



UNIVERSITÀ
DEGLI STUDI
FIRENZE

JOINT DOCTORATE BETWEEN
Università degli studi Firenze
Goethe Universität Frankfurt

DOTTORATO DI RICERCA IN FISICA E ASTRONOMIA
XXXV CICLO

Spin polarization in the strongly interacting QCD matter at global and local equilibrium

Settore scientifico Disciplinare FIS/02

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Years 2019-2022

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Notation

In this work, natural units are employed where $\hbar = c = k_B = 1$. The Minkowski metric tensor is $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Greek indices are assumed to run from 0 to 3, whereas Latin indices run from 1 to 3. Vectors in 3-space are denoted via their components (e.g. k^i) or with bold letters (e.g. \mathbf{k}). Sometimes components of three vectors will be denoted with the letter corresponding to the axis in a Cartesian reference; when this is done it is implied that $k_x = k^1$, $k_y = k^2$ and $k_z = k^3$. The Levi-Civita symbol is such that $\epsilon^{0123} = 1$. Contracted indices are intended to be summed over, and they are also denoted with a dot, e.g. $v \cdot w = v_\mu w^\mu = \sum_{\mu=0}^3 v_\mu w^\mu$ and $l : m = l_{\mu\nu} m^{\mu\nu} = \sum_{\mu,\nu=0}^3 l_{\mu\nu} m^{\mu\nu}$.

Operators are denoted with a wide hat $\widehat{T}^{\mu\nu}$ and versors with a smaller hat \hat{t} . An exception to this rule is the Dirac field operator, which is simply Ψ . The trace on the Hilbert space is denoted with a capital T, i.e. Tr, whereas traces on finite-dimensional spaces are simply tr. Other notation is introduced when needed in the text.

Zusammenfassung

Diese Doktorarbeit beschäftigt sich mit den Spin-Eigenschaften quantenmechanischer relativistischer Flüssigkeiten und versucht, einige Beobachtungen im sog. Quark-Gluon-Plasma (QGP) zu erklären. Das QGP ist ein in Schwerionenkollisionen erzeugter Zustand extrem heißer und dichter stark wechselwirkender Materie, der sich als quantenmechanische Flüssigkeit beschreiben lässt. Die Energiedichte in diesen Kollisionen ist derart hoch, dass die beteiligten Nukleonen schmelzen, sodass das resultierende Fluid, welches das extremste in der Natur bekannte darstellt, aus deren Konstituenten, den Quarks und Gluonen, besteht. Die Wissenschaft ist sich weitgehend einig, dass die Materie in unserem Universum kurz nach dem Urknall im Zustand eines QGP war, sodass die hier durchgeführten Studien auch einen Beitrag zur Physik des frühen Universums leisten können.

In peripheren Kollisionen wird das QGP mit einem nichtverschwindenden Bahndrehimpuls erzeugt. Während sich die Flüssigkeit ausdehnt, abkühlt, und schließlich zu Teilchen hadronisiert, wird erwartet, dass der Bahndrehimpuls des Fluids in Spin der emittierten Teilchen umgewandelt werden kann. Diese Vermutung wurde bestätigt, als die STAR Collaboration die Polarisation der Λ -Hyperonen gemessen hat. Diese Polarisation als Funktion der Kollisionsenergiedichte (die sog. *globale Polarisation*) verschwindet nicht, und konnte gut mit einem Modell beschrieben werden, das auf dem lokalen Gleichgewicht basiert. In diesem Modell geht die sog. thermische Vortizität, welche als die antisymmetrische Ableitung der Vierer-Temperatur definiert ist, als Quelle der Polarisation ein, sodass das QGP als das am schnellsten rotierende jemals erzeugte Fluid angesehen werden kann. An diesem Punkt muss angemerkt werden, dass der Polarisationsvektor zu diesem Zeitpunkt nur bis zur führenden Ordnung in der thermischen Vortizität berechnet werden konnte.

Überraschenderweise war die Polarisation als Funktion des Azimutwinkels bei festgehaltener Energiedichte (auch *lokale Polarisation* genannt) mit dieser Art Modell nicht vereinbar. Der Unterschied zwischen Theorie und Experiment war derart, dass zwischen der theoretischen Vorhersage und den experimentellen Daten gerade ein Vorzeichenunterschied bestand, weswegen sich für dieses Problem der Name *polarization sign puzzle* etabliert hat. Dieses Resultat sorgte für Aufruhr unter den beteiligten Wissenschaftlern, sodass viel Arbeit in die Aufklärung dieses Widerspruchs investiert wurde. Die Notwendigkeit einer Lösung war die Hauptmotivation für die vorliegende Arbeit.

Das QGP wird mit großem Erfolg als ein Fluid modelliert, welches sich quasi im Gleichgewicht befindet. Aus diesem Grund wurden in dieser Arbeit relativistische Flüssigkeiten im lokalen und globalen Gleichgewicht studiert. Insbesondere wurde das Dirac-Feld im globalen Gleichgewicht mit nichtverschwindender thermischer Vortizität analysiert, mit der Zielsetzung, einen exakten Ausdruck für die Polarisation zu finden. Um dieses Ziel zu erreichen, war es notwendig, eine neue Methode zur

Berechnung von Erwartungswerten in einem Gleichgewicht mit nichtverschwindender thermischer Vortizität anzuwenden, basierend auf einer iterativen Methode für die Berechnung des Erwartungswertes des Teilchenzahloperators $\hat{a}_s^\dagger(p)\hat{a}_t(p')$. Dieser Operator kann als eine Reihe von Funktionen ausgedrückt werden und erlaubt, jede thermische Zweipunktfunktion in der freien Feldtheorie zu berechnen. Auf diese Weise kann die exakte Wignerfunktion bestimmt werden, welche eine Hilfsfunktion für die Berechnung von Erwartungswerten wie z.B. des Energie-Impuls-Tensors, des Vektor- und Axialvektorstroms, sowie des Polarisationsvektors darstellt. Es war möglich, die genannten Größen für masselose Teilchen mit Hilfe einer Regularisierung namens *analytischer Destillation* zu bestimmen, welche darin besteht, den analytischen Teil einer Reihe von Funktionen zu extrahieren. Im Zuge dessen wurden asymptotische Reihenentwicklungen von Funktionenreihen gefunden, welche Anwendungen in der mathematischen Physik haben könnten. Die analytische Destillation reproduziert die aus der Literatur bekannten Ergebnisse, wurde aber auch auf bisher nicht untersuchte Fälle angewandt. So war es beispielsweise möglich, einen Gleichgewichts-zustand zu untersuchen, in dem Beschleunigung und Winkelgeschwindigkeit nicht orthogonal zueinander stehen, und so die Abhängigkeit bspw. der Energiedichte vom Skalarprodukt der Winkelgeschwindigkeit und der Beschleunigung zu bestimmen.

Schließlich wurde dank der iterativen Methode ein analytischer Ausdruck für die Polarisation von Dirac-Fermionen im globalen Gleichgewicht gefunden. Diese neue Formel enthält eine Funktion, welche als eine Verallgemeinerung der Fermi-Dirac-Verteilung für nichtverschwindende Vortizität aufgefasst werden kann. Die Formel wurde mit den experimentellen Daten zur globalen Polarisation verglichen, um die Bedeutung von Korrekturen höherer Ordnung in der thermischen Vortizität für die Physik der Polarisation zu verstehen. Allerdings hat sich herausgestellt, dass die thermische Vortizität, welche sich in Schwerionenkollisionen entwickelt, zu gering für einen nennenswerten Einfluss von Korrekturen höherer Ordnung ist. Mit ähnlichen Techniken war es möglich, die exakte Polarisation für Teilchen mit beliebigem Spin zu berechnen, wobei die lineare Näherung für fast alle Anwendungen ausreichend ist. Diese Untersuchungen waren dennoch notwendig um zu zeigen, dass die Vernachlässigung von Korrekturen höherer Ordnung nicht die Ursache für das polarization sign puzzle war.

Weiterhin wurde die Physik des lokalen Gleichgewichts untersucht, welches das globale Gleichgewicht verallgemeinert. Der Hauptunterschied zwischen globalem und lokalem Gleichgewicht liegt darin, dass in letzterem die symmetrische Ableitung der Vierer-Temperatur nicht verschwinden muss: diese Größe wird als thermischer Schertensor bezeichnet. Nun stellt sich die Frage, ob dieser Tensor, der in Schwerionenkollisionen von der gleichen Größenordnung wie die thermische Vortizität ist, zur Polarisation beitragen kann. Tatsächlich wurde herausgefunden, dass dies in der Tat der Fall ist: sowohl die thermische Vortizität als auch die thermische Scherung beeinflussen die Polarisation von Teilchen.

Nach der Berechnung einer neuen Quelle für die Polarisation war es notwendig, die neuen theoretischen Vorhersagen zu überprüfen. Diese Analyse wurde mittels einer Simulation der Entwicklung des QGP unter Verwendung von 3+1 dimensionalen hydrodynamischen Codes durchgeführt, die zunächst so eingestellt wurden, dass bekannte experimentell messbare Größen korrekt reproduziert werden konnten. Ein Resultat war, dass die thermische Scherung bei der Lösung des polarization sign puzzle hilft. Weiterhin beeinflusst die Geometrie der sog. Freeze-Out-Hyperfläche die

Ergebnisse stark: in Schwerionenkollisionen wird erwartet, dass diese Hyperfläche isotherm ist, da das chemische Potential vernachlässigt werden kann. Unter dieser Näherung kann die thermische Scherung das polarization sign puzzle lösen.

Weitergehende theoretische Untersuchungen sowie größere Datenmengen, besonders für Kollisionen niedriger Energie, werden benötigt, um das polarization sign puzzle ein für alle Mal zu lösen. Vom theoretischen Standpunkt aus sollte die Wichtigkeit des Anfangszustandes für die Physik der Polarisation bestimmt werden. Vorläufige Studien suggerieren, dass die Volumenviskosität eine Rolle spielen könnte, sodass sich thermische Vortizität und Scherung länger im Nichtgleichgewicht befinden können.

Introduction

The standard model of particle physics describes the Universe using a dozen matter particles and three fundamental interactions: the electromagnetic, weak and strong forces. The latter is responsible for interactions between hadrons and, more fundamentally, between quarks. The model describing the strong interaction is an $SU(3)$ gauge theory, where Dirac fields describe the quarks, whereas the gluons, which are the quanta of the $SU(3)$ gauge field, mediate the interaction. This theory is known as Quantum Chromo-Dynamics (QCD).

QCD exhibits many remarkable phenomena depending on the energy scale. At the scales we experience in everyday life, strongly interacting particles are tightly packed into hadrons. This behaviour is called “confinement”. In such a regime, perturbation theory is unreliable because the coupling constant of the strong interaction is too large. Therefore one has to use non-perturbative methods to study confinement. The strong interaction also exhibits asymptotic freedom [1]. Namely, the QCD coupling constant becomes smaller as we increase the energy scale. In the high-energy regime, perturbation theory is applicable, and its results are reliable. This behaviour testifies to how different the same theory can look at different energy scales, and is opposite compared to quantum electrodynamics.

One of the most compelling ways to study QCD is by colliding heavy ions at relativistic energies to probe the theory in extreme conditions. Particle colliders such as the Large Hadron Collider (LHC, at Geneva) and the Relativistic Heavy Ion Collider (RHIC, Brookhaven) have been built to this end. By colliding nuclei, the outcome of countless interactions among nucleons is observed in the form of particles that fly to the experimental detectors after the collision. These kinds of experiments were able to discover a new phase of QCD matter. The new state of matter was predicted to be well described as a relativistic fluid of strongly interacting particles and was named *Quark-Gluon Plasma* (QGP) [2].

The QGP is a phase of hadronic matter, in the same sense as the water vapour is a phase of water. Soon after the Big Bang, QGP was the dominant form of matter in our Universe [3] and investigating this exotic QCD plasma can shed some light on early Universe phenomena. The study of phase transition from hadronic matter to QGP is a very active field of research. In the small-baryon-density regime, lattice gauge theory is used quite successfully, and the phase transition appears to be a cross-over. At higher densities lattice QCD stops being reliable due to the infamous “sign problem”, which makes calculations unfeasible. Some speculations suggest that, for larger densities, the phase transition changes its character turning into a first-order phase transition. It is conjectured that, at some intermediate density, a critical point should appear where the order of the phase transition changes from cross-over to second-order before turning to first-order. The search for the critical point is one of the most active research areas in heavy-ion physics, both theoretically

and experimentally. Indeed, many new facilities, such as NICA (Dubna, Russia) and FAIR (Darmstadt, Germany), will soon start to probe the QGP in the baryon density range where the critical point is expected.

Furthermore, due to the high energy involved in the collision, topological gauge configurations with non-vanishing winding number may emerge in the QGP via sphaleron transitions. If this is the case, a fluctuating axial chemical potential could be measured experimentally, for example, via the chiral magnetic effect [4]. The chiral magnetic effect and its experimental detection represent another vibrant research area.

The QGP evolution is described successfully using quasi-ideal hydrodynamics, where the ratio of shear viscosity η to entropy density s is the smallest ever observed in Nature. Indeed, it is remarkably close to the theoretical limit of $\eta/s \sim 1/4\pi$, derived from the AdS/CFT correspondence [5]. Soon after its creation, the QGP expands, cooling down and rarefying until it eventually decays into new hadrons. This stage is called *hadronization* or *freeze-out*. After the freeze-out, particles produced fly towards the detectors, where they are measured.

The droplets of QGP formed experimentally, especially in non-central heavy-ion collisions, can carry a significant amount of orbital angular momentum, that should be taken into account in the theoretical description of the QGP and should be detectable experimentally [6]. One may surmise that a fraction of the orbital angular momentum will be transferred in the form of spin to the hadrons once they are formed during the freeze-out stage. This effect implies hadrons produced at freeze-out to have their spin polarized. Spin is a fundamental quantum number describing the internal angular momentum of particles. It plays a crucial role in classifying particles in bosons (integer spin) and fermions (half-integer spin) and dictating their properties via the spin-statistics theorem. Heavy-ion collisions provide a unique opportunity to study spin in the relativistic realm. The first theoretical calculations predicted spin polarization of particles produced at freeze-out to be driven by the thermal vorticity tensor, which is related to the angular velocity and the acceleration experienced by the fluid [7].

Such an effect was first detected in 2017 by the STAR experiment at RHIC, which measured for the first time a non-vanishing spin polarization of the Λ particle [8]. The experiment showed a signal of global polarization, which is the average spin of the Λ hyperon as a function of the collision energy. The reason why the Λ hyperon plays a prominent role in polarization physics is that it decays weakly into proton p and pion π : the weak interaction violates parity which causes the spin of the Λ to be mainly in the direction of the daughter proton. This finding was in agreement with the theoretical calculations and was a further success of the fluid-dynamic description of the QGP.

This milestone measurement stirred the community as it opened a new avenue of research in heavy-ion phenomenology: spin physics. After the first measurement, experiments collected more and more data, disclosing interesting phenomena and posing many intriguing challenges to theorists. The first to appear was the dependence of the polarization vector of the Λ particle on transverse momentum and the azimuthal angle, the so-called *local polarization*. The data show that the projection of the polarization vector predicted theoretically along the beam axis differs by a sign compared to the experimental measurements. Such inconsistency has since been known in the literature as the *polarization sign puzzle*. A massive theoretical effort was made to try to explain such a disagreement, and the present thesis is part of it.

Furthermore, novel spin-related phenomena have been observed experimentally. One such example is the so-called *spin alignment* for spin-one particles (in particular, of the ϕ and K^* mesons)[9]. The vorticity of the fluid cannot explain the magnitude (nor the sign) of the signal measured, which suggests that the origin of alignment has to be different. Even more recently, the evidence of alignment for a charmonium state such as the J/ψ has been reported.

From the theoretical standpoint, a density operator can be used to describe a quantum-relativistic fluid such as the QGP. The formalism of the *non-equilibrium density operator*, introduced by Zubarev [10] and reworked by other authors later [11], represents an ideal tool for this goal. The non-equilibrium density operator allows to describe a system close to (or at) a state of global equilibrium. Additionally, it provides a clear-cut distinction between dissipative and non-dissipative effects and allows deriving Kubo-like formulae for near-equilibrium corrections [12].

For relativistic systems, global equilibrium includes the familiar case of a static fluid at equilibrium with a constant temperature as well as less trivial examples that are more relevant to the physics of heavy-ion collisions. For example, global equilibrium can describe rigidly rotating and accelerating systems, where the values of angular velocity and acceleration are encoded into the (constant) thermal vorticity tensor, which was identified as the source of polarization. Computing thermal expectation values in systems at equilibrium with non-vanishing thermal vorticity has proved a formidable problem. Typically, it is tackled by solving the field equations in a curvilinear system of coordinates.

To calculate expectation values at global or local equilibrium an ancillary function can be used: the *Wigner function*. The reason why it is convenient to use it is that the Wigner function is a quantum generalisation of the familiar concept of the distribution function used in kinetic theory. Once the Wigner function is known, all other expectation values, such as the energy-momentum tensor, particle currents and the spin vector, can be calculated by integrating it. In the trivial case of equilibrium with constant temperature and vanishing thermal vorticity, the Wigner function has long been known. The form of the Wigner function at general global equilibrium with non-vanishing vorticity is one of the subjects of this thesis [13, 14].

In the case of local equilibrium, expectation values can rarely be computed exactly, the task being possible only if some symmetry is assumed [15]. However, we can use linear response theory to get corrections to the global equilibrium case. With the density-operator formalism, first-order corrections to the Wigner function, which are linear in the gradients of thermodynamic quantities (e.g. temperature, chemical potentials, and so on), can be systematically calculated. With this technique, novel contributions to the spin vector have been found that can provide the solution to the long-standing spin puzzle [16, 17]. With the density operator formalism, it is possible to show that a symmetric tensor, called *thermal shear*, produces a non-vanishing polarization.

This thesis is organized as follows. In chapter 1 we review Zubarev's formalism for systems at global and local equilibrium. We summarize the group-theoretical description of the Dirac field, which will be used in later chapters, and we introduce the Wigner function for such a field. Moreover, we brush up on the concept of spin in a relativistic field theory, both for massive and massless particles, and we obtain formulae expressing its expectation value in terms of the Wigner function for a gas of fermions.

Chapter 2 is devoted to a phenomenological description of the QGP. After a

brief historical introduction, we discuss the phase diagram and the current understanding of the evolution of matter in a heavy-ion collision. Then, we delve into the phenomenology of spin, describing the spin sign puzzle and a smattering of alignment physics, which is relevant for particles with spin larger than one-half.

The calculation of expectation values at global equilibrium is the main topic of chapter 3. We start by obtaining some asymptotic formulae for series of functions. These will be of capital importance for a new technique developed to compute expectation values at global equilibrium: the *analytic distillation*. To calculate mean values at global equilibrium, we use an iterative method to obtain the expectation value of the number operator and use this result to express the Wigner function for Dirac fermions at general global equilibrium with non-vanishing thermal vorticity. Finally, the expectation value of quantities such as the energy-momentum tensor, and axial and vector currents for massless fermions are obtained at equilibrium with acceleration, with rotation, and with both rotation and acceleration. The definition of a distribution function from the Wigner function in a chiral theory and mass corrections to the expectation values is also discussed.

In chapter 4, we apply the results of the previous chapter to polarization physics. We obtain the exact formula for the polarization vector of Dirac fermions at global equilibrium and we discuss its phenomenological importance. Lastly, we compute the exact spin density matrix and the spin vector for a gas of relativistic particles with any spin.

The last chapter, chapter 5, is devoted to local equilibrium physics and the study of thermal shear in phenomenological applications. Using linear response theory, we find the contribution of thermal shear to the polarization vector and we compare the prediction of the new formula with experimental data using realistic 3+1 hydrodynamics simulations. Our results show that the thermal shear can solve the polarization sign puzzle.

Chapter 1

Quantum statistical mechanics

In this chapter, the basic tools of relativistic quantum statistical mechanics are introduced. The formalism presented here will be employed throughout this work. First, we will review the quantum-mechanical meaning of the density operator and its properties. Then we will proceed by revising the formalism of the non-equilibrium density operator, introduced by Zubarev and co-authors [10] and later reworked by Van Weert [11], and discuss the global and local equilibrium and the case of dissipative fluids. After a quick review of the theory of the Dirac field from a group-theoretical perspective, the Wigner function for such a field will be introduced. We will disclose its properties and, after some discussion about spin physics in the relativistic regime, we will conclude this chapter by providing formulae connecting the Wigner function to the average spin of fermions.

1.1 Density operator

The density operator is a useful concept when dealing with statistical systems. Suppose we have a macroscopic system of which we do not know the precise micro-state [18]. What we mean by “the system” is actually an element of an ensemble of systems all prepared in the same way. Each sub-system can be in different micro-states ψ_λ so that the knowledge of the system cannot be complete, and we can only know the probability of the system being in the micro-state $|\psi_\lambda\rangle$ (for the moment we assume that these probabilities are somehow given). In this case, our system will not be a pure state but a *statistical mixture*. To describe a statistical mixture, the concept of state vector has to be generalized by introducing the density operator:

$$\hat{\rho} = \sum_{\lambda} P_{\lambda} |\psi_{\lambda}\rangle \langle \psi_{\lambda}|, \quad \sum_{\lambda} P_{\lambda} = 1. \quad (1.1)$$

In the above formulae, P_{λ} are real positive numbers, that are interpreted as the probability to observe the micro-state $|\psi_{\lambda}\rangle^*$. It is easy to realize that the density operator is hermitian, positive semi-definite and its trace is unitary, i.e. $\text{Tr}(\hat{\rho}) = 1$. Moreover, since the density operator is expressed in terms of the quantum states of the system, it follows that it is constant in the Heisenberg picture of quantum mechanics, which will be employed throughout this work:

$$\frac{d\hat{\rho}}{dt} = 0. \quad (1.2)$$

*If λ is a continuous degree of freedom, the summation is to be intended as an integral.

The density operator is very useful when dealing with expectation values. The expectation value of an operator \hat{O} on one of the micro-states of the system is given by $\langle \psi_\lambda | \hat{O} | \psi_\lambda \rangle$, and since the system is in the state ψ_λ with probability P_λ , the expectation value of the operator \hat{O} on the macro-state of the system is given by:

$$\langle \hat{O} \rangle = \sum_\lambda P_\lambda \langle \psi_\lambda | \hat{O} | \psi_\lambda \rangle.$$

Choosing any basis $|e_i\rangle$ of the Hilbert space and using the completeness relation $\sum_i |e_i\rangle \langle e_i| = I$, we have:

$$\langle \hat{O} \rangle = \sum_{\lambda, i} P_\lambda \langle \psi_\lambda | \hat{O} | e_i \rangle \langle e_i | \psi_\lambda \rangle = \sum_i \langle e_i | \left(\sum_\lambda P_\lambda | \psi_\lambda \rangle \langle \psi_\lambda | \hat{O} \right) | e_i \rangle,$$

therefore:

$$\langle \hat{O} \rangle = \text{Tr} \left(\hat{\rho} \hat{O} \right). \quad (1.3)$$

This formula will be used extensively in this work as it is very simple and also suitable for perturbative calculations.

Until now we have assumed that the micro-states and their relative probability are somehow given to us, and a procedure to identify the density operator shall be introduced. Commonly, the density operator is inferred by an axiomatic method that does not rely on the definition (1.1), but on assigning an a priori probability for the system to be in one of the possible micro-state. This is done by maximising the so-called *Von-Neumann entropy*, defined as:

$$\hat{S} = -\hat{\rho} \ln \hat{\rho}. \quad (1.4)$$

It is well-known that the entropy operator assigns a number to our “ignorance” about the system or, in other words, to the amount of lacking information. By this interpretation, as well as from explicit calculation, the density operator maximising the entropy is the one associated with a system where all micro-states are equiprobable, which corresponds to the maximal lack of a priori knowledge about the system.

Luckily, often times we have access to some properties that can help to better identify the density operator. For example, we may know the mean energy and/or the mean value of some conserved charge of the system. In such a case, the maximization of the entropy is a problem of maximisation of a function with additional constraints, where the method of Lagrange multipliers can be used. In the above-mentioned example of a system with fixed expectation values of energy and charge, one has to maximise the functional:

$$F[\hat{\rho}, \beta, \zeta, \ell] = -\hat{\rho} \ln \hat{\rho} - \beta \left[\text{Tr} \left(\hat{\rho} \hat{\mathcal{H}} \right) - E_0 \right] + \zeta \left[\text{Tr} \left(\hat{\rho} \hat{Q} \right) - Q_0 \right] + \ell \left(\text{Tr} \left(\hat{\rho} \right) - 1 \right),$$

where $\hat{\mathcal{H}}$ is the Hamilton operator, \hat{Q} is the charge operator and β , ζ and ℓ are Lagrange multipliers. Notice how setting the derivative with respect to the Lagrange multipliers to zero enforces the constraint of fixed mean energy, mean charge and unitary trace.

The density operator maximising such a functional is

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{\mathcal{H}} + \zeta \hat{Q}}, \quad (1.5)$$

where Z is the partition function required by normalization:

$$Z = \text{Tr} \left(e^{-\beta \hat{\mathcal{H}} + \zeta \hat{Q}} \right). \quad (1.6)$$

and the physical interpretation of the Lagrange multipliers β and ζ is $\beta = T^{-1}$ and $\zeta = \mu/T$, where T and μ are the temperature and the chemical potential of the system.

These concepts are well known in non-relativistic quantum statistical mechanics. For the rest of this manuscript, we will be studying the case of relativistic fluids not necessarily at equilibrium. The generalization of the density-operator formalism to this physical situation will be the subject of the next section.

1.2 Relativistic-quantum statistical mechanics

The formalism of the so-called relativistic non-equilibrium density operator was first introduced by Zubarev in 1979 and reworked by Van Weert in a later work [10, 11] (see also [19] for a modern perspective and derivation). It allows describing a relativistic fluid out of equilibrium, although it is often implied that we are not far from the equilibrium state. This formalism can be used to derive Kubo formulae for transport coefficients, both dissipative [12, 20–23] and non-dissipative (the latter sometimes called the *thermodynamic susceptibilities*) [24–28]. Furthermore, Zubarev’s formalism allows a clear-cut distinction between dissipative and non-dissipative effects. In this section, we will first review the general theory of the non-equilibrium density operator, and then consider the cases of global and local equilibrium. The procedure to obtain Kubo formulae for the dissipative and non-dissipative transport coefficients will also be addressed.

1.2.1 Non-equilibrium density operator

Let us consider a macroscopic continuum medium, for which a relativistic description is compulsory. Similarly to the non-relativistic case, where the state of the system was characterized by using the energy and a conserved charge, we may use densities such as the energy-momentum tensor and in some conserved current[†] to describe the state of the medium at hand. These densities are interpreted as the result of the thermal quantum average of the respective operator:

$$T_{true}^{\mu\nu} = \text{Tr} \left(\hat{\rho} \hat{T}^{\mu\nu} \right), \quad j_{true}^{\mu} = \text{Tr} \left(\hat{\rho} \hat{j}^{\mu} \right). \quad (1.7)$$

The energy-momentum tensor and the conserved current operators fulfil:

$$\partial_{\mu} \hat{T}^{\mu\nu} = 0, \quad \partial_{\mu} \hat{j}^{\mu} = 0.$$

Here and in what follows, operators are always assumed to be in the Heisenberg representation. It is worth spending some words on the definition of the energy-momentum tensor. In fact, in field theory, the energy-momentum and the angular momentum tensors are not univocally defined [29]. For example, the Noether theorem provides the so-called canonical energy-momentum tensor, whereas varying the

[†]Although one can consider more than one current, we will restrict ourselves to the treatment of a single one. The generalization is straightforward.

Lagrangian with respect to the metric yields the Belinfante energy-momentum tensor. Other notable examples of different definitions are the Hilgevoord-Wouthuysen and the de Groot-van Leeuwen-van Weert tensors [30, 31]. Although these tensors differ in general, they yield the same global charge (i.e. the four-momentum operator): this ambiguity in the choice of the energy-momentum tensor is called *pseudogauge*. The choice of a pseudogauge also affects the spin tensor $\widehat{S}^{\mu,\nu\rho}$, which is defined from

$$\widehat{J}^{\mu,\nu\rho} = x^\nu \widehat{T}^{\mu\rho} - x^\rho \widehat{T}^{\mu\nu} + \widehat{S}^{\mu,\nu\rho},$$

where $\widehat{J}^{\mu\nu\rho}$ is the total angular momentum density. A pseudogauge transformation of these tensors reads:

$$\begin{aligned} \widehat{T}'^{\mu\nu} &= \widehat{T}^{\mu\nu} + \frac{1}{2} \nabla_\alpha \left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu} \right), \\ \widehat{S}'^{\mu,\nu\rho} &= \widehat{S}^{\mu,\nu\rho} - \widehat{\Phi}^{\mu,\nu\rho} + \nabla_\alpha \widehat{Z}^{\mu\nu,\rho\alpha}. \end{aligned}$$

where the following properties hold:

$$\widehat{\Phi}^{\mu,\nu\rho} = -\widehat{\Phi}^{\mu,\rho\nu}, \quad \widehat{Z}^{\mu\nu,\rho\sigma} = -\widehat{Z}^{\nu\mu,\rho\sigma} = -\widehat{Z}^{\mu\nu,\sigma\rho}.$$

Despite yielding the same global charges, thermal expectation values at local equilibrium appear to be different if computed in different pseudogauges, and the choice of the “correct” pseudogauge is an open problem in the community [32]. Here, unless explicitly stated, we will use the energy-momentum tensor in the Belinfante pseudogauge, so that the energy-momentum tensor is symmetric and the spin tensor is identically vanishing.

In practice, the full knowledge of $\widehat{T}^{\mu\nu}$ and \widehat{j}^μ is neither required nor possible, and the state of the system can be specified equally well by using a subset of such densities: the energy density, the momentum flux and the charge density. To define these quantities, we have to introduce the mathematical concept of a space-time foliation [33]. Given a function $\tau(x)$, which is not necessarily the proper time of a given observer, such that space-time can be decomposed into disjoint space-like hypersurfaces defined by different constant values of τ , the collection of these hypersurfaces is called a foliation of space-time and each hypersurface is a level-hypersurface. Let $n^\mu(x)$ be the future-oriented time-like vector normal to the unique level-hypersurface in the foliation containing the point x , and $\Delta_n^{\mu\nu} = g^{\mu\nu} - n^\mu n^\nu$ the tangent projector at such a point. The vector n^μ can be identified with a four-velocity so that the choice of a foliation is equivalent to the choice of a reference frame. Given these definitions, the energy density, the momentum flux and the charge density are expressed as:

$$\varepsilon(x) = T^{\mu\nu} n_\mu n_\nu, \tag{1.8a}$$

$$w^\mu(x) = \Delta^{\mu\alpha} T_{\alpha\nu} n^\nu, \tag{1.8b}$$

$$\rho(x) = n_\mu j^\mu. \tag{1.8c}$$

If the state of the system is specified *only* by the above densities, we do not need the full energy-momentum and current density, but only their projection with respect to the hypersurface at hand. In other words, we do not need to specify the density operator satisfying Eq.(1.7), but only a density operator that reproduces (1.8). Notice that such a density operator is not uniquely defined, as different $T^{\mu\nu}$ and j^μ

may share the same relevant projections. Using the principle of maximum entropy on the hypersurface Σ_τ of the foliation, the generalised density operator reads:

$$\hat{\rho} = \frac{1}{Z} \exp \left[- \int_{\Sigma_\tau} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) \right]. \quad (1.9)$$

Here β_μ and ζ are Lagrange multipliers, and they are interpreted physically as the ratio between the four-velocity and the proper temperature and the ratio between the chemical potential and the proper temperature [11]:

$$\beta^\mu = \frac{u^\mu}{T}, \quad T = \frac{1}{\sqrt{\beta \cdot \beta}}, \quad \zeta = \frac{\mu}{T}. \quad (1.10)$$

Their values can be obtained, at least in principle, by solving the five constraints equations:

$$n_\mu(x) T^{\mu\nu}(x) = n_\mu(x) \langle \hat{T}^{\mu\nu}(x) \rangle, \quad (1.11a)$$

$$n_\mu(x) j^\mu(x) = n_\mu(x) \langle \hat{j}^\mu(x) \rangle, \quad (1.11b)$$

where $n_\mu(x)$ is the normal vector to Σ_τ . Z is the partition function of the system, defined as:

$$Z = \text{Tr} \left(\exp \left[- \int_{\Sigma_\tau} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) \right] \right), \quad (1.12)$$

and it is such that the trace of the density operator is one, $\text{Tr}(\hat{\rho}) = 1$.

Notice that the density operator (1.9) is not stationary, as it depends on time through the operators and the space-time foliation. In fact, Eq. (1.9) is sometimes called *local-equilibrium density operator* and it describes an ideal fluid in a non-equilibrium state. The reason why the fluid is ideal will be clarified at the end of the section, and by non-equilibrium we mean that the system is not in a global equilibrium state. On the other hand, the true density operator must be constant in the Heisenberg representation. If the system is in a state of local equilibrium at some initial time τ_0 , then the true non-equilibrium density operator is simply expressed by:

$$\hat{\rho}_{true} = \frac{1}{Z} \exp \left[- \int_{\Sigma_{\tau_0}} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) \right]. \quad (1.13)$$

It is interesting to notice that both (1.9) and (1.13) depend on a particular hypersurface in space-time, so expectation values computed with these operators may also depend thereon.

Equation (1.13) defines the true, stationary, non-equilibrium density operator describing the fluid. In most cases, however, the use of Eq. (1.13) is out of our reach since very little is known about the hypersurface at τ_0 , whereas it is easier to use Eq. (1.9) on a more convenient hypersurface. Using the Gauss theorem:

$$\begin{aligned} \hat{\rho}_{true} &= \frac{1}{Z} \exp \left[- \int_{\Sigma_{\tau_0}} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) \right] \\ &= \frac{1}{Z} \exp \left[- \int_{\Sigma_\tau} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) + \int_{\Omega} d\Omega \left(\hat{T}^{\mu\nu} \nabla_\mu \beta_\nu - \hat{j}^\mu \nabla_\mu \zeta \right) \right], \end{aligned} \quad (1.14)$$

where Ω is the volume encompassed by Σ_τ and Σ_0 and in full generality we used ∇ , the covariant derivative.

Equation (1.14) shows that the difference between the true density operator and the local equilibrium one is given by an integral over the volume Ω bounded by Σ_{τ_0} and Σ_τ . Interestingly, this difference is responsible for entropy production. It is possible to prove, that given the entropy:

$$S = -\text{Tr}(\hat{\rho} \ln \hat{\rho}) = \int d\Sigma_\mu s^\mu, \quad (1.15)$$

then:

$$\nabla_\mu s^\mu = (T_{true}^{\mu\nu} - \langle \hat{T}^{\mu\nu} \rangle_{LE}) \nabla_\mu \beta_\nu + (j_{true}^\mu - \langle \hat{j}^\mu \rangle_{LE}) \nabla_\mu \zeta, \quad (1.16)$$

where $\langle \bullet \rangle_{LE}$ is computed using the density operator (1.9) whereas $T_{true}^{\mu\nu}$ and j_{true}^μ are true values appearing in the constraints (1.7). We refer the reader to Refs. [11, 19] for the proof of equation (1.16).

The above equation shows clearly what is dissipative and what is not: using Eq. (1.9) to compute expectation values in Eq. (1.7) we will obtain non-equilibrium results for an ideal fluid as there is no entropy production, whereas if we use Eq. (1.13) dissipative effects are taken into account. This is the reason why Eq. (1.9) is called local-equilibrium density operator: it does not produce entropy.

1.2.2 Global equilibrium

A state of global equilibrium is defined not only by the requirement that the entropy production (1.16) vanishes but also by imposing that the density operator is globally defined, such that the dependence on the hypersurface Σ disappears. In other words, and referring to Eq. (1.14), the volume integral should vanish for any volume for $\hat{\rho}$ to be independent on Σ . Given that we are using a symmetric energy-momentum operator, this condition is met if:

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0, \quad \nabla_\mu \zeta = 0, \quad (1.17)$$

therefore the equilibrium state of the fluid is achieved if the four-temperature is a Killing vector and if the ratio of chemical potential over temperature is (covariantly) constant.

We will now focus our analysis on the Minkowski space-time in Cartesian coordinates, where the covariant derivative is equivalent to the partial derivative. In this case, global equilibrium is described by a constant ratio of chemical potential over temperature, whereas the most general expression for the four-temperature vector is:

$$\beta(x)^\mu = b^\mu + \varpi^{\mu\nu} x_\nu, \quad \varpi_{\mu\nu} = -\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu). \quad (1.18)$$

In the previous equation, b^μ is a constant four-vector and $\varpi^{\mu\nu}$ is a constant antisymmetric tensor called *thermal vorticity*. If we plug these expressions into the operator (1.9), we readily obtain:

$$\hat{\rho} = \frac{1}{Z} e^{-\hat{P} \cdot b + \frac{\varpi}{2} : \hat{J} + \zeta \hat{Q}}, \quad (1.19)$$

where only global charges appear: the four-momentum operator \hat{P} , the angular momentum-boost operator \hat{J} , and the global charge \hat{Q} :

$$\hat{P}^\mu = \int d\Sigma_\nu \hat{T}^{\nu\mu}, \quad \hat{J}^{\mu\nu} = \int d\Sigma_\rho (x^\mu \hat{T}^{\rho\nu} - x^\nu \hat{T}^{\rho\mu}), \quad \hat{Q} = \int d\Sigma_\mu \hat{j}^\mu. \quad (1.20)$$

We can see that, disregarding the global charge \widehat{Q} for a moment, Eq. (1.19) is the exponential of the generators of the Poincaré group. This observation will be useful for the purposes of Chapter 3.

One may wonder about the physical meaning of the thermal vorticity. First of all, since ϖ is an antisymmetric tensor, it is possible to decompose it along the four-velocity u^μ as follows:

$$\varpi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} w_\rho u_\sigma + (\alpha^\mu u^\nu - \alpha^\nu u^\mu), \quad (1.21)$$

where the components are:

$$w^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \varpi_{\nu\rho} u_\sigma, \quad \alpha^\mu = \varpi^{\mu\nu} u_\nu. \quad (1.22)$$

To make an analogy with the electro-magnetic tensor $F^{\mu\nu}$, here w_μ plays the role of the magnetic field, and α_μ is the electric field. It will be sufficient to give an interpretation to w_μ and α_μ to understand the meaning of the thermal vorticity. To give such an interpretation, we consider the case $\zeta = 0$ and discuss some notable examples of global equilibrium:

- $b^\mu = (\frac{1}{T_0}, \mathbf{0})$, $\varpi_{\mu\nu} = \frac{\omega}{T_0} (g_{\mu 1} g_{\nu 2} - g_{\mu 2} g_{\nu 1})$.

In this case, defining $r^2 = x^2 + y^2$, the four-temperature, the scalar temperature, the four-velocity and the vector w^μ read:

$$\begin{aligned} \beta^\mu &= \frac{1}{T_0} (1, -\omega y, \omega x, 0), & T &= \gamma T_0, \\ u^\mu &= \gamma (1, -\omega y, \omega x, 0), & w^\mu &= \frac{\gamma}{T_0} (0, 0, 0, \omega). \end{aligned} \quad (1.23)$$

with $\gamma = (1 - \omega^2 r^2)^{-1/2}$. From the four-velocity u^μ , it can be realized that we are describing a fluid at equilibrium rigidly rotating around the z axis with a constant angular velocity ω . The value of w^μ makes it clear that w^μ is nothing but the ratio between the angular velocity ω^μ and the proper temperature:

$$w^\mu = \frac{\omega^\mu}{T}. \quad (1.24)$$

- $b^\mu = (\frac{1}{T_0}, \mathbf{0})$, $\varpi_{\mu\nu} = \frac{a}{T_0} (g_{\mu 3} g_{\nu 0} - g_{\mu 0} g_{\nu 3})$.

In this case, we have:

$$\begin{aligned} \beta^\mu &= \frac{1}{T_0} (1 + az, 0, 0, at), & T &= \gamma T_0, \\ u^\mu &= \gamma (1 + az, 0, 0, at), & \alpha^\mu &= \frac{\gamma a}{T_0} (at, 0, 0, 1 + az). \end{aligned} \quad (1.25)$$

where $\gamma = [(1 + az)^2 - a^2 t^2]^{-1/2}$. To give a physical meaning to the parameter a we consider an observer comoving with the fluid, who measures as the proper temperature $T = T_0$. This observer moves along the trajectory $(1 + az)^2 - a^2 t^2 = 1$. If we compute the four-acceleration experienced by such an observer:

$$A^\mu = u \cdot \partial u^\mu = \frac{a}{T_0} (at, 0, 0, 1 + az) = T_0 \alpha^\mu, \quad A^2 = -a^2, \quad (1.26)$$

so we can interpret a as the proper acceleration experienced by the observer moving with the fluid cells such that $T = T_0$. In this case, we can see that α is the ratio between four-acceleration and proper temperature:

$$\alpha^\mu = \frac{A^\mu}{T}. \quad (1.27)$$

This analysis shows that the presence of non-vanishing thermal vorticity allows the description of non-trivial equilibria, where the fluid is rigidly rotating, or experiences a non-vanishing acceleration. Many systems can be described by this kind of equilibrium. Examples in astrophysics include rapidly rotating neutron stars, or even accretion disks around black holes (although to describe these systems a general-relativistic treatment is required). More relevant to our aim is the QGP produced in heavy-ion collisions, where the extreme violence of the collision produces a fluid with extremely large acceleration and angular velocity.

1.2.3 Local equilibrium and linear response theory

In this section, we come back to the study of the more general case of a fluid at local equilibrium described by the operator (1.9). Notice that the definition of local equilibrium adopted here differs from the one commonly used in kinetic theory. In kinetic theory, a system is said to be in local equilibrium if the distribution function is such that the collision term of the Boltzmann equation vanishes, although the equation itself would not in general be satisfied by such a distribution [34, 35]. The operator (1.9) can, on the other hand, describe a system where interactions may or may not be present, and no mention of their vanishing is made.

Although the local equilibrium density operator is an approximated form of the true non-equilibrium density operator, computing expectation values at local equilibrium represents a formidable challenge. It can be done exactly only if we assume some additional symmetries for the hypersurface Σ_τ , such as in the case of a boost-invariant fluid [15]. In more general cases, the exact calculation is simply not possible, and some approximation is needed.

One of the difficulties connected to the calculations of expectation values with the local-equilibrium density operator lies in the fact that the operators in the exponent are not globally defined, but are evaluated on the hypersurface Σ_τ (from now on, the subscript τ will be dropped). For the reader's convenience, we repeat the definition of the local equilibrium density operator, which we will denote from now on as $\hat{\rho}_{LE}$ to avoid confusion:

$$\hat{\rho}_{LE} = \frac{1}{Z} \exp \left[- \int_{\Sigma} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \zeta \hat{j}^\mu \right) \right]. \quad (1.28)$$

If one deals with the expectation values of local operators, an approximation can be found to cope with this difficulty. Let $\hat{O}(x)$ be a local operator, and consider the expectation value:

$$\langle \hat{O}(x) \rangle_{LE} = \text{Tr} \left(\hat{\rho}_{LE} \hat{O}(x) \right). \quad (1.29)$$

The approximation we are going to use is the so-called *hydrodynamic approximation*. The hydrodynamic regime is defined by the value of the *Knudsen number* Kn , that is the ratio between microscopic ℓ and macroscopic L length-scales of the system, $\text{Kn} = \ell/L$. In the hydrodynamic limit one has $\text{Kn} \ll 1$, so microscopic processes happen on a much smaller scale compared to macroscopic ones. The microscopic length scale can be identified with the mean free path between collisions in a kinetic approach, or with the correlations lengths in a quantum system. On the other hand, the typical macroscopic length-scale is given by the scale on which gradients are non-negligible.

In the hydrodynamic regime, we expect the main contribution to the expectation value of $\hat{O}(x)$ to come from the thermodynamic fields at the very same point x . In

other words, we anticipate thermodynamic fields to vary on a much larger scale compared to the relevant correlation and interaction lengths. Therefore, we can Taylor-expand the thermodynamic quantities around the point x :

$$\begin{aligned}
 \text{Tr} \left(\widehat{\rho}_{LE} \widehat{O}(x) \right) &\simeq \frac{1}{Z_{LE}} \text{Tr} \left(\exp \left[- \int_{\Sigma} d\Sigma_{\mu} \widehat{T}^{\mu\nu}(y) [\beta_{\nu}(x) + \partial_{\lambda} \beta_{\nu}(x) (y-x)^{\lambda}] \right. \right. \\
 &\quad \left. \left. - \widehat{j}^{\mu}(y) [\zeta(x) + \partial_{\lambda} \zeta(x) (y-x)^{\lambda}] \right] \widehat{O}(x) \right) \\
 &= \frac{1}{Z_{LE}} \text{Tr} \left(\exp \left[-\beta_{\nu}(x) \int_{\Sigma} d\Sigma_{\mu} \widehat{T}^{\mu\nu}(y) - \partial_{\lambda} \beta_{\nu}(x) \int_{\Sigma} d\Sigma_{\mu} (y-x)^{\lambda} \widehat{T}^{\mu\nu}(y) \right. \right. \\
 &\quad \left. \left. + \zeta(x) \int_{\Sigma} d\Sigma_{\mu} \widehat{j}^{\mu} + \partial_{\lambda} \zeta(x) \int_{\Sigma} d\Sigma_{\mu} (y-x)^{\lambda} \widehat{j}^{\mu} \right] \widehat{O}(x) \right) \\
 &= \frac{1}{Z_{LE}} \text{Tr} \left(\exp \left[-\beta_{\nu}(x) \widehat{P}^{\nu} + \zeta(x) \widehat{Q} \right. \right. \\
 &\quad \left. \left. - \partial_{\lambda} \beta_{\nu}(x) \int_{\Sigma} d\Sigma_{\mu} (y-x)^{\lambda} \widehat{T}^{\mu\nu}(y) + \partial_{\lambda} \zeta(x) \int_{\Sigma} d\Sigma_{\mu} (y-x)^{\lambda} \widehat{j}^{\mu} \right] \widehat{O}(x) \right). \quad (1.30)
 \end{aligned}$$

Notice that this expansion does not use any additional assumption other than the hydrodynamic hypothesis: this means that it is quite general, but may not be the best approximation of the statistical operator for some specific applications (see Chapter 5).

The first part of the approximated local-equilibrium density operator is essentially the same as the global-equilibrium density operator describing an equilibrium at constant b and ζ (see Eq. (1.19)), the only difference being that the thermodynamic fields have local values $\beta(x)$ and $\zeta(x)$. The remainder is a correction to such operator proportional to gradients of the four-temperature and chemical potential over temperature. In the zeroth-order approximation, one could neglect these gradient corrections completely, so that the expectation value would be:

$$\text{Tr} \left(\widehat{\rho}_{LE} \widehat{O}(x) \right) \simeq \frac{1}{Z_{\beta(x), \zeta(x)}} \text{Tr} \left(e^{-\beta(x) \cdot \widehat{P} + \zeta(x) \widehat{Q}} \widehat{O}(x) \right) \equiv \langle \widehat{O}(x) \rangle_{\beta(x), \zeta(x)}, \quad (1.31)$$

where we introduced the notation $\langle \bullet \rangle_{\beta(x), \zeta(x)}$ to indicate an expectation value computed at global equilibrium with local values of β and ζ , as in the above equation. The above equation shows that a rough estimate of expectation values at local equilibrium is given by the expectation values at global equilibrium but with x -dependent thermodynamic fields.

To carry out calculations beyond this level of approximation, we have to rely once more on the hydrodynamic hypothesis. Given the fact that the gradients in Eq. (1.30) are small, we can expand the exponential itself but we have to proceed with some care as the operators in the exponent do not commute. To cope with this difficulty, we use the identity [11, 36]:

$$e^{\widehat{A} + z \widehat{B}} = e^{\widehat{A}} \left(\mathbb{I} + \sum_{n=1}^{\infty} z^n \widehat{B}_n \right), \quad (1.32)$$

where

$$B_n = \int_0^1 d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \cdots \int_0^{\lambda_{n-1}} d\lambda_n \widehat{B}(\lambda_1) \cdots \widehat{B}(\lambda_n), \quad \widehat{B}(\lambda) = e^{-\lambda \widehat{A}} \widehat{B} e^{\lambda \widehat{A}}. \quad (1.33)$$

This formula allows expanding the density operator in Eq. (1.30) to any order in gradients. For the purposes of this work, however, we shall only keep the first-order expansion which leads to:

$$\begin{aligned} \widehat{\rho}_{LE} \simeq \frac{1}{Z} e^{-\beta(x) \cdot \widehat{P} + \zeta(x) \widehat{Q}(x)} & \left(\mathbb{I} - \partial_\lambda \beta_\nu(x) \int_0^1 dz \int_\Sigma d\Sigma_\mu (y-x)^\lambda \widehat{T}^{\mu\nu}(y-iz\beta(x)) \right. \\ & \left. + \partial_\lambda \zeta(x) \int_0^1 dz \int_\Sigma d\Sigma_\mu (y-x)^\lambda \widehat{j}^\mu(y-iz\beta(x)) \right) \end{aligned} \quad (1.34)$$

where we used the fact that \widehat{P} is the generator of translations to write:

$$\begin{aligned} e^{z\beta(x) \cdot \widehat{P}} \widehat{T}^{\mu\nu}(y) e^{-z\beta(x) \cdot \widehat{P}} &= \widehat{T}^{\mu\nu}(y-iz\beta(x)), \\ e^{z\beta(x) \cdot \widehat{P}} \widehat{j}^\mu(y) e^{-z\beta(x) \cdot \widehat{P}} &= \widehat{j}^\mu(y-iz\beta(x)). \end{aligned}$$

Notice that also the partition function should be expanded accordingly and its expansion in terms of gradients will be such that the correlators are the connected ones, as commonly happens in field theory [24]. For our purposes, it suffices to remind the reader that the connected two-point function is defined as:

$$\langle \widehat{A}\widehat{B} \rangle_c = \langle \widehat{A}\widehat{B} \rangle - \langle \widehat{A} \rangle \langle \widehat{B} \rangle, \quad (1.35)$$

and will be denoted with a subscript “c”, as shown in the above formula.

Finally, the first-order correction in $\partial_\lambda \beta_\nu$ and $\partial_\lambda \zeta$ to thermal expectation values at local equilibrium is given by:

$$\begin{aligned} \langle \widehat{O}(x) \rangle_{LE} \simeq \langle \widehat{O}(x) \rangle_{\beta(x), \zeta(x)} & - \partial_\lambda \beta_\nu(x) \int_0^1 dz \int_\Sigma d\Sigma_\mu (y-x)^\lambda \langle \widehat{T}^{\mu\nu}(y-iz\beta(x)) \widehat{O}(x) \rangle_{c, \beta(x), \zeta(x)} \\ & + \partial_\lambda \zeta(x) \int_0^1 dz \int_\Sigma d\Sigma_\mu (y-x)^\lambda \langle \widehat{j}^\mu(y-iz\beta(x)) \widehat{O}(x) \rangle_{c, \beta(x), \zeta(x)}. \end{aligned} \quad (1.36)$$

We have been able to map the calculation of expectation values at local equilibrium to the one of correlators between \widehat{O} and $\widehat{T}^{\mu\nu}$ and \widehat{j}^μ at global homogeneous equilibrium, significantly simplifying the task. These kinds of Kubo-like relations describe the response of the fluid to the presence of small gradients, that put it slightly off equilibrium. Notice however that, the striking difference between the relation (1.36) and the usual Kubo formulae, is that the former is non-dissipative as in a system described by the local equilibrium density operator the entropy production is vanishing according to equation (1.16).

The same approximation can be used to compute dissipative transport coefficients, as we briefly report here. In such a case we have to use the true non-equilibrium density operator (1.14) (in flat space-time for simplicity). Approximating the local equilibrium part at zeroth-order, we have:

$$\rho = \frac{1}{Z} \exp \left[-\beta(x) \cdot \widehat{P} + \zeta(x) \widehat{Q} + \int d\Omega \left(\widehat{T}^{\mu\nu} \partial_\mu \beta_\nu - \widehat{j}^\mu \partial_\mu \zeta \right) \right]. \quad (1.37)$$

Notice that we are including the integral over the volume Ω , which is the only difference between the true density operator (1.13) and the local-equilibrium one (1.28). Therefore, according to Eq. (1.16), the entropy production is not zero

and this is the reason why in this case linear-response theory provides dissipative corrections.

If the evolution of the system is hydrodynamic throughout the volume Ω , we can proceed exactly like before, obtaining:

$$\begin{aligned} \langle \widehat{O}(x) \rangle_{LE} &\simeq \langle \widehat{O}(x) \rangle_{\beta(x), \zeta(x)} \\ &- \int_0^1 dz \int d\Omega \partial_\mu \beta_\nu \langle \widehat{T}^{\mu\nu}(y - iz\beta(x)) \widehat{O}(x) \rangle_{c, \beta(x), \zeta(x)} \\ &+ \int_0^1 dz \int d\Omega \partial_\mu \zeta(x) \langle \widehat{j}^\mu(y - iz\beta(x)) \widehat{O}(x) \rangle_{c, \beta(x), \zeta(x)}. \end{aligned} \quad (1.38)$$

This equation, with some additional manipulations [12, 19] can be cast in the form of the standard Kubo relations. The above formula can be used to compute the dissipative correction to the expectation value of the local operator $\widehat{O}(x)$, leading for example to the Kubo formulae for the shear and bulk viscosities η and ζ in the case of $\widehat{O} = \widehat{T}^{\mu\nu}$.

This kind of linear response theory is very general and will be used in the later chapters to study polarization in relativistic fluids at local equilibrium.

1.3 Free Dirac field in the group-theory formalism

This work will be focused on the Dirac field and its properties at global and local equilibrium. Therefore, before undertaking such a study and especially for the purposes of Chapter 3, it is useful to review some key concepts. The quantum theory of the Dirac field has been extensively studied, and can be found in any quantum field theory textbook [37–39]. In this section, we will follow the construction due to S. Weinberg [40–42], which heavily uses the theory of the Lorentz group. Additional details concerning the group theory of the Lorentz group are relegated to Appendix A.

The free Dirac field in flat space-time can be expanded as:

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}} \sum_{s=-1/2}^{1/2} \int \frac{d^3p}{2\varepsilon} \left[\widehat{a}_s(p) u_s(p) e^{-ip \cdot x} + \widehat{b}_s^\dagger(p) v_s(p) e^{ip \cdot x} \right]. \quad (1.39)$$

Here $\widehat{a}_s(p)$ and its adjoint $\widehat{a}_s^\dagger(p)$ are the annihilation and creation operator for particle states of momentum p and spin state s . The analogues for antiparticle states are denoted as $\widehat{b}_s(p)$ and $\widehat{b}_s^\dagger(p)$. This expansion is completely general, and the quantum number s can describe either the spin or the helicity of the field. The integration variable is the on-shell momentum, and ε is the corresponding energy $\varepsilon = \sqrt{\mathbf{p}^2 + m^2}$. Creation and annihilation operators in the expansion (1.39) fulfil the anti-commutation rules:

$$\begin{aligned} \{\widehat{a}_s(p), \widehat{a}_t^\dagger(p')\} &= 2\varepsilon \delta_{st} \delta^3(\mathbf{p} - \mathbf{p}'), \\ \{\widehat{b}_s(p), \widehat{b}_t^\dagger(p')\} &= 2\varepsilon \delta_{st} \delta^3(\mathbf{p} - \mathbf{p}'), \end{aligned}$$

where all other possible anti-commutators vanish. The field expansion (1.39) can be written in a compact form by arranging the 4×1 column-spinors $u_s(p)$ and the annihilation operators $\widehat{a}_s(p)$ into vectors as:

$$U(p) = (u_{1/2}(p), u_{-1/2}(p)) \quad \widehat{A}(p) = \begin{pmatrix} \widehat{a}_{1/2}(p) \\ \widehat{a}_{-1/2}(p) \end{pmatrix}. \quad (1.40)$$

Using this notation both for $u_s(p)\widehat{a}_s(p)$ and $v_s(p)\widehat{b}_s^\dagger(p)$, the field expansion reads:

$$\Psi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{2\varepsilon} \left[U(p)\widehat{A}(p) e^{-ip \cdot x} + V(p)\widehat{B}^\dagger(p) e^{ip \cdot x} \right]. \quad (1.41)$$

Since $U(p)\widehat{A}(p)$ is a multiplication of a 4×2 matrix by a 2×1 one, this is effectively just a compact way to express the sum over s in Eq. (1.39).

The interesting feature of the spinors $U(p)$ and $V(p)$ is that they are related to the so-called standard Lorentz transformation, which is used to construct one-particle states in relativistic quantum mechanics. Indeed, the standard procedure to build the basis of the Fock space starts by constructing a “standard” state, with momentum \mathbf{p} , $|\mathbf{p}, \sigma\rangle$ where σ refers e.g. to the spin of the state. Then, a standard transformation, which we denote as $[p]$ is defined such that it transforms the standard four-momentum \mathbf{p} into the four-momentum p , i.e., $p^\mu = [p]^\mu{}_\nu \mathbf{p}^\nu$. The Fock space of particles with momentum p is built using the unitary representation of such a transformation $|p, \sigma\rangle = U([p])|\mathbf{p}\rangle$. This construction can be made both for massive and massless particles, although the standard vector \mathbf{p} and transformation $[p]$ differ in form in the two cases, as we will discuss later.

The form of the spinor in terms of the standard Lorentz transformations is fixed by requiring that the field $\Psi(x)$ transforms as the irreducible representation $(0, 1/2) \oplus (1/2, 0)$ of the orthochronous Lorentz group $\text{SO}(1, 3)^\uparrow$ [42]. Therefore, the spinor in the frame where the particle has momentum p can be obtained from the one in the standard frame (where the particle momentum is \mathbf{p}) by acting on it with the $(0, 1/2) \oplus (1/2, 0)$ representation of $[p]$ (see Appendix A):

$$U(p) = \begin{pmatrix} D([p]) & 0 \\ 0 & D([p])^{\dagger-1} \end{pmatrix} U(\mathbf{p}), \quad V(p) = \begin{pmatrix} D([p]) & 0 \\ 0 & D([p])^{\dagger-1} \end{pmatrix} V(\mathbf{p}), \quad (1.42)$$

where $D(\)$ is the 2-dimensional $(0, 1/2)$ representation $D^{(0,1/2)}$ of $\text{SO}(1, 3)^\uparrow$. In general, a Lorentz transformation in the $(0, 1/2) \oplus (1/2, 0)$ representation is given by:

$$S(\Lambda) = \begin{pmatrix} D(\Lambda) & 0 \\ 0 & D(\Lambda)^{\dagger-1} \end{pmatrix}, \quad (1.43)$$

and matrices of this kind are used to transform the Dirac field Ψ . The generators of $\text{SO}(1, 3)^\uparrow$ in the $(0, 1/2) \oplus (1/2, 0)$ representations are:

$$\begin{pmatrix} D^{(0,1/2)}(J^{\mu\nu}) & 0 \\ 0 & D^{(1/2,0)}(J^{\mu\nu}) \end{pmatrix} = \frac{i}{4} [\gamma^\mu, \gamma^\nu] \equiv \Sigma^{\mu\nu},$$

where the γ matrices are in the so-called Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (1.44)$$

and $\sigma_\mu = (I, \sigma_1, \sigma_2, \sigma_3)$, $\bar{\sigma}_\mu = (I, -\sigma_1, -\sigma_2, -\sigma_3)$, with σ_i being the Pauli matrices and $\sigma^\mu = g^{\mu\nu} \sigma_\nu$. A generic Lorentz transformation, like the one in Eq. (1.43), can be written as:

$$S(\Lambda) = \exp \left[-i \frac{\phi^{\mu\nu}}{2} \Sigma_{\mu\nu} \right]. \quad (1.45)$$

In the standard frame, the spinors $U(\mathbf{p})$, $V(\mathbf{p})$ are given by:

$$U(\mathbf{p}) = N \begin{pmatrix} \mathbf{p} \\ \widehat{\mathbf{p}} \end{pmatrix}, \quad V(\mathbf{p}) = N \begin{pmatrix} \mathbf{p} C^{-1} \\ \widehat{\mathbf{p}} C \end{pmatrix}. \quad (1.46)$$

Here, N is a normalization factor and $C = i\sigma_2^\dagger$; the arcs attached to the four-momentum \mathbf{p} represent the mapping of the vector \mathbf{p} from the Minkowski space to the space of 2×2 hermitian matrices. A matrix with a lower arc is given by:

$$\underline{X} \equiv X^\mu \sigma_\mu, \quad (1.47)$$

whereas the one with the upper arc is:

$$\widehat{X} = X^\mu \bar{\sigma}_\mu. \quad (1.48)$$

These definitions are commonly used to construct the $\text{SL}(2, \mathbb{C}) - \text{SO}(1, 3)^\dagger$ morphism [43]. See also Appendix A.

Up until now the discussion has been quite general and applies both to massive and massless fields. The differences between the two cases lie in the standard vector \mathbf{p} , the standard Lorentz transformation $[p]$ and the normalization factor N . In the massive case, one can choose $\mathbf{p}^\mu = (m, \mathbf{0})$, $N = 1/\sqrt{m}$ and we have:

$$U(\mathbf{p}) = \sqrt{m} \begin{pmatrix} I \\ I \end{pmatrix},$$

where I the 2×2 identity matrix. On the other hand, in the massless case, the standard vector is usually chosen to be $\mathbf{p}^\mu = (\kappa, 0, 0, \kappa)$, where $\kappa > 0$ is some fixed energy value. In this case, $N = 1/\sqrt{2\kappa}$ and the spinor $U(\mathbf{p})$ reads:

$$U(\mathbf{p}) = \frac{1}{\sqrt{2\kappa}} \begin{pmatrix} 2\kappa & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2\kappa \end{pmatrix} = \sqrt{2\kappa} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that, because of the above form, if we consider the column spinor of the massless field with helicity $+1/2$, which is the first column of the matrix $U(p)$ and will be denoted as $u_+(p)$, then only its upper two components are non-vanishing. For the spinor $u_-(p)$, in contrast, only the lower two components can be non-zero.

By construction, and by looking at the equations (1.40) and (1.46), it is apparent that the spinors depend on the choice of the standard vector \mathbf{p} and of the standard Lorentz transformation $[p]$, which are in principle arbitrary. However, also the creation and annihilation operators will depend on that choice [44] and the construction is such that the field operator is independent of such a choice.

The Dirac field obeys the Dirac equation

$$(i\rlap{/}\partial - m) \Psi = 0, \quad (1.49)$$

which in terms of the spinors reads:

$$(\rlap{/}\partial - m)U(p) = 0, \quad (\rlap{/}\partial + m)V(p) = 0,$$

[†]The appearance of the matrix C in the V spinor is dictated by the transformation properties of the \widehat{A} and \widehat{B} operators under charge conjugation.

with $\not{p} = \gamma^\mu p_\mu$. Using Eqs. (1.44), (1.47), and (1.48), \not{p} reads:

$$\not{p} = \begin{pmatrix} 0 & p \\ \widehat{p} & 0 \end{pmatrix}.$$

In the group formalism employed by Weinberg, the fact that spinors obey such relation is a by-product of requiring the S-matrix to be Lorentz invariant, and one can prove that the spinors in Eq. (1.42) identically fulfil the equation by taking into account the $\text{SL}(2, \mathbb{C}) - \text{SO}(1, 3)^\uparrow$ correspondence (see Appendix A for the proof). Quoting his words: “a free-field equation is nothing but an invariant record of which components are superfluous” [41].

Usually, some choices for the standard vector and the standard Lorentz boost are favoured compared to others, and they can lead to useful relations. For example in the massive case, choosing $\mathbf{p}^\mu = (m, \mathbf{0})$ and $[p]$ to be the pure boost taking \mathbf{p} to p one has:

$$U(p) = \frac{1}{\sqrt{2(\varepsilon + m)}} (\not{p} + m) \begin{pmatrix} I \\ I \end{pmatrix}.$$

Similarly, in the massless case, one chooses $\mathbf{p}^\mu = (\kappa, 0, 0, \kappa)$ and $[p]$ to be the composition of a Lorentz boost along the z axis and a rotation taking $\hat{\mathbf{z}}$ into $\hat{\mathbf{p}}$ with axis $\hat{\mathbf{z}} \times \hat{\mathbf{p}}$. In such a case it is possible to show (for the proof of these relations, see for example, Ref. [34]):

$$U(p) = \frac{1}{\sqrt{2p \cdot \mathbf{q}}} \not{p} \gamma^0 U(\mathbf{p}),$$

where $\mathbf{q} = (\kappa, 0, 0, -\kappa)$ is the parity conjugate of \mathbf{p} . Using these relations, or alternatively by explicit calculations from Eq. (1.42), it can be checked that both massive and massless spinors fulfil the following relations:

$$\begin{aligned} \bar{U}(p)U(p) &= 2mI, & U(p)\bar{U}(p) &= \not{p} + m, & \bar{U}(p)\gamma^\mu U(p) &= 2p^\mu I, \\ \bar{V}(p)V(p) &= -2mI, & V(p)\bar{V}(p) &= \not{p} - m, & \bar{V}(p)\gamma^\mu V(p) &= -2p^\mu I, \end{aligned} \quad (1.50)$$

where the Dirac conjugate of the spinor is defined as $\bar{U} = U^\dagger \gamma^0$. Similarly, one defines $\bar{\Psi} = \Psi^\dagger \gamma^0$ for the field itself.

To conclude this section, we introduce a matrix $\mathcal{C} = i\gamma^2$, which proves useful to deal with the $V(p)$ spinors, as it is related to charge conjugation. Indeed, one has:

$$V(p) = \mathcal{C}U(p)^*.$$

This matrix is also involved in the relations:

$$\gamma^0 \mathcal{C} S(\Lambda)^T \mathcal{C} \gamma^0 = S(\Lambda)^{-1}, \quad \gamma^0 \mathcal{C} \gamma^{\mu T} \mathcal{C} \gamma^0 = -\gamma^\mu. \quad (1.51)$$

1.4 The Wigner function

When concerned with the calculation of expectation values at local and global equilibrium two different approaches can be used: either one proceeds on a case-by-case study, computing each expectation value when needed, or uses the *Wigner-function* formalism. Indeed, one can express all expectation values in terms of integrals and traces of the Wigner function associated with the field. In this sense, although with some noteworthy differences, the Wigner function can be regarded as a quantum

analogue of the classical distribution function. Here, we will focus on the case of the free Dirac field, for which the covariant Wigner function reads [34]:

$$W_{AB}(x, k) = -\frac{1}{(2\pi)^4} \int d^4y e^{-ik \cdot y} \text{Tr} \left(\hat{\rho} : \Psi_A \left(x - \frac{y}{2} \right) \bar{\Psi}_B \left(x + \frac{y}{2} \right) : \right). \quad (1.52)$$

The normal ordering, expressed with the colons $: \bullet :$, is used to regularize expectation values by subtracting the vacuum contribution. We recall that $\bar{\Psi} = \Psi^\dagger \gamma^0$ is the Dirac conjugate of the field, and here A, B are spinor indices.

Notice that the Wigner function (1.52) is a 4×4 matrix which is not hermitian, but satisfies:

$$W^\dagger(x, k) = \gamma^0 W(x, k) \gamma^0.$$

It should also be stressed that in the definition (1.52) the variable k has the dimensions of momentum but it is in general not on-shell.

Being a 4×4 matrix, sometimes the Wigner function is decomposed with respect to the elements of the Clifford algebra $\{I, \gamma_5, \gamma^\mu, \gamma^\mu \gamma_5, \Sigma^{\mu\nu}\}$ as:

$$W = \frac{1}{4} (\mathcal{F} + \mathcal{P} \gamma_5 + \mathcal{V}_\mu \gamma^\mu + \mathcal{A}_\mu \gamma^\mu \gamma_5 + 2\mathcal{S}_{\mu\nu} \Sigma^{\mu\nu}), \quad (1.53)$$

where the components are given by:

$$\begin{aligned} \mathcal{F} &= \text{tr}(W) && \text{(scalar),} \\ \mathcal{P} &= \text{tr}(W \gamma_5) && \text{(pseudo-scalar),} \\ \mathcal{V}^\mu &= \text{tr}(W \gamma^\mu) && \text{(vector),} \\ \mathcal{A}^\mu &= \text{tr}(W \gamma^\mu \gamma_5) && \text{(axial-vector),} \\ \mathcal{S}^{\mu\nu} &= \text{tr}(W \Sigma^{\mu\nu}) && \text{(antisymmetric tensor).} \end{aligned}$$

Such a decomposition is often used to simplify the study of the Wigner function, especially in the context of kinetic theory [45–47].

We can expand the Wigner function in terms of creation and annihilation operators by plugging the field expansion (1.39) into Eq. (1.52), whence we find:

$$\begin{aligned} W(x, k) &= \frac{1}{(2\pi)^3} \sum_{s, t} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} \left\{ e^{ix \cdot (p' - p)} \left[\langle \hat{a}_t^\dagger(p') \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p') \delta^4 \left(k - \frac{p+p'}{2} \right) + \right. \right. \\ &\quad \left. \left. - \langle \hat{b}_s^\dagger(p') \hat{b}_t(p) \rangle v_s(p') \bar{v}_t(p) \delta^4 \left(k + \frac{p+p'}{2} \right) \right] - \left[e^{i(p+p') \cdot x} \langle \hat{b}_t^\dagger(p') \hat{a}_s^\dagger(p) \rangle v_t(p') \bar{u}_s(p) + \right. \right. \\ &\quad \left. \left. + e^{-i(p+p') \cdot x} \langle \hat{a}_t(p) \hat{b}_s(p') \rangle u_t(p) \bar{v}_s(p') \right] \delta^4 \left(k - \frac{p-p'}{2} \right) \right\}. \end{aligned} \quad (1.54)$$

From this expansion, we identify unambiguously particle-, antiparticle-, and space-like components of the Wigner function as:

$$W(x, k) = \theta(k^2) \theta(k^0) W_+(x, k) + \theta(k^2) \theta(-k_0) W_-(x, k) + \theta(-k^2) W_S(x, k), \quad (1.55)$$

and each component can be singled out by multiplying the Wigner function $W(x, k)$ by the appropriate combination of Heaviside θ functions.

Using the Dirac equation (1.49) and integrating by parts, one can show that W is a solution of the so-called *Wigner equation*, which for free Dirac fermions and restoring \hbar reads:

$$\left(\frac{i\hbar}{2} \not{\partial} + \not{k} - m \right) W(x, k) = 0. \quad (1.56)$$

This equation can be used as the starting point of a semi-classical expansion in \hbar (which is in fact an expansion in the gradients of the Wigner function) to provide quantum corrections to the Boltzmann equation [45, 48, 49]. The Wigner equation, however, is a consequence of the definition of the Wigner function and the Dirac equation, therefore *any* sensible Wigner function for free fermions will solve it. In other words, Eq. (1.56) does not provide any information on the density operator describing the state of the system. In this respect, the Wigner equation should rather be regarded as a constraint.

As mentioned before, using the Wigner function it is possible to express expectation values of local operators by means of integral relations involving the pseudo-momentum variable k . Considering the vector and axial current, and the canonical energy-momentum tensor, one has:

$$j^\mu(x) \equiv \langle : \hat{j}^\mu(x) : \rangle = \langle : \bar{\Psi}(x) \gamma^\mu \Psi(x) : \rangle = \text{tr} \left(\gamma^\mu \int d^4k W(x, k) \right), \quad (1.57a)$$

$$j_A^\mu(x) \equiv \langle : \hat{j}_A^\mu(x) : \rangle = \langle : \bar{\Psi}(x) \gamma^\mu \gamma_5 \Psi(x) : \rangle = \text{tr} \left(\gamma^\mu \gamma_5 \int d^4k W(x, k) \right), \quad (1.57b)$$

$$T_C^{\mu\nu}(x) \equiv \langle : \hat{T}_C^{\mu\nu}(x) : \rangle = \langle : \frac{i}{2} \bar{\Psi}(x) \gamma^\mu \overleftrightarrow{\partial}^\nu \Psi(x) : \rangle = \text{tr} \left(\gamma^\mu \int d^4k k^\nu W(x, k) \right), \quad (1.57c)$$

where $\overleftrightarrow{\partial}^\nu = \overrightarrow{\partial}^\nu - \overleftarrow{\partial}^\nu$ and the trace is on the spinor indices.

These expressions show that, once the Wigner function is known, it is possible to recover other relevant expectation values. For this reason, one would be tempted to draw an analogy between the Wigner function and the classical distribution function $f(x, p)$, but some differences prevent us from a direct identification. As we have pointed out, the Wigner function of the Dirac field is a non-hermitian matrix, and its momentum variable is not on-shell. Nonetheless, we can attempt to identify the distribution function as some integral of the Wigner function. To do so, we proceed like in Ref. [13], and study the vector current. The procedure proposed therein does in fact yield a definition of the distribution function in terms of the Wigner function for scalar fields. Confining ourselves to the particle part of the Wigner function, easily obtainable by using the decomposition (1.54), and using the Eq. (1.57a), we have:

$$j_+^\mu(x) = \frac{1}{(2\pi)^3} \sum_{s,t} \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} e^{i(p'-p)\cdot x} \langle \hat{a}_t^\dagger(p') \hat{a}_s(p) \rangle \bar{u}_t(p') \gamma^\mu u_s(p). \quad (1.58)$$

To identify the distribution function, we should be able to express Eq. (1.58) as the integral of $p^\mu f(x, p)$, like in kinetic theory. One can express the particle current (1.58) as the integral of a vector field depending on both momentum and space-time point:

$$j_+^\mu(x) = \int \frac{d^3p}{\varepsilon} \mathcal{J}^\mu(x, p). \quad (1.59)$$

The vector \mathcal{J} is not directed along the momentum p in general. We can *define* the distribution function $f(x, p)$ to be the component of $\mathcal{J}^\mu(x, p)$ along the momentum, so that:

$$\mathcal{J}^\mu(x, p) = p^\mu f(x, p) + N^\mu(x, p), \quad N \cdot p = 0.$$

To obtain such a relation in the massive case, one can simply employ the Gordon identity [39]. Although it is possible to find a similar decomposition also in the

massless case [14], the interpretation is not so straightforward because p^μ is orthogonal to itself so components of the distribution function can be found also in N^μ . This problem will be addressed in Section 3.7.

However, the difficulty in the identification of a properly-called distribution function is not really a problem, as one can compute mean values using Eq. (1.57), as we will do in Chapter 3.

1.5 Spin: the Pauli-Lubanski vector

One of the main topics of recent years' research in heavy-ion physics has been spin. As explained in the Introduction, the average spin polarization of Λ particles is accessible experimentally, and one of the objectives of this work is to give a theoretical description of the related phenomenology.

In special relativity, the concept of spin finds its covariant generalisation in the *Pauli-Lubanski operator*, which is defined as:

$$\widehat{W}^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}\widehat{P}_\sigma, \quad (1.60)$$

where $\widehat{J}^{\mu\nu}$ is the angular momentum-boost operator, and \widehat{P}^μ is the four-momentum operator. By construction, we can see that the Pauli-Lubanski operator is orthogonal to the four-momentum operator, $\widehat{W} \cdot \widehat{P} = 0$. The Pauli-Lubanski operator plays a significant role in the theory of the Lorentz group, as it is the generator of the *little group* of the four-momentum p , that is the subgroup of Lorentz transformations that leave p invariant, $\Lambda p = p$. Indeed, if we consider the action of a transformation generated by \widehat{W}^μ on p^μ , and we consider its infinitesimal expansion, we have:

$$\left(e^{-i\phi\cdot W}\right)^\mu_\nu p^\nu = \left(\mathbb{I} + \frac{i}{2}\phi_\alpha\epsilon^{\alpha\beta\rho\sigma}J_{\beta\rho}p_\sigma\right)^\mu_\nu p^\nu = p^\mu + \frac{1}{2}\phi_\alpha\epsilon^{\alpha\beta\rho\sigma}\delta_\rho^\mu g_{\beta\nu}p_\sigma p^\nu = p^\mu, \quad (1.61)$$

where we used the definition of the generators of Lorentz transformations in the four-vector representation:

$$(J_{\alpha\beta})^\mu_\nu = i\left(\delta_\alpha^\mu g_{\beta\nu} - \delta_\beta^\mu g_{\alpha\nu}\right). \quad (1.62)$$

Notice that since we are dealing with continuous transformations, the fact that an infinitesimal transformation leaves p invariant suffices to prove that also a finite transformation cannot change the four-momentum. From the Lie algebra of the Poincaré group, it is possible to infer the commutation rules of the Pauli-Lubanski operator:

$$[\widehat{W}_\mu, \widehat{P}_\nu] = 0, \quad (1.63a)$$

$$[\widehat{W}_\mu, \widehat{J}_{\alpha\beta}] = i\left(g_{\mu\beta}\widehat{W}_\alpha - g_{\mu\alpha}\widehat{W}_\beta\right), \quad (1.63b)$$

$$[\widehat{W}_\mu, \widehat{W}_\nu] = -i\epsilon_{\mu\nu\rho\sigma}\widehat{W}^\rho\widehat{P}^\sigma. \quad (1.63c)$$

From the commutation rule with \widehat{P}^μ , one sees that \widehat{W}^μ is invariant under translations, whereas from the commutation rule with $\widehat{J}^{\mu\nu}$ we realize that \widehat{W}^μ transforms as a vector operator under Lorentz transformations. The latter property implies that \widehat{W}^2 is a Lorentz invariant and as such it commutes with $J^{\mu\nu}$, much like \widehat{P}^2 does.

Since \widehat{W}^2 commutes also with \widehat{P}^μ , one sees that \widehat{W}^2 commutes with all generators of the Poincaré group, and therefore is a Casimir operator [43].

Given the Eqs. (1.63a) and (1.63c), it is possible to diagonalize simultaneously the four-momentum operator and one component of the Pauli-Lubanski vector. To begin with, one can restrict the action of \widehat{W} on the space with fixed momentum p :

$$\widehat{W}^\mu |p\rangle = \widehat{W}^\mu(p) |p\rangle, \quad (1.64)$$

Where $\widehat{W}(p)$ is simply:

$$\widehat{W}^\mu(p) = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \widehat{J}_{\nu\rho} p_\sigma. \quad (1.65)$$

Since:

$$\widehat{W}(p) \cdot p = 0, \quad (1.66)$$

we can decompose the Pauli-Lubanski operator along directions orthogonal to the four-momentum and for this purpose, we have to treat separately massive and massless particles. For massive particles, one can identify three space-like vectors $n_i(p)$, satisfying;

$$n_i(p) \cdot p = 0, \quad n_i(p) \cdot n_j(p) = -\delta_{ij}. \quad (1.67)$$

These vectors can be defined in the standard frame of the particle, which for massive particles coincides with the rest frame where $p = \mathbf{p} = (m, 0)$, as the versors of the x , y and z axis. Their expression in a generic frame is obtained by boosting them with the same standard boost $[p]$ defined by $p^\mu = [p]^\mu_\nu \mathbf{p}^\nu$. Explicitly one has $n_i^\mu(p) = [p]^\mu_\nu n_i(\mathbf{p})^\nu$. Consequently, the Pauli-Lubanski vector can be written as:

$$\widehat{W}^\mu(p) = \sum_{i=1}^3 n_i^\mu(p) \widehat{W}_i(p), \quad \widehat{W}_i = -\widehat{W}(p) \cdot n_i(p). \quad (1.68)$$

In this case, it is possible to define the spin operator as:

$$\widehat{S}_i(p) = \frac{\widehat{W}_i(p)}{m}. \quad (1.69)$$

These operators are in fact the generators of the $SO(3)$ group, as they obey:

$$[\widehat{S}_i(p), \widehat{S}_j(p)] = i\epsilon_{ijk} \widehat{S}_k(p). \quad (1.70)$$

This is of course a consequence of the fact that $SO(3)$ is the little group of massive particles (see Appendix A and, for example, Ref. [43]). In the rest frame of the particle, we can choose the basis of the physical Hilbert space in such a way that $\widehat{S}_3(\mathbf{p})$ is diagonal:

$$\widehat{S}_3(\mathbf{p}) |\mathbf{p}, s\rangle = s |\mathbf{p}, s\rangle, \quad (1.71)$$

and the eigenvalue s is interpreted as the spin of the particle in the particle's rest frame. Indeed, computing $\widehat{S}(\mathbf{p})$ using the definition (1.60), one realizes that the components $\widehat{S}(\mathbf{p})$ are just the generators of the rotation group, i.e., $\widehat{S}(\mathbf{p})_{x,y,z} = \widehat{\mathbf{J}}_{x,y,z}$.

Turning to the case of massless particles, the decomposition of the Pauli-Lubanski operator can involve a component along the momentum p^μ itself, as $p \cdot p = 0$. In this case, the decomposition reads:

$$\widehat{W}^\mu(p) = \widehat{h}(p) p^\mu + \widehat{W}_1(p) n_1^\mu(p) + \widehat{W}_2(p) n_2^\mu(p), \quad (1.72)$$

where $n_i \cdot p = 0$. We can define the standard frame as the frame where $\mathbf{p}^\mu = (\kappa, 0, 0, \kappa)$ and $n_i^\mu = \delta_i^\mu$, with $i = 1, 2$. Notice, however, that these vectors do not form a basis of the Minkowski space-time. To amend this, it is convenient to define an additional light-like vector $\mathbf{q}^\mu = (\kappa, 0, 0, -\kappa)$. Notice that this vector is orthogonal to the space-like $n_{1,2}^\mu(p)$ vectors, but not to p^μ . As in the previous case, in the frame where the particle momentum is p^μ , $n_{1,2}^\mu(p)$ and q^μ are defined via the same standard transformation $[p]$, that this time is taken as a boost along the z axis times a rotation along the axis $\hat{\mathbf{z}} \times \hat{\mathbf{p}}$, where $\hat{\mathbf{p}}$ is the direction of the spatial part of p^μ . It is known that any vector can be decomposed using $\{p, q, n_1, n_2\}$ as a basis, and for the Pauli-Lubanski vector the component along q is vanishing by virtue of $\widehat{W}(p) \cdot p = 0$. The components of the decomposition (1.72) are now given by:

$$\widehat{h}(p) = \frac{1}{q \cdot p} \widehat{W}(p) \cdot q, \quad \widehat{W}_{1,2}(p) = -\widehat{W}(p) \cdot n_{1,2}(p). \quad (1.73)$$

In this case, the operators obey the algebra:

$$\begin{aligned} [\widehat{h}(p), \widehat{W}_1(p)] &= i\widehat{W}_2, \\ [\widehat{h}(p), \widehat{W}_2(p)] &= -i\widehat{W}_1, \\ [\widehat{W}_i(p), \widehat{W}_j(p)] &= 0. \end{aligned}$$

The fact that an abelian sub-algebra exists is well known and comes together with a problem: if we allow the operators $\widehat{W}_i(p)$ to have non-zero eigenvalues on the physical Hilbert space, then it is possible to prove that they have a continuous spectrum of eigenvalues. This means that massless particles would be characterized by a continuous degree of freedom, at variance with experimental observations. The solution to this puzzle is to *require* the physical Hilbert space to be such that $\widehat{S}_i(p)|p\rangle = 0$ always. Notice, however, that this constraint has to be inserted by hand and it does not need to be satisfied for off-shell states.

In conclusion, we can construct the Hilbert space by diagonalizing just $\widehat{h}(p)$:

$$\widehat{h}(p)|p, h\rangle = h(p)|p, h\rangle. \quad (1.74)$$

The eigenvalues $h(p)$ represent the *helicity* of the particle.

Now that we have introduced the Pauli-Lubanski operator, we are interested in calculating its thermal expectation value. For this purpose, we will follow the theoretical derivation of Ref. [50].

Since to be interpreted as spin (or helicity) of a particle with momentum p the Pauli-Lubanski operator has to be evaluated at fixed momentum $\widehat{W}(p)$, we have to compute the expectation value on a *spin-only* Hilbert space. In this case, we will use the *spin-density operator* $\widehat{\Theta}(p)$, which can be seen as the standard density matrix where the momentum space has been traced out via a partial trace. As any other density operator, the spin density operator ought to be hermitian and with trace equal to one, and the mean Pauli-Lubanski vector is then computed as:

$$W^\mu(p) = \text{Tr}_s \left(\widehat{W}^\mu(p) \widehat{\Theta}(p) \right), \quad (1.75)$$

where with Tr_s we emphasize that the trace is to be computed only on spin degrees of freedom.

Let us start by considering the mean spin vector for massive particles of spin S . Recalling $\widehat{S}^\mu = \widehat{W}^\mu/m$ and using Eq. (1.68) and the completeness of the Hilbert space, we have:

$$S^\mu(p) = \sum_s \langle p, s | \widehat{S}^\mu(p) \widehat{\Theta}(p) | p, s \rangle = \sum_{s,t} \sum_i \langle p, s | \widehat{S}_i(p) | p, t \rangle \langle p, t | \widehat{\Theta}(p) | p, s \rangle n_i^\mu(p). \quad (1.76)$$

By virtue of Eq. (1.70), the matrices $\langle p, s | \widehat{S}_i(p) | p, t \rangle$ are the spin- S representation of the generators of the $SO(3)$ group, which we will denote as $D^{(S)}(\mathbf{J}^i)$. Therefore, defining the spin density matrix as:

$$\Theta_{rs}(p) = \langle p, r | \widehat{\Theta}(p) | p, s \rangle, \quad (1.77)$$

we have:

$$S^\mu(p) = \sum_i \text{tr} \left(D^{(S)}(\mathbf{J}^i) \Theta(p) \right) n_i^\mu(p), \quad (1.78)$$

where the trace is no longer on the Hilbert space but only on matrix indices. Recalling that $n_i^\mu(\mathbf{p}) = \delta_i^\mu$ and $n_i^\mu(p) = [p]^\mu_\nu n_i^\nu(\mathbf{p})$, we find:

$$S^\mu(p) = \sum_i [p]^\mu_i \text{tr} \left(D^{(S)}(\mathbf{J}^i) \Theta(p) \right), \quad (1.79)$$

where we have been able to express the mean spin vector in terms of the generators of rotations, the standard Lorentz transformation $[p]$ and the spin density matrix.

For massless particles, we can proceed in the same fashion but, instead of the spin-operator (1.69), we have to deal directly with the Pauli-Lubanski vector. In place of Eq. (1.76) we have:

$$\begin{aligned} W^\mu(p) &= \sum_s \langle p, s | \widehat{W}^\mu(p) \widehat{\Theta}(p) | p, s \rangle = p^\mu \sum_{s,t} \langle p, s | \widehat{h}(p) | p, t \rangle \langle p, t | \widehat{\Theta}(p) | p, s \rangle \\ &\quad + \sum_{s,t} \sum_{i=1}^2 n_i^\mu \langle p, s | \widehat{W}_i(p) | p, t \rangle \langle p, t | \widehat{\Theta}(p) | p, s \rangle. \end{aligned} \quad (1.80)$$

If we now enforce $\widehat{W}_{1,2}(p)$ to act trivially on the Hilbert space, as dictated by experimental observations, we end up with the very simple formula:

$$W^\mu(p) = p^\mu \sum_{s=\pm h} s \Theta_{ss}(p), \quad (1.81)$$

which shows that the physical states of massless particles can only be polarized in the direction of their momentum. Furthermore, interestingly, only the diagonal components of the spin-density matrix participate in the average Pauli-Lubanski vector.

Both expressions (1.79) and (1.81) depend on the spin density matrix. For distinguishable particles, as we have mentioned, it can be defined as the partial trace of $\widehat{\rho}$:

$$\widehat{\Theta}_{rs} = \text{Tr}_p(\widehat{\rho})_{rs} = \langle p, r | \widehat{\rho} | p, s \rangle. \quad (1.82)$$

If we consider quantum fields, however, its definition is taken to be:

$$\Theta_{rs}(p) = \frac{\text{Tr} \left(\widehat{\rho} \widehat{a}_s^\dagger(p) \widehat{a}_r(p) \right)}{\sum_t \text{Tr} \left(\widehat{\rho} \widehat{a}_t^\dagger(p) \widehat{a}_t(p) \right)} = \frac{\langle \widehat{a}_s^\dagger(p) \widehat{a}_r(p) \rangle}{\sum_t \langle \widehat{a}_t^\dagger(p) \widehat{a}_t(p) \rangle}, \quad (1.83)$$

where $\hat{a}_s^\dagger(p)$ and $\hat{a}_s(p)$ are the creation and annihilation operators of the quantum field, and Θ is clearly hermitian and has unitary trace. As has been discussed in the previous section, given the fact that the spin density matrix is expressed in terms of field operators, we expect that it would be possible to write the Eq. (1.83) in terms of the Wigner function, as will be done explicitly in the next section.

1.6 Spin vector and the Wigner function

The aim of this section is to write the spin vector for massive and massless particles in terms of the Wigner function. The procedure to obtain this connection has been established in Ref. [50] for massive fermions. Here we will repeat it, providing an extension for massless fermions.

Let us first prove a general property of the Wigner function of free fields, namely:

$$k^\mu \partial_\mu W_\pm(x, k) = 0. \quad (1.84)$$

The proof is completely general and holds for massive and massless particles. Consider the particle part of the Wigner function expanded as in Eq. (1.54). We have:

$$\begin{aligned} k^\mu \partial_\mu W_+(x, k) &\propto \\ &\propto \int \frac{d^3 p}{\varepsilon} \frac{d^3 p'}{\varepsilon'} k^\mu (p' - p)_\mu \delta^4(k - \frac{p+p'}{2}) \sum_{st} \langle \hat{a}_t^\dagger(p') \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p') e^{ix \cdot (p' - p)} \\ &\propto \int \frac{d^3 p}{\varepsilon} \int \frac{d^3 p'}{\varepsilon'} (p'^2 - p^2) \delta^4(k - \frac{p+p'}{2}) \sum_{st} \langle \hat{a}_t^\dagger(p') \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p') e^{ix \cdot (p' - p)} \\ &= 0, \end{aligned}$$

where we used the δ -function to set $2k^\mu = p^\mu + p'^\mu$ and the fact that both p and p' are on the mass-shell. It is easy to see that the same argument applies also for the antiparticle and for the space-like component W_S of the Wigner function and the identity (1.84) is proved.

This property is very important, as it implies the integral

$$\int_\Sigma d\Sigma_\mu k^\mu W(x, k).$$

to be independent of the hypersurface Σ where it is computed, as long as suitable boundary conditions are fulfilled. The above integral is, in this respect, a global quantity. Assuming this is the case, let us consider the integral over a surface of constant time $t = t_0$. Using Eq. (1.54) we find[§]:

$$\begin{aligned} &\int_{t=t_0} d^3 x k^0 W(x, k) = \\ &= k^0 \sum_{s,t} \int \frac{d^3 p}{2\varepsilon} \frac{d^3 p'}{2\varepsilon'} \left\{ \delta^3(p - p') \left[\langle \hat{a}_t^\dagger(p) \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p) \delta^4(k - p) + \right. \right. \\ &\quad \left. \left. - \langle \hat{b}_s^\dagger(p) \hat{b}_t(p) \rangle v_s(p) \bar{v}_t(p) \delta^4(k + p) \right] - \delta^3(p + p') \left[e^{2i\varepsilon t} \langle \hat{b}_t^\dagger(-p) \hat{a}_s^\dagger(p) \rangle v_t(-p) \bar{u}_s(p) + \right. \right. \\ &\quad \left. \left. + e^{-2i\varepsilon t} \langle \hat{a}_t(p) \hat{b}_s(-p) \rangle u_t(p) \bar{v}_s(-p) \right] \delta(k^0) \delta^3(k + p) \right\}. \end{aligned}$$

[§]In the integral, by $-p$ we intend $-\mathbf{p}$, as creation and annihilation operators only depend on the space-components of the momentum. The energy constrained by the on-shell condition.

Due to the presence of a term $k^0\delta(k^0)$, the space-like part of the Wigner function does not contribute to the integral and we end up with:

$$\int_{\Sigma} d\Sigma_{\mu} k^{\mu} W(x, k) = \frac{\delta(k^2 - m^2)}{2} \sum_{st} \theta(k^0) \langle \hat{a}_t^{\dagger}(k) \hat{a}_s(k) \rangle u_s(k) \bar{u}_t(k) - \theta(-k^0) \langle \hat{b}_s^{\dagger}(-k) \hat{b}_t(-k) \rangle v_s(-k) \bar{v}_t(-k), \quad (1.85)$$

where we switched back to a generic hypersurface Σ on the left-hand side of the equation, thanks to the property (1.84).

In practice, integrating $k^{\mu} W(x, k)$ over *any* hypersurface sets the variable k on-shell. Therefore one can define an on-shell version of the particle and of the antiparticle part of the Wigner function, denoted $w_+(k)$ and $w_-(k)$, as:

$$\begin{aligned} \frac{1}{2\varepsilon} \delta(k^0 - \varepsilon_k) w_+(k) &= \int_{\Sigma} d\Sigma_{\mu} k^{\mu} W_+(x, k), \\ \frac{1}{2\varepsilon} \delta(k^0 + \varepsilon_k) w_-(k) &= \int_{\Sigma} d\Sigma_{\mu} k^{\mu} W_-(x, k). \end{aligned} \quad (1.86)$$

Confining ourselves to the particle part, using Eq. (1.85) we find:

$$w_+(p) = \frac{1}{2} \sum_{st} \langle \hat{a}_t^{\dagger}(p) \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p). \quad (1.87)$$

We can already see that this expression highly resembles the spin density matrix (or at least its numerator) as defined in Eq. (1.83). Our task is now to invert the above equation to obtain the two-point function $\langle \hat{a}_t^{\dagger}(p) \hat{a}_s(p) \rangle$ in terms of the on-shell Wigner function.

The relations proved until now hold both in the case of massive and massless fermions, but for the inversion of Eq. (1.87) we have to take two different approaches. Let us start with the case of massive particles. By taking the trace of Eq. (1.87), using the cyclicity of the trace and the normalization of the spinors, Eq. (1.50), we have:

$$\text{tr}(w_+(p)) = \frac{1}{2} \sum_{st} \langle \hat{a}_t^{\dagger}(p) \hat{a}_s(p) \rangle \text{tr}(u_s(p) \bar{u}_t(p)) = m \sum_s \langle \hat{a}_s^{\dagger}(p) \hat{a}_s(p) \rangle. \quad (1.88)$$

Moreover, multiplying Eq. (1.87) by spinors with definite spin index r, l :

$$\bar{u}_r(p) w_+(p) u_l(p) = 2m^2 \langle \hat{a}_l^{\dagger}(p) \hat{a}_r(p) \rangle. \quad (1.89)$$

Therefore the spin density matrix can be expressed as:

$$\Theta_{rs}(p) = \frac{1}{2m} \frac{\bar{u}_r(p) w_+(p) u_s(p)}{\text{tr}(w_+(p))} = \frac{1}{2m} \frac{\int d\Sigma \cdot p \bar{u}_r(p) W_+(x, p) u_s(p)}{\int d\Sigma \cdot p \text{tr}(W_+(x, p))}. \quad (1.90)$$

By noticing that, as a consequence of Eq. (1.87), the on-shell Wigner function obeys the Dirac equation:

$$(\not{p} - m) w_+(p) = 0, \quad (1.91)$$

the spin density matrix can also be cast into the form:

$$\Theta_{rs}(p) = \frac{\int d\Sigma \cdot p \bar{u}_r(p) W_+(x, p) u_s(p)}{\int d\Sigma \cdot p \text{tr}(W_+(x, p) (\not{p} + m))}, \quad (1.92)$$

as was originally found in Ref. [50]. We can write the spin density matrix also using the 4×2 spinors introduced in Section 1.3:

$$\Theta(p) = \frac{1}{2m} \frac{\int d\Sigma \cdot p \bar{U}(p) W_+(x, p) U(p)}{\int d\Sigma \cdot p \text{tr}(W_+(x, p))}. \quad (1.93)$$

From this expression and using equation Eq. (1.79) and the Clifford algebra, one can finally write the thermal expectation value of the spin vector in terms of the Wigner function. The derivation is rather tedious and will not be repeated here, for further details we refer the reader to Ref. [50]. The final result reads:

$$S^\mu(p) = \frac{1}{2m} \frac{\int d\Sigma \cdot p \text{tr}(\gamma^\mu \gamma_5 W_+(x, p))}{\int d\Sigma \cdot p \text{tr}(W_+(x, p))}, \quad (1.94)$$

so that the mean spin operator is given by the integral over some space-like hypersurface of the axial component of the Wigner function.

It is worth dwelling a little longer on this formula. We have expressed the spin-density matrix and the spin vector in terms of the Wigner function, but the equation is quite unusual compared to Eq. (1.57): the Wigner function appears twice, and the integration is on the spacelike variable x and not on the pseudo-momentum. This latter difference poses a question regarding what hypersurface to choose for practical calculations. From the derivation above, it would seem that there are no special requirements that the hypersurface must fulfil, but we have to take into account that the derivation holds for free Dirac fields. In an idealized setting where we deal with a gas of non-interacting Dirac fermions, the hypersurface is indeed arbitrary, but in real systems the former is just an approximation. To clarify this point, let us consider a typical example of an application of this formula: the calculation of the polarization of Λ hyperons in heavy-ion collisions. Equation (1.94) represents the spin vector of the Dirac field, whence the spin polarization of a particle with spin S is defined as:

$$P^\mu(p) = \frac{S^\mu(p)}{S}, \quad (1.95)$$

so that the magnitude of polarization is between -1 and 1 . For Dirac fermions such as the Λ particle one has $P^\mu = 2S^\mu$. In heavy-ion collisions, the Λ s are generated when the Quark-Gluon Plasma (QGP) stops behaving as a fluid and hadronizes, meaning that quarks and gluons recombine in hadrons. Before this moment the Λ particles do not exist, so it is obvious that the hypersurface cannot be taken in the QGP phase. On the same footing, if we wait too long the Λ particles will have time to interact and even decay, and their description as free fields would not be applicable. Therefore, for heavy-ion applications, the best choice for the hypersurface of integration in Eq. (1.94) is the freeze-out hypersurface itself.

In the case of massless particles, the same procedure can be applied up until Eq. (1.87):

$$w_+(p) = \frac{1}{2} \sum_{st} \langle \hat{a}_t^\dagger(p) \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p).$$

In contrast, we should find a different way to extract the spin density matrix from this equation because in this case:

$$\text{tr}(w_+(p)) = \bar{u}_s(p) w_+(p) u_r(p) = 0. \quad (1.96)$$

We can overcome this difficulty by noticing that in Eq. (1.81) only the diagonal components of the spin density matrix appear. Using Eqs. (1.81) and (1.83), the

polarization vector in the massless case expressed in terms of the field operators reads:

$$W(p) = p^\mu \frac{\sum_{s=\pm h} s \langle \hat{a}_s^\dagger(p) \hat{a}_s(p) \rangle}{\sum_{t=\pm h} \langle \hat{a}_t^\dagger(p) \hat{a}_t(p) \rangle}. \quad (1.97)$$

Now let us compute the following trace:

$$\begin{aligned} \text{tr}(w_+(p)\gamma^\mu) &= \frac{1}{2} \sum_h \langle \hat{a}_l^\dagger(p) \hat{a}_h(p) \rangle \text{tr}(u_h(p) \bar{u}_l(p) \gamma^\mu) \\ &= \frac{1}{2} \sum_{hl} \langle \hat{a}_l^\dagger(p) \hat{a}_h(p) \rangle \bar{u}_l(p) \gamma^\mu u_h(p) = p^\mu \sum_h \langle \hat{a}_h^\dagger(p) \hat{a}_h(p) \rangle, \end{aligned} \quad (1.98)$$

where we used the cyclicity of the trace, along with the property $\bar{u}_l \gamma^\mu u_h(p) = 2p^\mu \delta_{lh}$ (see Eqs. (1.50)). Notice how this trace is just the denominator of the formula (1.97), multiplied by the four-momentum p^μ . We can extract the denominator of (1.97) simply by multiplying Eq. (1.98) by some vector ℓ^μ , provided that it is not orthogonal to p , i.e. $p \cdot \ell \neq 0$. In fact, decomposing ℓ^μ along the basis $\{p, q, n_1, n_2\}$ we see that only the component along q is physically relevant, as from Eq. (1.98) it follows $\text{tr}(w_+(k)\not{p}) = \text{tr}(w_+(k)\not{p}_{1,2}) = 0$. We find:

$$\sum_h \langle \hat{a}_h^\dagger(p) \hat{a}_h(p) \rangle = \frac{\text{tr}(w_+(k)\not{q})}{p \cdot q}. \quad (1.99)$$

Similarly, we can compute:

$$\text{tr}(w_+(p)\gamma^\mu\gamma_5) = 2p^\mu \sum_h h \langle \hat{a}_h^\dagger(p) \hat{a}_h(p) \rangle.$$

where we used the fact that, for massless particles, chirality is twice the helicity and the equation $\gamma_5 u_h(p) = 2h u_h(p)$ holds. With the same reasoning presented before, we have:

$$\sum_h h \langle \hat{a}_h^\dagger(p) \hat{a}_h(p) \rangle = \frac{\text{tr}(w_+(p)\not{q}\gamma_5)}{2p \cdot q}. \quad (1.100)$$

Using these results we can express the expectation value of the Pauli-Lubanski vector as:

$$W^\mu(p) = \frac{p^\mu}{2} \frac{\text{tr}(\not{q}\gamma_5 w_+(p))}{\text{tr}(w_+(p)\not{q})}, \quad (1.101)$$

and using Eq. (1.86) to give the spin vector in terms of the Wigner function, we have:

$$W^\mu(p) = \frac{p^\mu}{2} \frac{\int d\Sigma \cdot p \text{tr}(\not{q}\gamma^5 W_+(x, p))}{\int d\Sigma \cdot p \text{tr}(W_+(x, p)\not{q})}. \quad (1.102)$$

This formula is the massless analogue of Eq. (1.94).

Notice that in the massless case the denominator of the formula involves the vector part of the Wigner function and not the scalar one. This is necessary as the scalar part of the massless Wigner function vanishes. Furthermore, the Pauli-Lubanski vector is directed only along the momentum of the particle, as one would expect, and any additional components are projected out thanks to the presence of the vector q^μ . In fact, this projection is required to get rid of the unphysical components of the Pauli-Lubanski vector. In essence, it enforces the constraint that

$\widehat{W}_{1,2}$ vanish on physical states, which was otherwise not employed in the derivation. This is a very subtle point, that has gone unnoticed so far in the literature.

The connection between the spin vector and the Wigner function will be used in later chapters to compute the polarization of Dirac fermions in relativistic fluids at local and global equilibrium.

Chapter 2

Phenomenology of the Quark-Gluon Plasma

In this chapter, we discuss the phenomenology of the Quark-Gluon Plasma (QGP) produced in heavy-ion collisions [51]. The QGP is a phase of the QCD matter that is produced at high temperatures and is believed to resemble, on a much smaller scale, our own Universe microseconds after the Big Bang. We start with a brief overview about the history of the discovery of the QGP, and the observables that allow for its detection experimentally. After placing the QGP in the broader QCD phase diagram, we schematically describe the stages of a typical heavy-ion collision, where droplets of QGP are produced in the controlled setting of a laboratory. Finally, we discuss spin physics in this context, including the experimental challenges involved in the measurements, with a particular focus on the polarization of the Λ hyperon and the puzzles that it brought to the physics community.

2.1 A brief history of the most perfect fluid

The theory of strong interactions presents a peculiar characteristic compared to quantum electrodynamics: asymptotic freedom. The name asymptotic freedom refers to the strength of the interaction, which becomes weaker and weaker as the energy grows, or equivalently if particles (e.g. quarks) are put close to each other. On the other hand, in the low-energy regime, QCD should describe the confinement of quarks and gluons inside hadrons.

The existence of asymptotic freedom discovered by Gross, Politzer and Wilczek [1, 52, 53] in 1973 led people to think that, under extreme conditions, a state of matter where quarks and gluons are not bound into composite particles could be formed. The first location where such a “quark soup”, as the authors referred to it, was surmised to be present were astrophysical objects, such as the cores of neutron stars, exploding black holes, and the early Universe [3].

Even before the discovery of asymptotic freedom, Hagedorn put forth a model called “statistical bootstrap” [54]. In this model, denoting with $\rho(m)dm$ the number of particles with a mass between m and $m + dm$, the mass spectrum function $\rho(m)$ is found to have an asymptotic behaviour for large mass given by:

$$\lim_{m \rightarrow \infty} \rho(m) = m^\alpha e^{\frac{m}{T_0}},$$

where α is a negative parameter depending on the details of the model. The mass spectrum grows exponentially, and this leads to a divergence of the partition function

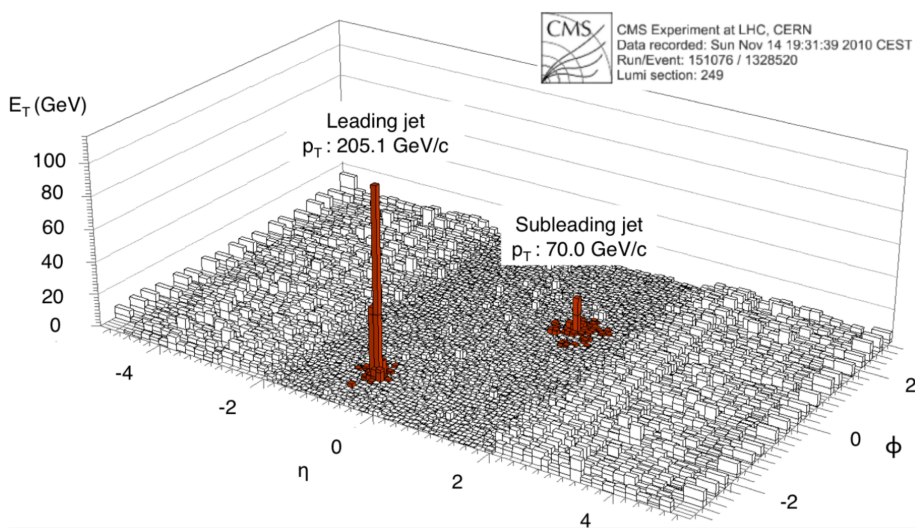


Figure 2.1: The energy deposition in the CMS calorimeter as a function of pseudorapidity and azimuthal angle. The difference in energy between the two jets suggests that one of the two lost a large fraction of its energy by interacting with the QGP. The figure is taken from Ref. [67].

at the temperature $T = T_0$. At the time, such a divergence was interpreted as the presence of a limiting temperature, which the system of hadrons cannot surpass. In 1975 it was realized by Cabibbo and Parisi [55] that such a divergence can be interpreted as the onset of a second-order phase transition from hadrons to a state where “quark liberation” happens. It was the first time that the phase diagram of QCD was addressed.

In later years many authors studied the collective and thermodynamic properties of this novel state of matter [56–62], which we call nowadays the Quark-Gluon Plasma (QGP), a name coined by Shuryak in 1978 [2].

Experimentally, the production of the QGP can be investigated via different probes. Here we list some of the most important [63–65]:

- **Strangeness enhancement:** it is expected that if the locally equilibrated QGP with temperature $T \sim 2m_s$ is formed in heavy-ion collisions (m_s being the strange quark mass), the number of strange-antistrange quarks would be higher compared to nucleon-nucleon collisions. The threshold temperature in the equilibrated system causes a more abundant production of strange and multi-strange hyperons, which are experimentally measurable [66]. This effect is compatible with the creation of a thermal hadronic gas.
- **Jet quenching:** jets of hadrons are produced by high- p_T partons via fragmentation. Interacting with the QCD medium, a parton can lose a significant fraction of its transverse momentum before fragmentation. This implies that in a pair of jets there can be an energy asymmetry if one of the jets interacts more with the QGP. A measurement of this effect is shown in Figure 2.1.
- **Hydrodynamics:** in non-central heavy-ion collisions, the shape of the QGP is of an ellipsoidal form due to the non-perfect overlap between the nucleons. This shape is commonly referred to as “almond shape”. Such an anisotropy

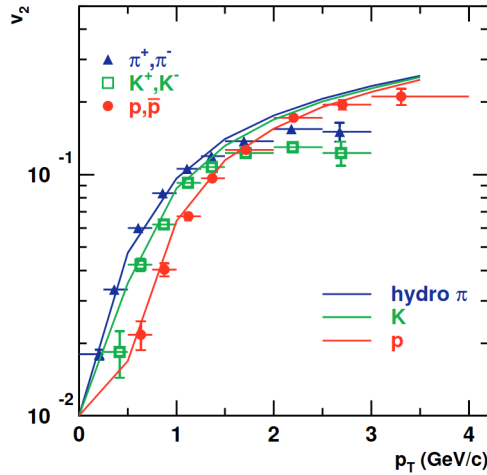


Figure 2.2: The elliptic flow of pions, kaons and protons as measured by the PHENIX experiment [68], compared with hydrodynamic simulation based on Ref. [70]. The figure is taken from Ref. [68].

in the initial state is reflected in the subsequent evolution. If we assume the evolution of the plasma to be described by relativistic hydrodynamics, the larger pressure exerted by the fluid in the reaction-plane direction implies that more particles are produced in rather than out-of-plane. This effect is quantified by the anisotropic flow coefficients v_n , which define the azimuthal expansion of the momentum spectrum of particles:

$$\frac{dN}{d^2p_T dy} = \frac{dN}{2\pi p_T^2 dp_T dy} \left[1 + 2 \sum_n v_n \cos(n(\phi - \Psi_{RP})) \right], \quad (2.1)$$

where p_T is the transverse momentum, y is the rapidity $y = \text{atanh}(\frac{p_z}{\varepsilon})$, z being the beam direction and ε the energy of the particle. The azimuthal angles ϕ and Ψ denote the angles of the particle and of the reaction plane, respectively. Given the ellipsoidal shape of the fluid, the dominant contribution is given by v_2 , which is called the elliptic flow. The elliptic flow is now a well-established experimental observable, and the data are well described by quasi-ideal hydrodynamics, as shown in Figure 2.2 [68, 69]. Interestingly, to match the elliptic-flow data, the equilibration time of the QGP should be less than 1 fm/c.

- **Quarkonium suppression:** in the QCD plasma, like in the electromagnetic one, there is a characteristic length scale above which the interaction becomes ineffective, the so-called Debye screening length. Bound states of heavy quarks (mostly charmed particles like J/ψ , ψ' ...) are sensitive to this screening length, that in the QGP can become smaller than the size of the particle itself. This leads to the melting of quarkonia in the QGP. This effect, however, can be very small as antagonist mechanisms also exist: quarkonium recombination and threshold enhancement [71].

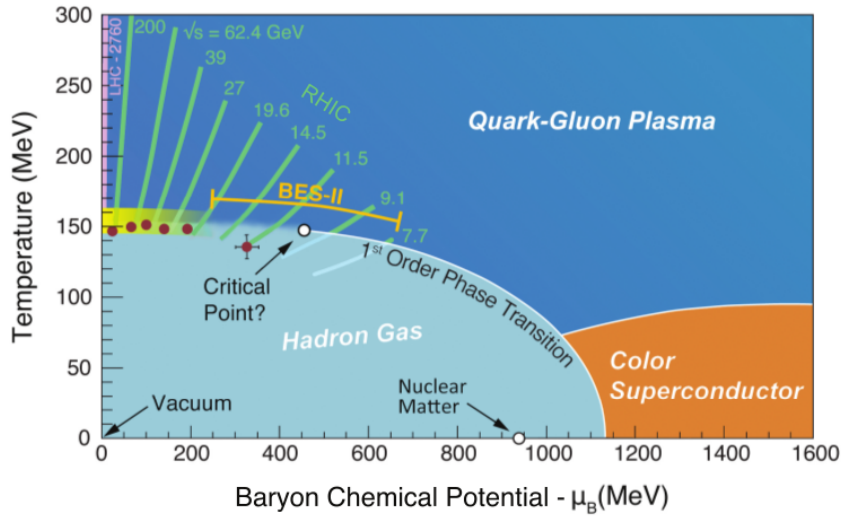


Figure 2.3: A schematic representation of the phase diagram of QCD is shown in the temperature-baryon chemical potential plane. The figure is taken from the NSAC 2015 Long Range Plan for Nuclear Physics.

2.2 The QCD phase diagram

The QGP represents one of the phases of QCD matter, the hadronic one being the most common and familiar. Here we give a very schematic description of the rest of the phase diagram of QCD, which is much richer than Cabibbo and Parisi originally believed [72–74].

The phase diagram of QCD is usually represented in terms of the baryon chemical potential and the temperature in a (μ_B, T) plot like Figure 2.3. Our knowledge about the phase diagram is most firm in the region $\mu_B \ll T$, where lattice QCD (LQCD) is fully applicable. In this regime, it is known that the transition from hadronic matter to QGP is a cross-over, happening at a temperature around 155 MeV. This temperature is measured, for example, from the inflexion point of the chiral condensate $\langle \bar{\Psi}\Psi \rangle$ or the peak of the chiral susceptibility $\chi = T/V \partial^2 \log Z / \partial m^2$, where ψ , m , T and V are the quark field, mass, temperature and volume, respectively. Notice that the above-mentioned quantities are associated with the restoration of the chiral symmetry of QCD at high temperatures and not to deconfinement itself. In fact, the true order parameter for the deconfinement transition in QCD is not known. This is in contrast with the pure $SU(3)$ gauge theory, where confinement and deconfinement are distinguishable, for example, using the Polyakov loop, which vanishes if the system is in the confined phase; in QCD the Polyakov loop is only an approximate order parameter for deconfinement [75]. It is possible that chiral restoration and deconfinement happen at the same temperature [76], even if a sequential transition with an intermediate state is not excluded [77]. We conclude this paragraph by mentioning that, in the regime $\mu_B \ll T$, LQCD can also provide a reliable equation of state for QCD matter, which is necessary to solve the system of relativistic hydrodynamic equations describing the QGP (see next section).

As the chemical potential increases, the transition is expected to switch to a first-order phase transition, possibly with a critical point in the region $\mu_B \sim T$. In this region of the phase diagram, LQCD gives less reliable results due to the

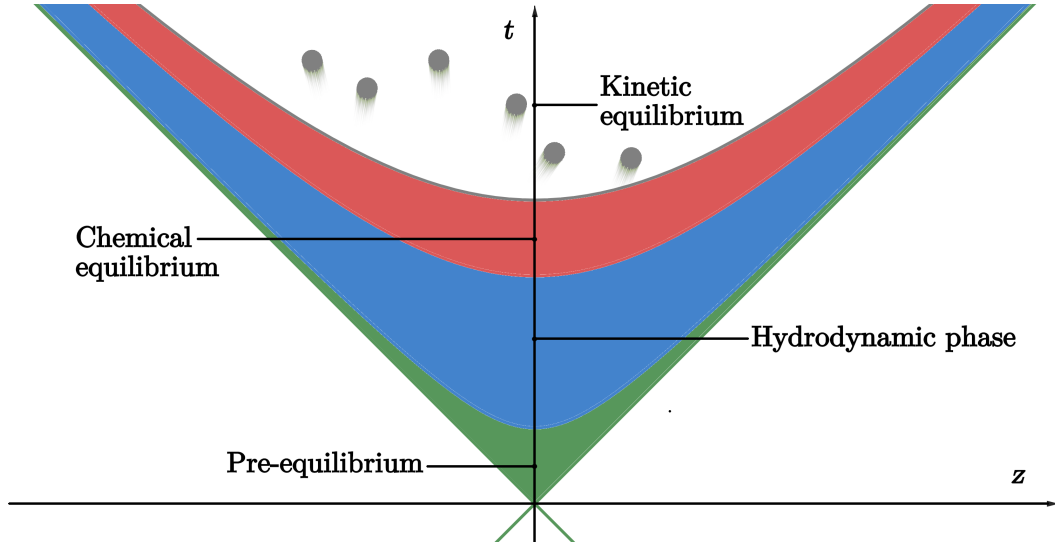


Figure 2.4: The stages of a typical heavy-ion collision, identified by different colours, are sketched. After a pre-equilibrium phase, the hydrodynamic evolution sets in. Once the fluid becomes too rarefied and cold, new hadrons are produced at the chemical freeze-out, and their elastic interaction ceases at the thermal freeze-out. After that, free hadrons can be observed in the detectors.

infamous sign problem. This problem originates from the presence of a real chemical potential in the action, that breaks hermiticity and leads to a complex-valued Dirac determinant which is very hard to calculate numerically. The techniques used to tackle this problem, which often rely on the analytic continuation of the chemical potential to imaginary values, fail or become incredibly expensive computationally if the chemical potential is too large. This region of the phase diagram, and in particular the location of the critical point, is one of the most actively researched topics in QCD physics, both theoretically and experimentally [78–83].

Another phase transition happens at much lower temperatures ($T \sim 0 - 20$ MeV) and for a chemical potential $\mu_B \sim 924$ MeV. This value of the baryon chemical potential represents the threshold where nuclei start being produced. This nuclear phase transition is first order, characterized by a coexistence of hadronic and nuclear matter, and as the temperature rises it ends in a critical endpoint, where the transition becomes of second order.

Finally, the large chemical potential region is characterized by the colour-superconducting phase [74]. The existence of this phase can be rigorously proven in the limit $\mu_B \gg \Lambda_{QCD}$, where, due to asymptotic freedom, the coupling constant is small. Such a phase could present itself inside neutron stars' cores, which provide an extraordinary setting, where the density of baryonic matter is much higher than the one achievable experimentally.

2.3 Stages of heavy-ion collisions

Here we describe the time evolution of the system once the collision takes place [51]. There are several stages after the collision until finally new hadrons are produced and can be detected experimentally. A schematic depiction is given in Figure 2.4.

In the very early stage of the collision, the highly coherent wave function of

the nuclei evolves into an incoherent distribution of quarks and gluons. The three main (and alternative) mechanisms proposed to describe this process are the QCD string breaking, the parton cascade and the colour glass condensate models. In the latter model, the density of gluons inside hadrons rises as a function of the parton's momentum fraction shared by gluons, up to a saturation scale. When such a momentum fraction is small, gluons are tightly packed inside hadrons and their interaction is weak. Once the Lorentz-contracted hadrons, in practice two sheets of gluons, collide and pass through each other, the gluonic field permeates the collision region producing the so-called *glasma*, which subsequently decays in the QGP. The time scale of this process is much less than 1 fm/c.

At this point, the QGP thermalizes to a state of local equilibrium and the subsequent evolution can be modelled with relativistic viscous hydrodynamics. Hydrodynamics is the effective theory describing the conservation of energy, momentum and charge. The corresponding densities, the energy-momentum tensor and the charge current, have vanishing four-divergence:

$$\begin{aligned}\partial_\mu T^{\mu\nu} &= 0, \\ \partial_\mu j^\mu &= 0.\end{aligned}$$

In dissipative hydrodynamics in the so-called Landau frame, the energy-momentum tensor and the current are parametrized as:

$$T^{\mu\nu} = \rho u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, \quad (2.2a)$$

$$j^\mu = nu^\mu + v^\mu. \quad (2.2b)$$

In the above equations, ρ , P and n are the energy density, the pressure and the charge density of the fluid, whereas Π , $\pi^{\mu\nu}$ and v^μ are the Bulk pressure, the shear stress and the diffusion current. The latter three quantities can be inferred from microscopic theories such as kinetic theory, and obey relaxation-type equations to make the hydrodynamic evolution causal [84–87]. The equations (2.2) are five equations, but the unknowns are six, namely u^μ , P , ρ and n : three components of the four-velocity (the fourth being fixed by $u^2 = 1$), the pressure, the energy and charge density. To solve the system of differential equations, an equation of state expressing the pressure as a function of temperature and chemical potential is required. For heavy-ion applications, the equation of state can be inferred from lattice QCD, so that the effective hydrodynamic description of the plasma takes directly into account fundamental QCD calculations.

During the hydrodynamic evolution, the QGP cools down and rarefies, eventually dissolving to form once again hadronic matter in the *hadronization* stage. At hadronization, particles can still be strongly interacting and are described by hydrodynamics until, eventually, they decouple. This last stage is called *freeze-out*. The freeze-out stage can be further divided into *chemical* and *thermal* freeze-out. This distinction is because inelastic interactions cease to be effective before the elastic ones. The chemical freeze-out is the moment when the last inelastic interaction takes place and the ratios of hadrons abundances are fixed from this point on. Nonetheless, elastic interactions can still take place and hadrons can exchange momentum. When the characteristic timescale of the collisions becomes comparable to or larger than the expansion timescale of the plasma, elastic collisions become ineffective too and particles are free to fly to the detectors: this stage is the thermal freeze-out. We can give an estimate of the time between two consecutive elastic

collisions by using the mean free path, whereas the expansion time of the fluid is given by one over the four-divergence of the four-velocity. The thermal freeze-out takes place when:

$$\tau_{coll} \sim \frac{1}{n\sigma} \geq \frac{1}{\partial_\mu u^\mu} \sim \tau_{exp}.$$

In practice, different models can be used to identify the freeze-out hypersurfaces. In hydrodynamic codes, the evolution is halted once some condition is met, for example when the local energy density or the local temperature reaches a critical value. The condition $\rho(x) = \rho_{crit}$ defines a 3D hypersurface which is identified with the chemical freeze-out hypersurface. After this stage, kinetic codes can be used as afterburners to further scatter the produced hadrons until the condition for thermal freeze-out is met.

Thermodynamic properties of the fluid are frozen on the freeze-out hypersurface, which therefore has a prominent role. For example, the spectrum of particles is calculated using the Cooper-Frye formula [88]:

$$\varepsilon \frac{dN}{d^3p} = \int_{\Sigma_{FO}} d\Sigma_\mu p^\mu f(p \cdot \beta(x), \zeta(x)),$$

where the distribution function f can be taken as the Bose-Einstein, the Fermi-Dirac or the Boltzmann distribution depending on the particle studied. The integral has to be computed numerically over the hypersurface produced by, for example, a hydrodynamic code. The polarization of fermions is also expressed as an integral over the freeze-out hypersurface, as we have seen in the previous chapter.

2.4 Polarization in the Quark-Gluon Plasma and the sign puzzle

If two nuclei collide with a finite impact parameter, it is natural to expect that the QGP is created with a sizeable angular momentum. In the early 2000s, it was realized that the orbital angular momentum can be transformed into spin polarization via spin-orbit coupling [6, 9], and that this effect could be detected by measuring particles produced at freeze-out.

The spin polarization of fermions is an observable of great phenomenological relevance. The most renowned example is the polarization of the Λ hyperon, although recently also the (global) polarization of Ω and Ξ has been measured [89, 90]. Another spin-related observable is the spin alignment of vector mesons, which will be discussed in the next section.

The first quantitative formula for the expectation value of spin of particles has been found in Ref. [7], and reads:

$$S^\mu(p) = -\frac{1}{8m} \epsilon^{\mu\nu\rho\sigma} p_\sigma \frac{\int d\Sigma \cdot p \varpi_{\nu\rho} n_F (1 - n_F)}{\int d\Sigma \cdot p n_F}. \quad (2.3)$$

Here n_F is the Fermi-Dirac distribution $n_F = [\exp(p \cdot \beta - \zeta) + 1]^{-1}$ and ϖ is the thermal vorticity (1.18). Both the thermal vorticity and the temperature are space-time dependent, and they must be integrated on the freeze-out hypersurface like for the Cooper-Frye formula (see also the discussion in Section 1.6).

The STAR experiment at RHIC was the first one performing polarization measurements for Λ hyperons in Au-Au collisions. According to the local-equilibrium model of the QGP, all Dirac fermions generated at freeze-out are expected to be polarized according to Eq. (2.3), but not all of them are measurable. Polarization can be measured for “self-analysing” particles, which decay via the weak interaction, and the Λ particle is the most abundant particle produced in heavy-ion collisions to do so.

The polarization of Λ is measured via the decay $\Lambda \rightarrow p + \pi$. In this process, the proton tends to be emitted in the direction of the polarization vector of the Λ in the hyperon’s rest frame. Denoting θ^* the angle between the proton’s momentum and the polarization vector of the Λ particle in the Λ ’s rest frame \mathbf{P} , the angular distribution of protons reads:

$$\frac{dN}{d \cos \theta^*} = \frac{1}{2} (1 + \alpha_H |\mathbf{P}_H| \cos \theta^*), \quad (2.4)$$

where H can represent the Λ or $\bar{\Lambda}$ hyperon, α_H is the decay parameter, that has the numerical value $\alpha_\Lambda = -\alpha_{\bar{\Lambda}} = 0.732 \pm 0.14$ [91]. In Eq. (2.4), P_H is the polarization averaged over momenta, therefore it is called global polarization.

After the first results, which put an upper bound on global polarization [92], a sizeable polarization was measured in 2017 [8]. Polarization was found to be of the order of some percent and from it the vorticity of the QGP could be inferred. This measurement showed that the QGP is the most vortical fluid in Nature, spinning with the astonishing angular velocity of $\omega \simeq 10^{22} \text{ s}^{-1}$. Global polarization is defined as the polarization vector averaged over the spectrum of Λ particles produced at freeze-out and it can be expressed by the equation:

$$P^\mu = \frac{\int \frac{d^3p}{\varepsilon} P^\mu(p) \left(\varepsilon \frac{dN}{d^3p} \right)}{\int \frac{d^3p}{\varepsilon} \left(\varepsilon \frac{dN}{d^3p} \right)}. \quad (2.5)$$

Recalling that for Dirac fermions $P^\mu(p) = 2S^\mu(p)$ (see Eq. (1.95)), and using the expression of the Lorentz-invariant spectrum of Dirac fermions

$$\varepsilon \frac{dN}{d^3p} = 2 \int d\Sigma \cdot p n_F, \quad (2.6)$$

$\varepsilon = \sqrt{m^2 + \mathbf{p}^2}$ being the energy, the global polarization simply reads:

$$P^\mu = 2 \frac{\int \frac{d^3p}{\varepsilon} S^\mu(p) \left(\int d\Sigma \cdot p n_F \right)}{\int \frac{d^3p}{\varepsilon} \int d\Sigma \cdot p n_F}, \quad (2.7)$$

where $S^\mu(p)$ is the spin vector, as given by Eq. (2.3). By symmetry, the global polarization must be directed along the global angular momentum of the QGP, which identifies the $-\hat{y}$ direction in the reference frame used experimentally.

Figure 2.5 reports the experimental results of the STAR and ALICE experiments [8, 92, 94, 95] together with different model predictions. In all these models, Eq. (2.3) is used to compute the spin vector, and they differ in how the thermal vorticity is calculated. All the hydrodynamic and transport codes are tuned to reproduce other observables (e.g. anisotropic flow). It can be seen that Eq. (2.3) can describe the data very well, regardless of the code used to simulate the QGP. It is interesting to

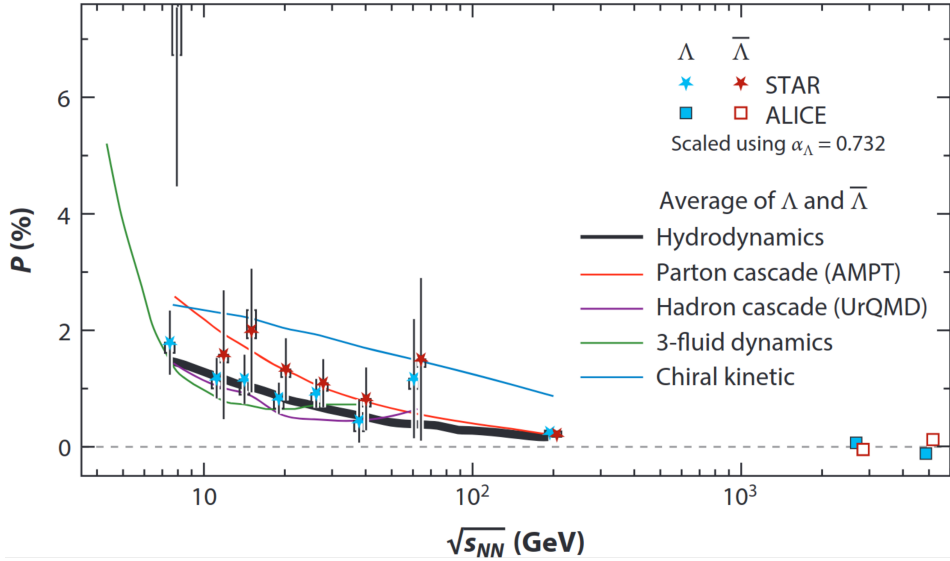


Figure 2.5: Figure taken from Ref. [93]. The experimental measurement of global polarization for Λ and $\bar{\Lambda}$ hyperons produced at midrapidity is shown at different collision energies, together with different model predictions using Eq. (2.3).

see that the polarization of Λ and $\bar{\Lambda}$ has the same sign, which is a clear indication of the hydrodynamic origin of the effect.

After the successful measurement of global polarization, a more detailed study of the properties of the Λ polarization was performed at 200 GeV by the STAR experiment. These measurements, commonly referred to as the local-polarization measurements, probe the dependence of polarization on the transverse momentum p_T and the azimuthal angle ϕ , and raised some important questions and challenges. The most puzzling results came from the azimuthal dependence of $P^J(\phi)$ and $P^z(\phi)$, which are the projections of the polarization vector along the angular momentum of the fluid and the beam axis respectively*.

Figure 2.6 shows the comparison between the experimental results and the hydrodynamic model calculations of local polarization. Concerning the polarization along the angular momentum P^J , it can be seen that the hydrodynamic model predicts a maximum polarization along the direction of the angular momentum, whereas polarization has a minimum along the reaction plane's direction. Strikingly, the experimental results show exactly the opposite behaviour. A similar discrepancy occurs for the polarization along the beam axis P^z , where the hydrodynamic model suggests an oscillation pattern opposite in sign compared with the experimental observation. All theoretical models shown in Figure 2.5 lead to similar conclusions.

The unexpected failure of Formula (2.3) to reproduce correctly the local polarization of the Λ particle, despite its excellent description of the global polarization has been known in the literature as the *polarization sign puzzle*. More recently, also the ALICE experiment was able to measure the second-order harmonic of the polarization along the beam direction $\langle P^z \sin(2\phi - 2\Psi_2) \rangle$, where ϕ is the azimuthal angle and Ψ_2 is the elliptic-flow plane angle. ALICE results lead to the same conclusion: Eq. (2.3) predicts the wrong sign for P^z .

*These directions correspond to the $-y$ - and the z -axis in the reference frame employed experimentally.

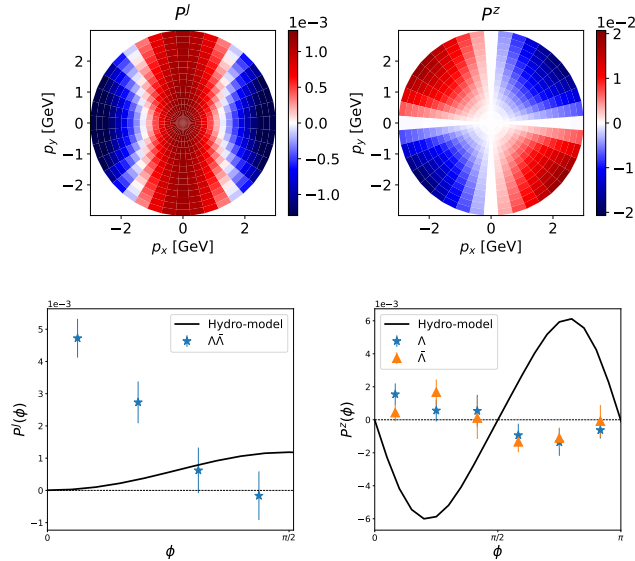


Figure 2.6: The results of a typical hydrodynamic simulation are compared to experimental data. The upper panels show the dependence of polarization on transverse momentum, for P^j (left) and P^z (right). The lower panels show the azimuthal-angle dependence of the same components of polarization, once they are integrated over p_T .

The attempts to solve the spin sign puzzle have been many. It has been proved that feed-down corrections are unable to explain the change of sign [96, 97]. Besides that, investigation on post-hadronization interactions and rescattering [98, 99], the role of the spin tensor and the spin potential [100–102], and dissipation in spin hydrodynamics [48, 103–108] have been undertaken. Moreover, the dependence on the initial-state configuration of hydrodynamics seems to play an important role in polarization physics [109, 110].

Other possible explanations concern Eq. (2.3) itself. This equation is only an approximation up to first order in thermal vorticity and higher-order corrections can affect the result. Even if it is unlikely that the quantitative modification will be so dramatic as to change the sign of P_z , since the vorticity is $\sqrt{\varpi} : \varpi < 1$ on the freeze-out hypersurface [17, 93], this possibility has not been thoroughly explored. On top of that, Eq. (2.3) might neglect some contributions to polarization, and the inclusion of other terms of first order in gradients may solve the puzzle.

2.5 Spin alignment

Other than polarization, another important spin-related observable is the *spin alignment* [9]. This observable is very important for particles with spin greater than $1/2$, the most important observations concerning vector mesons, i.e., spin-one particles.

The fact that alignment is only relevant for $S > 1/2$ particles comes directly from the spin density matrix. The spin density matrix being a $(2S + 1) \times (2S + 1)$ hermitian matrix with unit trace, it can be described completely through $4S(S + 1)$ real numbers. For $S = 1/2$ particles, three numbers suffice, and they are the three independent components of the spin vector (keeping in mind that $S^\mu p_\mu = 0$ constrains the remaining component). For $S = 1$ we need $8 = 3 + 5$ real numbers,

which are the components of the polarization vector and 5 numbers describing a symmetric 3×3 matrix with vanishing trace. For higher spin, additional tensor structures appear, whose physical significance is yet to be discovered.

The spin alignment is defined from the spin density matrix Θ . If there is no preferred spin state in the fluid, the spin density is proportional to the 3×3 identity matrix, that is $\Theta = 1/3\mathbb{I}$. The spin alignment is defined as the deviation of the 00-component[†] of the spin density matrix from that expectation:

$$\Theta_{00} - \frac{1}{3}. \quad (2.8)$$

This quantity is accessible experimentally from the angular distribution of the decay products of the vector mesons, which after integrating out the azimuthal angle reads [111]:

$$\frac{dN}{d(\cos\theta^*)} \propto 1 - \Theta_{00} + (3\Theta_{00} - 1) \cos^2\theta^*, \quad (2.9)$$

where θ^* is the angle between the polarization axis and the direction of the momentum of one of the decay products, as seen in the rest frame of the vector meson.

In this context, the most studied particles are the ϕ and K^* mesons [112–115], and more recently also the charmonium state J/ψ [116, 117]. Experimentally, the results show some puzzling behaviour. In particular, the measurements at STAR for energies $\sqrt{s_{NN}} \leq 200$ GeV show that $\Theta_{00} > 1/3$ for ϕ but $\Theta_{00} \lesssim 1/3$ for K^* . On the other hand, the measurements performed at 2.76 TeV indicate $\Theta_{00} \lesssim 1/3$ for both ϕ and K^* . At the same energy, J/ψ seems to have $\Theta_{00} \gtrsim 1/3$. This kind of behaviour is difficult to explain in a local-equilibrium framework, as one would expect similar results for different particles. Moreover, the change in sign of the alignment of the ϕ particle from 200 to 2760 GeV is puzzling in itself.

The contribution of vorticity to the spin alignment comes only at second order and in a rotating fluid one has [118]:

$$\Theta_{00}(p) \sim \frac{1}{3} + \frac{1 - 3(\hat{p} \cdot \hat{\omega})^2}{18} \frac{\omega^2}{T^2}. \quad (2.10)$$

This formula is not able to explain the experimental observations: not only the deviation from $1/3$ is too small compared to the measurements, but ϕ , K^* and J/ψ are predicted to behave in the same way.

Other models attempting to explain the alignment include quark-coalescence models [119] with intermediate fields [120, 121] and kinetic theory [122–124], but a definitive explanation of this effect is still missing.

[†]Here the components of the spin density matrix are labelled with indices ranging from -1 to 1 .

Chapter 3

Exact thermal expectation values at global equilibrium with rotation and acceleration

In this chapter, we address the calculation of thermal expectation values for non-interacting particles at global equilibrium with non-vanishing thermal vorticity using a new technique, which has been devised and employed in Refs. [13, 14]. The proposed procedure does not require the use of curvilinear coordinates and can be applied for equilibrium with any vorticity. This scheme consists of two main steps. The first one is the factorization of the density operator and its analytic continuation to an element of the Poincaré group [13, 50]. Thereafter, using group theory, it is possible to obtain recurrence relations for n-point functions of creation and annihilation operators and to express expectation values (including the Wigner function) as series of functions. Finally, we use a mathematical operation, dubbed *analytic distillation*, to extract from a series of functions only the analytic part, which is the physical result of the expectation value.

Since the mathematical tools are somewhat disconnected from the rest of the chapter, we will start by introducing them. Then, the method itself will be put forward for Dirac fermions, and the results will be presented. The analogue for the scalar field has been studied in Ref. [13], and will not be discussed here. This chapter is largely based on Ref. [14].

3.1 Mathematical tools: asymptotic power series

This section is devoted to the derivation of asymptotic power series of particular series of functions, which are essential to obtain the results of this chapter. A sum $\sum_n^N f(n, x)$ is said to be an asymptotic sum of a given function $g(x)$ if, for fixed N :

$$\lim_{x \rightarrow x_0} \left| g(x) - \sum_n^N f(n, x) \right| = 0.$$

The series associated to the sum is the asymptotic series of $g(x)$, which can be in general divergent. The symbol \sim is used to denote asymptotic equality: $g(x) \sim \sum_n f(n, x)$ [125]. A particular kind of asymptotic series is the asymptotic power series, in the form $\sum_n a_n (x - x_0)^n$. Notice that, at least in principle, it is always

possible to find the asymptotic power series of a given function, identifying a_n with the value of the limits:

$$a_n = \lim_{x \rightarrow x_0} \frac{g(x) - \sum_{k=N_{\min}}^{n-1} a_k (x - x_0)^k}{(x - x_0)^n}.$$

Asymptotic power series can be divergent too, and can involve negative powers like in Laurent series. A typical example is

$$g(x) = \int_x^\infty dt \frac{e^{x-t}}{t}, \quad g(x) \sim \sum_{n=1}^N \frac{(-1)^n n!}{x^{n+1}},$$

which is a divergent series with negative powers that approximates the exponential integral $g(x)$ for large values of x .

The tool enabling us to obtain asymptotic series for the series of functions we are interested in is the generalized Mellin transform, and the procedure was originally proposed by D. Zagier in the case of non-alternating series of functions [126, 127]. Here, this scheme will be generalized to alternating series, and for series which involve the sum on odd numbers only.

Before doing that, it is useful to review some general features of the Mellin transform. Further details can be found in Refs. [126, 127]. The Mellin transform of a function $\varphi(x)$ is defined as:

$$\{\mathcal{M}\varphi\}(s) = \int_0^\infty dt \varphi(t) t^{s-1}. \quad (3.1)$$

From this definition, one can show the important property

$$\{\mathcal{M}\varphi(\lambda x)\}(s) = \lambda^{-s} \{\mathcal{M}\varphi\}(s), \quad \lambda \in \mathbb{R}_{>0}. \quad (3.2)$$

According to the behaviour of the function $\varphi(x)$, the analytical structure of $\{\mathcal{M}\varphi\}$ changes. If $\varphi(x)$ decays to zero rapidly both at infinity and zero, then the Mellin transform is a holomorphic function on the complex plane, otherwise the region where $\{\mathcal{M}\varphi\}$ is holomorphic is smaller. For example, suppose $\varphi(t) \sim t^{-A}$ for $A \in \mathbb{R}$ as $t \rightarrow \infty$. In such a case the integral is convergent as long as $\Re(s) < A$, so that $\{\mathcal{M}\varphi\}$ is a holomorphic function only in that region. If, on the other hand, $\varphi(t) \sim t^{-B}$ with $B \in \mathbb{R}$ as $t \rightarrow 0$, then the Mellin transform is holomorphic in the region $\Re(s) > B$. If both behaviours are present, the region where the Mellin transform is holomorphic is the strip $B < \Re(s) < A$. In all the discussed cases, however, a meromorphic continuation of $\{\mathcal{M}\varphi\}$ can be provided out of the region where the integral (3.1) converges.

For instance, consider a function φ decaying rapidly at infinity, and whose asymptotic behaviour as $x \rightarrow 0$ is:

$$\varphi(x) \sim \sum_{n=0} a_n x^n. \quad (3.3)$$

From the above discussion, it follows that $\{\mathcal{M}\varphi\}$ is only defined in the region $\Re(s) > 0$. To give a continuation to the Mellin transform outside of the range of validity of Eq. (3.1), we consider, for $N > 0$

$$\{\mathcal{M}\varphi\}(s) = \int_0^1 dt \left(\varphi(t) - \sum_{n=0}^{N-1} a_n t^n \right) t^{s-1} + \sum_{n=0}^{N-1} \frac{a_n}{n+s} + \int_1^\infty dt \varphi(t) t^{s-1}. \quad (3.4)$$

In the above line, we have split the integral into two intervals, from 0 to 1 and from 1 to ∞ , and added and subtracted the asymptotic behaviour of φ about 0 in the first integral. Now the integral of the function between parenthesis converges for $s = 0$. In fact, with this simple trick we have been able to extend the Mellin transform to the larger portion of the complex plane $\Re(s) > -N$, as the function between parenthesis in the first integral has asymptotic behaviour t^N for $t \rightarrow 0$. Even more, we have found that such a continuation is a meromorphic function with simple poles at every negative integer $s = -n$, and residues a_n . The inverse statement is also true: if the generalized Mellin transform of a function φ is meromorphic, with poles at $s = -n$, then the residues represent the coefficients of the asymptotic expansion around $x = 0$ of φ .

With the above machinery, we can proceed to the calculation of the asymptotic power series of particular series of functions. We will study three cases separately.

Series of $f(nx)$

Let us start by reviewing the original proof by Zagier, for which we refer the reader to Refs. [126, 127].

Consider a function f with asymptotic expansion about $x = 0$ given by

$$f(x) \sim \sum_{k=-M, \neq -1}^{\infty} a_k x^k, \quad (3.5)$$

with M a positive integer. Then the series of functions:

$$g(x) = \sum_{n=1}^{\infty} f(nx) \quad (3.6)$$

has the asymptotic power expansion for $x \rightarrow 0$

$$g(x) \sim \frac{I_f}{x} + \sum_{n=0}^{\infty} a_n \zeta(-n) x^n, \quad I_f \equiv \int_0^{\infty} dt \left(f(t) - \sum_{k=-M}^{-2} a_k t^k \right). \quad (3.7)$$

Here, ζ is the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \quad (3.8)$$

The inclusion of the $k = -1$ term in the expansion (3.5) presents some additional difficulties which will be tackled later.

To prove Eq. (3.7), first, we remove the negative-power terms from f , defining:

$$\tilde{f}(x) = f(x) - \sum_{k=-M}^{-2} a_k x^k.$$

It follows that:

$$g(x) = \sum_{n=1}^{\infty} \tilde{f}(nx) + \sum_{n=1}^{\infty} \sum_{k=-M}^{-2} a_k n^k x^k \equiv \tilde{g}(x) + \sum_{n=1}^{\infty} \sum_{k=-M}^{-2} a_k n^k x^k, \quad (3.9)$$

where we have introduced \tilde{g} to define the series of the \tilde{f} functions.

The second term in the above equation is the series of a polynomial, and each term is convergent. Therefore it can be summed trivially by exchanging the order of the series and the finite sum:

$$\sum_{n=1}^{\infty} \sum_{k=-M}^{-2} a_k n^k x^k = \sum_{k=-M}^{-2} \sum_{n=1}^{\infty} a_k n^k x^k = \sum_{k=-M}^{-2} \zeta(-k) a_k x^k. \quad (3.10)$$

To compute the asymptotic expansion of \tilde{g} , let us consider its Mellin transform. From the property (3.2), it follows:

$$\{\mathcal{M}\tilde{g}\}(s) = \sum_{n=1}^{\infty} \{\mathcal{M}\tilde{f}(nt)\}(s) = \sum_{n=1}^{\infty} n^{-s} \{\mathcal{M}\tilde{f}\}(s) = \zeta(s) \{\mathcal{M}\tilde{f}\}(s). \quad (3.11)$$

We can now infer the analytic structure of $\{\mathcal{M}\tilde{g}\}$. Given the asymptotic behaviour of \tilde{f} , $\{\mathcal{M}\tilde{f}\}$ is a meromorphic function with poles at the non-positive integers (i.e., $s = -n = 0, -1, -2, \dots$), and residue a_n . It implies that $\{\mathcal{M}\tilde{g}\}$ also has the same poles, with residues $\zeta(s)a_n = \zeta(-n)a_n$. Moreover, the ζ -function has a pole at $s = 1$, which in turn implies that $\{\mathcal{M}\tilde{g}\}$ has a pole at $s = 1$, so that the residue $\{\mathcal{M}\tilde{f}\}(1) = I_f$ as defined in Eq. (3.7). Thus, the discussion presented earlier in this section implies that the asymptotic power expansion of \tilde{g} is given by:

$$\tilde{g}(x) \sim \frac{I_f}{x} + \sum_{n=0}^{\infty} \zeta(-n) a_n x^n. \quad (3.12)$$

Adding up all the terms in Eq. (3.9), we find that:

$$g(x) \sim \frac{I_f}{x} + \sum_{n=-M, \neq -1}^{\infty} a_n \zeta(-n) x^n, \quad (3.13)$$

which proves the result. \square

It is also interesting to study the case where the asymptotic expansion includes the term $k = -1$, although it will not be used in this chapter. Suppose:

$$f(x) \sim \sum_{k=-1}^{\infty} a_k x^k, \quad (3.14)$$

where the terms with $k < -1$ have been removed for sake of simplicity, as they can be summed as explained above. The ζ -function is singular in 1, and we ought to regularize the function f before we can compute the asymptotic series of g . Let us define the function:

$$h(x) = f(x) - a_{-1} \frac{e^{-x}}{x}, \quad (3.15)$$

which is regular at $x = 0$. We can compute:

$$g(x) = \sum_{n=1}^{\infty} f(nx) = \sum_{n=1}^{\infty} h(nx) + a_{-1} \sum_{n=1}^{\infty} \frac{e^{-nx}}{nx}. \quad (3.16)$$

The asymptotic expansion of g is found using the result (3.13) for the series of $h(nx)$, whereas the second sum can be computed exactly*

$$g(x) \sim \frac{1}{x} I_* + \sum_{n=0}^{\infty} h_n \zeta(-n) x^n - \frac{a_{-1}}{x} \ln \left(\frac{e^x - 1}{e^x} \right),$$

$$I_* = \int_0^{\infty} dt \left(f(t) - a_{-1} \frac{e^{-t}}{t} \right),$$

and from the definition of $h(x)$ one can see that the coefficients h_n are

$$h_n = a_n - a_{-1} \frac{(-1)^{n+1}}{(n+1)!}.$$

We can cast the result into the form:

$$g(x) \sim \frac{1}{x} I_* + \sum_{n=0}^{\infty} a_n \zeta(-n) x^n - a_{-1} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} \zeta(-n) x^n + \frac{a_{-1}}{x} \ln \left(\frac{e^x}{e^x - 1} \right),$$

where we used the properties of the logarithm to switch the numerator and the denominator in the last term. To further simplify the result, we have to rework the logarithm:

$$\frac{1}{x} \ln \left(\frac{e^x}{e^x - 1} \right) = -\frac{1}{x} \ln x + 1 + \frac{1}{x} \ln \left(\frac{x}{e^x - 1} \right). \quad (3.17)$$

It is possible to express the last logarithm on the right-hand side as a power series defining a function $l(x)$ and integrating the power expansion of its derivative as follows:

$$l(x) = \ln \left(\frac{x}{e^x - 1} \right) \Rightarrow l'(x) = \frac{1}{x} \left(1 - \frac{x e^x}{e^x - 1} \right) = - \sum_{r=0}^{\infty} \frac{B_{r+1}(1)}{(r+1)!} t^r$$

$$\Rightarrow l(x) = - \sum_{r=0}^{\infty} \frac{B_{r+1}(1)}{(r+1)!(r+1)} t^{r+1},$$

where the integration constant has been fixed requiring $l(0) = 0$ and we used the definition of the generating function of the Bernoulli polynomials $B_r(s)$:

$$\frac{x e^{sx}}{e^x - 1} = \sum_{r=0}^{\infty} \frac{B_r(s)}{r!} t^r.$$

Since $B_{r+1}(1) = -\zeta(-r)(r+1)$ for $r \geq 0$, the asymptotic power expansion of $g(x)$ reads:

$$g(x) \sim \frac{1}{x} (I_* - a_{-1} \ln x) + \sum_{n=0}^{\infty} a_n \zeta(-n) x^n + a_{-1} \left(1 + \sum_{n=0}^{\infty} \frac{1 - (-1)^{n+1}}{(n+1)!} \zeta(-n) x^n \right). \quad (3.18)$$

But $\zeta(-2n) = 0$ for $n > 0$, and one sees that the last term in parenthesis is vanishing.

The final result reads:

$$g(x) \sim \frac{1}{x} (I_* - a_{-1} \ln x) + \sum_{n=0}^{\infty} a_n \zeta(-n) x^n. \quad (3.19)$$

*If also negative coefficients from $-M$ to -2 appear in the expansion of f , they must be subtracted in the integral to ensure convergence, like in Eq. (3.7).

Series of $(-1)^{n+1}f(nx)$

Let us turn to the case of alternating series of functions, which is relevant for fermionic statistics. We will use a similar notation as in the previous subsection.

Consider the function $f(x)$ on the real axis, such that its asymptotic power series about $x = 0$ is given by:

$$f(x) \sim \sum_{k=-M}^{\infty} a_k x^k, \quad (3.20)$$

and consider the function $g(x)$ defined by the series:

$$g(x) = \sum_{n=1}^{\infty} (-1)^{n+1} f(nx). \quad (3.21)$$

Then the asymptotic power series of $g(x)$ about zero reads:

$$g(x) \sim \sum_{n=-M}^{\infty} a_n \eta(-n) x^n, \quad (3.22)$$

where $\eta(s)$ is the Dirichlet η -function, defined as:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s). \quad (3.23)$$

Notice that in this case the term $k = -1$ in the asymptotic series does not yield any additional complications as the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$ is convergent. To prove the formula, we proceed like in the previous section.

Once again, we realize that the negative powers in the asymptotic series (3.20) yield a convergent series and we can consider them separately and they result in a finite sum:

$$g(x) = \tilde{g}(x) + \sum_{k=-M}^{-1} a_k \eta(-k) x^k. \quad (3.24)$$

The calculation of the asymptotic series of g boils down to the one of \tilde{g} , where we define:

$$\tilde{f}(x) = f(x) - \sum_{k=-M}^{-1} a_k x^k, \quad \tilde{g}(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \tilde{f}(nx).$$

Thus, the Mellin transform of \tilde{g} is:

$$\{\mathcal{M}\tilde{g}\}(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \{\mathcal{M}\tilde{f}(nx)\} = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} \{\mathcal{M}\tilde{f}\}(s) = \eta(s) \{\mathcal{M}\tilde{f}\}(s). \quad (3.25)$$

Again we use the properties of the Mellin transform: $\{\mathcal{M}\tilde{f}\}$ is a meromorphic function, has simple poles at each non-positive integer and the residue in each pole is a_n . This implies that $\{\mathcal{M}\tilde{g}\}$ is a meromorphic function, with the simple poles located in all the non-positive integers, and residue $\eta(-n)a_n$. The important difference with

respect to the non-alternating case is that the η -function, in contrast with the ζ is holomorphic in the complex plane, so that there is no additional singularity at $s = 1$.

Therefore we have

$$\tilde{g}(x) \sim \sum_{n=0}^{\infty} a_n \eta(-n) x^n. \quad (3.26)$$

Combining this result with Eq. (3.24)

$$g(x) \sim \sum_{n=-M}^{\infty} a_n \eta(-n) x^n, \quad (3.27)$$

we have proved the formula (3.22). \square

Series of $f((2n + 1)x)$

The two above results can be combined to obtain the asymptotic power expansion of a series of functions where the summation is performed only on odd numbers. This result will not be used in this work but might be interesting for mathematical physics. To obtain such a formula, it suffices to consider:

$$\sum_{n=1}^{\infty} (-1)^{n+1} f(nx) = \sum_{n=0}^{\infty} (-1)^{2n+1} f(2nx) + \sum_{n=0}^{\infty} (-1)^{2n+2} f((2n + 1)x), \quad (3.28)$$

where we have split the series into the even- n and odd- n series. Now, using the asymptotic power series just derived, if the asymptotic expansion of f is given by (3.5) we have:

$$\sum_{n=0}^{\infty} f((2n + 1)x) \sim \sum_{n=-M}^{\infty} a_n \eta(-n) x^n + \frac{I_f}{2x} + \sum_{\substack{n=-M \\ n \neq -1}}^{\infty} a_n 2^n \zeta(-n) x^n. \quad (3.29)$$

If the term $k = -1$ is included in the asymptotic expansion of f :

$$\sum_{n=0}^{\infty} f((2n + 1)x) \sim \sum_{n=-M}^{\infty} a_n \eta(-n) x^n + \frac{1}{2x} (I_* - a_{-1} \ln 2x) + \sum_{\substack{n=-M \\ n \neq -1}}^{\infty} a_n 2^n \zeta(-n) x^n. \quad (3.30)$$

\square

3.2 Analytic distillation

The reason why it is important to know the asymptotic power expansion of these series of functions will become clear in the next sections: they are paramount to extracting analytic results from the Wigner-function formalism via a mathematical operation dubbed *analytic distillation*.

The analytic distillation has been defined in Ref. [13] as follows:

Definition. Let $f(z)$ be a function on a domain D of the complex plane and $z_0 \in \overline{D}$ (\overline{D} being the closure of D) a point where the function may not be analytic. Suppose

that the asymptotic power series of $f(z)$ in $z - z_0$ exist in subsets $D_i \subset D$ such that $\cup_i D_i = D$:

$$f(z) \sim \sum_n a_n^{(i)} (z - z_0)^n$$

where n can take integer negative values. If the series formed with the common coefficients in the subsets D_i restricted to $n \geq 0$ has a positive radius of convergence, the analytic function defined by this power series is called *analytic distillate* of $f(z)$ in z_0 and it is denoted by $\text{dist}_{z_0} f(z)$.

In practice, the definition prescribes a way to retain the analytic part of a function. One has to find a power expansion for the given function and remove coefficients in such a way that the new expansion is globally defined in the domain D , possibly the complex plane. The fact that asymptotic series may differ in different sectors of the complex plane is known as the Stokes phenomenon, and it is relevant in the theory of resurgence and trans-series [128–130]. Furthermore, one has to demand the series to be convergent and remove $n < 0$ terms so that, in practice, what is left is the analytic part of the original function.

In other words, the coefficients of the analytic distillate of a function f at a point z_0 are given by:

$$\bar{a}_n \equiv \begin{cases} a_n^{(i)} & \text{if } a_n^{(i)} = a_n^{(j)}, \forall i, j \text{ and } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{dist}_{z_0} f(z) \equiv \sum_{n=0}^{\infty} \bar{a}_n (z - z_0)^n, \quad (3.31)$$

where i, j denote different regions of the domain D . Notice that, if f is an analytic function at z_0 , then the distillate yields the function itself, whereas if the function is not analytic the result is vanishing. If the asymptotic expansion of f admits negative exponents, those are removed by the distillate.

As a simple example of analytic distillate, consider the function:

$$f(x) = \frac{e^{-x}}{x} + |x|.$$

The asymptotic power expansion of f is given by:

$$f(x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n-1} + |x|.$$

Since the absolute value has two different coefficients depending on the sign of x , it must be removed from the distillation. Moreover, the series has a negative power, that should also be removed. We find that the analytic distillate of $f(x)$ at $x = 0$ is given by:

$$\text{dist}_0 f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} x^{n-1} = \frac{e^{-x} - 1}{x},$$

which is analytic at $x = 0$. This procedure will be employed in the forthcoming sections in order to extract the exact expectation values computed recursively in the Wigner-function formalism, for a free field theory.

3.3 Factorization of the density operator at global equilibrium

Other than the asymptotic power series and the analytic distillation, there is another mathematical step which is critical for the calculation of exact thermal expectation values at global equilibrium: the factorization of the density operator.

The global equilibrium density operator, as discussed in Section 1.2.2, reads:

$$\hat{\rho} = \frac{1}{Z} e^{-\hat{P} \cdot b + \frac{\varpi \cdot \hat{J}}{2}}, \quad (3.32)$$

where b is a constant vector and ϖ the constant thermal vorticity, an antisymmetric tensor. As we have discussed in Section 1.2.2, the density operator is the exponential of the generators of the Poincaré group, \hat{P} being the generator of translations and \hat{J} of boosts and rotations.

The density operator would be far easier to handle were it factorized as the product of a complex translation times a complex Lorentz transformation, i.e.:

$$\hat{\rho} = \frac{1}{Z} e^{-b' \cdot \hat{P}} e^{\frac{\varpi' \cdot \hat{J}}{2}},$$

for some values of b' and ϖ' .

This factorization is always possible, as it was shown in Refs. [13, 50]. In particular, it is possible to factorize the density operator as:

$$\hat{\rho} = \frac{1}{Z} \exp \left[-b_\mu \hat{P}^\mu + \frac{\varpi_{\mu\nu}}{2} \hat{J}^{\mu\nu} \right] = \frac{1}{Z} \exp \left[-\tilde{b}_\mu(\varpi) \hat{P}^\mu \right] \exp \left[\frac{\varpi_{\mu\nu}}{2} \hat{J}^{\mu\nu} \right], \quad (3.33)$$

where the tilde transform of a vector is defined as:

$$\tilde{b}^\mu(\varpi) = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \underbrace{(\varpi_{\alpha_1}^{\mu} \varpi_{\alpha_2}^{\alpha_1} \dots \varpi_{\alpha_k}^{\alpha_{k-1}})}_{k \text{ times}} b^{\alpha_k}. \quad (3.34)$$

Notice how the tilde transformed vector \tilde{b} is in general not real if the vorticity is real, whereas it becomes real for imaginary vorticity. In the aforementioned references, the proof was put forth using the properties of translations and Lorentz transformations. However, it was also stated that the factorization just depends on the group structure and the Lie algebra of the Poincaré group, thus it is obtainable also via the Baker–Campbell–Hausdorff (BCH) formula. In this section, we will use this formalism to prove the factorization of the density operator.

The BCH formula allows us to express the product of the exponentials of two non-commuting operators as the exponential of a single operator given by a series of commutators. Explicitly, one has [131]:

$$\log(e^X e^Y) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_1+s_1>0 \\ \vdots \\ r_n+s_n>0}} \frac{[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n}]}{(\sum_{j=1}^n (r_j + s_j)) \cdot \prod_{i=1}^n r_i! s_i!}, \quad (3.35)$$

where:

$$[X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \dots X^{r_n} Y^{s_n}] = \underbrace{[X, \dots, [X]}_{r_1 \text{ times}}, \underbrace{[Y, \dots, [Y]}_{s_1 \text{ times}}, \underbrace{[X, \dots, [X]}_{r_2 \text{ times}}, \underbrace{[Y, \dots, [Y]}_{s_2 \text{ times}}, \dots]. \quad (3.36)$$

Notice that since $[Y, Y] = 0$, it follows that $s_n = 1$. Furthermore, if $s_n = 0$, which implies $r_n > 0$ the result vanishes for $[X, X] = 0$. For our purposes, we define:

$$X = -b_\alpha \widehat{P}^\alpha \quad Y = \frac{\varpi_{\mu\nu}}{2} \widehat{J}^{\mu\nu}. \quad (3.37)$$

To prove Eq. (3.33), we aim at finding an operator A such that:

$$e^{X+Y} = e^A e^Y.$$

Multiplying by the inverse Lorentz transform e^{-Y} and using Eq. (3.35), the operator A can be written as:

$$A = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_1+s_1>0 \\ \vdots \\ r_n+s_n>0}} \frac{[(X+Y)^{r_1}(-Y)^{s_1} \dots (X+Y)^{r_n}(-Y)^{s_n}]}{(\sum_{j=1}^n (r_j + s_j)) \cdot \prod_{i=1}^n r_i! s_i!}, \quad (3.38)$$

where to compute the commutators we have to use the algebra of the Poincaré group [42]:

$$[\widehat{J}^{\mu\nu}, \widehat{P}^\alpha] = i \left(g^{\alpha\nu} \widehat{P}^\mu - g^{\alpha\mu} \widehat{P}^\nu \right), \quad (3.39a)$$

$$[\widehat{P}^\mu, \widehat{P}^\nu] = 0. \quad (3.39b)$$

To help with the calculation, we can first prove an auxiliary result. Given the definitions of X and Y as Eq. (3.37), we can show that for any N :

$$\underbrace{[Y, [Y, \dots [Y, X]}_{N \text{ times}}] = -i^N \underbrace{(\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{N-1}\nu_N})}_{N \text{ times}} b^{\nu_N} \widehat{P}^\mu \quad (3.40)$$

Proof: The proof is done by induction. First consider $N = 1$:

$$[Y, X] = -b_\alpha \frac{\varpi_{\mu\nu}}{2} [\widehat{J}^{\mu\nu}, \widehat{P}^\alpha] = -ib_\alpha \frac{\varpi_{\mu\nu}}{2} \left(g^{\alpha\nu} \widehat{P}^\mu - g^{\alpha\mu} \widehat{P}^\nu \right) = -i\varpi_{\mu\nu} b^\nu \widehat{P}^\mu.$$

This simple line proves that the formula holds for the first step. Now, inductively, we assume the formula to hold for the N -th step, and prove that if this is the case the step $N + 1$ also holds. If we consider $N + 1$ commutators, we find:

$$\begin{aligned} \underbrace{[Y, [Y, \dots [Y, X]}_{N \text{ times}}] &= -i^N \underbrace{(\varpi_{\rho\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{N-1}\sigma})}_{N \text{ times}} b^\sigma [Y, P^\rho] \\ &= -i^N \frac{\varpi_{\mu\nu}}{2} \underbrace{(\varpi_{\rho\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{N-1}\sigma})}_{N \text{ times}} b^\sigma [\widehat{J}^{\mu\nu}, \widehat{P}^\rho] = \\ &= -i^{N+1} \frac{\varpi_{\mu\nu}}{2} \underbrace{(\varpi_{\rho\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{N-1}\sigma})}_{N \text{ times}} b^\sigma \left(g^{\rho\nu} \widehat{P}^\mu - g^{\rho\mu} \widehat{P}^\nu \right), \end{aligned}$$

and finally:

$$\underbrace{[Y, [Y, \dots [Y, X]}_{N \text{ times}}] = -i^{N+1} \underbrace{(\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_N\sigma})}_{N+1 \text{ times}} b^\sigma \widehat{P}^\mu.$$

We have proven that the step $N = 1$ holds and that if the step N holds it follows that the step $N + 1$ holds as well, and this concludes the proof by induction. \square

After this intermediate result, it is easy to compute the series (3.38). Let us consider each fixed n term of the series:

$$A = A_{(1)} - \frac{1}{2}A_{(2)} + \dots + \frac{(-1)^{n-1}}{n}A_{(n)} \dots$$

where each of the $A_{(n)}$ is given by:

$$A_{(n)} = \sum_{\substack{r_1+s_1>0 \\ \vdots \\ r_n+s_n>0}} \frac{[(X+Y)^{r_1}(-Y)^{s_1}(X+Y)^{r_2}(-Y)^{s_2} \dots (X+Y)^{r_n}(-Y)^{s_n}]}{(\sum_{j=1}^n (r_j + s_j)) \cdot \prod_{i=1}^n r_i! s_i!} \quad (3.41)$$

Starting with $A_{(1)}$, and since necessarily $s_n = 1$, we find:

$$A_{(1)} = \sum_{r_1=0}^{\infty} \frac{\overbrace{[X+Y, [X+Y, [\dots, [X+Y, -Y] \dots]]}^{r_1 \text{ times}}}{(r_1 + 1)!}.$$

On the other hand, since $[X, Y] \propto X$ all the terms like $[X, [\dots, [X, [X, -Y]]]$ vanish due to $[\widehat{P}^\mu, \widehat{P}^\nu] = 0$, and we are left with:

$$A_{(1)} = \sum_{r_1=0}^{\infty} \frac{\overbrace{[Y, [Y, [\dots, [Y, [X, -Y] \dots]]}^{r_1-1 \text{ times}}}{(r_1 + 1)!} = \sum_{r_1=0}^{\infty} \frac{\overbrace{[Y, [Y, [\dots, [Y, [Y, X] \dots]]}^{r_1 \text{ times}}}{(r_1 + 1)!}.$$

Using equation Eq. (3.40), we find:

$$A_{(1)} = - \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} (\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{k-1}\nu_k}) b^{\nu_k} \widehat{P}^\mu. \quad (3.42)$$

Now we turn to the $A_{(n>1)}$ terms, and we show that they vanish identically. Consider $A_{(2)}$:

$$\begin{aligned} A_{(2)} &= \sum_{r_1+s_1>0} \sum_{r_2=0}^{\infty} \frac{(-1)^{s_1}}{r_1 + s_1 + r_2 + 1} \frac{[(X+Y)^{r_1}, Y^{s_1}, (X+Y)^{r_2}, -Y]}{s_1! r_1! r_2!} = \\ &= \sum_{r_2=0}^{\infty} \sum_{r_1+s_1>0} \frac{(-1)^{s_1}}{r_1 + s_1 + r_2 + 1} \frac{[Y^{r_1}, Y^{s_1}, Y^{r_2}, X]}{s_1! r_1! r_2!}, \end{aligned}$$

where we removed the terms like $[X, [X, \dots [X, Y] \dots]]$ as they vanish. The sum takes place on fixed values of $r_1 + s_1$, so, changing variable $N = r_1 + s_1$, we write:

$$\begin{aligned} &\sum_{r_1+s_1>0} \frac{(-1)^{s_1}}{r_1 + s_1 + r_2 + 1} \frac{[Y^{r_1}, Y^{s_1}, Y^{r_2}, X]}{s_1! r_1! r_2!} \\ &= \sum_{N=1}^{\infty} \frac{[Y^{N+r_2}, X]}{r_2!(N+r_2+1)} \sum_{s_1=0}^{\infty} \frac{(-1)^{s_1}}{s_1!(N-s_1)!}. \end{aligned}$$

As s_1 grows, the factor $N - s_1$ eventually becomes negative. The term $(N - s_1)!$ in the second series becomes the factorial of a negative integer, which is infinite due

to the properties of the Euler Γ -function, the analytic continuation of the factorial. Therefore, the second series reduces to a finite, vanishing sum:

$$\sum_{s=0}^{\infty} \frac{(-1)^s}{s!(N-s)!} = \sum_{s=0}^N \frac{(-1)^s}{s!(N-s)!} = \frac{1}{N!} \sum_{s=0}^N \binom{N}{s} 1^{N-s} (-1)^s = \frac{(1-1)^N}{N!} = 0$$

and we have shown that $A_{(2)} = 0$.

The same steps can be applied to the other terms of Eq. (3.38), which turn out to be all vanishing. This is a very peculiar fact, characteristic of the Poincaré algebra. It is due to the fact that the commutator $[\hat{J}^{\mu\nu}, \hat{P}^\alpha]$ only results in combinations of the generators of translations \hat{P} , which form an abelian algebra.

Thus we finally have:

$$A = A_{(1)} = - \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} (\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{k-1}\nu_k}) b^{\nu_k} \hat{P}^\mu = -\tilde{b}(\varpi) \cdot \hat{P},$$

and the density operator can be factorized as shown in Eq. (3.33).

3.4 Recurrence method for expectation values

The procedure needed to calculate exact expectation values, as has been mentioned at the beginning of the chapter, consists of two steps. The first one is to use the factorized version of the global-equilibrium density operator, which has been addressed in the previous section. The operator (3.33) resembles the product of a translation and a Lorentz transformation, and this analogy can be made even more explicit by performing the analytic continuation:

$$\varpi_{\mu\nu} \mapsto -i\phi_{\mu\nu}. \quad (3.43)$$

This mapping will be very important to compute expectation values, as now the problem can be tackled by taking advantage of the properties of the Poincaré group. The analytic continuation of the density operator (3.33) is expressed as a product of the unitary representations of a (complex) translation and a Lorentz transformation, both acting on the Hilbert space:

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\tilde{b}_\mu(-i\phi) \hat{P}^\mu \right] \exp \left[-i \frac{\phi_{\mu\nu}}{2} \hat{J}^{\mu\nu} \right] \equiv \frac{1}{Z} \exp \left[-\tilde{b}_\mu(-i\phi) \hat{P}^\mu \right] \hat{\Lambda},$$

where we have defined $\hat{\Lambda} \equiv \exp[-i\phi : \hat{J}/2]$.

We can also allow for a chemical potential coupled to a conserved charge, say the electric charge, so that the density operator is:

$$\hat{\rho} = \frac{1}{Z} e^{\zeta \hat{Q}} \exp \left[-\tilde{b}_\mu(-i\phi) \hat{P}^\mu \right] \hat{\Lambda}. \quad (3.44)$$

One of the key ingredients to compute any expectation value is the two-point function of creation and annihilation operators $\langle \hat{a}_s^\dagger(p) \hat{a}_t(p') \rangle$, which we are now going to compute using group theory. The technique used here can be applied also for n -point functions involving more than two creation and annihilation operators, for example $\langle \hat{a}_s^\dagger(p) \hat{a}_t(p') \hat{a}_i^\dagger(q) \hat{a}_m(q') \rangle$. In this respect, it can be used also to compute exact correlators of operators bilinear in the fields such as $\langle \hat{T}^{\mu\nu} \hat{T}^{\rho\sigma} \rangle$. In full generality,

for the purpose of this section, we consider particles with spin S , whose spin state is labelled by the integer or half-integer number s ranging from $-S$ to S in steps of one. It is intended that the creation and annihilation operators obey commutation rules if the spin is integer and anticommutation rules if it is a half-integer. It is well known that the creation operator for a particle with spin S transforms under the Lorentz transformation $\widehat{\Lambda}$ as [42]:

$$\widehat{\Lambda}\widehat{a}_s^\dagger(p)\widehat{\Lambda}^\dagger = \sum_r D^S(W(\Lambda, p))_{rs}\widehat{a}_r^\dagger(\Lambda p), \quad (3.45)$$

where $W(\Lambda, p) = [\Lambda p]^{-1}\Lambda[p]$. This matrix is an element of the little group of the standard vector \mathbf{p} , as it transforms \mathbf{p} to p , then to Λp and finally back to \mathbf{p} . We recall that D^S stands for the $(0, S)$ -th finite-dimensional representation of the Lorentz group [42, 43].

As we have seen in Chapter 1, for massive particles one chooses as the standard vector the momentum of the particle in its rest frame $\mathbf{p} = (m, \mathbf{0})$. In such a case, it is straightforward to see that the little group is made of all rotations in three dimensions, $SO(3)$ and the $W(\Lambda, p)$ matrix is called *Wigner rotation*.

Equation (3.45) holds also for massless particles, although some care is necessary. In the case of massless particles, the usual choice for the standard momentum is $\mathbf{p} = (\kappa, 0, 0, \kappa)$, for some constant $\kappa > 0$. In Section 1.5 we established that two of the three generators of the little group must act trivially on the Hilbert space, and this requirement reduces the transformation $D(W(\Lambda, p))$ in Eq. (3.45) to a phase: $D(W(\Lambda, p))_{rs} = e^{-ir\theta(\Lambda, p)}\delta_{rs}$ [42–44]. Therefore, we realize that in both massive and massless cases, the matrix $W(\Lambda, p) = [\Lambda p]^{-1}\Lambda[p]$ involved in the transformation (3.45) is unitary (see also Appendix A for further details).

To calculate exactly $\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p) \rangle$ we start by using the transformation rule (3.45) and the cyclicity of the trace. Using the operator (3.44):

$$\begin{aligned} \langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p') \rangle &= \frac{1}{Z} e^\zeta \text{Tr} \left(e^{-\tilde{b}\cdot\widehat{P}} \widehat{\Lambda}\widehat{a}_s^\dagger(p)\widehat{a}_t(p') \right) \\ &= \frac{1}{Z} e^\zeta \sum_r D^S(W(\Lambda, p))_{rs} \text{Tr} \left(e^{-\tilde{b}\cdot\widehat{P}} \widehat{a}_r^\dagger(\Lambda p)\widehat{\Lambda}\widehat{a}_t(p') \right) \\ &= e^\zeta \sum_r D^S(W(\Lambda, p))_{rs} e^{-\tilde{b}\cdot\Lambda p} \langle \widehat{a}_t(p')\widehat{a}_r^\dagger(\Lambda p) \rangle. \end{aligned}$$

Finally, using the (anti)commutation rules of $\widehat{a}_s^\dagger(p)$ and $\widehat{a}_t(p)$, we find:

$$\begin{aligned} \langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p') \rangle &= (-1)^{2S} e^\zeta \sum_r D^S(W(\Lambda, p))_{rs} e^{-\tilde{b}\cdot\Lambda p} \langle \widehat{a}_r^\dagger(\Lambda p)\widehat{a}_t(p') \rangle + \\ &+ 2e^\zeta \varepsilon e^{-\tilde{b}\cdot\Lambda p} D^S(W(\Lambda, p))_{ts} \delta^3(\Lambda \mathbf{p} - \mathbf{p}'), \end{aligned} \quad (3.46)$$

where the factor $(-1)^{2S}$ originates from the use of commutation or anticommutation rules, depending on the spin of the particle. To solve Eq. (3.46) we can proceed iteratively. When dealing with equations to be solved iteratively, the solution consists of a particular solution of the inhomogeneous equation and a homogeneous solution, much like in the case of differential equations. It has been shown, however, that the homogeneous solution of equations like (3.46) is in general non-analytic for vanishing vorticity; since we anticipate thermal expectation values to be analytic when no vorticity is present, the homogeneous solution will be disregarded [13]. Also the

particular solution that we are about to find will contain non-analytic components, and those will be removed using the analytic distillation introduced in the previous section.

As the first step of the iterative procedure, we approximate the solution of Eq. (3.46) by the rightmost term, the one proportional to the δ -function:

$$\langle \widehat{a}_s^\dagger(p) \widehat{a}_t(p') \rangle \sim 2e^\zeta \varepsilon e^{-\tilde{b} \cdot \Lambda p} D^S(W(\Lambda, p))_{ts} \delta^3(\Lambda \mathbf{p} - \mathbf{p}'),$$

Now, we can plug this approximation back into the right-hand side of Eq. (3.46), so to obtain an updated version of the solution:

$$\begin{aligned} \langle \widehat{a}_s^\dagger(p) \widehat{a}_t(p') \rangle &\sim 2e^{2\zeta} \varepsilon (-1)^{2S} \times \\ &\sum_r D^S(W(\Lambda, p))_{rs} D^S(W(\Lambda, \Lambda p))_{tr} e^{-\tilde{b} \cdot (\Lambda p + \Lambda^2 p)} \delta^3(\Lambda^2 \mathbf{p} - \mathbf{p}') \\ &+ 2e^\zeta \varepsilon e^{-\tilde{b} \cdot \Lambda p} D^S(W(\Lambda, p))_{ts} \delta^3(\Lambda \mathbf{p} - \mathbf{p}'). \end{aligned}$$

The product of the two elements of the little group can be simplified:

$$D^S(W(\Lambda, \Lambda p)) D^S(W(\Lambda, p)) = D^S([\Lambda \Lambda p]^{-1} \Lambda [\Lambda p] [\Lambda p]^{-1} \Lambda [p]) = D^S(W(\Lambda^2, p)),$$

where we used the well-known property of representations $D^S(A) D^S(B) = D^S(AB)$. We obtain:

$$\begin{aligned} \langle \widehat{a}_s^\dagger(p) \widehat{a}_t(p') \rangle &\sim 2e^{2\zeta} \varepsilon (-1)^{2S} D^S(W(\Lambda^2, p))_{ts} e^{-\tilde{b} \cdot (\Lambda p + \Lambda^2 p)} \delta^3(\Lambda^2 \mathbf{p} - \mathbf{p}') + \\ &+ 2e^\zeta \varepsilon e^{-\tilde{b} \cdot \Lambda p} D^S(W(\Lambda, p))_{ts} \delta^3(\Lambda \mathbf{p} - \mathbf{p}'). \end{aligned}$$

If we keep this process going up to infinity, the result is given by the series:

$$\langle \widehat{a}_s^\dagger(p) \widehat{a}_t(p') \rangle = 2\varepsilon' \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\Lambda^n \mathbf{p} - \mathbf{p}') D^S(W(\Lambda^n, p))_{ts} e^{-\tilde{b} \cdot \sum_{k=1}^n \Lambda^k p + n\zeta}. \quad (3.47)$$

The expectation value $\langle \widehat{b}_s^\dagger(p) \widehat{b}_t(p') \rangle$ can be computed likewise, and the result would be just:

$$\langle \widehat{b}_s^\dagger(p) \widehat{b}_t(p') \rangle = 2\varepsilon' \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\Lambda^n \mathbf{p} - \mathbf{p}') D^S(W(\Lambda^n, p))_{ts} e^{-\tilde{b} \cdot \sum_{k=1}^n \Lambda^k p - n\zeta}. \quad (3.48)$$

Since the global-equilibrium density operator cannot change the number of particles in a given state, applying this procedure to $\widehat{a}_s^\dagger(p) \widehat{a}_t^\dagger(p')$, $\widehat{a}_s(p) \widehat{a}_t(p')$, $\widehat{a}_s(p) \widehat{b}_t(p)$ and alike combinations yields vanishing results. Indeed, those expectation values are computed with a homogeneous equation, which has only non-analytic solutions.

As a sanity check of the result, we can consider the case of vanishing thermal vorticity $\varpi = 0$, where the results are well-known. A vanishing thermal vorticity implies $\phi = 0$ and $\Lambda = I$, and Eq. (3.47) gives:

$$\begin{aligned} \langle \widehat{a}_s^\dagger(p) \widehat{a}_t(p') \rangle &= 2\varepsilon' \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ts} e^{-nb \cdot p + n\zeta} \\ &= 2\varepsilon \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ts} \frac{1}{e^{b \cdot p - \zeta} + (-1)^{2S+1}}, \end{aligned}$$

where the series converges as long as $b \cdot p > \zeta \Rightarrow m > \mu$ and correctly reproduces the Bose-Einstein and the Fermi-Dirac distributions for integer and half-integer S respectively. Given this result, one can interpret the series in n as the quantum statistics expansion, the $n > 1$ terms being corrections to the Boltzmann statistics, which in turn can be described by truncating the series at $n = 1$.

The results obtained here are series of δ -functions. In most of the cases, however, the two-point functions computed above will appear in momentum integrals, so that the δ -function will be integrated away. Furthermore, the result (3.47) makes sense only with the imaginary vorticity ϕ , but the physical mean value should be computed for the real vorticity ϖ . This continuation is not feasible in Eq. (3.47), because of the presence of the delta functions, but it will be made possible after the momentum integration and analytic distillation.

3.4.1 The Wigner function of Dirac fermions at general global equilibrium

Using the results of the previous section, we can calculate the exact Wigner function of the Dirac field for a system at global equilibrium with non-vanishing vorticity. Feeding Eq. (3.47) for $S = 1/2$ into Eq. (1.54), using the notation $D(\bullet) = D^{(1/2)}(\bullet)$ and recalling that at global equilibrium the space-like part of the Wigner function vanishes (see discussion after Eq. (3.48)), we get:

$$\begin{aligned}
 W(x, k) &= \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-ix \cdot (\Lambda^n p - p)} e^{-\tilde{b} \cdot \sum_{k=1}^n \Lambda^k p + n\zeta} \\
 &\times \sum_{rs} \left[e^{n\zeta} u_r(\Lambda^n p) D(W(\Lambda^n, p))_{rs} \bar{u}_s(p) \delta^4 \left(k - \frac{\Lambda^n p + p}{2} \right) \right. \\
 &\left. - e^{-n\zeta} v_r(p) D(W(\Lambda^n, p))_{sr} \bar{v}_s(\Lambda^n p) \delta^4 \left(k + \frac{\Lambda^n p + p}{2} \right) \right], \quad (3.49)
 \end{aligned}$$

or, with the compact 4×2 spinor notation (1.40):

$$\begin{aligned}
 W(x, k) &= \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-ix \cdot (\Lambda^n p - p)} e^{-\tilde{b} \cdot \sum_{k=1}^n \Lambda^k p} \times \\
 &\left[e^{n\zeta} U(\Lambda^n p) D(W(\Lambda^n, p)) \bar{U}(p) \delta^4 \left(k - \frac{\Lambda^n p + p}{2} \right) \right. \\
 &\left. - e^{-n\zeta} V(p) D(W(\Lambda^n, p))^T \bar{V}(\Lambda^n p) \delta^4 \left(k + \frac{\Lambda^n p + p}{2} \right) \right]. \quad (3.50)
 \end{aligned}$$

We can simplify the above expression by using the transformation rules of spinors under the Lorentz group (see Appendix A):

$$U(\Lambda p) D(W(\Lambda, p)) = U(\Lambda p) D(W(\Lambda, p))^{\dagger^{-1}} = S(\Lambda) U(p),$$

where $S(\Lambda)$ is the Lorentz transformation Λ in the Dirac representation $(0, 1/2) \oplus (1/2, 0)$ as given by Eq. (1.45), and in the second step we used the fact that both in the massive and the massless case (the latter by physical requirement) the little group consists of unitary matrices only. Equation (3.50) can be worked out using the Eq. (1.50) along with:

$$S(\Lambda^n) = S(\Lambda)^n,$$

thus:

$$U(\Lambda^n p) D(W(\Lambda^n, p)) \bar{U}(p) = S(\Lambda)^n U(p) \bar{U}(p) = S(\Lambda)^n (\not{p} + m).$$

The antiparticle's component can be put into a similar form. Keeping in mind that $V^* = \mathcal{C}U$ (see Section 1.3):

$$\begin{aligned} V(p)D(W(\Lambda^n, p))^T \bar{V}(\Lambda^n p) &= -(\gamma_0 \mathcal{C}U(\Lambda^n p)W(\Lambda^n, p)\bar{U}(p)\mathcal{C}\gamma^0)^T \\ &= -\gamma^0 \mathcal{C}(m + \not{p}^T)S(\Lambda)^n \mathcal{C}\gamma^0 = -(m - \not{p})S(\Lambda)^{-n}, \end{aligned}$$

where we have used the relations (1.51). Therefore, we can express the Wigner function as:

$$\begin{aligned} W(x, k) &= \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-ix \cdot (\Lambda^n p)} e^{-\tilde{b} \cdot \sum_{k=1}^n \Lambda^k p} \times \\ &\quad \left[e^{n\zeta} S(\Lambda)^n (m + \not{p}) \delta^4 \left(k - \frac{\Lambda^n p + p}{2} \right) + e^{-n\zeta} (m - \not{p}) S(\Lambda)^{-n} \delta^4 \left(k + \frac{\Lambda^n p + p}{2} \right) \right]. \end{aligned} \quad (3.51)$$

Notice that the properties used to derive the expression (3.51) hold both for massive and massless particles. Consequently, setting $m = 0$ in Eq. (3.51) immediately yields the Wigner function for massless particles. We can also easily recover the Wigner function for vanishing thermal vorticity by setting $\phi = 0$ and $\Lambda = I$:

$$W(x, k) = \frac{\delta(k^2 - m^2)}{(2\pi^3)} [\theta(k^0)(m + \not{p})n_F(b \cdot p - \zeta) + \theta(-k^0)(m - \not{p})n_F(b \cdot p + \zeta)] \quad (3.52)$$

We are now in the position to show that Eq. (3.51) is a solution of the Wigner equation for free Dirac fermions (1.56):

$$\left(\frac{i}{2} \not{\phi} + \not{k} - m \right) W(x, k) = 0. \quad (3.53)$$

We will confine ourselves to the particle part of the Wigner function since the calculations can be easily repeated for antiparticles. The action of the partial derivative on the Wigner function is trivial, as the only dependence on the space-time point x is in the exponent. We can write:

$$\frac{i}{2} \not{\phi} \mapsto \frac{1}{2} (\Lambda^n p - \not{p}) = \Lambda^n p - \not{k},$$

where we have taken advantage of the δ -function to write $\not{p} = 2\not{k} - \Lambda^n p$. We can now use the Lorentz transformation rules of the matrices \underline{p} and \widehat{p} which, denoting the transformation $D(\Lambda) = \Lambda$ for brevity, read:

$$\underline{\Lambda p} \Lambda^\dagger = \underline{\Lambda p}, \quad \Lambda^{\dagger-1} \widehat{p} \Lambda^{-1} = \widehat{\Lambda p},$$

and the identity $\underline{p} \widehat{p} = m^2$ (see appendix A), to work out the matrix product:

$$\begin{aligned} \Lambda^n p S(\Lambda)^n (m + \not{p}) &= \begin{pmatrix} 0 & \underline{\Lambda^n p} \\ \widehat{\Lambda^n p} & 0 \end{pmatrix} \begin{pmatrix} \Lambda^n & 0 \\ 0 & \Lambda^{n\dagger-1} \end{pmatrix} \begin{pmatrix} m & \underline{p} \\ \widehat{p} & m \end{pmatrix} \\ &= \begin{pmatrix} 0 & \underline{\Lambda^n p} \\ \Lambda^{n\dagger-1} \widehat{p} & 0 \end{pmatrix} \begin{pmatrix} m & \underline{p} \\ \widehat{p} & m \end{pmatrix} \\ &= \begin{pmatrix} \Lambda^n m^2 & m \underline{\Lambda^n p} \\ m \Lambda^{n\dagger-1} \widehat{p} & m^2 \Lambda^{n\dagger-1} \end{pmatrix} = m S(\Lambda)^n (m + \not{p}). \end{aligned}$$

In conclusion, we have:

$$\frac{i}{2}\not{\partial}W_+ = (m - \not{k})W_+,$$

which proves that Eq. (3.51) is indeed a solution of the Wigner equation (1.56).

The form in Eq. (3.51) can be further simplified. Defining the tilde transform of β as in Eq. (3.34)

$$\tilde{\beta}(\varpi) = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \underbrace{(\varpi^\mu \varpi^{\alpha_1} \dots \varpi^{\alpha_{k-1}})}_{k \text{ times}} \beta^{\alpha_k}, \quad (3.54)$$

where we recall that β in global equilibrium is given by Eq. (1.18)

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu} x_\nu,$$

and taking advantage of two identities proved in Ref. [13], namely:

$$\sum_{k=1}^n \Lambda^{-k} \tilde{b}(\varpi) = n \tilde{b}(-n\varpi)$$

and

$$-ix \cdot (\Lambda^n p - p) - n \tilde{b}(-n\varpi) \cdot p = -n \tilde{\beta}(-n\varpi) \cdot p, \quad (3.55)$$

we can rewrite Eq. (3.51) as:

$$W(x, k) = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n \tilde{\beta}_n \cdot p} \times \left[e^{n\zeta} S(\Lambda)^n (m + \not{p}) \delta^4 \left(k - \frac{\Lambda^n p + p}{2} \right) + e^{-n\zeta} (m - \not{p}) S(\Lambda)^{-n} \delta^4 \left(k + \frac{\Lambda^n p + p}{2} \right) \right], \quad (3.56)$$

where we have used, as we will from now on, the shorthand notation:

$$\tilde{\beta}_n \equiv \tilde{\beta}(-n\varpi).$$

Equation (3.56) is the final result for the covariant Wigner function of the free Dirac field in global equilibrium with non-vanishing thermal vorticity. As stated above, the result holds both for massive and massless particles once we set the mass m to zero. Similarly to the expression (3.47), the Wigner function (3.56) is only valid for imaginary vorticity. However, expectation values like (1.57) involve the integration over k and the result of such integration will be suitable to be continued analytically back to the real vorticity.

3.5 Currents at global thermodynamic equilibrium

Let us now consider the currents obtained from Eqs. (1.57) as integrals of the Wigner function. Thanks to the integration, the singular delta functions are removed from expectation values, and it will be possible to continue the final results back to the physical case of real thermal vorticity.

We start by studying the vector current $j^\mu(x)$. Plugging the Wigner function (3.56) into Eq. (1.57a) we obtain:

$$j^\mu(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} \times \left[e^{n\zeta} \text{tr}(\gamma^\mu S(\Lambda)^n \not{p}) - e^{-n\zeta} \text{tr}(\gamma^\mu \not{p} S(\Lambda)^{-n}) \right]. \quad (3.57)$$

The first term is the particle contribution, whereas the second term is the antiparticle one.

As a sanity check, we can compute the vector current in the case of vanishing chemical potential $\zeta = 0$, when the total current is bound to be zero. In fact, it is instructive to see how this comes about. The antiparticle contribution can be worked out using the properties of the $\gamma^0 \mathcal{C}$ matrix (1.51) and of the trace:

$$\begin{aligned} \text{tr}(\gamma^\mu \not{p} S(\Lambda)^{-n}) &= \text{tr}((\gamma^\mu \not{p} S(\Lambda)^{-n})^T) = \text{tr}\left(S(\Lambda)^{nT-1} \not{p}^T \gamma^{\mu T}\right) \\ &= \text{tr}(\gamma^0 \mathcal{C} S(\Lambda)^n \mathcal{C} \gamma^0 \gamma^0 \not{p} \mathcal{C} \gamma^0 \gamma^0 \mathcal{C} \gamma^\mu \mathcal{C} \gamma^0) = \text{tr}(\gamma^\mu S(\Lambda)^n \not{p}). \end{aligned} \quad (3.58)$$

We see that particles and antiparticles contribute equally to the vector current, yet with opposite signs, so when $\zeta = 0$ the overall current vanishes.

In the case of non-vanishing chemical potential, this is no longer true, and one can employ Eq. (3.58) to write the current (3.57) as:

$$j^\mu(x) = \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} \text{tr}(\gamma^\mu S(\Lambda)^n \not{p}) \sinh n\zeta. \quad (3.59)$$

Using the commutation rules of the gamma matrices, the trace in Eq. (3.59) can be written as:

$$\text{tr}(\not{p} \gamma^\mu S(\Lambda)^n) = p^\mu \text{tr}(S(\Lambda)^n) + 2ip_\nu \text{tr}(\Sigma^{\mu\nu} S(\Lambda)^n), \quad (3.60)$$

and the vector current reads:

$$j^\mu(x) = \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} [p^\mu \text{tr}(S(\Lambda)^n) + 2ip_\nu \text{tr}(\Sigma^{\mu\nu} S(\Lambda)^n)] \sinh n\zeta, \quad (3.61)$$

From this expression, we can see that the current density is no longer aligned with the four-momentum p as in kinetic theory, but quantum effects give rise to contributions orthogonal to p (see the discussion at the end of Section 1.4 and in Section 3.7).

Similarly, we can derive the series corresponding to the mean axial current (1.57b). In this case, it is possible to prove that the traces associated with particles and anti-particles contribute to the axial current with the same magnitude and sign, so that the total axial current can be non-vanishing even in the case of $\zeta = 0$. Using Eq. (3.56), we obtain the axial current:

$$j_A^\mu(x) = \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} \text{tr}(\gamma^\mu \gamma_5 S(\Lambda)^n \not{p}) \cosh n\zeta.$$

Like in Eq. (3.60) the trace can be split:

$$\text{tr}(\not{p} \gamma^\mu \gamma_5 S(\Lambda)^n) = p^\mu \text{tr}(\gamma_5 S(\Lambda)^n) + 2ip_\nu \text{tr}(\Sigma^{\mu\nu} \gamma_5 S(\Lambda)^n).$$

It is convenient to introduce a compact notation, defining:

$$A^{\mu\nu}(n) = \text{tr}(\gamma^\nu \gamma^\mu S(\Lambda)^n) = g^{\mu\nu} \text{tr}(S(\Lambda)^n) + 2i \text{tr}(\Sigma^{\mu\nu} S(\Lambda)^n), \quad (3.62)$$

$$A_5^{\mu\nu}(n) = \text{tr}(\gamma^\nu \gamma^\mu \gamma_5 S(\Lambda)^n) = g^{\mu\nu} \text{tr}(\gamma_5 S(\Lambda)^n) + 2i \text{tr}(\Sigma^{\mu\nu} \gamma_5 S(\Lambda)^n). \quad (3.63)$$

The vector and axial current can now be written as:

$$\begin{aligned} j^\mu(x) &= \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} \text{tr}(\gamma^\mu S(\Lambda)^n \not{p}) \sinh n\zeta \\ &= \frac{2}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^{\mu\nu}(n) \sinh n\zeta \left(-\frac{\partial}{\partial \tilde{\beta}_n^\nu} \right) \int \frac{d^3p}{2\varepsilon} e^{-n\tilde{\beta}_n \cdot p}, \end{aligned} \quad (3.64)$$

$$\begin{aligned} j_A^\mu(x) &= \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} \text{tr}(\gamma^\mu \gamma_5 S(\Lambda)^n \not{p}) \cosh n\zeta \\ &= \frac{2}{(2\pi)^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A_5^{\mu\nu}(n) \cosh n\zeta \left(-\frac{\partial}{\partial \tilde{\beta}_n^\nu} \right) \int \frac{d^3p}{2\varepsilon} e^{-n\tilde{\beta}_n \cdot p}, \end{aligned} \quad (3.65)$$

where we have written the four-momentum p as the derivative with respect to $\tilde{\beta}_n$ of the exponential function. The operations of integral and series have been exchanged because, as long as the vorticity is imaginary, the series is uniformly convergent [13].

From now on, we will focus on the massless case, where the result of the integral is:

$$\int \frac{d^3p}{2\varepsilon} e^{-n\tilde{\beta}_n \cdot p} = \int \frac{d^3p}{2\varepsilon} e^{-n\sqrt{\tilde{\beta}_n^2} \varepsilon} = 2\pi \int_0^\infty dp p e^{-n\sqrt{\tilde{\beta}_n^2} p} = \frac{2\pi}{n^2 \tilde{\beta}_n \cdot \tilde{\beta}_n}, \quad (3.66)$$

where to compute the integral we used the Lorentz invariance of the measure to go to the frame where $\tilde{\beta}_n = \left(\sqrt{\tilde{\beta}_n^2}, \mathbf{0} \right)$ and we used the relation $\varepsilon = p = \sqrt{\mathbf{p}^2}$ which holds for massless particles.

For massless particles, Eqs. (3.64) and (3.65) become:

$$j^\mu(x) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \frac{\tilde{\beta}_{n\nu}}{(\tilde{\beta}_n)^4} A^{\mu\nu}(n) \sinh n\zeta, \quad (3.67)$$

$$j_A^\mu(x) = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \frac{\tilde{\beta}_{n\nu}}{(\tilde{\beta}_n)^4} A_5^{\mu\nu}(n) \cosh n\zeta. \quad (3.68)$$

where $(\tilde{\beta}_n)^4 = (\tilde{\beta}_n \cdot \tilde{\beta}_n)^2$.

We can also obtain the series associated with the canonical energy-momentum tensor. By using Eq. (3.56) in the definition (1.57c), one has:

$$\begin{aligned} T_C^{\mu\nu} &= \int d^4k k^\nu \text{tr}(\gamma^\mu W(x, k)) = \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi)^3} \int \frac{d^3p}{2\varepsilon} e^{-n\tilde{\beta}_n \cdot p} \left[e^{n\zeta} \left(\frac{\Lambda^n p + p}{2} \right)^\nu \text{tr}(\gamma^\mu S(\Lambda)^n \not{p}) \right. \\ &\quad \left. + e^{-n\zeta} \left(\frac{\Lambda^n p + p}{2} \right)^\nu \text{tr}(\gamma^\mu \not{p} S(\Lambda)^{-n}) \right] \\ &= 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi)^3} \cosh n\zeta \int \frac{d^3p}{2\varepsilon} e^{-n\tilde{\beta}_n \cdot p} \left(\frac{\Lambda^n p + p}{2} \right)^\nu \text{tr}(\gamma^\mu S(\Lambda)^n \not{p}), \end{aligned}$$

where we used once again the identity (3.58). As we did before, we express the four-momentum as the derivative with respect to $\tilde{\beta}_n$ of the exponential and, using the shorthand notation (3.62), we have:

$$T_C^{\mu\nu} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cosh n\zeta}{(2\pi)^3 n^2} \times \left(A^{\mu\alpha}(n) \frac{\partial}{\partial \tilde{\beta}_{n\nu}} \frac{\partial}{\partial \tilde{\beta}_n^\alpha} + (\Lambda^n)^\nu{}_\rho A^{\mu\alpha}(n) \frac{\partial}{\partial \tilde{\beta}_{n\rho}} \frac{\partial}{\partial \tilde{\beta}_n^\alpha} \right) \int \frac{d^3 p}{2\varepsilon} e^{-n\tilde{\beta}_n \cdot p}.$$

In the massless case, using the integral (3.66), the above formula reduces to:

$$T_C^{\mu\nu} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cosh n\zeta}{2\pi^2} \frac{1}{n^4 \tilde{\beta}_n^4} \left[A^\mu{}_\gamma(n) \Delta_{\tilde{\beta}_n}^{\gamma\nu} + (\Lambda^n)^\nu{}_\rho A^\mu{}_\gamma(n) \Delta_{\tilde{\beta}_n}^{\gamma\rho} \right], \quad (3.69)$$

where:

$$\Delta_{\tilde{\beta}_n}^{\mu\nu} = 4 \frac{\tilde{\beta}_n^\nu \tilde{\beta}_n^\mu}{\tilde{\beta}_n^2} - g^{\nu\mu}. \quad (3.70)$$

We have been able to express the basic expectation values (1.57) as series of functions of the vorticity and chemical potential for massless Dirac fermions. The computation of the integral in k allowed us to get rid of the δ -function, providing non-singular results. These series, however, are convergent only as long as the vorticity is imaginary, and they become divergent in the physical case. The goal of the next section is to associate a finite result to these series, which should also be analytic when the vorticity is zero in order to allow for a smooth limit to the standard Fermi-Dirac statistics results.

3.6 Exact mean values of currents at global thermodynamic equilibrium

We will now study the series obtained in the previous section in three notable examples of global equilibrium: with acceleration, with rotation, and with both acceleration and angular velocity parallel to each other. From the latter case, we will obtain by covariance arguments formulae holding for any angular velocity and acceleration. To get finite results, we will use the asymptotic formula (3.22) and the analytic distillation procedure as introduced in Section 3.1. We will always consider the massless Dirac field.

3.6.1 Acceleration

As extensively studied elsewhere [24, 25, 27, 28, 132–134], and briefly recalled in Section 1.2.2, a system at equilibrium with a constant acceleration along the z -axis is characterized by the following thermal vorticity and four-temperature:

$$\varpi_{\mu\nu} = \frac{a}{T_0} (g_{3\mu} g_{0\nu} - g_{0\mu} g_{3\nu}), \quad \beta^\mu = \frac{1}{T_0} (1 + az, 0, 0, at). \quad (3.71)$$

This is the easiest case where to apply the techniques of Section 3.1. An important simplification is given by the fact that one can calculate all mean values of scalar operators at $t = z = 0$ and their expression at a generic space-time point is recovered

by the simple substitution $\beta(0) \rightarrow \beta(x)$, as was discussed in Ref. [13]. For imaginary thermal vorticity, setting $\phi = ia/T_0$, the series (3.54) yields:

$$n\tilde{\beta}_n^\mu(t=0, z=0) = \left(\frac{\sinh(n\phi)}{T_0\phi}, 0, 0, \frac{1 - \cosh(n\phi)}{T_0\phi} \right), \quad (3.72)$$

hence:

$$\beta(0)^2 = \frac{1}{T_0^2}, \quad n^2\tilde{\beta}_n^2 = \beta(0)^2 \frac{4 \sinh^2\left(\frac{n\phi}{2}\right)}{\phi^2}.$$

From the above definition of ϖ and ϕ , the Lorentz transformations Λ and $S(\Lambda)$ from (1.45) and the traces $A^{\mu\nu}$ and $A_5^{\mu\nu}$, Eqs. (3.62), (3.63), read:

$$\begin{aligned} (\Lambda^n)^\mu{}_\nu &= \begin{pmatrix} \cosh n\phi & 0 & 0 & \sinh n\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh n\phi & 0 & 0 & \cosh n\phi \end{pmatrix}, \\ S(\Lambda)^n &= \begin{pmatrix} e^{n\phi/2} & 0 & 0 & 0 \\ 0 & e^{-n\phi/2} & 0 & 0 \\ 0 & 0 & e^{-n\phi/2} & 0 \\ 0 & 0 & 0 & e^{n\phi/2} \end{pmatrix}, \\ A^{\mu\nu} &= \begin{pmatrix} 4 \cosh \frac{n\phi}{2} & 0 & 0 & -4 \sinh \frac{n\phi}{2} \\ 0 & -4 \cosh \frac{n\phi}{2} & 0 & 0 \\ 0 & 0 & -4 \cosh \frac{n\phi}{2} & 0 \\ 4 \sinh \frac{n\phi}{2} & 0 & 0 & -4 \cosh \frac{n\phi}{2} \end{pmatrix}, \\ A_5^{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4i \sinh \frac{n\phi}{2} & 0 \\ 0 & -4i \sinh \frac{n\phi}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.73)$$

These expressions have to be fed into Eqs. (3.67), (3.68) and (3.69). We start from the energy-momentum tensor at $x^\mu = 0$ and, for simplicity, consider the case of vanishing chemical potential first. Denoting, from now on, the series with imaginary acceleration ϕ with an additional subscript I , the 00-component of the energy-momentum tensor reads:

$$T_C^{00}(0)_I = \frac{3}{8\pi^2} \frac{1}{\beta^4} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^4 \frac{\sinh n\phi}{\sinh^5\left(\frac{n\phi}{2}\right)}. \quad (3.74)$$

The series uniformly converges when ϕ has a non-vanishing real part, however, if we try to use the real acceleration a in place of ϕ the series becomes divergent: the function to be summed is not decreasing for increasing n . Since we expect the result of the expectation value to be analytic in $\phi = 0$, the analytic distillation can be applied to get a finite result. The asymptotic power series of the functions in $T_C^{00}(0)_I$ can be found from Eq. (3.22). We proceed explicitly for the sake of clarity. To begin with, we obtain the power expansion of the function in the series. This is a Laurent series:

$$\frac{\sinh \phi}{\sinh^5 \frac{\phi}{2}} = \frac{32}{\phi^4} - \frac{4}{3\phi^2} - \frac{17}{180} + \sum_{n=1}^{\infty} a_n \phi^{2n},$$

where we have split the terms of order $n \leq 0$, and the ones with $n > 0$. The above function is an even function of ϕ , which implies that all the powers of ϕ are even,

including the positive ones. Under these circumstances, the formula (3.22) yields a polynomial:

$$\phi^4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sinh n\phi}{\sinh^5\left(\frac{n\phi}{2}\right)} \sim \phi^4 \left(\frac{32\eta(4)}{\phi^4} - \frac{4\eta(2)}{3\phi^2} - \frac{17\eta(0)}{180} \right),$$

because the η -function is vanishing for negative even numbers. The above formula is accurate to the order of $O(\phi^N)$ for any N , which means that the difference between the series and the polynomial is a function that is non-analytic at $\phi = 0$. Therefore, the analytic distillation simply reads:

$$\begin{aligned} \text{dist}_{\phi=0} T_C^{00}(0)_I &= \text{dist}_{\phi=0} \left(\frac{3}{8\pi^2} \frac{\phi^4}{\beta^4} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sinh n\phi}{\sinh^5\left(\frac{n\phi}{2}\right)} \right) \\ &= \frac{7\pi^2}{60\beta^4} - \frac{\phi^2}{24\beta^4} - \frac{17\phi^4}{960\pi^2\beta^4}. \end{aligned}$$

We can use the same procedure for the other components, and we find:

$$\begin{aligned} \text{dist}_{\phi=0} T_C^{\text{off diag}} &= 0, \\ \text{dist}_{\phi=0} T_C^{11;22;33} &= \frac{1}{3} \text{dist}_{\phi=0} T^{00} = \frac{7\pi^2}{180\beta^4} - \frac{\phi^2}{72\beta^4} - \frac{17\phi^4}{2880\pi^2\beta^4}. \end{aligned}$$

The components of the energy-momentum tensor result in polynomials in ϕ , which can be easily continued to real vorticity. This is simply done by replacing $\phi \rightarrow ia/T_0$.

We can decompose the energy-momentum tensor along the four-velocity $u^\mu = \beta^\mu/\sqrt{\beta^2}$ and the four-acceleration over temperature $\alpha^\mu = \varpi^{\mu\nu}u_\nu$ [13, 134]:

$$T_C^{\mu\nu} = \rho u^\mu u^\nu - p(g^{\mu\nu} - u^\mu u^\nu) + \mathcal{A} \alpha^\mu \alpha^\nu,$$

After setting $\phi \mapsto ia/T_0$, we find:

$$\begin{aligned} \rho &= \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4}, \\ p &= \frac{7\pi^2}{180\beta^4} - \frac{\alpha^2}{72\beta^4} - \frac{17\alpha^4}{2880\pi^2\beta^4}, \\ \mathcal{A} &= 0, \end{aligned} \tag{3.75}$$

where we used the covariant notation $\alpha^2 = \alpha^\mu \alpha_\mu = -a^2/T_0^2$. These results have been calculated in the origin, but as remarked above they hold for all space-time points once we allow the four-temperature and α to be coordinate dependent.

Now that the case with $\zeta = 0$ has been solved, we can tackle the case of a non-vanishing chemical potential. The series associated with the 00-component of the energy-momentum tensor reads:

$$T_C^{00}(0)_I = \frac{3}{8\pi^2} \frac{1}{\beta^4} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^4 \frac{\cosh n\zeta \sinh n\phi}{\sinh^5\left(\frac{n\phi}{2}\right)}. \tag{3.76}$$

The asymptotic expansion of the function associated with this series is more challenging to obtain, as it is not a series of $f(n\phi)$ alone, but of $f(n\phi)g(n\zeta)$. We

can cope with this problem with a simple trick. We write ζ in terms of ϕ as, for example, $\zeta = \phi/L$. We have:

$$T_C^{00}(0)_I = \frac{3}{8\pi^2} \frac{1}{\beta^4} \sum_{n=1}^{\infty} (-1)^{n+1} \lim_{L \rightarrow \phi/\zeta} \phi^4 \frac{\cosh\left(n\frac{\phi}{L}\right) \sinh n\phi}{\sinh^5\left(\frac{n\phi}{2}\right)} \quad (3.77)$$

In the domain where it converges ($L > 2/3$), the above series is a uniformly convergent series of continuous functions, so we are allowed to exchange the limit and the series:

$$T_C^{00}(0)_I = \lim_{L \rightarrow \phi/\zeta} \frac{3}{8\pi^2} \frac{1}{\beta^4} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^4 \frac{\cosh\left(n\frac{\phi}{L}\right) \sinh n\phi}{\sinh^5\left(\frac{n\phi}{2}\right)}. \quad (3.78)$$

Now the same procedure as before can be applied since we expressed the series as a series of functions of $n\phi$ only and L is regarded as a constant parameter. Without repeating the full derivation, we obtain the following results for the components of the energy-momentum tensor:

$$T_C^{00}(0)_I = \frac{7\pi^2}{60\beta^4} - \frac{\phi^2}{24\beta^4} - \frac{17\phi^4}{960\pi^2\beta^4} + \frac{\phi^2}{2\beta^4 L^2} + \frac{\phi^4}{4\pi^2\beta^4 L^4} - \frac{\phi^4}{8\pi^2\beta^4 L^2}, \quad (3.79)$$

$$T_C^{\text{off diag}}(0)_I = 0, \quad (3.80)$$

$$T_C^{11;22;33}(0)_I = \frac{1}{3} T^{00}, \quad (3.81)$$

and after taking the limit and performing the analytic distillation and continuation of the result to the real thermal vorticity we have:

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4} + \frac{\zeta^2}{2\beta^4} + \frac{\zeta^4}{4\pi^2\beta^4} - \frac{\alpha^2\zeta^2}{8\pi^2\beta^4}, \quad (3.82)$$

$$p = \frac{1}{3}\rho, \quad (3.83)$$

$$\mathcal{A} = 0. \quad (3.84)$$

The results (3.75) and (3.82) are in agreement with the perturbative expansion to second-order in the acceleration of Refs. [24, 25] and to fourth-order of Refs. [26–28]. In the case of vanishing chemical potential, the results also agree with the expectation from the Unruh effect, as they are vanishing at the Unruh temperature $T_U = a/2\pi$ [28, 133]. This is because we used the *exact* Wigner function, therefore the results are improved with respect to those of Ref. [135], where only an approximated version of the Wigner function at equilibrium was used. Such an approximation led to the puzzling result that the expectation value of the energy density vanishes at $T_0 = a/\pi$ instead of the Unruh temperature $a/2\pi$. In contrast, the exact Wigner function is fully consistent with the Unruh effect, and it is possible to show that this is a general feature of expectation values obtained from the exact Wigner function and analytic distillation [13, 14].

For what concerns the vector and axial current, we have:

$$j^0(0)_I = \frac{\phi^3}{2\pi^2\beta^3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sinh n\zeta}{\sinh^3\left(\frac{\phi}{2}\right)}, \quad (3.85)$$

$$j^{1,2,3}(0)_I = 0, \quad (3.86)$$

$$j_5^\mu(0)_I = 0. \quad (3.87)$$

The axial current vanishes identically and the vector current is:

$$j^\mu = \frac{\zeta}{\sqrt{\beta^2}} \left(\frac{1}{3\beta^2} + \frac{\zeta^2}{3\pi^2\beta^2} - \frac{\alpha^2}{4\pi^2\beta^2} \right) u^\mu, \quad (3.88)$$

which again is consistent with Refs. [24, 25]. These results show that there are no higher-order corrections in acceleration and chemical potential to currents and to the energy-momentum tensor with respect to the results (3.82) and (3.88).

3.6.2 Rotation

We now turn to the global thermodynamic equilibrium with pure rotation. The thermal vorticity and the four-temperature are:

$$\varpi_{\mu\nu} = \frac{\omega}{T_0} (g_{\mu 1} g_{\nu 2} - g_{\mu 2} g_{\nu 1}), \quad \beta^\mu = \frac{1}{T_0} (1, -\omega y, \omega x, 0), \quad (3.89)$$

where ω is the constant angular velocity. Thus, the density operator of rotational thermodynamic equilibrium reads:

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\frac{\hat{H}}{T_0} + \frac{\omega}{T_0} \hat{\mathbf{J}}_z \right], \quad (3.90)$$

where $\hat{\mathbf{J}}$ is the three-vector of the generators of rotations. The tilde-transformed four-temperature vector for imaginary angular velocity (i.e., $\omega/T_0 = -i\phi$) is:

$$n\tilde{\beta}(in\phi) = \left(\frac{n}{T_0}, ix(\cos(n\phi) - 1) + iy \sin(n\phi), iy(\cos(n\phi) - 1) - ix \sin(n\phi), 0 \right),$$

and we have:

$$n^2 \tilde{\beta}_n^2 = \frac{n^2}{T_0^2} + 4r^2 \sin^2 \left(\frac{n\phi}{2} \right), \quad (3.91)$$

where $r^2 = x^2 + y^2$. From Eqs. (1.45), (3.62), (3.63) and (3.89), we obtain:

$$\begin{aligned} (\Lambda^n)^\mu{}_\nu &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos n\phi & -\sin n\phi & 0 \\ 0 & \sin n\phi & \cos n\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ S(\Lambda)^n &= \begin{pmatrix} e^{-in\phi/2} & 0 & 0 & 0 \\ 0 & e^{in\phi/2} & 0 & 0 \\ 0 & 0 & e^{-in\phi/2} & 0 \\ 0 & 0 & 0 & e^{in\phi/2} \end{pmatrix}, \\ A^{\mu\nu} &= \begin{pmatrix} 4 \cos(\frac{n\phi}{2}) & 0 & 0 & 0 \\ 0 & -4 \cos(\frac{n\phi}{2}) & 4 \sin(\frac{n\phi}{2}) & 0 \\ 0 & -4 \sin(\frac{n\phi}{2}) & -4 \cos(\frac{n\phi}{2}) & 0 \\ 0 & 0 & 0 & -4 \cos(\frac{n\phi}{2}) \end{pmatrix}, \\ A_5^{\mu\nu} &= \begin{pmatrix} 0 & 0 & 0 & 4i \sin \frac{n\phi}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4i \sin \frac{n\phi}{2} & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.92)$$

We start again by computing the energy-momentum tensor, considering an equilibrium state with a non-vanishing chemical potential. We choose to compute the

Belinfante energy-momentum tensor, which for the Dirac field has a simple relation with the canonical one:

$$T_B^{\mu\nu}(x) = \frac{1}{2} (T_C^{\mu\nu}(x) + T_C^{\nu\mu}(x)), \quad (3.93)$$

and it is thus easy to derive it from Eq. (3.69). Let us consider the 00-component. Using Eqs. (3.91) and (3.92) in Eq. (3.69) we obtain:

$$T_B^{00}(x)_I = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 8T_0^4 \cos\left(\frac{n\phi}{2}\right) \cosh n\zeta [3n^2 + 2r^2 T_0^2 (\cos n\phi - 1)]}{\left(n^2 + 4r^2 T_0^2 \sin^2\left(\frac{n\phi}{2}\right)\right)^3}. \quad (3.94)$$

Like in the case of equilibrium with acceleration, the series is convergent for real ϕ , but it becomes divergent if ϕ has a non-vanishing imaginary part: the denominator has infinitely many zeroes so that the series becomes densely divergent on the imaginary axis.

Additionally, the case of equilibrium with rotation presents another complication. The series (3.94) is a series of functions of $n\phi$ and $n\zeta$, and on top of that, there are also n^2 terms. Similarly to what we did with the chemical potential in the previous section, to put the series in a form where the asymptotic expansion (3.22) is applicable we introduce another auxiliary real parameter B , so that the series (3.94) can be written as:

$$T_B^{00}(x)_I = \frac{4T_0^4}{\pi^2} \sum_{n=1}^{\infty} \lim_{\substack{B \rightarrow \phi \\ L \rightarrow \phi/\zeta}} \frac{1}{\left(n^2 \phi^2 + 4r^2 B^2 T_0^2 \sin^2\left(\frac{n\phi}{2}\right)\right)^3} \times \\ (-1)^{n+1} B^4 \cos\left(\frac{n\phi}{2}\right) \cosh\left(n\frac{\phi}{L}\right) [3n^2 \phi^2 + 2r^2 B^2 T_0^2 (\cos n\phi - 1)].$$

We exploit once more the fact that, as long as ϕ is real, the series is a uniformly convergent series of continuous functions to exchange the limit with the sum and obtain a series which is now suitable for the application of the formula (3.22):

$$T_B^{00}(x)_I = \lim_{\substack{B \rightarrow \phi \\ L \rightarrow \phi/\zeta}} \frac{4B^4 T_0^4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{\left(n^2 \phi^2 + 4r^2 B^2 T_0^2 \sin^2\left(\frac{n\phi}{2}\right)\right)^3} \times \\ (-1)^{n+1} \cos\left(\frac{n\phi}{2}\right) \cosh\left(n\frac{\phi}{L}\right) [3n^2 \phi^2 + 2r^2 B^2 T_0^2 (\cos n\phi - 1)].$$

Using the asymptotic formula (3.22) we end up with a polynomial in ϕ , with L - and B -dependent coefficients:

$$T_B^{00}(x)_I \sim \lim_{B \rightarrow \phi} \lim_{L \rightarrow \phi/\zeta} \frac{7\pi^2 B^4 T_0^4}{60\phi^4 (1 + B^2 r^2 T_0^2)^3} - \frac{7\pi^2 B^6 r^2 T_0^6}{180\phi^4 (1 + B^2 r^2 T_0^2)^3} \\ + \frac{7B^6 r^2 T_0^6}{36\phi^2 (1 + B^2 r^2 T_0^2)^4} - \frac{B^8 r^4 T_0^8}{72\phi^2 (1 + B^2 r^2 T_0^2)^4} \\ - \frac{B^4 T_0^4}{8\phi^2 (1 + B^2 r^2 T_0^2)^4} + \frac{17B^{10} r^6 T_0^{10}}{2880\pi^2 (1 + B^2 r^2 T_0^2)^5} \\ + \frac{247B^8 r^4 T_0^8}{2880\pi^2 (1 + B^2 r^2 T_0^2)^5} - \frac{137B^6 r^2 T_0^6}{576\pi^2 (1 + B^2 r^2 T_0^2)^5} + \frac{B^4 T_0^4}{64\pi^2 (1 + B^2 r^2 T_0^2)^5}$$

$$\begin{aligned}
& + \frac{1}{L^2} \left(-\frac{B^8 r^4 T_0^8}{6\phi^2 (1 + B^2 r^2 T_0^2)^4} + \frac{B^6 r^2 T_0^6}{3\phi^2 (1 + B^2 r^2 T_0^2)^4} + \frac{B^4 T_0^4}{2\phi^2 (1 + B^2 r^2 T_0^2)^4} \right. \\
& \quad - \frac{B^{10} r^6 T_0^{10}}{24\pi^2 (1 + B^2 r^2 T_0^2)^5} + \frac{13B^8 r^4 T_0^8}{24\pi^2 (1 + B^2 r^2 T_0^2)^5} \\
& \quad \left. + \frac{5B^6 r^2 T_0^6}{24\pi^2 (1 + B^2 r^2 T_0^2)^5} - \frac{3B^4 T_0^4}{8\pi^2 (1 + B^2 r^2 T_0^2)^5} \right) \\
& + \frac{1}{L^4} \left(-\frac{B^{10} r^6 T_0^{10}}{12\pi^2 (1 + B^2 r^2 T_0^2)^5} + \frac{B^8 r^4 T_0^8}{12\pi^2 (1 + B^2 r^2 T_0^2)^5} + \frac{5B^6 r^2 T_0^6}{12\pi^2 (1 + B^2 r^2 T_0^2)^5} \right. \\
& \quad \left. + \frac{B^4 T_0^4}{4\pi^2 (1 + B^2 r^2 T_0^2)^5} \right). \tag{3.95}
\end{aligned}$$

Now, one can take the limits $L \rightarrow \frac{\phi}{\zeta}$ and $B \rightarrow \phi$, obtaining a function which implicitly provides the asymptotic power series of the original function (3.94) about $\phi = 0$. Indeed, all coefficients are analytic in B , so they can be always expressed as a power series. After the analytic distillation and the analytic continuation $\phi \mapsto i\omega$ we obtain:

$$\begin{aligned}
T_B^{00}(x) &= \frac{(4\gamma^2 - 1)\gamma^4(7\pi^4 + 30\pi^2\zeta^2 + 15\zeta^4)T_0^4}{180\pi^2} \\
&+ \frac{(24\gamma^4 - 16\gamma^2 + 1)\gamma^4(\pi^2 + 3\zeta^2)T_0^2\omega^2}{72\pi^2} \\
&+ \frac{(960\gamma^6 - 1128\gamma^4 + 196\gamma^2 + 17)\gamma^4\omega^4}{2880\pi^2},
\end{aligned}$$

where we defined:

$$\gamma = \frac{1}{\sqrt{1 - r^2\omega^2}}.$$

The calculations for the other components will not be reported, but the results are listed in Appendix B.

We can decompose the energy-momentum tensor in terms of an orthogonal set of vectors. We introduce a tetrad of orthogonal non-normalized vectors that can be defined starting from the four-temperature and its derivatives:

$$u^\mu = \frac{\beta^\mu}{\sqrt{\beta^2}}, \quad \alpha^\mu = \varpi^{\mu\nu} u_\nu, \quad w^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}u_\sigma, \quad l^\mu = \epsilon^{\mu\nu\rho\sigma}w_\nu\alpha_\rho u_\sigma. \tag{3.96}$$

Using these vectors, the variables T_0 , ω and r^2 can be expressed in terms of Lorentz invariants:

$$\beta^2 = \frac{1}{\gamma^2 T_0^2}, \quad \alpha^2 = -(\gamma^2 - 1)\frac{\omega^2}{T_0^2}, \quad w^2 = -\gamma^2\frac{\omega^2}{T_0^2}, \quad l^2 = -\alpha^2 w^2.$$

The tetrad introduced can be used to decompose the energy-momentum tensor as:

$$\begin{aligned}
T_B^{\mu\nu}(x) &= \rho u^\mu u^\nu - p \Delta^{\mu\nu} + W w^\mu w^\nu + A \alpha^\mu \alpha^\nu + G^l l^\mu l^\nu \\
&+ G(l^\mu u^\nu + l^\nu u^\mu) + \mathbb{A}(\alpha^\mu u^\nu + \alpha^\nu u^\mu) + G^\alpha(l^\mu \alpha^\nu + l^\nu \alpha^\mu) \\
&+ \mathbb{W}(w^\mu u^\nu + w^\nu u^\mu) + A^w(\alpha^\mu w^\nu + \alpha^\nu w^\mu) + G^w(l^\mu w^\nu + l^\nu w^\mu). \tag{3.97}
\end{aligned}$$

It should be mentioned that the decomposition of the symmetric Belinfante tensor now involves 11 scalar quantities, and not 10 as one would expect. This happens because the tetrad (3.96) is not a basis on the rotation axis $r = 0$, as $\alpha^\mu = l^\mu = 0$ and some degeneracy has to be introduced to amend this problem [13].

Such an ambiguity reflects in an ambiguous identification of pressure, as we will only be able to identify its combinations with other coefficients. In Refs. [13, 14] the fact was used that $l^2 = -\alpha^2 w^2$ to fix the pressure from the relation:

$$T_B^{\mu\nu} \frac{l_\nu l_\mu}{l^2 l^2} = G^l l^2 - p.$$

However, this procedure produces a pressure which does not satisfy the equation $p = \rho/3$, expected for massless particles. This is again a consequence of the 11-components decomposition of the energy-momentum tensor. Here we choose to fix the pressure to $p = \rho/3$ (notice that only p , W , A and G^l suffer from this ambiguity, the other coefficients are unambiguously identifiable), to obtain agreement with the expectations for conformal fluids.

With this prescription, we find:

$$\begin{aligned} \rho &= \frac{7\pi^4 + 30\pi^2\zeta^2 + 15\zeta^4}{60\pi^2\beta^4} - \frac{\alpha^2(\pi^2 + 3\zeta^2)}{24\pi^2\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4} - \frac{w^2(\pi^2 + 3\zeta^2)}{8\pi^2\beta^4} \\ &\quad + \frac{23\alpha^2 w^2}{1440\pi^2\beta^4} + \frac{w^4}{64\pi^2\beta^4}, \\ p &= \frac{\rho}{3}, \\ G^l &= -\frac{2}{27\pi^2\beta^4}, \\ G &= \frac{\pi^2 + 3\zeta^2}{18\pi^2\beta^4} - \frac{31\alpha^2}{360\pi^2\beta^4} - \frac{13w^2}{120\pi^2\beta^4}, \\ W &= -\frac{\alpha^2}{27\pi^2\beta^4}, \\ A &= -\frac{w^2}{27\pi^2\beta^4}, \\ \mathbb{A} = \mathbb{W} = G^\alpha = G^w = A^w &= 0. \end{aligned} \tag{3.98}$$

The energy-momentum tensor obtained is the same as the one obtained in Refs. [136–138] by solving the Dirac equation in cylindrical coordinates. In Ref. [138], the same series as those in Eq. (3.94) and in Appendix B with $\zeta = 0$ were obtained, and they were regularized with different methods. Notice that if we set $w = 0$ in the above coefficients, the energy-momentum tensor boils down to the case of equilibrium with acceleration only.

We conclude this section by reporting the exact mean value of the vector and axial current. The series (3.67) and (3.68) for the massless field can be studied for imaginary vorticity in the same way as the ones for the energy-momentum tensor. Further details can be found in Appendix B, here we only give the final result after analytic distillation:

$$j^\mu = \frac{\zeta}{\sqrt{\beta^2}} \left(\frac{\pi^2 + \zeta^2}{3\pi^2\beta^2} - \frac{\alpha^2}{4\pi^2\beta^2} + \frac{w^2}{4\pi^2\beta^2} \right) w^\mu - \frac{\zeta}{\sqrt{\beta^2}} \frac{1}{6\pi^2\beta^2} l^\mu, \tag{3.99a}$$

$$j_A^\mu = \frac{1}{\beta^2} \left(\frac{1}{6} + \frac{\zeta^2}{2\pi^2} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2} \right) \frac{w^\mu}{\sqrt{\beta^2}}. \tag{3.99b}$$

Again, these results are in agreement with the previous literature, both with perturbative [24, 25, 139] and exact [136, 137, 140] results. Note that the time-reversal symmetry prevents the axial current to be directed along the acceleration vector, and parity prevents the vector current to be directed along the angular velocity.

The presence of an axial current directed along the angular velocity is a well-established effect, known as Axial Vortical Effect [141]. Such current is expected to be non-vanishing both for massless and massive particles [25] and has the interesting feature of containing coefficients which are independent of the axial chemical potential and that are not constrained by the chiral anomaly [142]. This is different from the vector current, which vanishes identically if the chemical potential is set to zero.

It should be mentioned that the results obtained by analytic distillation in the case of equilibrium with rotation, although finite, have a more difficult interpretation compared with the case of equilibrium with acceleration. In fact, without imposing boundary conditions on the field to ensure causality, i.e., $\omega r < 1$, the operator $\widehat{H} - \omega \widehat{J}^3$ in the density operator is not bounded from below, leading to unphysical divergences [13, 14].

3.6.3 Thermodynamic equilibrium with rotation and acceleration

After having successfully reproduced the known results in the literature for equilibrium with acceleration and with rotation, we can tackle the more general case when both rotation and acceleration are present. This will be a useful ground to test the analytic distillation method, as it is very hard to tackle such a density operator with the curvilinear coordinates approach. This challenge has only recently been coped with in the Anti-De Sitter space-time [143].

We will study the particular case where acceleration and rotation are both directed along the z -axis, and express expectation values in terms of the covariant combinations of the vectors of the tetrad $\{u^\mu, \alpha^\mu, w^\mu, l^\mu\}$ defined in Eq. (3.96). Written in this way, the result will hold for any angular velocity and acceleration.

The vorticity and the four-temperature of a system at equilibrium with acceleration and angular velocity parallel to the z -axis are given by:

$$\varpi_{\mu\nu} = \frac{\omega}{T_0}(g_{\mu 1}g_{\nu 2} - g_{\nu 1}g_{\mu 2}) + \frac{a}{T_0}(g_{\mu 3}g_{\nu 0} - g_{\nu 3}g_{\mu 0}), \quad \beta^\mu = \frac{1}{T_0}(1 + az, -\omega y, \omega x, at).$$

Using the above-defined vorticity and four-temperature, one can see that the tilde-transformed temperature reads:

$$\begin{aligned} n\widetilde{\beta}_n = & \left(it(\cos(\frac{an}{T_0}) - 1) + \frac{(1 + az) \sin(\frac{an}{T_0})}{a}, \right. \\ & - y \sinh\left(\frac{n\omega}{T_0}\right) + ix \left(\cosh\left(\frac{n\omega}{T_0}\right) - 1\right), x \sinh\left(\frac{n\omega}{T_0}\right) + iy \left(\cosh\left(\frac{n\omega}{T_0}\right) - 1\right), \\ & \left. t \sin\left(\frac{an}{T_0}\right) + \frac{i(1 + az) \left(\cos\left(\frac{an}{T_0}\right) - 1\right)}{a} \right), \end{aligned}$$

the additional matrices required for expectation values read:

$$\begin{aligned}
 \Lambda_{\nu}^{\mu} &= \begin{pmatrix} \cosh n\Phi & 0 & 0 & \sinh n\Phi \\ 0 & \cos n\phi & -\sin n\phi & 0 \\ 0 & \sin n\phi & \cos n\phi & 0 \\ \sinh n\Phi & 0 & 0 & \cosh n\Phi \end{pmatrix}, \\
 S(\Lambda)^n &= \begin{pmatrix} e^{n(\Phi-i\phi)/2} & 0 & 0 & 0 \\ 0 & e^{-n(\Phi-i\phi)/2} & 0 & 0 \\ 0 & 0 & e^{-n(\Phi+i\phi)/2} & 0 \\ 0 & 0 & 0 & e^{n(\Phi+i\phi)/2} \end{pmatrix}, \\
 A^{\mu\nu} &= \begin{pmatrix} 4\cos(\frac{n\phi}{2})\cosh(\frac{n\Phi}{2}) & 0 & 0 & -4\cos(\frac{n\phi}{2})\sinh(\frac{n\Phi}{2}) \\ 0 & -4\cos(\frac{n\phi}{2})\cosh(\frac{n\Phi}{2}) & 4\sin(\frac{n\phi}{2})\cosh(\frac{n\Phi}{2}) & 0 \\ 0 & -4\sin(\frac{n\phi}{2})\cosh(\frac{n\Phi}{2}) & -4\cos(\frac{n\phi}{2})\cosh(\frac{n\Phi}{2}) & 0 \\ 4\cos(\frac{n\phi}{2})\sinh(\frac{n\Phi}{2}) & 0 & 0 & -4\cos(\frac{n\phi}{2})\cosh(\frac{n\Phi}{2}) \end{pmatrix}, \\
 A_5^{\mu\nu} &= \begin{pmatrix} -4i\sin\frac{n\phi}{2}\sinh\frac{n\Phi}{2} & 0 & 0 & 4i\sin\frac{n\phi}{2}\cosh\frac{n\Phi}{2} \\ 0 & 4i\sin\frac{n\phi}{2}\sinh\frac{n\Phi}{2} & 4i\cos\frac{n\phi}{2}\sinh\frac{n\Phi}{2} & 0 \\ 0 & -4i\cos\frac{n\phi}{2}\sinh\frac{n\Phi}{2} & 4i\sin\frac{n\phi}{2}\sinh\frac{n\Phi}{2} & 0 \\ 4i\sin\frac{n\phi}{2}\cosh\frac{n\Phi}{2} & 0 & 0 & 4i\sin\frac{n\phi}{2}\sinh\frac{n\Phi}{2} \end{pmatrix}, \tag{3.100}
 \end{aligned}$$

where we performed the continuation $a/T_0 \rightarrow -i\Phi$, $\omega/T_0 \rightarrow -i\phi$.

Once again, we address the Belinfante tensor for a system of massless fermions. For this kind of equilibrium, the procedure of distillation is extremely long and tedious, therefore it is only briefly sketched in Appendix C. Here we just report the coefficients of the decomposition (3.97). We can define the tetrad $\{u^\mu, \alpha^\mu, w^\mu, l^\mu\}$ like we did in Eq. (3.96), however in this case the scalar product $\alpha \cdot w = -a\omega/T_0^2$ is non-vanishing. Imposing $p = \rho/3$ for the pressure, we find:

$$\begin{aligned}
 \rho &= \frac{7\pi^4 + 30\pi^2\zeta^2 + 15\zeta^4}{60\pi^2\beta^4} - \frac{\alpha^2(\pi^2 + 3\zeta^2)}{24\pi^2\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4} - \frac{w^2(\pi^2 + 3\zeta^2)}{8\pi^2\beta^4} \\
 &\quad + \frac{23\alpha^2w^2}{1440\pi^2\beta^4} + \frac{w^4}{64\pi^2\beta^4} + \frac{11(\alpha \cdot w)^2}{720\pi^2\beta^4}, \\
 p &= \frac{\rho}{3}, \\
 G^l &= -\frac{2}{27\pi^2\beta^4}, \\
 G &= \frac{\pi^2 + 3\zeta^2}{18\pi^2\beta^4} - \frac{31\alpha^2}{360\pi^2\beta^4} - \frac{13w^2}{120\pi^2\beta^4}, \tag{3.101} \\
 W &= -\frac{\alpha^2}{27\pi^2\beta^4}, \\
 A &= -\frac{w^2}{27\pi^2\beta^4}, \\
 A^w &= \frac{\alpha \cdot w}{27\pi^2\beta^4}, \\
 \mathbb{A} = \mathbb{W} = G^\alpha = G^w &= 0.
 \end{aligned}$$

These coefficients are consistent with the ones found in the case of equilibrium with pure-rotation, Eq. (3.98). For this equilibrium, we have an additional dependence

of the energy density and pressure on the scalar $\alpha \cdot w$, which was vanishing in both the previous cases. On top of that, the coefficient A^w is non-vanishing, and it is proportional to $\alpha \cdot w$. One can also study the vector and axial currents at equilibrium with acceleration and rotation. In this case, they turn out to be precisely equal to the ones obtained in equilibrium with rotation, Eq. (3.99), with no extra dependence on $\alpha \cdot w$. For a discussion about the series involved in this calculation, see Appendix C.

3.7 Massless particles and chiral kinetic theory

In recent times, the kinetic theory of massless fermions, known as chiral kinetic theory, has attracted much attention [144–147]. Usually, when addressing quantum corrections to classical kinetic theory, one performs a semi-classical expansion in \hbar of the Wigner equation, leading to Boltzmann-like equations for the distribution function. Now that the exact Wigner function at global equilibrium has been provided, we can try to infer the distribution function of massless Dirac fermions from it.

We focus on the particle part of the current, Eq. (1.58), and since different helicities do not mix in the massless case, we have:

$$j_+^\mu(x) = \frac{1}{(2\pi)^3} \sum_\lambda \int \frac{d^3p}{2\varepsilon} \frac{d^3p'}{2\varepsilon'} e^{i(p'-p)\cdot x} \langle \hat{a}_\lambda^\dagger(p') \hat{a}_\lambda(p) \rangle \bar{u}_\lambda(p') \gamma^\mu u_\lambda(p), \quad (3.102)$$

where λ is the helicity, taking values $-1/2$ and $1/2$.

One can identify the phase-space current-density $\mathcal{J}^\mu(p, x)$ as defined in Eq. (1.59) by reworking the integrand in Eq. (3.102). In principle, this can be done by calculating explicitly the value of $\bar{u}_\lambda(p') \gamma^\mu u_\lambda(p)$ (see Ref. [14]). Alternatively, we can proceed in a geometrical fashion, projecting \mathcal{J} along the tetrad $\{p, q, n_1, n_2\}$ employed in Section 1.5 for massless particles.

To begin with, we define the vector $L^\mu = \bar{u}_\lambda(p') \gamma^\mu u_\lambda(p)$ and we observe that it is a light-like vector for massless particles, as can be checked by explicit calculation. Denoting for brevity N^μ the vector space spanned by n_1 and n_2 , the decomposition of L^μ is:

$$L^\mu = \frac{L \cdot q}{q \cdot p} p^\mu + \frac{L \cdot p}{q \cdot p} q^\mu + N^\mu(p, q, L)$$

(this can be seen as the definition of N). Using the Dirac equation it is easy to see that $L \cdot p = 0$, therefore:

$$\bar{u}_\lambda(p') \gamma^\mu u_\lambda(p) = \frac{\bar{u}_\lambda(p') \not{q} u_\lambda(p)}{q \cdot p} p^\mu + N^\mu.$$

Now we can plug this decomposition into the current (3.102), obtaining:

$$\begin{aligned} j_+^\mu(x) &= \frac{1}{(2\pi)^3} \sum_\lambda \int \frac{d^3p}{\varepsilon} p^\mu \int \frac{d^3p'}{2\varepsilon'} \frac{\bar{u}_\lambda(p') \not{q} u_\lambda(p)}{2q \cdot p} e^{i(p'-p)\cdot x} \langle \hat{a}_\lambda^\dagger(p') \hat{a}_\lambda(p) \rangle \\ &+ \frac{1}{(2\pi)^3} \sum_\lambda \int \frac{d^3p}{\varepsilon} N^\mu \int \frac{d^3p'}{4\varepsilon'} e^{i(p'-p)\cdot x} \langle \hat{a}_\lambda^\dagger(p') \hat{a}_\lambda(p) \rangle. \end{aligned}$$

This equation defines the phase space current-density

$$\begin{aligned} \mathcal{J}^\mu &= p^\mu \int \frac{d^3p'}{2\varepsilon'} \frac{\bar{u}_\lambda(p') \not{q} u_\lambda(p)}{2q \cdot p} e^{i(p'-p)\cdot x} \langle \hat{a}_\lambda^\dagger(p') \hat{a}_\lambda(p) \rangle \\ &+ N^\mu \int \frac{d^3p'}{4\varepsilon'} e^{i(p'-p)\cdot x} \langle \hat{a}_\lambda^\dagger(p') \hat{a}_\lambda(p) \rangle. \end{aligned}$$

The decomposition depends on the choice of the vector q , but the current itself is independent thereof. In other words, the components and the decomposition of any vector depend on the chosen basis, but the vector is fixed in space-time. In chiral kinetic theory, the distribution function is given by the component along p^μ of the phase space density of the current. We can identify a q -dependent distribution function $f_\lambda(x, p)_{(q)}$ for particles with helicity λ as the coefficient of p^μ in $\mathcal{J}^\mu(p, x)$:

$$f_\lambda(x, p)_{(q)} \equiv \frac{1}{(2\pi)^3} \int \frac{d^3 p'}{2\varepsilon'} e^{i(p'-p)\cdot x} \langle \hat{a}_\lambda^\dagger(p') \hat{a}_\lambda(p) \rangle \frac{\bar{u}_\lambda(p') \not{q} u_\lambda(p)}{2q \cdot p}. \quad (3.103)$$

Using $\bar{u}_\lambda(p') \not{q} u_\lambda(p) = \text{tr}(\not{q} u_\lambda(p) \bar{u}_\lambda(p'))$, and introducing the projectors:

$$P_\lambda = \frac{\mathbb{I} + 2\lambda\gamma_5}{2}, \quad (3.104)$$

that select spinors with helicity λ we can write:

$$\begin{aligned} f_\lambda(x, p)_{(q)} &= \frac{1}{(2\pi)^3} \int \frac{d^3 p'}{2\varepsilon'} e^{i(p'-p)\cdot x} \langle \hat{a}_\lambda^\dagger(p') \hat{a}_\lambda(p) \rangle \frac{\text{tr}(\not{q} u_\lambda(p) \bar{u}_\lambda(p'))}{2q \cdot p} \\ &= \frac{1}{(2\pi)^3} \int \frac{d^3 p'}{2\varepsilon'} \frac{e^{i(p'-p)\cdot x}}{2q \cdot p} \sum_{s,t} \text{tr} \left(\not{q} P_\lambda \langle \hat{a}_t^\dagger(p') \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p') P_{-\lambda} \right) \\ &= \frac{1}{(2\pi)^3} \frac{1}{2q \cdot p} \text{tr} \left(\not{q} P_\lambda \int \frac{d^3 p'}{2\varepsilon'} e^{i(p'-p)\cdot x} \sum_{s,t} \langle \hat{a}_t^\dagger(p') \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p') \right), \end{aligned}$$

where we used the anticommutation rules of γ_5 together with $\gamma_5 u_\lambda(p) = 2\lambda u_\lambda(p)$, where 2λ is the chirality. Also notice that $(2\lambda)^2 = 1$.

The same product $\langle \hat{a}_t^\dagger(p') \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p')$ appears also in Eq. (1.54) and has been dealt with in Section 3.4.1. The same calculations yield:

$$\int \frac{d^3 p'}{2\varepsilon'} e^{i(p'-p)\cdot x} \sum_{s,t} \langle \hat{a}_t^\dagger(p') \hat{a}_s(p) \rangle u_s(p) \bar{u}_t(p') = \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} S(\Lambda)^n (\not{p} + m).$$

Plugging this result in the trace above, we have:

$$f_\lambda(x, p)_{(q)} = \frac{1}{(2\pi)^3} \frac{1}{2p \cdot q} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} \text{tr} \left(\frac{\mathbb{I} + 2\lambda\gamma_5}{2} S(\Lambda)^n \not{p} \not{q} \right).$$

The distribution function $f_\lambda(x, p)_{(q)}$ can be further simplified using the algebra of the γ -matrices to rewrite $\not{p} \not{q}$, finally obtaining:

$$\begin{aligned} f_\lambda(x, p)_{(q)} &= \text{dist}_{\varpi=0} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2(2\pi)^3} e^{-n\tilde{\beta}(-n\varpi) \cdot p} \times \\ &\quad \left\{ \text{tr} \left(\frac{I + 2\lambda\gamma_5}{2} \exp \left[n \frac{\varpi_{\rho\sigma}}{2} \Sigma^{\rho\sigma} \right] \right) + \frac{2iq_\mu p_\nu}{q \cdot p} \text{tr} \left(\frac{I + 2\lambda\gamma_5}{2} \Sigma^{\mu\nu} \exp \left[n \frac{\varpi_{\rho\sigma}}{2} \Sigma^{\rho\sigma} \right] \right) \right\}, \end{aligned} \quad (3.105)$$

where we have reintroduced the physical thermal vorticity, together with the operation of analytic distillation to regularise the series. This expression of the distribution function is in agreement with the general form (3.61), considering the more subtle decomposition of the current density in the massless case.

Notice how Eq. (3.105) differs from the usual *ansatz* of the equilibrium distribution function [47, 146–149], mostly for the exponential factor and for the absence of the frame vector in favour of the light-like vector q . The vector q has a geometrical origin, as it identifies *the only* light-like direction non-orthogonal to p sharing the same space-like orthogonal space.

3.8 Massive Dirac fermions

This chapter has been mostly concerned with massless particles. However, the Wigner function has been computed exactly also in the massive case so that we can, in principle, compute exact expectation values also for $m \neq 0$. This task appears much more difficult than in the massless case. Here we will comment on the matter, focusing on the vector and axial currents. Up until Eqs. (3.64) and (3.65), the relations found hold both for massive and massless particles. However, the value of the integral in those equations changes depending on the mass. Indeed, in the massive case we have:

$$\int \frac{d^3p}{2\varepsilon} e^{-n\tilde{\beta}_n \cdot p} = 2\pi \int_0^\infty dp \frac{p^2}{\sqrt{m^2 + p^2}} e^{-n\sqrt{\beta_n^2} \sqrt{m^2 + p^2}} = \frac{2\pi m K_1(mn\sqrt{\beta_n^2})}{n\sqrt{\beta_n^2}},$$

where to compute the integral we used the Lorentz invariance of the measure to boost $\tilde{\beta}_n$ to the frame where it has only the time component. Here K_1 is the modified Bessel function of second kind [150]:

$$K_n(z) = \frac{2^n n!}{(2n)!} \frac{1}{z^n} \int_z^\infty dy (y^2 - z^2)^{n-1/2} e^{-y}.$$

Computing the derivative in Eqs. (3.64) and (3.65), the vector and axial current read:

$$j^\mu(x) = \frac{m^2}{2\pi^2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} A_\nu^\mu \sinh(n\zeta) \frac{\tilde{\beta}_n^\nu}{\beta_n^2} K_2(mn\sqrt{\beta_n^2}), \quad (3.106a)$$

$$j_5^\mu(x) = \frac{m^2}{2\pi^2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} A_{5\nu}^\mu \cosh(n\zeta) \frac{\tilde{\beta}_n^\nu}{\beta_n^2} K_2(mn\sqrt{\beta_n^2}). \quad (3.106b)$$

In the limiting case of vanishing mass, given the asymptotic behaviour of the Bessel function:

$$\lim_{z \rightarrow 0} K_n(z) = \frac{2^{n-1}(n-1)!}{z^n}, \quad \text{for } n > 0,$$

these formulae reproduce Eqs. (3.67) and (3.68) as expected.

From the above expressions, we can see that the direction of the massive currents is given by the contraction of $\tilde{\beta}_n$ and $A_{(5)}^{\mu\nu}$, exactly like the massless case. Therefore, the components of the currents that were vanishing in the massless case will be vanishing also in the massive one.

The exact calculation of the analytic result of these series is, unfortunately, out of our reach at this point: the expansion of the modified Bessel function in terms of the thermal vorticity is too complicated. Nonetheless, it is possible to compute the mass corrections order by order expanding the Bessel function in terms of m . This will allow us to compare with the previous literature.

We focus on the first relevant mass correction. Using, for $n \in \mathbb{N}^+$:

$$K_n(z) \sim \frac{2^{n-1}(n-1)!}{z^n} - \frac{2^{n-2}(n-2)!}{z^{n-3}},$$

we find that the leading-order mass correction to the currents reads:

$$\delta_m j^\mu(x) = -\frac{m^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A^\mu{}_\nu \sinh(n\zeta) \frac{\tilde{\beta}_n^\nu}{\beta_n^2}, \quad (3.107)$$

$$\delta_m j_5^\mu(x) = -\frac{m^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} A_5^\mu{}_\nu \cosh(n\zeta) \frac{\tilde{\beta}_n^\nu}{\beta_n^2}. \quad (3.108)$$

The procedure to extract the results from these series has already been discussed in the previous sections, here we report only the results.

For equilibrium with acceleration the axial current is identically zero, whereas the mass correction to the vector current is:

$$\delta_m j^\mu = -\frac{m^2 \zeta}{2\pi^2} \frac{u^\mu}{\sqrt{\beta^2}}. \quad (3.109)$$

For equilibrium with rotation, as well as for equilibrium with rotation and acceleration, we find:

$$\delta_m j^\mu = -\frac{m^2 \zeta}{2\pi^2} \frac{u^\mu}{\sqrt{\beta^2}}, \quad (3.110)$$

$$\delta_m j_5^\mu = -\frac{m^2}{4\pi^2} \frac{w^\mu}{\sqrt{\beta^2}}. \quad (3.111)$$

These results are in agreement with the previous literature [24, 143, 151, 152]. In principle, we could get higher order corrections in mass further expanding the Bessel function, but we do not pursue such an improvement here. The full summation of the series (3.106) is postponed to future studies.

Chapter 4

Spin physics in global equilibrium

In the previous chapter, we introduced a new method to compute expectation values, in particular the Wigner function. Using the latter we can study the polarization vector to all orders in vorticity, which is the aim of the present chapter.

We begin by obtaining the expressions of the spin density matrix and the spin vector for Dirac fermions in global equilibrium, showing that they are consistent with the previous literature. Both quantities are expressed as series of functions, which we sum to obtain an analytic expression for the spin vector of spin- $\frac{1}{2}$ fermions. We compare the predictions of the exact formula with the experimental data and with the linear approximation usually employed in polarization studies, showing that higher-order corrections in the thermal vorticity cannot solve the sign puzzle.

We end the chapter by studying polarization for any spin and expressing the spin vector as a series of functions which can be summed for particles with generic spin S . The small-vorticity and the Boltzmann-statistics limits of our result agree with the previous literature.

4.1 Polarization of Dirac fermions

At this point of the present work, the most convenient way to compute spin-related effects for Dirac fermions in global equilibrium is to use the results of the previous chapter, together with the expressions of the spin density matrix and the spin vector (massive or massless) in terms of the Wigner function. For the free Dirac field, these relations read (see Eqs. (1.93), (1.94) and (1.102)):

$$\Theta_m(p) = \frac{1}{2m} \frac{\int d\Sigma \cdot p \bar{U}(p) W_+(x, p) U(p)}{\text{tr}_2 \left(\int d\Sigma \cdot p W_+(x, p) \right)}, \quad (4.1)$$

$$S^\mu(p) = \frac{1}{2} \frac{\int d\Sigma \cdot p \text{tr}(\gamma^\mu \gamma_5 W_+(x, p))}{\int d\Sigma \cdot p \text{tr}(W_+(x, p))}, \quad (4.2)$$

$$W^\mu(p) = \frac{p^\mu}{2} \frac{\int d\Sigma \cdot p \text{tr}(q \gamma_5 W_+(x, p))}{\int d\Sigma \cdot p \text{tr}(q W_+(x, p))}, \quad (4.3)$$

In this chapter, where global-equilibrium is considered, Σ is an arbitrary space-like hypersurface. Indeed, due to its very definition, global-equilibrium quantities do not depend on the hypersurface they are computed on, as the equilibrium state is

globally defined. We have also used the subscript m to make clear that the spin density matrix in Eq. (4.1) is in the massive case.

The Wigner function in global equilibrium has been found in the previous chapter, and is given by Eq. (3.56):

$$W(x, k) = \frac{1}{(2\pi)^3} \int \frac{d^3 p}{2\varepsilon} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{\beta}_n \cdot p} \times \left[e^{n\zeta} S(\Lambda)^n (m + \not{p}) \delta^4 \left(k - \frac{\Lambda^n p + p}{2} \right) + e^{-n\zeta} (m - \not{p}) S(\Lambda)^{-n} \delta^4 \left(k + \frac{\Lambda^n p + p}{2} \right) \right],$$

where $\Lambda = \exp \left[-i \frac{\phi}{2} : J \right]$, and the tilde-transformed four-temperature $\tilde{\beta}_n$ reads:

$$\tilde{\beta}_n = \sum_{k=0}^{\infty} \frac{n^k}{(k+1)!} \underbrace{(\phi_{\alpha_1}^{\mu} \phi_{\alpha_2}^{\alpha_1} \dots \phi_{\alpha_k}^{\alpha_{k-1}})}_{k \text{ times}} \beta^{\alpha_k}.$$

We recall that Λ is the Lorentz transformation with parameter ϕ corresponding to the analytic continuation of the density operator through the map $\varpi \mapsto -i\phi$. In this respect, ϕ can be considered as an imaginary thermal vorticity (see the previous chapter for details). All results obtained from these formulae have to be eventually continued back to the real physical vorticity ϖ .

Choosing $t = 0$ as an integration surface, the integral of the Wigner function over $d^3 x$ yields (here, we set $\zeta = 0$ for simplicity):

$$\int_{\Sigma} d\Sigma_{\mu} k^{\mu} W_{+}(x, k) = \delta(k^0 - \varepsilon_k) \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot k} S(\Lambda)^n (m + \not{k}) \delta^3(\Lambda^n k - k), \quad (4.4)$$

which makes k manifestly on-shell, as expected from Eq. (1.87). From this result, we can evaluate Eqs. (4.1), (4.2) and (4.3).

Considering massive particles to begin with, and using Eq. (4.4), the spin density matrix reads:

$$\Theta_m(p) = \frac{1}{2m^2} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} \text{tr}(\gamma^{\mu} \gamma_5 \exp[-in\phi : \Sigma/2] U(p) \delta^3(\Lambda^n p - p))}{\text{tr}_2 \left(\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} \exp[-in\phi : \Sigma/2] \delta^3(\Lambda^n p - p) \right)}, \quad (4.5)$$

where the Dirac equation for the spinors U , $\not{p}U(p) = mU(p)$, has been used. Similarly, the spin vector for massive particles (4.2) at global thermodynamic equilibrium is:

$$S^{\mu}(p) = \frac{1}{2m} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} \text{tr}(\gamma^{\mu} \gamma_5 \exp[-in\phi : \Sigma/2] \not{p}) \delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} \text{tr}(\exp[-in\phi : \Sigma/2]) \delta^3(\Lambda^n p - p)} \quad (4.6)$$

$$= \frac{1}{2m} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} p_{\nu} A_5^{\mu\nu}(n, \phi) \delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} \text{tr}(\exp[-in\phi : \Sigma/2]) \delta^3(\Lambda^n p - p)},$$

where we used the shorthand notation of $A_5^{\mu\nu}(n, \phi) = \text{tr} \left(\gamma^{\nu} \gamma^{\mu} \gamma_5 \exp \left[-i \frac{n\phi}{2} : \Sigma \right] \right)$, which was defined in Eq. (3.63). Notice that both Eqs. (4.5) and (4.6) are consistent with the Unruh effect. Indeed, for the global equilibrium with pure acceleration,

described in Subsection 3.6.1, we have:

$$e^{-in\phi:\Sigma/2} = \begin{pmatrix} e^{n\phi/2} & 0 & 0 & 0 \\ 0 & e^{-n\phi/2} & 0 & 0 \\ 0 & 0 & e^{-n\phi/2} & 0 \\ 0 & 0 & 0 & e^{n\phi/2} \end{pmatrix},$$

$$A_5^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 4i \sinh \frac{n\phi}{2} & 0 \\ 0 & -4i \sinh \frac{n\phi}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If we now set $\phi = i\frac{a}{T} = 2i\pi$, which corresponds to the Unruh temperature, the Lorentz transformation becomes $\exp[-in\phi : \Sigma/2] = (-1)^n \mathbb{I}$ and the A_5 tensor vanishes, so that the spin density matrix is proportional to the identity matrix and the mean spin vector is zero.

For massless particles, the spin vector (4.3) reads:

$$W^\mu(p) = p^\mu \frac{1}{2} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi)\cdot p} p_\nu q_\rho A_5^{\rho\nu}(n, \phi) \delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi)\cdot p} \text{tr}(\exp[-in\phi : \Sigma/2] \not{p} \not{q}) \delta^3(\Lambda^n p - p)} \quad (4.7)$$

and it is also consistent with the Unruh effect, for the same reason as before.

The results of the series (4.5), (4.6) and (4.7) look very difficult to compute exactly, as they are ratios of series of δ -functions; this problem will be tackled in the next section. For the time being, we consider the small-vorticity limit of the spin vectors (4.6) and (4.7), comparing with the previous literature. For this purpose, also the δ -function has to be expanded. Up to linear order, and using $\phi \mapsto i\varpi$, one has:

$$A_5 = \text{tr}(\gamma^\mu \gamma_5 S(\Lambda)^n \not{p}) \sim -n\varpi_{\alpha\beta} p_\nu \epsilon^{\mu\alpha\beta\nu} + \mathcal{O}(\varpi^2), \quad e^{-n\tilde{b}(-n\varpi)\cdot p} \sim e^{-b\cdot p} + \mathcal{O}(\varpi),$$

and only the 0-th order expansion of $\delta^3(\Lambda^n p - p) \sim \delta^3(0)$ has to be taken into account. Summing the series in n , the result is:

$$S^\mu(p) \sim -\frac{1}{8m} \epsilon^{\mu\alpha\beta\nu} \varpi_{\alpha\beta} p_\nu \frac{n_F(b\cdot p)(1 - n_F(b\cdot p))\delta^3(0)}{n_F(b\cdot p)\delta^3(0)}$$

$$= -\frac{1}{8m} \epsilon^{\mu\alpha\beta\nu} \varpi_{\alpha\beta} p_\nu (1 - n_F(b\cdot p)), \quad (4.8)$$

$$W^\mu(p) \sim -\frac{p^\mu}{8p\cdot q} \epsilon^{\rho\alpha\beta\nu} \varpi_{\alpha\beta} p_\nu q_\rho \frac{n_F(b\cdot p)(1 - n_F(b\cdot p))\delta^3(0)}{n_F(b\cdot p)\delta^3(0)}$$

$$= \frac{p^\mu}{8p\cdot q} \epsilon^{\alpha\beta\nu\rho} \varpi_{\alpha\beta} p_\nu q_\rho (1 - n_F(b\cdot p)), \quad (4.9)$$

where n_F is the Fermi-Dirac distribution function

$$n_F(b\cdot p) = \frac{1}{e^{b\cdot p} + 1}. \quad (4.10)$$

The polarization of massive fermions (4.8) is in agreement with the local-equilibrium expression found in Ref. [7], whereas the massless one depends on the vector q^μ : this is required to assign the physical meaning of helicity to the mean Pauli-Lubanski vector as argued in Section 1.6. Equation (4.9) differs from the spin vector of massless

particles usually reported in the literature, which in global equilibrium would read [149]:

$$W^\mu(p) = -\frac{1}{8p \cdot u} \epsilon^{\mu\alpha\beta\nu} \varpi_{\alpha\beta} p_\nu (1 - n_F(b \cdot p)), \quad (4.11)$$

but the above vector has unphysical components not aligned with p^μ .

It is interesting to rework the massless spin vector, computing explicitly the contraction of the Levi-Civita symbol with p and q . First of all, from the fact that in the standard frame $\mathbf{q}^\mu = (\kappa, 0, 0, -\kappa)$ is the parity conjugate of $\mathbf{p}^\mu = (\kappa, 0, 0, \kappa)$, it is possible to show that $q^\mu = [p]^\mu{}_\nu q^\nu = \kappa^2(1/\varepsilon, -\mathbf{p}/\varepsilon^2)$, where $\varepsilon = |\mathbf{p}|$ and we recall $p^\mu = [p]^\mu{}_\nu p^\nu$. Also, since $p \cdot q$ is a Lorentz invariant, we can compute the scalar product in the standard frame to find $p \cdot q = 2\kappa^2$. Thus, keeping the covariant notation (i.e. $p_i = g_{i\mu} p^\mu$):

$$\begin{aligned} \epsilon^{\alpha\beta\nu\rho} \varpi_{\alpha\beta} \frac{p_\nu q_\rho}{p \cdot q} &= \epsilon^{\alpha\beta\nu 0} \varpi_{\alpha\beta} \frac{p_\nu q_0}{2\kappa^2} + \epsilon^{\alpha\beta\nu i} \varpi_{\alpha\beta} \frac{p_\nu q_i}{2\kappa^2} = \frac{1}{2\varepsilon} \epsilon^{\alpha\beta i 0} \varpi_{\alpha\beta} p_i - \epsilon^{\alpha\beta 0 i} \varpi_{\alpha\beta} p_0 \frac{p_i}{2\varepsilon^2} \\ &= \frac{1}{\varepsilon} \epsilon^{\alpha\beta i 0} \varpi_{\alpha\beta} p_i = -2w \cdot \hat{p}, \end{aligned} \quad (4.12)$$

with

$$w^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho 0} \varpi_{\nu\rho} = \begin{pmatrix} 0 \\ \mathbf{w} \end{pmatrix}, \quad \hat{p}^\mu = \frac{p^\mu}{\varepsilon} = \begin{pmatrix} 1 \\ \hat{\mathbf{p}} \end{pmatrix}. \quad (4.13)$$

Therefore, the spin vector of massless particles at first order in vorticity can be written as:

$$W^\mu(p) = \frac{p^\mu}{4} \mathbf{w} \cdot \hat{\mathbf{p}} (1 - n_F(b \cdot p)), \quad (4.14)$$

where the scalar product in the three-space is used. Notice how the vector q no longer explicitly appears.

4.2 Exact polarization of Dirac fermions at global equilibrium

We are now in the position to compute exactly the polarization of massive and massless particles, using Eq. (4.6) and (4.7).

We start by considering the massive case with $\zeta = 0$, but we will see that the massless case can be treated similarly. The spin vector for massive particles, according to Eq. (4.6), is:

$$S^\mu(p) = \frac{1}{2m} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} p_\nu A_5^{\mu\nu}(n, \phi) \delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} \text{tr}(\exp[-in\phi : \Sigma/2]) \delta^3(\Lambda^n p - p)}.$$

The above expression is the ratio of two series of δ -functions. It looks quite daunting. However, one ought to consider how this ratio comes about, and for this purpose, it is useful to revise the definition of the spin density matrix together with the value of $\langle \hat{a}_s^\dagger(p) \hat{a}_r(p') \rangle$, which was obtained with the iterative method resulting in Eq. (3.47).

The spin density matrix has been *defined* via the ratio:

$$\Theta_{rs}(p) = \frac{\text{Tr}(\hat{\rho} \hat{a}_s^\dagger(p) \hat{a}_r(p))}{\sum_t \text{Tr}(\hat{\rho} \hat{a}_t^\dagger(p) \hat{a}_t(p))}, \quad (4.15)$$

and it involves the expectation value of the number operator of particles with momentum p . The denominator is such that the trace of the spin density matrix is one. The numerator of the spin density matrix has been calculated in Section 3.4, where the inhomogeneous solution of Eq. (3.46) was found. We recall that the homogeneous solutions of such an equation are non-analytic functions at $\phi = 0$, and are therefore ignored. We can use Eq. (3.47) to evaluate the numerator of the spin density matrix:

$$\text{Tr} \left(\widehat{\rho} \widehat{a}_s^\dagger(p) \widehat{a}_r(p) \right) = 2\varepsilon \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\Lambda^n \mathbf{p} - \mathbf{p}) D^S(W(\Lambda^n, p))_{rs} e^{-\tilde{b} \cdot \sum_{k=1}^n \Lambda^k p}. \quad (4.16)$$

This expression shows that the expectation value of the number operator is zero unless $\Lambda^n p = p$. We can look for solutions to this equation for a specific $n \neq 1$ or if $\Lambda p = p$, in which case the equation also holds for any $n > 1$. In the first instance, Λ is not generated by a continuous parameter, and the whole analytic continuation process would be debatable: we will disregard this set of solutions. Consequently, we focus on the case where between the Lorentz transformation Λ and the four-momentum of the particle p the relation $\Lambda p = p$ holds so that the expectation value (4.16) is not zero. Notice that, if this is not the case, the definition of the spin-density matrix becomes singular; if $\Lambda p \neq p$ there cannot be a non-zero average number of particles with momentum p and polarization cannot be generated.

The constraint given by $\Lambda p = p$ requires Λ to be in the little group of p (as p is left invariant under the action of Λ), and it is satisfied provided that ϕ , the parameter generating $\Lambda = \exp[-i\phi : J/2]$, obeys $\phi^{\mu\nu} p_\nu = 0$. The general solution can be expressed in terms of a vector $\theta_\mu^{(\phi)}$ as:

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \theta_\rho^{(\phi)} \frac{p_\sigma}{m}, \quad \theta^{(\phi)\rho} = -\frac{1}{2m} \epsilon^{\rho\mu\nu\sigma} \phi_{\mu\nu} p_\sigma. \quad (4.17)$$

If we use ϕ as in the above equation, the Lorentz transformation Λ can be cast into the form:

$$\Lambda = \exp \left(-\frac{i}{2} \phi : \widehat{J} \right) = \exp \left(i\theta^{(\phi)} \cdot \widehat{S}(p) \right),$$

and we see that the vector $\theta^{(\phi)}$ directly couples to the spin operator

$$\widehat{S}^\mu = \frac{\widehat{W}^\mu}{m} = -\frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} \widehat{J}_{\nu\rho} p_\sigma.$$

We have shown is that the constraint equation $\Lambda p = p$ allows polarization to develop only if Λ , which comes the density operator at imaginary vorticity, is generated by the Pauli-Lubanski vector.

Coming back to the calculation of the spin vector, if we consider Λ such that Eq. (4.17) is satisfied, then the δ -functions reduce to infinite constants that simplify in the ratio:

$$\begin{aligned} S^\mu(p) &= \frac{1}{2m} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} p_\nu A_5^{\mu\nu}(n, \phi) \delta^3(0)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} \text{tr}(\exp[-in\phi : \Sigma/2]) \delta^3(0)} \\ &= \frac{1}{2m} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} p_\nu A_5^{\mu\nu}(n, \phi)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-n\tilde{b}(in\phi) \cdot p} \text{tr}(\exp[-in\phi : \Sigma/2])}. \end{aligned}$$

The parameterization (4.17) yields more simplifications, which are essential to provide the analytic form of the polarization vector. First of all, the scalar product $\tilde{b}(in\phi) \cdot p$ gives:

$$\tilde{b}(in\phi) \cdot p = \sum_{k=0}^{\infty} \frac{(-n)^k}{(k+1)!} p_{\mu} \underbrace{(\phi_{\alpha_1}^{\mu} \phi_{\alpha_2}^{\alpha_1} \cdots \phi_{\alpha_k}^{\alpha_{k-1}})}_{k \text{ times}} b^{\alpha_k} = b \cdot p, \quad (4.18)$$

which is only the $k = 0$ term, as terms with $k > 0$ always involve $\phi_{\nu}^{\mu} p^{\nu}$ which vanishes in view of Eq. (4.17).

To compute the traces that appear in the series, we resort to the techniques used in Ref. [153]. We define the auxiliary variables:

$$z = \frac{\phi : \phi}{2} + i \frac{\phi : \tilde{\phi}}{2}, \quad \bar{z} = \frac{\phi : \phi}{2} - i \frac{\phi : \tilde{\phi}}{2}, \quad (4.19)$$

where

$$\tilde{\phi}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \phi_{\rho\sigma}$$

is the dual of the vorticity. It is possible to show that the following identities hold [153]:

$$\begin{aligned} \text{tr}(\Sigma^{\mu\nu}(\phi : \Sigma)^{2k+1}) &= (\phi^{\mu\nu} + i\tilde{\phi}^{\mu\nu})z^k + (\phi^{\mu\nu} - i\tilde{\phi}^{\mu\nu})\bar{z}^k, \\ \text{tr}(\gamma_5 \Sigma^{\mu\nu}(\phi : \Sigma)^{2k+1}) &= (\phi^{\mu\nu} + i\tilde{\phi}^{\mu\nu})z^k - (\phi^{\mu\nu} - i\tilde{\phi}^{\mu\nu})\bar{z}^k, \\ \text{tr}(\Sigma^{\mu\nu}(\phi : \Sigma)^{2k}) &= 0, \\ \text{tr}(\gamma_5 \Sigma^{\mu\nu}(\phi : \Sigma)^{2k}) &= 0. \end{aligned}$$

The above equations allow us to compute the traces we need for the spin vector. For example one has:

$$\begin{aligned} \text{tr}(\exp[-in\phi : \Sigma/2]) &= \sum_{k=0}^{\infty} \frac{n^k (-i)^k}{2^k k!} \phi_{\mu\nu} \text{tr}(\Sigma^{\mu\nu}(\phi : \Sigma)^{k-1}) \\ &= \sum_{k=-1}^{\infty} \frac{n^{2k+2} (-i)^{2k+2}}{2^{2k+2} (2k+2)!} \phi_{\mu\nu} \text{tr}(\Sigma^{\mu\nu}(\phi : \Sigma)^{2k+1}) \\ &= \sum_{k=-1}^{\infty} \frac{n^{2k+2} (-1)^{k+1}}{2^{2k+1} (2k+2)!} (z^{k+1} + \bar{z}^{k+1}) \\ &= 2 \cos\left(\frac{n\sqrt{z}}{2}\right) + 2 \cos\left(\frac{n\sqrt{\bar{z}}}{2}\right). \end{aligned}$$

Similarly:

$$\begin{aligned} \text{tr}(\gamma_5 \exp[-in\phi : \Sigma/2]) &= 2 \cos\left(\frac{n\sqrt{z}}{2}\right) - 2 \cos\left(\frac{n\sqrt{\bar{z}}}{2}\right), \\ \text{tr}(\Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]) &= -i(\phi^{\mu\nu} + i\tilde{\phi}^{\mu\nu}) \frac{\sin\left(\frac{n\sqrt{z}}{2}\right)}{\sqrt{z}} - i(\phi^{\mu\nu} - i\tilde{\phi}^{\mu\nu}) \frac{\sin\left(\frac{n\sqrt{\bar{z}}}{2}\right)}{\sqrt{\bar{z}}}, \\ \text{tr}(\gamma_5 \Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]) &= -i(\phi^{\mu\nu} + i\tilde{\phi}^{\mu\nu}) \frac{\sin\left(\frac{n\sqrt{z}}{2}\right)}{\sqrt{z}} + i(\phi^{\mu\nu} - i\tilde{\phi}^{\mu\nu}) \frac{\sin\left(\frac{n\sqrt{\bar{z}}}{2}\right)}{\sqrt{\bar{z}}}. \end{aligned}$$

These formulae hold for any ϕ , and in general z and \bar{z} are complex numbers. However, from Eq. (4.17), it follows that:

$$\tilde{\phi}^{\mu\nu} = \theta^{(\phi)\nu} \frac{p^\mu}{m} - \theta^{(\phi)\mu} \frac{p^\nu}{m},$$

whence:

$$\phi : \phi = -2\theta^{(\phi)\mu} \theta^{(\phi)}_{\mu} = -2\theta^{(\phi)^2}, \quad \phi : \tilde{\phi} = 0, \quad (4.20)$$

so that $z = \bar{z} = -\theta^{(\phi)^2} \in \mathbb{R}^+$. Notice that, since $\theta^{(\phi)} \cdot p = 0$ from Eq. (4.17), $\theta^{(\phi)\mu}$ is a space-like vector, and $\phi : \phi \geq 0$.

The above traces simplify, and one has:

$$\text{tr}(\exp[-in\phi : \Sigma/2]) = 4 \cos\left(\frac{n\sqrt{-\theta^{(\phi)^2}}}{2}\right), \quad (4.21a)$$

$$\text{tr}(\gamma_5 \exp[-in\phi : \Sigma/2]) = 0, \quad (4.21b)$$

$$\text{tr}(\Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]) = -2i\phi^{\mu\nu} \frac{\sin\left(\frac{n\sqrt{-\theta^{(\phi)^2}}}{2}\right)}{\sqrt{-\theta^{(\phi)^2}}}, \quad (4.21c)$$

$$\text{tr}(\gamma_5 \Sigma^{\mu\nu} \exp[-in\phi : \Sigma/2]) = 2\tilde{\phi}^{\mu\nu} \frac{\sin\left(\frac{n\sqrt{-\theta^{(\phi)^2}}}{2}\right)}{\sqrt{-\theta^{(\phi)^2}}}. \quad (4.21d)$$

We can use these relations to compute the traces in the polarization vector. Finally, recalling $A^{\mu\nu} = \text{tr}(\gamma^\nu \gamma^\mu \exp[-in\phi : \Sigma/2])$, we find:

$$A^{\mu\nu} = 4g^{\mu\nu} \cos\left(\frac{n\sqrt{-\theta^{(\phi)^2}}}{2}\right) + 4\phi^{\mu\nu} \frac{\sin\left(\frac{n\sqrt{-\theta^{(\phi)^2}}}{2}\right)}{\sqrt{-\theta^{(\phi)^2}}}, \quad (4.22a)$$

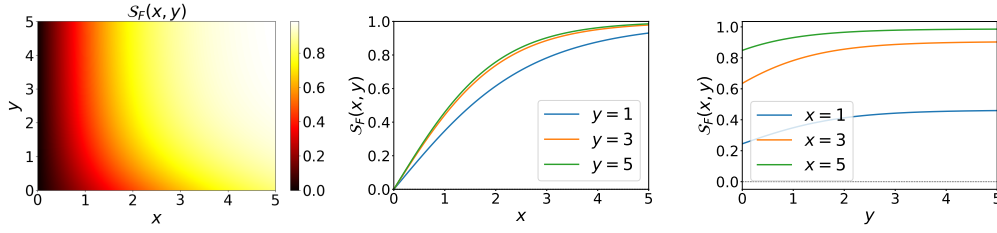
$$A_5^{\mu\nu} = 4i\tilde{\phi}^{\mu\nu} \frac{\sin\left(\frac{n\sqrt{-\theta^{(\phi)^2}}}{2}\right)}{\sqrt{-\theta^{(\phi)^2}}}. \quad (4.22b)$$

This is all we need to address the series for the exact spin vector of massive Dirac fermions in global equilibrium. Reintroducing the chemical potential $\zeta = \mu/T$, the spin vector reads:

$$\begin{aligned} S^\mu(p) &= \frac{1}{2m} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p + n\zeta} p_\nu A_5^{\mu\nu}(n, \phi)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p + n\zeta} \text{tr}(\exp[-in\phi : \Sigma/2])} \\ &= \frac{i\tilde{\phi}^{\mu\nu} p_\nu}{2m\sqrt{-\theta^{(\phi)^2}} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p + n\zeta} \sin\left(\frac{n\sqrt{-\theta^{(\phi)^2}}}{2}\right)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p + n\zeta} \cos\left(\frac{n\sqrt{-\theta^{(\phi)^2}}}{2}\right)}} \end{aligned}$$

Both series are convergent if $b \cdot p - \zeta > 0$, in which case the result is:

$$S^\mu(p) = \frac{i\epsilon^{\mu\nu\rho\sigma} \phi_{\rho\sigma} p_\nu}{4m\sqrt{-\theta^{(\phi)^2}} \cos\left(\frac{\sqrt{-\theta^{(\phi)^2}}}{2}\right) + e^{-b \cdot p + \zeta}} \frac{\sin\left(\frac{\sqrt{-\theta^{(\phi)^2}}}{2}\right)}{\cos\left(\frac{\sqrt{-\theta^{(\phi)^2}}}{2}\right)}. \quad (4.23)$$


 Figure 4.1: Plot of the dependence of the $\mathcal{S}_F(x, y)$ on its variables.

This is the final result of our calculation with imaginary thermal vorticity, and can be easily mapped to the physical case. Defining:

$$\theta^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma, \quad (4.24)$$

it is easy to see that mapping $\phi \mapsto i\varpi$ implies $\theta^{(\phi)} \mapsto i\theta$. Therefore the physical result reads:

$$S^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma \frac{1}{\sqrt{-\theta^2}} \frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right) + e^{-b\cdot p + \zeta}}. \quad (4.25)$$

Much like the case of the series associated with the Fermi-Dirac distribution function, the result is analytic in its variables, and we can analytically continue $S^\mu(p)$ making the above expression hold even when $b \cdot p - \zeta > 0$ is not satisfied. Furthermore the physical thermal vorticity is not required to satisfy the constraint as ϕ , as there is no dependence on it in the final result.

The polarization can be also expressed solely in terms of θ^μ , which points in the direction of the mean spin vector:

$$S^\mu(p) = \frac{1}{2} \frac{\theta^\mu}{\sqrt{-\theta^2}} \frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right) + e^{-b\cdot p + \zeta}} = \frac{1}{2} \frac{\theta^\mu}{\sqrt{-\theta^2}} \mathcal{S}_F\left(\sqrt{-\theta^2}, b \cdot p - \zeta\right). \quad (4.26)$$

In the above equation we have introduced the function:

$$\mathcal{S}_F(x, y) = \frac{\sinh x}{\cosh x + e^{-y}}, \quad (4.27)$$

which describes how much particles are polarized. Its behaviour is shown in Figure 4.1. The function is monotonically increasing both as a function of x and y , and eventually, saturates to an asymptotic limit, 1 for fixed y and $\tanh x$ for fixed x . Notice that, for $x \geq 0$, $0 \leq \mathcal{S}_F \leq 1$ so that \mathcal{S}_F is the percentage of polarization along the versor $\hat{\theta}^\mu = \theta^\mu / \sqrt{-\theta^2}$.

The limiting behaviours of the spin vector can be inferred from the ones of \mathcal{S}_F . It is easy to show that:

$$\lim_{x \rightarrow \infty} \mathcal{S}_F(x, y) = 1, \quad \lim_{x \rightarrow 0} \mathcal{S}_F(x, y) = \frac{x}{2}(1 - n_F(y)). \quad (4.28)$$

Using these formulae in the limiting cases $\sqrt{-\theta^2} \gg 1$ and $\sqrt{-\theta^2} \ll 1$ we find:

$$\begin{aligned} \lim_{\sqrt{-\theta^2} \rightarrow \infty} S^\mu(p) &= \frac{1}{2} \frac{\theta^\mu}{\sqrt{-\theta^2}}, \\ \lim_{\sqrt{-\theta^2} \rightarrow 0} S^\mu(p) &= \frac{\theta^\mu}{4}(1 - n_F(b \cdot p - \zeta)), \end{aligned}$$

where the second limit is in perfect agreement with the literature. It should be mentioned that the physical meaning of the first limit is dubious, as it describes a fluid experiencing an infinite angular velocity and/or acceleration.

The massless case can be tackled likewise, but some subtleties deserve to be commented on. The constraint equation $\Lambda p = p$ imposes that the parameter ϕ associated to the imaginary thermal vorticity is parametrized as:

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} h_\rho \frac{p_\sigma}{p \cdot q}, \quad (4.29)$$

Inverting such an equation in the massless case, to obtain h_ρ in terms of ϕ is a bit more challenging. Let us start by computing the dual of ϕ :

$$\tilde{\phi}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \phi_{\alpha\beta} = \frac{1}{p \cdot q} (h^\nu p^\mu - h^\mu p^\nu).$$

We can decompose h^μ along the tetrad $\{p, q, n_1, n_2\}$, but, as it will turn out, only the component of h^μ along q^μ is of physical relevance. Let us therefore write:

$$h^\mu = H^{(\phi)} q^\mu + h_\perp^\mu, \quad (4.30)$$

where the orthogonal space h_\perp is generated by $n_{1,2}$. One realizes that:

$$H^{(\phi)} = \frac{1}{2(p \cdot q)} \epsilon^{\mu\nu\alpha\beta} \phi_{\alpha\beta} p_\nu q_\mu = \frac{\tilde{\phi}^{\mu\nu} q_\mu p_\nu}{p \cdot q}, \quad \phi : \phi = 2 \frac{(h \cdot p)^2}{(p \cdot q)^2} = 2H^{(\phi)2}. \quad (4.31)$$

Reworking Eq. (4.3), keeping in mind the simplifications provided by $\phi_{\mu\nu} p^\nu = 0$, one obtains (here we have set $\zeta = 0$):

$$W^\mu(p) = p^\mu \frac{1}{2} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p} p_\nu q_\rho \text{tr}(\gamma^\nu \gamma^\rho \gamma_5 \exp[-in\phi : \Sigma/2])}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p} p_\lambda q_\sigma \text{tr}(\gamma^\lambda \gamma^\sigma \exp[-in\phi : \Sigma/2])}. \quad (4.32)$$

Using the identities (4.22) one obtains:

$$W^\mu = \frac{p^\mu}{2|H^{(\phi)}|} \frac{i\tilde{\phi}^{\alpha\beta} q_\alpha p_\beta}{p \cdot q} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p} \sin\left(\frac{n|H^{(\phi)}|}{2}\right)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p} \cos\left(\frac{n|H^{(\phi)}|}{2}\right)}. \quad (4.33)$$

Summing the series, which converges for $b \cdot p > 0$, one finds:

$$W^\mu(p) = \frac{p^\mu}{2} \frac{iH^{(\phi)}}{|H^{(\phi)}|} \frac{\sin\left(\frac{|H^{(\phi)}|}{2}\right)}{\cos\left(\frac{|H^{(\phi)}|}{2}\right) + e^{-b \cdot p}}. \quad (4.34)$$

The result can now be extended to real vorticity simply mapping $H^{(\phi)} \mapsto iH$ where

$$H = \frac{1}{2(p \cdot q)} \epsilon^{\mu\nu\rho\sigma} \varpi_{\mu\nu} p_\sigma. \quad (4.35)$$

One gets:

$$W^\mu(p) = -\frac{p^\mu}{2} \frac{H}{|H|} \frac{\sinh\left(\frac{|H|}{2}\right)}{\cosh\left(\frac{|H|}{2}\right) + e^{-b \cdot p}}, \quad (4.36)$$

and using the parity of the hyperbolic functions, $\sinh x = -\sinh(-x)$ and $\cosh x = \cosh(-x)$, we can also write

$$W^\mu(p) = -\frac{p^\mu}{2} \frac{\sinh\left(\frac{H}{2}\right)}{\cosh\left(\frac{H}{2}\right) + e^{-b \cdot p}}. \quad (4.37)$$

As we have shown in the previous section, in Eq. (4.12), $H = -\mathbf{w} \cdot \hat{p}$ and the mean Pauli-Lubanski vector can be written as:

$$W^\mu(p) = \frac{p^\mu}{2} \mathcal{S}_F\left(\frac{\mathbf{w} \cdot \hat{p}}{2}, b \cdot p\right), \quad (4.38)$$

where we made use of the function \mathcal{S}_F . The limits of very large and very small thermal vorticity, are easily obtained from Eq. (4.28):

$$\lim_{|\mathbf{w} \cdot \hat{p}| \rightarrow \infty} W^\mu = \frac{p^\mu}{2} \text{sgn}(\mathbf{w} \cdot \hat{p}), \quad (4.39)$$

$$\lim_{|\mathbf{w} \cdot \hat{p}| \rightarrow 0} W^\mu = \frac{p^\mu}{4} \mathbf{w} \cdot \hat{p} (1 - n_F(b \cdot p)), \quad (4.40)$$

and the limit of small vorticity is in agreement with Eq. (4.14).

4.3 Exact polarization in heavy-ion collisions

Now that a formula to exactly compute the spin vector of massive fermions has been derived, one may check whether it yields significant corrections to the polarization in relativistic heavy-ion collisions.

To this end, we study the global polarization of the Λ hyperon. We take experimental data points from Ref. [8] and use them to estimate the vorticity of the QGP, comparing the predictions of the linear approximation and the exact formula for polarization.

For the comparison, other than assuming that the QGP at freeze-out can be described as a fluid in global equilibrium, we work in the idealized hypothesis that the vorticity is only given by a uniform rotation along the $-y$ -axis, which corresponds to the direction of the global angular momentum in the reference frame employed in experiments. Considering particles at midrapidity, with four-momentum $p^\mu = (\varepsilon, p_x, p_y, 0)$ and $\varepsilon = \sqrt{m^2 + p_x^2 + p_y^2}$, we have:

$$\varpi_{\mu\nu} = \frac{\omega}{T_0} (g_{\mu 1} g_{\nu 3} - g_{\mu 3} g_{\nu 1}), \quad \theta^\mu = -\frac{\omega}{m T_0} (p_y, 0, \varepsilon, 0), \quad -\theta^2 = \frac{\omega^2}{T_0^2 m^2} (\varepsilon^2 - p_y^2). \quad (4.41)$$

Experimentally, the polarization vector in the rest frame of the particle is measured. The component of the θ vector along the y direction in the rest frame of the Λ particle is:

$$\theta_{0y} = -\frac{\omega}{m T_0} \left(\varepsilon - \frac{p_y^2}{m + \varepsilon} \right). \quad (4.42)$$

This is the only component of θ_0 that yields a non vanishing integral over d^3p . The spectrum of particles is:

$$\varepsilon \frac{dN}{d^3p} = 2 \int d\Sigma_\mu p^\mu n_F = 2\varepsilon V n_F, \quad (4.43)$$

where the integral has been computed on a constant-time hypersurface of volume V , due to the global-equilibrium hypothesis and n_F is the Fermi-Dirac distribution function:

$$n_F = \frac{1}{e^{\frac{\varepsilon-\mu}{T}} + 1}. \quad (4.44)$$

Finally, the global polarization can be computed using Eq. (2.7), where we consider only particles at midrapidity:

$$P_y^{L/E}(\sqrt{s_{NN}}) = \frac{\int \frac{d^3p}{2\varepsilon} \delta(p_z) \varepsilon \frac{dN}{d^3p} 2S_y^{L/E}}{\int \frac{d^3p}{2\varepsilon} \delta(p_z) \varepsilon \frac{dN}{d^3p}} = -\frac{\omega}{mT_0} \frac{\int d^2p n_F 2\left(\varepsilon - \frac{p_y^2}{\varepsilon+m}\right) f^{L/E}}{\int d^2p n_F}. \quad (4.45)$$

In the above formula, polarization depends on collision energy through the temperature and chemical potential. The function $f_{L/E}$ are defined from the linearised and exact formula for the spin vector through $S_y^{L/E} = \theta_y f^{L/E}$. Explicitly:

$$f^L = \frac{1 - n_F}{4}, \quad f^E = \frac{\mathcal{S}_F}{2\sqrt{-\theta^2}}, \quad (4.46)$$

as dictated by Eqs. (4.8) and (4.26) respectively. The integrals in the linear and exact case read:

$$P_y^L = -\frac{\omega}{mT_0} \frac{\int d^2p n_F \left(\varepsilon - \frac{p_y^2}{\varepsilon+m}\right) \frac{1-n_F}{2}}{\int dp n_F}, \quad (4.47)$$

$$P_y^E = -\frac{\int d^2p n_F \left(\varepsilon - \frac{p_y^2}{\varepsilon+m}\right) \frac{\mathcal{S}_F\left(\frac{\omega}{mT_0} \sqrt{\varepsilon^2 - p_y^2}, \frac{\varepsilon-\mu}{T_0}\right)}{\sqrt{\varepsilon^2 - p_y^2}}}{\int dp n_F}. \quad (4.48)$$

The dependence on the collision energy $\sqrt{s_{NN}}$ of chemical potential and temperature in Eq. (4.45) are taken from Ref. [154], and read:

$$\begin{aligned} \mu_B(\sqrt{s_{NN}}) &= 2.06 \frac{\ln \sqrt{s_{NN}}}{(\sqrt{s_{NN}})^{1.13}} [\text{GeV}], \\ T(\sqrt{s_{NN}}) &= (0.1675 - 0.1583 \mu_B^2(\sqrt{s_{NN}})) [\text{GeV}], \end{aligned}$$

where $\sqrt{s_{NN}}$ is in GeV.

To assess the impact of the exact formula compared to the linear one, we compute the integrals (4.47) and (4.48), changing the value of the constant angular velocity ω . Our goal, is to establish for which ω the formulae yield the central value of the global polarization data of the Λ particle from Ref. [8]. We will not consider the experimental uncertainties due to the very rough estimate we are after.

The vorticities resulting from this analysis are reported in Table 4.3. It can be seen that the relative difference between the values of vorticity obtained from the linear and the exact formulae is at most 10^{-3} , which is very small. The reason for this is that the thermal vorticity is itself small in relativistic heavy-ion collisions. In fact, restoring \hbar and for $\omega \sim 10^{22} \text{s}^{-1}$, $T_0 \sim 150 \text{ MeV}$, one has:

$$\frac{\omega \hbar}{2T_0} \sim 0.02. \quad (4.49)$$

$\sqrt{s_{NN}}$ [GeV]	P_{Λ} [%]	$\omega_L[\times 10^{22}s^{-1}]$	$\omega_E[\times 10^{22}s^{-1}]$	$(\omega_E - \omega_L)/\omega_E$
7.7	1.8	2.1101	2.1122	1×10^{-3}
11.5	1.19	1.4347	1.4354	4×10^{-4}
14.5	1.17	1.4252	1.4258	4×10^{-4}
19.6	0.84	1.0325	1.0327	2×10^{-4}
27	0.93	1.1499	1.1502	2×10^{-4}
39	0.45	0.5586	0.5586	1×10^{-4}
62.4	1.2	1.4933	1.4939	4×10^{-4}
200	0.1	0.1247	0.1247	$< 10^{-4}$

Table 4.1: Estimates of the QGP's angular velocity for a given collision energy and experimentally measured polarization. The experimental polarization of the Λ particle is taken from Ref. [8]. The subscript L and E refer to the linear approximation and the exact formula respectively.

In conclusion, we have found that the exact formula for polarization leads only to very small corrections at low energies for what concerns the global polarization, and it is indistinguishable from the linear approximation in heavy-ion physics within experimental uncertainties. On the other hand, it is hard to fathom a significant contribution in the local-polarization sector; we can definitively state that higher-order contributions in vorticity cannot solve the sign puzzle of polarization. The linear formula for the spin vector should be sufficiently accurate in almost all applications.

4.4 Spin physics in global equilibrium for any spin

Using the results of the previous chapter and the discussion of Section 4.2, we can evaluate the spin vector of a field with generic spin S . In this case, we avoid using the Wigner function, as this would require a generalization of Eq. (1.93) to any spin. Using the definition of the spin density matrix (1.83) and the exact two-point function (3.47), the spin density matrix can be straightforwardly written as:

$$\Theta_{rs}(p) = \frac{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-\tilde{b}(in\phi) \cdot p} W_{rs}(\Lambda^n, p) \delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-\tilde{b}(in\phi) \cdot p} \text{tr}(W(\Lambda^n, p)) \delta^3(\Lambda^n p - p)}. \quad (4.50)$$

Given the discussion of Section 4.2, the Dirac δ -function constrains the imaginary vorticity ϕ such that $\phi_{\mu\nu} p^\nu = 0$. Focusing on massive particles first, this means that Λ , as well its continuation to real vorticity, is the exponential of the generators of rotations because the Pauli-Lubanski vector generates $\text{SO}(3)$ for massive fields. In such a case, the Wigner rotation $W(\Lambda, p) = [\Lambda p]^{-1} \Lambda [p]$ can be simplified. Indeed, it is possible to show that if Λ is a rotation, then the Wigner rotation is the same rotation as Λ , i.e. $W(\Lambda^n, p) = D^{(S)}(\Lambda^n)$ where we are considering a generic S -representation of the rotation group [42]. Therefore

$$\Theta(p) = \frac{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb \cdot p} D^{(S)}(\Lambda^n)}{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb \cdot p} \text{tr}(D^{(S)}(\Lambda^n))},$$

where we have simplified $\tilde{b}(in\phi) \cdot p = nb \cdot p$, see Eq. (4.18). All possible spin observables can be derived from this matrix. For instance, the spin polarization

vector of a spin- S field is given by Eq. (1.79):

$$S^\mu(p) = \sum_i [p]^\mu_i \text{tr}(\Theta(p) J_i^{(S)}), \quad (4.51)$$

$J_i^{(S)}$ being the i -th generator of the rotation group in the S -representation. Such a trace can be computed for any representation using the *character* $\chi(S)$ of the representation, that is the trace of an element of the representation.

Let us start by rewriting $\exp\left(-in\frac{\phi:J}{2}\right)$ in terms of the generators of rotations. Using the notation of section 4.2, we write:

$$e^{-in\frac{\phi:J}{2}} = e^{in\theta^{(\phi)} \cdot S} = e^{in\theta_0^{(\phi)} \cdot \mathbf{J}^{(S)}} = \exp[-in(\theta_0^{(\phi)} \mathbf{J}_x + \theta_0^{(\phi)} \mathbf{J}_y + \theta_0^{(\phi)} \mathbf{J}_z)], \quad (4.52)$$

where we used Lorentz invariance to write the rotation in the standard frame, where $p \rightarrow \mathbf{p} = (m, \mathbf{0})$ and $\theta \rightarrow \theta_0$, and the components of the spin vector boil down to the generators of rotations $W^i = mJ^i$ (see Section 1.5). This transformation corresponds to a rotation of an angle $n\sqrt{-\theta_0^{(\phi)2}}$ around the axis $\hat{\theta}_0^{(\phi)\mu} = \theta_0^{(\phi)\mu} / \sqrt{-\theta_0^{(\phi)2}}$. The trace of such a rotation in a spin- S representation is given by [43]:

$$\chi(S) = \text{tr}\left(D^{(S)}(\Lambda^n)\right) = \frac{\sin\left[n\left(S + \frac{1}{2}\right)\sqrt{-\theta_0^{(\phi)2}}\right]}{\sin\left(\frac{n\sqrt{-\theta_0^{(\phi)2}}}{2}\right)}. \quad (4.53)$$

To compute the polarization vector from Eq. (4.51) we can write:

$$\text{tr}\left(e^{in\theta_0 \cdot \mathbf{J}} \mathbf{J}_{x,y,z}^{(S)}\right) = \frac{i}{n} \frac{\partial}{\partial \theta_0^{(\phi)}{}_{x,y,z}} \text{tr}\left(e^{in\theta_0^{(\phi)} \cdot \mathbf{J}}\right) = \frac{i}{n} \frac{\partial}{\partial \theta_0^{(\phi)}{}_{x,y,z}} \chi(S),$$

and the explicit derivation yields:

$$\begin{aligned} \text{tr}(D^{(S)}(\Lambda^n) \mathbf{J}_{x,y,z}^{(S)}) &= i \frac{\theta_0^{(\phi)}{}_{x,y,z}}{2\sqrt{-\theta_0^{(\phi)2}}} \text{csc}^2\left(\frac{n\sqrt{-\theta_0^{(\phi)2}}}{2}\right) \times \\ &\quad \left[S \sin\left(n(1+S)\sqrt{-\theta_0^{(\phi)2}}\right) - (1+S) \sin\left(nS\sqrt{-\theta_0^{(\phi)2}}\right) \right]. \end{aligned} \quad (4.54)$$

The calculation of the spin vector is now straightforward; the standard boost in Eq. (4.51) is such that $\theta_0^{(\phi)}$ is boosted to $\theta^{(\phi)}$ and we use Lorentz invariance to write $\theta_0^{(\phi)2} = \theta^{(\phi)2}$. Therefore, the polarization vector for a massive field of generic spin S reads:

$$\begin{aligned} S^\mu(p) &= \frac{i\theta^{(\phi)\mu}}{2\sqrt{-\theta^{(\phi)2}}} \frac{1}{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb \cdot p} \text{csc}\left(\frac{n\sqrt{-\theta^{(\phi)2}}}{2}\right) \sin\left(n\left(S + \frac{1}{2}\right)\sqrt{-\theta^{(\phi)2}}\right)} \\ &\quad \times \sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb \cdot p} \frac{S \sin\left(n(1+S)\sqrt{-\theta^{(\phi)2}}\right) - (1+S) \sin\left(nS\sqrt{-\theta^{(\phi)2}}\right)}{\sin^2\left(\frac{n\sqrt{-\theta^{(\phi)2}}}{2}\right)}. \end{aligned} \quad (4.55)$$

Notice that the case $S = 1/2$ yields, after using some elementary trigonometric identities:

$$\begin{aligned} S^\mu(p) &= \frac{-i\theta(\phi)^\mu}{2\sqrt{-\theta(\phi)^2}} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p} \sin\left(\frac{n\sqrt{-\theta(\phi)^2}}{2}\right)}{\sum_{n=1}^{\infty} (-1)^{n+1} e^{-nb \cdot p} \cos\left(\frac{n\sqrt{-\theta(\phi)^2}}{2}\right)} \\ &= \frac{-i\theta(\phi)^\mu}{2\sqrt{-\theta(\phi)^2}} \frac{\sin\left(\frac{\sqrt{-\theta(\phi)^2}}{2}\right)}{\cos\left(\frac{\sqrt{-\theta(\phi)^2}}{2}\right) + e^{-b \cdot p}}, \end{aligned}$$

which precisely reproduces Eq. (4.23).

After performing the summation it is possible to map the parameter ϕ back to the real vorticity via $\phi \mapsto i\varpi$. Here, however, we do not pursue such goal. We confine ourselves to a comparison of our results and the previous literature. For example, it is easy to check that in the limit of small $\sqrt{-\theta(\phi)^2}$ the spin vector reads:

$$\begin{aligned} S^\mu(p) &= -i\theta(\phi)^\mu \frac{S(S+1)}{3} \frac{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb \cdot p} n}{\sum_{n=1}^{\infty} (-1)^{2S(n+1)} e^{-nb \cdot p}} \\ &= -i\theta(\phi)^\mu \frac{S(S+1)}{3} (1 - (-1)^{2S} n_{F/B}(b \cdot p)), \end{aligned}$$

such that after the continuation to real vorticity:

$$S^\mu(p) \mapsto \theta^\mu \frac{S(S+1)}{3} (1 - (-1)^{2S} n_{F/B}(b \cdot p)) \quad (4.56)$$

which is in agreement with expectations [97, 155].

Furthermore, we can consider the case of Boltzmann statistics, which consists in truncating the series to the $n = 1$ term. The result is:

$$\begin{aligned} S^\mu(p) &= \frac{-i\theta(\phi)^\mu}{2\sqrt{-\theta(\phi)^2}} \operatorname{csc}\left(\frac{\sqrt{-\theta(\phi)^2}}{2}\right) \times \\ &\quad \frac{\left[S \sin\left((1+S)\sqrt{-\theta(\phi)^2}\right) - (1+S) \sin\left(S\sqrt{-\theta(\phi)^2}\right) \right]}{\sin\left(\left(S + \frac{1}{2}\right)\sqrt{-\theta(\phi)^2}\right)}, \end{aligned}$$

which, after the continuation, can be written in terms of the character:

$$S^\mu(p) = \frac{\theta^\mu}{\sqrt{-\theta^2}} \frac{\chi'(\sqrt{-\theta^2})}{\chi(\sqrt{-\theta^2})}, \quad (4.57)$$

where with χ' we denote the derivative of χ with respect to $\sqrt{-\theta^2}$. This expression is consistent with the results of Refs. [118, 156].

Similarly, we can take on the polarization of massless particles. Using the definition of the Pauli-Lubanski vector (1.60) and Eqs. (4.29), (4.30) and (4.31) we can write:

$$\exp\left(-in \frac{\phi \cdot \hat{J}}{2}\right) = \exp\left(in \frac{h_\rho}{p \cdot q} \widehat{W}^\rho\right) = \exp\left(in \frac{h \cdot p}{q \cdot p} \widehat{h}\right) = \exp\left(in H^{(\phi)} \widehat{h}\right), \quad (4.58)$$

where $H^{(\phi)} = \epsilon^{\mu\nu\alpha\beta} \phi_{\alpha\beta} p_\nu q_\mu / (2p \cdot q)$ and, since only the component of the Pauli-Lubanski vector along p^μ is physically relevant, we used $\widehat{W}^\mu = \widehat{h} p^\mu$. Now, taking into account Eq. (1.74), and that only two helicity states ($+h$ and $-h$) are possible for massless particles, it is easy to see:

$$W^\mu(p) = p^\mu \frac{\sum_{s=\pm h} s \Theta_{ss}}{\sum_{s=\pm h} \Theta_{ss}} = i p^\mu h \frac{\sum_{n=1}^{\infty} (-1)^{2h(n+1)} e^{-b \cdot p} \sin(n H^{(\phi)} h)}{\sum_{n=1}^{\infty} (-1)^{2h(n+1)} e^{-b \cdot p} \cos(n H^{(\phi)} h)}, \quad (4.59)$$

where with h we denote the helicity of the particle (i.e., $h = 1/2, 1, 3/2 \dots$).

The above series can be straightforwardly summed, and after mapping $\phi \mapsto i\varpi$ and $H^{(\phi)} \mapsto iH$ it yields:

$$W^\mu(p) = -p^\mu h \frac{\sinh(H h)}{\cosh(H h) - (-1)^{2h} e^{-b \cdot p}}. \quad (4.60)$$

This formula reproduces Eq. (4.37) for $h = 1/2$.

Chapter 5

Local-equilibrium spin polarization

This chapter delves deeper into the phenomenology of polarization in relativistic fluids. The goal we pursue here is to explain the measurements of the polarization of the Λ produced in heavy-ion collisions at the Relativistic Heavy-Ion Collider (RHIC) in Brookhaven and the Large Hadron Collider (LHC) in Geneva. The QGP produced in the collisions is believed to hadronize not in global equilibrium, but in a state of local equilibrium. Thus, the results of the previous section have to be generalized. In this chapter, we linger on the difference, to first order in gradients, between global and local equilibrium, namely on the fact that the symmetric derivative of the four-temperature vector can be non-vanishing. Such a tensor is called thermal shear. We compute explicitly the thermal expectation value of the spin vector of the Dirac field using the local-equilibrium density operator and show that the thermal shear yields a non-vanishing contribution to polarization. Finally, after discussing an improved approximation for linear-response theory suitable for high-energy heavy-ion collisions, we perform realistic hydrodynamic simulations to compare the predictions of the newly found formula to experimental data.

This chapter is based on Refs. [16, 17].

5.1 Thermal shear in relativistic fluids at local equilibrium

If we consider a fluid described by the local-equilibrium density operator (1.28), such as the QGP appears to be, the Killing condition (1.17) does not hold. In such circumstances, the symmetric derivative of the four-temperature may be non-zero. Such a tensor is called the *thermal shear* and it is defined as:

$$\xi_{\mu\nu} = \frac{1}{2} (\partial_\mu \beta_\nu + \partial_\nu \beta_\mu). \quad (5.1)$$

The thermal shear is an out-of-equilibrium property of the fluid, as the Killing equation in the equilibrium conditions (1.17) makes $\xi_{\mu\nu}$ vanish. Therefore, we expect that the closer the fluid is to a state of global equilibrium, the smaller the thermal shear will be.

In the previous chapter, our focus was on the properties of systems in global equilibrium with non-vanishing thermal vorticity. As we have extensively discussed, the thermal vorticity informs us about the angular velocity and acceleration experienced

by the fluid. One may wonder what is the physical interpretation of the thermal-shear tensor. To answer this question, we devise some particular four-temperature vectors, taking the shear tensor as a constant (which is in general not the case). To begin with, let us study a fluid where the thermal shear is proportional to the metric tensor:

$$\xi_{\mu\nu} = \frac{\mathcal{X}}{T_0} g_{\mu\nu}. \quad (5.2)$$

This shear is generated by the four-temperature vector:

$$\beta^\mu(x) = b^\mu + \frac{\mathcal{X}}{T_0} x^\mu. \quad (5.3)$$

Assuming $b^\mu = \frac{1}{T_0}(1, \mathbf{0})$, the invariant temperature and the four-velocity of the fluid are given by:

$$T = \frac{T_0}{\sqrt{(1 + \mathcal{X}t)^2 - \mathcal{X}^2 \mathbf{x}^2}}, \quad u^\mu = \frac{1}{\sqrt{(1 + \mathcal{X}t)^2 - \mathcal{X}^2 \mathbf{x}^2}}(1 + \mathcal{X}t, \mathcal{X} \mathbf{x}). \quad (5.4)$$

Similarly to what we have done for the vorticity in Section 1.2.2, we can consider an observer who measures T_0 as the proper temperature. Such an observer moves on the trajectory defined by $(1 + \mathcal{X}t)^2 - \mathcal{X}^2 \mathbf{x}^2 = 1$, so that $T = T_0$. The expansion scalar computed on this curve is given by:

$$\theta = \partial \cdot u = 4\mathcal{X}. \quad (5.5)$$

We see that \mathcal{X} is nothing but the constant expansion scalar measured by this observer.

We can also consider a non-diagonal thermal shear, for example:

$$\xi_{\mu\nu} = \frac{Q}{T_0}(\delta_\mu^1 \delta_\nu^2 + \delta_\mu^2 \delta_\nu^1). \quad (5.6)$$

This particular tensor yields:

$$\beta^\mu(x) = b^\mu + \frac{Q}{T_0}(g^{\mu 1} y + g^{\mu 2} x), \quad (5.7)$$

and the proper temperature and velocity are:

$$T = \frac{T_0}{\sqrt{1 - Q^2(x^2 + y^2)}}, \quad u^\mu = \frac{1}{\sqrt{1 - Q^2(x^2 + y^2)}}(1, -Qy, -Qx, 0). \quad (5.8)$$

The fluid shear tensor is defined as:

$$\sigma^{\mu\nu} = \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\Delta^{\mu\nu}\theta, \quad (5.9)$$

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ and $\nabla^\mu = \Delta^{\mu\nu} \partial_\nu$. If we compute the shear tensor in the origin, where $T = T_0$ we find:

$$\sigma^{xy}|_{x^\mu=0} = Q, \quad (5.10)$$

and all other components vanish. We showed that a non-diagonal thermal shear indicates the presence of the properly-called shear tensor in the fluid.

5.2 Local-equilibrium density operator to first order in gradients

Since not only the thermal vorticity but also the thermal shear is present in local equilibrium, it is compulsory to estimate the effects of the latter on expectation values. In the next section, we will compute the polarization coming from gradients of temperature in local equilibrium, accounting for both the thermal vorticity and the thermal shear and show that the latter indeed yields a non-vanishing polarization [16, 17]. We will employ the density-operator formalism as developed in the previous chapters, although different approaches also lead to similar (yet different) conclusions [157, 158].

Before venturing into the calculations, it is useful to study the role of thermal vorticity and thermal shear in the density operator itself. We start by recalling once again the local-equilibrium form of the density operator:

$$\hat{\rho}_{LE} = \frac{1}{Z} \exp \left[- \int d\Sigma_\mu \hat{T}^{\mu\nu} \beta_\nu \right]. \quad (5.11)$$

For applications to heavy-ion collisions, one can choose the integration hypersurface to be the freeze-out, or decoupling, hypersurface. In fact, the statistical-hadronization model provided countless indications supporting the picture that the quark-gluon plasma freezes out close to thermal equilibrium.

In this section, we will not discuss the effects of chemical potential and its gradients. It is known that they contribute to the polarization via the so-called *spin-Hall effect* [157, 159, 160], but they are expected to be significant at low collision energy, which we will not consider here.

Expanding the β field up to first order in gradients, following the hydrodynamic approximation, we can identify three different terms, two being familiar from global-equilibrium studies, involving \hat{P}^μ and $\hat{J}^{\mu\nu}$, and one being present only out of equilibrium and depending on a new operator $\hat{Q}^{\mu\nu}$:

$$\begin{aligned} \hat{\rho}_{LE} &\simeq \frac{1}{Z_{LE}} \exp \left[- \int_\Sigma d\Sigma_\mu \hat{T}^{\mu\nu}(y) [\beta_\nu(x) + \partial_\lambda \beta_\nu(x) (y-x)^\lambda] \right] \\ &\simeq \frac{1}{Z_{LE}} \exp \left[-\beta(x) \cdot \hat{P} + \frac{1}{2} \varpi_{\nu\lambda}(x) \hat{J}_x^{\nu\lambda} - \frac{1}{2} \xi_{\nu\lambda} \hat{Q}_x^{\nu\lambda} \right], \end{aligned}$$

where ϖ and ξ are the thermal vorticity and thermal shear:

$$\varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu), \quad \xi_{\mu\nu} = \frac{1}{2}(\partial_\mu \beta_\nu + \partial_\nu \beta_\mu). \quad (5.12)$$

These two tensors couple to similar, but fundamentally different operators. On the one hand, the thermal vorticity couples with the angular-momentum boost operator (centred at the point x), exactly as in global equilibrium:

$$\hat{J}_x^{\lambda\nu} = \int_\Sigma d\Sigma_\mu \left[(y-x)^\lambda \hat{T}^{\mu\nu}(y) - (y-x)^\nu \hat{T}^{\mu\lambda}(y) \right] = \int_\Sigma d\Sigma_\mu \hat{J}^{\mu,\lambda\nu}. \quad (5.13)$$

The density of angular momentum $\hat{J}^{\mu,\lambda\nu}$ is conserved, therefore $\hat{J}^{\lambda\nu}$ does not depend on the integration hypersurface.

On the other hand, the thermal shear couples to the tensor:

$$\hat{Q}_x^{\lambda\nu} \equiv \int_\Sigma d\Sigma_\mu \left[(y-x)^\lambda \hat{T}^{\mu\nu}(y) + (y-x)^\nu \hat{T}^{\mu\lambda}(y) \right] = \int_\Sigma d\Sigma_\mu \hat{Q}^{\mu,\lambda\nu}. \quad (5.14)$$

Graphically, the difference between $\widehat{J}^{\mu\nu}$ and $\widehat{Q}^{\mu\nu}$ is just a sign, which may look like a small difference. However, it has a dramatic consequence: $\widehat{Q}^{\mu,\lambda\nu}$ is not conserved,

$$\partial_\mu \widehat{Q}^{\mu,\lambda\nu} = \partial_\mu \left((y-x)^\lambda \widehat{T}^{\mu\nu}(y) + (y-x)^\nu \widehat{T}^{\mu\lambda}(y) \right) = 2\widehat{T}^{\lambda\nu}(y). \quad (5.15)$$

Therefore, it ensues that the $\widehat{Q}^{\mu\nu}$ operator depends on the integration hypersurface. As a consequence, $\widehat{Q}^{\mu\nu}$ is not a tensor operator, since if we act on it with a Lorentz transformation we have:

$$\begin{aligned} \widehat{\Lambda} \widehat{Q}_x^{\lambda\nu} \widehat{\Lambda}^{-1} &= \Lambda^{-1\lambda}{}_\rho \Lambda^{-1\nu}{}_\sigma \int_{\Sigma_\Lambda} d\Sigma_\mu \left[(y-\Lambda x)^\rho \widehat{T}^{\mu\sigma}(y) + (y-\Lambda x)^\sigma \widehat{T}^{\mu\rho}(y) \right] \\ &\neq \Lambda^{-1\lambda}{}_\rho \Lambda^{-1\nu}{}_\sigma \widehat{Q}_{\Lambda x}^{\rho\sigma}, \end{aligned}$$

where the last inequality follows from the fact that the operator calculated on the Lorentz-transformed hypersurface Σ_Λ is not the same as the one computed on Σ due to Eq. (5.14). We anticipate that *all* thermal expectation values arising from the thermal shear will unavoidably depend on the decoupling hypersurface, thus breaking covariance.

The previous statement may sound disquieting, but given the form of the local-equilibrium density operator, it should not come as a surprise. Indeed, let \widehat{V}^μ be a vector operator, meaning that $\widehat{\Lambda} \widehat{V}^\mu \widehat{\Lambda}^{-1} = \Lambda^{-1\mu}{}_\nu \widehat{V}^\nu$. The local-equilibrium expectation value of \widehat{V}^μ transforms as:

$$\begin{aligned} \Lambda^\mu{}_\nu \langle \widehat{V}^\nu \rangle_{LE,\beta} &= \text{Tr} \left(\widehat{\rho} \widehat{\Lambda}^{-1} \widehat{V}^\mu \widehat{\Lambda} \right) = \text{Tr} \left(\widehat{\Lambda} \widehat{\rho} \widehat{\Lambda}^{-1} \widehat{V}^\mu \right) \\ &= \frac{1}{Z_{LE}} \text{Tr} \left(\exp \left[- \int_{\Sigma_\Lambda} d\Sigma_\mu \widehat{T}^{\mu\nu}(x) \Lambda_{\nu\rho} \beta^\rho(x) \right] \widehat{V}^\mu \right) \neq \langle \widehat{V}^\mu \rangle_{LE,\Lambda\beta} \end{aligned} \quad (5.16)$$

and it is therefore not a vector since the integration hypersurface has changed. This is a feature of the local-equilibrium density-operator formalism, and it is neither avoidable nor a problem.

5.3 Spin and thermal shear tensor

To compute the polarization at local equilibrium, it is convenient to use Eq. (1.94) to express the spin vector of particles in terms of the particle part of the Wigner function:

$$S^\mu(k) = \frac{1}{2} \frac{\int_\Sigma d\Sigma \cdot k \text{tr} [\gamma^\mu \gamma^5 W^+(x, k)]}{\int_\Sigma d\Sigma \cdot k \text{tr} [W^+(x, k)]}. \quad (5.17)$$

Here, as pointed out in Section 1.6, the integration happens on the freeze-out surface, where considering hadronic fields as free is a good approximation. We can introduce a Wigner operator \widehat{W} in such a way that the Wigner function is its expectation value. Comparing with Eq. (1.54), the particle component of the Wigner operator is:

$$\widehat{W}_{ab}^+(x, k) = \theta(k^0) \theta(k^2) \frac{1}{(2\pi)^4} \int d^4s e^{-ik \cdot s} : \overline{\Psi}_b(x + s/2) \Psi_a(x - s/2) :, \quad (5.18)$$

where Ψ is the free Dirac field (the effective hadronic field in heavy ion applications), the colons $: \bullet :$ denote the normal ordering and θ is the Heaviside step

function. To calculate the corrections to polarization arising from the gradients of temperature, we can compute the same corrections for the Wigner operator using the local-equilibrium density operator (5.11) $W_{ab}^+(x, k) = \text{Tr} \left(\widehat{\rho}_{LE} \widehat{W}_{ab}^+(x, k) \right)$. Since the Wigner operator is a local operator, we can employ the hydrodynamic approximation to obtain Kubo-like formulae for the response of the Wigner function to gradients of the four-temperature, as discussed in Section 1.2.3. Using Eq. (1.36) for the Wigner operator we find:

$$\langle \widehat{W}_{ab}^+(x, k) \rangle_{LE} \simeq \langle \widehat{W}_{ab}^+(x, k) \rangle_{\beta(x)} + \Delta W_{ab}^+(x, k), \quad (5.19)$$

with:

$$\Delta W_{ab}^+(x, k) = - \int_0^1 dz \int_{\Sigma} d\Sigma_{\lambda}(y) \Delta \beta_{\rho}(x, y) \langle \widehat{W}_{ab}^+(x, k) \widehat{T}^{\lambda\rho}(y - iz\beta(x)) \rangle_{c, \beta(x)}, \quad (5.20)$$

where we are using the short-hand notation $\Delta \beta_{\rho}(x, y) = \partial_{\alpha} \beta_{\rho}(y - x)^{\alpha}$. The subscript c stands for the connected part of the two-point function and $\beta(x)$ denotes the thermal expectation value computed with the density operator:

$$\widehat{\rho}_0 = \frac{1}{Z} \exp[-\beta(x) \cdot \widehat{P}].$$

The 0-th order in gradients of the Wigner function can be read from Eq. (3.52) replacing b with $\beta(x)$:

$$W_0^+(x, k) = \langle \widehat{W}^+(x, k) \rangle_{\beta(x)} = (m + \not{k}) \delta(k^2 - m^2) \theta(k_0) \frac{1}{(2\pi)^3} n_F(\beta(x) \cdot k),$$

and is used to compute the denominator of Eq. (5.17).

The calculation of the first-order correction (5.20) in temperature gradients can be carried out as follows. First, we write the Belinfante energy-momentum tensor in terms of the Wigner operator. Using the relation between the Wigner function and the mean energy-momentum tensor (1.57c) together with the expression for the Belinfante tensor of the Dirac field in terms of the canonical one (3.93), we have:

$$\widehat{T}^{\lambda\rho}(y) = \frac{1}{2} \int d^4 k' \left(k'^{\rho} \text{tr} \left[\gamma^{\lambda} \widehat{W}(y, k') \right] + k'^{\lambda} \text{tr} \left[\gamma^{\rho} \widehat{W}(y, k') \right] \right),$$

hence Eq. (5.20) becomes

$$\begin{aligned} \Delta W_{ab}^+(x, k) = & - \frac{1}{2} \int_0^1 dz \int_{\Sigma} d\Sigma_{\lambda}(y) \Delta \beta_{\rho}(x, y) \times \\ & \int d^4 k' \sum_{cd} \left(k'^{\rho} \gamma_{dc}^{\lambda} + k'^{\lambda} \gamma_{dc}^{\rho} \right) \langle \widehat{W}_{ab}^+(x, k) \widehat{W}_{cd}(y - iz\beta(x), k') \rangle_{c, \beta(x)}, \end{aligned} \quad (5.21)$$

where a , b , c , and d are spinor indices. Therefore, we only need to calculate the correlator between two Wigner functions to find the correction (5.20)

To compute this correlator, we express the Wigner operator in terms of the Dirac fields. From Eq. (1.54) we find:

$$\begin{aligned} \widehat{W}_{ab}^+(x, k) = & \frac{1}{(2\pi)^3} \sum_{\tau, \tau'} \int \frac{d^3 p}{2\varepsilon_p} \int \frac{d^3 p'}{2\varepsilon_{p'}} \delta^4 \left(k - \frac{p+p'}{2} \right) \times \\ & e^{-ix \cdot (p' - p)} u_{\tau'}(p')_a \bar{u}_{\tau}(p)_b \widehat{a}_{\tau}^{\dagger}(p) \widehat{a}_{\tau'}(p'). \end{aligned}$$

Furthermore, the connected part of the thermal expectation value of four creation and annihilation operators is easily computable using Wick's theorem:

$$\langle \widehat{a}_1^\dagger \widehat{a}_2 \widehat{a}_3^\dagger \widehat{a}_4 \rangle_c = \langle \widehat{a}_1^\dagger \widehat{a}_2 \widehat{a}_3^\dagger \widehat{a}_4 \rangle - \langle \widehat{a}_1^\dagger \widehat{a}_2 \rangle \langle \widehat{a}_3^\dagger \widehat{a}_4 \rangle = \langle \widehat{a}_1^\dagger \widehat{a}_4 \rangle \langle \widehat{a}_2 \widehat{a}_3^\dagger \rangle,$$

where \widehat{a}_1^\dagger and \widehat{a}_2 come from the first Wigner operator and \widehat{a}_3^\dagger and \widehat{a}_4 from the second one. The two-point functions involving \widehat{a} and \widehat{a}^\dagger are well-known quantities, and they can also be recovered from Eq. (3.47) in the case of homogeneous equilibrium:

$$\begin{aligned} \langle \widehat{a}_\tau^\dagger(k) \widehat{a}_\sigma(q) \rangle_{\beta(x)} &= \delta_{\tau\sigma} 2\varepsilon_q \delta^3(\mathbf{k} - \mathbf{q}) n_F(k), \\ \langle \widehat{a}_{\tau'}(k') \widehat{a}_{\sigma'}^\dagger(q') \rangle_{\beta(x)} &= \delta_{\tau'\sigma'} 2\varepsilon_{q'} \delta^3(\mathbf{k}' - \mathbf{q}') (1 - n_F(k')), \end{aligned}$$

Using the property

$$\sum_{\sigma} u_{\sigma}(p) \bar{u}_{\sigma}(p) = \not{p} + m,$$

the correlator of two Wigner operators turns out to be:

$$\begin{aligned} \langle \widehat{W}_{ab}^+(x, k) \widehat{W}_{cd}(y - iz\beta(x), k') \rangle_{c, \beta(x)} &= \\ \frac{1}{(2\pi)^6} \int \frac{d^3 p}{2\varepsilon_p} \frac{d^3 p'}{2\varepsilon_{p'}} \delta^4\left(k - \frac{p+p'}{2}\right) \delta^4\left(k' - \frac{p+p'}{2}\right) & \\ \times (\not{p}' + m)_{ad} (\not{p} + m)_{cb} e^{i(p-p') \cdot (x-y)} e^{-z(p-p')\beta} n_F(p) (1 - n_F(p')). & \end{aligned}$$

Plugging this expression into Eq. (5.21) we find:

$$\begin{aligned} \Delta W_{ab}^+(x, k) &= - \int_0^1 dz \int_{\Sigma} d\Sigma_{\lambda}(y) \Delta \beta_{\rho}(x, y) \frac{1}{(2\pi)^6} \int \frac{d^3 p}{2\varepsilon_p} \int \frac{d^3 p'}{2\varepsilon_{p'}} \delta^4\left(k - \frac{p+p'}{2}\right) \times \\ \mathcal{T}^{\lambda\rho}(p, p')_{ab} e^{i(p-p') \cdot (x-y)} e^{-z(p-p')\beta} n_F(p) (1 - n_F(p')), & \end{aligned} \quad (5.22)$$

where \mathcal{T} is

$$\mathcal{T}^{\lambda\rho}(p, p')_{ab} = \frac{1}{4} \left[(\not{p}' + m) \gamma^{\lambda} (\not{p} + m) \right]_{ab} (p^{\rho} + p'^{\rho}) + \frac{1}{4} \left[(\not{p}' + m) \gamma^{\rho} (\not{p} + m) \right]_{ab} (p^{\lambda} + p'^{\lambda}). \quad (5.23)$$

Writing explicitly $\Delta\beta$ we have:

$$\begin{aligned} \Delta W_{ab}^+(x, k) &= - \int_0^1 dz \int_{\Sigma} d\Sigma_{\lambda}(y) \partial_{\kappa} \beta_{\rho} \frac{(y-x)^{\kappa}}{(2\pi)^6} \int \frac{d^3 p}{2\varepsilon_p} \int \frac{d^3 p'}{2\varepsilon_{p'}} \delta^4\left(k - \frac{p+p'}{2}\right) \\ \times \mathcal{T}^{\lambda\rho}(p, p')_{ab} e^{i(p-p') \cdot (x-y)} e^{-z(p-p')\beta} n_F(p) (1 - n_F(p')). & \end{aligned} \quad (5.24)$$

To evaluate Eq. (5.24), we use the Gauss theorem to move the integral from the freeze-out hypersurface to a constant-time hypersurface, which we denote by Σ_B . In addition, also a 4D integral over the region encompassed by the two hypersurfaces will be considered, see Fig. 5.1. The correction to the spin vector can be written as the sum of a term coming from the volume Ω and another one coming from the hypersurface Σ_B

$$\begin{aligned} S^{\mu}(k) \simeq S_{\partial\beta, \Omega}^{\mu}(k) + S_{\partial\beta, B}^{\mu}(k) &= \frac{1}{2} \frac{\int_{\Sigma} d\Sigma \cdot k \operatorname{tr} [\gamma^{\mu} \gamma^5 \Delta_{\Omega} W^+(x, k)]}{\int_{\Sigma} d\Sigma \cdot k \operatorname{tr} [W_0^+(x, k)]} \\ &+ \frac{1}{2} \frac{\int_{\Sigma} d\Sigma \cdot k \operatorname{tr} [\gamma^{\mu} \gamma^5 \Delta_B W^+(x, k)]}{\int_{\Sigma} d\Sigma \cdot k \operatorname{tr} [W_0^+(x, k)]}, \end{aligned} \quad (5.25)$$

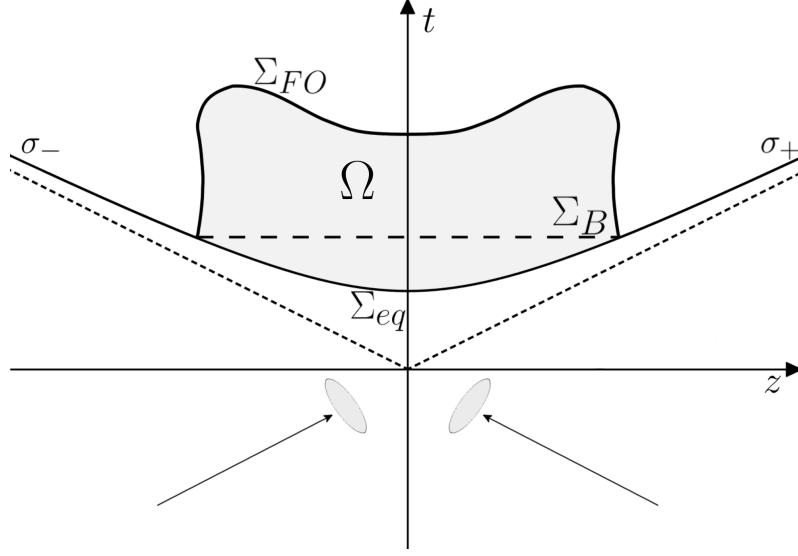


Figure 5.1: Sketch of a heavy-ion collision with the hypersurfaces mentioned in the text. Σ_{eq} represents the hypersurface where the true density operator (1.13) should be computed, and Σ_{FO} is the freeze-out hypersurface. The integration domain of the local-equilibrium density operator should be Σ_{FO} together with the hypersurfaces σ_{\pm} , but their contribution would be negligible. The hypersurface Σ_B and the 4D region Ω are also shown in the figure.

where:

$$\begin{aligned} \Delta_{\Omega} W_{ab}^+(x, k) = & - \int_0^1 dz \int_{\Omega} d^4 y \frac{\partial_{\kappa} \beta_{\rho}(x)}{(2\pi)^6} \int \frac{d^3 p}{2\varepsilon_p} \int \frac{d^3 p'}{2\varepsilon_{p'}} \delta^4 \left(k - \frac{p + p'}{2} \right) \\ & \times \mathcal{T}^{\rho\kappa}(p, p')_{ab} e^{i(p-p') \cdot (x-y)} e^{-z(p-p')^{\beta}} n_F(p) (1 - n_F(p')), \end{aligned}$$

and

$$\begin{aligned} \Delta_B W_{ab}^+(x, k) = & - \int_0^1 dz \int_{\Sigma_B} d\Sigma_{\lambda}(y) \partial_{\kappa} \beta_{\rho}(x) \frac{(y-x)^{\kappa}}{(2\pi)^6} \int \frac{d^3 p}{2\varepsilon_p} \int \frac{d^3 p'}{2\varepsilon_{p'}} \delta^4 \left(k - \frac{p + p'}{2} \right) \\ & \times \mathcal{T}^{\lambda\rho}(p, p')_{ab} e^{i(p-p') \cdot (x-y)} e^{z(p-p')^{\beta}} n_F(p) (1 - n_F(p')). \end{aligned}$$

Let us start with the 4D integral. Assuming that the region Ω is large enough in space, we can use the approximation:

$$\int_{\Omega} d^4 y e^{i(p-p') \cdot (x-y)} \simeq \delta t (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'),$$

where δt is the temporal extent of the region Ω . Thus:

$$\Delta_{\Omega} W_{ab}^+(x, k) = - \delta t \partial_{\kappa} \beta_{\rho}(x) \frac{1}{(2\pi)^3} \int \frac{d^3 p}{4\varepsilon_p^2} \delta^4(k - p) \mathcal{T}^{\rho\kappa}(p, p)_{ab} n_F(p) (1 - n_F(p)). \quad (5.26)$$

Given the definition of \mathcal{T} , Eq. (5.23), and using the fact that the trace

$$\text{tr} \left[\gamma^{\mu} \gamma^5 (\not{p}' + m) \gamma^{\lambda} (\not{p} + m) \right] = 4i \epsilon^{\mu\lambda\tau\sigma} p_{\tau} p'_{\sigma},$$

we readily find that

$$\text{tr} [\gamma^\mu \gamma^5 \Delta_\Omega W^+(x, k)] = 0,$$

so that there is no contribution to the polarization vector coming from the volume integral. This result was expected for the part coupling with the angular-momentum boost operator because it is conserved and therefore independent of the integration hypersurface. For the term coupling with \widehat{Q}_x this is not obvious, and in fact, it is a consequence of the hypersurfaces and the volume considered, and of the observable, the spin vector.

We are left with:

$$S^\mu(k) \simeq \frac{1}{2} \frac{\int_\Sigma d\Sigma \cdot k \text{tr} [\gamma^\mu \gamma^5 \Delta_B W^+(x, k)]}{\int_\Sigma d\Sigma \cdot k \text{tr} [W_0^+(x, k)]}, \quad (5.27)$$

We write down this contribution explicitly. The numerator of Eq. (5.27) reads:

$$\begin{aligned} \mathcal{N}^\mu &\equiv \int_\Sigma d\Sigma \cdot k \text{tr} [\gamma^\mu \gamma^5 \Delta_B W^+(x, k)] \\ &= \frac{-2}{(2\pi)^6} \int_\Sigma d\Sigma \cdot k \int_0^1 dz \int \frac{d^3 p}{2\varepsilon_p} \int \frac{d^3 p'}{2\varepsilon_{p'}} \delta^4 \left(k - \frac{p + p'}{2} \right) \times \\ &\quad \int_{\Sigma_B} d\Sigma_\lambda(y) \partial_\kappa \beta_\rho(x) (y - x)^\kappa e^{-z(p-p')^\beta} n_F(p) (1 - n_F(p')) \times \\ &\quad e^{i(p-p') \cdot (x-y)} \left[i \epsilon^{\mu\lambda\tau\sigma} p_\tau p'_\sigma k^\rho + i \epsilon^{\mu\rho\tau\sigma} p_\tau p'_\sigma k^\lambda \right] \end{aligned}$$

and the denominator is simply:

$$\mathcal{D} \equiv \int_\Sigma d\Sigma \cdot k \text{tr} [W_0^+(x, k)] = \frac{4m}{(2\pi)^3} \int d\Sigma \cdot k \delta(k^2 - m^2) \theta(k_0) n_F(k).$$

If we assume the hypersurface Σ_B to be much larger compared to the other scales, we can approximate it as a hyperplane. In such a case we recover a Dirac delta function and its derivative:

$$\begin{aligned} \int_{\Sigma_B} d\Sigma_\lambda(y) (y - x)^\kappa e^{i(p-p') \cdot (x-y)} &= \int_{\Sigma_B} d^3 y \hat{t}_\lambda (y - x)^\kappa e^{i(p-p') \cdot (x-y)} \\ &\simeq -\hat{t}_\lambda \Delta_{\kappa'}^\kappa (2\pi)^3 \frac{\partial}{\partial p'_{\kappa'}} \delta^3(\mathbf{p} - \mathbf{p}') + \hat{t}_\lambda \hat{t}^\kappa (2\pi)^3 \Delta t \delta^3(\mathbf{p} - \mathbf{p}'), \end{aligned} \quad (5.28)$$

where $\hat{t} = (1, \mathbf{0})$ is the unit vector normal to Σ_B , corresponding to the time direction in the QGP frame of Fig. 5.1, and $\Delta^{\mu\nu} = g^{\mu\nu} - \hat{t}^\mu \hat{t}^\nu$, $\Delta t = (y - x) \cdot \hat{t}$ and $y \cdot \hat{t}$ is constant in Σ_B by definition. The second term in Eq. (5.28), much like the volume contribution, contains a $\delta^3(\mathbf{p} - \mathbf{p}')$, which implies once again that its contribution to the polarization vector is vanishing, much like it happened for ΔW_Ω .

We are left with:

$$\begin{aligned} \mathcal{N}^\mu &\simeq -\frac{2}{(2\pi)^3} \int_\Sigma d\Sigma \cdot k \int_0^1 dz \int \frac{d^3 p}{2\varepsilon_p} \int \frac{d^3 p'}{2\varepsilon_{p'}} \delta^4 \left(k - \frac{p + p'}{2} \right) \partial_\kappa \beta_\rho(x) \\ &\quad \times \hat{t}_\lambda \Delta_{\kappa'}^\kappa \frac{\partial}{\partial p'_{\kappa'}} \delta^3(\mathbf{p} - \mathbf{p}') \left[\epsilon^{\mu\lambda\tau\sigma} p_\tau p'_\sigma k^\rho + \epsilon^{\mu\rho\tau\sigma} p_\tau p'_\sigma k^\lambda \right] e^{-z(p-p')^\beta} n_F(p) (1 - n_F(p')). \end{aligned}$$

Given that the square bracket is vanishing for $p = p'$, the integration by parts yields:

$$\begin{aligned} \mathcal{N}^\mu &= \frac{1}{(2\pi)^3} \int_\Sigma d\Sigma \cdot k \partial_\kappa \beta_\rho(x) \int \frac{d^3 p}{2\varepsilon_p^2} \delta^4(k - p) n_F(p) (1 - n_F(p)) \\ &\quad \times \hat{t}_\lambda \Delta_{\kappa'}^\kappa \left[\epsilon^{\mu\lambda\tau\sigma} p_\tau \frac{\partial p_\sigma}{\partial p_{\kappa'}} k^\rho + \epsilon^{\mu\rho\tau\sigma} p_\tau \frac{\partial p_\sigma}{\partial p_{\kappa'}} k^\lambda \right]. \end{aligned}$$

We can then integrate over p getting:

$$\begin{aligned} \mathcal{N}^\mu &= \frac{1}{(2\pi)^3} \int_{\Sigma} d\Sigma \cdot k \partial_\kappa \beta_\rho(x) \theta(k_0) \delta(k^2 - m^2) n_F(k) (1 - n_F(k)) \\ &\quad \times \hat{t}_\lambda \Delta_{\kappa'}^\kappa \frac{\partial k_\sigma}{\partial k_{\kappa'}} \frac{k_\tau}{\varepsilon_k} \left[\epsilon^{\mu\lambda\tau\sigma} k^\rho + \epsilon^{\mu\rho\tau\sigma} k^\lambda \right]. \end{aligned}$$

We have to be careful in computing the derivatives: due to the presence of the projector, κ is a spatial index and the derivatives are only with respect to the spatial components of the momentum k . But since k is now on-shell, the k^0 component has a non-trivial dependence on $k^{i=1,2,3}$. Therefore:

$$\frac{\partial k_\sigma}{\partial k_{\kappa'}} = \Delta_\sigma^{\kappa'} - \hat{t}_\sigma \frac{k^{\kappa'}}{\varepsilon_k}.$$

Using this expression, it is easy to see:

$$\partial_\kappa \beta_\rho \hat{t}_\lambda \Delta_{\kappa'}^\kappa \frac{\partial k_\sigma}{\partial k_{\kappa'}} = \partial_\sigma \beta_\rho \hat{t}_\lambda - \frac{k^\kappa}{\varepsilon} \partial_\kappa \beta_\rho \hat{t}_\lambda \hat{t}_\sigma,$$

and we can compute explicitly:

$$\begin{aligned} &\partial_\kappa \beta_\rho \hat{t}_\lambda \Delta_{\kappa'}^\kappa \frac{\partial k_\sigma}{\partial k_{\kappa'}} \left[\epsilon^{\mu\lambda\tau\sigma} k^\rho + \epsilon^{\mu\rho\tau\sigma} k^\lambda \right] \\ &= \left(\partial_\sigma \beta_\rho \hat{t}_\lambda - \frac{k^\kappa}{\varepsilon} \partial_\kappa \beta_\rho \hat{t}_\lambda \hat{t}_\sigma \right) \left[\epsilon^{\mu\lambda\tau\sigma} k^\rho + \epsilon^{\mu\rho\tau\sigma} k^\lambda \right] \\ &= \partial_\sigma \beta_\rho \hat{t}_\lambda k^\rho \epsilon^{\mu\lambda\tau\sigma} + \varepsilon \epsilon^{\mu\rho\tau\sigma} \partial_\sigma \beta_\rho - \partial_\kappa \beta_\rho \hat{t}_\lambda k^\kappa \epsilon^{\mu\rho\tau\lambda} \\ &= \varepsilon \epsilon^{\mu\rho\tau\sigma} \partial_\sigma \beta_\rho + (\partial_\sigma \beta_\rho + \partial_\rho \beta_\sigma) \hat{t}_\lambda k^\rho \epsilon^{\mu\lambda\tau\sigma}, \end{aligned}$$

where in the last step we have changed the name of the indices and used the properties of the Levi-Civita symbol. Using the definitions of the thermal vorticity and the thermal shear (5.12), we can write:

$$\partial_\kappa \beta_\rho \hat{t}_\lambda \Delta_{\kappa'}^\kappa \frac{\partial k_\sigma}{\partial k_{\kappa'}} \left[\epsilon^{\mu\lambda\tau\sigma} k^\rho + \epsilon^{\mu\rho\tau\sigma} k^\lambda \right] = -\varepsilon \epsilon^{\mu\rho\sigma\tau} \varpi_{\rho\sigma} - 2\xi_{\sigma\rho} \hat{t}_\lambda k^\rho \epsilon^{\mu\lambda\sigma\tau},$$

so that we can identify the contribution coming from thermal vorticity and the one from thermal shear. For the thermal vorticity we have:

$$\mathcal{N}_\varpi^\mu = \frac{-1}{(2\pi)^3} \int_{\Sigma} d\Sigma \cdot k \theta(k_0) \delta(k^2 - m^2) n_F(k) (1 - n_F(k)) \epsilon^{\mu\nu\sigma\tau} \varpi_{\nu\sigma} k_\tau,$$

whence:

$$S_\varpi^\mu(k) = -\frac{1}{8m} \epsilon^{\mu\nu\sigma\tau} k_\tau \frac{\int_{\Sigma} d\Sigma \cdot k n_F (1 - n_F) \varpi_{\nu\sigma}}{\int_{\Sigma} d\Sigma \cdot k n_F}, \quad (5.29)$$

which is the well-known expression (2.3) for polarization. Similarly for the thermal shear:

$$\mathcal{N}_\xi^\mu = -\frac{2}{(2\pi)^3} \int_{\Sigma} d\Sigma \cdot k \theta(k_0) \delta(k^2 - m^2) n_F(k) (1 - n_F(k)) \epsilon^{\mu\nu\sigma\tau} k_\tau \hat{t}_\nu \xi_{\sigma\rho} \frac{k^\rho}{\varepsilon_k},$$

and we have an additional contribution to polarization:

$$\boxed{S_\xi^\mu(p) = -\frac{1}{4m} \epsilon^{\mu\nu\sigma\tau} \frac{p_\tau p^\rho}{\varepsilon} \frac{\int_{\Sigma} d\Sigma \cdot p n_F (1 - n_F) \hat{t}_\nu \xi_{\sigma\rho}}{\int_{\Sigma} d\Sigma \cdot p n_F}}, \quad (5.30)$$

which is, in general, non-vanishing. This term represents a non-dissipative, non-equilibrium contribution to the spin polarization vector in local thermodynamic equilibrium, and it should be added to Eq. (5.29) so that the total polarization in local equilibrium up to first order in gradients of the four-temperature reads:

$$S_{LE}^\mu(k) = -\frac{1}{8m} \epsilon^{\mu\nu\sigma\tau} k_\tau \frac{\int_\Sigma d\Sigma \cdot k n_F (1 - n_F) \left(\varpi_{\nu\sigma} + 2\hat{t}_\nu \xi_{\sigma\rho} \frac{p^\rho}{\varepsilon} \right)}{\int_\Sigma d\Sigma \cdot k n_F}. \quad (5.31)$$

The most remarkable difference between Eqs. (5.29) and (5.30) is that the latter depends on a particular vector \hat{t}^μ , and thus breaks covariance. This is, however, to be expected because the local-equilibrium density operator itself depends on the choice of the integration hypersurface. To linear order in gradients, the $\hat{Q}^{\mu\nu}$ operator, to which the thermal shear couples, makes this dependence explicit as it is not conserved, but the dependence would still be present at higher orders, as has been discussed in Section 5.2. The vector \hat{t}^μ should be interpreted as an approximation of the unit vector perpendicular to the hypersurface Σ_{FO} .

5.4 High energy heavy-ion collisions and isothermal local equilibrium

The newly found formula for polarization should be tested numerically and its predictions compared with experimental data. For such a comparison, we have in mind a particular experimental setting: very high-energy heavy-ion collisions. Indeed, the most reliable measurements of local polarization have been performed at 200 GeV and above, whereas for lower collision energies the statistics are still poor. It is worth asking if Eq. (5.31) is the most appropriate formula to describe polarization in high-energy heavy-ion collisions. The formula (5.31) has been derived under the very general assumption of hydrodynamic regime, and we have not used any additional hypothesis to simplify or improve the formula. In fact, if we aim at computing polarization in high-energy nuclear collisions we can do better.

If the heavy ions collide at very high energy, it is well known that the chemical potentials are vanishing with very good approximation, and the only relevant intensive thermodynamic parameter is the temperature T . This observation has a very important consequence. Consider the expanding QGP created in a high-energy collision: we know that at a certain point the QGP will stop behaving as a fluid, crossing the freeze-out hypersurface. It is very reasonable to assume that the parameterization of the decoupling hypersurface can be given in terms of the thermodynamic variables characterising the QGP, but for sufficiently high collision energy the only parameter at our disposal is the temperature. Thus, any parameterization of the decoupling hypersurface boils down to $T(x) = T_{\text{dec}}$ for some constant value of the decoupling temperature T_{dec} .

With this in mind, let us consider the local-equilibrium density operator. As we have stressed above, if the collision energy is high enough, the freeze-out hypersurface in the local-equilibrium operator is just a constant-temperature hypersurface:

$$\hat{\rho}_{LE} = \frac{1}{Z} \exp \left[- \int_{\Sigma_{FO}} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \hat{\zeta} j^\mu \right) \right] \simeq \frac{1}{Z} \exp \left[- \int_{T=T_{\text{dec}}} d\Sigma_\mu \hat{T}^{\mu\nu} \beta_\nu \right],$$

where the chemical potential is negligible. Furthermore, since $\beta^\mu = u^\mu / T$, u^μ being

the four-velocity, the isothermal local-equilibrium density operator reads:

$$\hat{\rho}_{ILE} = \frac{1}{Z} \exp \left[-\frac{1}{T_{\text{dec}}} \int_{T=T_{\text{dec}}} d\Sigma_{\mu} \hat{T}^{\mu\nu} u_{\nu} \right]. \quad (5.32)$$

We have shown that, in the conditions described above, the local-equilibrium density operator naturally reduces to the isothermal local-equilibrium density operator (5.32). Although $\hat{\rho}_{LE}$ and $\hat{\rho}_{ILE}$ are precisely equivalent in an isothermal decoupling scenario, the linear-response theory results differ depending on whether we use one or the other. The reason is that in linear-response theory we have employed so far, the expansion was made in terms of the gradients of the four-temperature vector β^{μ} , thus involving both the gradients of temperature and of the four-velocity. Notice that, despite the decoupling hypersurface being parametrized by a constant temperature, the gradients of temperature are not zero, but are directed in the orthogonal direction to the hypersurface. Therefore, the gradients of temperature would play a role in such an expansion, as it can be readily seen in Eq. (5.31). In contrast, since the temperature is a constant in the operator (5.32), the linear-response theory formulae applied to this operator would involve only the gradients of u^{μ} .

If the mean spin vector could be computed exactly independently of the hypersurface, then the hypothesis of isothermal decoupling could be used both at the beginning of the calculation (i.e., when choosing between $\hat{\rho}_{LE}$ and $\hat{\rho}_{ILE}$) or at the end, and the result would be the same. But having to resort to linear-response theory, it is paramount to choose the best density operator before computing the gradient expansion. In the case of high-energy nuclear collisions, the best choice is (5.32).

Without repeating the full calculation, it is easy to see what the result of polarization at linear order of gradients would be had we used the density operator (5.32). It suffices substitute in Eq. (5.31)*:

$$\partial_{\mu} \beta_{\nu} \mapsto \frac{1}{T_{\text{dec}}} \partial_{\mu} u_{\nu}.$$

Defining the kinematic vorticity $\omega_{\mu\nu}$ and shear $\Xi_{\mu\nu}$ as:

$$\omega_{\mu\nu} = -\frac{1}{2}(\partial_{\mu} u_{\nu} - \partial_{\nu} u_{\mu}), \quad \Xi_{\mu\nu} = \frac{1}{2}(\partial_{\mu} u_{\nu} + \partial_{\nu} u_{\mu}), \quad (5.33)$$

the spin-polarization vector corresponding to an isothermal local equilibrium reads:

$$S_{ILE}^{\mu}(k) = -\frac{1}{8m} \epsilon^{\mu\nu\sigma\tau} k_{\tau} \frac{\int_{\Sigma} d\Sigma \cdot k n_F (1 - n_F) \left(\omega_{\nu\sigma} + 2\hat{t}_{\nu} \Xi_{\sigma\rho} \frac{v^{\rho}}{\varepsilon} \right)}{T_{\text{dec}} \int_{\Sigma} d\Sigma \cdot k n_F}. \quad (5.34)$$

Notice that Eq. (5.31) *does not* reduce to Eq. (5.34) if we integrate on the isothermal freeze-out hypersurface. The reason is that Eq. (5.31) includes the gradients of temperature, which are non-vanishing in the orthogonal direction to the hypersurface and contribute to polarization. In Eq. (5.34) such gradients are absent.

From the above discussion, we expect S_{ILE} (5.34) to give better results than S_{LE} (5.31) for high-energy collisions. On the other hand, and for the same reasons, we expect S_{LE} to perform better at lower collision energy, but a better approximation than Eq. (5.31) might be worked out also in that case.

*We would like to stress that this is just an *effective* replacement, and the gradients of temperature are not zero neither in Eq. (5.31) nor in Eq. (5.34).

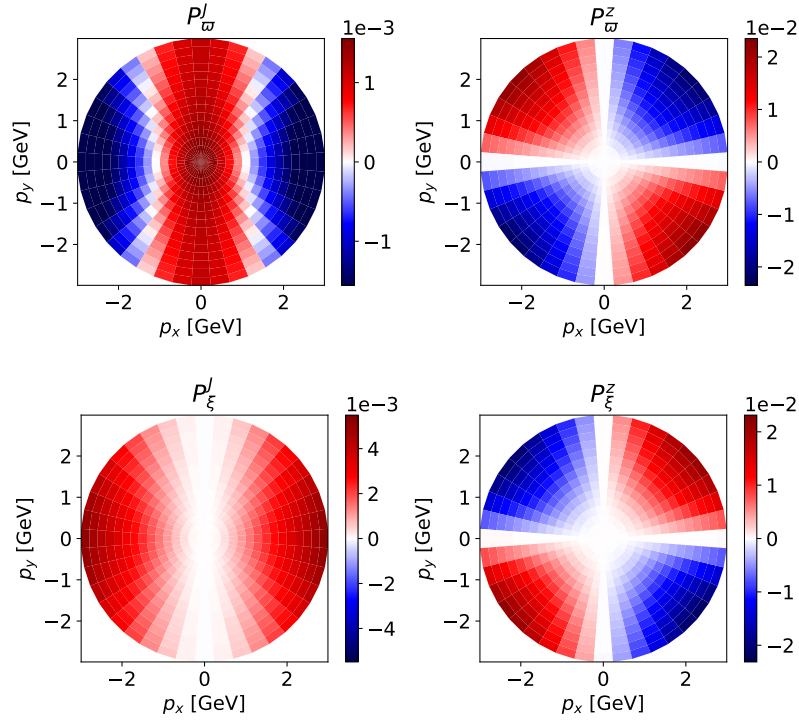


Figure 5.2: The components of the polarization vector along the angular-momentum direction P^J and the beam axis P^z . The upper and lower panels show respectively the calculations using only thermal vorticity and only thermal shear.

5.5 Polarization in Au-Au collisions at 200 GeV

As shown in the previous section, the thermal shear does contribute to polarization and cannot be neglected for a reliable prediction of polarization of Λ particles.

Formulae (5.31) and (5.34) require the use of a viscous-hydrodynamic code for the evaluation of the gradients of the four-temperature (or velocity) and numerical integration over the freeze-out hypersurface.

The chain of calculations used in this work is the following. The initial state is an averaged entropy-density profile, from a Monte-Carlo Glauber model generated by Glissando v. 2.702 [161]. For the simulation of the hydrodynamic stage of the evolution of the QGP, we used vHLLE[†] [163]. In particular, the branches “polarization_dbeta_dec2021_noTgrad” and “polarization_dbeta_dec2021” are used. These branches produce a file containing the relevant variable on the freeze-out hypersurface: the former stores the gradients of the four-velocity, the latter of the four-temperature. Finally, the freeze-out files are fed into a code called “particlizationCalc” that computes the integral over the hypersurface[‡], yielding either the result of Eq. (5.34) or (5.31) depending on the branch of vHLLE used.

The formulae obtained in the previous chapter are used in the final step of the chain, and here we test their results compared to experimental data for Au-Au

[†]There have been some changes in how the initial state is fed into the hydrodynamic code after the publication [17], that’s why the results presented here do not match exactly those of that paper. The modifications of vHLLE are briefly described in Ref. [162]

[‡]link to the repository <https://github.com/yukarpenko/particlizationCalc>

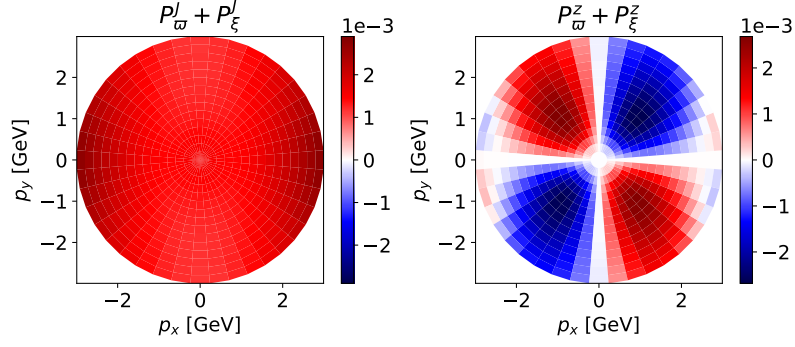


Figure 5.3: Polarization along the angular momentum direction P^J and the beam axis P^z as predicted by Eq. (5.31).

collision at 200 GeV. The initial state is generated by averaging 2×10^4 Glissando events, and the centrality range is set to 20-60% to match the measurements [164, 165]. The shear viscosity over entropy density is set to $\eta/s = 0.08$ and we assume the bulk viscosity is vanishing. When not stated otherwise, the critical energy density used to identify the freeze-out hypersurface is $\varepsilon_{\text{crit}} = 0.32 \text{ GeV}/\text{fm}^3$, corresponding to a decoupling temperature of about 155 MeV.

We start by calculating polarization with Eq. (5.31). Figure 5.2 shows in the upper panel the polarization coming from the thermal vorticity (5.29) and in the lower one the polarization from the thermal shear (5.30). The upper panel exhibits the typical behaviour which gave rise to the sign puzzle: the polarization along the beam direction is first negative and then positive measuring the azimuthal angle rotating counter-clockwise from the reaction plane, and the polarization along the angular momentum is maximal along the angular-momentum direction. Both these results are at variance with experimental data, as already discussed. Promisingly, the lower panel shows the opposite trend, which agrees with the experimental data.

The total polarization is given by formula (5.31), where both the thermal vorticity and the thermal shear are taken into account. The results of the calculation of Eq. (5.31) are shown in Fig. 5.3, where we can see that the total polarization is still incompatible with experimental data: P^J is almost constant, and P^z has the wrong sign.

However, as argued in the previous section, Eq. (5.31) might not be the best approximation for high-energy heavy-ion collisions. Therefore, we evaluate the results of the isothermal-equilibrium approximation and Eq. (5.34). We can see, in Fig. 5.4, that again the contribution to polarization coming from the kinematic vorticity has the wrong sign compared to the experimental data, whereas the polarization from shear is consistent with them. Furthermore, if the isothermal freeze-out prescription is used, the contribution originating from the kinematic shear is slightly larger than the one from the kinematic vorticity.

The upper panel of Fig. 5.5 shows the total isothermal local-equilibrium polarization, computed using Eq. (5.34). The plots show qualitative agreement with the data, as the polarization along the angular momentum is maximal along the reaction plane and the pattern of the polarization along the beam axis is consistent with the experimental findings.

To give a more quantitative statement, we compare the prediction of Eq. (5.34) with the local-polarization data from the STAR experiment [164, 165]. The lower

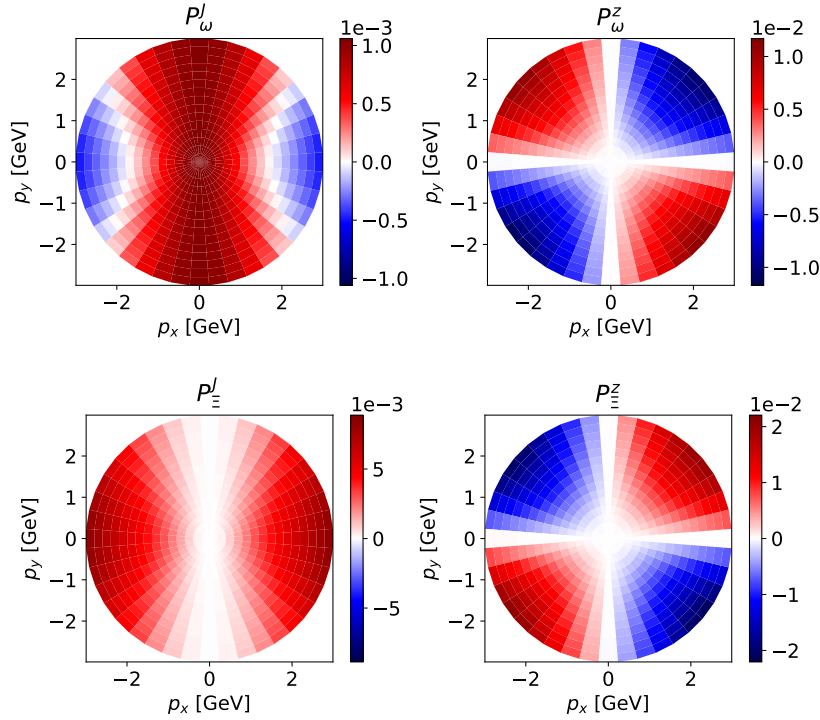


Figure 5.4: The components of the polarization vector along the angular momentum direction P^J and along the beam axis P^z in the isothermal freeze-out scenario. The upper and lower panels show respectively the calculations using only kinematic vorticity and only kinematic shear.

panel of Fig. 5.5 shows the azimuthal dependence of the polarization along the angular-momentum direction P^J and the beam axis P^z . To show the azimuthal dependence, the p_T spectrum of particles is also integrated over the range p_T 0.5 – 6 GeV, which is the same range of integration as the data. We can see that the agreement with experimental data is restored for a decoupling temperature of about 155 MeV. Polarization, in particular along the beam direction, exhibits a strong dependence on the decoupling temperature, as shown in the lower-right panel of Fig. 5.5. For a too high decoupling temperature, the oscillation is too large, whereas if the decoupling happens at temperatures as low as $T_{\text{dec}} = 130$ MeV, the sign-puzzle presents itself once again. For what concerns P^J , we can see that its temperature dependence is much milder and, although it is now predicted to be maximal along the reaction plane, its dependence on ϕ is flatter compared to the data.

Furthermore, it is important to notice that the addition of the thermal shear does not spoil the agreement with the data for what concerns the global polarization, as has been shown for example in Ref. [166].

Thus, the contribution of thermal shear to polarization, alongside the isothermal-decoupling prescription, is able to restore agreement between the hydrodynamic model of heavy-ion collisions and experimental data. This finding further supports the idea that the QGP behaves like a locally equilibrated fluid at hadronization, even in the spin sector. On the other hand, the quantitative impact of dissipative corrections is still to be investigated.

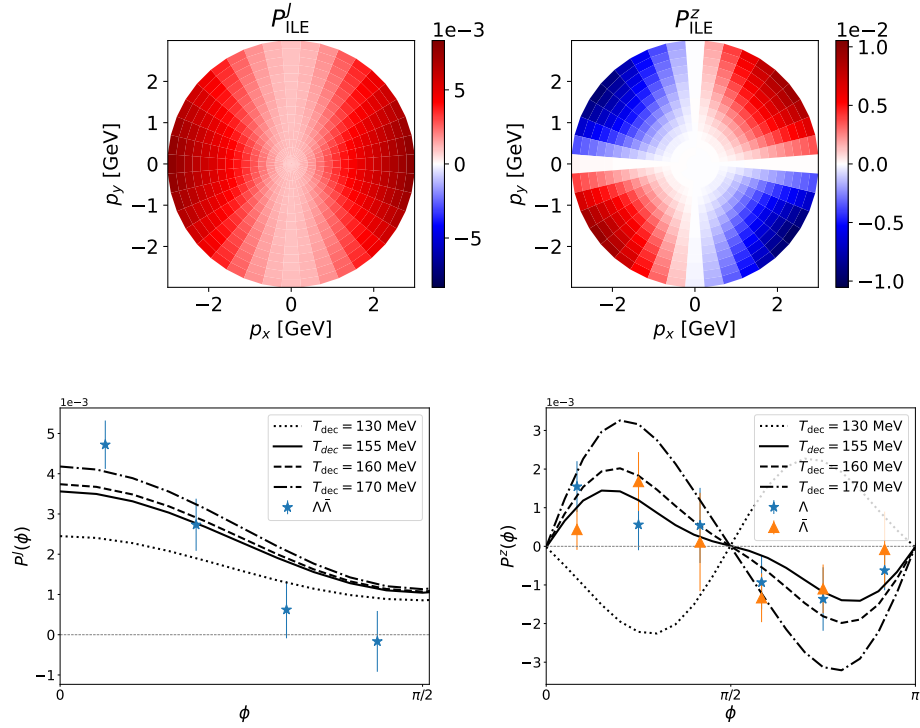


Figure 5.5: Top panel: total polarization along the angular-momentum direction P^J and the beam axis P^z in the isothermal freeze-out scenario, as predicted by Eq. (5.34). Bottom panel: Azimuthal dependence of the polarization along the angular-momentum direction (left) and the beam axis (right). Data points are taken from Refs. [164] and [165]. For $P^z(\phi)$, to convert from the experimentally measured $\langle \cos \theta_p^* \rangle$ to polarization, we used $\alpha_H = 0.732$ and $\langle (\cos \theta_p^*)^2 \rangle = 1/3$.

Conclusions

The spin physics of relativistic quantum fluids has been studied in detail, focusing on Dirac fermions. The tool used in our analysis has been Zubarev's density operator, which allows for a fully quantum description of the statistical system. Both the cases of global and local equilibrium have been considered. In the former, we have developed a new technique to compute thermal expectation values based on an iterative procedure and the regularization of particular series of functions using analytic distillation. For the distillation of our results, we derived two new asymptotic power expansions of series of functions using the method of the generalized Mellin transform. The iterative procedure allowed us to obtain an exact expression of the Wigner function in global equilibrium, the mean energy-momentum tensor, and the vector and axial currents in the case of massless fermions in equilibrium with a generic non-vanishing vorticity. With the same technique, we have studied mass corrections to the expectation values of currents up to the leading order. The calculation of exact expectation values with the analytic distillation proves quite challenging for massive particles, and more work is needed in this respect.

We have computed the exact polarization vector for particles with any spin in global equilibrium using the iterative method proposed to calculate expectation values. The formula extends the results of the previous literature to all order in thermal vorticity. We have estimated the quantitative importance of the exact polarization formula in heavy-ion phenomenology by studying the global polarization and comparing it to STAR data. Our results show that the correction to the linear approximation, which has been exclusively used in polarization studies so far, is negligible: the two formulae are indistinguishable within experimental uncertainties. Therefore we conclude that corrections of higher-order terms in thermal vorticity cannot explain the discrepancy between the theoretical predictions and the experimental data for the local polarization.

The polarization of massless particles has also been addressed. We have provided a general formula connecting the Pauli-Lubanski vector of massless Dirac fermions to the Wigner function. This formula differs from the ones proposed in the literature in that the formula derived here predicts the polarization vector to be directed only along the particle's momentum. Indeed, this is a characteristic feature of massless particles, which can only have the spin parallel or anti-parallel to their momentum. We have also provided an analytical expression for the polarization vector of massless particles with any helicity in global equilibrium.

Finally, we have investigated fluids in local equilibrium. In this framework, the importance of the thermal shear tensor, which is vanishing in global equilibrium, has been realized. We have computed the leading-order contribution of the thermal shear to the polarization vector and showed that, alongside the thermal vorticity, it provides an additional source of polarization. Thermal shear-induced polarization

represents a non-dissipative non-equilibrium effect because it originates from the local-equilibrium density operator. Furthermore, it breaks covariance, as the final formula for polarization depends on the hypersurface where the fluid is in local equilibrium. The thermal shear can have an impact on other phenomenological observables in the locally equilibrated QGP, as its magnitude is comparable to, or even larger than, the thermal vorticity on the freeze-out hypersurface.

The polarization of the Λ hyperon has been calculated numerically, including the newly found thermal shear contribution. The QGP evolution has been simulated using a 3+1 hydrodynamic code until the freeze-out stage, where another code was used to compute the integrals relevant to polarization. The results show that the geometry of the freeze-out hypersurface strongly affects the polarization. We have studied two scenarios: a generic one, where no information about the freeze-out hypersurface is assumed, and an isothermal one, which should be a good approximation for high-energy collisions. The former could not solve the sign puzzle for collisions at 200 GeV, but the latter performed much better. In the isothermal freeze-out scenario the kinematic vorticity and shear are used in place of the thermal ones. This scenario restores the agreement between experimental data and theoretical predictions for a decoupling temperature very close to the chemical freeze-out temperature of the Λ particles. Solving the spin-polarization sign puzzle using the local-equilibrium density operator represents a further success of the quasi-ideal fluid paradigm of the Quark-Gluon Plasma.

Appendix A

Spinors and group theory

This appendix is a review of the spinor and Dirac formalism from a group-theoretical perspective. We will focus on the spin-1/2 case, but the same construction can be applied quite generally to fields of arbitrary spin [40, 41].

The Dirac field describes particles and antiparticles transforming under the $(0, 1/2) \oplus (1/2, 0)$ projective representation of $\text{SO}(1, 3)^\uparrow$, the orthochronous component of the Lorentz group. Therefore we start by reviewing these representations individually. To begin with, we consider the one-to-one correspondence between four-vectors and 2×2 hermitian matrices, defined by:

$$\underline{x} = x^\mu \sigma_\mu = x^0 \mathbb{I} + x^1 \sigma_1 + x^2 \sigma_2 + x^3 \sigma_3 = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \quad (\text{A.1})$$

Here we use the same notation introduced in Section 1.3: $\sigma_\mu = (\mathbb{I}, \sigma_1, \sigma_2, \sigma_3)$ and $\bar{\sigma}_\mu = (\mathbb{I}, -\sigma_1, -\sigma_2, -\sigma_3)$, where σ_i are the Pauli matrices. Lorentz transformations on this matrix space are represented by $\text{SL}(2, \mathbb{C})$ matrices, which we denote as $D(\Lambda)$, acting on the four vector \underline{x} as:

$$\Lambda \underline{x} = D(\Lambda) \underline{x} D(\Lambda)^\dagger,$$

This rule fixes the matrix $D(\Lambda)$ up to a sign, defining a two-to-one correspondence between $\text{SL}(2, \mathbb{C})$ and $\text{SO}(1, 3)^\uparrow$.

The above equation describes the $(0, 1/2)$ projective representation $D^{(0,1/2)}$ of $\text{SO}(1, 3)^\uparrow$, where the generators of boosts and rotations are respectively $D^{(0,1/2)}(K_i) = i\sigma_i/2$ and $D^{(0,1/2)}(J_i) = \sigma_i/2$. Explicitly:

$$D(\Lambda) = D^{(0,1/2)}(\Lambda).$$

A different map between four vectors and hermitian matrices can be defined:

$$\widehat{x} \equiv x^\mu \bar{\sigma}_\mu = x^0 \mathbb{I} - x^1 \sigma_1 - x^2 \sigma_2 - x^3 \sigma_3 = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix}.$$

The \widehat{x} and \underline{x} matrices have the property:

$$\widehat{x} \underline{x} = \underline{x} \widehat{x} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x \cdot x \mathbb{I} = x^2 \mathbb{I}. \quad (\text{A.2})$$

Since this product is Lorentz-invariant, we can infer the transformation rule of \widehat{x} from that of \underline{x} . For Eq. (A.2) to be valid in any reference frame, one needs:

$$\widehat{\Lambda x} = D(\Lambda)^{\dagger-1} \widehat{x} D(\Lambda)^{-1}.$$

The representation corresponding to this rule is the $(1/2, 0)$ representation of $\text{SO}(1, 3)^\dagger$. Its generators are $D^{(1/2,0)}(K_i) = -i\sigma_i/2$ for boosts and $D^{(1/2,0)}(J_i) = \sigma_i/2$ for rotations. Hence we have the relation:

$$D^{(1/2,0)}(\Lambda) = D^{(0,1/2)}(\Lambda)^{\dagger-1} = D(\Lambda)^{\dagger-1}.$$

We can apply this construction to one-particle states and their momentum. As we have also remarked in the main text, the construction of one-particle states starts from the definition of a standard vector \mathbf{p} and a standard Lorentz transformation $[p]$ (depending on p and \mathbf{p}) transforming the standard vector \mathbf{p} to the momentum p . In the $(0, 1/2)$ and $(1/2, 0)$ representations, this statement reads:

$$D([p])\underset{\sim}{\mathbf{p}}D([p])^\dagger = \underset{\sim}{p}, \quad D([p])^{\dagger-1}\widehat{\mathbf{p}}D([p])^{-1} = \widehat{p}. \quad (\text{A.3})$$

Dealing with momentum vectors, Eq. (A.2) now gives the identity $\underset{\sim}{p} \widehat{p} = m^2\mathbb{I}$, where m is the mass of the particle.

The identification of \mathbf{p} and $[p]$ changes depending if the one-particle state is massive or massless. For massive particles, it is more convenient to choose $\mathbf{p} = (m, \mathbf{0})$, so that the standard momentum identifies the rest frame. In this case, the standard transformation is a pure boost. For massless particles a typical choice is $\mathbf{p} = (\kappa, 0, 0, \kappa)$, where $\kappa > 0$. The standard transformation is the product of a boost along the z axis and a rotation around the $\hat{\mathbf{z}} \times \hat{\mathbf{p}}$ axis.

The subgroup of Lorentz transformations that leave the standard vector invariant, called the *little group*, plays a crucial role in the transformation rules of particle states and creation and annihilation operators. For massive particles, the little group is given by the group of rotations in three dimensions, $\text{SO}(3)$. Although this is almost obvious, it is instructive to check it in the $(1/2, 0)$ representation. Using Eq. (A.1) the standard momentum reads $\underset{\sim}{\mathbf{p}} = m\mathbb{I}$, so that Lorentz transformations that leave it invariant must obey $D(\Lambda)^\dagger D(\Lambda) = \mathbb{I}$, meaning $D(\Lambda)$ is unitary. Since the generators of rotations are hermitian in this representation (the generators of boosts are anti-hermitian), unitary matrices in $\text{SL}(2, \mathbb{C})$ correspond to rotations.

On the other hand, for massless particles with $\mathbf{p} = (\kappa, 0, 0, \kappa)$, we have:

$$\underset{\sim}{\mathbf{p}} = \begin{pmatrix} 2\kappa & 0 \\ 0 & 0 \end{pmatrix},$$

and transformations leaving this matrix invariant can be parametrized as:

$$D(\Lambda) = \begin{pmatrix} e^{-i\phi/2} & Ze^{-i\phi/2} \\ 0 & e^{i\phi/2} \end{pmatrix}. \quad (\text{A.4})$$

The real parameter ϕ is the rotation angle around the z -axis, whereas the complex Z parameter corresponds to a translation in the Euclidean x - y plane [44].

Given a generic Lorentz transformation Λ , it is possible to build an element of the little group of \mathbf{p} as $W(\Lambda, p) = [\Lambda p]^{-1}\Lambda[p]$: this transformation maps \mathbf{p} to p , then to Λp and finally back to \mathbf{p} , so that \mathbf{p} is unchanged. Such a matrix, known as the *Wigner rotation*, dictates how creation and annihilation operators transform under Lorentz transformations. In fact, one has:

$$\widehat{\Lambda}\widehat{a}_r(p)\widehat{\Lambda}^{-1} = \sum_s D(W(\Lambda, p))_{rs}^\dagger \widehat{a}_s(\Lambda p), \quad (\text{A.5})$$

where we denote $\widehat{\Lambda}$ as the representation of Λ on the Hilbert space. As we have shown above, for massive particles $W(\Lambda, p)$ is a unitary matrix (i.e., a rotation), but this does not seem to be the case for massless particles (see Eq. (A.4)), unless the parameter Z is set to zero. However, this is a physical requirement for representations of physical states: known massless particle states transform according to representations with $Z = 0$ [40, 43, 44]. Therefore, when acting on on-shell states, the Wigner rotation can always be considered unitary.

We can now study the transformation rule of the Dirac field. To prove that it transforms in the $(0, 1/2) \oplus (1/2, 0)$ representation we should show:

$$\widehat{\Lambda}\Psi(x)\widehat{\Lambda}^{-1} = S(\Lambda)^{-1}\Psi(\Lambda x), \quad (\text{A.6})$$

where $S(\Lambda)$ is given in the Eq. (1.43). We show this for the particle term only, using the notation (1.41) with the spinors (1.42):

$$\Psi_+(x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3p}{2\varepsilon} e^{-ip \cdot x} U(p) \widehat{A}(p).$$

The transformation rule (A.5) in compact notation reads:

$$\widehat{\Lambda}\widehat{A}(p)\widehat{\Lambda}^{-1} = D(W(\Lambda, p))^\dagger \widehat{A}(\Lambda p),$$

so that the field transforms as:

$$\widehat{\Lambda}\Psi_+(x)\widehat{\Lambda}^{-1} = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3p}{2\varepsilon} e^{-ip \cdot x} U(p) D(W(\Lambda, p))^\dagger \widehat{A}(\Lambda p).$$

Writing the spinor $U(p)$ as Eq. (1.42), and using the invariance of $\widehat{\mathbf{p}}$ under the Wigner rotation, the transformation rule (A.3), and the unitarity of the Wigner rotation, we have:

$$\begin{aligned} U(p)D(W(\Lambda, p))^\dagger &= \begin{pmatrix} D([\mathbf{p}]) & 0 \\ 0 & D([\mathbf{p}])^{\dagger-1} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{p}} D(W(\Lambda, p))^\dagger \\ \widehat{\mathbf{p}} D(W(\Lambda, p))^{-1} \end{pmatrix} = \\ &= \begin{pmatrix} D([\mathbf{p}])D([\mathbf{p}]^{-1}\Lambda^{-1}[\Lambda p])\widehat{\mathbf{p}} \\ D([\mathbf{p}]^{\dagger-1})D([\mathbf{p}]^\dagger\Lambda^\dagger[\Lambda p]^\dagger)\widehat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} D(\Lambda^{-1}) & 0 \\ 0 & D(\Lambda^{-1})^{\dagger-1} \end{pmatrix} U(\Lambda p) \\ &= S(\Lambda)^{-1}U(\Lambda p), \end{aligned}$$

so that changing the integration variable we see that:

$$\widehat{\Lambda}\Psi_+(x)\widehat{\Lambda}^{-1} = S(\Lambda)^{-1}\Psi_+(\Lambda x)$$

in agreement with Eq. (A.6). For the antiparticle term, the proof is similar. From this proof it also follows that the generators of the Lorentz group in the Dirac representation can be written:

$$\Sigma^{\mu\nu} = \begin{pmatrix} D^{(0,1/2)}(J^{\mu\nu}) & 0 \\ 0 & D^{(1/2,0)}(J^{\mu\nu}) \end{pmatrix}.$$

Writing the generators in terms of the Pauli matrices it can be seen that this form coincides with the commutator $(i/4)[\gamma^\mu, \gamma^\nu]$, where the γ matrices are in the so-called Weyl representation (1.44).

We can also show easily that the Dirac equation is satisfied. Considering the state with standard momentum, and using Eq. (1.42), we have:

$$\not{p}U(\mathbf{p}) = N \begin{pmatrix} 0 & \mathbf{p} \\ \widehat{\mathbf{p}} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \widehat{\mathbf{p}} \end{pmatrix} = Nm \begin{pmatrix} m\mathbb{I} \\ m\mathbb{I} \end{pmatrix},$$

N being a normalization factor. In the massive case $\mathbf{p} = \widehat{\mathbf{p}} = m\mathbb{I}$, and in the massless one $\widehat{\mathbf{p}} \mathbf{p} = m^2 = 0$, so that the Dirac equation is satisfied for $p = \mathbf{p}$. From the transformation rules of the spinors:

$$U(p)D(W(\Lambda, p))^\dagger = S(\Lambda)^{-1}U(\Lambda p)$$

it follows that the equation is fulfilled for any p obtained from \mathbf{p} by means of a Lorentz transformation. The proof is identical for the $V(p)$ spinors.

Appendix B

Analytic distillation for pure rotation

We report some intermediate steps for the calculation of the energy-momentum tensor and the vector and axial currents of the massless Dirac field in global equilibrium with rotation.

The matrices and tensors Λ^n , $S(\Lambda)^n$, A and A_5 are reported in Eq. (3.92) and will be used to compute the expectation values.

Starting from the energy-momentum tensor, using Eq. (3.69) we find the series corresponding to the exact canonical energy-momentum tensor, that is used in Eq. (3.93) to find the exact Belinfante tensor. The latter is given by:

$$T_B^{\mu\nu}(x)_I = \lim_{B \rightarrow \phi} \lim_{L \rightarrow \phi/\zeta} \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cosh\left(\frac{n\phi}{L}\right)}{\left(n^2\phi^2 + 4B^2r^2T_0^2 \sin^2\left(\frac{n\phi}{2}\right)\right)^3} \Theta_n^{\mu\nu}(x),$$

where the tensor $\Theta_n(x)$ has the following components:

$$\begin{aligned} \Theta_n^{00}(x) &= 8B^4T_0^4 \cos\left(\frac{n\phi}{2}\right) [3n^2\phi^2 + 2B^2r^2T_0^2(\cos n\phi - 1)], \\ \Theta_n^{11}(x) &= 8B^4T_0^4 \cos\left(\frac{n\phi}{2}\right) \left(n^2\phi^2 + 4B^2T_0^2(r^2 - 4y^2) \sin^2\left(\frac{n\phi}{2}\right)\right), \\ \Theta_n^{22}(x) &= 8B^4T_0^4 \cos\left(\frac{n\phi}{2}\right) \left(n^2\phi^2 + 4B^2T_0^2(r^2 - 4x^2) \sin^2\left(\frac{n\phi}{2}\right)\right), \\ \Theta_n^{33}(x) &= 8B^4T_0^4 \cos\left(\frac{n\phi}{2}\right) \left(n^2\phi^2 + 4B^2r^2T_0^2 \sin^2\left(\frac{n\phi}{2}\right)\right), \\ \Theta_n^{01}(x) &= 16iB^5T_0^5yn\phi \sin\left(\frac{n\phi}{2}\right) (\cos(n\phi) + 3), \\ \Theta_n^{02}(x) &= -16iB^5T_0^5xn\phi \sin\left(\frac{n\phi}{2}\right) (\cos(n\phi) + 3), \\ \Theta_n^{12}(x) &= 64B^6T_0^6xy \sin\left(\frac{n\phi}{2}\right) \sin(n\phi), \\ \Theta_n^{03} &= \Theta_n^{13} = \Theta_n^{23} = 0. \end{aligned}$$

Notice that the auxiliary real parameters L and B have been introduced in order to make the series of functions of the form discussed in Sec. 3.1.

One ought to proceed as outlined in Sec. 3.6.2: we obtain the asymptotic power series associated with each component with B - and L - dependent coefficients, then we perform the limits in B and L , to obtain an implicit asymptotic expansion of the series of functions about $\phi = 0$. Then, we compute the analytic distillate and perform the mapping $\phi \rightarrow i\omega/T_0$ back to real vorticity. The results of such a procedure for the non-vanishing components of the energy-momentum tensor can be cast as follows:

$$\begin{aligned}
 T_B^{00} &= \frac{(960\gamma^6 - 1128\gamma^4 + 196\gamma^2 + 17) \gamma^4 \omega^4}{2880\pi^2} + \frac{(24\gamma^4 - 16\gamma^2 + 1) \gamma^4 (3\zeta^2 + \pi^2) T_0^2 \omega^2}{72\pi^2} \\
 &+ \frac{(4\gamma^2 - 1) \gamma^4 (15\zeta^4 + 30\pi^2\zeta^2 + 7\pi^4) T_0^4}{180\pi^2}, \\
 T_B^{11} &= \frac{\gamma^4 (15\zeta^4 + 30\pi^2\zeta^2 + 7\pi^4) T_0^4}{180\pi^2} \\
 &+ \frac{\gamma^4 \omega^4 (120\gamma^4 (24\zeta^2 T_0^2 y^2 + 8\pi^2 T_0^2 y^2 + 1) - 8\gamma^2 (60\zeta^2 T_0^2 y^2 + 20\pi^2 T_0^2 y^2 + 11) - 17)}{2880\pi^2} \\
 &+ \frac{\gamma^4 T_0^2 \omega^2 (15\zeta^2 (\gamma^2 (8\zeta^2 T_0^2 y^2 + 4) - 1) + 5\pi^2 (\gamma^2 (48\zeta^2 T_0^2 y^2 + 4) - 1) + 56\pi^4 \gamma^2 T_0^2 y^2)}{360\pi^2} \\
 &+ \frac{\gamma^6 (240\gamma^4 - 132\gamma^2 - 17) y^2 \omega^6}{720\pi^2}, \\
 T_B^{22} &= \frac{\gamma^4 (15\zeta^4 + 30\pi^2\zeta^2 + 7\pi^4) T_0^4}{180\pi^2} \\
 &+ \frac{\gamma^4 \omega^4 (120\gamma^4 (24\zeta^2 T_0^2 x^2 + 8\pi^2 T_0^2 x^2 + 1) - 8\gamma^2 (60\zeta^2 T_0^2 x^2 + 20\pi^2 T_0^2 x^2 + 11) - 17)}{2880\pi^2} \\
 &+ \frac{\gamma^4 T_0^2 \omega^2 (15\zeta^2 (\gamma^2 (8\zeta^2 T_0^2 x^2 + 4) - 1) + 5\pi^2 (\gamma^2 (48\zeta^2 T_0^2 x^2 + 4) - 1) + 56\pi^4 \gamma^2 T_0^2 x^2)}{360\pi^2} \\
 &+ \frac{\gamma^6 (240\gamma^4 - 132\gamma^2 - 17) x^2 \omega^6}{720\pi^2} \\
 T_B^{33} &= \frac{(120\gamma^4 - 88\gamma^2 - 17) \gamma^4 \omega^4}{2880\pi^2} + \frac{(4\gamma^2 - 1) \gamma^4 (3\zeta^2 + \pi^2) T_0^2 \omega^2}{72\pi^2} \\
 &+ \frac{\gamma^4 (15\zeta^4 + 30\pi^2\zeta^2 + 7\pi^4) T_0^4}{180\pi^2}, \\
 T_B^{01} &= -\frac{(3\gamma^2 - 1) \gamma^6 (3\zeta^2 + \pi^2) T_0^2 y \omega^3}{9\pi^2} \\
 &- \frac{\gamma^6 (15\zeta^4 + 30\pi^2\zeta^2 + 7\pi^4) T_0^4 y \omega}{45\pi^2} - \frac{(80\gamma^4 - 64\gamma^2 + 15) \gamma^6 y \omega^5}{240\pi^2} \\
 T_B^{02} &= \frac{(3\gamma^2 - 1) \gamma^6 (3\zeta^2 + \pi^2) T_0^2 x \omega^3}{9\pi^2} + \frac{\gamma^6 (15\zeta^4 + 30\pi^2\zeta^2 + 7\pi^4) T_0^4 x \omega}{45\pi^2} \\
 &+ \frac{(80\gamma^4 - 64\gamma^2 + 15) \gamma^6 x \omega^5}{240\pi^2}, \\
 T_B^{12} &= -\frac{(6\gamma^2 - 1) \gamma^6 (3\zeta^2 + \pi^2) T_0^2 x y \omega^4}{18\pi^2} \\
 &- \frac{\gamma^6 (15\zeta^4 + 30\pi^2\zeta^2 + 7\pi^4) T_0^4 x y \omega^2}{45\pi^2} - \frac{(240\gamma^4 - 132\gamma^2 - 17) \gamma^6 x y \omega^6}{720\pi^2},
 \end{aligned}$$

where $\gamma = 1/\sqrt{1 - r^2 \omega^2}$. To obtain the Lorentz-invariant coefficients appearing in the decomposition (3.97) we employ the tetrad (3.96), that in the case of pure

rotation reads:

$$u^\mu = \gamma(1, -y\omega, x\omega, 0), \quad \alpha^\mu = \left(0, -\frac{\gamma x\omega^2}{T_0}, -\frac{\gamma y\omega^2}{T_0}, 0\right),$$

$$w^\mu = \left(0, 0, 0, \frac{\gamma\omega}{T_0}\right), \quad l^\mu = \left(\frac{\gamma(\gamma^2 - 1)\omega^2}{T_0^2}, -\frac{\gamma^3 y\omega^3}{T_0^2}, \frac{\gamma^3 x\omega^3}{T_0^2}, 0\right).$$

Using the relations

$$\beta^2(x) = \frac{1}{T_0^2 \gamma^2}, \quad \alpha^2(x) = -(\gamma^2 - 1) \frac{\omega^2}{T_0^2}, \quad w^2(x) = -\gamma^2 \frac{\omega^2}{T_0^2},$$

we can write the results as in Eqs. (3.98).

The mean values of the vector and axial current are obtained similarly. In this case, the only non-vanishing components of the series (3.67) and (3.68) are:

$$j^0(x)_I = \lim_{B \rightarrow \phi} \lim_{L \rightarrow \phi/\zeta} \frac{4B^3 T_0^3}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n\phi \cos\left(\frac{n\phi}{2}\right) \sinh\left(\frac{n\phi}{L}\right)}{\left(n^2 \phi^2 + 4B^2 T_0^2 r^2 \sin^2\left(\frac{n\phi}{2}\right)\right)^2},$$

$$j^1(x)_I = \lim_{B \rightarrow \phi} \lim_{L \rightarrow \phi/\zeta} \frac{8iB^4 T_0^4 y}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left(\frac{n\phi}{2}\right) \sinh\left(\frac{n\phi}{L}\right)}{\left(n^2 \phi^2 + 4B^2 T_0^2 r^2 \sin^2\left(\frac{n\phi}{2}\right)\right)^2},$$

$$j^2(x)_I = \lim_{B \rightarrow \phi} \lim_{L \rightarrow \phi/\zeta} -\frac{8iB^4 T_0^4 x}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left(\frac{n\phi}{2}\right) \sinh\left(\frac{n\phi}{L}\right)}{\left(n^2 \phi^2 + 4B^2 T_0^2 r^2 \sin^2\left(\frac{n\phi}{2}\right)\right)^2},$$

$$j_A^3(x)_I = \lim_{B \rightarrow \phi} \lim_{L \rightarrow \phi/\zeta} -\frac{4iB^3 T_0^3}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n\phi \sin\left(\frac{n\phi}{2}\right) \cosh\left(\frac{n\phi}{L}\right)}{\left(n^2 \phi^2 + 4B^2 T_0^2 r^2 \sin^2\left(\frac{n\phi}{2}\right)\right)^2}.$$

Repeating the steps we performed for the energy-momentum tensor we find:

$$j^0(x)_I = \text{dist}_0 j^0(x)_I \Big|_{\phi=i\omega/T_0} = \frac{\gamma^4 \zeta (\zeta^2 + \pi^2) T_0^3}{3\pi^2} + \frac{(4\gamma^2 - 1) \gamma^4 \zeta T_0 \omega^2}{12\pi^2},$$

$$j^1(x)_I = \text{dist}_0 j^1(x)_I \Big|_{\phi=i\omega/T_0} = -\frac{\gamma^4 \zeta (\zeta^2 + \pi^2) T_0^3 y \omega}{3\pi^2} - \frac{(4\gamma^2 - 3) \gamma^4 \zeta T_0 y \omega^3}{12\pi^2},$$

$$j^2(x)_I = \text{dist}_0 j^2(x)_I \Big|_{\phi=i\omega/T_0} = \frac{\gamma^4 \zeta (\zeta^2 + \pi^2) T_0^3 x \omega}{3\pi^2} + \frac{(4\gamma^2 - 3) \gamma^4 \zeta T_0 x \omega^3}{12\pi^2},$$

$$j_A^3(x) = \text{dist}_0 j_A^3(x)_I \Big|_{\phi=i\omega/T_0} = \frac{(4\gamma^2 - 3) \gamma^4 \omega^3}{24\pi^2} + \frac{\gamma^4 (3\zeta^2 + \pi^2) T_0^2 \omega}{6\pi^2}.$$

Decomposing the currents along the tetrad the vector and axial currents can be written as Eqs. (3.99a) and (3.99b).

Appendix C

Analytic distillation for rotation and acceleration

This appendix collects some steps to achieve the distillation of the series for equilibrium with rotation and acceleration, the results of which are reported in Section 3.6.3. The relevant tensors Λ^n , $S(\Lambda^n)$, $A^{\mu\nu}$ and $A_5^{\mu\nu}$ can be read off Eq. (3.100).

We start from the Belinfante energy-momentum tensor. First, we need to map the acceleration and the angular velocity to imaginary values, $a/T_0 \rightarrow -i\Phi$ and $\omega/T_0 \rightarrow -i\phi$. Then, to obtain the asymptotic power expansion, we have to introduce auxiliary parameters, some of which depend on the component of the energy-momentum tensor. The Belinfante tensor reads:

$$T_B^{\mu\nu}(x)_I = \lim_{B,C,D \rightarrow \bar{B}, \bar{C}, \bar{D}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cosh(n\zeta)}{2 \left(B \sinh^2\left(\frac{n\Phi}{2}\right) + C \sin^2\left(\frac{n\phi}{2}\right) \right)^3} \Theta_n^{\mu\nu}(x, B, C, D),$$

where (t, x, y, z) are Cartesian coordinates and $r^2 = x^2 + y^2$:

$$\bar{B} = \Phi^2 T_0^2 t^2 + (1 - i\Phi T_0 z)^2, \quad \bar{C} = r^2 \Phi^2 T_0^2, \quad (C.1)$$

and \bar{D} is component-dependent:

$$\bar{D}_{00} = (T_0 \Phi z + i), \quad \bar{D}_{11} = T_0 \Phi y, \quad \bar{D}_{22} = T_0 \Phi x, \quad \bar{D}_{33} = T_0 \Phi t. \quad (C.2)$$

As already discussed, the limits and the series can be exchanged as the series are uniformly convergent series of continuous functions of arguments B , C , D , as long as ϕ and Φ are real*. The tensor Θ reads:

$$\begin{aligned} \Theta_n^{00} &= -T_0^4 \Phi^4 \cosh\left(\frac{n\Phi}{2}\right) \cos\left(\frac{n\phi}{2}\right) \left(B \sinh^2\left(\frac{n\Phi}{2}\right) + C \sin^2\left(\frac{n\phi}{2}\right) + 4D_{00}^2 \sinh\left(\frac{n\Phi}{2}\right) \right), \\ \Theta_n^{11} &= T_0^4 \Phi^4 \cosh\left(\frac{n\Phi}{2}\right) \cos\left(\frac{n\phi}{2}\right) \left(B \sinh^2\left(\frac{n\Phi}{2}\right) + C \sin^2\left(\frac{n\phi}{2}\right) - 4D_{11}^2 \sin\left(\frac{n\phi}{2}\right) \right), \\ \Theta_n^{33} &= T_0^4 \Phi^4 \cosh\left(\frac{n\Phi}{2}\right) \cos\left(\frac{n\phi}{2}\right) \left(B \sinh^2\left(\frac{n\Phi}{2}\right) + C \sin^2\left(\frac{n\phi}{2}\right) - 4D_{33}^2 \sinh\left(\frac{n\Phi}{2}\right) \right), \\ \Theta_n^{01} &= T_0^5 \Phi^5 y (i + T_0 \Phi z) \sinh\left(\frac{n\Phi}{2}\right) \sin\left(\frac{n\phi}{2}\right) (\cosh(n\Phi) + \cos(n\phi) + 2), \\ \Theta_n^{03} &= -T_0^5 \Phi^5 t (i + T_0 \Phi z) \sinh\left(\frac{n\Phi}{2}\right) \sinh(n\Phi) \sin(n\phi) \csc\left(\frac{n\phi}{2}\right), \\ \Theta_n^{12} &= T_0^6 \Phi^6 xy \sinh(n\Phi) \operatorname{csch}\left(\frac{n\Phi}{2}\right) \sin\left(\frac{n\phi}{2}\right) \sin(n\phi) \\ \Theta_n^{13} &= T_0^6 \Phi^6 ty \sinh\left(\frac{n\Phi}{2}\right) \sin\left(\frac{n\phi}{2}\right) (\cosh(n\Phi) + \cos(n\phi) + 2), \\ \Theta_n^{22} &= \Theta_n^{11}(y \mapsto x), \quad \Theta_n^{02} = \Theta_n^{01}(y \mapsto -x), \quad \Theta_n^{23} = \Theta_n^{12}(y \mapsto -x). \end{aligned}$$

*Indeed the series does not converge for $B = 0$, but since $T_0 > 0$, \bar{B} is always different from zero.

Even with the introduction of the auxiliary parameters, we are dealing with series of functions of $n\Phi$, $n\phi$ and $n\zeta$, which are all independent parameters. Our next task is to express them as functions of a single one. We start by introducing polar coordinates for Φ and ϕ :

$$\Phi = \xi \cos \theta, \quad \phi = \xi \sin \theta. \quad (\text{C.3})$$

Now we can parametrize $\zeta = \xi/L$ as we did before, moving the limit out of the series for the same reasons already discussed. After these steps, the series is a series of functions of $n\xi$, whereas θ and L are parameters, and we can finally use the asymptotic expansion (3.22).

We sketch the steps for the 00-component of the Belinfante tensor:

$$T_B^{00}(x)_I = \lim_{B,C,D \rightarrow \bar{B}, \bar{C}, \bar{D}} -\frac{T_0^4 \Phi^4}{4\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \cosh\left(\frac{n\Phi}{2}\right) \cos\left(\frac{n\phi}{2}\right) \cosh(n\zeta) \times \\ \frac{B \sinh^2\left(\frac{n\Phi}{2}\right) + C \sin^2\left(\frac{n\phi}{2}\right) + 2(\cosh(n\Phi) - 1)D^2}{\left(B \sinh^2\left(\frac{n\Phi}{2}\right) + C \sin^2\left(\frac{n\phi}{2}\right)\right)^3},$$

where $\bar{B}, \bar{C}, \bar{D}$ are given in Eqs. (C.1) and (C.2) ($D = D_{00}$ and $\bar{D} = \bar{D}_{00}$ is implied). Parametrizing Φ and ϕ in polar coordinates as in Eq. (C.3), and writing $\zeta = \xi/L$ we obtain a series of functions $f(n\xi)$:

$$T_B^{00}(x)_I = \lim_{\substack{B,C,D \rightarrow \bar{B}, \bar{C}, \bar{D} \\ L \rightarrow \xi/\zeta}} -\frac{T_0^4 \xi^4 \cos^4 \theta}{4\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cosh\left(\frac{n\xi \cos \theta}{2}\right) \cos\left(\frac{n\xi \sin \theta}{2}\right) \cosh\left(\frac{n\xi}{L}\right)}{\left(B \sinh^2\left(\frac{n\xi \cos \theta}{2}\right) + C \sin^2\left(\frac{n\xi \sin \theta}{2}\right)\right)^3} \times \\ \left(B \sinh^2\left(\frac{n\xi \cos \theta}{2}\right) + C \sin^2\left(\frac{n\xi \sin \theta}{2}\right) + 2(\cosh(n\xi \cos \theta) - 1)D^2\right) \\ \equiv \lim_{\substack{B,C,D \rightarrow \bar{B}, \bar{C}, \bar{D} \\ L \rightarrow \xi/\zeta}} -\frac{T_0^4 \xi^4 \cos^4 \theta}{4\pi^2} G(\theta, \xi, B, C, D).$$

Using the results of Section 3.1, we can obtain an asymptotic series in powers of ξ :

$$G \sim \frac{1}{720L^4 (Bc^2 + Cs^2)^5} \left\{ -B^2c^4 [c^4 (139CL^4s^2 - 120D^2L^2 (L^2s^2 - 4)) \right. \\ - 30c^2 (CL^2s^2 (4 - 5L^2s^2) + 2D^2 (L^4s^4 - 24L^2s^2 + 16)) + 68c^6D^2L^4 \\ + Cs^2 (107L^4s^4 + 600L^2s^2 - 720)] \\ + B^3c^6 [30c^2L^2 (L^2s^2 - 4) - 17c^4L^4 + 15 (L^4s^4 - 24L^2s^2 + 16)] \\ - Bc^2Cs^2 [c^4 (107CL^4s^2 + 240D^2L^2 (5L^2s^2 - 8)) + 664c^6D^2L^4 \\ + 6c^2 (25CL^2s^2 (L^2s^2 - 4) + 4D^2 (33L^4s^4 - 80)) \\ + Cs^2 (139L^4s^4 + 120L^2s^2 - 720)] \\ + C^2s^4 [15c^4 (CL^4s^2 + 40D^2L^2 (L^2s^2 + 4)) + 364c^6D^2L^4 \\ + 6c^2 (5CL^2s^2 (L^2s^2 + 12) + 2D^2 (9L^4s^4 + 120L^2s^2 + 80)) \\ \left. + Cs^2 (-17L^4s^4 + 120L^2s^2 + 240)] \right\}$$

$$\begin{aligned}
& - \frac{\pi^2}{18L^2\xi^2 (Bc^2 + Cs^2)^4} \{ B^2c^4 (c^2L^2 + 3L^2s^2 - 12) \\
& + 2B [c^4 (6D^2 (L^2s^2 - 4) - CL^2s^2) + c^2Cs^2 (L^2s^2 - 12) + 2c^6D^2L^2] \\
& - Cs^2 (3c^2 (CL^2s^2 + 4D^2 (L^2s^2 + 4)) + 20c^4D^2L^2 + Cs^2 (L^2s^2 + 12)) \} \\
& + \frac{7\pi^4 (Bc^2 + 4c^2D^2 + Cs^2)}{45\xi^4 (Bc^2 + Cs^2)^3}
\end{aligned}$$

where

$$c = \cos \theta, \quad s = \sin \theta.$$

Now that the asymptotic expansion of G in powers of ξ , which we will denote as G_A , has been computed, we can calculate the limits in the auxiliary parameters and transform the variables ξ, θ back to Φ, ϕ . Hence:

$$\text{dist}_0 T_B^{00}(x)(\phi, \Phi)_I = \text{dist}_0 \frac{T_0^4 \Phi^4}{4\pi^2} G_A(\Phi, \phi, \zeta, \bar{B}, \bar{C}, \bar{D}).$$

The last step consists of setting $\Phi \rightarrow ia/T_0$ and $\phi \rightarrow i\omega/T_0$, to finally obtain the physical thermal expectation value:

$$T_B^{00}(x) = \text{dist}_0 \frac{a^4}{4\pi^2} G_A(ia/T_0, i\omega/T_0, \bar{B}, \bar{C}, \bar{D}),$$

where the arguments of $\bar{B}, \bar{C}, \bar{D}$ in Eqs. (C.1), (C.2) are also affected by the mapping $\Phi \rightarrow ia/T_0$ and $\phi \rightarrow i\omega/T_0$. Similarly, the other components can be computed. One eventually finds that the decomposition (3.97) involves the coefficients in Eq. (3.101).

The tetrad (3.96) for equilibrium with rotation and acceleration along the z -axis used to obtain Eq. (3.101) is given by:

$$\begin{aligned}
w^\mu &= \gamma (1 + az, -\omega y, \omega x, at), \\
\alpha^\mu &= \frac{\gamma}{T_0} (a^2 t, -\omega^2 x, -\omega^2 y, a(1 + az)), \\
w^\mu &= \frac{\gamma \omega}{T_0} (at, ax, ay, 1 + az), \\
l^\mu &= \frac{\gamma^3 \omega (a^2 + \omega^2)}{T_0^2} (\omega r^2 (1 + az), -y((1 + az)^2 - a^2 t^2), x((1 + az)^2 - a^2 t^2), r^2 at),
\end{aligned}$$

with:

$$\begin{aligned}
\beta^2 &= \frac{a^2 (z^2 - t^2) + 2az - r^2 \omega^2 + 1}{T_0^2}, \\
\alpha^2 &= -\frac{a^4 (z^2 - t^2) + 2a^3 z + a^2 + r^2 \omega^4}{T_0^4 \beta^2}, \\
w^2 &= -\frac{\omega^2 (a^2 (r^2 - t^2 + z^2) + 2az + 1)}{T_0^4 \beta^2}, \\
l^2 &= -\frac{r^2 \omega^2 (a^2 (t^2 - z^2) - 2az - 1) (a^2 + \omega^2)^2}{T_0^4 \beta^2},
\end{aligned}$$

where $\gamma = ((1 + az)^2 - a^2 t^2 - \omega^2 r^2)^{-1/2}$. The scalar product $\alpha \cdot w = -a\omega/T_0^2$ is also non-vanishing.

For what concerns the vector and axial currents, they are expressed as series via Eqs. (3.67) and (3.68):

$$j^\mu(x)_I = \lim_{B,C \rightarrow \bar{B}, \bar{C}} \frac{\Phi^3 T_0^3}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sinh(n\zeta)}{\left(B \sinh\left(\frac{n\Phi}{2}\right) + C \sin\left(\frac{n\phi}{2}\right)\right)^2} \Upsilon_n^\mu,$$

$$j_A^\mu(x)_I = \lim_{B,C \rightarrow \bar{B}, \bar{C}} \frac{\Phi^3 T_0^3}{2\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cosh(n\zeta)}{\left(B \sinh\left(\frac{n\Phi}{2}\right) + C \sin\left(\frac{n\phi}{2}\right)\right)^2} \Upsilon_n^\mu,$$

where \bar{B} and \bar{C} are the same as in Eqs. (C.1) and (C.2) and the vectors Υ and Γ read:

$$\begin{aligned} \Upsilon^0 &= -\Phi t T_0 \sin\left(\frac{n\phi}{2}\right) \sinh\left(\frac{n\Phi}{2}\right), \\ \Upsilon^1 &= \Upsilon^0(t \mapsto x), \\ \Upsilon^2 &= \Upsilon^0(t \mapsto y), \\ \Upsilon^3 &= -(i + \Phi T_0 z) \sin\left(\frac{n\phi}{2}\right) \sinh\left(\frac{n\Phi}{2}\right), \\ \Gamma^0 &= (1 - i\Phi T_0 z) \cos\left(\frac{n\phi}{2}\right) \sinh\left(\frac{n\Phi}{2}\right), \\ \Gamma^1 &= iy\Phi T_0 \sin\left(\frac{\phi}{2}\right) \cosh\left(\frac{n\Phi}{2}\right), \\ \Gamma^2 &= \Gamma^1(y \mapsto -x), \\ \Gamma^3 &= -it\Phi T_0 \cos\left(\frac{n\phi}{2}\right) \sinh\left(\frac{n\Phi}{2}\right). \end{aligned}$$

Repeating the same steps as we did for the energy-momentum tensor, we find the expressions (3.99a) and (3.99b), without any additional dependence on $\alpha \cdot w$.

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