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RESEARCH ARTICLE

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Groups of piecewise isometric permutations of lattice points, or Finitary rearrangements of tessellations

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Abstract

Through the glasses of didactic reduction, we consider a (periodic) tessellation Δ of either Euclidean or hyperbolic n-space M. By a piecewise isometric rearrangement of Δ we mean the process of cutting M along corank-1 tile-faces into finitely many convex polyhedral pieces, and rearranging the pieces to a new tight covering of the tessellation Δ. Such a rearrangement defines a permutation of the (centers of the) tiles of Δ, and we are interested in the group $PI(\Delta)$ of all piecewise isometric rearrangements of Δ. In this paper, we offer (a) an illustration of piecewise isometric rearrangements in the visually attractive hyperbolic plane, (b) an explanation on how this is related to Richard Thompson's groups, (c) a section on the structure of the group $\mathrm{pei}(\mathbb{Z}^n)$ of all piecewise Euclidean rearrangements of the standard cubically tessellated $\mathbb{R}^n$, and (d) results on the finiteness properties of some subgroups of $\mathrm{pei}(\mathbb{Z}^n)$.

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CHAPTER 1. INTRODUCTION

1 | GENERALITIES AND MAIN RESULT

1.1 | The groups

Let M denote either Euclidean or hyperbolic n-space, $N \in \mathbb{N}$, and $\Gamma \leq \text{Isom}(M)$ a discrete group of isometries of M with the property that Γ admits a finite sided convex fundamental polyhedron D with finite volume.[†] We aim to study certain groups of permutations of the orbit $\Omega := \Gamma p$, for a given point $p \in M$. The major part of this paper is concerned with the most down-to-earth case when $\Omega := \mathbb{Z}^n$, viewed as the set of tile centers of the tessellation dual to the standard tessellation of Euclidean $\mathbb{R}^n$ by unit cubes.

To define the notion of a piecewise Γ-isometric permutation $\pi : \Omega \to \Omega$ requires a notion of Γ-*polyhedral pieces* of Ω on which π should be isometric, and it is reasonable to require that the geometry of these pieces be related to the geometry of Γ. Thus, together with the base point $p \in M$ we choose a finite set $\mathcal{H}$ of 'Γ-*relevant*' *closed half-spaces* of M, and the resulting groups will — to some extent — depend on this choice: We fix a (finite-sided convex) fundamental polyhedron D and take $\mathcal{H}$ to be an irredundant finite set of half spaces with the property that D is the intersection $D = \bigcap_{H \in \mathcal{H}} H$ and each member of $\mathcal{H}$ has its boundary spanned by a side of D.

By a *convex* Γ-*polyhedral* subset P of M we mean any finite intersection of Γ-translates $H\gamma$, where $\gamma \in \Gamma$ and $H \in \mathcal{H}$. And a general Γ-*polyhedral* subset of M is a finite union of convex ones. By abuse of language, we call the intersection $S = \Omega \cap P$ a *(convex)* Γ-*polyhedral piece of* Ω whenever $P \subseteq M$ is a (convex) Γ-polyhedral subset.

Definition. Let $S \subseteq \Omega$ be a Γ-polyhedral set, and $\Gamma^* \leqslant \Gamma$ a subgroup. A permutation $g : S \to S$ is said to be *piecewise* Γ^*-*isometric* if S can be written as a disjoint union of finitely many Γ-polyhedral pieces $S = S_1 \cup S_2 \cup \cdots \cup S_k$ with the property that the restriction of g to each S_i is also the restriction of an isometry $\varphi_i \in \Gamma^*$.

We write $G_{\Gamma^*}(S)$ for the group of all piecewise isometric Γ^*-permutations of S. The permutations in $G_{\Gamma^*}(S)$ with finite support form a normal subgroup of $G_{\Gamma^*}(S)$ which we denote by sym(S); the quotient group

$$G_{\Gamma^*}(S)/\text{sym}(S)$$

is often particularly interesting.

[†] In the hyperbolic case this implies that D is actually a generalized polytope; see [30, Theorem 6.4.8].

Remark.

(1) Particularly nice is the situation when M comes with a regular tessellation. In that case we take Ω as the centers of mass of the tiles, and $G(\Omega)$ could be viewed (and termed) as the *group of all piecewise isometric tile-rearrangements*: Here, Γ is the group of all isometries of M compatible with the tessellation, and $\mathcal{H}$ as the set of half-spaces bounded by the span of a corank-1 face of a tile-fundamental domain.
(2) In a recent preprint [19] Farley and Huges present a promising general abstract approach to the finiteness properties of what they call *locally defined groups*. In their terminology, our piecewise isometric permutations are locally defined by isometries and hence appear as a special case. The authors obtain unified proofs for (the positive direction of) type F_n, for several generalized Thompson groups, but they add the remark that our examples [10] appear to pose a more substantial challenge.

In this paper we consider the group $G_{\Gamma^*}(S)$ in two special cases:

(a) When $M = \mathbb{H}^2$ is the hyperbolic plane we consider *triangle groups* and their orientation preserving subgroups $\Gamma^* \leqslant \Gamma$ acting on the tessellation Δ by the Γ-translates of a hyperbolic triangle D. In the special case when D is the *ideal* triangle (all three vertices at infinity) the quotient $G_{\Gamma^*}(\Omega)/\mathrm{sym}(\Omega)$ is Richard Thompson's groups V. In the more general case when D has at least one vertex at infinity we can assume that one of these corresponds to the point $\infty \in \partial\mathbb{H}^2$ in the upper half plane model, and that all tile-vertices of Δ in $\partial\mathbb{H}^2$ correspond to rational numbers.
    We show that in this situation the group $G_{\Gamma^*}(S)$ has a description in terms of the spine T of the tessellation Δ, which is a bipartite tree. This description can be used to prove finiteness properties of $G_{\Gamma^*}(\Omega)/\mathrm{sym}(\Omega)$ if and only if Γ contains no hyperbolic elements with rational fixed points. We also outline how one could attack the general case.
(b) Our main concern then is the case when $M = \mathbb{R}^n$ is Euclidean n-space, $\Gamma = \mathrm{Isom}(\mathbb{Z}^n)$, and Γ^* either equal to Γ or its translation subgroup $T \leqslant \Gamma$. We call the Γ-polyhedral pieces $S \subseteq \mathbb{Z}^n$ the *orthohedral* subsets of $\mathbb{Z}^n$, and consider the *piecewise Euclidean isometry* groups $\mathrm{pei}(S) = G_\Gamma(S)$ and its subgroup $\mathrm{pet}(S) \leqslant \mathrm{pei}(S)$, the *piecewise Euclidean translation* groups $G_T(S)$ of arbitrary orthohedral subsets $S \subseteq \mathbb{Z}^n$.

If S is the disjoint union of h copies of $\mathbb{N}$ then the pet-group $\mathrm{pet}(S) = G_T(S)$ is Houghton's group H_n [22]. Known for more than 38 years was also the pet-group $\mathrm{pet}(S)$ when $S = \bigcup_{1\leqslant i\leqslant h} \mathbb{N}^2$ is a disjoint union of h quadrants: This was the topic of the second author's diploma thesis [34] in which she proved, among other things, that $\mathrm{pet}(S)$ is of type F_{h-1} (see Section 1.5 for more information).

The fact that our groups have prominent relatives is not our only motivation: In Chapter 3 we make an effort to analyze the structure of $\mathrm{pei}(S)$, and this culminates at the end of Section 4 with full information on the normal subgroup lattice of $\mathrm{pei}(S)$. And in Chapter 4 we get concrete information on finiteness properties (finite presentability and high finiteness length — see Section 1.2) of $\mathrm{pet}(S)$ and $\mathrm{pei}(S)$. Thus, here is a new playground — prominently located in a good neighborhood — to studying the interaction between structure and finiteness properties. From the tree-hyperbolic world where the monsters live (like Thompsons' group V), we have gotten used to seeing many examples which are simple groups of type F_∞. What we find in our Euclidean analogue is similar, but interestingly different: Instead of simplicity we find the *Bottleneck theorem* (Theorem 4.18) which excludes hidden normal subgroups;

and instead of F_∞ we find, for example, that pei$(\mathbb{Z}^n)$ is of type F_{2^n-1} — for the main results, see Section 1.3.

1.2 | The finiteness length of a group

Every group is *of type* F_0; every finitely generated group is *of type* F_1; every finitely presented group (equivalently: every fundamental group $\pi_1(X)$ of a finite cell complex X) is *of type* F_2; and $\pi_1(X)$ is *of type* F_m $(m \geqslant 2)$ if X is a finite cell complex and $\pi_1(X) = 0$, for all i with $2 \leqslant i < m$.

Ten years after Wall introduced these finiteness properties, Borel and Serre [12, 13] showed that all semi-simple S-arithmetic groups have special homological features; in particular they are of type F_∞ (equivalently, type F_m for all $m \in \mathbb{N}$). And this was only the first of a number of important infinite families of groups that turned out to be of type F_∞ in the following decades; many of them, just like arithmetic groups, in the center of mainstream group theory: automorphism groups of free groups [18], Thompson's groups [15], etc. More recent results in this direction are based on Brown's topological discrete Morse theory technique [14] and its powerful CAT(0)-version of Bestvina-Brady [3].

The insight that many important groups have much further reaching finiteness properties than finite presentability is great progress — but having 'good' finiteness properties is only one side of the concept: The focus on the *finiteness length* function $fl : \mathbf{Gr} \to \mathbb{N} \cup \{0, \infty\}$, defined on all groups G by

$$fl(G) := \sup\{m \mid G \text{ is of type } F_m\}$$

takes both sides into account. Analogous *algebraic length functions* afl_A are defined for every G-module A, to be the supremum of all non-negative integers m with the property that A admits a free resolution which is finitely generated in all dimensions at most m. The functions afl_A have the considerable advantage that they extend immediately to monoids G. We write afl for $afl_{\mathbb{Z}}$, where $\mathbb{Z}$ stands for the infinite cyclic group with the trivial G-action; by the Hurewicz theorem we know that afl coincides with fl on all finitely presented groups (that is, whenever $fl(G) \geqslant 2$). An important feature of both fl and afl_A is that they are constant on commensurability classes of groups.

In general, the finiteness length of a group is notoriously difficult to compute. Nevertheless, to study and interpret accessible parts of the pattern that these functions carve into group theory can be very fruitful. A convincing example is the following: If we fix a finitely generated group G, then the function $Hom(G, \mathbb{R}_{add}) \to \mathbb{N} \cup \{0, \infty\}$, which associates with each homomorphism $\chi : G \to \mathbb{R}_{add}$ the value of afl_A on the submonoid $\chi^{-1}([0, \infty)) \subseteq G$, imposes in the finite-dimensional $\mathbb{R}$-vector space $Hom(G, \mathbb{R}_{add})$ the pattern exhibited by the homological Σ-invariants $\Sigma^k(G; A)$ of [9]. On the other hand, we can also evaluate fl and afl_A on the commensurability classes of subgroups containing G', and this yields patterns on the rational Grassmann space of $\mathbb{Q}$-linear subspaces of $G/G' \otimes \mathbb{Q}$ (which parametrizes these classes). The core of the main Σ-results of [6, 8, 9], [33] consists then of exhibiting the precise relationship between the two patterns.

An intriguing point is that in all computable examples the finiteness length patterns have a polyhedral flavor: they turn out to be expressible in terms of finitely many inequalities. One of the few general results here, polyhedrality of $\Sigma^0(G;A)$ when G is Abelian, was proved in [7] by methods which were later partly re-detected in tropical geometry. But polyhedrality questions on $\Sigma^k(G;A)$ for non-Abelian G and $k > 0$ are wide open.

1.3 | The results

Chapter 2 has two goals: it illustrates piecewise isometric permutations in the visually attractive area of two-dimensional hyperbolic geometry, and it links our piecewise isometric permutations to the group theory revolving around Thompson's groups.

We mentioned already that for non-cocompact triangle groups Γ we can express $G_{\Gamma^*}(\Omega)$ in terms of the spine T of Δ. This exhibits $G_{\Gamma^*}(\Omega)$ as a permutation group on the vertices of the tree T. We discuss whether this action respects almost all edges and cyclic star-orderings of T (following the terminology of [11], [28, 29], [31], this would be an action by *quasi-planar-tree automorphisms*).

We find: If Γ has signature $[\infty,\infty,\infty]$ then T is the dyadic tree and the action of $G_{\Gamma^*}(\Omega)$ on it is the one that has always been used to describe the elements of Thompson's groups in terms of generators (and is obviously quasi isometric). In the general case the action is always by piecewise planar-tree isometries, and by quasi-isometries if and only if the signature is $[p,q,\infty]$, with at least one of p,q infinite or odd.

Chapter 3 and 4 are about the Euclidean case — more precisely, we restrict attention to the case when M is Euclidean and carries the standard tessellation by unit cubes, that is, $\Gamma = \mathrm{Isom}(\mathbb{Z}^n)$, and Γ^* is either equal to Γ or its translation subgroup $T \leqslant \Gamma$, and the goal is to make first steps toward evaluating the finiteness length functions *fl* and *afl* on what we like to view as the *pei- and pet-clouds* around $\mathrm{Isom}(\mathbb{Z}^n)$, respectively, $\mathbb{Z}^n$: the groups $\mathrm{pei}(S) = G_\Gamma(S)$, respectively, $\mathrm{pet}(S) = G_T(S)$, as S runs through all orthohedral subsets of some $\mathbb{Z}^N$.

To state the main results requires the following notation: By an *orthant of rank n* ($n \in \mathbb{N}$) we mean any subset $L \subseteq \mathbb{Z}^N$ isometric to the standard rank-n orthant $\mathbb{N}^n$. Each orthohedral set $S \subseteq \mathbb{Z}^N$ is the disjoint union of finitely many orthants $S = L_1 \cup L_2 \cup \cdots \cup L_k$.

Definition (Rank and height). By the *rank* of S, denoted by $n = \mathrm{rk}\, S$, we mean the maximum rank of the orthants L_i; and the *height* of S, denoted by $h(S)$, is the number of orthants of rank $\mathrm{rk}\, S$ among the L_i.

One observes that the orthohedral sets with the piecewise Euclidean-isometric maps between them (called *pei-maps*) form a category. Clearly, the pei-isomorphisms are the bijective pei-isometries; and in Section 3.4 we observe that orthohedral sets are pei-isomorphic if and only if their rank and height agree.

Chapter 3 starts with introducing these basic concepts then turns to analyzing the group theoretic structure of $G := \mathrm{pei}(S)$ for an arbitrary orthohedral subset $S \subseteq \mathbb{Z}^N$. The results are summarized in some detail at the end of Section 4.1. The key here is a structure at infinity of the orthoedral set S — analogous to the structure at infinity of the tessellated hyperbolic plane which was used above to relate groups of piecewise hyperbolic isometries to Thompson's groups. This structure at infinity of S consists of

(1) a rank-graded G-set $\Gamma^*(S) = \bigcup_k \Gamma^k(S)$, called the set of *germs of S*;

(2) a family of rank-k cosets $\langle\gamma\rangle \subset \mathbb{Z}^N$ attached at the germs $\gamma \in \Gamma^k(S)$ which we call the *rank-k tangent coset of* γ. (The product $T^k := \prod_{\gamma\in\Gamma^k(S)}\langle\gamma\rangle$ can be viewed as the rank-k part of the *tangent space* T of S at infinity);
(3) an induced action of G on $\Gamma^k(S)$, and an induced action of G on each $\langle\gamma\rangle$ by isometries $g_\gamma : \langle\gamma\rangle \to \langle\gamma g\rangle$.

The definition of germs is in Section 3.2, and their tangent cosets crop up first in Section 4.3. Here we mention merely:

- two orthants $L, L' \subset \mathbb{Z}^N$ are *commensurable* if L, L' and $L \cap L'$ have the same rank, and the commensurability classes represented by rank-k orthants $L \subset S$ are the germs $\gamma \in \Gamma^k(S)$;
- the tangent coset $\langle\gamma\rangle$ of $\gamma \in \Gamma^k(S)$ is the union of all members of the commensurability class γ and thus $\langle\gamma\rangle \cong \mathbb{Z}^k$;
- for any given pair $(\gamma, g) \in \Gamma^k(S) \times G$, an orthant L representing γ can be chosen sufficiently far out to make that the restriction of g to L is an isometry $g|_L : L \to Lg$ (see Lemma 3.3). This restriction defines the action on both $\Gamma^k(S)$, and T.

Single elements $g \in G$ are supported on a finite set of orthants, and the maximum rank of these orthants is the *rank of* g, denoted by $\mathrm{rk}(g)$. From this we infer that g_γ is the identity if and only if $\mathrm{rk}(g) < \mathrm{rk}(\gamma)$.

Now we consider the normal series

$$1 = G_{-1} \leqslant G_0 \leqslant G_1 \leqslant \cdots \leqslant G_k \leqslant \cdots \leqslant G_n = G,$$

where the *rank-k subgroup* G_k consist of all elements $g \in G$ of rank at most k.

Its factors G_k/G_{k-1} exhibit clear footprints of the structure of $\mathrm{Isom}(\mathbb{Z}^n)$ which is exhibited by the refinement

$$G_{k-1} \leqslant C^{\mathrm{ord}}(\Gamma^k(S)) \leqslant C(\Gamma^k(S)) \leqslant G_k,$$

where $C(\Gamma^k(S))$ consists of all elements $g \in G$ which fix all rank-k germs of S, and $C^{\mathrm{ord}}(\Gamma^k(S))$ consists of all $g \in G$ with the property that, in addition, for all germs $\gamma \in \Gamma^k(S)$ the isometry $g_\gamma : \langle\gamma\rangle \to \langle\gamma g\rangle$ is a translation.

We prove that the quotient $A_k := C^{\mathrm{ord}}(\Gamma^k(S))/G_{k-1}$ is free-Abelian (of rank ∞ for $k < n$); $C(\Gamma^k(S))$ is the direct product of symmetric groups of degree k, and $G_k/C(\Gamma^k(S))$ is the finitary symmetric group of degree $|\Gamma^k(S)|$ — for more details, see Theorem 4.8.

In particular, G is elementary amenable, and in the $G_k/C^{\mathrm{ord}}(\Gamma^k(S))$-module A_k we can track a congruence- subgroup type property — see Section 4.7.

The Bottleneck theorem (Theorem 4.18) in Section 4.9 finally shows that the rank-subgroups G_k are characteristic; and, up to an index 2, all normal subgroups of G can be tracked by the ones in the quotients G_k/G_{k-1}.

In Chapter 4 we use Brown's approach in [14] to compute the finiteness lengths of $\mathrm{pet}(S)$ and a lower bound on the finiteness lengths of $\mathrm{pei}(S)$. Just as in Brown's paper each *fl*-result comes together with a parallel *afl*-result, hence our results have the same feature. We found

Theorem A. *Rank and height of an orthohedral set S determines the group* $\mathrm{pei}(S)$ *up to isomorphism, and we have fl*$(\mathrm{pei}(S)) \geqslant h(S) - 1$*; in particular, fl*$(\mathrm{pei}(\mathbb{Z}^n)) \geqslant 2^n - 1$.

For a more precise result see Section 5.1.

The exact value of $fl(\mathrm{pei}(\mathbb{Z}^n)) < \infty$ remains a challenging open problem. In the pet-case we know more: The isomorphism class of pei(S) is not determined by rank and height of the orthohedral set S. But we do have a precise result for the special case when S is a stack of h parallel orthants of the same rank:

Theorem B. *If S is a stack of h rank-n orthants then* $fl(\mathrm{pet}(S)) = h(S) - 1$.

A generalization to a stack of k-skeletons of an orthant is in Theorem 7.5.

Remark. Highly complex elementary amenable groups with high finiteness length can also be constructed in terms of permutational wreath products $A \wr_X B$. Bartholdi, de Cornulier, and Kochloukova [2] provide the technique to compute $fl(A \wr_X B)$ in favorable situations; and Kropholler–Martino [26] apply this to construct a sequence of wreath powers of Houghton's group H_n, $P(m) := H_n(\wr_X H_n)^m$ with constant finiteness length $fl(P(m)) = fl(H_n) = n - 1$ for all m. Thus they take $fl(H_n)$ for granted and provide a method to increase the complexity of H_n, whereas in the present work we extend Brown's computation of $fl(H_n)$ to new groups which are poly-(locally Houghton-by-finite) and analyze their structure.

1.4 | Outlook

Let Γ be a discrete group of (Euclidean or hyperbolic) isometries with polyhedral fundamental domain of finite volume. By generalizing the definition of the group pei($\mathbb{Z}^n$) to the groups $G_\Gamma(\Omega)$ of all piecewise Γ-isometric permutations of the orbit $\Omega = \Gamma p$, we have endowed each such group Γ with the G_Γ-*cloud* of all piecewise Γ-isometric permutation groups $G_\Gamma(S)$ where S runs through the Γ-polyhedral subsets of Ω. The success with evaluating the finiteness length function on the clouds around Isom($\mathbb{Z}^n$) and $\mathbb{Z}^n$, together with the observation that the groups around $SL_2(\mathbb{Z})$ are closely related to the highly respected Thompson groups, indicates that finding more of this might be a difficult but worthwhile program.

Particularly promising projects would be

(i) Finding the phi(Ω) when Ω is given by a regular tessellation of the hyperbolic plane, and the precise relationship between the induced group on the boundary, phi(Ω)/sym(Ω), and Thompson's groups. First steps in this direction based on (a slight generalization of) Theorem 2.3 are suggested in Section 2.6.

(ii) There are strong indications that $fl(\mathrm{pei}(S)) > h(S) - 1$. In particular, Thomas Kilcoyne has a proof that pei(S)/sym(S) is finitely presented if S is a stack of at least 2 quadrants. Thus, progress in this direction seems accessible — whether pei(S)/sym(S) is better behaved than pei(S) itself remains to be seen. For the Houghton groups this is trivially true, and the subtle difference between QV and $\tilde{QV}$ (see [31]) might indicate that this is indeed the case.

(iii) Defining and studying a group pal($\mathbb{Z}^n$) of *piecewise affine-linear permutations* on $\mathbb{Z}^n$, and find the footprints of the structure of $SL_n(\mathbb{Z}^n)$ in its structure.

1.5 | Remark on the history of this paper

Houghton originally introduced his groups in [22]. Theorem B, in the Houghton group case, that is, when S is a stack of rays, is due to Brown [14], and we follow his footsteps.

The inequality $fl(\mathrm{pet}(S)) \geqslant h(S) - 1$ in the rank-2 case when S is a stack of quadrants (as well as the equality for a certain 'diagonal subgroup' of $\mathrm{pet}(S)$) is due to the second author and appears in her diploma thesis (Frankfurt 1992 [34]), to which the first author contributed little more than the definition[†] of the group. Her diploma thesis could have been the starting point of a promising PhD project — but she preferred starting a true-to-life career in software development.

Back then, hunting for further generalizations of such groups was not the first author's priority either — they looked artificial and in those days only of use as counterexamples to questions that nobody asked. Therefore the project went dormant for 22 years, until an increasing number of publications on Houghton's groups ([1], [16], [28], [35], [38], etc.) suggested that Sach's diploma thesis [34] should be published, translated, and generalized. We started our collaboration in 2014.

The (back then surprising) insight that our groups are not only *generalized Houghton groups* but fit in a more interesting general (Euclidean or hyperbolic) geometric framework, which could be described as the groups of *tile-permutations induced by finitary rearrangements of tessellations*, was added when we put the preprint [10] (together with the original diploma thesis [34]) on the arXiv in June 2016, and we submitted [10] to the LMS in May 2017. Sadly, our collaboration ended in 2018 due to serious health issues. Substantial results on the group structure (Section 4) and the hyperbolic triangle groups (Section 2) were added during the refereeing process, and the expanded paper with the new title was accepted for publication in October 2021.

CHAPTER 2. ON THE HYPERBOLIC CASE

2 | PLANAR HYPERBOLIC EXAMPLES

2.1 | Piecewise Γ-hyperbolic triangle groups

Let D be a hyperbolic triangle of finite area in the compactified (Poincaré disk model of the) hyperbolic plane $\mathbb{H}^2 \cup \partial\mathbb{H}^2$. We write v_i for the vertices and e_i for edges of D, $1 \leqslant i \leqslant 3$, and use the convention that v_i is opposite to e_i. We write $H_i \subset \mathbb{H}^2$ for the half-plane which contains D and is bounded by the line spanned by e_i, and we put $\mathcal{H} := \{H_1, H_2, H_3\}$ to be the irredundant finite set of half-spaces as in the definition in Section 1.1.

Regardless of whether some of the v_i are on $\partial\mathbb{H}^2$, the triangle D has a unique inscribed circle (exhibited in the fundamental triangle in Figures 1 and 3). We will take its hyperbolic center as the base point $p \in D$ and call it the tile center of D. Let $t_i \in e_i$ denote the touching point of the inscribed circle on the edge e_i. Elementary geometric arguments show that if e_i, e_j are the two edges emanating from v_k then there is a hyperbolic disk B_k centered at v_k which has t_i and t_j on its boundary (if $v_k \in \partial\mathbb{H}^2$ then B_k is understood to be a horodisk centered at v_k).

We assume that the angle of D at v_i is $\frac{\pi}{q_i}$, where $q_i \in \mathbb{N} \cup \{\infty\}$ with $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} < 1$. Then the hyperbolic reflections σ_i over the edges e_i define a particularly nice tessellation on $\mathbb{H}^2$ and generate the isometry group Γ. The triple $[q_1, q_2, q_3]$ is the *signature* of the triangle group Γ. $\mathbb{H}^2$ is now endowed with a simplicial Γ-complex Δ. Here are some elementary facts:

[†] Influenced by Greenberg's courage to define $SL_2(\mathbb{Z})$-geometry [21] — which is similar to but different from ours.

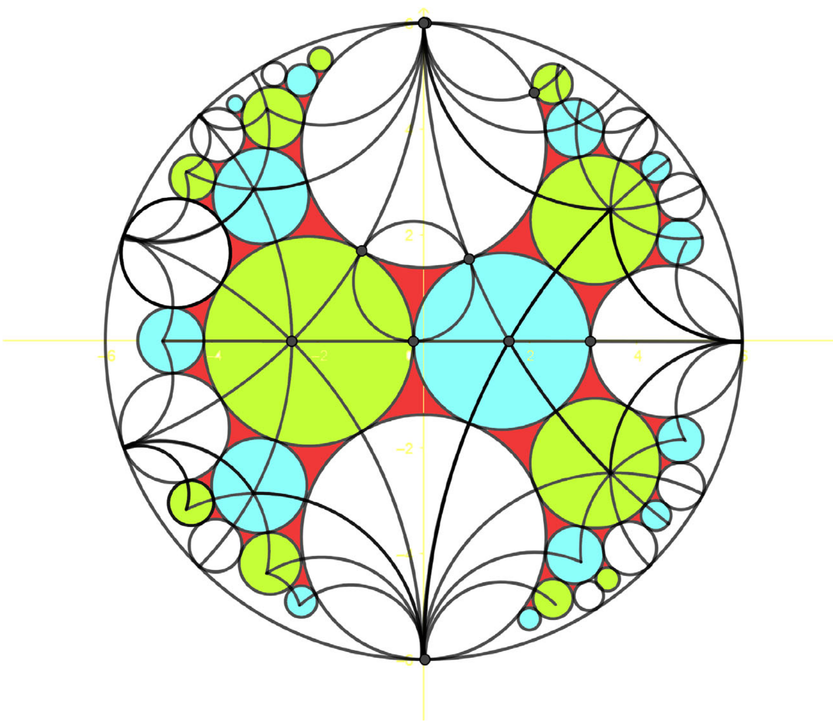

FIGURE 1 The signature $[3, 4, \infty]$ case

(1) if $P \subset \mathbb{H}^2$ is Γ-polyhedral, so is the closure of its complement;
(2) each edge e of Δ spans a hyperbolic line $h[e]$ in the 1-skeleton Δ^1;
(3) a hyperbolic line $h \subset \Delta^1$ is tessellated by infinitely many finite edges if and only if h is a *hyperbolic axis* (the axis of a hyperbolic element $g \in \Gamma$).

We put $\Omega = \Gamma p$ and are interested in the group $G(\Omega) = \text{phi}(\Omega)$ of all piecewise Γ-isometric permutations, and in subgroups $G_{\Gamma^*}(\Omega) \leqslant G(\Omega)$, when $\Gamma^* \leqslant \Gamma$ is a specified subgroup of Γ. $\mathbb{H}^2$ is now equipped with three Γ-orbits ΓB_i of disks centered at the tile-vertices gv_i; these disks touch each other, and their mutual touching points coincide with the points gt_i. The hyperbolic segments connecting the tile centers of edge-neighboring tiles cross vertically at the points gt_i through the edge ge_i of Δ and constitute the edges of the dual tessellation Δ^* of Δ. The dual tile with center $gv_i \in \mathbb{H}^2$ is a convex $2q_i$-gon around the inscribed disk gB_i (in the case when $gv_j \in \partial\mathbb{H}^2$ this is the area bounded by a doubly infinite sequence of finite dual edges tangent to the horodisk gB_j with center gv_j).

The charm of Δ^* is that Ω stands for the tiles, and the edges of Δ^* indicate how tiles are glued together. This opens the possibility that the Γ-polyhedral pieces can be described in terms of the 1-skeleton of Δ^*. If D is compact this remains a challenge to be addressed somewhere else.

2.2 | The case when Γ is a non-cocompact triangle group

The situation is simpler when D is not compact, and from now on we assume that at least $q_3 = \infty$: The point is that we have now a Γ-equivariant retraction of the hyperbolic plane $\mathbb{H}^2$ along the hyperbolic lines emanating out of the horodisk centers gv_j for $g \in \Gamma$ and $v_j \in \partial\mathbb{H}^2$, and

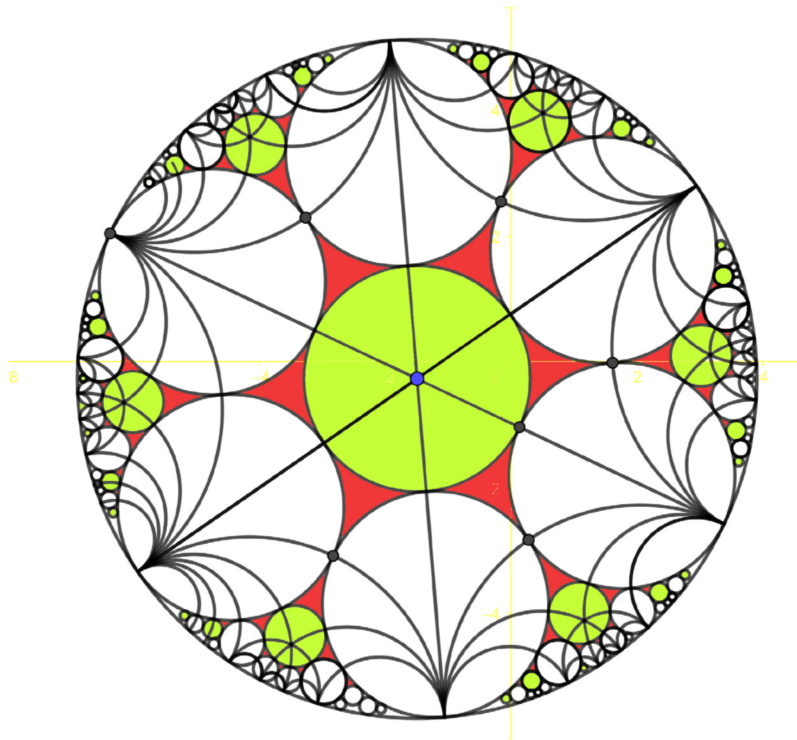

FIGURE 2 The signature $[3, \infty, \infty]$ case

terminating on the boundaries of the horodisks gB_j. The truncated hyperbolic plane

$$\mathbf{T} := \mathbb{H}^2 - \left(\bigcup_{g \in \Gamma,\, v_j \in \partial\mathbb{H}^2} \mathrm{Int(gB_j)} \right),$$

obtained by excision of all open horodisks gB_j, closely approximates the finite part Δ^*_{fin} of the complex Δ^*. $\mathbf{T}$ is a tree-shaped union of bands meandering between the horodisks gB_l toward $\partial\mathbb{H}^2$, and it contains the finite part of the dual tessellation $\Delta^*_{\text{fin}} \subset \mathbf{T}$ is the cell complex with vertex set $\Omega = \Gamma p$, all dual edges of length equal to the diameter of the inscribed circle of D, and 2-cells semi-regular $2q_i$-gons around the finite disks gB_i.

That $\mathbf{T}$ is tree-shaped can be seen by referring to either the retraction of $\mathbb{H}^2$ onto $\mathbf{T}$, or to the fact that whenever a band enters an area bounded by two horodisks through its tight entrance there is no escape on a different route through another exit.

Illustrations. With signature $[2, 3, \infty]$, $[3, \infty, \infty]$, $[\infty, \infty, \infty]$ are exhibited in Figures 1–3. The colored part of the pictures exhibits the set $\mathbf{T}$ (the finite disks B_i in green/blue, and the rest of $\mathbf{T}$ consists of little triangular red shapes, each containing exactly one point of Ω). As the neighboring ones touch each other at a kissing point this red part of $\mathbf{T}$ actually contains and outlines the 1-skeleton of Δ^*_{fin}.

The retraction of $\mathbb{H}^2$ onto $\mathbf{T}$ (or Δ^*_{fin}) can be prolongated to a retraction $\rho : \mathbb{H}^2 \to T$ onto a Γ-invariant tree $T \subseteq \mathbf{T}$ which we call the *spine* (of X). If D has a finite edge e this retraction pushes

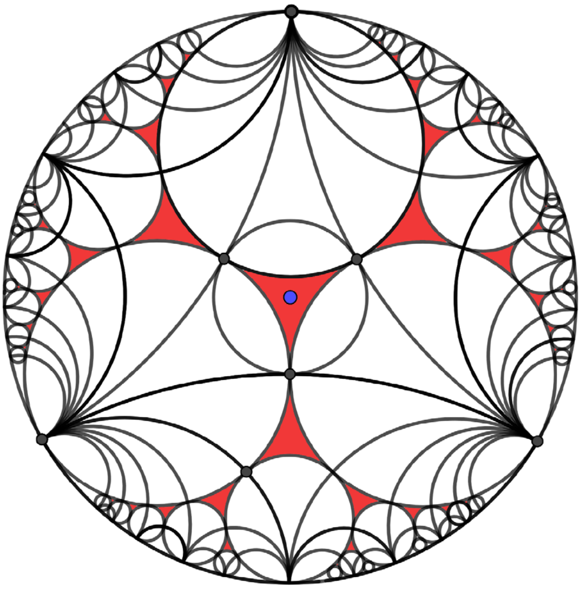

FIGURE 3 The signature $[\infty, \infty, \infty]$ case

each triangle gD onto ge. Hence T is just the finite part of the 1-skeleton of Δ. It contains vertices of degrees q_1 and vertices of degree q_2.

If Γ has signature $[q, \infty, \infty]$ only one vertex v of D is in $\mathbb{H}^2$ and its opposite edge e is a line. We write h for the hyperbolic segment connecting v with the nearest point t on e. Then centered at the endpoints of each ge we have two horodisks gB_j, gB_k which touch each other in the point gt. The prolongated retraction sends the whole of gD to the segment gh which is one half of an edge of T — the second half is its reflection over the axis ge. Hence, as above, the vertex set of T is the set of vertices of Δ in $\mathbb{H}^2$, but now the edges of T are the geodesic segments connecting vertices with their images under reflection over the opposite sides of their tiles.

If Γ is of type $[\infty, \infty, \infty]$ then $T = \Delta^*_{\text{fin}}$ and all vertices are of degree 3.

2.3 | Tessellated sectors and rooted subtrees of the spine

By a (open or closed) *sector* $S \subset \mathbb{H}^2$ we mean a subset bounded by two rays emanating from $v \in \mathbb{H}^2$. The two rays are the *legs* and their endpoints the *feet* of S. Occasionally it is convenient to include *ideal sectors* which have line-legs and their tip and feet in $\partial\mathbb{H}^2$. S is a *tessellated sector* (or a *sector of* Δ) if its tip is a vertex and its legs lie in the 1-skeleton of Δ and hence are tessellated by edges of Δ.

We call a tessellated sector S *small* if its legs are single edges of Δ and no proper subsector with the same tip has the same property. One observes that the star of a small sector consists of two tiles if the triangle D has exactly one vertex at infinity, and of a single tile in all other cases. In any case two small sectors are Γ-translates of each other if and only if their tips are in the same Γ-orbit.

Closely related to sectors of Δ are the rooted subtrees of the spine T. If $v \in \text{ver}R$ is a vertex of a subtree $R \subset T$ we write $\deg_T(v)$ for the degree of v as a vertex in T. If $\deg_R(v) = 1$ then v is a *leaf* of R, and if $\deg_R(v) = \deg_T(v)$ then v is an *inner vertex* of R. The subforest spanned by all inner vertices of R is the *inner part* of R, denoted by $\mathring{R}$, and the set of vertices of R which are not inner is the *boundary* of R in T, denoted by $\partial R \subset \mathrm{ver}R$. A *rooted subtree* $R \subset T$ is a subtree whose boundary in T consists of a single vertex r, the *root* of R; and if the root is a leaf of R we say that R is a *leaf-rooted* subtree. The inner part of a leaf-rooted subtree is a rooted tree and best described as a *half-tree* of T, that is, one of the two connected components obtained by removing the interior of an edge of T.

Lemma 2.1. *Under the assumption that Γ is a hyperbolic triangle group with signature $[q_1, q_2, \infty]$ we have a 1–1-correspondence between the small tessellated sectors S of Δ and the leaf-rooted subtrees of the spine T. This correspondence associates to S the maximal subtree of $S \cap T$ and to R the minimal subcomplex of Δ containing the convex closure $\overline{R}$ of R.*

Proof. Elementary and left to the reader. □

2.4 | The limit sets of Γ-polyhedral pieces

Let $P \subset \mathbb{H}^2$ be an arbitrary Γ-polyhedral subset. We consider the boundary of P in the *completed* hyperbolic plane $\mathbb{H}^2 \cup \partial\mathbb{H}^2$, and denote it by $\partial P =: \partial_{\mathrm{fin}} P \cup \partial_\infty P$, where $\partial_{\mathrm{fin}} P$ consists of finitely many edge paths, and $\partial_\infty P$ is the *limit-set of* P and consists of finitely many segments of $\partial\mathbb{H}^2$ – the connected components of $\partial_\infty P$ with respect to the S^1-topology of the disk model.

We need a slightly stronger connectivity concept: A segment $[x, y] \subset \partial_\infty P$ is *strongly connected*, if none of its inner points is a limit point of the complement $P^{\mathrm{c}} := \mathbb{H}^2 - P$; and the maximal strongly connected segments are the *strongly connected components* of $\partial_\infty P$. We claim that each connected component of $\partial_\infty P$ is ‘tessellated’ by (that is, the union with pairwise disjoint interiors of) its finite set of strongly connected components. Indeed, the only reason why a connected component might not be strongly connected is the possibility that it might contain a tip of the complement $P^{\mathrm{c}} := \mathbb{H}^2 - P$. As we know that Γ-polyhedrality of P implies that the closure of P^{c} is also Γ-polyhedral (see Section 2.1), P^{c} has only finitely many tips. This proves the claim; and at the same time also the following.

Observation. If P is convex Γ-polyhedral then $\partial_\infty P$ is not necessarily connected, but all its connected components are strongly connected.

Lemma 2.2. *Let P be a convex Γ-polyhedral set, $C \subset \partial_\infty P$ a connected component of its limit set, and assume that C has two endpoints $x, y \in C$, but at neither of them ∂P continues with a ray in Δ which is part of a hyperbolic axis. Then there is a canonical finite gallery $G[C] \subset P$, consisting of small sectors and single tiles, which tessellates a neighborhood of C in P.*

Proof. By assumption ∂P continues at x and y with a ray-edge or a line-edge e_x, e_y of Δ. As T is a tree, we have a canonical shortest path in Δ^1 starting in x, passing through a unique reduced edge path $\omega \subset T$ of length $m \geqslant 0$ from $e_x \cap T$ to $e_y \cap T$, and ending at y. As $T \cap C = \emptyset$ the simple closed path $C \cup \omega$ is the boundary of a topological disk B.

For simplicity we first deal with the case when T is contained in the 1-skeleton of Δ, that is, the triangle D has a unique vertex v_∞ in $\partial\mathbb{H}^2$. In that case $B \cap \mathbb{H}^2$ is a subcomplex of Δ, and we claim that this is the gallery we are looking for.

Each of the m finite edges of ω is now the edge of a uniquely defined tile with its opposite vertex in C, and we can describe the union of these tiles as a finite set of pairwise disjoint *fans* F_z (= finite gallery of tiles with the common vertex z), where z runs through a finite subset of $C \cap \Gamma v_\infty$. As C is strongly connected each F_z is contained in P. The closure of each connected component of the complement $B - \bigcup_z F_z$ is a sector of Δ with tip in ω and both legs ray-edges. Such sectors are finite unions of small sectors with the same tip. This shows that $G[C] := B \cap \mathbb{H}^2$ is the gallery along C as asserted.

In the signature $[q_1, \infty, \infty]$ case the vertices of T are in Δ but not the edges. Nevertheless, the argument follows the same line: Instead of considering tiles with an edge $e \in \omega$ we now consider the quadrilaterals $\square$ consisting of tile pairs sharing a line-edge of Δ, with the edge $e \in T$ on the short diagonal and a vertex of $\square$ in C. The signature $[\infty, \infty, \infty]$ case is even simpler. Here ω connects the centers of the tiles with vertices in the endpoints of C, and we argue by considering the gallery of tiles covering ω. $\square$

We will now consider arbitrary Γ-polyhedral sets, that is, finite unions of convex Γ-polyhedral pieces, $U := \bigcup_i P_i$, and we can assume that the convex polyhedral sets P_i have pairwise disjoint interiors. We find it convenient to express this by saying U is *tessellated* by the *pieces* P_i, or that the family $\mathcal{P} := (P_i)_i$ is a *tessellation* of U. But we will avoid calling these pieces 'tiles' – they are infinite unions of the original tiles.

By a *refinement* of $\mathcal{P}$ we mean a tessellation $\mathcal{P}'$ of U with the property that each $P' \in \mathcal{P}'$ is contained in some $P \in \mathcal{P}$.

Theorem 2.3. *Each tessellation $\mathcal{P}$ of $\mathbb{H}^2$ by a finite set of convex Γ-polyhedral pieces admits a refinement of the following kind: There is a tessellation $\mathcal{P}'$ of $\mathbb{H}^2$ by finitely many single tiles and small sectors, which turns into a refinement of $\mathcal{P}$ when we take all pieces $P' \in \mathcal{P}'$ which are small sectors whose middle ray m is a hyperbolic axis, cut them along m in two, and replace P' by the two fragments.*

Proof. The boundary $\partial\mathbb{H}^2$ is tessellated by the connected components of the limit set $\partial_\infty P$, with P running through $\mathcal{P}$. The points on $\partial\mathbb{H}^2$ which are endpoints of these connected components are finite in numbers, and we consider the subset X consisting of those points x which are corner points of some $P \in \mathcal{P}$, in a position where a section of $\partial_\infty P$ turns into a section of $\partial_{\text{fin}} P$ which lies on a hyperbolic axis $h_x \subset \Delta^1$. In order to apply Lemma 2.2 we have to get around the points $x \in X$. Note that X is empty unless the signature of Γ has two finite entries.

Each vertex on h_x is the tip of a unique small sector with its middle ray on h_x and with limit point x. These small sectors are nested and their intersection is the singleton set $\{x\}$. Since the legs of each sector S isolate x from the complement of S we find that infinite edge paths of Δ^1 can only reach x through a ray on h_x. This shows: 1) x is a limit point of only the two convex polyhedral corner pieces at x, $P_x^+, P_x^- \in \mathcal{P}$; and 2) if the tip of such a small sector S is sufficiently close to x then $S \subset U = P_x^+ \cup P_x^-$.

Thus, we have a canonical choice by taking the one small sector S_x with its middle line on the axis h_x, its tip in a vertex of e_x, and the property that S_x is maximal with respect $S_x \cap (\mathbb{H}^2 - U) = \emptyset$ Now we excise the interior of S_x from the corner pieces P_x^+ and P_x^-, noting that cutting off the corner x along a ray-edge preserves both convexity and Γ-polyhedrality. Therefore, we can replace in $\mathcal{P}$ the pieces P_x^+, P_x^+ by the remaining fractions. The result is a tessellation of $\mathbb{H}^2 - S_x$. It avoids the corner x as its boundary turns into the legs of S_x before it meets the remaining finite fragment of the ray e_x.

When we have removed all corners $x \in X$ in this way we have transformed the tessellation $\mathcal{P}$ into a tessellation $\mathcal{P}^*$ of $\mathbb{H}^2 - \bigcup_{x\in X} S_x$. Then we can apply Lemma 2.2 to all convex Γ-polyhedral pieces $P \in \mathcal{P}^*$ to find, along the connected components of their limit sets $\partial_\infty P$, galleries consisting of single tiles and small sectors. Together with the small sectors $S_x, x \in X$ they provide a tessellation $\mathcal{P}'$ of a neighborhood of $\mathbb{H}^2$. As all our tiles have a vertex at infinity, such a tessellation must cover all tiles of Δ; hence $\mathcal{P}'$ is the tessellation of $\mathbb{H}^2$ claimed to exist in the assertion of the theorem. □

Remark (The scaly spider). Retrospectively, the type of tessellation that we can always achieve is easy to describe: We pick a finite subtree T_0 of the spine T (the spider's body). Attached to the body are (1) all rays which emanate in the 1-skeleton of Δ out of T_0 and are not a single ray-edge (the spider's legs); and (2) all tiles of Δ that contain a one-dimensional part of an edge of T (the spider's scales). Then the complement of the scaled body of the spider is the disjoint union of small sectors with the spider's legs on their middle line.

Theorem 2.3 shows that modulo finite permutations each piecewise isometric pemutation of the set Ω of all tile centers is given by isometries restricted to small tessellated sectors and halfs of small sectors (split along a hyperbolic axis), which form a finite gallery along the $\partial\mathbb{H}^2$. Helpful is the simple fact that describing the restriction of an isometry φ to a small sector S requires only the image of the edge emanating at the root together with the information whether φ preserves the orientation. Restricted to the spine, this corresponds to displacing a leaf-rooted subtree R to another position (at a vertex with the same degree in T). To describe the half-sector moves in the spine is more subtle: They are not quasi-autmorphisms of *T;* rather one has to split the rooted subtree along its tree trunk (which lies on the hyperbolic axis) in two, and accept a copy of the trunk in both fragments to keep them connected.

2.5 | Piecewise planar tree isometric permutations

For simplicity we will from now on restrict the focus to the case when Δ^1 contains no hyperbolic axis, and when the acting group is the orientation preserving subgroup of $\Gamma^* \leqslant \Gamma$. Then Theorem 2.3 asserts that each tessellation $\mathcal{P}$ of $\mathbb{H}^2$ by a finite set of convex Γ-polyhedral pieces admits a refinement consisting of single tiles and small sectors.

Thus, a piecewise Γ^*-isometric permutation $g \in G_{\Gamma^*}(\Omega)$ is now given by the restriction of isometries to finitely many tiles and sectors. As isometries respect the spine T, Lemma 2.1 tells us how to translate this information to the corresponding tessellation of the spine T by a finite set of edges and a subforest F of finitely many leaf-rooted subtrees R_i (the maximal subtrees of $S \cap T$ as S runs through the small sectors of the gallery along $\partial\mathbb{H}^2$). As T and R_i intersect the boundary of small sectors only in their tips the subtrees R_i tessellate T outside a finite subgraph. More precisely: The restriction of g embeds each R_i by a planar-tree isomorphism onto the trees of a subforest F' such that $T - F$ and $T - F'$ have the same number of vertices.

In the present framework it is natural to say that g induces on the spine T a *piecewise planar-tree isometric* (ppti-isometric) vertex permutation. Thus we found a homomorphism of $G_{\Gamma^*}(\Omega)$ into the group of all a piecewise planar-tree isometric vertex permutation of T which we term ppti(T).

Conversely: Using Lemma 2.1 one observes that each tessellation of T by finitely many leaf-rooted subtrees and single edges can be refined to a tessellation by finitely many single edges and leaf-rooted R_i whose convex closure are (or at least contained in) small sectors S_i; and each planar-tree isometry on R_i can be represented by a metric-tree isometry on R_i and then extended to an isometry on S_i. Hence $G_{\Gamma^*}(\Omega) \cong \mathrm{ppti}(T)$.

By definition piecewise planar-tree isometric vertex permutations respect not only the pairs of endpoints of almost all edges of T but also the cyclic ordering of the stars at almost all vertices of T. Hence, g can also be referred to as a quasi-(planar-tree) automorphism of the spine T; see [11], [28, 29], [31].

Hence we can also summarize:

Corollary 2.4. *Let $\Gamma^* \leqslant \Gamma$ be the orientation preserving subgroup of a triangle group with signature $[q_1, q_2, \infty]$. If at least one of q_1 or q_2 is odd or infinite, then the group of all piecewise Γ^*-isometric permutations of the tile centers, $G_{\Gamma^*}(\Omega)$ coincides with the quasi-(planar-tree) automorphism group ppti(ver(T)) of the spine T of the tessellation Δ.*

2.6 | Connection with Thompson's groups

Interpreting the statement of Corollary 2.4 for the triangle group Γ with signature $[\infty, \infty, \infty]$ yields the connection with Thompson's groups: in this case the spine T is the infinite binary (planar) tree T_2 that has always been around when Thompson's groups (and their generalizations) have been investigated in terms of generators or as groups of piecewise linear homeomorphisms of the Cantor set on the real line or on $\partial\mathbb{H}^2$.

Thus, we infer that for triangle groups with signature $[q_1, q_2, \infty]$, and at least one of q_1, q_2 odd or infinite, the quotients $G_{\Gamma^*}(\Omega)/\mathrm{sym}(\Omega)$ are straightforward generalizations of Thompson's group V. Hence roofs in the literature (for example, [37]), showing that these generalized Thompson groups are of type F_∞, also apply in our situation.

However, to extend the results on $G_{\Gamma^*}(\Omega)/\mathrm{sym}(\Omega)$ for more general Γ, or on the groups $G_{\Gamma^*}(\Omega)$ themselves, it seems more rewarding to skip the detour to the spine T but rather try to modify tools that were successful for Thompson's groups: instead of tree-parameters and the partially ordered set of rooted subtrees of T one might be able to use hyperbolic plane parameters and the partially ordered set of small sectors (or halves of small sectors) of the tessellated hyperbolic plane to find some understanding of $G_\Gamma(\Omega)$ when the 1-skeleton of Δ contains a hyperbolic axis — with a bit of luck even in the case when Γ is a cocompact triangle group.

Instead of trying to do this one-handedly it would be interesting to know how much of that can already be covered (or promoted) by the cloning systems of [37] or the abstraction of [19].

CHAPTER 3. THE EUCLIDEAN CASE I: THE STRUCTURE OF pei(S)

3 | ORTHOHEDRAL SETS

3.1 | Integral orthants in $\mathbb{Z}^N$

In the standard N-dimensional Euclidean integral lattice $\mathbb{Z}^N$ we consider affine-orthogonal transformations

$$\tau_{a,A} : \mathbb{Z}^N \to \mathbb{Z}^N$$

$$\tau_{a,A}(x) = a + Ax,$$

where $A \in O(N, \mathbb{Z})$ is an integral orthogonal matrix and $a \in \mathbb{Z}^N$. Inside $\mathbb{Z}^N$ we have the *standard orthant of rank* N, $\mathbb{N}^N \subseteq \mathbb{Z}^N$, and all images of its k-dimensional faces, $0 \leqslant k \leqslant N$, under affine-orthogonal transformations. More precisely: the subsets $L = \tau_{a,A}\langle Y\rangle \subseteq \mathbb{Z}^N$, where $\langle Y\rangle$ stands for the monoid generated by the k-element set Y of canonical basis vectors. We call L an *integral orthant* (of rank-k, and based at $a \in L$) of $\mathbb{Z}^N$ or just a *rank-k* orthant.

We write Ω^k for the set of all rank-k orthants of $\mathbb{Z}^N$ and Ω^* for the union $\bigcup_k \Omega^k$. Ω^* is partially ordered by inclusion, with $\Omega^0 = \mathbb{Z}^N$. The subset of all orthants based at the origin 0 will be denoted by $\Omega_0^* \subseteq \Omega^*$; it retracts the order preserving projection $\tau : \Omega^* \to \Omega_0^*$ which associates to each orthant $L \in \Omega^*$ based at $a \in \mathbb{Z}^N$ its unique parallel translate $\tau(L) = -a + L \in \Omega_0^*$. $\tau(L)$ is characterized by its canonical basis $Y = \{y \in \pm X \mid a + \mathbb{N}y \subseteq L\}$ which indicates the directions of L; hence we call $\tau(L)$ the indicator of L. Note that Y is given by the function $f\colon X \to \{0, 1, -1\}$ with $f(x) = \epsilon \in \{1, -1\}$ if $\epsilon x \in Y$, and $f(x) = 0$ if $\{x, -x\} \cap Y = \emptyset$; hence $|\Omega_0^*| = 3^N$.

We call a subset $S \subseteq \mathbb{Z}^N$ *orthohedral* if it is the union of a finite set of orthants — without losing generality we can assume that the union is disjoint. The *rank* of S, denoted by $\mathrm{rk}\, S$, is the maximum rank of an orthant contained in S. If S is isometric to $\mathbb{N}^k \times \{1, 2, \dots, h\}$, we call it a *stack of orthants of height h*. The terminology agrees with the *height* $h(S)$ of an arbitrary orthohedral set $S \subseteq \mathbb{Z}^N$, defined as the number of orthants of maximal rank, $\mathrm{rk}\, S$, which participate in a pairwise disjoint finite decomposition of $S = L_1 \cup L_2 \cup \dots \cup L_m$ — see Section 1.1 in the introduction.

Lemma 3.1. *Orthohedrality of subsets $S \subseteq \mathbb{Z}^N$ is closed under the set-theoretic operations of taking intersections, unions, and complements.*

Proof. The main observation here is that the intersection of a finite set of half-spaces of $\mathbb{Z}^N$ (each defined by an upper or lower bound on one coordinate) is a disjoint union of finitely many orthants. We prove this by induction. If two of these half-spaces, H, H', are bounded by parallel hyperplanes then either one of them is redundant or their intersection $H \cap H'$ is a (possibly empty) finite union of lower dimensional subspaces. In both cases we are reduced to the intersection of fewer half-spaces. If no pair of the half spaces have parallel boundary there are only $k \leqslant N$ of them, and their intersection is isometric to $\mathbb{Z}^{N-k} \oplus \mathbb{N}^k$ and hence is a finite union of 2^{N-k} rank-N orthants. The assertion of the lemma follows now by set-theoretic tautologies. □

Remark. As a consequence we note that the orthohedral subsets of $\mathbb{Z}^N$ are precisely the $\mathbb{Z}^N$-polyhedral subsets of the lattice $\mathbb{Z}^N$ as defined in Section 3.1.

We write $\Omega^k(S) = \{L \in \Omega^k | L \subseteq S\}$ for the set of all rank-k orthants of S, $\Omega^*(S)$ for the disjoint union over k, and $\Omega_0^*(S)$ for the set of all orthants of S based at the origin 0. We consider the restriction of the indicator map $\tau : \Omega^*(S) \to \Omega_0^*$. We write $S_\tau \subseteq \mathbb{Z}^N$ for the union of all orthants in $\tau(\Omega^*(S))$ and call this the indicator image of S. Note that $\tau(\Omega^*(S)) = \Omega_0^*(S_\tau)$, and we can view the indicator map as a rank preserving surjection $\tau : \Omega^*(S) \twoheadrightarrow \Omega_0^*(S_\tau)$.

3.2 | Germs of orthants

Two orthants L, L' in Ω^* are said to be *commensurable* if $\mathrm{rk}\, L = \mathrm{rk}(L \cap L') = \mathrm{rk}\, L'$. We write $\gamma(L)$ for the commensurability class of L and call it the *germ* of L. The union of all members of $\gamma(L)$

is a coset of a subgroup of $\mathbb{Z}^N$; we denote it by $\langle L\rangle \subseteq \mathbb{Z}^N$ and call it the tangent coset of S at γ. The germs inherit from their representing orthants L the *rank*, relations like *parallelism and orthogonality*, and also a *partial ordering* defined as follows: given two germs γ, γ' we put $\gamma \leqslant \gamma'$ if they can be represented by orthants $L, L' \in \Omega^*$ with $L \subseteq L'$. Note that if L and L' are arbitrary orthants representing γ and γ', respectively, then $\gamma \leqslant \gamma'$ if and only if (1) L' contains an orthant parallel to L and (2) $L \subseteq \langle L'\rangle$.

We write $\Gamma^*(S) = \bigcup_k \Gamma^k(S)$ for the set of all germs of orthants in S and $\Gamma^*_0(S)$ for the set of all germs represented by an orthant of S based at the origin 0. $\Gamma^*(\mathbb{Z}^N)$ and $\Gamma^*_0(\mathbb{Z}^N)$ are abbreviated as Γ^* and Γ^*_0, respectively. As Γ^*_0 and Ω^*_0 are canonically bijective, we will identify them when this is convenient. Note that $\Gamma^*(S)$ is a convex subset of Γ^* in the sense that if $\gamma \in \Gamma^*(S)$ then $\{\gamma' \in \Gamma^* \mid \gamma' \leqslant \gamma\} \subseteq \Gamma^*(S)$. We can interpret the indicator map as an order and rank preserving surjection $\tau : \Gamma^*(S) \to \Gamma^*_0$ with $\tau(\Gamma^*(S)) = \Gamma^*_0(S_\tau)$. By $\max \Gamma^*(S)$ we mean the set of all maximal germs of S.

Exercise. Observe that $\tau(\max \Gamma^*(S)) \supseteq \max \Gamma^*_0(S_\tau)$, but this is not, in general, an equality.

Lemma 3.2. $\max \Gamma^*(S)$ *is finite for each orthohedral set S. The set of all germs of rank* $n = \operatorname{rk} S$ *is a subset of* $\max Gamma^*(S)$*, whose cardinality coincides with the height* $h(S)$*. Hence* $h(S)$ *is independent of the particular decomposition of S.*

Proof. Let $S = \bigcup_j L_j$ be an arbitrary decomposition of S as a finite pairwise disjoint union of orthants L_j. Each orthant $L \subseteq$ is the disjoint union of the orthants $M_j = L \cap L_j$, and exactly one of them is commensurable to L. Hence $\gamma(L) = \gamma(M_j) \subseteq \gamma(L_j)$. This shows that each germ $\gamma \in \Gamma^*(S)$ is smaller than or equal to one of the $\gamma(L_j)$. In particular, $\max \Gamma^*(S)$ is contained $\{\gamma(L_j) \mid j\}$ and hence finite. The orthants L_j of rank n form a complete set of representatives of all orthants of rank n. □

Remark. We leave it to the reader to deduce that $h(S \cup S') = h(S) + h(S')$, if S and S' are orthohedral sets with $\operatorname{rk}(S) = \operatorname{rk}(S') > \operatorname{rk}(S \cap S')$.

3.3 | Piecewise isometric maps

Let $S \subseteq \mathbb{Z}^N$ be an orthohedral subset. We call a map $f : S \to \mathbb{Z}^N$ *piecewise-Euclidean-isometric* (abbreviated as *pei-map*), if S is covered by a finite set Λ of pairwise disjoint orthants, with the property that the restriction of f to each orthant $L \in \Lambda$ is an isometric embedding $f_{|L} : L \to S$. The *support* of $f \in G(S)$, $\operatorname{supp}(f) = \{a \in S \mid ag \neq a\}$, is orthohedral, and we refer to its rank also as *the rank of* f, denoted by $\operatorname{rk}(f)$.

Analogously, we call f a *piecewise Euclidean-translation map* (abbreviated as *pet-map*), if S is a finite disjoint union of orthants with the property that the restriction of f to each of them is a parallel shift.

If a bijection $f : S \to S'$ is a pei-map (respectively, a pet-map), so is f^{-1} and we say that S and S' are *pei-isomorphic* (respectively, a *pet-isomorphic*).

By the argument used in the proof of Lemma 3.2 one shows that if f is a pei-map, then each orthant $L \subseteq S$ contains a commensurable suborthant on which f restricts to an isometric embedding. In fact, we leave it to the reader to observe the following.

Lemma 3.3. *Let $f : S \to \mathbb{Z}^N$ be an injective map on an orthohedral set $S \subseteq \mathbb{Z}^N$. Then f is a pei(respectively, pet)-injection if and only if every orthant L of S contains a commensurable suborthant $L' \subseteq L$ on which f is given by an isometry (respectively, a translation) onto $f(L') \subseteq \mathbb{Z}^N$.*

It follows that every injective pei-map $f : S \to \mathbb{Z}^N$ induces a rank preserving injection $f_* : \Gamma^*(S) \to \Gamma^*(f(S))$. f_* does not preserve the ordering — not even if f is a pet-map. But since it is rank-preserving, it does induce a bijection between the germs of maximal rank of $\Gamma^*(S)$ and $\Gamma^*(f(S))$, when $h(f(S)) = h(S)$. The following observations can be left as an exercise:

Lemma 3.4. *If $f : S \to \mathbb{Z}^N$ is a pet-map, then $f_*(\gamma)$ is parallel to γ for each $\gamma \in \Gamma^*(S)$. Hence $\tau(f_*(\gamma)) = \tau(\gamma)$, and $S_\tau = f(S)_\tau$. In other words we have the commutative diagram*

$$\begin{tikzcd} \Gamma^*(S)_\tau \arrow[r, "f_*"] \arrow[d] & \Gamma^*(f(S))_\tau \arrow[d] \\ \Gamma_0^*(S) \arrow[r, equal] & \Gamma_0^*(f(S)_\tau). \end{tikzcd}$$

3.4 | Normal forms

Consider the disjoint union of orthants

$$S = L_1 \cup L_2 \cup \ldots \cup L_m$$

in $\mathbb{Z}^N$. Assuming that $\operatorname{rk} S < N$ we have enough space to parallel translate each L_i to an orthant L_i' in such a way that the L_i' are still pairwise disjoint, but that each (oriented) parallelism class of the orthants L_i' is assembled to a stack. This describes a pet-bijection $S \to S' = \bigcup_j S_j$, where the S_j stand for pairwise disjoint and non-parallel stacks of orthants. We can go one step further by observing that when the maximal orthants of a stack S_i are parallel to suborthants of the stack S_j, then there is a pet-bijection $S_i \cup S_j \to S_j$ which feeds S_i into S_j. Hence we can delete all stacks S_i of orthants that are parallel to a suborthant of some other S_j and find

Proposition 3.5 (pet-Normal form). *Each orthohedral set S is pet-isomorphic to a disjoint union of stacks of orthants $S' = \bigcup_j S_j$, with the property that no maximal orthant of any S_j is parallel to a suborthant in some S_k, if $k \neq j$.*

Corollary 3.6 (pei-Normal form). *Each orthohedral set S is pei-isomorphic to a stack of orthants.*

As rank $\operatorname{rk} S$ and height $h(S)$ are pei-invariant; hence they can be read off from the pei-normal form; and the pair $(\operatorname{rk} S, h(S))$ characterizes S up to pei-isomorphism. For the corresponding pet-result we consider the height function

$$h_S : \Gamma_0^* \longrightarrow \mathbb{N} \cup \{0\}, \tag{1}$$

which assigns to each 0-based orthant $L \in \Omega_0^* = \Gamma_0^*$ the number of maximal germs $\gamma \in \max \Gamma^*(S)$ with $\tau(\gamma) = L$, which is finite by Lemma 3.2. The support $\operatorname{supp}(h_S) \subseteq \Gamma_0^*$ is the set of all 0-based orthants L with $h_S(L) > 0$. From the Exercise in Section 3.2 we infer that $\max \Gamma_0^*(S_\tau) \subseteq \mathrm{supp}(h_S)$, and that this is not, in general an equality. One observes easily that the equality

$$\tau(\max \Gamma^*(S)) = \max \Gamma_0^*(S_\tau) \text{ or equivalently: } \max \Gamma_0^*(S_\tau) = \mathrm{supp}(h_S) \tag{2}$$

is a necessary condition for S to be in pet-normal form. Thus we call S *quasi-normal* if the equation (2) holds. Of course, a quasi-normal orthohedral set is not necessarily in pet-normal form. But as quasi-normality implies that τ restricts to a surjection $\tau : \max \Gamma^*(S) \twoheadrightarrow \max \Gamma_0^*(S_\tau)$, $\max \Gamma^*(S)$ is the pairwise disjoint union of the fibers $f^{-1}(\gamma)$, which consist of $h_S(\gamma)$ germs parallel to γ. This can be viewed as a weak germ-version of the pet-normal form.

Lemma 3.7. *If $f : S \to \mathbb{Z}^N$ is a pet-injection of a quasi-normal orthohedral set $S \subseteq \mathbb{Z}^N$, then $f_*(\max \Gamma^*(S)) \subseteq \max \Gamma^*(f(S))$.*

Proof. By Lemma 3.3 f induces a rank preserving bijection

$$f_* : \Gamma^*(S) \to \Gamma^*(f(S)),$$

and by Lemma 3.4 $f(S)_\tau = S_\tau$. Let $\gamma \in \max \Gamma^*(S)$. Then we know that $\tau(\gamma)$ is maximal in $\Gamma_0^*(S_\tau)$. Since f is a pet map, we also know that $\tau(f_*(\gamma)) = \tau(\gamma)$; hence $\tau(f_*(\gamma))$ is maximal in $\Gamma_0^*(S_\tau) = \Gamma_0^*(f(S)_\tau)$. We claim that $f_*(\gamma)$ is maximal in $\Gamma^*(f(S)_\tau)$. Indeed, if $f_*(\gamma)$ is not in $\max \Gamma^*(f(S)_\tau)$, then $\tau(f_*(\gamma))$ cannot be maximal in $\Gamma_0^*(f(S)_\tau)$. This shows that $f_*(\max \Gamma^*(S)) \subseteq \max \Gamma^*(f(S))$, as asserted. □

Corollary 3.8. *If $f : S \to S'$ is a pet-isomorphism between quasi-normal orthohedral sets, then $f_*(\max \Gamma^*(S)) = \max \Gamma^*(S')$ and $h_S = h_{S'}$.*

This shows, in particular, that the stack heights in a pet-normal form are uniquely determined and characterize S up to pet-isomorphism.

4 | PERMUTATION GROUPS SUPPORTED ON ORTHOHEDRAL SETS

4.1 | pei- and pet-Permutation groups

Let $G = \mathrm{pei}(\mathbb{Z}^N)$ denote the group of all pei-permutations of $\mathbb{Z}^N$. From now on it will be convenient to follow the permutation-group tradition to have the permutation group G act on its set $\Omega = \mathbb{Z}^N$ from the *right* and interpret the product gf of elements $g, f \in G$ as g *followed by* f.

The *support* of an element $g \in G$ is defined as the union of all orthants on which g restricts to a non-trivial isometry:

$$\mathrm{supp}(g) := \bigcup \{L \in \Omega^\star \mid g_{|L} \text{ is an isometric embedding} \neq \mathrm{id}_L\}.$$

To say that a given subset $S \subseteq \mathbb{Z}^N$ *supports* g merely means that that $\mathrm{supp}(g) \subseteq S$.

Exercise. Prove that $\mathrm{supp}(g)$ is the minimal orthohedreal subset containing the set $\{a \in \mathbb{Z}^N \mid ag \neq a\}$.

As we know that the support of an element $g \in G$ is orthohedral it makes sense to put $\mathrm{rk}(g) := \mathrm{rk}(\mathrm{supp}(g))$, and call this the *rank* of the element g.

The *support* of a subgroup $H \leqslant G$ is the union of the supports of its elements.

The product of a finite number of elements $g_i \in G$ is *disjoint* if $\mathrm{supp}(g_i) \cap \mathrm{supp}(g_j) = \emptyset$ for all $i \neq j$.

If $S \subseteq \mathbb{Z}^N$ is orthohedral we write $G(S) := \{g \in G \mid \mathrm{supp}(g) \subseteq S\}$ for the subgroup of G supported on S. As we know, by Lemma 3.1, that the complement of S is also orthohedral each pei-bijection of S extends to an element of G; hence the subgroup $G(S) \leqslant G$ is also the pei-automorphism group of S. We write also $\mathrm{pei}(S)$ for $G(S)$ when this is convenient.

As an immediate consequence of Corollary 3.6 we have

Corollary 4.1. *If $S \subseteq \mathbb{Z}^N$ is an orthohedral subset, then* $\mathrm{pei}(S)$ *is isomorphic to* $\mathrm{pei}(S')$*, where S' is a stack of orthants of rank* $\mathrm{rk}\, S$ *and height $h(S)$.*

The set of all pet-permutations on the orthohedral set S is the pet-subgroup $\mathrm{pet}(S) \leqslant G(S)$. As an immediate consequence of Proposition 3.5 and Corollary 3.8 we find

Corollary 4.2. *If $S \subseteq \mathbb{Z}^N$ is an orthohedral subset and $S' = \bigcup_j S_j$ its pet-normal form, then* $\mathrm{pet}(S)$ *is isomorphic to* $\mathrm{pet}(S')$.

Definition 4.3 (The rank groups G_k). As conjugation in $G = \mathrm{pei}(\mathbb{Z}^{\mathbb{N}})$ preserves the rank of the elements, putting $G_{-1} := 1$, and for $k \geqslant 0$

$$G_k := \{g \in G \mid \mathrm{rk}(g) \leqslant k\},$$

yields the normal series $1 = G_{-1} \leqslant G_0 \leqslant \cdots \leqslant G_k \leqslant \cdots \leqslant G_N = G$ which plays the key role to understanding the structure of G. For each orthohedral subset $S \subseteq \mathbb{Z}^N$, $G_k(S) := G_k \cap G(S)$ yields the corresponding normal sequence for $G(S)$.

Note that by the pei-normal form we have

$$\mathrm{pei}(S) = G_{\mathrm{rk}\, S}(S) \cong G_{\mathrm{rk}\, S}\left(\bigcup_{1 \leqslant i \leqslant h(S)} \mathbb{N}^{\mathrm{rk}\, S}\right),$$

for every orthohedral set $S \subseteq \mathbb{Z}^N$.

In this section we are aiming for insight into the group theoretic structure of $\mathrm{pei}(S)$, are now in a position to outline its main results in a nutshell:

Theorem 4.4. *If $G =$ $\mathrm{pei}(S)$, with S an orthohedral set of rank* $\mathrm{rk}\, S = n$ *then the following holds:*

(i) G_k/G_{k-1} is an extension of a free-Abelian normal subgroup (of infinite rank when $1 \leqslant k <$ $\mathrm{rk}(S)$) with a locally finite factor group. In particular G is elementary amenable.

(ii) The rank-groups G_k are characteristic in G. Each normal subgroup $N \leqslant G$ is contained in some G_k and intersects G_{k-1} in a subgroup of index at most 2. Consequently every Abelian-by-locally-finite section of G is a section of $G_k/\mathrm{alt}\, G_{k-1}$ for some $k \leqslant n$, and $n+1$ is the minimum length of normal series of G with Abelian-by-locally-finite factors.

(iii) G satisfies the maximal condition for normal subgroups.

For the proofs see Theorems 4.8 and 4.18, and Corollary 4.19.

4.2 | The action of G on the germs

By Lemma 3.3 we know that given a pei-permutation $g \in G$, each germ $\gamma \in \Gamma^*$ represented by an orthant L contains a suborthant L' commensurable with L on which g restricts to an isometric embedding of L' into Lg. Hence putting $\gamma g := \gamma(L'g) \in \Gamma^*$ well defines a rank preserving action of G on Γ^*. For each orthohedral subset $S \subseteq \mathbb{Z}^N$ this action restricts to an action of $G(S)$ on $\Gamma^*(S)$.

Lemma 4.5. *G_k acts on the set Γ^k of rank-k germs by finite permutations; and for each orthohedral set S, the restricted action of $G_k(S)$ on $\Gamma^k(S)$ is highly transitive in the sense that each bijection $f : F \to F'$ between finite subsets of $\Gamma^k(S)$ is induced by the action of some $g \in G_k(S)$.*

Proof. An element $g \in G_k$ can only dislocate the rank-k germs in

$$\Gamma^k(\mathrm{supp}(g)) \subseteq \Gamma^k,$$

and these are finite in number.

We claim that the bijection f extends to a permutation π of $F \cup F'$. To see this consider the graph $\mathfrak{G}$ with vertex set ver$\mathfrak{G} = F \cup F'$, and the oriented edge set edg$\mathfrak{G} = \{(a, f(a)) \mid a \in F\}$. Then one observes that $\mathfrak{G}$ can be completed to a permutation graph since $|F - F \cup F'| = |F' - F \cup F'|$.

Now we represent the elements of $F \cup F'$ by a set of pairwise disjoint orthants $\{L_\gamma \mid \gamma \in F \cup F'\}$, and we lift the graph $\mathfrak{G}$ as follows: we choose for each $\gamma \in F \cup F'$ an isometry $\tilde{\pi}_\gamma : L_\gamma \to L_{f(\gamma)}$, but ensure that along each simple closed path the product of the chosen isometries is the identity. The union of these isometries is an element of $G_k(S)$, and induces the map f. □

4.3 | Stabilizers of rank-k germs

Next we consider the stabilizer

$$C(\gamma) := \{g \in G_k \mid \gamma g = \gamma\}, \quad \gamma \in \Gamma^k. \tag{3}$$

We attach to γ the union $\langle\gamma\rangle \subseteq \mathbb{Z}^N$ of all orthants of $\mathbb{Z}^N$ representing γ. Thus $\langle\gamma\rangle$ is a coset of a coordinate subgroup of $\mathbb{Z}^N$ and isometric to $\mathbb{Z}^k$; we call it the *tangent coset of S at γ*. The stabilizer $C(\gamma)$ acts canonically on $\langle\gamma\rangle$: Indeed, given $g \in C(\gamma)$, we find an orthant L' representing γ with the property that g maps L' isometrically to $L'g$ which is commensurable to L, and that isometry extends canonically to an isometry of $\langle\gamma\rangle$ onto itself. This yields a homomorphism

$$\varphi_\gamma : C(\gamma) \to \mathrm{Isom}\langle\gamma\rangle. \tag{4}$$

$\langle\gamma\rangle$ carries additional $C(\gamma)$-invariant structure: As commensurable orthants are canonically linked by a unique parallel translation we can endow the canonical basis of the orthants representing γ with compatible orderings. Hence $\langle\gamma\rangle$ comes endowed with a canonical $C(\gamma)$-invariant set $X(\gamma)$ of k pairwise orthogonal coordinate directions. $C(\gamma)$ acts k-transitively on $X(\gamma)$; and the homomorphism (4) factors, modulo translations, through an epimorhism onto the symmetric permutation group on $X(\gamma)$,

$$\overline{\varphi}_\gamma : C(\gamma) \twoheadrightarrow \mathrm{sym}_k(X(\gamma)). \tag{5}$$

By an *ordered germ* we mean a germ γ together with an ordering on the canonical basis-directions of $\langle\gamma\rangle$; and we write $C^{\mathrm{ord}}(\gamma)$ for the stabilizer of the ordered germ γ. By choosing an ordering of the canonical monoid basis of $\mathbb{Z}^{\mathbb{N}}$ we can impose germ orderings simultaneously on all germs of $\mathbb{Z}^{\mathbb{N}}$; these orderings are preserved by all maps induced by inclusions and parallel translations, and in this situation we say that the germs are endowed with *compatible orderings*. It is easy to observe

Lemma 4.6.

(i) $G_{k-1} \leqslant C^{\mathrm{ord}}(\gamma) \leqslant C(\gamma)$;
(ii) $\ker(\overline{\varphi}_\gamma) = C^{\mathrm{ord}}(\gamma)$;
(iii) $\ker(\varphi_\gamma) = \{g \in G \mid g$ *fixes an orthant representing* γ *pointwise*$\}$.

Exercise. The following conditions are equivalent for an orthohedral set S:

(1) S has a germ γ with $\mathrm{coker}(\varphi_\gamma)$ non-zero:
(2) S has a germ γ with $\mathrm{coker}(\varphi_\gamma) = \mathbb{Z}$;
(3) there is a number $k \in \mathbb{N}$ with $\Gamma^k(S)$ a singleton set;
(4) S is pei-isomorphic to $\mathbb{N}^k$ with $k \geqslant 1$.

By Lemma 4.5 the stabilizers of all rank-k germs are conjugates of one another; hence their intersection $C(\Gamma^k(S)) := \bigcap_{\gamma\in\Gamma^k(S)} C(\gamma)$ is a normal subgroup of $G(S)$, and so is $C^{\mathrm{ord}}(\Gamma^k(S)) := \bigcap_{\gamma\in\Gamma^k(S)} C^{\mathrm{ord}}(\gamma)$. We claim that this yields the following refinement of the normal series based on ranks:

$$G_{k-1}(S) \leqslant C^{\mathrm{ord}}(\Gamma^k(S)) \leqslant C(\Gamma^k(S)) \leqslant G_k(S), \tag{6}$$

for all $k \leqslant rkS$.

Indeed, the first two inclusions immediate from the first part of Lemma 4.6, while the remaining inclusion is the following observation: Given $g \in G(S)$, any rank-$(k+1)$ germ γ is represented by an orthant L on which g restricts to an isometric embedding $f = g\,|_L \colon L \to S$, and f maps each rank-k face F of L to a face Fg of Lg. Assuming $g \in C(\Gamma^k(S))$ implies that each Fg is commensurable to F hence f parallel shifts each F to Fg, and these shifts can be interpreted in the tangent coset $\langle\gamma\rangle$. Since f is an isometry it follows that f can only be the identity of L, when $\mathrm{rk}(g) \leqslant k$.

4.4 | Dynamics of the action of G_k on Γ^{k-1}

If S is an orthohedral set of rank n then $\Gamma^n(S)$ is finite. $\Gamma^{n-1}(S)$ is infinite and comes with a rank-1 orthohedral structure: Each rank-n orthant $L \subset S$ contributes $n-1$ maximal parallelism classes of rank-$(n-1)$ germs represented by parallel cross-sections of L; we call these the rays of $\Gamma^{n-1}(S)$. As S is orthohedral we find that the union of $(n-1)h(S)$ such rays is cofinite in $\Gamma^{n-1}(S)$. Thus $\Gamma^{n-1}(S)$ has a one-dimensional piecewise isometric structure and one observes readily the induced action of $G(S)$ is piecewise isometric.

Let $g \in C(\gamma)$ for some $\gamma \in \Gamma^k$. Let L be a rank-k orthant representing γ, with the property that g restricted to L is isometric. Then Lg is a rank-k orthant commensurable to L; we put $fl_\gamma(g) := h(L - Lg) - h(Lg - L)$ and note that this is an integer which does not depend on the particular choice of L. Thus, $fl_\gamma : C(\gamma) \to \mathbb{Z}$ is a well-defined homomorphism for all $\gamma \in \Gamma^k$. It measures the balance of trading rank-$(k-1)$ germs toward and away from Γ, and we call it the *corank-1 germ flow of* $g \in C(\gamma)$ at $\gamma \in \Gamma^k$. As the action of $g\mid_L$ can be monitored in the tangent coset $\langle\gamma\rangle$ via the homomorphism (4) we have $fl_\gamma(g) = fl_\gamma(\varphi_\gamma(g))$.

If $fl_\gamma(g)$ is positive γ is a *sink* of g; if $fl_\gamma(g)$ is negative it is a *source* of g. Clearly, $fl_\gamma(g)$ vanishes when $\gamma \notin \Gamma^k(\mathrm{supp}(g))$. This shows that $g \in G$ has, if any, only finitely many sources and sinks in $\Gamma^{\mathrm{rk}(g)}$. Hence collecting the flow maps fl_γ as γ runs through Γ^k yields the *global flow homomorphism*

$$fl : C(\Gamma^k) \to \oplus_{\gamma\in\Gamma^k}\mathbb{Z}. \tag{7}$$

We say that the elements of its kernel are *stagnant*, call ker fl the *stagnant* subgroup of G_k, denoted by $ST_k(S)$, and note that $G_{k-1} \leqslant ST_k \leqslant G_k$. Next we claim that the total flow-sum function vanishes on $C(\Gamma^k(S))$, that is, we have for each k,

$$\sum_{\gamma\in\Gamma^k(S)} fl_\gamma(g) = 0. \tag{8}$$

Proof. We choose, for a given $g \in C(\Gamma^k(S))$, a finite set Λ of pairwise disjoint orthants representing the rank-k germs of supp(g), and with the property that g restricted to each $L \in \Lambda$ is an isometric embedding $L \to S$. As g fixes all rank-k germs Lg is commensurable to L for each $L \in \Lambda$.

We claim that without loss of generality we can assume that as L runs through Λ, the sets $L \cup Lg$ are pairwise disjoint. Indeed, the intersections $L'g \cap L$ are necessarily of rank less than k when L' and L are different members of Λ; hence we find in L a commensurable suborthant K that avoids intersecting any of the $L'g$ with $L \neq L'$. Replacing L by such a suborthant K for all $L \in \Lambda$ justifies the claim.

Let $T := \bigcup_{L\in\Lambda} L$. The complements of both T and Tg in supp(g) are of rank $\leqslant k-1$ and since g yields a pet-isomorphism between them we have $h(\mathrm{supp}(g) - T) = h(\mathrm{supp}(g) - Tg)$. On the other hand, the two complements have decompositions into disjoint unions

$$\mathrm{supp}(g) - T \;=\; (\mathrm{supp}(g) - T \cup Tg) \;\cup\; (Tg - T),$$

$$\mathrm{supp}(g) - Tg \;=\; (\mathrm{supp}(g) - T \cup Tg) \;\cup\; (T - Tg),$$

from which we infer that $h(T - Tg) = h(Tg - T)$. This establishes equation (8). □

4.5 | Generation in G_k

We start by introducing special elements $g \in G = G(\mathbb{Z}^N)$.

- We call g a *single-orthant-isometry* if supp(g) is a single-orthant L on which g restricts to an isometry of L.

- We call g an *orthant-n-cycle* if we are given a set of pairwise disjoint orthants, cyclically connected by a sequence of n isometries

$$L_1 \xrightarrow{f_1} L_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} L_n \xrightarrow{f_n} L_1, \ \ with \ \ f_1 f_2 \cdots f_n = Id_{L_1},$$

and g is the union $g = \bigcup_{1 \leqslant i \leqslant n} f_i$. An orthant-2-cycle is also called an *orthant-transposition*.
- We call g a *pei-translation from L to L' (or between L and L')* if $\mathrm{supp}(g) = L \cup L'$ is the union of two disjoint orthants containing commensurable suborthants $K \subseteq L$, $K' \subseteq L'$ such that g restricted to K is the parallel shift that sends K to L and g restricted to L' is the parallel shift that sends L' to K'. This implies that $\mathrm{supp}(g)$ has exactly two rank-k germs (a source and a sink), fixes them, and restricts to a pei-isomorphism $g\,|_{L-K} : (L-K) \to (L'-K')$, when $h(L-K) = h(L'-K')$, as is also seen from the vanishing of the total flow function, cf. (7).
- We call g a *an endotranslation* if it is supported on an orthant L and parallel shifts a commensurable suborthant $K \subseteq L$ to a commensurable suborthant $Kg \subseteq L$. This implies that $h(L-K) = h(L-Kg)$. Note that this includes all elements $g \in G_{k-1}$ as the special case when $G\,|_K = \mathrm{id}_K$.

  One observes easily that endotranslations are stagnant, cf. (7), and that products of endotranslations with commensurable supports are again endotranslations. Hence, the set of all endotranslations supported on orthants with one and the same germ $\gamma \in \Gamma^k(S)$ forms a subgroup $E_k(\gamma)$ with $G_{k-1} \leqslant E_k(\gamma) \leqslant ST_k(S)$, for all γ.

  We write $E_k(S)$ for the group generated by all $E_k(\gamma)$ with $\gamma \in \Gamma^k(S)$.

Exercise.

1. Prove that g is a rank-k endotranslation if and only if $g \in C$ is supported on a rank-k orthant L, $\Gamma^k(L)$ is the and singleton set $\{\gamma(L)\}$, and g fixes the ordering of its boundary directions.
2. Prove that $E_k(L)$ is a normal subgroup of $\mathrm{pei}(L)$, and $\mathrm{pei}(L)$ is the semi-direct product of is the semi-direct product of $E_k(L)$ with the subgroup $\mathrm{Isom}(L) \leqslant \mathrm{pei}(L)$.
   - Special pei-translations $g : L \to L'$ are those when $L - K$ and $L' - K'$ are different corank-1 faces F, F' of L, L' and the restriction of g to F is an order preserving isometry. We call these the unit-pei-translations from L into L' and note that they are uniquely determined by the face pair (F, F') and a given ordering on the canonical basis of $\mathbb{N}^N$. For simplicity we will often use '(unit)-tanslation' for '(unit)-pei-translation' when this is unambiguous.

     Similarly, we consider the special endotranslations $g : L \to L$ with the property that K and K and Kg are the complements of two different corank-1 faces $F = L - K$ and $F' = L - Kg$. We call these the unit-endotranslations, noting that they are uniquely given by the pair (F, F') of different faces and the isometry $g|_{F \cup F'}$. There are two possible canonical requirements that we can ask $g|_{F \cup F'}$ to fulfill: (1) $g|_{F \cup F'}$ is the orthant-transposition given by the restriction of a reflection on L, or (2) $g|_{F \cup F'}$ is given by the uniquely defined order preserving isometry $f : F \to F'$. We will always use the first option unless making the statement to the contrary. Thus the unit-endotranslations on L are uniquely determined by their face pair (F, F').

For later reference, we collect some elementary facts on the arithmetics of these special elements.

Lemma 4.7. (A) ***Orthant-transpositions and pei-translations***

(i) If $\tau : L \to L'$ is an orthant-transposition and $\alpha : L \to L$ a single-orthant-isometry then $\alpha = \alpha\tau \cdot \tau$ exhibits α as the product of two orthant-transpositions.

If σ, σ' are single-orthant-reflections of $L.L'$, respectively, then there is an orthant-transposition $\tau : L \to L'$ with $\sigma\sigma' = [\sigma, \tau]$.

(ii) Every unit-pei-translation λ of rank-k is the product of two orthant-transpositions $\lambda = \tau\tau'$ of rank-k. Related to this is the observation that $\lambda^2 = \tau\tau^\lambda = [\tau, \lambda]$. The translation λ itself is not necessarily a commutator (cf. Theorem 4.10). However, if $|\Gamma^k| \geqslant 3$ then there is an orthant-transposition τ and a unit-pei-translation μ, both of rank-k, with $\lambda = [\mu, \tau]$.

(iii) Assume that we are given two disjoint rank-k orthants L, L', together with rank-k suborthants $K \subseteq L$ and $K' \subseteq L'$. If $h(L - K) = h(L' - K')$. Then there is a pei-translation λ from L to L' which parallel shifts K to L and L' to K'; and λ can be chosen as a product of unit-pei-translations.

Moreover, each pei-translation of rank-k is equal, modulo G_{k-1}, to a product of unit-pei-translations of rank-k.

(B) *Single-orthant-reflections and endotranslations*

(iv) Assume that we are given a rank-k orthant L with two rank-k suborthants $K, K' \subseteq L$. If $h(L - K) = h(L - K')$ then there is an endotranslation η on L which parallel shifts K to K', and η can be chosen as a product of unit-endotranslations.

Moreover, each endotranslation of rank-k is equal, modulo G_{k-1}, to a product of unit-endotranslations of rank-k.

(v) Let $\sigma_{xy} : L \to L$ be the reflection of the orthant L interchanging the canonical axes x, y (orthogonal to faces F_x, F_y) and fixing the remaining ones. Let t_y denote the parallel shift of L in direction y by one unit into itself, and $\sigma_{xy}^{t_y}$ the corresponding reflection of Lt_y. Putting $\eta_{xy} := \sigma_{xy}\sigma_{xy}^{t_y} : L \to L$ yields an explicit description of the unit-endotranslation of L defined on the face pair (F_x, F_y) by the restriction of σ_{xy}: We have $L = Lt_x \cup F_x$; on Lt_x, η_{xy} is the diagonal shift by one (diagonal) unit in direction $y - x$ onto $Lt_x = L - F_y$, and on F_x it is the restriction $\sigma_{xy}|_{F_x}$ which maps F_x onto F_y. Then we have

$$\eta_{xy}^{\sigma_{xy}} = \eta_{yx} = (\eta_{xy})^{-1}, \ [\sigma_{xy}, \eta_{xy}] = \eta_{xy}^2, \ \textit{and} \ \eta_{xy}\eta_{yx}^{t_y} = \sigma_{xy}|_{F_x \cup F_y}.$$

We observe that if t_y is induced by a unit-pei-translation λ, then η_{xy} is the commutator $\eta_{xy} = [\sigma_{xy}, \lambda]$ and $\sigma_{xy}|_{F_x \cup F_y} = [\eta_{xy}, \lambda]$.

Moreover, if x, y, z are three pairwise different canonical basis elements of L then

$$\eta_{xy}\eta_{yz}\eta_{zx} = \sigma_{yz}|_{F_x} \quad \textit{and} \quad \eta_{xz}\eta_{yx}\eta_{zy} = \sigma_{yz}|_{F_x \cup F_y}, \tag{9}$$

and note that σ_{yz} is a reflection of the face F_x of L and $\sigma_{yz}|_{F_x \cup F_y}$ is a canonical orthant-transposition of the form $(F_x - F_x \cap F_y, F_y - F_x \cap F_y)$.

Proof. Assertion (i) is obvious.

(ii) Let λ be a unit-translation from K to L which maps the face F of K isometrically onto the face $F\lambda$ of L. Then the isometry $\lambda|_F : F \to F\lambda$ extends uniquely to an isometry $K \to L$ which fixes $F\lambda$ pointwise, and thus defines an orthant-transposition $\tau = (K, L)$. Correspondingly, the restriction of $\lambda^2|_F : F \to F\lambda^2$ extends uniquely to an isometry $K \to L - F\lambda$ and hence defines a pei-transposition $\tau' = (K, L - F\lambda)$. Both $\tau\tau' = (L, K)(K, L - F\lambda) = \lambda$ and the formula $\lambda^2 = \tau\tau^\lambda = [\tau, \lambda]$ are easily seen by inspection.

If there is a third rank-k orthant M disjoint to both K and L we consider a new pei-transposition $\tau := (L, M)$. Then λ^τ is a unit-translation from K to M. $\lambda^{-1}\lambda^\tau = [\lambda, \tau]$ is a unit-pei-translation from M to L. One checks that $[\lambda, \tau]$ is conjugate, by an appropriate choice of an orthant-3-cycle of the form $\pi = (M, L, K)$, to $\lambda = [\lambda, \tau]^\pi$. This shows that $\lambda = [\mu, (K, L)]$ with $\mu = \lambda^{(K.L.M)}$.

(iii) This assertion is easy to accept by viewing unit-translations from L to L' as the process of cutting a rank-$(k-1)$ orthant off from a face of L and pushing it down onto a face of L'. By repeating this process with changing face pairs one constructs an orthant- translation λ from L to L' which parallel shifts K to L, and since arbitrary face pairings are possible we can achieve that λ parallel shifts L' onto an arbitrary given rank-k suborthant K' with $h(L' - K') = h(L - K)$.

If λ is an arbitrary pei-translation of rank-k the procedure above constructs a product π of unit-pei-translations of rank-k that coincides with λ on the one rank-k orthant on which λ is a non-zero isometry. π depends on the special procedure, but $\mathrm{supp}(\lambda\pi^{-1})$ is always of rank-$(k-1)$. This shows that modulo G_{k-1}, λ is equal to π.

(iv) The argument for (iv) is similar to the one in (iii) above: instead of moving K' to L by sequence of parallel shifts along coordinate axes we have to move K' directly to L' by a sequence of pushing/pulling pairs along two axis — details are left to the reader.

(v) All formulae are proved by inspection which can be left to the reader as an exercise. In the case of formulae (9), start by showing that the restriction of $\eta_{xy}\eta_{yz}\eta_{zx}$ to $(1,0,0) + L$ is the identity, and so is the restriction of $\eta_{xz}\eta_{yx}\eta_{zy}$ to $(1,0,0) + L$. $\square$

The following classifies $G_k(S)/G_{k-1}(S)$ for an arbitrary orthohedral set S up to extensions.

Theorem 4.8.

(i) Each transposition (γ, γ') of germs in $\Gamma^k(S)$ lifts to an orthant-transposition of representing orthants in S, and the action of $G_k(S)$ on $\Gamma^k(S)$ induces an isomorphism onto the finitary symmetric group,

$$G_k(S)/C(\Gamma^k(S)) \;\cong\; \mathrm{sym}(\Gamma^k(S)).$$

(ii) The action of $C(\Gamma^k(S)$ on the canonical coordinate directions $X(\gamma)$ in each $\langle\gamma\rangle$, $\gamma \in \Gamma^k(S)$ defines an isomorphism $C(\Gamma^k(S)$ onto the finitary direct product of the symmetric permutation groups of degree k,

$$C(\Gamma^k(S))/C^{\mathrm{ord}}(\Gamma^k(S)) \;\cong \bigoplus_{\gamma\in\Gamma^k(S)} \mathrm{sym}_k X(\gamma).$$

(iii) The homomorphism $\varphi = \oplus_\gamma \varphi_\gamma$ restricted to $C^{\mathrm{ord}}(\Gamma^k(S))$ induces a short exact sequence

$$0 \to C^{\mathrm{ord}}(\Gamma^k(S))/G_{k-1}(S) \xrightarrow{\varphi} \bigoplus_{\gamma\in\Gamma^k(S)} Trans\langle\gamma\rangle \xrightarrow{\oplus_\gamma fl_\gamma} \mathbb{Z} \to 0.$$

In particular, $A^k(S) := C^{\mathrm{ord}}(\Gamma^k(S))/G_{k-1}(S)$ is free-Abelian of rank $|\Gamma^k(S)| - 1$ (which is infinite for $k < \mathrm{rk}(S)$). Each $a \in A^k(S)$ can be represented by an element $g \in C^{\mathrm{ord}}(\Gamma^k(S))$ with *the property that* supp(g) *is the union of a finite set of pairwise disjoint orthants that represent the non-trivial components of* $\varphi(a)$*; and* $A^k(S)$ *is generated by unit-translations and unit-endotranslations.*

Proof.

(i) As γ, γ' are different germs they can be represented by a pair of disjoint orthants, and any orthant-transposition between those lifts the transposition of the germs. The rest of assertion (i) is immediate from Lemmas 4.5 and 4.6.

(ii) Combining the homomorphisms (5) with γ running through $\Gamma^k(S)$ yields a homomorphism $\overline{\varphi} = \prod_\gamma \overline{\varphi}_\gamma : C(\Gamma^k(S)) \to \prod_\gamma \mathrm{sym}(X(\gamma))$, and by Lemma 4.6 its kernel is $\bigcap_\gamma C^{\mathrm{ord}}(\gamma) = C^{\mathrm{ord}}(\Gamma^k(S))$. If γ is an arbitrary rank-k germ with $\overline{\varphi}_\gamma(g) \neq Id$ then any orthant representing γ is commensurable to an orthant in supp(g). This shows that $\overline{\varphi}_\gamma(g)$ has only finitely many non-vanishing components; hence we can infer that the image of $\overline{\varphi}$ is in the finitary product. As each permutation in sym$(X(\gamma))$ can be lifted by a single-orthant-isometry, the image of $\overline{\varphi}$ is, in fact, the full finitary product.

(iii) Combining the homomorphisms (4) with γ running through $\Gamma^k(S)$ yields a homomorphism φ of $C(\Gamma^k(S))$ into the product $\prod_\gamma \mathrm{Isom}(\langle\gamma\rangle)$. As above in (ii) one argues that for g of rank-k, $\varphi(g)$ has only finitely many non-vanishing components; hence we can infer that the restriction of φ to the ordered germs yields a homomorphism into the direct sum

$$\varphi : C^{\mathrm{ord}}(\Gamma^k(S)) \to \bigoplus_{\gamma\in\Gamma^k(S)} \mathrm{Trans}\langle\gamma\rangle. \tag{10}$$

By Lemma 4.6 the kernel of φ consists of the elements g that pointwise fix in each rank-k orthant a commensurable suborthant; that means supp(g) contains no rank-k orthant. Hence $\ker(\varphi) = G_{k-1}(S)$ and φ induces an embedding of $C^{\mathrm{ord}}(\Gamma^k(S))/G_{k-1}(S)$ into the Abelian group $\bigoplus_\gamma \mathrm{Trans}(\langle\gamma\rangle)$.

We choose, for a given $g \in C^{\mathrm{ord}}(\Gamma^k(S))$, a finite set Λ of pairwise disjoint orthants representing the rank-k germs of supp(g), and with the property that g restricted to each $L \in \Lambda$ is an isometric embedding $L \to S$. As g fixes all rank-k germs Lg is commensurable to L for each $L \in \Lambda$.

As we saw in the proof of equation (8) we can assume, without loss of generality, that as L runs through Λ, the sets $L \cup Lg$ are pairwise disjoint, and as in that proof we put $T := \bigcup_{L\in\Lambda} L$. Then we observe that $h(T - Tg) - h(Tg - T)$ is the total flow *fl* and deduce from equation (8) that $T - Tg$ and $Tg - T$ are pei-isometric.

Thus, by the pei-normal form, there is an pei-bijection $\beta : Tg - T \to T - Tg$. Let $\alpha : T \to Tg$ denote the restriction of g to T and put $K := \alpha^{-1}(T \cap Tg)$. The composition of α with the union $\mathrm{id}_{T\cap Tg} \cup \beta : Tg \to T$ now yields a pei-permutation $\mu : T \to T$ which coincides on K with the restriction of g. Thus, the composition $g\mu^{-1}$ is supported in supp(g) and fixes K pointwise, when $\mathrm{rk}(g\mu^{-1}) < k$. This shows that $g = \mu$ modulo $G_{k-1}(S)$. Hence every element of $C^{\mathrm{ord}}(\Gamma^k(S))/G_{k-1}(S)$ can be represented by an element $g \in C^{\mathrm{ord}}(\Gamma^k(S))$ with the property that g is supported on a set of pairwise disjoint orthants L_γ each of which represents its index $\gamma \in \Gamma^k(\mathrm{supp}(g))$.

From here it is easy to prove that $C^{\mathrm{ord}}(\Gamma^k(S))/G_{k-1}(S)$ is represented by a product of translations. We use induction on the number $|\{\gamma \mid \varphi_\gamma(g) \neq 0\}|$: Pick a pair of germs γ, γ', both with $\varphi_\gamma(g) \neq 0$. Then multiply g with a sequence of translation μ_i from L_γ to L'_γ in coordinate directions such that the product $\mu = \prod_i \mu_i$ reverses the restriction of g to a commensurable suborthant $K \subseteq L_\gamma$ that g parallel shifts within L_γ. Then $g\mu$ fixes K pointwise, hence modulo $G_{k-1}(S)$, $g\mu$ is equal to a an element of $C^{\mathrm{ord}}(\Gamma^k(S))$ with smaller number $|\{\gamma \mid \varphi_\gamma(g\mu) \neq 0\}|$. The procedure ends when the $\varphi_\gamma(g) = 0$ except for one germ γ, and then g is mod $G_{k-1}(S)$ is an endo-translation. In view of parts (v) and (vi) of Lemma 4.7, this proves (iii). $\square$

As a consequence of Theorem 4.8 we obtain economical generation properties. The obvious crucial fact is that if S is an orthohedral set of rank $rkS = n$ then $|\Gamma^n(S)| \in \mathbb{N}$, while $|\Gamma^k(S)| = \infty$ when $k < n$. The *exceptional case* when $|\Gamma^n(S)| = 1$ — or equivalently: G_n contains no rank-n orthant-transpositions — requires special treatment. In that case all rank-n elements of G_n are rank-n stagnant, and this is a serious restriction on the rank-$(n-1)$ elements that are products of rank-n orthant-transpositions. For example, non-trivial pei-translations cannot be products of single-orthant-reflections.

Corollary 4.9. *Let S be a stack of $h(S)$ rank-n orthants, $G_k := G_k(S) \leqslant \mathrm{pei}(S)$, and $\Gamma^k = \Gamma^k(S)$, with $0 \leqslant k \leqslant n \in \mathbb{N}$.*

(i) If $|\Gamma^k| \geqslant 2$, then G_k is generated by its orthant-transpositions of rank-k.
If $|\Gamma^k| = 1$, then $k = n$ and $G_n = \mathrm{pei}(\mathbb{N}^n)$ is the normal subgroup generated by all single-orthant-reflections of rank n.
(ii) If $|\Gamma^k| \geqslant 2$ then $C^{\mathrm{ord}}(\Gamma^k)$ is generated by its pei-translations of rank-k.
If $|\Gamma^k| = 1$ then $C^{\mathrm{ord}}(\Gamma^k)$ is the normal subgroup generated by the endotranslations of rank-k.
(iii) If $|\Gamma^k| \geqslant 5$ then every product of two orthant-transpositions $g = \tau\tau'$, where $\tau, \tau' \in G_k$, can be written as a product $g = \upsilon_1\upsilon_2\upsilon_3$, where each υ_i is either trivial or a product $\upsilon_i = \tau_i\tau_i'$ of two disjoint *orthant-transpositions (that is, $\mathrm{supp}(\tau_i) \cap \mathrm{supp}(\tau_i') = \emptyset$, for each $1 \leqslant i \leqslant 3$).*

Proof.

(i) We start by proving that the claim holds true modulo G_{k-1}. By Theorem 4.8 this amounts to lift generators of the three sections $Q_1 := G_k/C(\Gamma^k)$, $Q_2 := C(\Gamma^k)/C^{\mathrm{ord}}(\Gamma^k)$, and $Q_3 := C^{\mathrm{ord}}(\Gamma^k)/G_{k-1}$. Now, Q_1 is generated by germ transpositions, and those lift to orthant-transpositions. Q_2 is generated by transpositions of face directions, and those lift to single-orthant-reflections. Q_3 is generated by unit-pei-translations and unit-endotranslations. By Lemma 4.7(iii) and (iv) we can thus infer that G_k/G_{k-1} is generated by orthant-transpositions, single-orthant-reflections, unit-pei-translations, and unit-endotranslations.
In the *exceptional case* where $G_k = \mathrm{pei}(\mathbb{N}^n)$ contains neither orthant-transpositions nor pei-translations of rank-k, G_k/G_{k-1} is thus generated by single-orthant-reflections and unit-endotranslations. Moreover, we know from Lemma 4.7(v) that unit-endotranslations are products of two single-orthant-reflections and that single-orthant-reflections actually suffice to generate G_k/G_{k-1} in that case.
The case when $|\Gamma^k| \geqslant 2$ is similar: here the existence of a rank-k orthant-transposition $\tau \in G_k$ allows to apply Lemma 4.7(i) showing that all single-orthant-reflections of rank k can now be replaced by of products of two orthant-transpositions. Hence G_k/G_{k-1} is generated by its rank-k orthant-transpositions in this case.
Next we prove that if $|\Gamma^k| \geqslant 2$ then every rank-$(k-1)$ orthant-transposition (F, F') is a product of rank-k orthant transpositions. This is easy when F and F' are contained in disjoint rank-k orthants, for then they are, in fact, faces of disjoint rank-k orthants (L, L'), and $(F, F') = (L, L')(L - F, L' - F')$. And if F and F' are contained in the same rank-k orthant L,

we find a rank-$(k-1)$ orthant F'' supported in a rank-k orthant disjoint to L, and therefore $(F', F'')(F, F'')(F', F'') = (F, F')$. The corresponding weaker result in the exceptional case $|\Gamma^k| = 1$ is obvious: If (F, F') is an arbitrary rank-$(k-1)$ orthant-transposition then we find a rank-k orthant L disjoint to $F \cup F'$, and by Lemma 4.7(v) a face-transpositions (F_x, F_y) which is a product single-orthant-reflections of rank-k. As $|\Gamma^{k-1}| = \infty$ any two rank-$[k-1)$ orthant-transpositions are conjugate in pei(S) hence the normal subgroup generated by (F_x, F_y) contains (F, F').

Now assertion (i) follows by induction on k: Let $H \leqslant G_k$ be the subgroup generated by all rank-k orthant-transpositions (respectively, the normal subgroup generated by all rank-k reflections). In the case $k = 0$, we have $H = G_0$ because G_0 is the finitary countable symmetric group and hence generated by its transpositions. If $k > 1$ we have seen that the subgroup generated by rank-k transpositions (respectively, the normal subgroup generated by all rank-k reflections) contains all rank-$(k-1)$ orthant-transpositions, and by induction those generate G_{k-1}. Thus the rank-k orthant-transpositions (respectively, single-orthant-reflections generate both G_k/G_{k-1} and G_{k-1}, and hence G_k.

(ii) The proof along the lines of assertion (i) and can be left for the reader.

(iii) Let $\tau = (K, L)$, $\tau' = (M, N)$. If $|\Gamma^k(S)| = \infty$ one finds rank-k orthants X, Y such that K, L, X, Y and M, N, X, Y are pairwise disjoint quadruples, and $(K, L)(M, N) = (K, L)(X, Y)(X, Y)(M, N)$ as needed. As S is a stack of orthants we infer that if k is finite at least 5 then $k = rkS$ and any two rank-n orthants are either disjoint or commensurable. If both K and L are commensurable to M or N we find two rank-k orthants X, Y as above, the argument above applies. In the remaining case we may assume that K is commensurable to M but $L \cap N = \emptyset$; then we find two rank-n orthants X, Y such that both K, L, N, X, Y and M, L, N, X, Y are pairwise disjoint quintuples, and $(K, L)(M, N) = (K, L)(X, N)(X, N)(Y, L)(Y, L)(M, N)$. $\square$

Exercise. Prove that (a) The stagnant subgroup of G_k,

$$ST_k = \ker\left(fl \,:\, C(\Gamma^k) \to \bigoplus_{\gamma \in \Gamma^k(S)} \mathbb{Z} \right),$$

is generated by G_{k-1} together with all single-orthant-isometries of rank-k, and also equal to the normal subgroup generated by all single-orthant-reflections.

(b) The stagnant subgroup of $C^{\mathrm{ord}}(\Gamma^k(S))$, that is,

$$E_k(S) := ST_k \cap C^{\mathrm{ord}}(\Gamma^k(S)),$$

is the normal subgroup of G_k generated by all endotranslations of rank-k. As a group it is generated by G_{k-1} together with all endotranslations of rank-k.

4.6 | Conjugation, Abelianization, and alternation

As before, S is a stack of orthants of rank-n and $G :=$ pei(S), and we recall that $|\Gamma^k|$ is finite if and only if $k = n$. We say an element $g \in G_k$, $k \leqslant n$, is *even* if g is equal to the product of an even number of rank-k orthant-transpositions — by Lemma 4.7 this includes all single-orthant-isometries of rank-k; and corank-1 faces of a rank-k orthants can be written as products of two such. We write $\operatorname{alt} G_k \leqslant G_k$ for the subgroup consisting of all even elements and observe that $\operatorname{alt} G_k$ is the kernel of the *rank-k parity homomorphism*, $\operatorname{par}_{\Gamma^k} : G_k \to \mathbb{Z}_2$, which sends $g \in G_k$ to the parity of the permutation that g induces on the rank-k germs Γ^k. Hence $\operatorname{alt} G_k$ is of index 2 in G_k. Moreover, by Corollary 4.9(iii) $\operatorname{alt}(G_k)$ is generated by products of pairs of disjoint orthant-transpositions if $|\Gamma^k(S)| \geqslant 5$.

We will also need a refinement of the action of G_k on $\Gamma^k(S)$. When $g \in G_k$ sends the tangent coset $\langle\gamma\rangle$ to $\langle\gamma g\rangle$ then it also induces a map $g : X(\gamma) \to X(\gamma g)$ between the canonical axes directions of $\langle\gamma\rangle$ and $\langle\gamma g\rangle$. Given an orthant L representing γ on which g is isometric, and a canonical axis-direction $x \in X(\gamma)$, we have x orthogonal to a unique corank-1 face F of L and xg is the canonical axis direction orthogonal to Fg. Thus G_k acts on the disjoint union $Y := \bigcup_{\gamma\in\Gamma^k} X(\gamma)$ by finite permutations; and we have a corresponding parity homomorphism $\operatorname{par}_{Y^k} : G_k \to \mathbb{Z}_2$.

Restricted to $C(\Gamma^k)$ the parity map par_{Y^k} is easy to compute: On $C^{\mathrm{ord}}(\Gamma^k)$ even the action on Y is trivial, hence we need consider it only on

$$C(\gamma^k)/C^{\mathrm{ord}}(\Gamma^k) = ST_k/E_k.$$

ST_k is generated by all single-orthant-reflections, and as those are the transpositions of the symmetric groups $\operatorname{sym}(X(\gamma))$ an element of ST_k has parity 0 (or is even) if and only if it is the product of an even number of single-orthant-reflections. This is a subgroup of index 2 in ST_k, we call it the alternating subgroup $\operatorname{alt} ST_k \leqslant ST_k$, and have the normal series

$$G_{k-1} \leqslant E_k \leqslant [ST_k, ST_k] \leqslant \operatorname{alt} ST_k \leqslant ST_k \leqslant C(\Gamma^k) \leqslant \operatorname{alt} G_k \leqslant G_k.$$

Theorem 4.10.

(i) If $|\Gamma^k(S)| \geqslant 3$ then the following holds:

(a) In G_k all unit-pei-translations of rank-k are conjugate, together they generate $C^{\mathrm{ord}}(\Gamma^k(S))$, and $(G_k)_{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (generated by an orthant-transposition and a single-orthant-reflection).

(b) In G_k all products $\sigma_1\sigma_2$ of pairs of disjoint single-orthant-reflections of rank-k are conjugate, together they generate $\operatorname{alt} ST_k \leqslant ST_k$, and $(C(\Gamma^k(S)))_{ab} \cong (ST_k)_{ab} \cong \bigoplus_{\Gamma^k(S)} \mathbb{Z}_2$ (generated by single-orthant-reflections).

(ii) If $|\Gamma^k(S)| \geqslant 4$ then all orthant-3-cycles $p = (L_1, L_2, L_3)$ of rank-k are conjugate (more generally: If $|\Gamma^k(S)| \geqslant m$ then any two orthant-$(m-1)$-cycles are conjugate) in G_k and together they generate $\operatorname{alt} G_k$.

(iii) If $|\Gamma^k(S)| \geqslant 5$ then all products $\tau_1\tau_2$ of pairs of disjoint rank-k orthant-transpositions are conjugate and together they generate $\operatorname{alt} G_k$.

(iv) If $|\Gamma^k(S)| = 2$ then $(G_k)_{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (generated by an orthant-transposition, a single-orthant-reflection, and a unit-pei-translation).

(v) If $|\Gamma^k(S)| = 1$ (hence $G_k \cong \operatorname{pei}(\mathbb{N}^k)$) then $(G_k)_{ab} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (generated by a single-orthant-reflection, and a unit-endotranslation).

Proof. (i)(a) Let λ, λ' be two rank-k unit-pei-translations from K to L, and K to L' respectively. If L and L' are disjoint then λ and λ' are conjugate by an orthant-transposition (L, L'). If L and L' are nested (and hence commensurable) an auxiliary rank-k orthant is available to construct a translation that sends L to L' or vice versa, and thus a conjugation between λ and λ'. In the general case one finds inside L a rank-k suborthant which is either disjoint to or contained in L' and obtains the required conjugation in two steps. The general conjugation assertion is now obvious, and that the unit-pei-translations generate all of $C^{\mathrm{ord}}(\Gamma^k(S))$ was established in Lemma 4.7.

The action of the two parity homomorphisms yields an epimorphism $\mathrm{par}_{Y^k} \times \mathrm{par}_{\Gamma^k} : G_k \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$, whose kernel is generated by all translations together with all products of two single-orthant-reflections. By Lemma 4.7(i) and (ii) both are commutators.

(i)(b) The proof is analogous and easier than the one of (i)(a).

(ii) To prove this we start by observing that if $p = (L_1, L_2, L_3)$ is an orthant-3-cycle of rank-k, given by given by the pair of isometries $L_1 \xrightarrow{\varphi_1} L_2 \xrightarrow{\varphi_2} L_3$, then the fact that there is an auxiliary rank-k orthant K disjoint to $\mathrm{supp}(p)$ provides the existence of translations $\vartheta_i \in \mathrm{Isom}(K \cup L_i)$, with the property that $\vartheta_i(K)$ is an arbitrary given commensurable suborthant of L_i. We can put them together to an element $\vartheta \in \mathrm{Isom}(K \cup \bigcup_i L_i)$. Hence we find that p is conjugate to orthant 3-cycles $p' = (L_1', L_2', L_3')$, where the L_i' are arbitrary given commensurable suborthants of L_i.

If $q = (M_1, M_2, M_3)$ is an arbitrary second orthant-3-cycle we can choose the suborthants $L_i' \subseteq L_i$ to be either contained in M_i (if L_i and M_i are commensurable) or disjoint to M_i (if L_i and M_i are disjoint). Thus, in order to prove that p and q are commensurable we can now assume, without loss of generality, that each L_i is either contained in or disjoint to M_i.

Now we complete the proof of (ii) in two steps: First we choose, for all indices i with $L_i \cap M_i = \emptyset$, an arbitrary orthant-transposition $\tau_i = (M_i, L_i)$. Conjugation with these τ_i shows that we find a conjugate of p which replaces L_i by M_i whenever L_i is not contained in M_i. In other words we are now reduced to a case when $L_i \subseteq M_i$ for all i. Repeating the first step completes the proof. It is clear that the argument proves, in fact, the general statement for orthant-3-cycles generate all pairs of orthant-transpositions.

(iii) The argument is exactly like that of i(a). Let $\tau_1\tau_2 = (K, L)(M, N)$. We show first that $\tau_1\tau_2$ is conjugate to $(K, L)(M, N')$ for each choice of N' disjoint to K, L, M. This is done by the same case distinction as in i(a). Then one can repeat the argument with K, L, and M. The generation assertion is covered by Corollary 4.9(iii).

(iv) If $|\Gamma^k(S)| = 2$ then S is the disjoint union of two rank-k orthants K, L, and we consider in G an orthant-transposition $\tau = (K, L)$, a unit-pei-translation λ from K to L, and a single-orthant-reflection σ of K. We have $C(\Gamma^k(S)) = C^{\mathrm{ord}}(\Gamma^k(S))ST_k$, and since the stagnant normal subgroup ST_k contains all of E_k but no non-trivial pei-translations $C(\Gamma^k(S)) = gp(ST_k)$ is the semi-direct product of the normal ST_k with the infinite cyclic group $gp(\lambda)$. As $G_k/C(\Gamma^k(S)$ is cyclic of order 2 generated by τ it follows that G_k/ST_k is isomorphic to the infinite dihedral group $gp(\lambda, \tau)$, and its Abelianization is the Klein-4-group generated by λ and τ. As $G_k/\mathrm{alt}\, G_k$ is the Klein-4-group generated by s and τ, this shows that all three elements λ, τ, σ are needed to generate $(G_k)_{ab}$; and as $\lambda^2 = [\tau, \lambda]$ we find the asserted result.

(v) $|\Gamma^k(S)| = 1$. In that case $G_k = \mathrm{pei}(L)$, for a single rank-k orthant L, and this is easily seen to be the semi-direct product $E_k \rtimes \mathrm{Isom}(L)$. The symmetric group Isom(L) acts transitively on the k-axes and hence on the unit-endotranslations supported on L: Using the notation of Lemma 4.7(v) we have, for example, $\eta_{xy}^{\sigma_{xz}} = \eta_{zy}$, and can infer that $[\sigma_{xz}, \eta_{xy}^{-1}] = \eta_{zy}\eta_{xy}^{-1}$. It follows that all endo-translation of L coincide in the Abelianization, and by Lemma 4.7(v) their square is a commutator.

Lemma 4.7(v) also shows that G_k' contains an orthant-transposition of rank equal to $(k-1)$. As $|\Gamma^{k-1}(S)| = \infty$ all rank-$(k-1)$ orthant-transpositions are conjugate and hence by Corollary 4.9 all of G_{k-1} is contained in G_k'. This shows that $(G_k)_{ab}$ is the Klein-4 group generated by σ_{xy} and η_{xy}. □

4.7 | The $G_k/C^{\mathrm{ord}}(\Gamma^k)$-module structure of $C^{\mathrm{ord}}(\Gamma^k(S))/G_{k-1}$

Using Theorem 4.8(iii) we can consider $A^k(S) := C^{\mathrm{ord}}(\Gamma^k(S))/G_{k-1}$ as the kernel of the corank-1 germ-flow homomorphism sum in the direct sum

$$\bigoplus_{\gamma\in\Gamma^k(S)} \mathrm{Trans}\langle\gamma\rangle.$$

Thus, each $a \in A^k(S)$ is given by a finitely supported family of translations indexed by the (in general infinite) rank-k germs, $a = (t_\gamma)_{\gamma\in\Gamma^k(S)}$, and each t_γ is uniquely determined by its translation vector in $\mathbb{Z}^k$ with respect to the canonical basis of the tangent coset $\langle\gamma\rangle$. Hence we can write t_γ as a row-vector $(a_{(\gamma,1)}, \dots, a_{(\gamma,k)}) \in \mathbb{Z}^k$, and note that the sum of its entries is the flow value $fl_\gamma(a)$.

Thus, in this section we organize the elements of $A^k(S)$ as the additive group of integral $(\Gamma^k(S) \times k)$-matrices $\mathbf{a} = (a_{(\gamma,i)})$ with only finitely many non-zero entries that add up to 0. The row indices are the rank-k germs $\gamma \in \Gamma^k(S)$ for a fixed number k, and they are endowed with a compatible ordering of the canonical bases of $\langle\gamma\rangle$. The column index $1 \leqslant i \leqslant k$ stands for the ith canonical basis element in this ordering.

The quotient group $Q_k(S) := G_k/C^{\mathrm{ord}}(\Gamma^k)$ is the (finitary permutational) wreath product $S_k(\beta) \wr \mathrm{sym}(\Gamma^k(S))$, where $\beta \in \Gamma^k(S)$ is a chosen base germ and $S_k(\beta)$ the symmetric group on the canonical basis of $\langle\beta\rangle$. We interpret $\mathrm{sym}(\Gamma^k(S))$ as the permutation group on the entries which stabilizes all columns and acts diagonally by the symmetric group the set of all rank-k germs.

The flow $fl_\gamma(\mathbf{a}) \in \mathbb{Z}$ of a matrix $\mathbf{a} \in A^k(S)$ at γ is the sum of the entries in the γ-row. The total flow $fl(\mathbf{a})$ is the sum of all entries of $\mathbf{a}$ and by (8) we have $fl(\mathbf{a}) = 0$.

Row-subgroups

A general γ-row-matrix represents an element of $A^k(S)$ if and only if its row-sum is zero, and then it is represented by an endotranslation on any γ-representing orthant. We write $E_\gamma \leqslant A_k(S)$ for the subgroup of all γ-row-matrices. The unit-endotranslation at γ represent the γ-row-matrices consisting of a *lone pair of entries* $(1, -1)$, by which we mean that all other entries are zero. Let $E \leqslant A^k(S)$ denote the subgroup of $A^k(S)$ generated by all row-subgroups, and observe that $E = \ker(fl : A^k(S) \to \oplus_{\gamma\in\Gamma^k(S)}\mathbb{Z})$. In particular, E is a $Q_k(S)$-submodule of $A^k(S)$, and as any finite set of germs can be represented by pairwise disjoint orthants we have $E = \bigoplus_{\gamma\in\Gamma^k(S)} E_\gamma$.

Column-subgroups

Lei i be a natural number $\leqslant k$. A general finite ith-column-matrix $(n_\gamma)_{\gamma\in\Gamma^k}$ is given by a finitely supported map $f_i : \Gamma^k(S) \to \mathbb{Z}$, $n_i := f_i(\gamma)$ and defines the parallel shift of a finite set of pairwise disjoint orthants L_γ of $\langle\gamma\rangle$ in the direction of their ith axis onto $n_i L_\gamma$. This defines an element of $A^k(S)$ if and only if the column sum $\sum_{\gamma\in\Gamma^k} n_\gamma$ is zero, and we write $C_i \leqslant A^k(S)$, $i = 1, 2, \dots, k$, for the subgroup of all ith-column matrices. Note that every matrix $\mathbf{a}$ with a lone pair of unit entries $1, -1$ in different rows (γ, γ') is represented by a unit-pei-translation λ; and $\mathbf{a}$ is a column matrix if and only if the two unit shifts of λ are anti-parallel.

The diagonal subgroup $D \leqslant A^k(S)$

The column subgroups C_i are invariant under the order preserving action of $\mathrm{sym}(\gamma^k(S))$ but not, of course, under all of $Q_k(S)$ — nor is the direct sum $\bigoplus_{1\leqslant i\leqslant k} C_i$. Only the diagonally embedded copy of C_1 into $A^k(S)$, that is, the group $D \leqslant A^k(S)$ of all matrices with constant rows and zero column sums is actually a $Q_k(S)$-submodule of $A^k(S)$.

Lemma 4.11. *For every $Q_k(S)$-submodule $M \leqslant A^k(S)$ we have*

(i) If $\mathbf{m}$ is a matrix in M, so is the matrix $(fl_1(\mathbf{m}), \dots, fl_k(\mathbf{m})) \in D$ which has in each of its columns the flow-column of $\mathbf{m}$.
(ii) If $\Gamma^k(S)$ is infinite then we have for every choice of a pair (γ, γ') of different germs in $\Gamma^k(S)$, M is generated by $E_\gamma \cap M$ together with the lone-pair-of-rows matrices supported on the (γ, γ')-rows. (In other words: M is generated by its endotranslations of γ-orthants and its pei-translations between γ and γ').
(iii) Either $M \leqslant D$ or there is a unique minimal natural number q with $qE \leqslant M$.
(iv) Either $M \leqslant E$ or there is a unique minimal natural number p with $pD \leqslant M$.

Proof.

(i) As $\mathbf{m}$ has only finitely many non-zero rows we find an n element $\vartheta \in Q_k(S)$ which permutes its columns cyclically. It follows that $\mathbf{m} + \mathbf{m}\vartheta + \cdots + \mathbf{m}\vartheta^{k-1}$ is contained in M and has the required form.
(ii) As every matrix $0 \neq \mathbf{m} \in M$ has only finitely many non-zero rows, $\mathbf{m}$ contains both a non-zero row and a zero row, and $Q^k(S)$ contains a transposition that interchanges the two. Thus, M contains a lone-row-pair matrix of the form $\mathbf{m}(1-\tau) = \binom{-\alpha}{\alpha}$ for every row α of $\mathbf{m}$. As $Q^k(S)$ acts 2-transitively on the rows we can assume that here the entry $-\alpha$ stands in a pre-chosen γ-row while α has its original position. If we subtract all these lone-pair-of-rows matrices for all non-zero rows $\neq \gamma$ from $\mathbf{m}$ we find the γ-row matrix whose entries are the column sums of $\mathbf{m}$, that is, $\mathbf{m}' := (\sum_{\gamma \in \Gamma^k} n_{\gamma 1}, \dots, \sum_{\gamma \in \Gamma^k} n_{\gamma k}) \in M$. The flow of $\mathbf{m}'$ is the total flow of $\mathbf{m}$ and hence zero. This shows that $\mathbf{m}' \in E_\gamma$. Thus M is generated by $E_\gamma \cap M$ together with all lone-pair-of-rows matrices in M. As $Q_k(S)$ acts k-transitively, each lone-pair-of-rows matrix of M is conjugate to a lone (γ, γ')-pair-of-rows matrix of M.
(iii) We assume that M is not contained in D, that is, it contains a matrix $\mathbf{a}$ which contains a γ-row with two different entries $x \neq y \in \mathbb{Z}$. Let $\tau \in S_k(\gamma)$ be the transposition that interchanges those two entries. Then $\mathbf{a}(1-\tau)$ is a matrix in $E_\gamma \cap M$ with a *lone pair of entries* of the form $(z, -z)$. We consider the smallest natural number q with the property that $E_\gamma \cap M$ contains a matrix with a lone pair of entries of the form $(q, -q)$, and call this a *minimal lone-pair-of-entries matrix* $E_\gamma \cap M$. Since $Q_k(S)$ acts highly transitively on the rows this applies to each rank-k germ γ. With familiar arguments, one observes that each row matrix of M with a lone pair of entries is a multiple of a minimal lone-pair-of-entries row matrix; hence the latter generate in each E_γ the subgroup $q(E_\gamma \cap M) \leqslant M$) and in $E \cap M$ the $Q_k(S)$-submodule $qE \leqslant M$.
(iv) Now we assume that M is not contained in E. Then there is a matrix $\mathbf{a} \in M$ with non-zero flow-column $\mathbf{fl}(\mathbf{a}) := (fl_\gamma(\mathbf{a}))_{\gamma \in \Gamma}$. As the sum of the entries of $\mathbf{fl}(\mathbf{a})$ is zero, its entries cannot be constant; hence $\mathbf{fl}(\mathbf{a})$ contains a pair of non-equal entries. Therefore $\mathbf{a}$ contains a pair of non-equal rows, and we find a row-transposition τ which interchanges these two rows. $\mathbf{a}(1-\tau)$ is then a *lone-pair-of-rows* matrix in M of the form $\binom{\alpha}{-\alpha}$ which has a lone pair of non-zero entries in its flow column $\mathbf{fl}(\mathbf{a}(1-\tau))$. It follows that there is a smallest natural number p with the property that M contains a lone-pair-of-rows matrix $\mathbf{m} = \binom{\alpha}{-\alpha}$ with a lone-pair-of-entries flow-column of the form $\binom{\pm p}{\mp p}$. We call $\mathbf{m}$ a *flow-minimal* lone-pair-of-rows matrix M.

By part (i) it follows that the constant lone-pair-of-rows matrix

$$\pm p\begin{pmatrix} 1, \dots, 1 \\ -1, \dots, -1 \end{pmatrix}$$

is also a flow-minimal lone-pair-of-rows matrix in M, when $pD \leqslant M$. □

We are now in a position to describe all $Q_k(S)$-submodules of $A^k(S)$.

Theorem 4.12. *The $Q_k(S)$-submodules $0 \neq M \leqslant A^k(S)$ are of the following types:*

- *$pD \leqslant M \leqslant D$ for some $p \in \mathbb{N}$, D/pD contains only finitely many $Q_k(S)$-submodules, and if p is minimal and $\Gamma^k(S)$ is infinite then $M = pD$;*
- *$qE \leqslant M \leqslant E$ for some $q \in \mathbb{N}$, E/qE contains only finitely many $Q_k(S)$-submodules, and if q is minimal and $\Gamma^k(S)$ is finite then $M = \mathbb{Z}Q_k(S)(E_\gamma \cap M)$;*
- *$pD + qE \leqslant M \leqslant A^k(S)$ for some $p, q \in \mathbb{N}$, and $A^k(S)/pD + qE$ contains only finitely many $Q_k(S)$-submodules.*

Proof. Note that D is free-Abelian with a countable basis X, and $Q := Q^k(S)$ acts on $D = \mathbb{Z}[X]$ via the symmetric group $\mathrm{sym}(X)$. If M is contained in D it cannot be contained in E hence by Lemma 4.11(vi) there is a unique minimal $p \in \mathbb{N}$ with $pD \leqslant M$. Thus, $D/pD \cong \mathbb{Z}_p[X]$. If X is finite so is D/pD. If X is infinite and $x, y \in X$ are two different elements then (a special case of) Lemma 4.11(ii) implies that every Q-submodule of $M/pD \leqslant \mathbb{Z}_p[X]$ is generated by lone-pair-of-row matrices in $\{tx - ty \mid t \in \mathbb{Z}_p\}$, which is a finite subset of D/pD independent of M. Thus, in both cases we find that D/pD contains only finitely many Q-submodules.

The case when $M \leqslant E$ is similar: M cannot be in D; hence Lemma 4.11(iii) applies and asserts that there is a unique minimal $q \in \mathbb{N}$ with $qE \leqslant M$. If $\Gamma^k(S)$ is finite, so is E/qE. If $\Gamma^k(S)$ is infinite then (a special case of) Lemma 4.11 (ii) implies that for any chosen $\gamma \in \Gamma^k(S)$ all Q-submodules $M \leqslant E$ are generated by $E_\gamma \cap M$. Hence every submodule of E/qE is generated by elements in the finite set E_γ/qE_γ. Thus, in both cases we find that D/pD contains only finitely many Q-submodules.

If M is neither contained in D nor in E then $pD + qE \leqslant M$, and therefore $t(D + E) \leqslant M$ for $t := \gcd(p, q)$. In this situation we fix a germ $\gamma \in \Gamma^k(S)$ and consider the γ-row matrices $\mathbf{a} := (t, t, \dots, t)$, $\mathbf{b} := (t, -t, 0, \dots, 0)$, both elements of $D + E \leqslant M$, and the element $\vartheta \in Q$ which cyclically permutes the entries of the γ-rows. By observing that

$$\mathbf{a} - \mathbf{b}(1 + \vartheta + 2\vartheta^2 + 3\vartheta^3 + \cdots + k\vartheta^k) = (kt, 0, \dots, 0) \in t(D + E) \leqslant M$$

we can infer that $ktA^k(S) \leqslant D + E \leqslant M$. Hence it remains to show that $A^k(S)/ktA^k(S)$ contains only finitely many submodules. This is again obvious when $\Gamma^k(S)$ is finite, for in that case $A^k(S)/ktA^k(S)$ is a finite Abelian group. If $\Gamma^k(S)$ is infinite Lemma 4.11(ii) asserts that for any two different germs $\gamma, \gamma' \in \Gamma^k(S)$ we know that all submodules M of $\Gamma^k(S)$ are generated by $E_\gamma \cap M$ together with all lone pairs of (γ, γ')-rows in M. Modulo kt, this shows that all submodules of $A^k(S)/ktA^k(S)$ are generated by a subset of a finite set which depends only on D and E. Hence $A^k(S)/ktA^k(S)$ contains only finitely many submodules. □

The fact that $A^k(S)$ is generated by two $Q_k(S)$-orbits — unit-pei-translations and unit-endotranslations — shows that $A^k(S)$ is a finitely generated $Q_k(S)$-module. From Theorem 4.12 we infer that all submodules of $A^k(S)$ are finitely generated; in other words:

Corollary 4.13. *The $Q^k(S)$-module $A^k(S)$ is Noetherian.*

4.8 | The finite subgroups

Our next results will be used in Section 4.9 to classify all normal subgroups of pei(S). On the side it also yields all finite subgroups.

Lemma 4.14. *Let S be orthohedral, $L \subset S$ an orthant of rank-k, and $g \in G_k$. If g has the property that its image in G_k/G_{k-1} of finite order m then we can find a commensurable suborthant $K \subset L$ with the property that the sequence*

$$K \xrightarrow{g} Kg \xrightarrow{g} Kg^2 \xrightarrow{g} \cdots \xrightarrow{g} Kg^{|\Lambda|-1} \xrightarrow{g} Kg^{|\Lambda|} = K \tag{11}$$

goes through a set Λ of pairwise disjoint orthants representing the germs $\gamma(L)g^j$, $j \geqslant 0$, and ends with an isometry $\alpha := g^{|\Lambda|}|_K : K \to K$.

We call the sequence (11) the *covering orthant-orbit* of the germ-orbit $\gamma(L)gp(g)$.

Remark. This applies also in the case when $\gamma g = \gamma$ (and even when $\mathrm{rk}(g) < k$): Then the assertion is γ is represented by a rank-k orthant K pointwise fixed by g.

Proof. By Lemma 3.3 we can assume that the restriction of g to L is an isometric embedding. If Lg is not commensurable to L then the intersection $L \cap Lg^j$ is of smaller rank for all $1 \leqslant j < t =$ length of the length of the g-orbit of the germ of $L \in \Gamma^k$. Hence we find a commensurable suborthant $K \subset L$ with the property that $Kg^j \cap K = \emptyset$ for all $1 \leqslant j < t$. This implies that we have a sequence like the one asserted to exist in the lemma, except that it ends with an isometry $\alpha : K \to Kg^t$ onto a commensurable orthant Kg^t. But by assumption α' cannot be of infinite order, hence further powers g^J will, after finitely many steps, come back to K. Taking their intersections yields the claimed assertion. ☐

The following theorem extends the lemma from a single element g to a finitely generated subgroup H.

Theorem 4.15. *Let $H \leqslant G_k$ be a finitely generated subgroup whose rank-$(k-1)$ subgroup $N := H \cap G_{k-1}$ is of finite index in H. Then we find a set Λ of pairwise disjoint orthants representing the germs in $\Gamma^k(\mathrm{supp}(H))$ with the property that H acts on $S' := \bigcup_{L \in \Lambda} L$ by isometries (on and between the members of Λ) — with the understanding this includes the assertion that N fixes S' pointwise.*

Proof. The set of rank-k germs $\Gamma^k(\mathrm{supp}(H))$ is finite and permuted by H, and we can represent the germs $\gamma \in \Gamma^k(S)$ by pairwise disjoint rank-k orthants L_γ. Moreover, by passing, if necessary, to commensurable suborthants we may, for a given finite set $P \subseteq H$, assume that the restrictions $f|_{L_\gamma}$ are isometric for all $f \in P$ and all $\gamma \in \Gamma^k(S)$. The proof of Lemma 4.14 shows that this is true for a single element g, and the general case follows by induction on $|P|$: For the inductive step we can assume that the orthants $L\gamma$ we start with already satisfy the conclusion for a proper subset of P and then go through arguments in the proof of Lemma 4.14 for an additional element. Note also that each $n \in N \cap P$ will have the feature to act trivially on S'. Now we take advantage of this by applying it to a set P which we choose as follows: First we pick a transversal $T \subset H$ of H/N which contains the unit element of H and put $X := T^{\pm 1}$; then we consider all triple products $XXX \subset H$ and pick a finite subset $Y \subset N$ which contains the set $N \cap XXX$ and generates N as a monoid; finally we put $P := X \cup Y$ and note that P generates H as a monoid. Now we observe:

- As $Y \subset P$ we have $\mathrm{supp}(N) \cap L_\gamma = \emptyset$ (in particular $L_\gamma y = L_\gamma$) for all $\gamma \in \Gamma^k(S)$.
- As $T^{\pm 1} \subseteq P$ the translates $L_\gamma t$ are orthants commensurable to $L_{\gamma t}$ for all $t \in T$, and the $T_{\gamma t} t^{-1}$ are orthants commensurable to L_γ.
- For each $\gamma \in \Gamma^k(S)$ we now consider the intersection

$$K_\gamma := \bigcap_{t \in T} L_{\gamma t} t^{-1}.$$

  This is a finite intersection of orthants commensurable to L_γ and hence is a suborthant contained in and commensurable to L_γ. Thus $\Lambda := \{K_\gamma \mid \gamma \in \Gamma^k(S)\}$ is a pairwise disjoint set of representatives of the germs in $\Gamma^k(S)$ on which all restrictions of elements in P are isometric injections.

We aim to prove that the elements of P act on and/or permute the members of Λ by isometries. For the elements in Y we know this already. To prove it for $x \in X = T^{\pm 1}$ we note that for each pair $(t, x) \in T \times X$ there is a unique $s \in T$ with $n := t^{-1}xs \in N$. From here we find, on the one hand, $L_{\gamma t} t^{-1} x = L_{\gamma t}(t^{-1}x) = L_{\gamma t} n s^{-1} = L_{\gamma t} s^{-1}$ since $n =\in P$; and on the other hand, $\gamma t = \gamma x s n^{-1} = x s n^{-1}(xs)^{-1} xs = \gamma xs$ since $xsn^{-1}(xs)^{-1} \in N$ which acts trivially on the rank-k germs. Hence $L_{\gamma t} t^{-1} x = L_{\gamma xs} s^{-1}$, and we find

$$K_\gamma x = \left(\bigcap_{t \in T} L_{\gamma t} t^{-1} \right) x = \bigcap_{t \in T} L_{\gamma t}(t^{-1}x) = \bigcap_{t \in T} L_{\gamma xs} s^{-1} = \bigcap_{s \in T} L_{\gamma xs} s^{-1} = K_{\gamma x}.$$

This shows that the monoid generators of H and hence H itself acts on the union $S'' := \bigcup_{L \in \Lambda} L$ as asserted. Thus S has the H-invariant decomposition of S as the union of S'' and its complement $S' := S - S''$; and as S'' covers all rank-k germs of S its complement is of rank at most $(k-1)$. □

Corollary 4.16. *A subgroup $H \leqslant G_k$ is finite if and only if* $\mathrm{supp}(H)$ *is the union of a finite set Λ of pairwise disjoint orthants $\bigcup_{L \in \Lambda} L$ on which H acts faithfully by means of isometries on and between the members of Λ.*

4.9 | The normal subgroups of pei(S)

Throughout this section S is an orthohedral set of rank $rkS = n$, germs $\Gamma^k := \Gamma^k(S)$, and height $h(S) = |\Gamma^n|$; and $G := \mathrm{pei}(S)$. The most important normal subgroups of G we have met so far are the *rank subgroups*, and between them the (ordered and unordered) germ stabilizers $C^{\mathrm{ord}}(\Gamma^k) \leqslant C(\Gamma^k)$. But in addition to those we found also the stagnant subgroups $ST_k \leqslant C(\Gamma^k)$ (with $C^{\mathrm{ord}}(\Gamma^k) ST_k = C(\Gamma^k)$, see (4)), the endotranslation subgroup $E_k := ST_k \cap C^{\mathrm{ord}}(\Gamma^k)$ (see the

Exercise at the end of Section 4.5), and the alternating subgroups alt G_k (see Theorem 4.10). The lattice of these normal subgroups is exhibited in the diagram

$$\begin{array}{ccc} & \leqslant \quad C^{\mathrm{ord}}(\Gamma^k) \quad \leqslant & \\ G_{k-1} \leqslant E_k & & C(\Gamma^k) \leqslant \mathrm{alt}\, G_k \leqslant G_k. \\ & \leqslant \mathrm{alt}\, ST_k \leqslant ST_k \leqslant & \end{array} \tag{12}$$

The G-module structure of $A^k(S) = G_k/G_{k-1}$ exhibited in Section 4.7 provides detailed information on the normal subgroups between G_{k-1} and C^{ord}. The goal is now to show that what we have seen so far covers essentially all normal subgroups of G.

To prove this requires detailed information on the normal subgroups $gp_G(g)$ of G generated by specific elements $g \in G_k$, $k \leqslant n$, and we start by investigating the special case when the canonical image of g in G_k/G_{k-1} is of positive finite order m. In this case Lemma 4.14 describes the covering orthant-orbit Λ of a given g-orbit of Γ^k, and how g acts on the union $\bigcup \Lambda$ by isometries on and between its members $L \in \Lambda$.

Lemma 4.17. *Let $g \in G_k$, with $k \leqslant n = rkS$, be an element whose image in G_k/G_{k-1} is of finite order. Then the following holds (but note that if $k = n$ then $|\Gamma^k|$ is finite, and if $k < n$ then $|\Gamma^k| = \infty$ and only assertion* (v) *is relevant).*

(i) $\mathrm{rk}(g) = k$ *alone implies already that $gp_G(g)$ contains* alt G_{k-1}.
(ii) If g acts non-trivially on Γ^k then $gp_G(g)$ contains in addition to alt G_{k-1} certain *products of pairs of disjoint rank-k elements of the form $\varphi\varphi^g$, where $\varphi \in G_k$ is a single-orthant-reflection or an endotranslation of rank-k.*
(iii) If g acts non-trivially on Γ^k and $|\Gamma^k| \geqslant 3$ then $gp_G(g)$ contains alt ST_k *(which includes G_{k-1} and E_k). In addition we have: $gp_G(g)$ contains either an orthant-3-cycles of rank-k, or the product of a pair of disjoint orthant-transpositions. (In the second case we have $|\Gamma^k| \geqslant 4$ and further consequences below apply).*
(iv) If g acts non-trivially on Γ^k and $|\Gamma^k| \geqslant 4$ then $gp_G(g)$ contains

$$C^{\mathrm{ord}}(\Gamma^k)^2 \mathrm{alt}\, ST_k$$

which is a subgroup of finite index in G. If g acts on Γ^k by a 3-cycle or by a single transposition then $gp_G(g)$ contains the commutator subgroup G'.
(v) If g acts non-trivially on Γ^k and $|\Gamma^k| \geqslant 5$ then $gp_G(g)$ contains alt G_k.

Proof. (i) Let $L \in \Lambda$ be an orthant contained in $\mathrm{supp}(g)$. Then — regardless of whether $Lg = L$ or $Lg \neq L$ — there is a rank-$(k-1)$ face F of L with $F \neq Fg$. We claim that one can choose an orthant-transposition $\tau = (K, K')$ of rank-$(k-1)$ supported on suborthants of L parallel to F and of distance 1 to each other, with the feature that τ and τg are disjoint. Indeed, if $Lg \neq L$ then taking $K := F$ and K' its parallel neighbor of distance 1 will do; and if $Lg = L$ then we can take K to be the orthant obtained by shifting F diagonally into itself by two diagonal units, and K' its parallel neighbor of distance 1 (which has distance > 1 from all other faces). Due to Lemma 3.1 $\tau \in G_k$. Then the commutator $[\tau, g] = \tau\tau^g$ is the product of two disjoint rank-$(k-1)$ orthant-transpositions and contained in $gp_{G_k}(g)$. As $|\Gamma^{k-1}| = \infty$ we know from Theorem 4.10(iii) that the conjugates of $[\tau, g]$ generate alt G_{k-1}.

(ii) By assumption Γ^k has a g-orbit of length at least 2, and we consider the corresponding covering orthant-orbit Λ. Let $L \in \Lambda$, $K \subseteq L$ an arbitrary rank-k suborthant, and σ a single-orthant-reflection of K. Due to Lemma 4.6 $\sigma \in G_k$, hence $[g,\sigma] = \sigma^g \sigma \in gp_{G_k}(g)$, and $K \cap Kg = \emptyset$. Thus, $gp_{G_k}(g)$ contains a product of two disjoint g-conjugate single-orthant-reflections of rank-k.

We can apply this to the suborthant $Kt_y \subset K$ and recall from Lemma 4.7(v) that $\eta = \sigma\sigma^{t_y}$ is a unit-endotranslation. As both $\sigma^g\sigma$ and ${\sigma^{t_y}}^g\sigma^{t_y}$ are in $gp_{G_k}(g)$, so is their product $(\sigma^g\sigma)({\sigma^{t_y}}^g\sigma^{t_y}) = \eta^g\eta$. This shows that $gp_G(g)$ also contains products of pairs of disjoint g-conjugate (unit)-endotranslations.

(iii) We assume that g acts non-trivially on Γ^k and $|\Gamma^k| \geqslant 3$. Three cases occur:

Case 1: Γ^k contains a g-orbit of length $\geqslant 3$. Let Λ be the corresponding covering orthant-orbit, and $K \subseteq L$ an arbitrary rank-k suborthant of some $L \in \Lambda$. Then K, Kg, Kg^2 are pairwise disjoint. We put $\tau := (K, Kg)$ to be the orthant-transposition defined by the restriction $g|_K : K \to Kg$, and observe that the commutator $[g,\tau] = \tau^g\tau = g^{-1}g^\tau$ is contained in $gp_G(g)$ and is the orthant-3-cycle

$$K \xrightarrow{g} Kg \xrightarrow{g} Kg^2 \xrightarrow{g^{-2}} K.$$

Case 2: Γ^k contains a g-orbit of length 2 (with covering orthant-orbit $L \xrightarrow{g} Lg \xrightarrow{g} L$), and disjoint to it is a g-invariant rank-k orthant L' (with covering orthant-orbit $L' \xrightarrow{g} L'$). We choose arbitrary rank-k suborthants $K \subseteq L$, $K' \subseteq L'$, pick an orthant-transposition $\tau := (K', K)$, and observe that the commutator $[g,\tau] = \tau^g\tau = g^{-1}g^\tau$ is contained in $gp_G(g)$ and is the orthant-3-cycle

$$K \xrightarrow{\tau} K' \xrightarrow{\tau^g} Kg \xrightarrow{\tau^g\tau} K.$$

Appropriate products of two orthant-3-cycles are products of pairs of disjoint orthant-transpositions. By Theorem 4.10(i)(b) all products of pairs of disjoint orthant-transpositions are conjugate and generate alt ST_k. Hence $gp_G(g)$ contains the unique subgroup of index 2 in $C(\Gamma^k)$ and all alternating finite orthant-permutations; together these generate the commutator subgroup of G_k.

Case 3: Γ^k contains two g-orbit of length 2 (with corresponding covering orthant-orbits $L_i \xrightarrow{g} L_ig \xrightarrow{g} L_i$, i=1, 2). We choose arbitrary rank-k suborthants $K_1 \subseteq L_1$, $K_2 \subseteq L_2$, pick an orthant-transposition $\tau := (K_1, K_2)$, and observe that the commutator $[g,\tau] = \tau^g\tau = g^{-1}g^\tau$ is contained in $gp_G(g)$ and is the product $\tau_1\tau_2$ of two disjoint g-conjugate orthant-transpositions. By Theorem 4.10(i)(b) all products of pairs of disjoint orthant-transpositions are conjugate and generate alt ST_k.

(iv) Now we assume $|\Gamma^k| \geqslant 4$. By Theorem 4.10(ii) we know that all orthant-3-cycles are conjugate. Thus, if $gp_G(g)$ contains an orthant-3-cycle of rank-k then $|\Gamma^k| \geqslant 4$ implies that it contains all of them. We claim that this implies $C^{\text{ord}}(\Gamma^k) \leqslant gp_G(g)$.

To prove this recall that every unit-pei-translation λ is the product of two orthant-transpositions $\lambda = \tau\tau'$ of the form $(K,L)(K,L-F)$ (see the proof of Lemma 4.7(ii)). As an additional rank-k orthant M disjoint to $K \cup L$ is available we can write $\tau\tau'$ as the product $\lambda = (K,L)(K,L-F) = (K,L)(K,M)(K,M)(K,L-F) = (K,L,M)(K,M,L-F)$ of two orthant-3-cycles, when $\lambda \in gp_G(g)$. Our claim follows since the unit-pei-translations generate $C^{\text{ord}}(\Gamma^k)$.

By (iii) we know that $gp_G(g)$ contains also alt ST_k, and together with $C^{\mathrm{ord}}(\Gamma^k)$ this yields the unique subgroup of index 2 in $C(\Gamma^k)$. Moreover the orthant-3-cycles generate, in the symmetric group $G/C^{\mathrm{ord}}(\Gamma^k)$, the alternating subgroup of index 2. Hence $gp_G(g)$ contains a normal subgroup of index 4; this can only be the commutator subgroup G'.

It remains to consider the case $gp_G(g)$ contains no orthant-3-cycle — this happens only when $|\Gamma^k| = 4$ (which implies that $k = n = \mathrm{rk}(S)$) and Γ^k is the union of two g-orbits of length 2. In that case assertion (iii) still tells us that $gp_G(g)$ contains all of E_k and products $\tau_1\tau_2$ of pairs of disjoint orthant-transpositions of rank-k.

From pairs of disjoint orthant-transpositions we can obtain the result for pairs of disjoint translations: We use our disjoint orthant-transpositions $\tau_i = (K_i, L_i)$ to construct the disjoint unit-pei-translations $\lambda_j = (K_i, L_i)(K_i, L_i - F_i)$, where F_i stands for a rank-$(k-1)$-face of L_i. Then we have $\lambda_1\lambda_2 = (K_1, L_1)(K_2, L_2)(K_1, L_1 - F_1)(K_2, L_2 - F_2)$ which shows that $\lambda_1\lambda_2 \in gp_G(g)$. It follows that all products of pairs of disjoint unit-pei-translations are in $gp_G(g)$. By multiplying two appropriate such pairs we find that $gp_G(g)$ contains all squares of unit-pei-translations and therefore all translations of even length, that is, $C^{\mathrm{ord}}(\Gamma^k)^2 \leqslant gp_G(g)$. Hence $gp_G(g)$ contains $C^{\mathrm{ord}}(\Gamma^k)^2 \mathrm{alt}\, ST_k$ and all products of disjoint pairs of orthant-transpositions. We leave it to the reader to prove that this is a finite index subgroup of G_k is a subgroup of finite index in G_k.

(v) We assume that g acts non-trivially on Γ^k and $|\Gamma^k| \geqslant 5$. We know by assertion (iii) that $gp_G(g)$ contains either an orthant-3-cycle or a product of two disjoint orthant-transpositions. As $(K, L, M)(L, M, N) = (K, M)(L, N)$ we have products of disjoint orthant-transpositions in either case and we know, by Theorem 4.10, that they generate alt $G_k \leqslant G_k$ as a normal subgroup. □

We can now prove that the index-2 pairs alt $G_k < G_k$ are 'bottlenecks' for the normal subgroups N of G, that is, either $N \leqslant G_k$ or alt $G_k \leqslant N$. Or, equivalently:

Theorem 4.18 (Bottleneck Theorem). *For every normal subgroup $N \leqslant G$ of rank* $\mathrm{rk}(N) = k$ *we have* alt $G_{k-1} \leqslant N \leqslant G_k$; *(recall that $G_{-1} := 1$).*

Proof. The key here is proving that the assertion i) of Lemma 4.17, that is, $\mathrm{rk}(g) = k$ alone implies alt $G_{k-1} \leqslant gp_G(g)$, holds without the assumption that the image of $g \in G_k$ in G_k/G_{k-1} be of finite order. To prove this, we can now assume that g is of infinite order.

As $\Gamma^k(\mathrm{supp}(g))$ is finite and $G_k/C^{\mathrm{ord}}(\Gamma^k)$ is a torsion group, some power g^p is a non-trivial pei-isometry of rank-k which fixes the germ $\gamma(L)$ of some rank-k orthant L. Hence g^p parallel shifts t L to an orthant $Lg^p \neq L$ commensurable to L. Then one finds inside L rank-$(k-1)$ othants $K \subset L$ with the property that $K, Kg^p, Kg^{2p}, Kg^{3p}, \ldots$ are sequences of length $\geqslant 3$ of pairwise disjoint parallel orthants. As in the proof of Lemma 4.17(iii), Case 1. We find in $gp_G(g)$ an orthant-3-cycle of rank-$(k-1)$. Since $|\Gamma^{k-1}| = \infty$ assertion v) of Lemma 4.17 applies in rank-$(k-1)$ and yields alt $G_{k-1} \leqslant gp_G(g)$. □

We use the Bottleneck theorem to recover the rank of elements $g \in G = \mathrm{pei}(S)$ as a group theoretic property. For this we introduce the *translation-rank* of the elements $g \in G$, by putting

$$\mathrm{trk}(g) := min_{0 \leqslant k \leqslant \mathrm{rk}(g)}\{k \mid G_k \cap gp(g) \neq 1\}.$$

Note that $\mathrm{trk}(g) \leqslant \mathrm{rk}(g)$, and $\mathrm{trk}(g) = 0$ if and only if g is a torsion element. And if g is torsion-free then $trk(g)$ is the maximal $t \in \mathbb{N}$ with the property that $\mathrm{supp}(g)$ contains a rank-t orthant L on which the restriction $g^p\,|_L : L \to S$ of some $g^p \in gp(g)$ is also induced by a non-trivial translation $\vartheta : \langle L \rangle \to \langle L \rangle$.

Now we put $t = trk(g)$, choose a generator g^q of $G_k \cap gp(g)$ and consider the sequence of normal subgroups $N_p := gp_G(g^p)$, $p \in \mathbb{N}$. As $t \leqslant rk(g^p)$ for all $p \in \mathbb{N}$ we know from the Bottleneck theorem that $\operatorname{alt} G_{t-1} \leqslant N_p$ for all $p \in \mathbb{N}$. On the other hand, as $G_t/C^{\mathrm{ord}}(\Gamma^t)$ is a torsion group and $g^q \in G_t$ we know some power g^p of g is contained in $C^{\mathrm{ord}}(\Gamma^t)$. Since $C^{\mathrm{ord}}(\Gamma^t)$ is a characteristic subgroup of G it follows that $N_p = gp_G(g^p)$ is also contained in $C^{\mathrm{ord}}(\Gamma^t)$.

Now we can use that $C^{\mathrm{ord}}(\Gamma^t)/G_{t-1}$ is free-Abelian and hence residually finite: the subgroup N_p/G_{t-1} is generated by the G-translates of g^p/G_{t-1}, hence for each $i \in \mathbb{N}$, we have $N_{pi}/G_{t-1} = gp_G(g^{pi}/G_{t-1}) = (N_p/G_{t-1})^i$, when $\bigcap_{i\in\mathbb{N}} N_{pi} \leqslant G_{t-1}$.

Putting things together we find that the intersection of all normal subgroups N_p (a quantity that depends only on the group structure of G) satisfies, for each $g \in G$,

$$\operatorname{alt} G_{\operatorname{trk}(g)-1} \leqslant \bigcap_{p\in\mathbb{N}} gp_G(g^p) \leqslant G_{\operatorname{trk}(g)-1}.$$

Since $\operatorname{alt} G_{t-1}$ and G_{t-1} uniquely determine one another this shows that they are characterized in terms of the group structure of $G = \operatorname{pei}(S)$.

Summarizing we have

Corollary 4.19. *$G = \operatorname{pei}(S)$ satisfies the maximal condition for normal subgroups.*

The rank-groups G_k are uniquely determined by the group structure of $\operatorname{pei}(S)$. In particular, the poly-(Abelian-by-locally-finite) length of G which is equal to $\operatorname{rk} S + 1$, is an invariant of the group structure.

Exercise. In [32, 36] Osin and Wesolek–Williams define fine-meshed (ordinal valued) ranks which measure the complexity of elementary amenable groups G. Use the Bottleneck theorem to compute these ranks for $G = \operatorname{pei}(S)$.

CHAPTER 4. THE EUCLIDEAN CASE II: THE FINITENESS LENGTH

5 | A LOWER BOUND FOR THE FINITENESS LENGTH OF pei(S)

In this section we will define a certain 'diagonal' subgroup, $\operatorname{pei}_{\mathrm{dia}}(S) \leqslant \operatorname{pei}(S)$, and prove

Theorem 5.1. *For every orthohedral set S we have*

$$fl(\operatorname{pei}(S)) \geqslant fl(\operatorname{pei}_{\mathrm{dia}}(S)) = h(S) - 1.$$

We follow the strategy of Brown's proof in the influential paper [14] which covers the case when S is a stack of rays; and we also take full advantage of the technical results and the insight provided by the second author's diploma thesis [34].

5.1 | The height of a pei-injection $f : S \to S$

We start with a general observation on the set of germs, when an orthohedral set S comes with a decomposition of a disjoint union $S = A \cup B$ of two orthohedral subsets. In that case every orthant

$L \subseteq S$ inherits the decomposition $L = (A \cap L) \cup (B \cap L)$, which shows that one of the orthants of either A or B is commensurable to L. This shows that the germs of S have an induced disjoint decomposition $\Gamma^k(S) = \Gamma^k(A) \cup \Gamma^k(B)$ for each k.

Now let S be an orthohedral set of rank $\operatorname{rk} S = n$. We can represent the rank-n germs of S by pairwise disjoint orthants $L_1, \dots, L_h$, $h = h(S)$, with the property that the restriction of f to each L_i is an isometric embedding into S. $f(L_i)$ is then commensurable to some L_j, and since f is injective it follows: f permutes the germs $\gamma(L_1), \dots, \gamma(L_p)$, and $\operatorname{rk}(S - f(S)) < \operatorname{rk} S$.

As $S - f(S)$ is an orthohedral set, we now obtain that the number of rank-$(n-1)$ germs in $S - f(S)$ is finite. We call this number *the height of* f, denoted by $h(f) = h(S - f(S)) = h(S - Sf)$.

Lemma 5.2.

(i) *If $g, f : S \to S$ are two pei-injections, then $h(gf) = h(g) + h(f)$.*
(ii) *If $A \subseteq S$ is an orthohedral subset whose complement $A^{\mathrm{c}} = S - A$ has rank $\operatorname{rk} A^{\mathrm{c}} < n = \operatorname{rk} S$, then the height of any pei-injection $f : S \to S$ is given by $h(f) = h(A \cap f(A)^{\mathrm{c}}) - h(A^{\mathrm{c}} \cap f(A))$.*

Proof. (i) Consider the disjoint union $S = (S - Sg) \cup Sg$. As f is injective $Sf = (Sf - Sgf) \cup Sgf$ is also a disjoint union. Hence so is

$$S = (S - Sf) \cup Sf = (S - Sf) \cup (Sf - Sgf) \cup Sgf,$$

and we find

$$S - Sgf = (S - Sf) \cup (Sf - Sgf).$$

Now, f is a pei-bijection between $(S - Sg)$ and $(S - Sg)f = (Sf - Sgf)$; and a pei-bijection of a an orthohedral set induces a pei-bijection on its germs. Thus the number of rank-$(n-1)$ germs of $(S - Sg)$ and $(Sf - Sgf)$ are the same. This proves (i).

(ii) Each pei-injection $f : S \to S$ induces an injection $f^* : \Gamma^{n-1}(S) \to \Gamma^{n-1}(S)$. We abbreviate $B = A^{\mathrm{c}}$ and know from $\operatorname{rk} B < n$ that $\Gamma^{n-1}(B)$ is finite. Hence f^* restricts to a bijection $f^* : \Gamma^{n-1}(B) \to \Gamma^{n-1}(f(B))$. On the complement we find the induced injection $f^* : \Gamma^{n-1}(A) \to \Gamma^{n-1}(f(A))$.

We use the abbreviation $P^* := \Gamma^{n-1}(P)$ for $P = S, A, B$, and consider the disjoint union

$$\begin{aligned} S^* - f^*(S^*) &= (A^* - A^* \cap f^*(S^*)) \cup (B^* - B^* \cap f^*(S^*)) \\ &= (A^* - A^* \cap f^*(A^*) - A^* \cap f^*(B^*)) \\ &\quad \cup (B^* - B^* \cap f^*(A^*) - B^* \cap f^*(B^*)). \end{aligned}$$

By definition, $h(f) = h(S^* - f^*(S^*))$. Using the fact that B^* as well as $A^* - f^*(A^*)$ are finite, we find

$$\begin{aligned} h(f) &= h(A^* - A^* \cap f^*(A^*)) - h(A^* \cap f^*(B^*)) \\ &\quad + h(B^*) - h(B^* \cap f^*(A^*)) - h(B^* \cap f^*(B^*)). \end{aligned}$$

Now we apply that $h(B^*) = h(f(B^*))$ and observe that

$$\begin{aligned} -h(A^* \cap f^*(B^*)) + h(B^*) - h(B^* \cap f^*(B^*)) \\ &= -h(A^* \cap f^*(B^*)) + h(f(B^*)) - h(B^* \cap f^*(B^*)) \\ &= h(f(B^*) - h((A^* \cap f^*(B^*)) \cup (B^* \cap f^*(B^*))) = 0. \end{aligned}$$

Hence our expression for $h(f)$ simplifies to

$$\begin{aligned} h(f) &= h(A^* - A^* \cap f^*(A^*)) - h(B^* \cap f^*(A^*)) \\ &= h(A^* \cap f^*(A^*)^{\mathrm{c}}) - h(B^* \cap f^*(A^*)) \end{aligned}$$

as asserted. □

5.2 | Monoids of pei-injections

Let S be an orthohedral set of rank $n = \operatorname{rk} S$ in pet-normal form. In particular S is the pairwise disjoint union of finitely many specified stacks of orthants. By Lemma 3.2 the set of all maximal germs of S, $\max \Gamma^*(S)$, is finite. We write $M(S)$ for the monoid of all pei-injections $S \to S$. It is endowed with the height function $h : M(S) \to \mathbb{N}$ of Section 5.1. Let $M_0(S)$ be the submonoid of all pei-endoinjections of S, which fix all maximal germs of S. $M_0(S)$ is of finite index in $M(S)$ since $\max \Gamma^*(S)$ is finite. Just as we have observed for pei-permutations, each $f \in M_0$ induces an isometry $\tau_{(f,\gamma)} : \langle\gamma\rangle \to \langle\gamma\rangle$ on the tangent coset of each germ $\gamma \in \max \Gamma^*(S)$. Thus we have a homomorphism

(5.1) $$\kappa : M_0(S) \twoheadrightarrow \bigoplus\nolimits_{\gamma\in\max\Gamma^*(S)} \operatorname{Isom}(\langle\gamma\rangle), \text{ given by}$$

$$\kappa(f) = \bigoplus_{\gamma\in\max\Gamma^*(S)} \tau_{(f,\gamma)}.$$

The *translation submonoid* $M_{\mathrm{tr}}(S) \subseteq M_0(S)$ consists of all $f \in M_0(S)$ with the property that the induced maps $\tau_{(f,\gamma)} : \langle\gamma\rangle \to \langle\gamma\rangle$ are translations for each $\gamma \in \max \Gamma^*(S)$. Since the translation subgroup of $\operatorname{Isom}(\langle\gamma\rangle)$ is of finite index, $M_{\mathrm{tr}}(S)$ has finite index in $M_0(S)$. And restricting (5.1) yields a surjective homomorphism

(5.2) $$\kappa : M_{\mathrm{tr}}(S) \twoheadrightarrow \bigoplus_{\gamma\in\max\Gamma^*(S)} \mathbb{Z}^{\operatorname{rk}(\gamma)} = \mathbb{Z}^N,$$

with $N = \Sigma_{\gamma\in\max\Gamma^*(S)} \operatorname{rk}(\gamma)$.

Every orthant L contains a characteristic *diagonal element* $u_L \in L$: the sum of the canonical basis of L. We write $t_L : L \to L$ for the translation given by addition of u_L and call this the *diagonal unit-translation* of L. The general diagonal translations on L (that is, on $\langle L\rangle$) are given by addition of an integral multiple of u_L. By the *diagonal submonoid* $M_{\mathrm{dia}}(S) \subseteq M_{\mathrm{tr}}(S)$ we mean the set of all elements $f \in M_{\mathrm{tr}}(S)$ with the property that for each $\gamma \in \max \Gamma^*(S)$ the induced isometry $\tau_{(f,\gamma)} : \langle\gamma\rangle \to \langle\gamma\rangle$ is a diagonal translation. Restricting (5.2) yields the homomorphism

(5.3) $$\kappa : M_{\mathrm{dia}}(S) \twoheadrightarrow \bigoplus\nolimits_{\gamma\in\max\Gamma^*(S)} \mathbb{Z} = \mathbb{Z}^{|\max\Gamma^*(S)|}.$$

We write $\max \Omega^*(S)$ for the set of all maximal orthants of the stacks of S, and consider the set $T = \{t_L \mid L \in \max \Omega^*(S)\}$ of all diagonal unit-translations of these orthants. Each $t_L \in T$ extends canonically to a pei-injection on $t_L : S \to S$, which is the identity on $S - L$. We denote it by the same symbol t_L, and with this interpretation T generates a free-Abelian submonoid $\mathrm{mon}(T) \leqslant M_{\mathrm{dia}}(S)$.

Following the strategy of [14] we put

Definition 5.3. Given $f, f' \in M_{\mathrm{dia}}(S)$ we define $f \leqslant f'$ if there is some $t \in \mathrm{mon(T)}$ with $tf = f'$.

Observation. $M_{\mathrm{dia}}(S)$ is a *directed partially ordered set.*

It is an important fact that the height function $h : M(S) \to \mathbb{N}$ is order preserving and its restrictions to totally ordered subsets of $M(S)$ are injective. We will also have to consider slices of $M_{\mathrm{dia}}(S)$. For given $r, s \in \mathbb{N}_0$, $r \leqslant s$ we put

$$M^{[r,s]} := \{f \in M_{\mathrm{dia}}(S) \mid r \leqslant h(f) \leqslant s\} \text{ and}$$
$$M^{[r,\infty]} := \{f \in M_{\mathrm{dia}}(S) \mid r \leqslant h(f)\}.$$

$M^{[r,\infty]}$ inherits the partial ordering from $M_{\mathrm{dia}}(S)$ and is also a directed set.

5.3 | Maximal elements less than f in $M_{\mathrm{dia}}(S)$

From now on we assume that all maximal orthants of the stacks of S have the same finite rank $n = \mathrm{rk}\, S$. We put $\Lambda := \max \Omega^*(S)$. We write

$$M_{<f} = \{a \in M_{\mathrm{dia}}(S) \mid a < f\}, \quad M_{\leqslant f} = \{a \in M_{\mathrm{dia}}(S) \mid a \leqslant f\}$$

for the ‘open, respectively, closed cones below f” and aim to understand the set of all maximal elements of $M_{<f}$. For this it will be convenient to introduce an abbreviation for the points on the (finite) boundary of the maximal orthants L; so we set $\partial L := L - Lt_L$.

Lemma 5.4. *Let b be a maximal element of $M_{<f}$. Then there is a unique maximal orthant $L \in \Lambda$ with the property that $f = t_L b$ and $h(f) = h(b) + n$. Furthermore b is given as the union $b = b' \cup b''$, where $b' : \partial L \to (S - Sf)$ is a pei-injection, and $b'' : (S - \partial L) \to Sf$ is the restriction $(t_L^{-1} f) \mid_{(S-\partial L)}$. Conversely, if $c' : \partial L \to (S - Sf)$ is an arbitrary pei-injection distinct from b', then the union $c = c' \cup b''$ is a maximal element of $M_{<f}$ distinct from b.*

Proof. For each element $b \in M_{<f}$ there is some $t \in \mathrm{mon}(T)$ with $f = tb$. t has a unique reduced expansion as a product of elements of T; let $l(t)$ denote the length of this expansion. As $h(t_L) = L = n$ for each $L \in \Lambda$ we have $h(t) = n\, l(t)$. It follows that if b is maximal, then $h(t) = n$ and $t = t_L \in T$ for some $L \in \Lambda$. The maximal orthant L is uniquely determined by the fact that the restriction of f and b coincide on $S - L$. The restriction b'' of b to $(S - \partial L)$ coincides with $(t_L^{-1} f) \mid_{(S-\partial L)}$, and has its image in Sf. The restriction b' of b to ∂L is not determined by f and L. As b and f are

injective we know that

$$\begin{aligned}\emptyset = (\partial L)b \cap (S - \partial L)b &= (\partial L)b \cap ((S - L)b \cup Lt_L b) \\ &= (\partial L)b \cap ((S - L)f \cup Lf) \\ &= (\partial L)b \cap Sf.\end{aligned}$$

Hence b' can be viewed as a pei-injection $\partial L \to (S - Sf)$. If we replace b' by another pei-map $c' : \partial L \to (S - Sf)$, the union $c = c' \cup b''$ will still satisfy $f = t_L c$ and $h(f) = h(c) + n$. This shows that c will also be maximal in $M_{<f}$. □

Lemma 5.5. *Let $B \subseteq M_{<f}$ be a finite set of maximal elements of $M_{<f}$. Then the following conditions are equivalent:*

(i) the elements of B have a common lower bound δ in $M_{<f}$;
(ii) for every pair $(b, b') \in B \times B$ with $b \neq b'$ and $tb = f = t'b'$ for diagonal unit-translations t, t', we have
  (a) $t \neq t'$ and
  (b) $b(\partial L) \cap b'(\partial L') = \emptyset$, where L, respectively, L' are the maximal orthants of S on which t, respectively, t' acts non-trivially.

Proof. (i)⇒(ii). Let δ be a common lower bound of the elements of B. Then for every pair $(b, b') \in B \times B$ there are diagonal translations $d, d' \in \mathrm{mon}(T)$ with $d\delta = b$ and $d'\delta = b'$. From $tb = f = t'b'$ we obtain $td\delta = t'd'\delta$ and conclude $td = t'd'$. The assumption $t = t'$ would now imply $d = d'$ and hence $b = b'$.

Let L, respectively, L' denote the maximal orthants of S on which t, respectively, t' acts non-trivially. As d, d' are diagonal translations, we have $d(L) \subseteq L$ and $d'(L') \subseteq L'$. From $t \neq t'$ we know $L \cap L' = \emptyset$, and hence $(\partial L)d \cap (\partial L')d' = \emptyset$. Since δ is injective, this implies $\emptyset = (\partial L)d\delta \cap (\partial L')d'\delta = (\partial L)b \cap (\partial L')b'$, as asserted.

(ii)⇒ (i). For each $b \in B$ we have some diagonal unit-translation $t_b \in T$ with $t_b b = f$, and we put

$$t_B := \prod\nolimits_{b \in B} t_b. \tag{5.4}$$

By assumption (i) the maximal orthants L_b on which t_b is a diagonal unit-translation are pairwise disjoint. Thus $|B| \leqslant h(S)$, and S decomposes in the disjoint union $S = (\bigcup_{b \in B} L_b) \cup S'$. We define the pei-injection $\delta_B : S \to S$ as follows:

$$\delta_B := \begin{cases} t_b^{-1} f & \text{on each } L_b t_b \\ b & \text{on the complements } \partial L_b = L_b - L_b t_b \\ f & \text{on } S'. \end{cases}$$

Assumption (ii) guarantees that the restriction of δ_B to the union

$$\bigcup_{b \in B} \partial L_b = \bigcup_{b \in B} (L_b - L_b t_b) = (S - St_B)$$

is injective. And since the image of each ∂L_b is disjoint to $f(S)$, we also find that the image of $(S - St_B)$ is disjoint to $f(S)$, and also to $f(S') \subseteq f(S)$. This shows that δ is a pei-injection. It remains to prove that δ_B is a common lower bound for the elements of B. By commutativity we find elements $s_b \in$ mon(T) with $t_B\delta_B = t_b s_b \delta_B$, where

$$s_b = \prod\nolimits_{x\in(B-\{b\})} t_x. \tag{5.5}$$

One observes that $s_b\delta_B$ and b agree on $(S - St_b) = (L_b - L_b t_b)$, and that $t_B\delta_B = f = t_b b$. Hence b and $s_b\delta_B$ agree on S. □

Lemma 5.6. *In the situation of Lemma 5.5 we have for the lower bound δ_B defined in the proof:*

(i) δ_B is, in fact, a largest common lower bound of the elements of B;
(ii) $h(\delta_B) \geqslant h(f) - h(S)n$.

Proof. (i) We compare an arbitrary common lower bound γ with δ_B, the lower bound constructed in the proof above. Thus for each $b \in B$ we are given $u_b \in$ mon(T) with $u_b\gamma = b$. We fix a base element $b' \in B$ and define the diagonal translation $t' \in$ mon(T) by its action on S as

$$t' := \begin{cases} u_{b'} & \text{on } S' \\ u_b & \text{on each } L_b. \end{cases}$$

We use the elements s_b of (5.5) and observe that

$$\begin{aligned} xt'\gamma &= xu_{b'}\gamma = xb' = xs_{b'}\delta_B = x\delta_B \quad \text{for } x \in S' \\ xt'\gamma &= xu_b\gamma = xb = xs_b\delta_B = x\delta_B \quad \text{for } x \in L_b. \end{aligned}$$

This shows that $t'\gamma = \delta_B$, hence $\delta_B \geqslant \gamma$.

(ii) For the translation t of (5.4) we have $t\delta_B = f$ and can deduce that $h(\delta_B) = h(f) - h(t) = h(f) - |B|n \geq h(f) - h(S)n$. □

5.4 | The simplicial complex of $M_{\mathrm{dia}(S)}$

We consider the simplicial complex $|M_{\mathrm{dia}}(S)|$, whose vertices are the elements of $M_{\mathrm{dia}}(S)$ and whose chains of length $k, a_0 < a_1 < \cdots < a_k$, are the k-simplices. As the partial ordering on $M_{\mathrm{dia}}(S)$ is directed, $|M_{\mathrm{dia}}(S)|$ is contractible.

In this section we aim to prove

Lemma 5.7. *If $h(f) \geqslant 2 \cdot \mathrm{rk}\, S \cdot h(S)$, then $|M_{<f}|$ has the homotopy type of a bouquet of $(h(S) - 1)$-spheres.*

The first step toward proving Lemma 5.7 is to consider the covering of $|M_{<f}|$ by the subcomplexes $|M_{\leqslant b}|$, where b runs through the maximal elements of $M_{<f}$. We write $N(f)$ for the nerve of this covering. Lemmas 5.5 and 5.6 show that all finite intersections of such subcomplexes $|M_{\leqslant b}|$ are again cones and hence contractible. It is a well-known fact that in this situation the space is

homotopy equivalent to the nerve of the covering. Hence we have

$$|M_{<f}| \text{ is homotopy equivalent to the nerve } N(f)$$

and it remains to compute the homotopy type of $N(f)$.

The next step is to use the results of Section 5.3 to find a combinatorial model for the nerve $N(f)$. The vertices of $N(f)$ are the maximal elements of $M_{<f}$, and hence, by Lemma 5.4, in 1–1-correspondence to the disjoint union $A = \bigcup_{L\in\Lambda} A_L$, where A_L stands for the set of all pei-injections $a : \partial L \to (S - Sf)$. Lemma 5.4 allows to translate the simplicial structure of $N(f)$ into a simplicial complex $\Sigma(f)$ on A: the p-simplices of $N(f)$ are the p-element sets of maximal elements $B \subseteq M_{<f}$ with a common lower bound, and the corresponding p-simplices of $\Sigma(f)$ are the sequences $(a_L)_{L\in\Lambda'}$, where Λ' is a p-element subset of Λ and $a_L \in A_L$ with the property

(∗) *The intersections of the images $a_L(\partial L), L \in \Lambda'$, are pairwise disjoint.*

The next Lemma 5.8 on colored graphs will enable us to determine the homotopy type of $\Sigma(f)$. This is a natural generalization of the second author's (Sach's) Lemma 4.7 [34], where it served as a major technical key in extending computation of *fl*(pet(S)) from the case when S is a stack of rays (the Houghton group result of [14], to the case when S is a stack of quadrants. Sach's lemma and its proof were based on but are in parts rather different from Brown's lemmas 5.2 or 5.3. Ken Brown, in turn, remarks that the inductive proof of his Lemma 5.3 uses 'a method due to K. Vogtmann [private communication]'.

Let $\Gamma = (V, E)$ be a combinatorial graph, given by a set V of *vertices* and set E of *edges*, where an edge is a set consisting of two non-equal vertices. A *clique* of Γ is any subset $C \subseteq V$ with the property that any two vertices of C are joined by an edge of Γ. The *flag-complex* $K(\Gamma)$ is the simplicial complex on V whose p-simplices are the cliques consisting of $p+1$ vertices of V. Our main example here is the complex $\Sigma(f)$, which is easily seen to be the flag-complex of its 1-skeleton $\Gamma(f)$.

Let h be a natural number. We say that the graph $\Gamma_h = (V, E)$ is *h-colored* if its vertex set V is the pairwise disjoint union of h subsets $V_1, \dots, V_h$ (where the index i is the color of the vertices in V_i), and no edge has endpoints with the same color.

Lemma 5.8. *If all colors $i \in \{1, 2, \dots, h\}$ of an h-colored graph $\Gamma_h = (V, E)$ satisfy the two properties*

(1) V_i contains at least two distinct elements, and
(2) for any choice of $2(h-1)$ vertices $u^1, \dots, u^{2(h-1)}$ in $V - V_i$ there are two vertices v, w in V_i which are adjacent to each u^j; in other words, for each $j \in \{1, 2, \dots, 2(h-1)\}$ there is an edge path of length 2 in Γ_h joining v and w via u^j.

Then the flag-complex $K(\Gamma_h)$ has the homotopy type of a bouquet of $(h-1)$-spheres.

Remark. Note that (2) holds vacuously if h = 1; and if $h > 1$ then (1) actually follows from (2).

Proof. We use induction on h, starting with the observation that the statement is trivial when $h = 1$. For $h \geqslant 2$ we assume that $K(\Gamma_{h-1})$ is homotopy equivalent to a bouquet of $(h-2)$-spheres, if Γ_{h-1} is an $(h-1)$-colored graph which satisfies the properties (1) and (2). We construct $K(\Gamma_h)$

in several steps, similar to the method applied in Brown's proof for Hougthon's groups [14]. We start with choosing a base vertex $v_1 \in V_1$ and consider its star in $K(\Gamma_h)$,

$$K_0 := \mathrm{st}_{K(\Gamma h)}(v_1).$$

Then we proceed with $i = 1, 2, \dots, h$ by taking the union of $K_{i-1} \cup V_i'$, where V_i' is the set of all vertices of V_i which are not joined with the base vertex v_1 by an edge. And we put

$$K_i := \text{ full subcomplex of } K(\Gamma_h) \text{ generated by } K_{i-1} \cup V_i'.$$

One observes that $(V_1 \cup \dots \cup V_i) \subseteq K_i$ and $(V_{i+1} \cup \dots \cup V_h) \cap K_i = K_0$. In particular, $K_h = K(\Gamma_h)$. K_i is obtained from K_{i-1} by adjoining vertices $v \in V_i$ that are not connected to the base vertex v_1 by an edge; then taking the full subcomplex of $K(\Gamma_h)$. Thus K_i is obtained from K_{i-1} by adjoining for these vertices v the cone over

$$\mathrm{lk}(K_{i-1}, v) := \text{ the link of } v \text{ in } K_{i-1}.$$

The 1-skeleton of $\mathrm{lk}(K_{i-1}, v)$ has vertex set

$$\begin{aligned} W \ &= \ W_1 \cup \dots \cup W_{i-1} \cup W_{i+1} \cup \dots \cup W_h \ \text{ with} \\ W_j \ &:= \text{set of vertices of } V_j \text{ which are joined with } v \text{ by an edge,} \\ &\quad\ \text{for } j = 1, \dots, i-1 \\ W_j \ &:= \text{ set of vertices of } V_j \text{ which are joined with } v \text{ and } v_1 \text{ by an} \\ &\quad\ \text{edge, for } j = i+1, \dots, h. \end{aligned}$$

Thus, the 1-skeleton of $\mathrm{lk}(K_{i-1}, v)$ is an $(h-1)$-colored subgraph Γ_{h-1} of Γ_h with vertex set W and colors $\{1, 2, \dots, h\} - \{i\}$, and $\mathrm{lk}(K_{i-1}, v)$ is the flag-complex $K(\Gamma_{h-1})$. Now we consider any $2(h-2)$ vertices $u^1, \dots, u^{2(h-2)}$ of $W - W_j$ with colors in $\{1, 2, \dots, h\} - \{i, j\}$ for some $j \in \{1, 2, \dots, h\} - \{i\}$.

Together with the vertices v_1 and v, we obtain $2(h-1)$ vertices $u^1, \dots, u^{2(h-2)}, v_1, v$ of $V - V_j$. By property (2) of Γ_h, there exists two vertices w, w' in V_j , which can be joined by an edge path of length 2 via u^k for each $k \in \{1, 2, \dots, 2(h-2)\}$, and additionally via v_1, v. In particular, w and w' can be joined by an edge with v_1 and v, and so they are vertices of W_j. Hence Γ_{h-1} satisfies the two properties of the lemma, and in view of the inductive hypothesis, $\mathrm{lk}(K_{i-1}, v)$ is homotopy equivalent to a bouquet of $(h-2)$-spheres.

From here we can use the same arguments as in the proof of [14, in Lemma 5.3]: Starting with the contractible complex K_0, K_1 is obtained from K_0 by adjoining for each vertex $v \in V_1'$ a cone over $\mathrm{lk}(K_0, v)$. Using the homotopy type of $\mathrm{lk}(K_0, v)$, we can deduce that K_1 is homotopy equivalent to a bouquet of $(h-1)$-spheres. For the next steps in the construction of K_h, we know that K_i is obtained from K_{i-1} by adjoining for each vertex $v \in V_i'$ a cone over $\mathrm{lk}(K_{i-1}, v)$. In view of the homotopy type of $\mathrm{lk}(K_{i-1}, v)$, we see that, up to homotopy, the passage from K_{i-1} to K_i consists of the adjunction of $(h-1)$-cells to a bouquet of $(h-1)$-spheres. □

We will now apply Lemma 5.8 to the 1-skeleton $\Gamma(f)$ of $\Sigma(f)$. By definition its vertex set is the disjoint union $A = \bigcup_{L \in \Lambda} A_L$, and we regard the various A_L as the coloring of $\Gamma(f)$. The edges of Γ are the pairs of such pei-injections $\{a_L, a'_{L'}\}$ with disjoint images. Thus $\Gamma(f)$ is an $h(S)$-colored graph $\Gamma(f)_{h(S)}$ in the sense above, and in order to establish Lemma 5.7 it remains to prove the following.

Lemma 5.9. *If $h(f) \geqslant 2 \cdot \operatorname{rk} S \cdot h(S)$, then $\Gamma(f)_{h(S)}$ satisfies the assumptions of Lemma 5.8.*

Proof. Let $n := S$ and $h := h(S)$. By the remark following Lemma 5.8 we can assume $h > 1$ and have to prove (2). For this we fix $L \in \Lambda$ and consider a set of $2(h-1)$ elements $F \subseteq A - A_L$. We have to show that there are two elements $a, b \in A_L$ with the property that for each $c \in F$ the image $\operatorname{im}(c) = c(\partial L(c))$ is disjoint to both $a(\partial L)$ and $b(\partial L)$. In other words: there are two pei-injections

$$a, b : \partial L \to (S - Sf) - \left(\bigcup_{c \in F} \operatorname{im}(c) \right).$$

To show this it suffices to compare the height function — that is, the number of rank-$(n-1)$ germs — of domain and target. Clearly, $h(a(\partial L))) = h(\partial L) = n$, and the same applies to every vertex of A. Hence $h(\bigcup_{c \in F} \operatorname{im}(c)) \leqslant 2(h-1)n$. By assumption $h(S - Sf) \geqslant 2hn$, and so the target orthohedral set has height at least $2hn - 2(h-1)n = 2n$, which is more than the height $h(\partial L) = n$ of the domain. In this situation one observes easily that there are arbitrarily many different pei-injections in A_L whose image is disjoint to $\bigcup_{c \in F} \operatorname{im}(c)$. This proves the lemma. □

Remark. If we replace $M_{<f}$ by the subset $M_{r,f} := \{a \in M_{\mathrm{dia}}(S) \mid r \leq h(a) \text{ and } a < f\}$, the assertion of Lemma 5.7 is true, provided f satisfies the additional condition $h(f) \geqslant r + h(S)$. In this case we know by Lemma 5.6 that $h(\delta_B) \geqslant r$, where δ_B stands for the largest lower bound of a finite set B of maximal elements of $M_{r,f}$. Thus δ_B is an element of $M_{r,f}$ and the proof of Lemma 5.6 works the same way for the reduced simplicial complex $|M_{r,f}|$.

5.5 | Stabilizers and cocompact skeletons of $M(S)$

The group $G(S)$ of all pei-permutations acts on $M(S)$ from the right, and as $h(g) = 0$ for all $g \in G(S)$ the height function $h : M(S) \to \mathbb{N}$ is invariant under this action. Correspondingly, $G_\#(S) := G(S) \cap M_\#(S)$ acts on $M_\#(S)$, where # stands for 0, tr, or dia. We will also restrict attention to the various $G_\#(S)$-invariant subsets $M_\#^{[r,s]}(S) = \{f \in M_\#(S) \mid r \leqslant h(f) \leqslant s\}$ for prescribed numbers $r \leqslant s$ in $\mathbb{N}_0$. And also, mutatis mutandis, for the corresponding pet-groups $\operatorname{pet}_\#(S)$, note that $\operatorname{pet}_0(S) = \operatorname{pet}_{\mathrm{tr}}(S)$.

We start with the following simple observation:

Lemma 5.10. *Two elements $f, f' \in M(S)$ are in the same* pei(S)*-orbit if and only if $(S - Sf)$ and $(S - Sf')$ are pei-isomorphic.*

Proof. As both Sf and Sf' are pei-isomorphic to S there is always a pei-isomorphism $g' : Sf \to Sf'$. Assuming there is also a pei-isomorphism $g'' : (S - Sf) \to (S - Sf')$ implies that the union $g = g' \cup g''$ is a pei-permutation of S with $fg = f'$. Conversely, $fg = f'$ implies $(S - Sf') = (Sg - Sfg) = (S - Sf)g$, hence $(S - Sf')$ is pei-isomorphic to $(S - Sf)$. □

Since orthohedral sets of the same rank and height are pei-isomorphic by Corollary 3.6, it follows that pei(S) acts transitively on the set of all pei-injections of a given $\operatorname{rk}(S - Sf)$ and height k. The very same can be said for the action of $G_\#(S)$ on $M_\#(S)$.

Let $\Delta = (a_0 < a_1 < \cdots < a_{k-1} < a_k)$ be a k-simplex of $|M(S)|$. By definition there are elements $t_1, t_2, \ldots, t_k \in \mathrm{mon}(T)$, with $a_i = t_i a_0$ for all i; they are uniquely defined and form a k-simplex $\Delta' = (\mathrm{id} < t_1 < \cdots < t_{k-1} < t_k) \in |\mathrm{mon}(T)|$. Moreover, putting $\sigma(\Delta) := (\Delta, a_0)$ defines a bijection

$$\sigma : |M(S)| \longrightarrow |\mathrm{mon}(T)| \times M(S).$$

The action of pei(S) on $|M(S)|$ is given by $(a_0 < a_1 < \cdots < a_k)g = (a_0 g < a_1 g < \cdots < a_{k-1} g < a_k g)$. We can leave it to the reader to observe that this action induces, via σ, on $|\mathrm{mon}(T)| \times M(S)$ the $G(S)$-action given by simple right action on $M(S)$.

The simple structure of the $G_\#(S)$-action on $|M_\#(S)|$ has two immediate consequences:

Corollary 5.11.

(i) The stabilizer of a k-simplex of $|M_\#(S)|$ coincides with the stabilizer of its minimal vertex f and is isomorphic to $G_\#(S - Sf)$.

(ii) For every numbers $r \leqslant s$ in $\mathbb{N} \cup \{0\}$ the simplicial complex of

$$M_\#^{[r,s]}(S) = \{f \in M_\#(S) \mid r \leqslant h(f) \leqslant s\}$$

is cocompact under the $G_\#(S)$-action.

Proof.

(i) One observes that right action of $g \in G_\#(S)$ on $M_\#(S)$ fixes an element $f \in M_\#(S)$ if and only if g restricted to Sf is the identity. In other words, the stabilizer of the vertex $f \in M_\#(S)$ is isomorphic to $G_\#(S - Sf)$.

(ii) We use the interpretation of a simplex $\Delta = (a_0 < a_1 < \cdots < a_{k-1} < a_k) \in |M(S)|$ in $|\mathrm{mon}(T)| \times M(S)$. Since $G_\#(S)$ acts transitively on the set of all pei-injections in $M_\#(S)$ of a given $\mathrm{rk}(S - Sf)$ and height k, the bound on $h(a_0)$ allows only finitely many $G_\#(S)$-orbits on the second component $M(S)$. The bound on $h(a_i)$ for $i = 1, \ldots, k$ allows only finitely many simplices in the first component $|\mathrm{mon}(T)|$. □

5.6 | The conclusion

Here we put things together to prove Theorem 5.1, that is, $fl(G(S)) \geqslant fl(G_{\mathrm{dia}}(S)) = h(S) - 1$.

Proof. We will first show, by induction on $n = S$, that $fl(G_{\mathrm{dia}}(S)) = h(S) - 1$. If $n = 1$, then the group $G_0(S)$ is the Houghton group on $h(S)$ rays and has finite index in $G(S)$. In that case the assertion is due to Brown [14]. Now we assume $n > 1$. Here we use $M^{[r,s]} = \{f \in M_{\mathrm{dia}}(S) \mid r \leqslant h(f) \leqslant s\}$, $r, s \in \mathbb{N}$. Since $f \in M^{[r,s]}$ is a diagonal pei-injection, the height of f is a multiple of n. So we fix the lower bound $r = nk_0$, $k_0 \in \mathbb{N}$, and consider the filtration of $M := M^{[r,\infty]}$ in terms of $M^k := M^{[r,nk]}$, with $k \to \infty$. Then we follow the argument of Brown [14].

- First we note that M is a directed partially ordered set and hence $|M|$ is contractible.
- $|M^{k+1}|$ is obtained from $|M^k|$ by adjoining cones over the subcomplexes $|M_{<f}|$ for each f with $h(f) = k + 1$. By Lemma 5.7 and the Remark at the end of Section 5.4, we know that the sub-

complexes $|M_{<f}|$ have the homotopy type of a bouquet of $(h(S)-1)$-spheres for k sufficiently large. This shows that the embedding $|M^k| \subseteq |M^{k+1}|$ is homotopically trivial in all dimensions less than $h(S)$.

- By Corollary 5.11 we know that the $|M^k|$ have cocompact skeleta.
- The stabilizers, $\mathrm{stab}_{G(S)}(f)$, of the vertices $f \in M$ — in fact of all simplices — are of the form $G(S - Sf)$. As $\mathrm{rk}(S - Sf) < \mathrm{rk}\, S$ the inductive hypothesis applies. The assumption that M contains only injections f with $h(f) \geqslant r$ implies now, that $fl(\mathrm{stab}_{G(S)}(f)) \geqslant r - 1$ for each $f \in M$.

We can choose r arbitrarily; if we choose $r \geqslant h(S) + 1$ the main results of [14] apply and it follows that $fl(G_{\mathrm{dia}}(S)) = h(S) - 1$. This completes the inductive step.

In order to prove that $fl(G(S)) \geqslant fl(G_{\mathrm{dia}}(S))$ we note that $fl(G(S)) = fl(G_{\mathrm{tr}}(S))$, since $G_{\mathrm{tr}}(S)$ is of finite index in $G(S)$. Then we observe that $G_{\mathrm{dia}}(S)$ is a normal subgroup of $G_{\mathrm{tr}}(S)$ with $Q = G_{\mathrm{tr}}(S)/G_{\mathrm{dia}}(S)$ finitely generated Abelian. As $fl(Q) = \infty$ this implies $fl(G_{\mathrm{tr}}(S)) \geqslant fl(G_{\mathrm{dia}}(S))$. $\square$

6 | A LOWER BOUND FOR THE FINITENESS LENGTH OF pet(S) FOR A STACK OF ORTHANTS

In this section we will show

Theorem 6.1. *If S is a stack of orthants then* $fl(\mathrm{pet}(S)) \geqslant h(S) - 1$.

The steps to prove this lower bound of $fl(\mathrm{pet}(S))$ are similar to those in Section 5 for the corresponding pei-result. We will use a certain poset of injective pet-maps $f : S \to S$ to form a simplicial complex, and we will choose a diagonal subgroup of pet(S) for the action on the complex. However, the part concerning the finiteness length of the stabilizers of f is more difficult here, because the set $(S - Sf)$ is generally not pet-isomorphic to a stack of orthants with lower rank (there are different parallelism classes of rank-$(n-1)$ germs in S if $\mathrm{rk}\, S = n$). So even if the stabilizers are isomorphic to $\mathrm{pet}(S - Sf)$, there is no base for an induction argument.

In order to set up an inductive proof we need a version of Theorem 6.1, which makes the assertion not only for stacks of orthants but also for stacks S of parallel copies of a 'rank-n-skeleton' of an orthant. In combination with special injective pet-maps f (the 'super-diagonal' maps), such a stack S leads to a set $(S - Sf)$, which has the structure of a stack of rank-$(n-1)$-skeletons.

6.1 | Stack of skeletons of an orthant

Let X be the canonical basis of the standard orthant $\mathbb{N}^N$. Every orthant L is of the form $a + \bigoplus_{y \in Y} \mathbb{N}y$, where Y is a subset of X. L carries the structure of a simplex whose faces, indexed by the subsets $Z \subseteq Y$, are the suborthants $L_Z = a + \bigoplus_{z \in Z} \mathbb{N}z \subseteq L$. We refer to L_Z as a *rank-k-face* of L if $|Z| = k$. By the *rank-k-skeleton* of L, denoted by $L^{(k)}$, we mean the union of all rank-k-faces of L. Thus the skeleta of L form an ascending chain of orthohedral set

$$\{a\} = L^{(0)} \subseteq L^{(1)} \subseteq \cdots \subseteq L^{(k)} \subseteq \cdots \subseteq L^{(\mathrm{rk}\, L)} = L.$$

Let $L^{(n)}$ be the rank-n-skeleton of a rank-r orthant $L = a + \bigoplus_{y\in Y} \mathbb{N}y$. Then $L^{(n)}$ is the union of $h(L^{(n)}) = \binom{r}{n}$ pairwise non-parallel rank-n orthants.

Now we consider a stack S of parallel copies of the rank-n-skeleton $L^{(n)}$ of an rank-r orthant — in other words, $S = R^{(n)}$ is the rank-n-skeleton of a stack R of rank-r orthants. We call each copy of $L^{(n)}$ in such a stack S a *component of* S, and we write $c(S)$ for the number of components of S. Note that $h(S) = c(S)\binom{r}{n}$. The next proposition shows a lower bound for $fl(\mathrm{pet}(S))$, and the case $n = r$ yields the assertion of Theorem 6.1.

Proposition 6.2. *If S is a stack of rank-n-skeletons of an orthant then* $fl(\mathrm{pet}(S)) \geqslant c(S) - 1$.

For later purpose in this section we consider the subset $\mathring{S} \subseteq S$ of all regular points of S, which is defined as follows: If S is an orthant, then $\mathring{S}$ is the image $t_S(S)$ of S under the diagonal unit-translation; and if S is a stack of rank-n-skeletons of an orthant, a point $p \in S$ is regular if S contains a maximal suborthant of rank equal to S, which contains p as a regular point. The complement, denoted by $\mathrm{sing}(S) = S - \mathring{S}$, is the set of all *singular points* of S. In the case when S is a stack of orthants, we will also use the geometrically more suggestive notation ∂S for $\mathrm{sing}(S)$. If $S = R^{(n)}$ is the n-skeleton of a stack of rank-r orthants L, then $\mathrm{sing}(S) = R^{(n-1)}$ and $\mathring{S}$ has the canonical decomposition as the disjoint union of the regular points of the maximal orthants of S. By a *component* of $\mathring{S}$ we mean $C \cap \mathring{S}$, the intersection of $\mathring{S}$ with a component C of S. Note that $c(S) = c(\mathring{S})$.

Lemma 6.3. *For the sets S and $\mathring{S}$ the following holds:*

(i) S and $\mathring{S}$ are pet-isomorphic. Hence $\mathrm{pet}(S)$ *is isomorphic to* $\mathrm{pet}(\mathring{S})$*;*
(ii) $h(\mathrm{sing}(\mathring{S})) = h(\mathrm{sing}(S))(r - n + 1)$*, where r is the rank of the stack R with $S = R^{(n)}$.*

Proof.

(i) S is the disjoint union of $\mathring{S}$ and $(S - \mathring{S})$. As each maximal orthant of $(S - \mathring{S})$ is parallel to a subortant of $\mathring{S}$, the assertion follows from the pet-normal form.
(ii) Since $\mathring{S} = R^{(n)} - R^{(n-1)}$, $\mathrm{sing}(\mathring{S})$ is the disjoint union of $h(R^{(n)}) \cdot n$ rank-$(n-1)$ orthants. So $h(\mathrm{sing}(\mathring{S})) = h(R^{(n)})n$. For the height of S and $\mathrm{sing}(S)$ we have $h(S) = h(R^{(n)}) = c(S)\binom{r}{n}$ and $h(\mathrm{sing}(S)) = h(R^{(n-1)}) = c(S)\binom{r}{n-1}$. As $\binom{r}{n}n = \binom{r}{n-1}(r - n + 1)$, we get $h(R^{(n)})n = h(R^{(n-1)})(r - n + 1)$. □

6.2 | Reduction to the diagonal subgroup

From now on we assume that S is a stack of rank-n-skeletons of an orthant. Since S and $\mathring{S}$ are pet-isomorphic, it suffices to establish Proposition 6.2 for the set $\mathring{S}$, which is more suitable for some parts of the proof. As noted above $\mathring{S}$ is canonically in pet-normal form. In particular, every maximal germ of S (or $\mathring{S}$) is represented by a unique maximal orthant of $\mathring{S}$. Thus we can conceptually simplify matters by replacing the set of all maximal germs, $\max \Gamma^*(S) = \max \Gamma^*(\mathring{S})$, by the set of the canonical representatives $\max \Omega^*(\mathring{S})$, the set of all maximal orthants of $\mathring{S}$.

Let $M_{\mathrm{dia}}(\mathring{S})$ denote the monoid of all diagonal pei-injections of $\mathring{S}$ introduced in Section 5.2. $M_{\mathrm{dia}}(\mathring{S})$ is a submonoid of $M_{\mathrm{tr}}(\mathring{S})$, the translation submonoid of $M(\mathring{S})$. Its elements f have the property that they induce, for each $L \in \max \Omega^*(\mathring{S})$, a diagonal translation $\tau_{(f,L)} : \langle L\rangle \to \langle L\rangle$. Now we consider the submonoid $M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S}) \subseteq M_{\mathrm{dia}}(\mathring{S})$ consisting of all diagonal pet-injections $f : \mathring{S} \to \mathring{S}$ which satisfy the additional *super-diagonality* condition:

(6.1) When two maximal orthants $L, L' of \mathring{S}$ are contained in the same component of S, then the diagonal translations $\tau_{(f,L)}$ and $\tau_{(f,L')}$ have the same translation length.

The restriction of the homomorphism (5.2) of Section 5.2 to $M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})$ can thus be interpreted as a map

$$\lambda : M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S}) \twoheadrightarrow \bigoplus_{C \in \mathrm{Comp}(\mathring{S})} \mathbb{Z} = \mathbb{Z}^{c(S)}, \tag{6.2}$$

which associates to each super-diagonal pet-injection f the translation length $\lambda(f, C)$ on each component C of $\mathring{S}$.

The group of all invertible elements of $M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})$ is the *super-diagonal pet-group* $\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})$.

Let be $\mathrm{pet}_{\mathrm{tr}}(\mathring{S})$ the group of all invertible elements of $M_{\mathrm{tr}}(\mathring{S})$. It is a subgroup of $\mathrm{pet}(\mathring{S})$, which has finite index in $\mathrm{pet}(\mathring{S})$. Analogous to (5.2) in Section 5.2 is a homomorphism

$$\kappa : \mathrm{pet}_{\mathrm{tr}}(\mathring{S}) \twoheadrightarrow \bigoplus_{L \in \max \Omega^*(\mathring{S})} \mathrm{Tran}(\langle L \rangle), \quad \text{given by } \kappa(g) = \bigoplus_{L \in \max \Omega^*(\mathring{S})} \tau(g, L) \tag{6.3}$$

which associates to each pet-injection $g \in \mathrm{pet}_{\mathrm{tr}}(\mathring{S})$ the translation length $\tau(g, L)$ on each maximal orthant L of $\max \Omega^*(\mathring{S})$. We observe that a permutation $g \in \mathrm{pet}_{\mathrm{tr}}(\mathring{S})$ is in $\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})$, if and only if the translations $\tau_{(g,L)}$ are diagonal for each L and its translation length constant as L runs through the maximal orthants of a component C of $\mathring{S}$.

Given a component C of $\mathring{S}$, we consider the set $\Lambda(C) := \max \Omega^*(C)$ of all $h(C) = \binom{r}{n}$ rank-n orthants of C. For each orthant $L \in \Lambda(C)$, we write $Y(L)$ for its canonical basis. The translation $\tau_{(g,L)} : \langle L \rangle \to \langle L \rangle$ has the canonical decomposition into the direct sum of translations $\tau^y_{(g,L)}$ in the directions $y \in Y(L)$, and we write $l^y(g, L) \in \mathbb{Z}$ for the corresponding translation lengths.

Therefore, for $g \in \mathrm{pet}(\mathring{S})$ to be super-diagonal, means that the numbers $l^y(g, L) \in \mathbb{Z}$ coincide for all pairs in $P(C) := \{(y, L) \mid y \in L \in \Lambda(C)\}$ — and this is so for all components C. Hence, associating to g the sequence

$$\left(l^y(g, L) - l^{y'}(g, L')\right)_{(i(C),C)},$$

with $i(C)$ running through all pairs $((y, L), (y', L')) \in P(C)$, and C through the components of $\mathring{S}$, exhibits the super-diagonal pet-group $\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})$ as the kernel of a homomorphism of $\mathrm{pet}_{\mathrm{tr}}(\mathring{S})$ into a finitely generated Abelian group. It is well known that in this situation $fl(\mathrm{pet}_{\mathrm{tr}}(\mathring{S})) \geqslant fl(\mathrm{pet}_{\mathrm{sdia}}(\mathring{S}))$. Since $\mathrm{pet}_{\mathrm{tr}}(\mathring{S})$ has finite index in $\mathrm{pet}(\mathring{S})$, we have $fl(\mathrm{pet}(\mathring{S})) = fl(\mathrm{pet}_{\mathrm{tr}}(\mathring{S}))$, hence

$$fl\big(\mathrm{pet}(\mathring{S})\big) \geqslant fl\big(\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})\big).$$

The proof of Proposition 6.2 is thus reduced to a proof of $fl(\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})) = c(S) - 1$. To show this, we follow the arguments in the proof of the corresponding pei-result: $fl(\mathrm{pei}_{\mathrm{dia}}(S)) = h(S) - 1$, where S was a stack of $h(S)$ orthants of rank n. In the present situation, where $\mathring{S}$ is the set of regular points of the n-skeleton of the stack R of $h(R)$ orthants, the components C of $\mathring{S}$ have to take over the role previously played by the orthants L of the stack S. Correspondingly we now have to work with the multiplicative submonoid $\mathrm{mon}(T) \subseteq M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})$ freely generated by the set T of all super-diagonal unit-translations $t_C : C \to C$ as C runs through the components of $\mathring{S}$, where each $t_C = \prod_{L\in\Lambda(C)} t_L$ is the composition of the diagonal unit-translations t_L defined in Section 5.2. As at the end of Section 5.2 we use the action of mon(T) by left multiplication to endow $M^{\text{pet}}_{\text{sdia}}(\mathring{S})$ with a partial ordering; and we observe that this partial ordering is directed.

6.3 | Maximal elements $< f$ in $M^{\text{pet}}_{\text{sdia}}(\mathring{S})$

To adapt notation to the one used in the corresponding pei-situation in Section 5, we write Λ for the set of all components C of $\mathring{S}$, and $\partial C := C - Ct_C$ for each component $C \in \Lambda$. Note that C is the disjoint union of $h(C) = \binom{r}{n}$ rank-n orthants, using the notation of Section 6.1 one for each n-element set $Z \subseteq Y$. Hence $h(\partial C) = n\binom{r}{n}$. We are still in the situation that all maximal orthants of $\mathring{S}$ have the same finite rank $n = rkS$. And given $f \in M^{\text{pet}}_{\text{sdia}}(\mathring{S})$ we write

$$M_{<f} = \{a \in M^{\text{pet}}_{\text{sdia}}(\mathring{S}) \mid a < f\}, \quad M_{\leqslant f} = \{a \in M^{\text{pet}}_{\text{sdia}}(\mathring{S}) \mid a \leqslant f\},$$

for the 'open, respectively, closed cones below f', aiming to understand the set of all maximal elements of $M_{<f}$.

Lemma 6.4. *Let b be a maximal element of $M_{<f}$. Then there is a unique component C of $\mathring{S}$ with the property that $f = t_C b$, and $h(f) = h(b) + n$. Furthermore, b is given as the union $b = b' \cup b''$, where b': $\partial C \to (\mathring{S} - \mathring{S}f)$ is a pet-injection, and $b'' : (\mathring{S} - \partial C) \to \mathring{S}f$ is the restriction $t_C^{-1} f|_{(\mathring{S}-\partial C)}$. Conversely, if $c' : \partial C \to (\mathring{S} - \mathring{S}f)$ is an arbitrary pet-injection distinct to b', then the union $c = c' \cup b''$ is a maximal element of $M_{<f}$ distinct to b.*

Proof. See argument in Lemma 5.4. □

Lemma 6.5. *Let $B \subseteq M_{<f}$ be a finite set of maximal elements of $M_{<f}$. Then the following conditions are equivalent:*

(i) The elements of B have a common lower bound δ in $M_{<f}$.
(ii) For every pair $(b, b') \in B \times B$, with $b \neq b'$ and $tb = f = t'b'$ for super-diagonal unit-translations t, t', we have
   (a) $t \neq t'$, and
   (b) $b(\partial C) \cap b'(\partial C') = \emptyset$, where C, respectively, C' are the components of $\mathring{S}$ on which t, respectively, t' acts non-trivially.

Proof. For each $b \in B$ we have a super-diagonal unit-translation $t_b \in T$ with $t_b b = f$, and we put

$$t_B := \prod_{b\in B} t_b. \tag{6.4}$$

By assumption (a) the components C_b, on which t_b acts non-trivially, are pairwise disjoint. Thus $|B| \leqslant c(S)$, and S decomposes in the disjoint union $S = (\bigcup_{b\in B} C_b) \cup S'$. We define the pet-injection $\delta_B : S \to S$ as follows:

$$\delta_B := \begin{cases} t_b^{-1} f & \text{on each } C_b t_b \\ b & \text{on the complements } \partial C_b = C_b - C_b t_b \\ f & \text{on } S'. \end{cases}$$

To show that δ_B is a common lower bound, see arguments in Lemma 5.5. □

Lemma 6.6. *In the situation of Lemma 6.5 we have for the lower bound δ_B defined in the proof:*

(i) δ_B is, in fact, a largest common lower bound of the elements of B;
(ii) $h(\delta_B) \geqslant h(f) - h(S)n$.

Proof. For (i) see argument in Lemma 5.6. For (ii) we use the translation $t = \Pi_{b\in B} t_b$ of (6.4) which satisfies $t\delta_B = f$ and yields:

$$\begin{aligned} h(\delta_B) = h(f) - h(t) &= h(f) - |B| \cdot h(t_C) \\ &= h(f) - |B| \cdot h(C)n \\ &\geqslant h(f) - c(S)h(C)n = h(f) - h(S)n. \end{aligned}$$ □

6.4 | The simplicial complex of $M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})$

We consider the simplicial complex $|M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})|$, whose vertices are the elements of $M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})$ and whose chains of length $k, a_0 < a_1 < \cdots < a_k$, are the k-simplices. As the partial ordering on $M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})$ is directed, $|M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})|$ is contractible.

In this section we aim to prove

Lemma 6.7. *If $h(f) \geqslant 2 \cdot \operatorname{rk} S \cdot h(S)$ then $|M_{<f}|$ has the homotopy type of a bouquet of $(c(S) - 1)$-spheres.*

The first step toward proving Lemma 6.7 is to consider the covering of $|M_{<f}|$ by the subcomplexes $|M_{\leqslant b}|$, where b runs through the maximal elements of $M_{<f}$. We write $N(f)$ for the nerve of this covering. Lemma 6.6(i) asserts that all finite intersections of such subcomplexes $|M_{\leqslant b}|$ are again cones and hence contractible. It is a well-known fact that in this situation the space is homotopy equivalent to the nerve of the covering. Hence we have

$$|M_{<f}| \text{ is homotopy equivalent to the nerve } N(f),$$

and it remains to compute the homotopy type of $N(f)$.

The next step — replacing the nerve $N(f)$ by the combinatorial complex $\Sigma(f)$ — follows the arguments in Section 3: We find that the set of vertices of $\Sigma(f)$ is the disjoint union $A = \bigcup_{C\in\Lambda} A_C$, where A_C stands for the set of all pet-injections $a : \partial C \to (\mathring{S} - \mathring{S}f)$; and the p-simplices of $\Sigma(f)$ are the sequences $(a_C)_{C\in\Lambda'}$, where Λ' is a p-element subset of Λ whose entries $a_C \in A_C$ satisfy the condition

(6.5) The intersections of the images $a_C(\partial C), C \in \Lambda'$, are pairwise disjoint.

The homotopy type of $\Sigma(f)$ can again be computed by Lemma 5.7, which we apply to the 1-skeleton $\Gamma(f)$ of $\Sigma(f)$, viewed as a $c(S)$-colored graph $\Gamma(f)_{c(S)}$. At the end it remains to prove

Lemma 6.8. *If $h(f) \geqslant 2 \cdot \mathrm{rk}\, S \cdot h(S)$ then $\Gamma(f)_{c(S)}$ satisfies the assumptions of Lemma 5.8.*

Proof. Let $n := \mathrm{rk}\, S$ and $h := c(S)$. Assumption (1) is a consequence of assumption (2) except in the trivial case $h = 1$.

To prove (2) we fix $C \in \Lambda$ and consider a set of $2(h-1)$ elements $F \subseteq A - A_C$. We have to show that there are two elements $a, b \in A_C$ with the property that for each $d \in F, d : \partial C_d \to (\mathring{S} - \mathring{S}f), \mathrm{im}(d) = d(\partial C_d)$ is disjoint to both $a(\partial C)$ and $b(\partial C)$. In other words: there are two pet-injections

$$a, b : \partial C \to (\mathring{S} - \mathring{S}f) - (\textstyle\bigcup_{d \in F} \mathrm{im}(d)). \tag{6.6}$$

For this it suffices to compare the height function, that is, the number of rank-$(n-1)$ germs, of domain and target. Clearly, $h(a(\partial C)) = h(\partial C) = nh(C)$, and the same applies to every vertex of A. Hence $h(\bigcup_{d \in F} \mathrm{im}(d)) \leqslant 2(h-1)nh(C)$. By assumption $h(\mathring{S} - \mathring{S}f) \geqslant 2nh(S)$, and so the target orthohedral set has height at least $2nh(S) - 2(h-1)nh(C) = 2nh(C)$, which is more than at least twice the height $h(\partial C) = nh(C)$ of the domain when $h(C)$ is positive. Moreover, by Lemma 5.10, the set $(\mathring{S} - \mathring{S}f)$ is pet-isomorphic to a stack of copies of ∂C. In this situation one observes that the two different pet-injections required in (6.6) above certainly do exist. This proves the Lemma 6.8 and hence Lemma 6.7. $\square$

Remark. By the same argument as in the remark at the end of Section 5.4, the assertion of Lemma 6.7 remains to hold true if $M_{<f}$ is replaced with the subset $M_{r,f} := \{a \in M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S}) \mid h(a) \geqslant r$ *and* $a < f\}$ and f satisfies the additional condition $h(f) \geqslant r + h(S)$.

6.5 | Stabilizers and cocompact skeletons of $|M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})|$

Here we consider the monoid $M^{\mathrm{pet}}(S)$ of all pet-injections endowed with the height function $h : M^{\mathrm{pet}}(S) \to \mathbb{Z}$ inherited from $M(S)$ and the pet(S)-action induced by right multiplication. Our main interest, however, is the super-diagonal submonoid $M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S}) \subseteq M^{\mathrm{pet}}(S)$ acted on by the super-diagonal pet-group $\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})$.

Lemma 6.9.

(i) *$f, f' \in M^{\mathrm{pet}}(S)$ are in the same* pet(S)*-orbit if and only if $(S - Sf)$ and $(S - Sf')$ are pet-isomorphic.*

(ii) *Let S be a stack of copies of $L^{(n)}$, where L is a rank-r orthant. If $f \in M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})$ with $h(f) > 0$, then $h(f)$ is a multiple of $(r-n+1)\binom{r}{n-1}$ and $(\mathring{S} - \mathring{S}f)$ is pet-isomorphic to a stack of $h(f)/\binom{r}{n-1}$ copies of $L^{(n-1)}$.*

(iii) *Two elements $f, f' \in M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S})$ with $h(f) = h(f') > 0$ are in the same* $\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})$*-orbit.*

Proof. The proof of (i) is analogous to the proof of its pei-version wich is Lemma 5.10.

(ii) The key here is a pet-version of Lemma 5.2(i). The set of germs $\Gamma^{n-1}(\mathring{S})$ decomposes into its parallelism classes, and as these are pet$(\mathring{S})$-invariant, the height function $h : M^{\text{pet}}_{\text{sdia}}(\mathring{S}) \to \mathbb{Z}$ can be written as the sum of functions $h_Y : M^{\text{pet}}_{\text{sdia}}(\mathring{S}) \to \mathbb{Z}$, with Y running through all $(n-1)$-element subsets of the canonical basis of L, that count the number of germs in $\Gamma^{n-1}(\mathring{S} - \mathring{S}f)$ parallel to $\langle Y \rangle$.

We need the h_Y-version of Lemma 5.2(i), asserting that we have, for all rank-$(n-1)$ faces Y of L,

(6.7) $h_Y(f) = h_Y(A \cap (Af)^{\text{c}}) - h_Y(A^{\text{c}} \cap Af)$ *for every orthohedral subset* $A \subseteq S$ *with* rk $A^{\text{c}} < n$.

The proof is a straightforward adaptation of the one in Section 5.1 and can be left to the reader.

We can refine (6.7) by exhibiting A as the disjoint union of rank-n orthants K_i on which f acts by (super-diagonal) translations. Since f is super-diagonal, the corresponding translation lengths $\lambda_{C(i)}$ depend only on the component $C(i)$ of $\mathring{S}$ containing K_i. One observes that $f(K_i)$ is contained in the uniquely defined maximal orthant of $\mathring{S}$ containing K_i. This has the consequence that for $i \neq j, K_i \cap (K_j f)^{\text{c}} = K_i$ and $K_i^{\text{c}} \cap K_j f = K_j f$, from which one finds

$$\begin{aligned} h_Y(f) &= h_Y\Big(\bigcup_i (K_i \cap (K_i f)^{\text{c}}\Big) - h_Y\Big(\bigcup_i (K_i^{\text{c}} \cap K_i f)\Big) \\ &= \sum_i h_Y(K_i \cap (K_i f)^{\text{c}} - h_Y(K_i^{\text{c}} \cap K_i f) = \sum_i \lambda_{C(i)}. \end{aligned} \tag{6.8}$$

Clearly, for each component C of S, sing(C) contains exactly one orthant parallel to $\langle Y \rangle$. By Lemma 6.3(ii) this orthant is parallel to a face of exactly $(r-n+1)$ maximal orthants K_i in C, and each of them gives rise to a summand $\lambda_{C(i)}$. Hence summation over all K_i contained in a single-component C of S yields $\lambda_C(r-n+1)$; and summation over all I, finally, $h_Y(f) = \lambda(r-n+1)$, where λ is the sum of λ_C, with C running through all components of S.

This shows, in particular, that $h_Y(f)$ is independent of Y. As we are assuming that $h(f) > 0$ it follows that $\lambda > 0$ and $h(f) = \lambda \cdot (r-n+1) \cdot \binom{r}{n-1}$.

It follows that $(\mathring{S} - \mathring{S}f)$ is pet-isomorphic to disjoint union $S' \cup S''$, where S' is a stack of $h_Y(f) = \lambda(r-n+1)$ copies of $L^{(n-1)}$ and S'' a subset of rank $< n-1$. As $\lambda > 0, S'$ contains at least one copy of $L^{(n-1)}$. In this situation S' contains orthants parallel to any given maximal orthant of S''. In view of the pet-normal form of $S' \cup S''$ it follows that $S' \cup S''$ is pet-isomorphic to S', that is, to a stack of $\lambda(r-n+1)$ copies of $L^{(n-1)}$.

(iii) Part (ii) shows that $h(f) = h(f') > 0$ implies that $\mathring{S} - \mathring{S}f$ and $\mathring{S} - \mathring{S}f'$ are pet-isomorphic. Hence, by assertion (i), there is a pet-permutation $g \in$ pet$(\mathring{S})$ with $f' = fg$. The assumption that f and f' are in $M^{\text{pet}}_{\text{sdia}}(\mathring{S})$ implies that $g \in \text{pet}_{\text{sdia}}(\mathring{S})$. □

Corollary 6.10.

(i) The stabilizer of $f \in M^{\text{pet}}_{\text{sdia}}(\mathring{S})$ *in* $\text{pet}_{\text{sdia}}(\mathring{S})$ *is isomorphic to* pet$(\mathring{S} - \mathring{S}f)$.

(ii) For every number $r, s \in \mathbb{N} \cup \{0\}$, *the simplicial complex of* $M^{[r,s]} := \{f \in M^{\text{pet}}_{\text{sdia}}(\mathring{S}) \mid r \leqslant h(f) \leqslant s\}$ *is cocompact under the* $\text{pet}_{\text{sdia}}(\mathring{S})$*-action.*

Proof. (i) Is immediate from Lemma 6.9(i).

(ii) Lemma 6.9(iii) asserts that $\text{pet}_{\text{sdia}}(S)$ acts cocompactly on the vertices of a given height in the simplicial complex $|M^{\text{pet}}_{\text{sdia}}(S)|$. Just as in the proof of Corollary 5.11(ii), this yields the claimed assertion. □

6.6 | The conclusion

It is an elementary observation that right action of $g \in \mathrm{pet}(S)$ on $M^{\mathrm{pet}}(S)$ fixes an element $f \in M^{\mathrm{pet}}(S)$ if and only if g restricted to $f(S)$ is the identity. In other words, the stabilizer of a vertex $f \in M^{\mathrm{pet}}(S)$ is isomorphic to $\mathrm{pet}(S - Sf)$. This will be crucial for the inductive step in the following inductive.

Proof of Proposition 6.2. In Section 6.2 we proved already that $fl(\mathrm{pet}(\mathring{S})) \geqslant fl(\mathrm{pet}_{\mathrm{sdia}}(\mathring{S}))$; hence it suffices to show $fl(\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})) \geqslant c(S) - 1$. We will argue by induction on $n = \mathrm{rk}\, S$. If $n = 1$, then $h(S) = c(S) \cdot r$, and the group $\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})$ is the Houghton group on $h(S)$ rays. By Brown [14] this implies that $fl(\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})) \geqslant h(S) - 1 \geqslant c(S) - 1$. This establishes the case $n = 1$ of the induction.

Now we assume $n > 1$. By induction we can assume that $fl(\mathrm{pet}(S')) \geqslant c(S') - 1$ holds for every stack S' of copies of a rank-$(n-1)$-skeleton of an orthant L. To prove the inductive step we start with restricting attention to the subgroup $\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})$ acting on the super-diagonal monoid $M^{[u,v]} = \{f \in M^{\mathrm{pet}}_{\mathrm{sdia}}(\mathring{S}) \mid u \leqslant h(f) \leqslant v\}$, $u, v \in \mathbb{N}$. By Lemma 6.9(ii) the $h(f)$ is a multiple of $s := (r - n + 1)\binom{r}{n-1}$. So we fix a lower bound $u = sk_0, k_0 \in \mathbb{N}$, and consider the filtration of $M := M^{[u,\infty]}$ in terms of $M^k =: M^{[u,sk]}$, with $k \to \infty$. Then we argue as follows.

- First, we note that M is a directed partially ordered set and hence $|M|$ is contractible.
- $|M^{k+1}|$ is obtained from $|M^k|$ by adjoining cones over the subcomplexes $|M_{<f}|$ for each f with $h(f) = k + 1$. By Lemma 6.7 and the remark at the end of Section 6.4, we know that the subcomplexes $|M_{<f}|$ have the homotopy type of a bouquet of $(c(S) - 1)$-spheres for k sufficiently large. This shows that the embedding $|M_k| \subseteq |M_{k+1}|$ is homotopically trivial in all dimensions $< c(S)$.
- By Corollary 6.10(ii) we know that each $|M^k|$ has cocompact skeleta.
- The stabilizer of the $f \in M$ under the action of $\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})$ on M coincides with $\mathrm{pet}(\mathring{S} - \mathring{S}f)$ by Corollary 6.10(i). Lemma 6.9(ii) asserts that if $h(f) > 0$ then $(\mathring{S} - \mathring{S}f)$ is pet-isomorphic to a stack of copies of the rank-$(n-1)$-skeleton of an orthant. The stack height here is $c(\mathring{S} - \mathring{S}f) = h(f)/\binom{r}{n-1}$. We can choose u arbitrarily; if we choose $u = (c(S) + 1)\binom{r}{n-1}$ the inductive hypothesis together with the assumption that $h(f) \geqslant u$ yields

$$\mathrm{fl}\big(\mathrm{pet}(\mathring{S} - \mathring{S}f)\big) \geqslant c(\mathring{S} - \mathring{S}f) - 1 = h(f)\Big/\binom{r}{n-1} - 1 \geqslant c(S),$$

  for all $f \in M$.
- The main result of [14] now establishes $fl(\mathrm{pet}_{\mathrm{sdia}}(\mathring{S})) \geqslant c(S) - 1$. This completes the inductive step. □

7 | THE UPPER BOUNDS OF $fl(\mathrm{pet}(S))$ WHEN $S \subseteq \mathbb{N}^N$

7.1 | More structure at infinity

Here we assume, for simplicity, that our orthohedral sets S are contained in $\mathbb{N}^N$. The pei-normal form Corollary 3.5 this is not a restriction for the pei-group $\mathrm{pei}(S)$, and it is a basic special case for the pet-group $\mathrm{pet}(S)$:

Given an element $x \in X$ (that is, a coordinate axis), we write $\Gamma^1_x(S)$ for the set of all germs of rank-1 orthants of S parallel to $\mathbb{N}x$. We have a canonical embedding $\kappa : \Gamma^1_x(S) \to \mathbb{N}^{N-1}$ defined as follows: Each $\gamma \in \Gamma^1_x(S)$ is represented by a unique maximal orthant $L \in \Omega^1(S)$; we delete the x-coordinate of the base point of L and put $\kappa(\gamma)$ to the remaining coordinate vector. We write $\partial_x S$ for the image $\kappa(\Gamma^1_x(S))$, and we will often identify $\Gamma^1_x(S)$ with $\partial_x S$ via κ. $\partial_x S$ can be viewed as the boundary of S *at infinity in direction* x.

Lemma 7.1. *For $\partial_x S$ the following hold.*

(i) $\partial_x S$ is an orthohedral subset of $\mathbb{N}^{N-1}$.
(ii) For very rank-$(k-1)$ orthant $L \subseteq \partial_x S$, there is a unique rank-k orthant $L' \subseteq S$ which is maximal with respect to the property that for each point of $p \in L\, \kappa^{-1}(p)$ is represented by a suborthant of L.

Proof. Easy. □

7.2 | Short exact sequences of pet-groups

From now on we assume that $S = \bigcup_j S_j \subseteq \mathbb{N}^m$ where m is minimal and S is in pet-normal form as defined after Proposition 3.5. Given $x \in X$ arbitrary we note that S is the disjoint union $S = S(x) \cup S^\perp(x)$, where $S(x)$ collects the stacks S_j which contain a rank-1 orthant parallel to $\mathbb{N}x$, and $S^\perp(x)$ the stacks S_j which are perpendicular to x. We note that $\partial_x S = \partial_x S(x)$, and we have an obvious projection $\pi_x : S(x) \twoheadrightarrow \partial_x S$. Moreover, there is a canonical injection $\sigma_x : \partial_x S \to S(x)$ which maps each germ $\gamma \in \partial_x S$ to the base point of the unique maximal rank-1 orthant representing γ, and is right-inverse to $\pi_x : S(x) \twoheadrightarrow \partial_x S$.

As the action of pet(S) on the $\Omega^1(S)$ preserves directions it induces, for each coordinate axis $x \in X$, an action on $\Gamma^1_x(S) = \partial_x S$, and one observes that this is an action by pet-permutations. This yields an induced homomorphism $\vartheta_x : \mathrm{pet}(S) \to \mathrm{pet}(\partial_x S)$. The kernel of ϑ_x is the set of all pet-permutations fixing all rank-1 germs parallel to x. And we note the following:

(i) $\sigma_x : \partial_x S \to S(x)$ induces an embedding of pet$(\partial_x S)$ as a subgroup of pet$(S(x))$, which splits the surjective homomorphism

$$\vartheta_x : \mathrm{pet}(S(x)) \twoheadrightarrow \mathrm{pet}(\partial_x S)$$

induced by π_x.

(ii) Every pet-permutation $g \in \mathrm{pet}(S(x))$ extends to a pet-permutation of S by the identity on $S^\perp(x)$. This exhibits pet$(S(x))$ as a canonical subgroup of pet(S). Even though we do not have pet(S) acting on $S(x)$, we do have that the surjective homomorphism

$$\vartheta_x : \mathrm{pet}(S) \twoheadrightarrow \mathrm{pet}(\partial_x S)$$

splits by the embedding $\mathrm{pet}(\partial_x S) \leqslant \mathrm{pet}(S(x)) \leqslant \mathrm{pet}(S)$.

Summarizing we have

Proposition 7.2. pet$(\partial_x S)$ *is a retract both of* pet(S) *and of* pet$(S(x))$. *In other words, we have split exact sequences*

$$1 \to K \to \mathrm{pet}(S) \to \mathrm{pet}(\partial_x S) \to 1$$

and

$$1 \to K^{\dagger} \to \mathrm{pet}(S(x)) \to \mathrm{pet}(\partial_x S) \to 1.$$

7.3 | An upper bound of the finiteness length of pet(S)

To deduce an upper bound for the finiteness lengths of the pet-groups we need the following elementary lemma which was overlooked in [5]; and is now folklore.

Lemma 7.3. *Let G be a group. If a subgroup $H \leqslant G$ is a retract of G then $fl(H) \geqslant fl(G)$.*

Proof. The assertion $fl(G) \geqslant s$ is equivalent to saying that on the category of G-modules the homology functors $H_k(G; -)$ commute with direct products for all $k < s$. That this is inherited by retracts follows from the fact that $H_k(-; -)$ is a functor on the appropriate category of pairs (G, A), with G a group and A a G-module. □

If $S \subseteq \mathbb{N}^N$ is an orthohedral set of rank $\mathrm{rk}\, S = n$ in pet-normal form, then $\Omega_0^*(\mathbb{N}^m)$ is canonically bijective to the set $P(X)$ of all subsets of X. Hence we can view the height function (2) of Section 3.4 as a map

$$h_S : P(X) \to \mathbb{N} \cup \{0\},$$

and organize stacks of maximal orthants of S as follows: For every subset $Y \subseteq X$ we have the (possibly empty) stack $S(Y) \subseteq S$ of $h_S(Y)$ orthants parallel to the orthant $\langle Y \rangle$ defined by Y.

For each $(n-1)$-element subset $Y \subseteq X$ we consider the link $\mathrm{lk}(Y)$ of Y in S_τ, by this we mean the set of all n-element sets $Y' \subseteq Y$ with the property that $\langle Y' \rangle \subseteq S_\tau$, noting that $\langle Y' \rangle \in \max \Omega_0^*(S_\tau)$. Then we put $S(\mathrm{lk}(Y)) \subseteq S$ to be the union of the stacks $S(Y')$ with Y' running through $\mathrm{lk}(Y)$. The height, $h(S(\mathrm{lk}(Y)))$, is the sum of all stack heights $h_S(\langle Y' \rangle)$ as Y' runs through $\mathrm{lk}(Y)$.

Theorem 7.4. *If $S \subseteq \mathbb{N}^m$ is orthohedral of rank n in pet-normal form, then each indicator $(n-1)$-subset $Y \subseteq X$ with non-empty link $\mathrm{lk}(Y)$ imposes an upper bound $fl(\mathrm{pet}(S)) \leqslant h(S(\mathrm{lk}(Y))) - 1$.*

Proof. We choose any $y \in Y$ and consider the projection $\pi_y : S \twoheadrightarrow \partial_y S \cup \{\emptyset\}$, where the symbol $\{\emptyset\}$ is the image of $S - S(y)$. Proposition 7.2 asserts that $\mathrm{pet}(\partial_y S)$ is a retract of $\mathrm{pet}(S)$. We have $\mathrm{rk}\, \partial_y S = \mathrm{rk}\, S - 1$; in fact, $\partial_y S$ is the disjoint union of stacks $S(Z)$, with Z running through all subsets of X avoiding y and satisfying $\langle Z \cup \{y\} \rangle \in \max \Omega_0^*(S_\tau)$. Thus note that $S(Z)$ is a stack of rank-$(n-1)$ orthants with unchanged stack height $h(S(Z)) = h_S(Z \cup \{y\})$.

We can choose the next element $y' \in Y - \{y\}$, consider the projection $\pi_{y'} : \partial_y S \twoheadrightarrow \partial_{y'} \partial_y S \cup \{\emptyset\}$. Upon putting $\pi_y(\emptyset) = \emptyset$ for all $y \in Y$, we can iterate the argument with all elements of $Y = \{y, \ldots, z\}$, noting that only the stacks in $S(\mathrm{lk}(Y))$ survive all these projections. The composition

$$\pi_y = \pi_z \ldots \pi_y : S \twoheadrightarrow \partial_z \ldots \partial_y \cup \{\emptyset\}$$

projects the stacks of $S(\mathrm{lk}(Y))$ onto stacks of rank-1 orthants with the original stack heights. This shows that $\mathrm{pet}(S)$ admits a retract isomorphic to the pet-group $\mathrm{pet}(S')$ of a disjoint union of $h(S(\mathrm{lk}(Y))$ rank-1 orthants. But $\mathrm{pet}(S')$ contains Houghton's group on $h(S(\mathrm{lk}(Y)))$ copies of $\mathbb{N}$ as a subgroup of finite index. Hence Lemma 7.3 together with Brown's result [14] yields $fl(\mathrm{pet}(S)) \leqslant fl(\mathrm{pet}(S')) = h(S(\mathrm{lk}(Y)) - 1$, as asserted. □

7.4 | Application to stacks of the n-skeleton of an orthant

Let S be a stack of $c(S)$ copies of the rank-n-skeleton $K^{(n)}$ of a rank-r orthant K. The link of each cardinality-$(n-1)$ subset Y of the cardinality-r set X contains exactly $(r-n+1)$ cardinality-n subsets Y'. And S contains exactly $c(S)$ orthants parallel to $\langle Y'\rangle$. Hence the height of disjoint union of the stacks of S over the link $\mathrm{lk}(Y)$ is $h(S(\mathrm{lk}(Y)) = c(S)(r-n+1)$. Combining Proposition 6.2 with Theorem 7.4 thus yields

Theorem 7.5. *If S is the rank-n-skeleton of a stack of rank-r orthants then*

$$c(S) - 1 \leqslant fl(\mathrm{pet}(S)) \leqslant c(S)(r-n+1) - 1.$$

Corollary 7.6. *If S is a stack of orthants then* $fl(\mathrm{pet}(S)) = c(S) - 1$.

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REFERENCES

1. Y. Atolin, J. Burillo, and A. Martino, *Conjugacy in Houghton's groups*, Publ. Mat. **59** (2015), 3–16.
2. L. Bartholdi, Y. de Cornulier, and D. Kochloukova, *Homological finiteness properties of wreath products*, Quarterly J. of Math. **66** (2015), 437–457.
3. M. Bestvina and N. Brady, *Morse theory and finiteness properties of groups*, Invent. Math. **129** (1997), 445–470.
4. R. Bieri, *Mayer-Vietoris sequences for HNN-groups and homological duality*, Math. Z. **143** (1975), 123–130.
5. R. Bieri and B. Eckmann, *Finiteness properties of duality groups*, Comment. Math. Helv. **49** (1974), 74–83.
6. R. Bieri and R. Geoghegan, *Connectivity properties of group actions on non-positively curved spaces*, Memoirs of the American Mathematical Society, vol. 765, American Mathematical Society, Providence, RI, 2003.
7. R. Bieri and J. R. J. Groves, *The geometry of the set of characters induced by valuations*, J. reine angew. Math. **347** (1984), 168–195.
8. R. Bieri, W. D. Neumann, and R. Strebel, *A geometric invariant of discrete groups*, Invent. Math. **90** (1987), 451–477.
9. R. Bieri and B. Renz, *Valuations on free resolutions and higher geometric invariants of groups*, Comm. Math. Helv. **63** (1988), 464–497.
10. R. Bieri and H. Sach, *Groups of piecewise isometric permutations of lattice points*, 2016. arXiv:1606:07728[math.GR].

11. C. Bleak, F. Matucci, and M. Neunhoeffner, *Embeddings into Thompson's group V and $coCF$ groups*, J. Lond. Math. Soc. **94** (2016), 583–597.
12. A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comm. Math. Helv. **48** (1973), 436–491.
13. A. Borel and J.-P. Serre, *Cohomologie d'immeubles et de groupes S-arithmetiques*, Topology **15** (1976), 211–232.
14. K. S. Brown, *Finiteness properties of groups*, J. Pure Appl. Algebra **44** (1987), 45–75.
15. K. S. Brown and R. Geoghegan, *An infinite dimensional FP_∞-group*, Invent. Math. **77** (1984), 367–381.
16. J. Burillo, S. Cleary, A. Martino, and C. E. Röver, *Commensurations and metric properties of Houghton's groups*, Pacific J. Math. **285** (2016). https://msp.org/pjm/2016/285-2/pjm-v285-n2-p03-p.pdf.
17. J. Burillo, B. Nucinkis, and L. Reeves, *An irrational-slope Thompson's group*, Publ. Mat. **65** (2021), 809–839.
18. M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, Invent. Math. **84** (1986), 91–119.
19. D. S. Farley and B. Hughes, *Finiteness properties of locally defined groups*. arXiv:2010.08035v1 [math.GR] (2020).
20. L. Funar, C. Kapoudjian, and V. Sergiescou, *Asymptotically rigid mapping class groups and Thompson's groups*, Handbook of Teichmüller theory, A. Papadopoulos (ed.), vol. 3, EMS Publishing House, Zurich, 2011.
21. P. Greenberg, *Piecewise $SL_2(\mathbb{Z})$-geometry*, Trans. Amer. Math. Soc. **335** (1993), 705–720.
22. C. H. Houghton, *The first cohomology of a group with permutation module coefficients*, Arch. Math. **31** (1978), 254–258.
23. C. Kopoudjian, *From symmetries of the modular tower of genus zero real stable curves to a euler class for the dyadic circle*, Compos. Math. **13** (2003), 49–73.
24. P. H. Kropholler, *On groups of type $(FP)_\infty$*, J. Pure Appl. Algebra **90** (1993), 55–67.
25. P. H. Kropholler, C. Martinez-Pérez, and B. E. A. Nucinkis, *Cohomological finiteness conditions for elementary amenable groups*, J. reine angew. Math. **637** (2009), 49–62.
26. P. H. Kropholler and C. Martino, *Graph wreath products and finiteness conditions for elementary amenable groups*, J. Pure Appl. Algebra **220** (2016), 422–434.
27. S. R. Lee, *Geometry of Houghton's groups*, arXiv:1212.0257 (2012).
28. J. Lehnert, *Gruppen von quasi-Automorphismen*, Goethe-Universität Frankfurt, Frankfurt am Main, 2008.
29. J. Lehnert and P. Schweitzer, *The co-word problem for the Higman-Thompson group is context-free*, Bull. Lond. Math. Soc. **39** (2007), no. 2, 235–241.
30. J. Ratcliffe, Foundations of Hyperbolic Manifolds, *Graduate Texts in Mathematics*, vol 149, Springer, 1994.
31. B. Nucinkis and S. St. John-Green, *Quasi automorphisms of the infinite rooted 2-edge coloured binary tree*, Group. Geom. Dyn. **6** (2018), 529–570.
32. D. V. Osin, *Elementary classes of groups*, Mat. Zametki **72** (2002), no. 1, 84–93.
33. B. Renz, *Geometrische Invarianten und Endlichkeitseigenschaften von Gruppen*, Dissertation, Frankfurt am Main, 1988.
34. H. Sach, *$(FP)_n$-Eigenschaften der verallgemeinerten Houghton-Gruppen*, Diplomarbeit, Goethe-Universität Frankfurt, 1992.
35. S. St. John-Green, *Centralisers in Houghton's groups*, Proc. Edinb. Math. Soc. **58** (2015), 769–785.
36. P. Wesolek and J. Williams, *Chain conditions, elementary amenable groups, and descriptive set theory*, Group. Geom. Dyn. **11** (2017), 649–684.
37. S. Witzel and M. C. B. Zaremsky, *Thompson groups for systems of groups, and their finiteness properties*, Group. Geom. Dyn. **12** (2018), 289–358.
38. M. C. B. Zaremsky, *On the Σ-invariants of generalized Thompson groups and Houghton groups*, Int. Math. Res. Not. IMRN **2017** (2017), 5861–5896.