Comparative Dynamics of Atmosphere-Ocean-Models within the Description of the Perturbed Global Carbon Cycle

E. O. Siré, G. H. Kohlmaier, G. Kratz, U. Fischbach, and H. Bröhl

Institut für Physikalische und Theoretische Chemie,
Johann Wolfgang Goethe-Universität, Frankfurt (Main)

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For a certain class of ocean models describing the exchange of inorganic carbon between the atmosphere and the surface layer of the ocean as well as between the surface layer and the deep sea the dynamical airborne fraction is evaluated analytically under the assumption that the growth rate of the atmospheric source term (fossil fuel plus net biogenic carbon input into the atmosphere) is slowly variable with time. Each of these models exhibits a certain uptake capacity of the deep ocean which is quantified. Considerations are made as to whether the terrestrial biota are to be regarded as a source or a sink for additional atmospheric CO_2 depending on the modelling of the deep ocean. It is shown that a global one-dimensional box-diffusion ocean model with a depth dependent eddy diffusivity $K(z) = K(0) \exp[-z/z^*]$, with an adjustable parameter set $\{K(0), z^*\}$, provides a fairly well fit to the prebomb ^{14}C ocean distribution and to an appreciable net biogenic carbon transfer into the atmosphere. The range of future atmospheric CO_2 partial pressures is estimated for a given fossil input.

1. Introduction

Mainly two different ways have been chosen to approach the problem of the atmospheric carbon dioxide budget. First, carbon exchange fluxes between the atmosphere and the oceans as well as the biota have been estimated for a certain year or as cumulative amounts for certain ranges of time. These estimates contain information on changes of different biospheric zones, e.g. forest clearing, desertifications, fires, forestry, and agriculture. Some of them also account for the amount of carbon taken up additionally by the biosphere through new growth and possible fertilization by the increased atmospheric content of carbon dioxide and nitrous oxides [1–6]. The majority of the quoted authors agree in that the biosphere is most probably a source for atmospheric CO_2.

On the other hand models have been constructed which simulate the carbon exchange between the atmosphere and its adjacent reservoirs [7–16]. These models, if realistic, should offer the possibility to verify the time behaviour of certain observables, e.g. atmospheric CO_2 partial pressure, $^{13}C/^{12}C$-ratios as well as ^{14}C activities in the atmosphere and in different depths of the oceans. Until now the more complex models have been developed for the oceans, reflecting our more serious lack of knowledge concerning the response (kinetic) behaviour of the biosphere to anthropogenic influences [17]. Most of these models allow carbon to enter the deep sea only by penetration through a well-mixed surface layer, and they mostly result in that the biosphere is suggested to be a sink for additional atmospheric carbon dioxide.

The effects of possible sinks within the oceans, e.g. additional dissolution of $CaCO_3$ and increased primary production by increased PO_4^{3-} river discharge are still uncertain [18–20].

In view of a possible global climatic change due to carbon dioxide released by human activities [21, 22], it is of severe importance to put foreward the development of reliable carbon cycle models in order to predict the future atmospheric CO_2 increase.

The objective of this paper is twofold: (1) to derive general analytical expressions for the airborne fraction, expressing the variation of the atmospheric CO_2 content in relation to the fossil input and an effective (net) transfer of carbon between the land biota and the atmosphere, and (2) to attempt to give a quantitative description of the controversal views of the atmospheric carbon budget (so-called „sink-source problem").

In Section 2 the carbon exchange between atmosphere and surface ocean is formulated in a general non-linear way (i.e. by using a dynamic buffer

Reprint requests to Prof. Dr. G. H. Kohlmaier, Institut für Physikalische und Theoretische Chemie, Universität Frankfurt, Robert-Mayer-Straße 11, D-6000 Frankfurt (Main) 1.

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factor, while the uptake of inorganic carbon by the deep sea depends on the special model employed. For a persistently exponentially rising atmospheric CO_2 input the linear response theory predicts a constant airborne fraction for sufficiently large observation times [9]. As we shall show one can generalize such well-known results by assuming that the time variation of the growth rate of the atmospheric source term can be considered as a "slow" variable of the system. This source term is regarded as composed of carbon dioxide released by combustion of fossil fuels ($F_f(t)$) plus the net biogenic flux into the atmosphere ($F_b^{eff}(t)$; negative, if the terrestrial biota act as a sink). Such approximations are usefull to achieve estimates of atmospheric CO_2 increase resulting from different carbon cycle models as well as from different scenarios of future anthropogenic impact [23].

In Section 3 the method of calculating the coefficient of the dynamic coupling between the mixed surface layer and the deep sea (which enters into our general expression for the airborne fraction) is presented.

The formulation for a discrete box-chain model is presented in Section 3a; as special cases it incorporates the 2-box-ocean model [7, 8] as well as the 1-box-diffusion model of Oeschger et al. [10] (continuum limit for the homogeneous case). The possible enhancement of carbon transfer from the mixed surface to the deep sea by an advection-diffusion mechanism proposed by Hoffert et al. [12] is discussed in Section 3b. In Section 3c we present a straight forward generalization of the 1-dimensional box-diffusion model namely by assuming a depth dependent eddy diffusivity coefficient $K(z)$, which has been chosen to have the shape of a decreasing exponential function. By calibration of the two parameters entering into $K(z) = K(0) \exp[-z/z^*]$, namely $\{K(0), z^*\}$, a consistency of the field data (i.e. average preindustrial (prebomb) radiocarbon distribution and larger amounts of net biospheric reduction fluxes) can be obtained.

Although the box-diffusion model with an adjustable constant eddy diffusivity coefficient provides a good fit to the average distribution of natural and bomb produced radiocarbon, the major criticism of this model comes from the fact that it cannot remove sufficient amounts of excess CO_2 from the atmosphere. Such models were forced to introduce a biospheric growth coefficient (parameter) in order to balence the fossil fuel CO_2 budget.

In Section 4a we discuss the "sink-source problem" in terms of the analytical airborne fraction and uncertainties of field data.

For matter of illustration in the final section 4b we present the typical time dependence of the analytical airborne fraction and other system observables (e.g. atmospheric CO_2 fraction, dynamical buffer factor, etc.) for the case of a logistic fossil fuel input in the time range of 1860—2260.

2. The Adiabatic Approximation for the Airborne Fraction

Taking the variable t as the difference between a year A.D. and the year 1860 we denote

$$F(t) := F_f(t) + F_b^{eff}(t), \tag{1a}$$

$$N(t) := \int_0^t F(t')\,dt' = N_f(t) + N_b^{eff}(t), \tag{1b}$$

where $F_f(t)$ is the fossil input rate, $N_f(t)$ the corresponding cumulated fossil input, $F_b^{eff}(t)$ the *net* flux of carbon from or into the biosphere according to the sign of $F_b^{eff}(t)$, and $N_b^{eff}(t)$ is the corresponding cumulative biota change. The total input flux of CO_2 into the atmosphere ($\dot{F}(t) > 0$) is denoted, according to Eq. (1a), by $F(t)$. For the relative change of the total cumulated carbon input $N(t)$ we use the following abbreviation:

$$a(t) := \frac{\dot{N}(t)}{N(t)}, \quad (\dot{N}(t) := F(t),\ t > 0). \tag{2}$$

The basic model structure of our system is shown in Fig. 1, where $N_a(t)$ denotes the carbon mass of the atmosphere and $N_m(t)$ stands for the mass of dissolved inorganic carbon compounds in the surface layer of the ocean. Both compartments are considered to be well-mixed (homogeneous) with respect to the carbon mass distribution. The kinetic fluxes of carbon from the atmosphere to the mixed sea surface and vice versa are $F_{am}(t)$ and $F_{ma}(t)$, resp.; the net flux to the deep sea, as a response to the atmospheric input $F(t)$, is denoted by $\Delta F_{md}(t)$. The possibility of direct ventilation of carbon to the deep sea by way of its high latitude source regions under consideration of the carbon exchange between atmosphere and polar waters are excluded in our treatment.

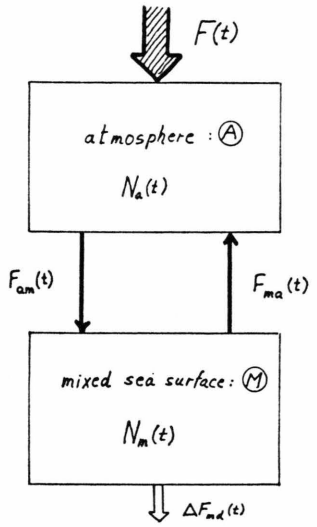

Fig. 1. Atmosphere — sea surface layer — subsystem of the carbon cycle models described.

In accordance with other existing models we shall assume steady state conditions (subscript "0") at preindustrial times ($t < 0$):

$$F_{am0} = F_{ma0}, \quad \Delta F_{md}(0) = 0, \quad F(-0) = 0, \qquad (3a-c)$$

i.e. there is no net flux between the compartments A and M nor is between the mixed sea surface and the rest of the ocean (deep sea or deep sea plus polar waters). The rate equations for carbon mass changes in the atmosphere and the surface layer are written as:

$$\begin{aligned} dN_a/dt &= F_{ma}(t) - F_{am}(t) + F(t) \qquad (4a)\\ &= k_{ma} N_{m0}(p_m/p_{m0}) - k_{am} N_a + F(t), \end{aligned}$$

$$\begin{aligned} dN_m/dt &= F_{am}(t) - F_{ma}(t) - \Delta F_{md}(t), \\ (k_{am} N_{a0} &= k_{ma} N_{m0} = F_{am0}), \qquad (4b) \end{aligned}$$

where p_m denotes the partial pressure of the physically dissolved carbon dioxide in the mixed sea surface and k_{am} and k_{ma} are the corresponding rate constants of the kinetic fluxes. The ratio p_m/p_{m0} is associated to fractional changes in the mixed layer carbon mass N_m by the so-called buffer factor [7]:

$$\xi := \frac{(p_m - p_{m0})/p_{m0}}{(N_m - N_{m0})/N_{m0}}. \qquad (5)$$

With Eqs. (1, 2) and (5) and introducing the fractions related to fossil plus effective biospheric input

$$\alpha(t) := \frac{N_a(t) - N_{a0}}{N(t)} = \frac{\Delta N_a(t)}{N(t)}, \qquad (6a)$$

$$\mu(t) := \frac{N_m(t) - N_{m0}}{N(t)} = \frac{\Delta N_m(t)}{N(t)}, \qquad (6b)$$

we can reformulate the dynamic Eqs. (4a—b):

$$d\alpha/dt = -[a(t) + k_{am}]\alpha + k_{ma}\xi(t)\mu + a(t), \qquad (7a)$$

$$d\mu/dt = k_{am}\alpha - \{a(t)[1 + \Theta(t)] + k_{ma}\xi(t)\}\mu, \qquad (7b)$$

where in the last Eq. (7b) we have defined a dimensionless function $\Theta(t)$ (which gives a measure of the relative strength of carbon mass transfer between mixed surface and deep ocean) by:

$$\frac{\Delta F_{md}(t)}{\Delta N_m(t) a(t)} := \Theta(t). \qquad (8)$$

The system (7a—b) of non-autonomous first order differential equations for the new variables $\{\alpha(t), \mu(t)\}$ is in several aspects formal:

1) non-linear couplings can be contained in ξ as well as in Θ,
2) the function $\Theta(t)$ will generally contain memory effects (i.e. it depends on the history of the mixed layer perturbations).

In order to achieve an analytical approximation for the airborne fraction defined by Eq. (6a) we propose the following formulation for the variable buffer factor

$$\begin{aligned} \xi(t) &= \xi_0 \frac{1 + c_1 \Delta N_m(t)/N_{m0}}{1 - c_2 \Delta N_m(t)/N_{m0}} \\ &= \xi_0 \frac{\xi(t) + c_1[p_m(t) - p_{m0}]/p_{m0}}{\xi(t) - c_2[p_m(t) - p_{m0}]/p_{m0}}. \qquad (9) \end{aligned}$$

The parameters of Eq. (9) can e.g. be obtained by fitting Bacastow and Keeling's [8] numerical results for the curve $\xi(p_m/p_{m0})$ in the range of

$$1 \leqq p_m/p_{m0} \leqq 9 : c_1 = 0.18, \quad c_2 = 4.72 \quad \text{and} \quad \xi_0 = 9.$$

We shall assume in the following that the effective growth function $a(t)$, Eq. (2), can be considered for sufficiently large times t ($t \gtrsim 100$ yr) as a slowly changing function. This suggests that the variables $\{\alpha(t), \mu(t)\}$ of Eqs. (7a—b) can approximately be computed by using the "adiabatic" approximation of $\dot{\alpha}(t), \dot{\mu}(t) \approx 0$. When specifying a concrete model for the deep ocean we will determine the coupling function $\Theta(t)$, Eq. (8), by using similar arguments

(see Section 3). By this method, the quality of reproducing the inventories, computed by a numerical integration of corresponding differential equations, depends on the structure of the relaxation spectrum of the model system as well as on the smoothness of the perturbing input function. Assuming the adiabatic approximation for Eqs. (7a—b), we obtain a theoretical airborne fraction (related to fossil plus net biospheric input):

$$\alpha(t) = \frac{1}{1 + \Phi_{0c}(t)}, \quad (0 \leqq \alpha(t) \leqq 1), \quad (10a)$$

with the relative uptake capacity of the ocean approximated by ($\Delta N_d(t)$ denotes the amount of transfered carbon mass to the deep ocean for the given CO_2 history):

$$\Phi_{0c}(t) := \frac{\Delta N_m(t) + \Delta N_d(t)}{\Delta N_a(t)} \approx \frac{k_{am}}{a(t)} \cdot \left[1 + \frac{k_{ma}}{a(t)} \xi(t) \frac{1}{1 + \Theta(t)}\right]^{-1}. \quad (10b)$$

Using the constraint of mass conservation

$$N(t) = \Delta N_a(t) + \Delta N_m(t) + \Delta N_d(t)$$

together with $\dot{\alpha} = \dot{\mu} \approx 0$, the coupling function $\Theta(t)$, Eq. (8), will have the property:

$$\Theta(t) \approx \Delta N_d(t)/\Delta N_m(t). \quad (10c)$$

The variable buffer factor $\xi(t)$ in this model, Eq. (9), is determined by using the following equation:

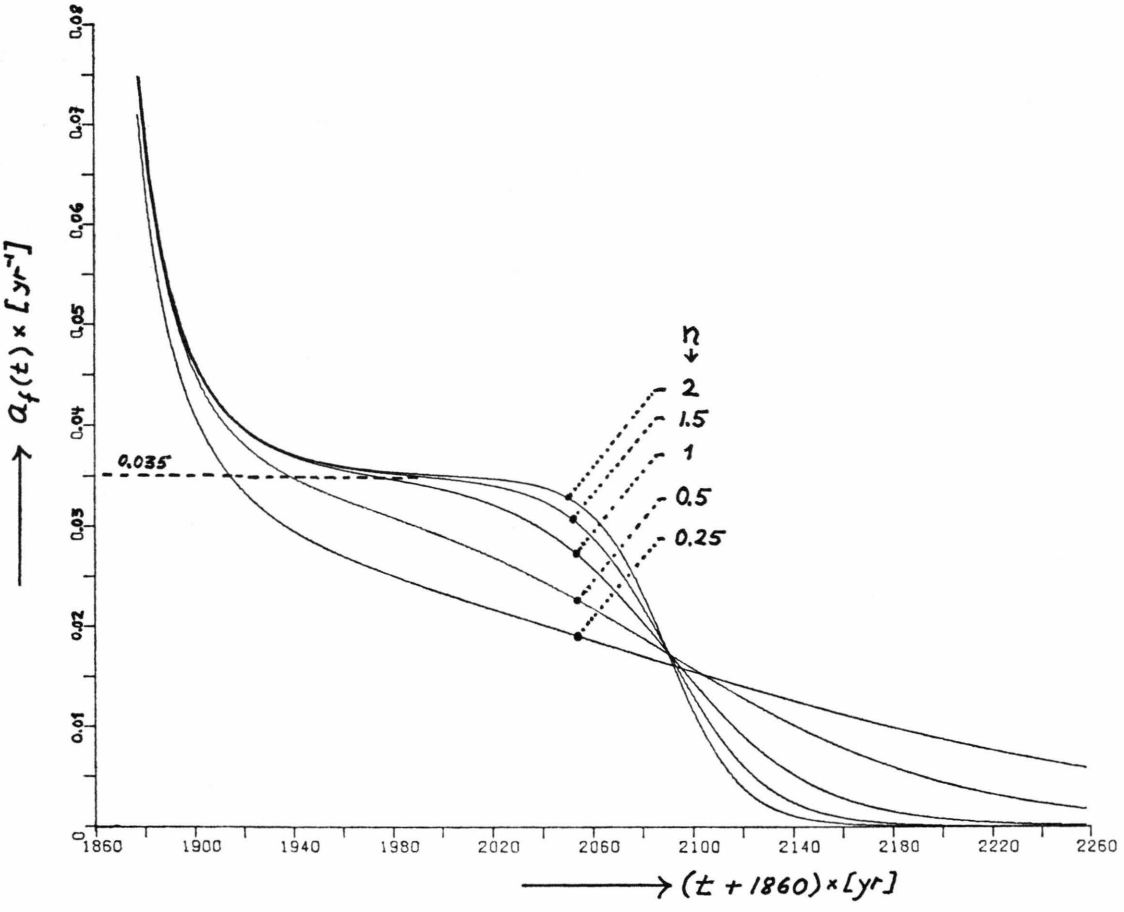

Fig. 2. Relative rates of fossil input for different cut-off-parameters n vs. time ($N_{f\infty} = 7000$ Gt C; $\Delta_0 = 2.62$ Gt C, $k_f = 3.5\%$/yr).

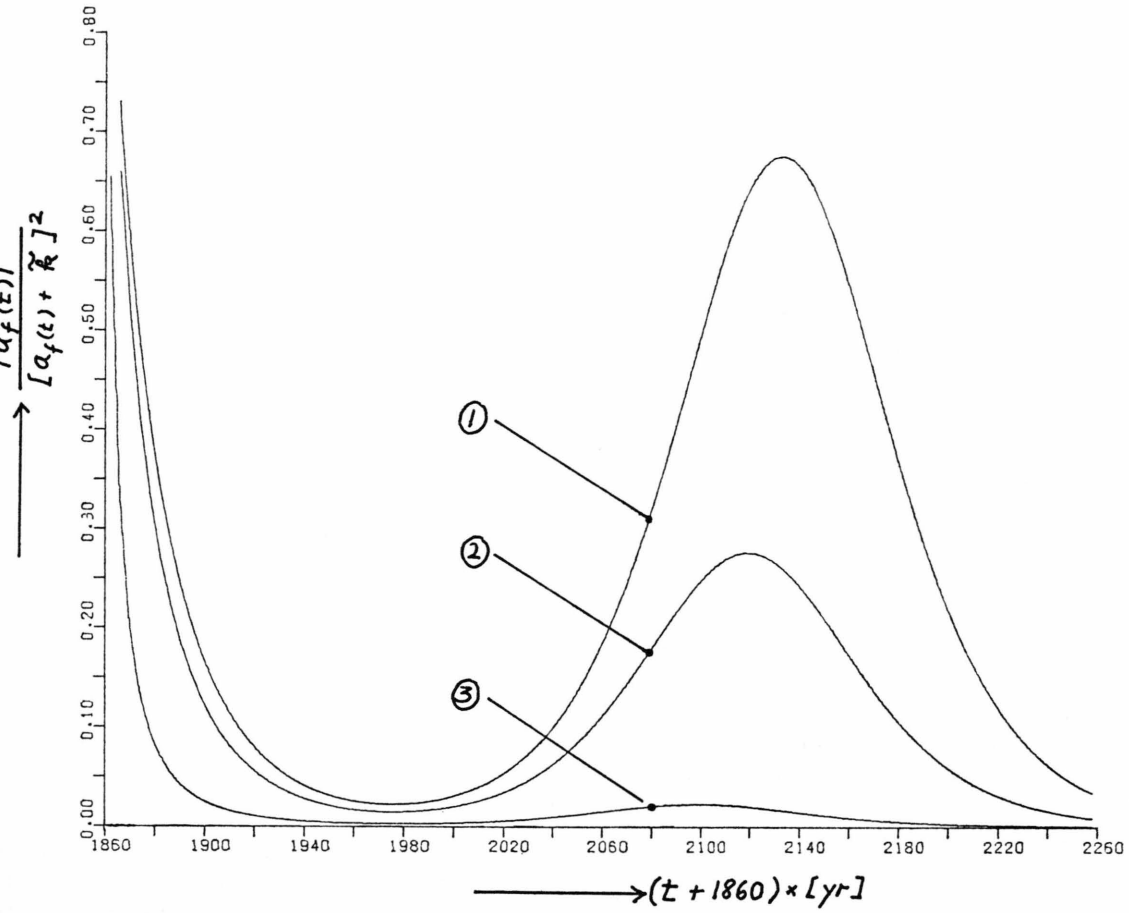

Fig. 3. Function $|\dot{a}_f(t)|/[a_f(t)+\tilde{k}]^2$ for the logistic input ($n=1$) vs. time for different $\tilde{k}=\mathrm{Min}(k_{ij})$; (1) $\tilde{k}=0.01$; (2) $\tilde{k}=0.02$; (3) $\tilde{k}=0.1$ ($N_{f\infty}=7000$ Gt C; $\Delta_0=2.62$ Gt C, $k_f=3.5\%$/yr).

$$\frac{\Delta N_m(t)}{N_{m0}} \approx \frac{\left[(1-c_2\mu_0(t)N(t)/N_{m0})^2+4\mu_0(t)\dfrac{N_{a0}N(t)}{N_{m0}^2}\xi_0(c_1+c_2)\right]^{1/2}-\left(1+c_2\mu_0(t)\dfrac{N(t)}{N_{m0}}\right)}{2\left[\mu_0(t)\dfrac{N_{a0}}{N_{m0}}\xi_0(c_1+c_2)-c_2\right]}, \tag{10d}$$

with

$$\mu_0(t) := \frac{k_{am}/a(t)}{\dfrac{k_{am}N_{a0}}{a(t)N_{m0}}\xi_0+[1+\Theta(t)](1+k_{am}/a(t))}. \tag{10e}$$

$\mu_0(t)$ corresponds to the fraction cumulated in the mixed layer when using a constant buffer factor $\xi(t)=\xi_0$. Since the accommodation of the driven system is more or less determined by a set of time dependent rates of the form $[a(t)+k_{ij}]$ (k_{ij}: a rate constant of the system) it is plausible that the adiabatic approximation is best fulfilled for sufficient long observation times, i.e. if the inequality

$$t \gtrsim [a(t)+\mathrm{Min}(k_{ij})]^{-1} \tag{11a}$$

holds *and* if the smoothness condition is guaranted:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}[(a(t)+\mathrm{Min}(k_{ij})]^{-1}\right| = \frac{|\dot{a}(t)|}{[a(t)+\mathrm{Min}(k_{ij})]^2} \ll 1. \tag{11b}$$

For continuum models for the deep ocean $\mathrm{Min}(k_{ij})$ in Eqs. (11a—b) has to be replaced by some average of the spectrum. Even if the inequalities are not fulfilled over the whole time scale, the model result for the airborne fraction $\alpha(t)$, Eqs. (10a—e), is meaningfull to achieve lower bounds to the exact numerical version. This can be proved for the linearly driven case by taking into acount that $N(t)$ is a monotonically increasing function ($F(t) \geqq 0$).

In Fig. 2 we have plotted the effective fossil fuel growth rate $a_f(t) := F_f(t)/N_f(t)$ for different modified logistic input strategies (see Appendix A) and for simplicity using a fixed growth coefficient k_f. Because of $N_f(+0) = 0$, the functions $a_f(t)$ have a simple pole at $t = 0$: $a_f(t) \sim t^{-1}$ for $t \to +0$.

Since the cumulative fossil input $N_f(t)$ has been fitted to the recent historical period by an exponential input rate, $a_f(t)$ approaches a plateau ($a_f(t) \approx k_f$) after about $t \approx 100$ yr ($n \geqq 1$). However for $n < 1$ there are non-negligible deviations from the exponential behaviour. For illustration we have plotted in Fig. 3 the type of function defined by the left hand side of Eq. (11b) and using the logistic function ($n = 1$); $\mathrm{Min}(k_{ij})$ are taken from the set $\{0.1, 0.02, 0.01\}$ whereas k_f, Δ_0 and $N_{f\infty}$ are those used in Figure 2. For $\mathrm{Min}(k_{ij}) = 0.01$, as can be seen in Fig. 3, the inequality Eq. (11b) is no more fulfilled on the whole time scale after $t = 60$ yr, thus violating the adiabatic conditions.

3. Ocean Models

In this section we shall study explicitly the parametric and time dependent structure of Θ for different one-dimensional ocean models. Dephth dependent turbulent diffusivity as well as a simple advection model proposed by Hoffert et al. [12] is considered. The using of a more refined ocean model is probably an important element in the existing controversy about global carbon budgeting. As has already been pointed out in Section 2, we simplify the computation of Θ by using the adiabatic assumption for the deep sea fractions (concentrations related to the cumulated anthropogenic input). The ocean is regarded as extending to its mean depth ($h_m + h_d$) with equal transversal area of $A = 3.34 \cdot 10^{14}$ m^2.

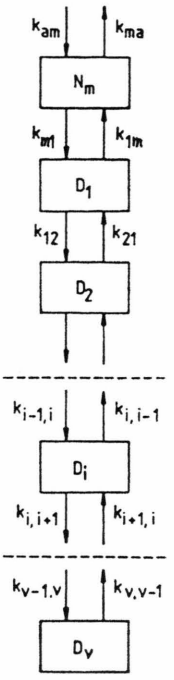

Fig. 4. Linear chain model for carbon exchange within the sea.

3a) Chain Models

We first assume, according to Fig. 4, a chain of v adjacent linearly interacting compartments for the deep sea reservoir, with also linear coupling of the surface layer with the first compartment of the deep sea. D_i ($i = 1, 2, \ldots, v$) denote the amount of carbon in the compartment i; the net flux from the reservoir i to reservoir $i+1$, assuming linear kinetics, is given by:

$$j_{i,i+1} = k_{i,i+1} D_i - k_{i+1,i} D_{i+1}, \quad (i = 1, \ldots, v-1). \tag{12}$$

The set of linear differential equations for the carbon inventories D_i is accordingly given by

$$\mathrm{d}D_1/\mathrm{d}t = \Delta F_{md} - j_{12}, \tag{13a}$$

$$\mathrm{d}D_i/\mathrm{d}t = j_{i-1,i} - j_{i,i+1}, \quad (i = 2, \ldots, v-1), \tag{13b}$$

$$\mathrm{d}D_v/\mathrm{d}t = j_{v-1,v}, \tag{13c}$$

whereas the disturbed flux from the mixed surface layer to the deep sea compartment D_1 is written as:

$$\Delta F_{md} = k_{m1} N_m - k_{1m} D_1. \tag{14}$$

Defining the relative perturbations:

$$\vartheta_i := \frac{D_i(t) - D_{i0}}{N(t)}, \tag{15}$$

and making use of the steady state conditions:

$$k_{i,i+1} D_{i0} = k_{i+1,i} D_{i+1,0}, \tag{16}$$

we obtain the new set of differential equations:

$$\dot{\vartheta}_1 = k_{m1}\mu - [a(t) + k_{1m} + k_{12}]\vartheta_1 + k_{21}\vartheta_2, \tag{17a}$$

$$\dot{\vartheta}_i = k_{i-1,i}\vartheta_{i-1} - [a(t) + k_{i,i-1} + k_{i,i+1}]\vartheta_i + k_{i+1,i}\vartheta_{i+1}, \quad (i = 1, \ldots, v-1), \tag{17b}$$

$$\dot{\vartheta}_v = k_{v-1,v}\vartheta_{v-1} - [a(t) + k_{v,v-1}]\vartheta_v. \tag{17c}$$

Mass conservation holds through the relationship:

$$\alpha(t) + \mu(t) + \sum_{i=1}^{v} \vartheta_i(t) = 1. \tag{18}$$

Invoking the adiabatic approximation of $\dot{\vartheta}_i \approx 0$, straightforward algebra for Eqs. (17a–c) and using Eq. (8) gives:

$$\Theta(t) \approx \frac{k_{m1}}{a(t)} \frac{1}{1 + \frac{k_{1m}}{a(t)} \frac{1}{1 + \Theta_1(t)}}, \tag{19}$$

with the recursion formula:

$$\Theta_i(t) = \frac{k_{i,i+1}}{a(t)} \frac{1}{1 + \frac{k_{i+1,i}}{a(t)} \frac{1}{1 + \Theta_{i+1}(t)}} \tag{20}$$

and the boundary condition of $\Theta_{v+1} \equiv 0$. This means that $\Theta(t)$ can be expressed as a finite fractional series. This method is illustrated in Appendix B by considering the case of a homogenous chain structure. We obtain:

$$\Theta(t) = \frac{h_d}{h_m} \frac{\tanh p}{p}, \quad p := \sqrt{\frac{a(t)}{K}}\, h_d, \tag{21a–b}$$

where K is a constant eddy diffusivity parameter. This result can be regarded as a slight generalisation of the result of Oeschger et al. [10] by replacing the e-folding time k_f^{-1} of an exponential fossil input by the time-dependent function $a(t)$, Equation (2). Since for $t \to \infty$ the function $a(t)$ decreases to zero, we have the asymptotic limit of $\Theta(t \to \infty) = h_d/h_m$, corresponding to a homogeneous distribution of the perturbed carbon profile in the ocean. However, this full uptake capacity of the deep sea, as a comparison of Figs. 2 and 5 reveals, is reached only after centuries.

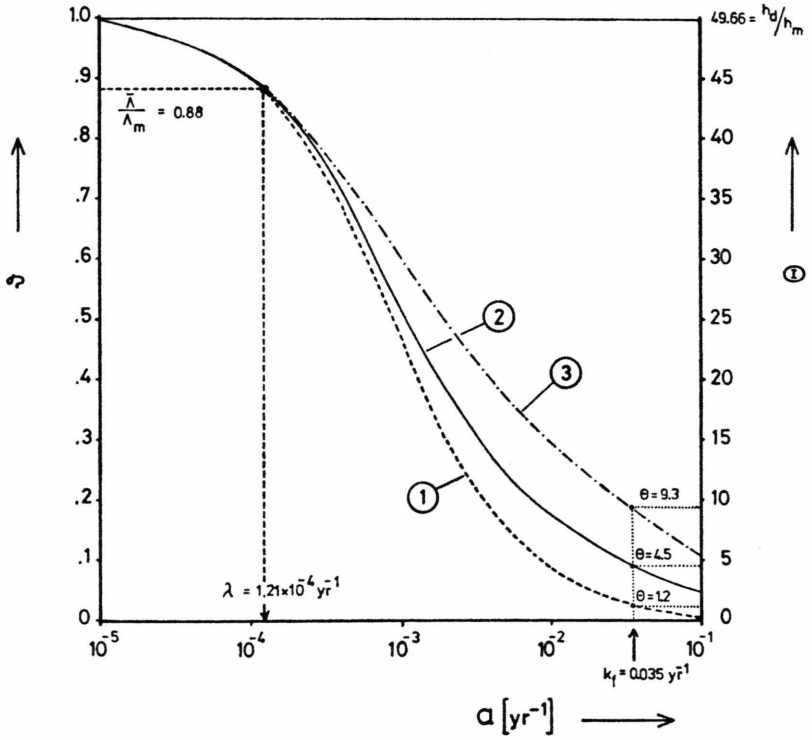

Fig. 5. The coupling function Θ and the ratio $\vartheta = \Theta/(h_d/h_m)$ vs. a for three different non-advective ocean models.
1: 2-Box-Ocean model ($T_d = 1/k_{dm} = 1127$ yr);
2: 1-Box-Diffusion model (K = 4000 m²/yr), see Sect. 3a;
3: 1-Box-Diffusion model with depth dependent eddy diffusivity ($z^* = 749$ m, $K(0) = 26192$ m²/yr), see Sect. 3c.

For comparison we consider the simple case of a 2-box ocean model; according to Eq. (19) by identifying $k_{m1} = k_{md}$, $k_{1m} = k_{dm}$ and $\Theta_1 \equiv 0$ we have:

$$\Theta(t) = \frac{k_{md}}{a(t) + k_{dm}} = \frac{h_d}{h_m} \frac{1}{1 + a(t)/k_{dm}}, \qquad (22)$$

where we have used equations similar to Eqs. (B11 — 12) of Appendix B. The coupling function Θ and the corresponding ratio $\vartheta := \Theta/(h_d/h_m)$ $(0 \leqq \vartheta \leqq 1)$ for both models, Eqs. (21a) and (22), have been plotted in Fig. 5 in dependence on $a \in [10^{-5}, 10^{-1}]$. The parameters used, namely $K = 4000\ m^2/yr$ (1.27 cm^2/s) for the box-diffusion model and $1/k_{dm} = 1127$ yr (turnover time) for the 2-box ocean model is consistent with the observed global averaged pre-bomb $^{14}C/C$ ratio in the oceans ($\overline{\Delta}$) [10]:

$$\vartheta[a = \lambda = 1.21 \cdot 10^{-4}/yr] = \overline{\Delta}/\Delta_m = 0.88,$$

with $\overline{\Delta} = 0.836$ and $\Delta_m = 0.95$ (Δ: ^{14}C depletion factor, $\Delta = 1 + \Delta\ ^{14}C/1000$). As can be seen from Fig. 5, the dynamic transfer strenght Θ with $a = k_f = 3.5\%/yr$ has the values of 1.2 and 4.5, corresponding to the parameters adopted for the 2-box and 1-box diffusion ocean model, resp. A comparative discussion of these two models can be found in [9—10].

3b) The Advection-diffusion Model

Hoffert et al. [12] have recently presented an extension of the Oeschger et al. [10] box-diffusion model including upwelling water masses in the low latitude oceans. In their model water masses leave the mixed surface layer towards high latitudes being transfered almost instantaneously to the bottom of the deep sea. In their first report the model has been run under steady-state conditions to achieve reasonably good agreement with observed ^{14}C carbon profiles. The perturbed fluxes and carbon inventories of this model are shown in Fig. 6, whereas the perturbed carbon concentrations at the lateral boundaries of the mixed layer in polar waters correspond to $Aw\Delta c_p$. Aw is the volumetric flow and w the uppwelling velocity. Accordingly the perturbed flux $\Delta F_{md}(t)$ is given by:

$$\Delta F_{md}(t) = A \left[w(\Delta c_p - \Delta c_m) - K \frac{\partial \Delta c_d}{\partial z} \bigg|_0 \right], \qquad (23)$$

where K is the constant eddy diffusivity and $\Delta c_d(z, t)$ is the perturbed carbon concentration profile of the deep ocean. The time-dependent concentration profile $\Delta c_d(z, t)$ is governed by the transport equation:

$$\frac{\partial \Delta c_d}{\partial t} = \frac{\partial}{\partial z} \left\{ w \Delta c_d + K \frac{\partial \Delta c_d}{\partial z} \right\}. \qquad (24)$$

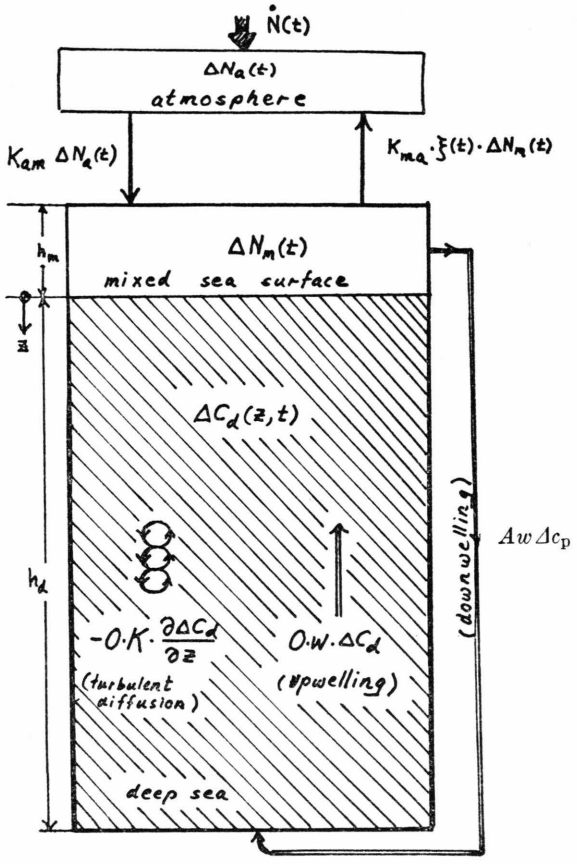

Fig. 6. Perturbed carbon fluxes in the advection-diffusion model. (In the Figure read A instead of O.)

Continuity of the perturbed concentrations at the boundary between mixed surface layer and the top of the deep ocean gives the first boundary condition:

$$\Delta c_d(0, t) = \Delta N_m(t)/(A h_m) = \Delta c_m(t). \qquad (25)$$

The boundary condition at the deep sea bottom can be expressed as:

$$K \frac{\partial \Delta c_d}{\partial z} \bigg|_{z = h_d} + w \Delta c_d(h_d, t) = w \Delta c_p(t). \qquad (26)$$

Which means that the diffusive plus upwelling carbon flux at the sea floor must equal the flux $Aw\Delta c_p(t)$ injected at the bottom from polar surface waters. We introduce the following dimension-

less variables and parameters:

$$y := z/h_d\,, \quad p := \sqrt{a(t)/K}\, h_d\,,$$
$$\varrho := w h_d/K\,. \qquad (27\,a-c)$$

With these notations inserted into Eq. (24) and taking the adiabatic limit, i.e.

$$\frac{\partial}{\partial t}\,[\Delta c_d(y,t)/N(t)] \approx 0\,,$$

we obtain the homogeneous second order differential equation:

$$\frac{\partial^2 \Delta c_d}{\partial y^2} + \varrho\,\frac{\partial \Delta c_d}{\partial y} - p^2\,\Delta c_d = 0\,. \qquad (27\,d)$$

Inserting Eq. (23) into Eq. (8), by considering the boundary condition Eq. (25) and the definitions Eqs. (27 a—c), we obtain:

$$\vartheta[a(t)] := \frac{\Theta}{(h_d/h_m)} = \frac{1}{p^2} \qquad (28)$$
$$\cdot\left\{\varrho\left[\frac{\Delta c_p}{\Delta c_m} - 1\right] - \frac{\partial}{\partial y}\ln \Delta c_d\Big|_{y=0}\right\},$$

where ϑ is the relative dynamical coupling coefficient (Θ related to the static value of $\mathrm{Max}(\Theta) = h_d/h_m$) which can now be evaluated by solving Eq. (27 d) together with the boundary conditions. As we have indicated in Eq. (28), in the adiabatic limit ϑ is a functional of $a(t)$. At this place we had to make some assumptions concerning the concentration ratio $\Delta c_p/\Delta c_m$; if this ratio is determined according to $\Delta c_p/\Delta c_m \approx c_{p0}/c_{m0} \approx 1$, we can neglect the advective term in Equation (28). For this case we can give the following structure for ϑ [23]:

$$\vartheta = \frac{1}{p^2}\,\frac{1}{1 + \dfrac{\varrho}{2\Delta}\tanh \Delta} \qquad (29\,a)$$
$$\cdot\left\{\varrho\left(1 - \frac{\exp[\varrho/2]}{\cosh \Delta}\right) + \frac{p^2 + \frac{1}{2}\varrho^2}{\Delta}\tanh \Delta\right\}$$

with the abbreviation

$$\Delta := \sqrt{p^2 + (\varrho/2)^2}\,. \qquad (29\,b)$$

In this advection-diffusion model the *perturbed* diffusion flux is directed *downwards* at the upper layers of the deep sea whereas at the bottom layers it is directed *upwards*. The full capacity of the deep sea system for uptake of excess carbon can formally be achieved from Eq. (21 a) for three limiting cases:

$$\lim_{t\to\infty} \Theta = \lim_{K\to\infty} \Theta = \lim_{\substack{w\to\infty \\ (K\neq 0)}} \Theta = h_d/h_m\,. \qquad (30)$$

For $\varrho = 0$ (no advection) we obtain again the result of Equation (21 a—b). In Fig. 7 we have plotted the analytical Θ according to Eqs. (29 a—b) versus the eddy diffusivity coefficient K for various upwelling velocities w.

Using the standard set of $K = 4000\ \mathrm{m^2/yr}$, $w = 4\ \mathrm{m/yr}$, $h_d = 3725\ \mathrm{m}$, $h_m = 75\ \mathrm{m}$ and $a(1980) = k_f = 0.035/\mathrm{yr}$ (an average exponential growth coefficient between 1860 and the present time) we have $\Theta = 5.3$, whereas the case without advection ($w = 0$) corresponds to the value of $\Theta = 4.5$. For this parameter set we achieve only a 15% increase in Θ. Anticipating a more general result of Section 3 c, we can relate $\vartheta[a(t)]$ (evaluated in the adiabatic limit) to the average prebomb radiocarbon distribution in the ocean by the relationship $\bar{\Delta}/\Delta_m = \vartheta[\lambda]$. Thus, if the perturbed carbon concentrations of Eqs. (25, 26, 27 d) are replaced by the corresponding ^{14}C depletion factors together with the substitution $a(t) \to \lambda$ (radiocarbon decay constant), a calibration of the parameter entering this model can be achieved [12]. For the set $K = 4000\ \mathrm{m^2/yr}$, $w =$

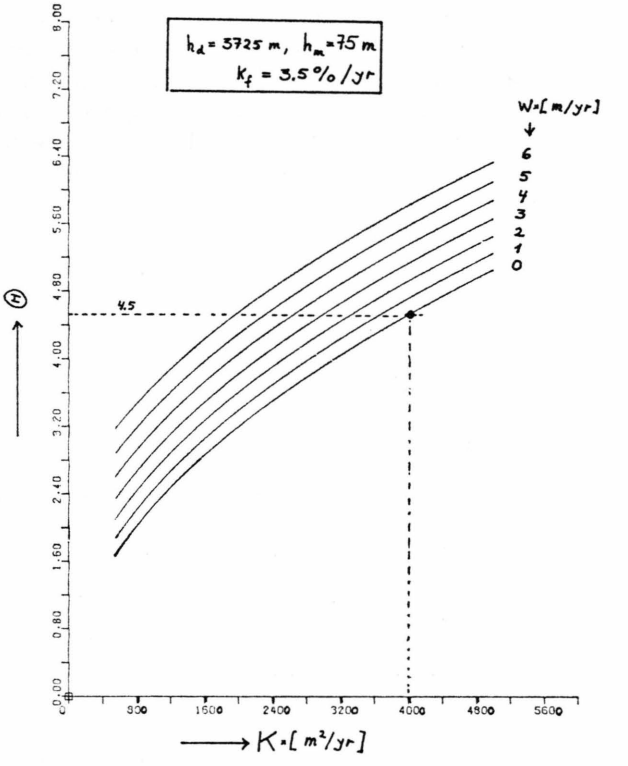

Fig. 7. Θ vs. K for different upwelling velocities w according to Eq. (29 a—b); $a = k_f$.

4 m/yr, $\Delta_p = 0.86$ and $\Delta_m = 0.95$ we obtain:

$$\Delta(y) = \tfrac{3}{4} e^{\lambda_1 y} + \tfrac{1}{5} e^{-\lambda_2 y} \quad (\lambda_1 = 0.1086, \quad \lambda_2 = 3.8336). \tag{31}$$

The average value of $\overline{\Delta} = 0.842$ and the deep sea bottom value of $\Delta_d = 0.839$ and a minimum value of $\Delta_{min} = 0.799$ located at $z = 2120$ m is obtained from Equation (31).

Hoffert et al. [12] used $K = 2000$ m^2/yr and obtained $\overline{\Delta} = 0.82$, $\Delta_d = 0.8475$ and a minimum value of $\Delta_{min} \approx 0.8$ located at $z = 1380$ m. However, for this parameter set there results a slight reduction of $\Theta[k_t = 3.5\%]$ against Oeschger's value of $\Theta = 4.5$ ($w = 0$), as can be seen from Figure 7. Although this simple advection-diffusion model is able to reproduce the main features of the global prebomb ^{14}C distribution, it cannot account for a sufficiently enhanced absorption of additional atmospheric carbon into the deep sea (see Section 4a).

3c) *The advection-diffusion Model with Depth-dependent Eddy Diffusivity*

We now generalize the model of the preceeding section assuming a 1-dimensional vertical turbulent mixing mechanism with a depth-dependent eddy diffusivity $K(z)$, including a constant upwelling velocity w. Invoking the adiabatic approximation we have to consider the following homogeneous linear differential equation for the perturbed concentration profile $\Delta c_d(y, t)$:

$$\frac{\partial}{\partial y}\left\{\varkappa(y)\frac{\partial \Delta c_d}{\partial y} + \varrho_0 \Delta c_d\right\} - p^2(0,t)\,\Delta c_d = 0 \tag{32}$$

with the boundary conditions

$$\Delta c_d(0,t) = \Delta c_m(t), \tag{33a–b}$$

$$\left.\frac{\partial \Delta c_d}{\partial y}\right|_{y=1} = \varkappa(1)\,\varrho_0[\Delta c_p(t) - \Delta c_d(1,t)],$$

where we have introduced the following notations:

$$y := \frac{z}{h_d}, \quad \varrho_0 := \frac{w}{K(0)} \cdot h_d, \tag{34a–d}$$

$$p(y,t) := \sqrt{\frac{a(t)}{K(y)}}\, h_d, \quad \varkappa(y) := \frac{K(y)}{K(0)}.$$

The dynamical coupling coefficient can be computed according to relationship:

$$\vartheta[a(t)] = \frac{\Theta}{(h_d/h_m)} = \frac{1}{p_0^2} \cdot \left\{\varrho_0\left[\frac{\Delta c_p}{\Delta c_m} - 1\right] - \frac{\partial}{\partial y} \ln \Delta c_d \Big|_{y=0}\right\}, \tag{35}$$

with $p_0 := p(0,t)$. The steady state advection-diffusion model applied to the vertical distribution of the prebomb radiocarbon depletion factor $\Delta = 1 + \Delta^{14}C/1000$ (the prebomb level of $\Delta^{14}C = 0^0/_{00}$ is assumed for the atmosphere) in the ocean gives a differential equation which is isomorphic to Equation (32). Thus, with the substitutions

$$\Delta c_d(y,t) \to \Delta(y), \quad a(t) \to \lambda$$

and after an integration of Eq. (32) in the interval $y \in [0, 1]$ adopting boundary conditions corresponding to Eqs. (33a–b) for the depletion factors, the simple relationship

$$\overline{\Delta}/\Delta_m = \vartheta[\lambda], \quad (\Delta(0) = \Delta_m), \tag{36}$$

results. This means that once determined the function $\vartheta[a(t)]$ in the adiabatic limit, Eq. (35), we can obtain the depth averaged value $\overline{\Delta}$ for the radiocarbon distribution by a simple substitution:

$$a(t) \to \lambda, \quad \Delta c_m \to \Delta_m, \quad \Delta c_p \to \Delta_p.$$

In order to give an analytical example we tentatively use the following 2-parameter eddy diffusivity distribution:

$$\varkappa(y) = K(y)/K(0) = e^{-qy}, \quad (q \geqq 0). \tag{37}$$

To fit the tritium profile in the central North Atlantic and He^3 age versus depth profile for the Sargasso Sea, models with depth dependent diffussivity become important [28]. The attenuation distance $z^* = h_d/q$, through which the eddy diffusivity decays to $1/e = 0.368$ of its original value $K(0)$ at the boundary between mixed layer and deep sea, is considered as a further adjustable parameter of this model. Without going into mathematical details and for simplicity considering only the non-advective case ($\varrho_0 \equiv 0$), we can solve Eq. (32) and obtain the following functional form for $\vartheta[a]$, Eq. (35):

$$\vartheta[a(t)] = \frac{1}{p_0} \cdot \frac{I_0\left(\frac{2p_0}{q} e^{q/2}\right) K_0\left(\frac{2p_0}{q}\right) - I_0\left(\frac{2p_0}{q}\right) K_0\left(\frac{2p_0}{q} e^{q/2}\right)}{I_0\left(\frac{2p_0}{q} e^{q/2}\right) K_1\left(\frac{2p_0}{q}\right) + I_1\left(\frac{2p_0}{q}\right) K_0\left(\frac{2p_0}{q} e^{q/2}\right)}, \tag{38}$$

where I_n, K_n are the modified Bessel-functions of order $n \in \{0, 1\}$. The limit of vanishing q reproduces the result of Eq. (21), if we note that $K(0) \equiv K$. If we tentatively set $K(0) = 16000$ m²/yr ($\triangleq 5.1$ cm²/sec) and $q = 4$ ($z^* = 931$ m) we obtain a depth averaged value of $\bar{K} = 4000$ m²/yr. For this parameter set we obtain for the radiocarbon ratio according to Equation (36): $\vartheta[\lambda] = 0.914$ ($\lambda = 1.21 \cdot 10^{-4}$/yr). Since the actual values for the depletion factors fall within the uncertainty ranges [10]

$$\Delta_m \in [0.93, 0.95], \quad \bar{\Delta} \in [0.81, 0.86],$$

thus $\vartheta[\lambda] \in [0.844, 0.925]$, the result is acceptable. A better parameter set, which reproduces the most probable value of $\vartheta[\lambda] = 0.88$ ($\Delta_m = 0.95$, $\bar{\Delta} = 0.836$), is given by $q = 4.9698$ ($z^* = 745$ m) and $K(0) = 26192$ m²/yr ($\triangleq 8.32$ cm²/sec) and corresponds to a depth averaged value of

$$\bar{K} = 5270 \text{ m}^2/\text{yr} \quad (\triangleq 1.62 \text{ cm}^2/\text{sec}).$$

The dependence of Θ and ϑ on $a(t)$ in the range of $a \in [10^{-5}, 10^{-1}]$ for this parameter set is shown in Figure 5. For the present situation $a = k_f \approx 3.5\%$/yr we can duplicate the perturbed flux ΔF_{md} against the value of Oeschger et al. [10]: $\Theta[k_f] \cong 9.3$. In order to reproduce the history of the atmospheric CO_2 content since the beginning of the industrialization as well as to give predictions for the coming decades there is no need to consider processes in the ocean which operate on a time-scale much larger than $1/k_f \approx 30$ yr ($\approx$ mean age of fossil fuel CO_2). On this time-scale the deep sea acts as an infinite reservoir and the coupling coefficient Θ does not depend on h_d. Equation (38) simplifies in this time range to give

$$\Theta[k_f] = \frac{z^*}{h_m} \frac{1}{p^*} \frac{K_0(2p^*)}{K_1(2p^*)}, \quad p^* := \sqrt{\frac{k_f}{K(0)}}\, z^*. \tag{39}$$

We should have obtained this result also by using the short time-scale boundary condition of

$$\lim_{y \to \infty} \partial \Delta c_d / \partial y = 0.$$

Since the parameter set $z^* = 745$ m, $K(0) = 8.32$ cm²/sec gives a relatively high radiocarbon age at the bottom of the sea ($\Delta_d \approx 0.69$), it would be worthwile to study the case of finite advection with the possibility of deep sea ventilation to obtain a satisfactory prebomb radiocarbon distribution together with the constraint of a large ocean absorption capacity ($\Theta(k_f) \gtrsim 9$).

4. Results and Discussion

4a) The Source-Sink Controversy

Now being able to quantify the sink strenghts of certain ocean models, we shall proceed in this chapter to the question of which biospheric net inputs $F_b^{eff}(t)$ are compatible with the above mentioned different model assumptions concerning the uptake capacity of the deep sea. The net biospheric input can be split up into an anthropogenic contribution $F_b(t)$ due to the cutting and burning of forests as well as to soil modifications, and a biospheric uptake flux $\Delta F_{ab}(t)$ induced by CO_2 increase in the atmosphere and the subclimax state of the biota. $\Delta F_{ab}(t)$ can be considered as the response of the biospheric system due to the anthropogenic input $[F_f(t) + F_b(t)]$. Claims have been made that the biosphere is a major *source* of CO_2 [1—3], however those engaged in global carbon budgeting have been calling for a modest *increase* in the size of the terrestial biosphere [10, 24—26] order to achieve a balance in the carbon budget. This sink-source problem has recently been reviewed by some authors [6, 18, 23, 26]. Most of this discussion is based on the simple box-diffusion model [10], with model parameters calibrated by using isotopic data. A recent review by Oeschger et al. [27] states that the net biospheric carbon dioxide flux F_b^{eff} into the atmosphere at present time is probably at most about 10% of the fossil fuel input rate F_f. Larger releases from the biosphere into the atmosphere would therefor imply a remarkable stimulation of oceanic carbon uptake.

We assume in the following that the net biospheric input $F_b^{eff}(t)$ has not strongly deviated from an exponential growth, with an average growth coefficient $k_f = 3.5\%$ for the period 1860—1980. We define a parameter ε, as a measure of the source ($\varepsilon > 0$) or sink ($\varepsilon < 0$) strenght of the actual biosphere:

$$\varepsilon(t) := F_b^{eff}(t)/F_f(t). \tag{40}$$

This parameter should not be confused with the so called biota growth factor, which has been employed in carbon cycle models [8, 10, 27]. However, under certain assumptions both types of parameters can be related.

Since by assumption ε does not strongly depend on t for the period between 1860 ($t = 0$) and the present, we have after Eqs. (1—2) the following implications:

$$N(t) \approx (1+\varepsilon) N_f(t),$$
$$a(t) \approx a_f(t) = \frac{F_f(t)}{N_f(t)} \approx k_f. \qquad (41a-b)$$

According to the Mauna Loa record, covering the period 1958—1978, the apparent airborne fraction related to fossil fuel input is determined by:

$$\hat{\alpha}_f := \frac{N_a(1978) - N_a(1958)}{N_f(1978) - N_f(1958)} = 0.56 \pm 0.06. \qquad (42)$$

The apparent airborne fraction related to the fossil input for the whole period since the beginning of industrialisation is defined equivalently by

$$\alpha_f(t) := (N_a(t) - N_{a0})/N_f(t), \qquad (43)$$

and can be determined only within large error limits, since the initial atmospheric inventary N_{a0} is not well known. Since by assumption the theoretical airborne fraction α, Eqs. (10a—b), is nearly constant for the period 1860—1978 ($a(t) \approx k_f$), we can tentatively determine the following relationship between the empirical quantity of Eq. (42) and the theoretical airborne fraction α under consideration of Eqs. (6a), (10a—b), (41a—b) and (43):

$$\alpha \approx \frac{1}{1+\varepsilon} \hat{\alpha}_f. \qquad (44)$$

In Eq. (44) the theoretical airborne fraction α, which is related to the cumulated fossil plus net biospheric input, is computed in the adiabatic approximation by using Eqs. (10a—e), chosing a deep ocean model, as for example described in Section 3a—c, and using for simplicity a constant growth coefficient k_f. Appart from the coupling strength Θ between the mixed layer and the deep ocean, there are still some uncertainties concerning the other quantities entering the theoretical expression of Equation (10b). The following set we refer to as a standard $k_{am} = 1/(7.7\text{ yr})$, $N_{a0} = 624$ Gt C ($\triangleq 293$ ppm), $N_{m0} = 618$ Gt C ($\triangleq$: $h_m = 75$ m, $A = 3.34 \cdot 10^{14}$ m^2, $c_{m0} = 0.0247$ kg C/m^3), $\xi = 10$, $k_f = 0.035/\text{yr} = 1/(28\text{ yr})$. This parameter set determines a theoretical airborne fraction α, according to Eqs. (10a—b), and its dependence on Θ for the interval [0, 12] is shown in Fig. 8 (dashed curve: $\alpha_{st.}(\Theta)$). We define an upper bound function $\alpha_{max}(\Theta)$ for the parameters $k_{am} = 1/(10\text{ yr})$, $k_f = 4.3\%/\text{yr}$, $\xi = 10$, $N_{m0} = 618$ Gt C and $N_{a0} = 624$ Gt C and equivalently a lower bound function $\alpha_{min}(\Theta)$ by using the set: $k_{am} = 1/(4\text{ yr})$, $k_f = 3.5\%/\text{yr}$, $\xi = 9$, $N_{a0} = 575$ Gt C, $N_{m0} = 666$ Gt C. Both curves are shown in Figure 8 as well. The sign of the dimensionless parameter ε, as can been seen in Fig. 8, is deter-

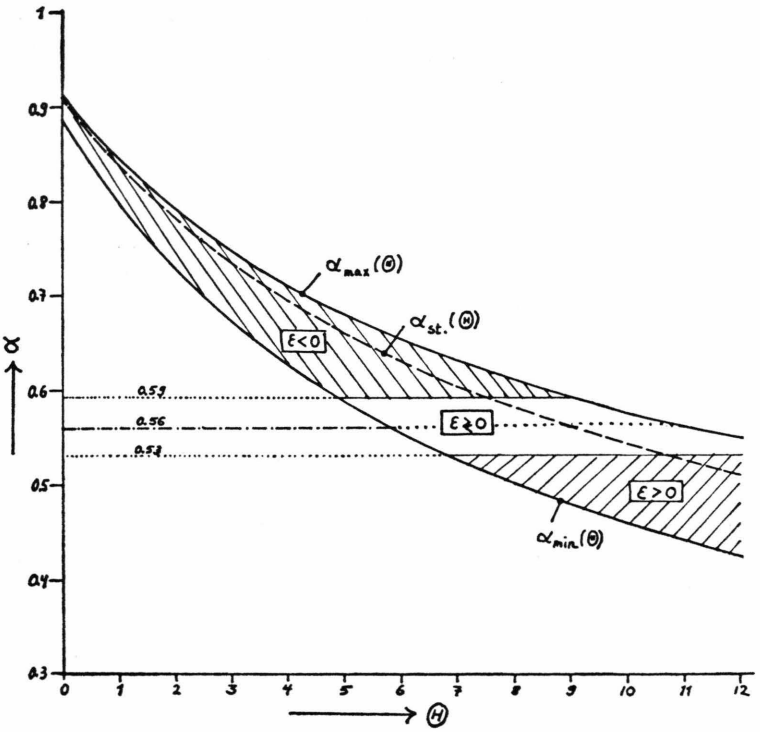

Fig. 8. Airborne fraction α vs. coupling function Θ for different data sets:

α_{max}: $k_{am} = 1/10$ yr, $k_f = 1/23$ yr, $\xi = 10$, $N_{ao} = 624$ Gt C (293 ppm), $N_{mo} = 618$ Gt C ($h_m = 75$ m);

α_{st}: $k_{am} = 1/7.7$ yr, $k_f = 1/23$ yr, $\xi = 10$, $N_{ao} = 624$ Gt C (293 ppm), $N_{mo} = 618$ Gt C ($h_m = 75$ m);

α_{min}: $k_{am} = 1/4$ yr, $k_f = 1/28$ yr, $\xi = 9$, $N_{ao} = 575$ Gt C (270 ppm), $N_{mo} = 666$ Gt C.

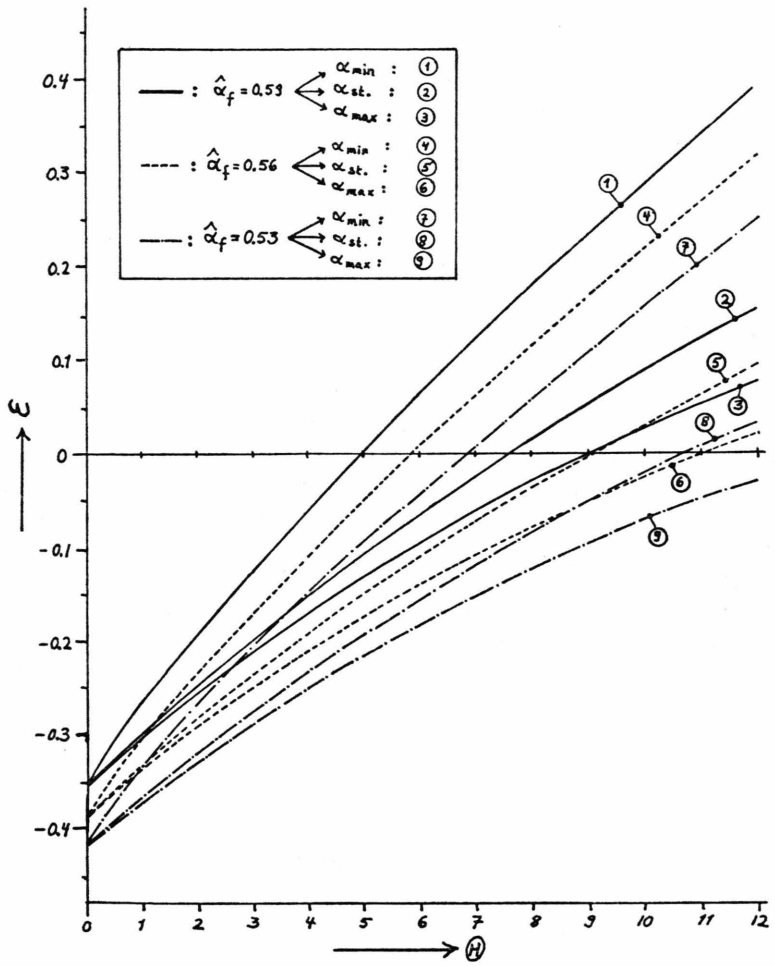

Fig. 9. Uncertainty ranges of biospheric transfer parameter ε in dependence on coupling function Θ.

mined both from the value of Θ and the apparent airborne fraction $\hat{\alpha}_f$, Equation (42). Assuming only a variation of 5% for $\hat{\alpha}_f$, thus taking $\hat{\alpha}_f \in [0.53, 0.59]$, and assuming $\Theta > 5$, there will be a region (between the hatched regions in Fig. 8) of uncertainty where the biosphere is a net source ($\varepsilon > 0$) or a sink ($\varepsilon < 0$) for CO_2. The functional dependence of the parameter ε on Θ for different combinations of apparent airborne fractions $\hat{\alpha}_f$ and the three already defined theoretical airborne fractions α have been plotted in Figure 9. The standard curve ⑤ of Fig. 9 predicts a net release of biospheric CO_2 into the atmosphere for rather large coupling strenghts, namely for $\Theta > 9$, thus illucidating the dramatic controversy between biologists, claiming for ε to be between [0.1, 0.5], and oceanographers, claiming that the biosphere is a net sink for atmospheric CO_2. The simple box-diffusion model after Oeschger et al. [10] with an eddy diffusivity of $K = 4000$ m^2/yr gives ($h_m = 75$ m, $k_f = 3.5\%/yr$): $\Theta = 4.46$. Even if one assumes that K can be taken from the interval $[2, 6] \times 10^3$ m^2/yr there is a moderate change of Θ in between the interval [3.1, 5.46]. However, for the case of curve ①, using a larger apparent airborne fraction of $\hat{\alpha}_f = 0.59$ and a drastically changed rate constant of $k_{am} = 1/(4\,yr)$, the theory predicts positive biospheric parameters ε for $\Theta > 5$.

Taking into account depth dependent turbulent diffusion which is consistent with the stationary $^{14}C/C$ isotopic depth profile one can arrive at $\Theta = 9.3$, giving $\varepsilon \approx 0$ for standard curve ⑤, Figure 9. It can be expected that ε may be even larger, if advection is considered additionally. So, it can be shown with such analytical carbon cycle models, that a mild biospheric CO_2 source is compatible with a refined

model ocean (depth dependent turbulent diffusivity and advection).

4b) Analytical Time-dependent Airborne Fraction

It has become clear, that the global fossil fuel reserve, estimated to contain more than ten times the amount of carbon in today's atmosphere ($N_{f\infty} \cong 7000$ Gt C), could raise the present concentrations of atmospheric CO_2 by large factors as a result of its combustion. There is serious concern about the possible global climatic changes as an effect of such an increase. Model projections for the atmospheric variations during the next 100 years or so, using a variable buffer factor $\xi(t)$ and a logistic ($n = 1$) fossil-fuel CO_2 source function with a final cumulated limit of $N_{f\infty} = 7000$ Gt C, have shown that the CO_2 fraction $N_a(t)/N_{a0}$ will be about 10 [11, 25]. Such model projections can also be estimated by using the analytic method described in Section 2. This method has several advantages: (a) quick estimates of future atmospheric CO_2 fractions for different anthropogenic input strategies are achieved, (b) the effects of using different carbon exchange mechanisms between the mixed sea surface layer and the upper layers of the deep ocean can be estimated, (c) different internal deep sea exchange mechanism can be considered by this method, (d) the effects of different model parameters of a concrete system become transparent.

To conclude this section, we present the time dependence of different functions for the advection-diffusion ocean model described in Section 3b over a 400 years range of time. For reasons of simplicity we neglect any net biospheric input into the atmosphere and use a logistic fossil-fuel input with parameters (see appendix A): $k_f = 0.035$/yr, $N_{f\infty} = 7000$ Gt C, $\Delta_0 = 2.63$ Gt. In Fig. 10 the assumed CO_2 fossil-fuel production rate $\dot{N}_f(t) = F_f(t)$ and the corresponding cumulative production $N_f(t)$ is plotted. The rate peak is located in the vicinity of the year 2060 corresponding to a cumulative depletion of 50%. In Fig. 11 we have plotted the analytical airborne fraction $\alpha_f(t)$, according to Eqs. (10a—e) and using the coupling function $\Theta(t)$

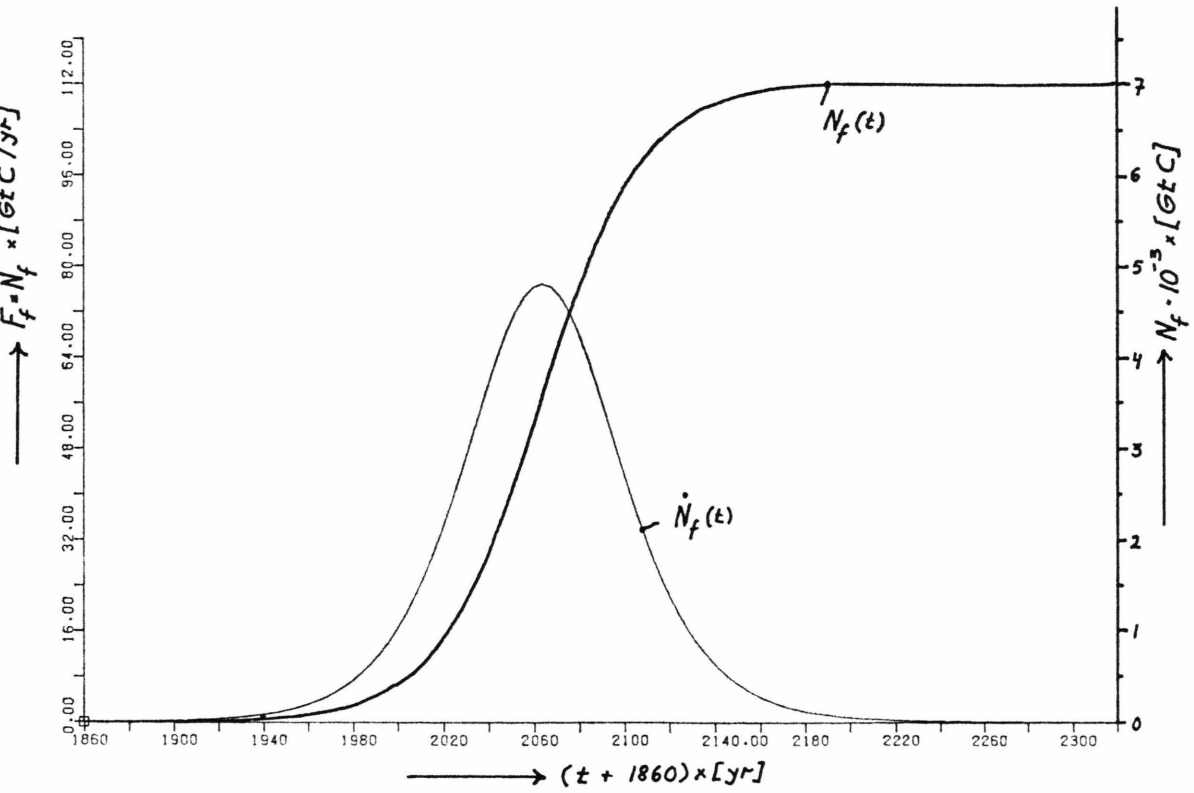

Fig. 10. Differential ($\dot{N}_f$) and cumulated (N_f) fossil input with $k_f = 3.5\%$/yr and $N_{f\infty} = 7000$ Gt C.

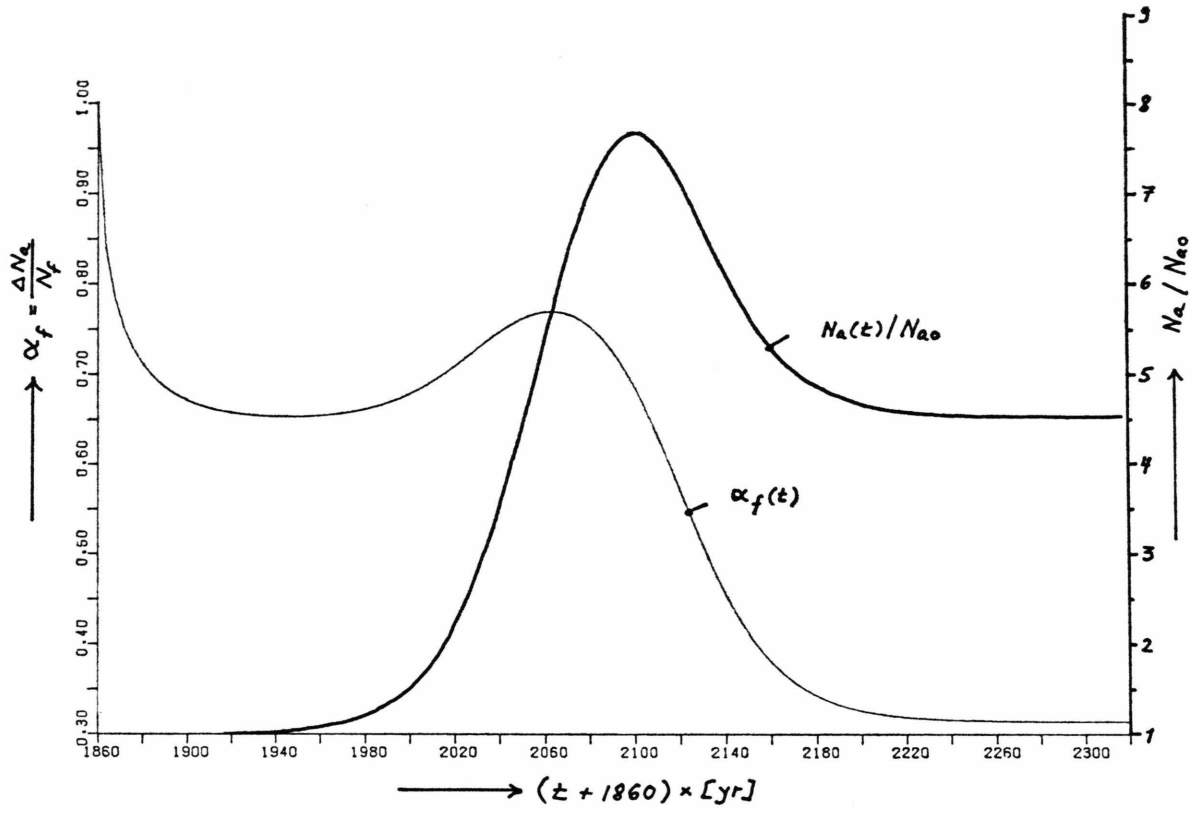

Fig. 11. Airborne fraction related to fossil input (Fig. 10) a_f and CO_2 factor $N_a(t)/N_{a0}$ for a 400 years time range.

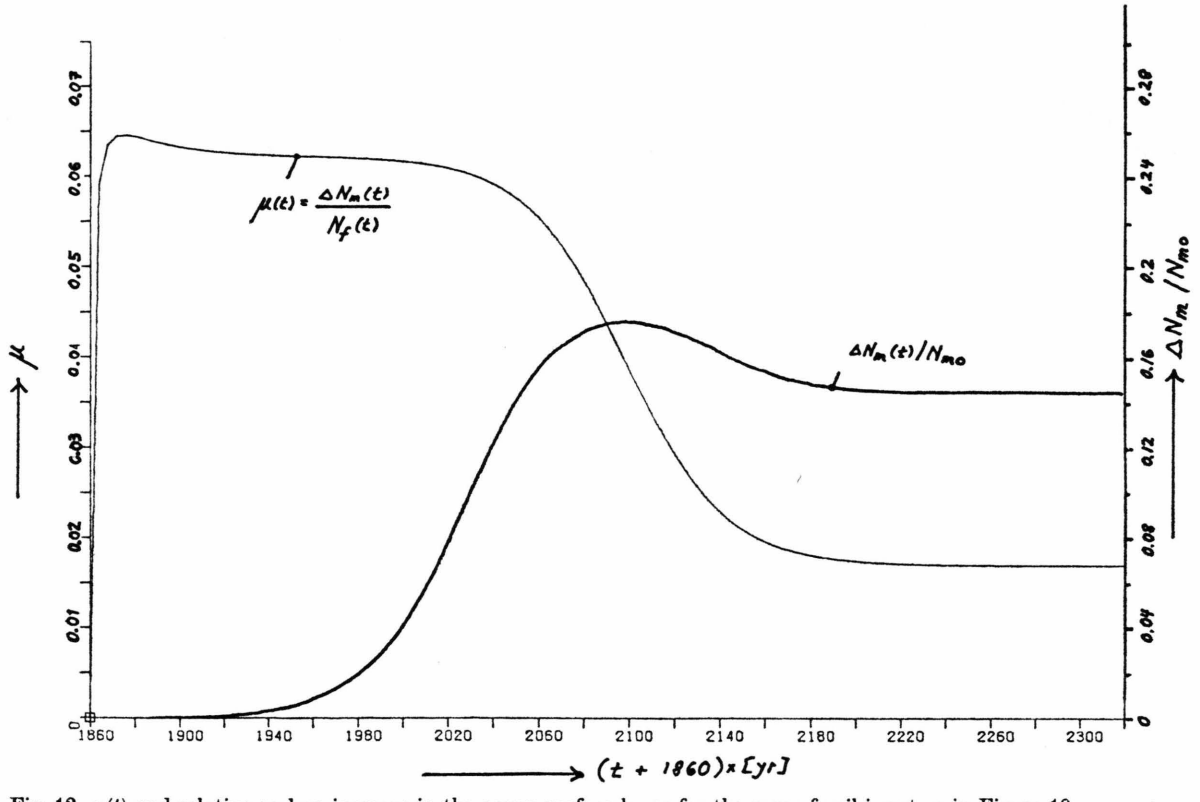

Fig. 12. $\mu(t)$ and relative carbon increase in the ocean surface layer for the same fossil input as in Figure 10.

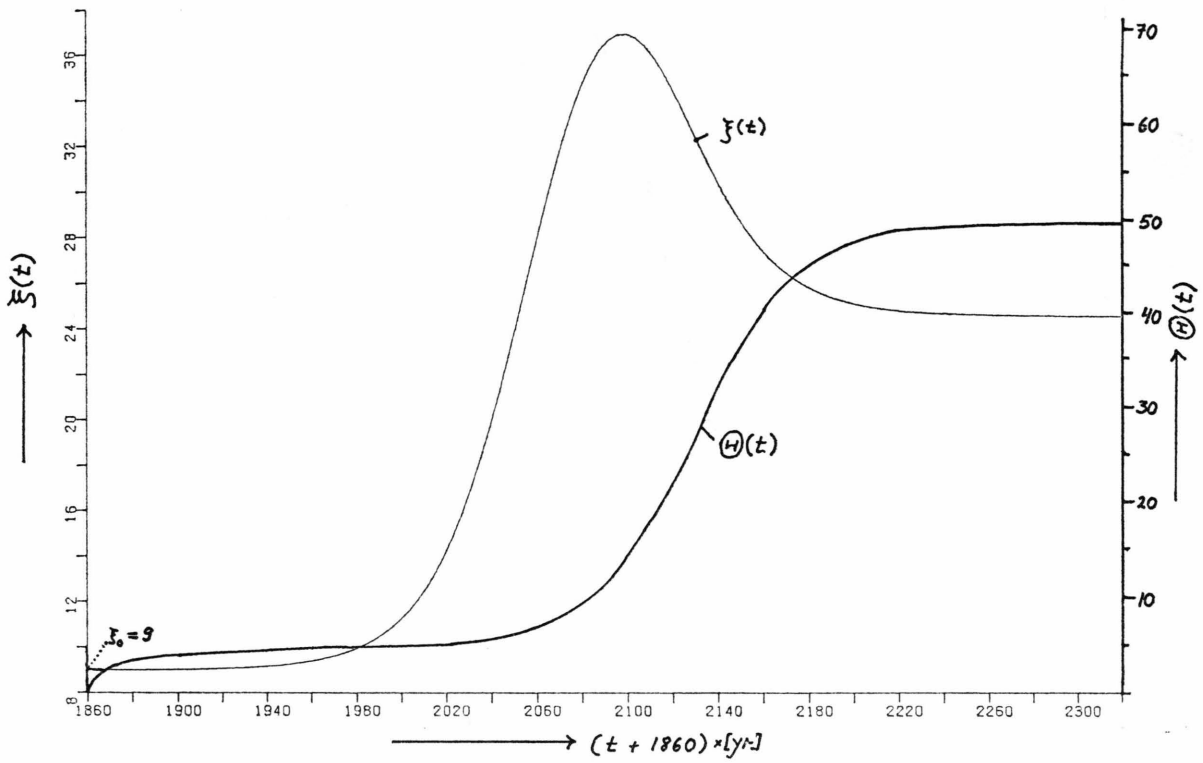

Fig. 13. Buffer factor ξ and deep sea coupling function Θ vs. time again for a fossil input according to Figure 10.

of Eq. (29), for the period 1860—2300. The following parameter set was used: $k_{am} = 1/(7.7\,\text{yr})$, $h_m = 75$ m, $h_d = 3725$ m, $K = 4000\ \text{m}^2/\text{yr}$, $w = 4$ m/yr, $N_{a0} = 624$ Gt C, $N_{m0} = 618$ Gt C and finally the variable buffer factor, Eq. (9), as described in Section 2. As has been expected, the theoretical $\alpha_f(t)$ does not reproduce the Mauna Loa data for the period 1958—1980, invoking either a net biospheric sink for additional atmospheric CO_2 or an inappropriate ocean model. The corresponding CO_2 fraction is also shown in Fig. 11, with a maximum of 8 at the year 2100. The airborne fraction $\alpha_f(t)$ is not monotonically decreasing with time since we have taken into account the nonlinearity introduced by a variable buffer factor; it namely shows a maximum located at the year 2060. The dynamical behaviour of the mixed layer is demonstrated in Fig. 12, showing the small uptake capacity of this reservoir. At the year 2100, when the atmospheric and mixed layer inventories reach their maximum values, 90% of the fossil fuel reserve will be depleted and the buffer factor (see Fig. 13) has the maximum-value of around $\xi(2100) \approx 36$. The time dependence of the coupling function $\Theta(t)$, Eq. (29), and the buffer factor $\xi(t)$ is plotted in Figure 13. Obviously, the dynamic uptake capacity of the deep ocean is moderate for the period 1860—2100 and the ocean acts as an ideal sink for excess CO_2 in the atmosphere only after the year 2200.

Acknowledgement

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The calculations have been carried out on the university computers at Frankfurt (Main) and Darmstadt.

Appendix

A) Modeling the Fossil Input

To assess the future CO_2 burden, long-term considerations must take into account that the total amount of fossil fuels are limited. We propose a slight generalisation for commonly used fossil source

functions [9]:

$$a_f(t) := \frac{\dot{N}_f(t)}{N_f(t)} \tag{A1}$$
$$= k_f(t)\frac{N_f(t)+\Delta_0}{N_f(t)}\left[1-\left(\frac{N_f(t)}{N_{f\infty}}\right)^n\right].$$

The positive parameter Δ_0 in Eq. (A1) considers that $N_f = 0$ is not a fixed point of the differential equation (A1) ($\dot{N}_f(+0) = F_f(+0) = k_f(+0)\Delta_0$). The solution of Eq. (A1) corresponds to a modified logistic fossil source function (under appropriate conditions on $k_f(t)$) with an adjustable positive cut-off parameter n; $N_{f\infty}$ is the ultimately available fossil fuel. The growth function $k_f(t)$ serves to simulate the empirical data on fossil fuel combustion for the period 1860—1980. One finds for the periods 1860—1910 and 1945—1980 a differential growth coefficient of $\Delta F_f/\Delta N_f = 4.3\%/\text{yr}$ but a disturbed growth in between [15, 16]. Introducing the dimensionsless function $u(t) := \{N_{f\infty}/[N_f(t)+\Delta_0)\}^n$ and noting that $\Delta_0 \ll N_{f\infty}$ we have the simple linear differential equation:

$$\mathrm{d}u/\mathrm{d}t = n\,k_f(t)\,[u-1]. \tag{A2}$$

Integration of Eq. (A2) finally gives for the cumulated fossil input:

$$N_f(t) = \frac{N_{f\infty}}{\left\{1+\left[\left(\frac{N_{f\infty}}{\Delta_0}\right)^n - 1\right]\exp\left[-n\int_0^t k_f(t')\,\mathrm{d}t'\right]\right\}^{1/n}} - \Delta_0. \tag{A3}$$

Since $k_f(t)$ is positive and at least piecewise continuous for the period of 1860—1945 (we set $k_{f0} \equiv k_f(t) = \text{const}$ for $t \geqq 85$ yr) we have the correct asymptotic limit of $N_f(\infty) = N_{f\infty}$.

For the sake of completeness we can deduce from the analytical result of Eq. (A3) some interesting relations assumming $k_f(t) = k_{f0} = \text{const}$ for

$$t \geqq 85\text{ yr} \quad \text{and} \quad N_f(85\text{ yr}) \ll N_{f\infty}.$$

The time t_0 of the occurence of the fossil rate peak for different fuel combustion patterns is given by ($\Delta_0 \ll N_f(85\text{ yr}) \ll N_{f\infty}$):

$$t_0 \approx \frac{1}{K_{f0}}\left[\ln\left(\frac{N_{f\infty}}{\Delta_0}\right) - \ln(n^{1/n})\right], \tag{A4}$$

with a peak maximum

$$F_f(t_0) = \dot{N}_f(t_0) \approx \frac{k_{f0}N_{f\infty}}{(1+n)^{1/n}}\frac{n}{1+n}. \tag{A5}$$

The ratio of fossil fuel combusted until the time $t = t_0$ to the ultimately available one, is accordingly given by:

$$N_f(t_0)/N_{f\infty} = (1+n)^{-1/n}. \tag{A6}$$

B) The Continuum Limit for the Homogeneous Chain

Denoting the depth of the deep sea compartments by h, we introduce the following abbreviations:

$$p(t) := \sqrt{\frac{a(t)}{K}}\,h_d \tag{B1}$$

$$K := kh^2, \quad h_d := \nu h, \tag{B2—3}$$

$$k := k_{1m} = k_{1,2} = k_{2,1} = \cdots = k_{\nu-1,\nu} = k_{\nu,\nu-1}. \tag{B4}$$

From Eq. (20) we obtain the following difference equations:

$$\Delta\Theta_i := \Theta_{i+1} - \Theta_i = \frac{\Theta_i^2 + \Theta_i - (\nu/p)^2}{(\nu/p)^2 - \Theta_i}, \tag{B5}$$

which can also be written in the form:

$$\left\{\frac{(\nu/p)^2-\Theta^+}{\Theta_i-\Theta^+} - \frac{(\nu/p)^2-\Theta^-}{\Theta_i-\Theta^-}\right\}\Delta\Theta_i = \sqrt{1+\left(\frac{2\nu}{p}\right)^2}, \tag{B6}$$

where

$$\Theta^{+/-} := \tfrac{1}{2}[-1 \pm \sqrt{1+(2\nu/p)^2}] \tag{B7}$$

denotes the roots of the numerator of Equation (B5). We now analyse the continuum limit ($\nu \to \infty$) with the constraint of finite K and h_d, Equations (B2—5). Transforming Eq. (B6) to an integral

$$\int_{\Theta_1}^{\Theta_\nu}\left\{\frac{(\nu/p)^2-\Theta^+}{\Theta-\Theta^+} - \frac{(\nu/p)^2-\Theta^-}{\Theta-\Theta^-}\right\}\mathrm{d}\Theta = \sqrt{1+\left(\frac{2\nu}{p}\right)^2}\,(\nu-1), \tag{B8}$$

with $\Theta_\nu = (\nu/p)^2/[1+(\nu/p)^2]$, and considering that the parameter p is finite and that $\nu/p \gg 1$ holds, we

achieve after integration:

$$\ln\left\{\frac{(1+\Theta_1 p/\nu)}{(1-\Theta_1 p/\nu)}\right\} = 2p\,, \tag{B9}$$

or

$$\Theta_1(t) = \nu\,\frac{\tanh[p(t)]}{p(t)} \gg 1\,. \tag{B10}$$

Inserting the relationship of Eq. (B10) into Eq. (19) of Section 3a and using the conditions

$$k_{m1} N_{m0} = k_{1m} D_{10}\,,$$

$$\frac{N_{m0}}{A\,h_m} = \frac{D_{10}}{A\,h_d}\,, \tag{B11–12}$$

we finally obtain for the dynamic coupling function (Eqs. (21) of Section 3a):

$$\Theta(t) = \frac{h_d}{h_m}\,\frac{\tanh p(t)}{p(t)}\,. \tag{B13}$$

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