_1^2\left(\mathsf{k}_2^2 - \mathsf{q}\right)}{\mathsf{k}_2^2\left(\mathsf{k}_1^2 - \mathsf{k}_2^2\right)}\frac{\exp(\mathsf{k}_2\mathsf{r})}{\mathsf{r}} - \frac{\exp(\mathsf{d}_2\mathsf{r})}{\mathsf{r}}, \quad \text{where } \mathsf{r} = \|\vec{x}\|. \tag{A.3}$$

Then

$$\mathcal{U}_j^{(i)} = \frac{\delta_{ij}}{4\pi\rho\mathsf{c}_2^2}\frac{\exp(\mathsf{d}_2\mathsf{r})}{\mathsf{r}} + \frac{1}{4\pi\rho s^2}\frac{\partial^2}{\partial x_i \partial x_j}\mathcal{H}(\mathsf{r}, s), \quad i, j \in \{1, 2, 3\}, \tag{A.4}$$

$$\mathcal{T}^{(i)} = \frac{\mathsf{qe}}{4\pi\beta\left(k_1^2 - \mathsf{k}_2^2\right)}\frac{\partial}{\partial x_i}\left(\frac{\exp(\mathsf{k}_1\mathsf{r})}{\mathsf{r}} - \frac{\exp(\mathsf{k}_2\mathsf{r})}{\mathsf{r}}\right), \quad i \in \{1, 2, 3\}, \tag{A.5}$$

$$\mathcal{U}_j^{(4)} = -\frac{\beta}{4\pi\frac{\kappa}{T_{\mathrm{ref}}}\rho\mathsf{c}_1^2\left(\mathsf{k}_1^2 - \mathsf{k}_2^2\right)}\frac{\partial}{\partial x_j}\left(\frac{\exp(\mathsf{k}_1\mathsf{r})}{\mathsf{r}} - \frac{\exp(\mathsf{k}_2\mathsf{r})}{\mathsf{r}}\right), \quad j \in \{1, 2, 3\}, \tag{A.6}$$

$$\mathcal{T}^{(4)} = -\frac{1}{4\pi\frac{\kappa}{T_{\mathrm{ref}}}\left(\mathsf{k}_1^2 - \mathsf{k}_2^2\right)}\left(\left(\mathsf{k}_1^2 - \mathsf{d}_1^2\right)\frac{\exp(\mathsf{k}_1\mathsf{r})}{\mathsf{r}} - \left(\mathsf{k}_2^2 - \mathsf{d}_1^2\right)\frac{\exp(\mathsf{k}_2\mathsf{r})}{\mathsf{r}}\right). \tag{A.7}$$

For the convenience of the reader, we remark that for any $\mathsf{k} \in \mathbb{C}$,

$$\frac{\partial}{\partial x_i}\frac{\exp(\mathsf{kr})}{\mathsf{r}} = \frac{x_i}{\mathsf{r}^3}\left(\mathsf{kr} - 1\right)\exp(\mathsf{kr}), \tag{A.8}$$

$$\frac{\partial}{\partial x_i \partial x_j}\frac{\exp(\mathsf{kr})}{\mathsf{r}} = \left[\delta_{ij}\frac{\mathsf{kr} - 1}{\mathsf{r}^3} + \frac{x_i x_j}{\mathsf{r}^5}\left(3 - 3\mathsf{kr} + \mathsf{k}^2\mathsf{r}^2\right)\right]\exp(\mathsf{kr}). \tag{A.9}$$

As we are in the Laplace domain, the fundamental solution for Q is obtained from those for $\mathcal{T}$ by multiplying with s.

Finally, the fundamental solution is given by

$$\vec{\vec{G}}_s = \begin{pmatrix} \mathcal{U}_j^{(i)} & \mathcal{T}^{(i)} \\ \mathcal{U}_j^{(4)} & \mathcal{T}^{(4)} \end{pmatrix}. \tag{A.10}$$

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