Online Appendix

6.1. Detailed Dynare Topology - Underlying Equations

Here we follow Villemot (2011) and develop in detail the matrices involved using the typology of variables from Dynare. In contrast to Villemot (2011), however, we develop the matrices explicitly, detailing the submatrices and their dimensions. While this first subsection contains nothing new, this alternate presentation might be of independent interest, hopefully increasing the approachability of the dimension reductions associated with the typology developed for Dynare. Additionally, it lays down the underlying typology needed to bring the matrix quadratic and elements of the Newton algorithms from the main text in line with Dynare. The first-order approximation of (1) at the steady state, where we only derive the homogenous - that is, in y_t - component necessary for the solution of the matrix quadratic equation (5), is

$$
\mathbf{f}_{\mathbf{y}_{t+1}}\mathbf{y}_{t+1} + \mathbf{f}_{\mathbf{y}_t}\mathbf{y}_t + \mathbf{f}_{\mathbf{y}_{t+1}}\mathbf{y}_{t-1} = \mathbf{0}
$$

The vector y_t is subdivided into $\mathbf{y}_t^s$, "static" variables with only nonzero derivatives at t, $\mathbf{y}_t^{--}$, "purely backward looking" variables with only nonzero derivatives at t and $t-1$, $\mathbf{y}_t^m$, "mixed" variables with nonzero derivatives at $t+1$, t, and $t-1$, and $\mathbf{y}_t^{++}$, "purely forward looking" variables with only nonzero derivatives at $t+1$ and t. The lengths of the subvectors in y_t satisfy the following equalities

$$
n^d = n^{--} + n^m + n^{++}, \quad n^+ = n^m + n^{++}, \quad n^- = n^{--} + n^m, \quad n = n^s + n^d = n^s + n^{--} + n^m + n^{++}
$$

where n^d is the number of dynamic variables, the sum of number of purely backward-looking, n^{--}, mixed n^m, and purely forward-looking variables, n^{++}. The number of forward-looking variables, n^+, is the sum of the number of mixed, n^m, and purely forward-looking variables, n^{++}, and the number of backward-looking variables, n^-, is the sum of the number of purely backward-looking, n^{--} and mixed variables n^m. Hence, the number of endogenous variables is the sum of the number of static, n^s, and dynamic variables, n^d, or the sum of the number of static, n^s, purely backward-looking, n^{--}, mixed n^m, and purely forward-looking variables, n^{++}. Arranging the matrices $\mathbf{f}_{\mathbf{y}_{t+1}}$, $\mathbf{f}_{\mathbf{y}_t}$, and $\mathbf{f}_{\mathbf{y}_{t-1}}$ accordingly gives

$$
\begin{array}{cc}
 & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\
\begin{array}{c} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} &
\left[\begin{array}{cc|c}
\mathbf{0} & \mathbf{0} & \\
\mathbf{0} & \mathbf{0} & \underset{n\times n^+}{\mathbf{f}_{\mathbf{y}^+}} \\
\mathbf{0} & \mathbf{0} & \\
\mathbf{0} & \mathbf{0} &
\end{array}\right]
\end{array}
\begin{array}{cc}
 & 1 \\
\begin{array}{c} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} &
\left[\begin{array}{c}
\mathbf{y}_{t+1}^s \\ \mathbf{y}_{t+1}^{--} \\ \mathbf{y}_{t+1}^m \\ \mathbf{y}_{t+1}^{++}
\end{array}\right]
\end{array}
+
\begin{array}{cc}
 & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\
\begin{array}{c} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} &
\left[\begin{array}{c}
\\ \underset{n\times n}{\mathbf{f}_{\mathbf{y}^0}} \\ \\ \\
\end{array}\right]
\end{array}
\begin{array}{cc}
 & 1 \\
\begin{array}{c} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} &
\left[\begin{array}{c}
\mathbf{y}_t^s \\ \mathbf{y}_t^{--} \\ \mathbf{y}_t^m \\ \mathbf{y}_t^{++}
\end{array}\right]
\end{array}
$$

$$
+
\begin{array}{cc}
 & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\
\begin{array}{c} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} &
\left[\begin{array}{c|c|c}
\mathbf{0} & & \mathbf{0} \\
\mathbf{0} & \underset{n\times n^-}{\mathbf{f}_{\mathbf{y}^-}} & \mathbf{0} \\
\mathbf{0} & & \mathbf{0} \\
\mathbf{0} & & \mathbf{0}
\end{array}\right]
\end{array}
\begin{array}{cc}
 & 1 \\
\begin{array}{c} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} &
\left[\begin{array}{c}
\mathbf{y}_{t-1}^s \\ \mathbf{y}_{t-1}^{--} \\ \mathbf{y}_{t-1}^m \\ \mathbf{y}_{t-1}^{++}
\end{array}\right]
\end{array}
= \underset{n\times 1}{\mathbf{0}}
$$

Subdividing the matrix into the columns associated with static variables and the remaining variables, also referred to as "dynamic" variables $\mathbf{y}_t^d$ - having nonzero at $t+1$ and/or $t-1$ yields

$$\mathbf{f}_{\mathbf{y}^0} = \begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++} \end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{array}{c|c} \underset{n\times n^s}{\mathbf{S}} & \underset{n\times n^d}{\mathbf{S}^-} \end{array}\right] \end{array}$$

Performing a QR decomposition on $\mathbf{S}$, $\underset{n\times n^s}{\mathbf{S}} = \underset{n\times n}{\mathbf{Q}}\ \underset{n\times n^s}{\mathbf{R}}$, where $\mathbf{R} = \begin{array}{cc} & n^s \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{matrix} \breve{\mathbf{A}}^{0s} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{matrix}\right] \end{array}$ and premultiplying the system of equations with the inverse of the unitary Q, Q^*, gives

$$\mathbf{Q}^*\mathbf{f}_{\mathbf{y}_{t+1}}\mathbf{y}_{t+1} + \mathbf{Q}^*\mathbf{f}_{\mathbf{y}_t}\mathbf{y}_t + \mathbf{Q}^*\mathbf{f}_{\mathbf{y}_{t+1}}\mathbf{y}_{t+1} = \mathbf{0}$$

$$\begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++} \end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \underset{n\times n^+}{\mathbf{A}^+} \\ \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \end{array}\right] \end{array} \begin{array}{cc} & 1 \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{matrix} \mathbf{y}_{t+1}^s \\ \mathbf{y}_{t+1}^{--} \\ \mathbf{y}_{t+1}^m \\ \mathbf{y}_{t+1}^{++} \end{matrix}\right] \end{array} + \begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++} \end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{matrix} \\ \underset{n\times n}{\mathbf{A}^0} \\ \\ \end{matrix}\right] \end{array} \begin{array}{cc} & 1 \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{matrix} \mathbf{y}_t^s \\ \mathbf{y}_t^{--} \\ \mathbf{y}_t^m \\ \mathbf{y}_t^{++} \end{matrix}\right] \end{array}$$

$$+ \begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++} \end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{array}{c|c|c} \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \underset{n\times n^-}{\mathbf{A}^-} & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} \end{array}\right] \end{array} \begin{array}{cc} & 1 \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{matrix} \mathbf{y}_{t-1}^s \\ \mathbf{y}_{t-1}^{--} \\ \mathbf{y}_{t-1}^m \\ \mathbf{y}_{t-1}^{++} \end{matrix}\right] \end{array} = \underset{n\times 1}{\mathbf{0}}$$

where $\mathbf{A}^+ = \mathbf{Q}^*\mathbf{f}_{\mathbf{y}^+}$, $\mathbf{A}^0 = \mathbf{Q}^*\mathbf{f}_{\mathbf{y}^0} = \left[\underset{n\times n^s}{\mathbf{Q}^*\mathbf{S}} \quad \underset{n\times n^d}{\mathbf{Q}^*\mathbf{S}^-}\right] = \begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++} \end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{array}{c|c} \breve{\mathbf{A}}^{0s} & \\ \mathbf{0} & \underset{n\times n^d}{\mathbf{Q}^*\mathbf{S}^-} \\ \mathbf{0} & \\ \mathbf{0} & \end{array}\right] \end{array}$, and $\mathbf{A}^- = \mathbf{Q}^*\mathbf{f}_{\mathbf{y}^-}$. Subdividing the system of equations in accordance with the QR decomposition yields

$$\begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++} \end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \underset{n^s\times n^+}{\breve{\mathbf{A}}^+} \\ \hline \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^+} \\ \mathbf{0} & \mathbf{0} & \end{array}\right] \end{array} \begin{array}{cc} & 1 \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{matrix} \mathbf{y}_{t+1}^s \\ \mathbf{y}_{t+1}^{--} \\ \mathbf{y}_{t+1}^m \\ \mathbf{y}_{t+1}^{++} \end{matrix}\right] \end{array} + \begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++} \end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{array}{c|c} \breve{\mathbf{A}}^{0s} & \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \\ \hline \mathbf{0} & \\ \mathbf{0} & \underset{n^d\times n^d}{\tilde{\mathbf{A}}^0} \\ \mathbf{0} & \end{array}\right] \end{array} \begin{array}{cc} & 1 \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++} \end{matrix} & \left[\begin{matrix} \mathbf{y}_t^s \\ \mathbf{y}_t^{--} \\ \mathbf{y}_t^m \\ \mathbf{y}_t^{++} \end{matrix}\right] \end{array}$$

$$
+\begin{array}{r}
\\ n^s \\ n^{--} \\ n^m \\ n^{++}
\end{array}
\begin{array}{c}
\begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\
\left[\begin{array}{c|cc|c}
\mathbf{0} & \multicolumn{2}{c|}{\underset{n^s\times n^-}{\breve{\mathbf{A}}^-}} & \mathbf{0} \\
\hline
\mathbf{0} & & & \mathbf{0} \\
\mathbf{0} & \multicolumn{2}{c|}{\underset{n^d\times n^-}{\tilde{\mathbf{A}}^-}} & \mathbf{0} \\
\mathbf{0} & & & \mathbf{0}
\end{array}\right]
\end{array}
\begin{array}{r}
\\ n^s \\ n^{--} \\ n^m \\ n^{++}
\end{array}
\begin{array}{c}
1 \\
\begin{bmatrix} \mathbf{y}^s_{t-1} \\ \mathbf{y}^{--}_{t-1} \\ \mathbf{y}^m_{t-1} \\ \mathbf{y}^{++}_{t-1} \end{bmatrix}
\end{array}
= \underset{n\times 1}{\mathbf{0}}
$$

The matrix $\tilde{\mathbf{A}}^0$ is

$$
\underset{n^d\times n^d}{\tilde{\mathbf{A}}^0}\underset{n^d\times 1}{\mathbf{y}^d_t} = \begin{bmatrix} \underset{n^d\times n^{--}}{\tilde{\mathbf{A}}^{0--}} & \underset{n^d\times n^m}{\tilde{\mathbf{A}}^{0m}} & \underset{n^d\times n^{++}}{\tilde{\mathbf{A}}^{0++}} \end{bmatrix} \begin{bmatrix} \underset{n^{--}\times 1}{\mathbf{y}^{--}_t} \\ \underset{n^m\times 1}{\mathbf{y}^m_t} \\ \underset{n^{++}\times 1}{\mathbf{y}^{++}_t} \end{bmatrix}
$$

The vectors for forward and backward-looking variables can be assembled depending on how the mixed variables are assigned according to either the first or second equality in the following

$$
\begin{cases}
\overset{1}{=} \underbrace{\begin{bmatrix} \underset{n^d\times n^{--}}{\tilde{\mathbf{A}}^{0--}} & \underset{n^d\times n^m}{\tilde{\mathbf{A}}^{0m}} \end{bmatrix}}_{\underset{n^d\times n^-}{\tilde{\mathbf{A}}^{0-}}} \underbrace{\begin{bmatrix} \underset{n^{--}\times 1}{\mathbf{y}^{--}_t} \\ \underset{n^m\times 1}{\mathbf{y}^m_t} \end{bmatrix}}_{\underset{n^-\times 1}{\mathbf{y}^-_t}} + \underbrace{\begin{bmatrix} \underset{n^d\times n^m}{\mathbf{0}} & \underset{n^d\times n^{++}}{\tilde{\mathbf{A}}^{0++}} \end{bmatrix}}_{\underset{n^d\times n^+}{\tilde{\mathbf{A}}^{0+}}} \underbrace{\begin{bmatrix} \underset{n^m\times 1}{\mathbf{y}^m_t} \\ \underset{n^{++}\times 1}{\mathbf{y}^{++}_t} \end{bmatrix}}_{\underset{n^+\times 1}{\mathbf{y}^+_t}} \\
\overset{2}{=} \underbrace{\begin{bmatrix} \underset{n^d\times n^{--}}{\tilde{\mathbf{A}}^{0--}} & \underset{n^d\times n^m}{\mathbf{0}} \end{bmatrix}}_{\underset{n^d\times n^-}{\tilde{\mathbf{A}}^{0-}}} \underbrace{\begin{bmatrix} \underset{n^{--}\times 1}{\mathbf{y}^{--}_t} \\ \underset{n^m\times 1}{\mathbf{y}^m_t} \end{bmatrix}}_{\underset{n^-\times 1}{\mathbf{y}^-_t}} + \underbrace{\begin{bmatrix} \underset{n^d\times n^m}{\tilde{\mathbf{A}}^{0m}} & \underset{n^d\times n^{++}}{\tilde{\mathbf{A}}^{0++}} \end{bmatrix}}_{\underset{n^d\times n^+}{\tilde{\mathbf{A}}^{0+}}} \underbrace{\begin{bmatrix} \underset{n^m\times 1}{\mathbf{y}^m_t} \\ \underset{n^{++}\times 1}{\mathbf{y}^{++}_t} \end{bmatrix}}_{\underset{n^+\times 1}{\mathbf{y}^+_t}}
\end{cases}
$$

The mixed variables can then be selected out of the vectors of forward and backward-looking variables via

$$
\mathbf{y}^m_t = \mathbf{y}^m_t
$$

$$
\underbrace{\begin{bmatrix} \underset{n^m\times n^{--}}{\mathbf{0}} & \underset{n^m\times n^m}{\mathbf{I}} \end{bmatrix}}_{\underset{n^m\times n^-}{\mathbf{I}^-}} \underbrace{\begin{bmatrix} \underset{n^{--}\times 1}{\mathbf{y}^{--}_t} \\ \underset{n^m\times 1}{\mathbf{y}^m_t} \end{bmatrix}}_{\underset{n^-\times 1}{\mathbf{y}^-_t}} = \underbrace{\begin{bmatrix} \underset{n^m\times n^m}{\mathbf{I}} & \underset{n^m\times n^{++}}{\mathbf{0}} \end{bmatrix}}_{\underset{n^m\times n^+}{\mathbf{I}^+}} \underbrace{\begin{bmatrix} \underset{n^m\times 1}{\mathbf{y}^m_t} \\ \underset{n^{++}\times 1}{\mathbf{y}^{++}_t} \end{bmatrix}}_{\underset{n^+\times 1}{\mathbf{y}^+_t}}
$$

$$
\underset{n^m\times n^-}{\mathbf{I}^-}\underset{n^-\times 1}{\mathbf{y}^-_t} = \underset{n^m\times n^+}{\mathbf{I}^+}\underset{n^+\times 1}{\mathbf{y}^+_t}
$$

These are the “selection” matrices of Villemot (2011).

6.2. Detailed Dynare Topology - Matrix Quadratic

We now continue with the topology from Dynare and apply it to the underlying matrix quadratic. The transition matrix, P, from (4) that solves the matrix equation (13) can be subdivided in accordance to

Dynare's typology as

$$\mathbf{P}=\begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++}\end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++}\end{matrix} & \left[\begin{array}{c|c|c|c} \mathbf{P}_{s,s} & \mathbf{P}_{s,--} & \mathbf{P}_{s,m} & \mathbf{P}_{s,++} \\ \hline \mathbf{P}_{--,s} & \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{P}_{--,++} \\ \hline \mathbf{P}_{m,s} & \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{P}_{m,++} \\ \hline \mathbf{P}_{++,s} & \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{P}_{++,++}\end{array}\right]\end{array} = \begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++}\end{matrix} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{P}_{\bullet,s} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{P}_{\bullet,++}\end{array}\right]\end{array} = \begin{array}{cc} & n \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++}\end{matrix} & \left[\begin{array}{c} \mathbf{P}_{s,\bullet} \\ \hline \mathbf{P}_{--,\bullet} \\ \hline \mathbf{P}_{m,\bullet} \\ \hline \mathbf{P}_{++,\bullet}\end{array}\right]\end{array}$$

The matrix quadratic can be expressed as

$$\begin{aligned}\underset{n\times n}{\mathbf{M}(\mathbf{P})} &= \underset{n\times n}{\mathbf{A}}\mathbf{P}^2+\underset{n\times n}{\mathbf{B}}\mathbf{P}+\underset{n\times n}{\mathbf{C}} \\ &= \underbrace{(\mathbf{AP}+\mathbf{B})}_{\equiv\mathbf{G}}\mathbf{P}+\mathbf{C}\end{aligned}$$

For a solvent P of the matrix quadratic, taking the structure of C from the Dynare typology above into account yields

$$\begin{aligned}\mathbf{M}(\mathbf{P}) &= 0 = \mathbf{GP}+\mathbf{C} \\ &= \mathbf{G}\begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++}\end{matrix} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{P}_{\bullet,s} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{P}_{\bullet,++}\end{array}\right]\end{array} + \begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++}\end{matrix} \\ \begin{matrix} n^s \\ n^{--} \\ n^m \\ n^{++}\end{matrix} & \left[\begin{array}{c|c|c} \mathbf{0} & \underset{n^s\times n^-}{\breve{\mathbf{A}}^-} & \mathbf{0} \\ \hline \mathbf{0} & & \mathbf{0} \\ \hline \mathbf{0} & \underset{n^d\times n^-}{\breve{\mathbf{A}}^-} & \mathbf{0} \\ \hline \mathbf{0} & & \mathbf{0}\end{array}\right]\end{array}\end{aligned}$$

From corollary 4.5 of Lan and Meyer-Gohde (2014), $\mathbf{G}$ has full rank if P is the unique solvent of $M(P)$ stable with respect to the closed unit circle, hence the columns of P associated with nonzero columns in C, the static and forward-looking variables are zero $\rightarrow \mathbf{P}_{\bullet,s} = \underset{n\times n^s}{\mathbf{0}}$, $\mathbf{P}_{\bullet,++} = \underset{n\times n^{++}}{\mathbf{0}}$, whence P is $\mathbf{P} = \begin{array}{cc} & \begin{matrix} n^s & n^{--} & n^m & n^{++}\end{matrix} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{0}\end{array}\right]\end{array}$ and $\mathbf{M}(\mathbf{P}) = \left[\begin{array}{cccc} \underset{n\times n^s}{\mathbf{0}} & \underset{n\times n^{--}}{\mathbf{M}(\mathbf{P})^{--}} & \underset{n\times n^m}{\mathbf{M}(\mathbf{P})^m} & \underset{n\times n^{++}}{\mathbf{0}}\end{array}\right]$. Consequentially, the first n^s rows of the matrix quadratic are

$$\begin{aligned}&\underset{n^s\times n^+}{\breve{\mathbf{A}}^+}\begin{array}{cc} & \begin{matrix} n^{--} & n^m\end{matrix} \\ \begin{matrix} n^m \\ n^{++}\end{matrix} & \left[\begin{array}{c|c} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m}\end{array}\right]\end{array}\begin{array}{cc} & \begin{matrix} n^{--} & n^m\end{matrix} \\ \begin{matrix} n^{--} \\ n^m\end{matrix} & \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m}\end{array}\right]\end{array} + \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}}\begin{array}{cc} & \begin{matrix} n^{--} & n^m\end{matrix} \\ \begin{matrix} n^{--} \\ n^m \\ n^{++}\end{matrix} & \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m}\end{array}\right]\end{array} \\ &+ \underset{n^s\times n^-}{\breve{\mathbf{A}}^-} + \underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}}\begin{array}{cc} & \begin{matrix} n^{--} & n^m\end{matrix} \\ n^s & \left[\begin{array}{c|c} \mathbf{P}_{s,--} & \mathbf{P}_{s,m}\end{array}\right]\end{array} = \underset{n^s\times n^-}{\mathbf{0}}\end{aligned}$$

$$\text{Given}\quad \begin{array}{cc} & n \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c} \mathbf{P}_{--,\bullet} \\ \hline \mathbf{P}_{m,\bullet} \\ \hline \mathbf{P}_{++,\bullet} \end{array}\right] \end{array},\quad \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^{s} & \left[\begin{array}{c|c} \mathbf{P}_{s,--} & \mathbf{P}_{s,m} \end{array}\right] \end{array} \quad \text{solves}$$

$$\begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^{s} & \left[\begin{array}{c|c} \mathbf{P}_{s,--} & \mathbf{P}_{s,m} \end{array}\right] \end{array} = -\left[\underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}}\right]^{-1} \left(\underset{n^s\times n^+}{\breve{\mathbf{A}}^{+}} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \end{array} & \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \end{array}\right] \end{array} \right.$$

$$\left. + \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} \end{array}\right] \end{array} + \underset{n^s\times n^-}{\breve{\mathbf{A}}^{-}} \right)$$

and the first n^s rows of P are $\underset{n^s\times n}{\mathbf{P}_{s,\bullet}} = \begin{array}{cc} & \begin{array}{cccc} n^{s} & n^{--} & n^{m} & n^{++} \end{array} \\ n^{s} & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}_{s,--} & \mathbf{P}_{s,m} & \mathbf{0} \end{array}\right] \end{array}$.

The last n^d columns and rows of P solve the reduced matrix quadratic equation

$$\begin{array}{cc} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} \mathbf{0} & \\ \hline \mathbf{0} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}} \\ \hline \mathbf{0} & \end{array}\right] \end{array} \underbrace{\begin{array}{cc} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} & \mathbf{P}_{--,++} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{P}_{m,++} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{P}_{++,++} \end{array}\right] \end{array}}_{\underset{n^d\times n^d}{\tilde{\mathbf{P}}}} \cdot \underset{n^d\times n^d}{\tilde{\mathbf{P}}} + \underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}} \underset{n^d\times n^d}{\tilde{\mathbf{P}}}$$

$$+ \begin{array}{cc} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} & \mathbf{0} \\ \hline \underset{n^d\times n^-}{\tilde{\mathbf{A}}^{-}} & \mathbf{0} \\ \hline & \mathbf{0} \end{array}\right] \end{array}$$

$$= \underset{n^d\times n^d}{\tilde{\mathbf{M}}(\tilde{\mathbf{P}})} = \begin{array}{cc} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ n^{d} & \left[\begin{array}{c|c|c} \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{--} & \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{m} & \mathbf{0} \end{array}\right] \end{array} = \underset{n^d\times n^d}{\mathbf{0}}$$

6.3. Detailed Dynare Topology - Newton Step

The Newton-based methods in the main text all require solving a Sylvester equation for the iterative Newton step, $d\mathbf{P}$. This can be broken down using the typology from above as follows:

$$\mathbf{A}\cdot d\mathbf{P}\cdot\mathbf{P} + (\mathbf{AP}+\mathbf{B})d\mathbf{P} + \mathbf{M}(\mathbf{P}) = \mathbf{0}$$

As was shown above, $\underset{n\times n}{\mathbf{M}(\mathbf{P})} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{M}(\mathbf{P})^{\bullet,--} & \mathbf{M}(\mathbf{P})^{\bullet,m} & \mathbf{0}\end{array}\right]\end{array}$ and $\mathbf{P} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{0}\end{array}\right]\end{array}$, hence it follows that

$$d\mathbf{P} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & d\mathbf{P}_{\bullet,--} & d\mathbf{P}_{\bullet,m} & \mathbf{0}\end{array}\right]\end{array}$$

To see this, develop the expression $\mathbf{A}\cdot d\mathbf{P}\cdot\mathbf{P}$

$$d\mathbf{P}\mathbf{P} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & d\mathbf{P}\mathbf{P}_{\bullet,--} & d\mathbf{P}\mathbf{P}_{\bullet,m} & \mathbf{0}\end{array}\right]\end{array}$$

$$\mathbf{A}d\mathbf{P}\mathbf{P} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{A}d\mathbf{P}\mathbf{P}_{\bullet,--} & \mathbf{A}d\mathbf{P}\mathbf{P}_{\bullet,m} & \mathbf{0}\end{array}\right]\end{array}$$

The first and last block columns give

$$\begin{array}{cc} & \begin{array}{cc} n^s & n^{++}\end{array} \\ n & \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0}\end{array}\right]\end{array} + (\mathbf{A}\mathbf{P}+\mathbf{B}) \begin{array}{cc} & \begin{array}{cc} n^s & n^{++}\end{array} \\ n & \left[\begin{array}{c|c} d\mathbf{P}^{\bullet,s} & d\mathbf{P}^{\bullet,++}\end{array}\right]\end{array} + \begin{array}{cc} & \begin{array}{cc} n^s & n^{++}\end{array} \\ n & \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0}\end{array}\right]\end{array} = \begin{array}{cc} & \begin{array}{cc} n^s & n^{++}\end{array} \\ n & \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0}\end{array}\right]\end{array}$$

and $\mathbf{A}\mathbf{P}+\mathbf{B}=\mathbf{G}$ is full rank (see above), $d\mathbf{P}^{\bullet,s}$ and $d\mathbf{P}^{\bullet,++}$ are zero matrices.

As the first block columns of $\mathbf{A}$ and $d\mathbf{P}$ are zero and the first block column of $\mathbf{B} = \begin{array}{cc} & n^s \\ \begin{array}{c} n^s \\ n^{--} \\ n^m \\ n^{--}\end{array} & \left[\begin{array}{c} \breve{\mathbf{A}}^{0s} \\ \hline \mathbf{0} \\ \hline \mathbf{0} \\ \hline \mathbf{0}\end{array}\right]\end{array}$, $d\mathbf{P}_{s,\bullet}$ is given by $d\mathbf{P}_{s,\bullet} = \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++}\end{array} \\ n^s & \left[\begin{array}{c|c|c|c} \mathbf{0} & d\mathbf{P}^{s,--} & d\mathbf{P}^{s,m} & \mathbf{0}\end{array}\right]\end{array}$, where

$$\begin{aligned}
\begin{array}{cc} & \begin{array}{cc} n^{--} & n^m\end{array} \\ n^s & \left[\begin{array}{c|c} d\mathbf{P}^{s,--} & d\mathbf{P}^{s,m}\end{array}\right]\end{array} = -\underset{n^s\times n^s}{[\breve{\mathbf{A}}^{0s}]^{-1}} \Bigg(& \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m\end{array} \\ n^s & \left[\begin{array}{c|c} \mathbf{M}^{s,--}(\mathbf{P}) & \mathbf{M}^{s,m}(\mathbf{P})\end{array}\right]\end{array} + \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m\end{array} \\ \begin{array}{c} n^{--} \\ n^m \\ n^{++}\end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{--,--} & d\mathbf{P}_{--,m} \\ \hline d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m}\end{array}\right]\end{array} \\
& + \underset{n^d\times n^+}{\breve{\mathbf{A}}^{+}} \Bigg\{ \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m\end{array} \\ \begin{array}{c} n^m \\ n^{++}\end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m}\end{array}\right]\end{array} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m\end{array} \\ \begin{array}{c} n^{--} \\ n^m\end{array} & \left[\begin{array}{c|c} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \hline \mathbf{P}_{m,--} & \mathbf{P}_{m,m}\end{array}\right]\end{array} \\
& + \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m\end{array} \\ \begin{array}{c} n^m \\ n^{++}\end{array} & \left[\begin{array}{c|c} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \\ \hline \mathbf{P}_{++,--} & \mathbf{P}_{++,m}\end{array}\right]\end{array} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^m\end{array} \\ \begin{array}{c} n^{--} \\ n^m\end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{--,--} & d\mathbf{P}_{--,m} \\ \hline d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m}\end{array}\right]\end{array} \Bigg\} \Bigg)
\end{aligned}$$

given a solution for
$$\begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \begin{bmatrix} d\mathbf{P}_{--,--} & d\mathbf{P}_{--,m} \\ d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{bmatrix} \end{array}$$

Hence the remaining equations are (where zero columns of $\mathbf{P}$, $d\mathbf{P}$, $\mathbf{M}(\mathbf{P})$ have been eliminated where appropriate)

$$\underset{n^d \times n^+}{\tilde{\mathbf{A}}^+} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ n^{++} \end{array} & \begin{bmatrix} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \end{array} & \begin{bmatrix} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \end{bmatrix} \end{array}$$

$$+ \left(\underset{n^d \times n^+}{\tilde{\mathbf{A}}^+} \begin{array}{cc} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ \begin{array}{c} n^{m} \\ n^{++} \end{array} & \begin{bmatrix} \mathbf{P}_{m,--} & \mathbf{P}_{m,m} & \mathbf{0} \\ \mathbf{P}_{++,--} & \mathbf{P}_{++,m} & \mathbf{0} \end{bmatrix} \end{array} + \underset{n^d \times n^d}{\tilde{\mathbf{A}}^0} \right) \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \\ n^{++} \end{array} & \begin{bmatrix} d\mathbf{P}_{--,--} & d\mathbf{P}_{--,m} \\ d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{bmatrix} \end{array}$$

$$+ \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^d & \begin{bmatrix} \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{--} & \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{m} \end{bmatrix} \end{array}$$

$$= \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^d & \begin{bmatrix} \mathbf{0} & \mathbf{0} \end{bmatrix} \end{array}$$

Defining $\underset{n^d \times n^d}{\tilde{\mathbf{A}}^0} = \begin{array}{cc} & \begin{array}{ccc} n^{--} & n^{m} & n^{++} \end{array} \\ n^d & \begin{bmatrix} \tilde{\mathbf{A}}^{0--} & \tilde{\mathbf{A}}^{0m} & \tilde{\mathbf{A}}^{0++} \end{bmatrix} \end{array}$, the foregoing is

$$\underbrace{\left(\underset{n^d \times n^+}{\tilde{\mathbf{A}}^+} \begin{array}{cc} & n^{--} \\ \begin{array}{c} n^{m} \\ n^{++} \end{array} & \begin{bmatrix} \mathbf{P}_{m,--} \\ \mathbf{P}_{++,--} \end{bmatrix} \end{array} + \underset{n^d \times n^{--}}{\tilde{\mathbf{A}}^{0--}} \right)}_{\equiv \underset{n^d \times n^{--}}{\boldsymbol{\alpha}}} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^{--} & \begin{bmatrix} d\mathbf{P}_{--,--} & d\mathbf{P}_{--,m} \end{bmatrix} \end{array}$$

$$+ \underbrace{\begin{array}{cc} & \begin{array}{cc} n^{m} & n^{++} \end{array} \\ n^d & \left[\underset{n^d \times n^+}{\tilde{\mathbf{A}}^+} \begin{bmatrix} \mathbf{P}_{m,m} \\ \mathbf{P}_{++,m} \end{bmatrix} + \underset{n^d \times n^m}{\tilde{\mathbf{A}}^{0m}} \quad \underset{n^d \times n^{++}}{\tilde{\mathbf{A}}^{0++}} \right] \end{array}}_{\equiv \underset{n^d \times n^+}{\boldsymbol{\beta}}} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ n^{++} \end{array} & \begin{bmatrix} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{bmatrix} \end{array}$$

$$+ \underbrace{\underset{n^d \times n^+}{\tilde{\mathbf{A}}^+}}_{\equiv \underset{n^d \times n^+}{\boldsymbol{\gamma}}} \begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ n^{++} \end{array} & \begin{bmatrix} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{bmatrix} \end{array} \underbrace{\begin{array}{cc} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{--} \\ n^{m} \end{array} & \begin{bmatrix} \mathbf{P}_{--,--} & \mathbf{P}_{--,m} \\ \mathbf{P}_{m,--} & \mathbf{P}_{m,m} \end{bmatrix} \end{array}}_{\equiv \underset{n^- \times n^-}{\boldsymbol{\delta}}}$$

$$
\underbrace{+\begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^{d} & \left[\begin{array}{c|c} \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{--} & \tilde{\mathbf{M}}(\tilde{\mathbf{P}})^{m} \end{array}\right] \end{array}}_{\equiv \underset{n^d\times n^-}{-\boldsymbol{\theta}}}
$$

$$
= \begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^{d} & \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \end{array}\right] \end{array}
$$

written more compactly as

$$
\underset{n^d\times n^-}{\boldsymbol{\theta}} = \underset{n^d\times n^{--}}{\boldsymbol{\alpha}} \begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^{--} & \left[\begin{array}{c|c} d\mathbf{P}_{--,--} & d\mathbf{P}_{--,m} \end{array}\right] \end{array} + \underset{n^d\times n^+n^{++}}{\boldsymbol{\beta}} \begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ \\ \end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{array}\right] \end{array}
$$

$$
+ \underset{n^d\times n^+n^{++}}{\boldsymbol{\gamma}} \begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ \\ \end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{array}\right] \end{array} \underset{n^-\times n^-}{\boldsymbol{\delta}}
$$

Performing a QR decomposition $\underset{n^d\times n^{--}}{\boldsymbol{\alpha}} = \underset{n^d\times n^d}{\mathbf{U}} \begin{array}{c c} & n^{--} \\ \begin{array}{c} n^{--} \\ n^{+} \end{array} & \left[\begin{array}{c} \mathbf{T} \\ \hline \mathbf{0} \end{array}\right] \end{array}$, $\tilde{\mathbf{U}} = \mathbf{U}^* = \begin{array}{c c} & n^{d} \\ \begin{array}{c} n^{--} \\ n^{+} \end{array} & \left[\begin{array}{c} \tilde{\mathbf{U}}_1 \\ \hline \tilde{\mathbf{U}}_2 \end{array}\right] \end{array}$ and premultiplying with $\tilde{\mathbf{U}}$ gives two sets of equations. First

$$
\underset{n^+\times n^d}{\tilde{\mathbf{U}}_2} \underset{n^d\times n^+n^{++}}{\boldsymbol{\beta}} \begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ \\ \end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{array}\right] \end{array} + \underset{n^+\times n^d}{\tilde{\mathbf{U}}_2} \underset{n^d\times n^+n^{++}}{\boldsymbol{\gamma}} \begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ \\ \end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{array}\right] \end{array} \underset{n^-\times n^-}{\boldsymbol{\delta}} = \underset{n^+\times n^d}{\tilde{\mathbf{U}}_2} \underset{n^d\times n^-}{\boldsymbol{\theta}}
$$

A generalized Sylvester equation in $\begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ n^{++} \end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{array}\right] \end{array}$. Given its solution, the remaining elements of $d\mathbf{P}$ are given by

$$
\begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ n^{--} & \left[\begin{array}{c|c} d\mathbf{P}_{--,--} & d\mathbf{P}_{--,m} \end{array}\right] \end{array} = \underset{n^{--}\times n^{--}}{\mathbf{T}^{-1}} \underset{n^{--}\times n^d}{\tilde{\mathbf{U}}_1} \left(\underset{n^d\times n^-}{\boldsymbol{\theta}} - \underset{n^d\times n^+n^{++}}{\boldsymbol{\gamma}} \begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ \\ \end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{array}\right] \end{array} \underset{n^-\times n^-}{\boldsymbol{\delta}} \right.
$$

$$
\left. - \underset{n^d\times n^+n^{++}}{\boldsymbol{\beta}} \begin{array}{c c} & \begin{array}{cc} n^{--} & n^{m} \end{array} \\ \begin{array}{c} n^{m} \\ \\ \end{array} & \left[\begin{array}{c|c} d\mathbf{P}_{m,--} & d\mathbf{P}_{m,m} \\ \hline d\mathbf{P}_{++,--} & d\mathbf{P}_{++,m} \end{array}\right] \end{array} \right)
$$

6.4. Detailed Dynare Topology - Line Search

The line search methods in the text require finding zeros of the polynomial

$$
g'(x) = 2\alpha(x-1) + \beta\left(2x - 3x^2\right) + 4\gamma x^3 \tag{A1}
$$

where $\alpha = \|M(P)\|_F^2$, $\beta = trace\left(M(P)^* A(\Delta P)^2 + \left(A(\Delta P)^2\right)^* M(P)\right)$ and $\gamma = \left\|A(\Delta P)^2\right\|_F$.

Using the typology from Dynare and the results above

$$\alpha = ||\mathbf{M}(\mathbf{P})||_F^2 = \operatorname{tr}\left(\mathbf{M}(\mathbf{P})^*\mathbf{M}(\mathbf{P})\right)$$

$$= \operatorname{tr}\left(\begin{array}{c} n \\ \begin{array}{r} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} \left[\begin{array}{c} \mathbf{0} \\ \hline \mathbf{M}(\mathbf{P})^*_{--} \\ \hline \mathbf{M}(\mathbf{P})^*_m \\ \hline \mathbf{0} \end{array}\right] \end{array} \; \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{M}(\mathbf{P})_{--} & \mathbf{M}(\mathbf{P})_m & \mathbf{0} \end{array}\right] \end{array} \right)$$

$$= \operatorname{tr}\left(\mathbf{M}(\mathbf{P})^*_{--}\mathbf{M}(\mathbf{P})_{--}\right) + \operatorname{tr}\left(\mathbf{M}(\mathbf{P})^*_m\mathbf{M}(\mathbf{P})_m\right)$$

$$\mathbf{M}(\underset{n\times n}{\mathbf{P}}) = \underset{n\times n}{\mathbf{A}}\mathbf{P}^2 + \underset{n\times n}{\mathbf{B}}\mathbf{P} + \underset{n\times n}{\mathbf{C}}$$

$$= \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{r} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} & \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{0} & \underset{n^s\times n^+}{\breve{\mathbf{A}}^+} \\ \hline \mathbf{0} & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^+} \\ \mathbf{0} & \mathbf{0} & \end{array}\right] \end{array} \; \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}\mathbf{P}_{\bullet,--} & \mathbf{P}\mathbf{P}_{\bullet,m} & \mathbf{0} \end{array}\right] \end{array}$$

$$+ \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{r} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} & \left[\begin{array}{c|c} \breve{\mathbf{A}}^{0s} & \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \\ \hline \mathbf{0} & \\ \mathbf{0} & \underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}} \\ \mathbf{0} & \end{array}\right] \end{array} \; \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ n & \left[\begin{array}{c|c|c|c} \mathbf{0} & \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} & \mathbf{0} \end{array}\right] \end{array}$$

$$+ \begin{array}{cc} & \begin{array}{cccc} n^s & n^{--} & n^m & n^{++} \end{array} \\ \begin{array}{r} n^s \\ n^{--} \\ n^m \\ n^{++} \end{array} & \left[\begin{array}{c|c|c} \mathbf{0} & \underset{n^s\times n^-}{\breve{\mathbf{A}}^-} & \mathbf{0} \\ \hline \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \underset{n^d\times n^-}{\tilde{\mathbf{A}}^-} & \mathbf{0} \\ \mathbf{0} & & \mathbf{0} \end{array}\right] \end{array}$$

$$= \begin{array}{cc} & \begin{array}{ccc} n^s & n^- & n^{++} \end{array} \\ n & \left[\begin{array}{c|c|c} \mathbf{0} & \left[\begin{array}{ccc} \underset{n^s\times n^s}{\mathbf{0}} & \underset{n^s\times n^{--}}{\mathbf{0}} & \underset{n^s\times n^+}{\breve{\mathbf{A}}^+} \\ \underset{n^{--}\times n^s}{\mathbf{0}} & \underset{n^{--}\times n^{--}}{\mathbf{0}} & \\ \underset{n^m\times n^s}{\mathbf{0}} & \underset{n^m\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^+} \\ \underset{n^{++}\times n^s}{\mathbf{0}} & \underset{n^{++}\times n^{--}}{\mathbf{0}} & \end{array}\right] \underset{n\times n}{\mathbf{P}} \left[\begin{array}{cc} \underset{n\times n^{--}}{\mathbf{P}_{\bullet,--}} & \underset{n\times n^m}{\mathbf{P}_{\bullet,m}} \end{array}\right] & \mathbf{0} \end{array}\right] \end{array}$$

$$
+\begin{array}{cc}
 & \begin{array}{ccc} \quad n^s \quad & \qquad\qquad\qquad n^- \qquad\qquad\qquad & n^{++} \end{array}\\
n & \left[\begin{array}{c|c|c}
\mathbf{0} &
\begin{bmatrix}
\underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}} & \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \\
\underset{n^{--}\times n^s}{\mathbf{0}} & \\
\underset{n^m\times n^s}{\mathbf{0}} & \underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}} \\
\underset{n^{++}\times n^s}{\mathbf{0}} &
\end{bmatrix}
\begin{bmatrix} \underset{n\times n^{--}}{\mathbf{P}_{\bullet,--}} & \underset{n\times n^m}{\mathbf{P}_{\bullet,m}} \end{bmatrix}
& \mathbf{0}
\end{array}\right]
\end{array}
+\begin{array}{cc}
 & \begin{array}{ccc} n^s & \quad n^- \quad & n^{++} \end{array}\\
n & \left[\begin{array}{c|c|c}
\mathbf{0} &
\begin{bmatrix}
\underset{n^s\times n^-}{\breve{\mathbf{A}}^{-}} \\
\underset{n^d\times n^-}{\tilde{\mathbf{A}}^{-}}
\end{bmatrix}
& \mathbf{0}
\end{array}\right]
\end{array}
$$

$$
\begin{bmatrix}
\underset{n^s\times n^s}{\mathbf{0}} & \underset{n^s\times n^{--}}{\mathbf{0}} & \underset{n^s\times n^+}{\breve{\mathbf{A}}^{+}} \\
\underset{n^{--}\times n^s}{\mathbf{0}} & \underset{n^{--}\times n^{--}}{\mathbf{0}} & \\
\underset{n^m\times n^s}{\mathbf{0}} & \underset{n^m\times n^{--}}{\mathbf{0}} & \underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}} \\
\underset{n^{++}\times n^s}{\mathbf{0}} & \underset{n^{++}\times n^{--}}{\mathbf{0}} &
\end{bmatrix}
\underset{n\times n}{\mathbf{P}}
\begin{bmatrix} \underset{n\times n^{--}}{\mathbf{P}_{\bullet,--}} & \underset{n\times n^m}{\mathbf{P}_{\bullet,m}} \end{bmatrix}
=
\begin{bmatrix}
\underset{n^s\times n^+}{\breve{\mathbf{A}}^{+}} \\
\underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}}
\end{bmatrix}
\underset{n^+\times n}{\mathbf{P}_{m/++,\bullet}}
\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}}
$$

where

$$
\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} =
\begin{array}{cc}
 & \begin{array}{cc} n^{--} & n^m \end{array}\\
n & \left[\begin{array}{c|c} \mathbf{P}_{\bullet,--} & \mathbf{P}_{\bullet,m} \end{array}\right]
\end{array}
\quad \text{and} \quad
\underset{n\times n^+}{\mathbf{P}_{\bullet,m/++}} =
\begin{array}{cc}
 & \begin{array}{cc} n^m & n^{++} \end{array}\\
n & \left[\begin{array}{c|c} \mathbf{P}_{\bullet,m} & \mathbf{P}_{\bullet,++} \end{array}\right]
\end{array}
$$

$$
\mathbf{M}(\underset{n\times n}{\mathbf{P}}) =
\begin{array}{cc}
 & \begin{array}{ccc} n^s & \qquad\qquad\qquad n^- \qquad\qquad\qquad & n^{++} \end{array}\\
n & \left[\begin{array}{c|c|c}
\mathbf{0} &
\left(\left(
\begin{bmatrix}
\underset{n^s\times n^+}{\breve{\mathbf{A}}^{+}} \\
\underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}}
\end{bmatrix}
\underset{n^+\times n}{\mathbf{P}_{m/++,\bullet}}
+
\begin{bmatrix}
\underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}} & \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \\
\underset{n^{--}\times n^s}{\mathbf{0}} & \\
\underset{n^m\times n^s}{\mathbf{0}} & \underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}} \\
\underset{n^{++}\times n^s}{\mathbf{0}} &
\end{bmatrix}
\right)
\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}}
+
\begin{bmatrix}
\underset{n^s\times n^-}{\breve{\mathbf{A}}^{-}} \\
\underset{n^d\times n^-}{\tilde{\mathbf{A}}^{-}}
\end{bmatrix}
\right)
& \mathbf{0}
\end{array}\right]
\end{array}
$$

$$
=\begin{array}{cc}
 & \begin{array}{ccc} n^s & n^- & n^{++} \end{array}\\
n & \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{X} & \mathbf{0} \end{array}\right]
\end{array}
$$

$$
\underset{n\times n^-}{\mathbf{X}} =
\begin{array}{cc}
 & n^- \\
\begin{array}{c} n^s \\ n^d \end{array} & \left[\begin{array}{c} \mathbf{X}_1 \\ \hline \mathbf{X}_2 \end{array}\right]
\end{array}
=
\begin{array}{cc}
 & n^- \\
\begin{array}{c} n^s \\ n^d \end{array} &
\left[\begin{array}{c}
\left(\underset{n^s\times n^+}{\breve{\mathbf{A}}^{+}}\underset{n^+\times n}{\mathbf{P}_{m/++,\bullet}} + \begin{bmatrix} \underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}} & \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \end{bmatrix}\right)\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} + \underset{n^s\times n^-}{\breve{\mathbf{A}}^{-}} \\ \hline
\underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}}\underset{n^+\times n}{\mathbf{P}_{m/++,\bullet}}\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} + \begin{bmatrix} \underset{n^d\times n^s}{\mathbf{0}} & \underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}} \end{bmatrix}\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} + \underset{n^d\times n^-}{\tilde{\mathbf{A}}^{-}}
\end{array}\right]
\end{array}
$$

$$
=
\begin{array}{cc}
 & n^- \\
\begin{array}{c} n^s \\ n^d \end{array} &
\left[\begin{array}{c}
\left(\underset{n^s\times n^+}{\breve{\mathbf{A}}^{+}}\underset{n^+\times n}{\mathbf{P}_{m/++,\bullet}} + \begin{bmatrix} \underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}} & \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}} \end{bmatrix}\right)\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} + \underset{n^s\times n^-}{\breve{\mathbf{A}}^{-}} \\ \hline
\underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}}\underset{n^+\times n}{\mathbf{P}_{m/++,\bullet}}\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} + \underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}}\underset{n^d\times n^-}{\mathbf{P}_{d,--/m}} + \underset{n^d\times n^-}{\tilde{\mathbf{A}}^{-}}
\end{array}\right]
\end{array}
$$

$$
=
\begin{array}{cc}
 & n^- \\
\begin{array}{c} n^s \\ n^d \end{array} &
\left[\begin{array}{c}
\underset{n^s\times n^s}{\breve{\mathbf{A}}^{0s}}\underset{n^s\times n^-}{\mathbf{P}_{s,--/m}} + \left(\underset{n^s\times n^+}{\breve{\mathbf{A}}^{+}}\underset{n^+\times n^d}{\mathbf{P}_{m/++,d}} + \underset{n^s\times n^d}{\breve{\mathbf{A}}^{0d}}\right)\underset{n^d\times n^-}{\mathbf{P}_{d,--/m}} + \underset{n^s\times n^-}{\breve{\mathbf{A}}^{-}} \\ \hline
\underset{n^d\times n^+}{\tilde{\mathbf{A}}^{+}}\underset{n^+\times n}{\mathbf{P}_{m/++,\bullet}}\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} + \underset{n^d\times n^d}{\tilde{\mathbf{A}}^{0}}\underset{n^d\times n^-}{\mathbf{P}_{d,--/m}} + \underset{n^d\times n^-}{\tilde{\mathbf{A}}^{-}}
\end{array}\right]
\end{array}
$$

$$
\operatorname{tr}\Big(\mathbf{M}(\mathbf{P})^*\mathbf{M}(\mathbf{P})\Big) = \operatorname{tr}(\mathbf{X}^*\mathbf{X}) = \operatorname{tr}(\mathbf{X}_1^*\mathbf{X}_1) + \operatorname{tr}(\mathbf{X}_2^*\mathbf{X}_2) \qquad \text{by construction,}\ \underset{n^s\times n^-}{\mathbf{X}_1} = \underset{n^s\times n^-}{\mathbf{0}}
$$

$$= \mathrm{tr}(\mathbf{X}_2^*\mathbf{X}_2) = \mathrm{tr}\left(\tilde{\mathbf{M}}(\tilde{\mathbf{P}})^*\tilde{\mathbf{M}}(\tilde{\mathbf{P}})\right)$$

$$\gamma = ||\mathbf{A}d\mathbf{P}^2||_F^2 = \mathrm{tr}\left(\left(\mathbf{A}d\mathbf{P}^2\right)^*\mathbf{A}d\mathbf{P}^2\right)$$

$$\mathbf{A}d\mathbf{P}^2 = \begin{array}{c} \\ n \end{array}\begin{array}{ccc} n^s & n^- & n^{++} \\ \left[\;\mathbf{0}\right. & \underset{n\times n}{\mathbf{A}}\, d\,\underset{n\times n}{\mathbf{P}}\, d\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} & \left.\mathbf{0}\;\right] \end{array}$$

$$= \begin{array}{c} \\ n \end{array}\begin{array}{ccc} n^s & n^- & n^{++} \\ \left[\;\mathbf{0}\right. & \begin{bmatrix} \underset{n^s\times n^+}{\breve{\mathbf{A}}^+} \\ \underset{n^d\times n^+}{\tilde{\mathbf{A}}^+} \end{bmatrix} d\underset{n^+\times n}{\mathbf{P}_{m/++,\bullet}}\, d\underset{n\times n^-}{\mathbf{P}_{\bullet,--/m}} & \left.\mathbf{0}\;\right] \end{array}$$

$$= \begin{array}{c} \\ n \end{array}\begin{array}{ccc} n^s & n^- & n^{++} \\ \left[\;\mathbf{0}\right. & \begin{bmatrix} \underset{n^s\times n^+}{\breve{\mathbf{A}}^+} \\ \underset{n^d\times n^+}{\tilde{\mathbf{A}}^+} \end{bmatrix} d\underset{n^+\times n^-}{\mathbf{P}_{m/++,--/m}}\, d\underset{n^-\times n^-}{\mathbf{P}_{--/m,--/m}} & \left.\mathbf{0}\;\right] \end{array}$$

$$= \begin{array}{c} \\ n \end{array}\begin{array}{ccc} n^s & n^- & n^{++} \\ \left[\;\mathbf{0}\right. & \begin{bmatrix} \underset{n^s\times n^-}{\mathbf{Y}_1} \\ \underset{n^d\times n^-}{\mathbf{Y}_2} \end{bmatrix} & \left.\mathbf{0}\;\right] \end{array}$$

$$\delta\mathbf{P} = d\underset{n^+\times n^-}{\mathbf{P}_{m/++,--/m}}\, d\underset{n^-\times n^-}{\mathbf{P}_{--/m,--/m}}$$

$$\gamma = \mathrm{tr}(\mathbf{Y}_1^*\mathbf{Y}_1) + \mathrm{tr}(\mathbf{Y}_2^*\mathbf{Y}_2)$$

$$= \mathrm{tr}\left(\underset{n^s\times n^+}{\breve{\mathbf{A}}^+}\;\underset{n^+\times n^-}{\delta\mathbf{P}}\;\underset{n^-\times n^+}{\delta\mathbf{P}^*}\;\underset{n^+\times n^s}{\breve{\mathbf{A}}^{+*}}\right) + \mathrm{tr}\left(\underset{n^d\times n^+}{\tilde{\mathbf{A}}^+}\;\underset{n^+\times n^-}{\delta\mathbf{P}}\;\underset{n^-\times n^+}{\delta\mathbf{P}^*}\;\underset{n^+\times n^d}{\tilde{\mathbf{A}}^{+*}}\right)$$

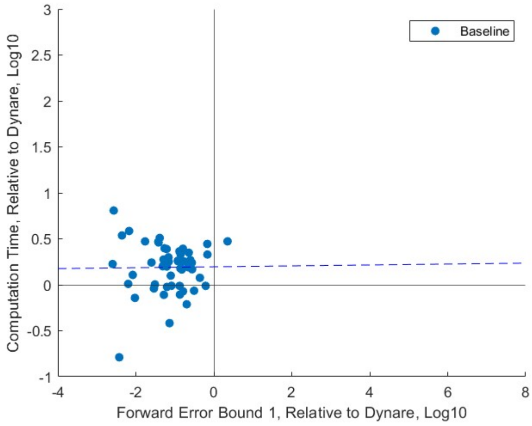
$$= \mathrm{tr}\left(\underset{n^-\times n^+}{\delta\mathbf{P}^*}\left[\underset{n^+\times n^s}{\breve{\mathbf{A}}^{+*}}\;\underset{n^s\times n^+}{\breve{\mathbf{A}}^+} + \underset{n^+\times n^d}{\tilde{\mathbf{A}}^{+*}}\;\underset{n^d\times n^+}{\tilde{\mathbf{A}}^+}\right]\underset{n^+\times n^-}{\delta\mathbf{P}}\right)$$

$$\mathrm{tr}\left(\mathbf{A}d\mathbf{P}^2\cdot\mathbf{M}(\mathbf{P})^*\right) = \mathrm{tr}\left(\begin{array}{c} \\ n \end{array}\begin{array}{ccc} n^s & n^- & n^{++} \\ \left[\;\mathbf{0}\right. & \begin{bmatrix} \underset{n^s\times n^-}{\mathbf{Y}_1} \\ \underset{n^d\times n^-}{\mathbf{Y}_2} \end{bmatrix} & \left.\mathbf{0}\;\right] \end{array} \cdot \begin{array}{c} \\ n^s \\ n^- \\ n^{++} \end{array}\begin{array}{c} n \\ \left[\begin{array}{c} \mathbf{0} \\ \begin{bmatrix} \underset{n^-\times n^s}{\mathbf{0}} & \underset{n^-\times n^d}{\mathbf{X}_2^*} \end{bmatrix} \\ \mathbf{0} \end{array}\right] \end{array}\right)$$

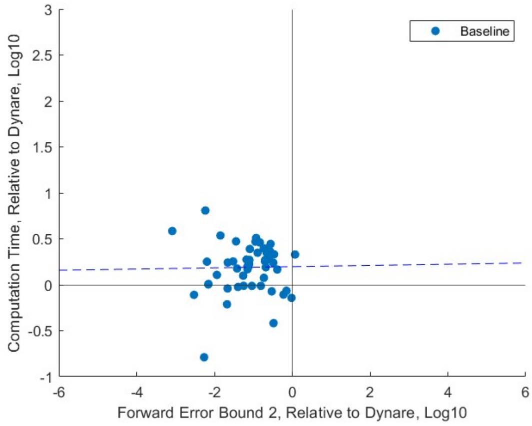
$$= \mathrm{tr}\left(\begin{bmatrix} \underset{n^s\times n^-}{\mathbf{Y}_1} \\ \underset{n^d\times n^-}{\mathbf{Y}_2} \end{bmatrix}\begin{bmatrix} \underset{n^-\times n^s}{\mathbf{0}} & \underset{n^-\times n^d}{\mathbf{X}_2^*} \end{bmatrix}\right)$$

$$= \mathrm{tr}\left(\underset{n^d\times n^-}{\mathbf{Y}_2}\;\underset{n^-\times n^d}{\mathbf{X}_2^*}\right)$$

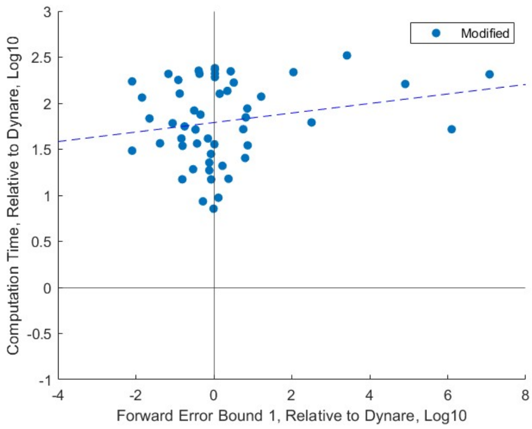
6.5. Additional Figures



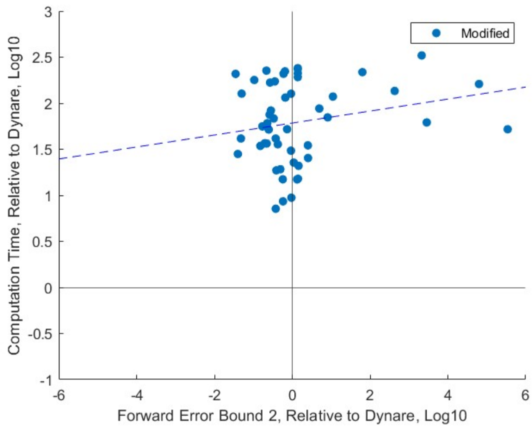
(A) Forward Error 1, Baseline Relative to Dynare



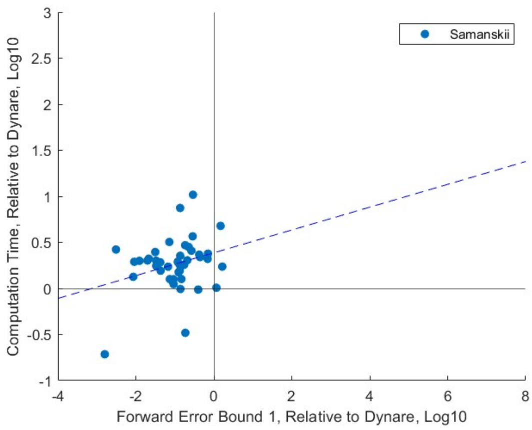
(B) Forward Error 2, Baseline Relative to Dynare



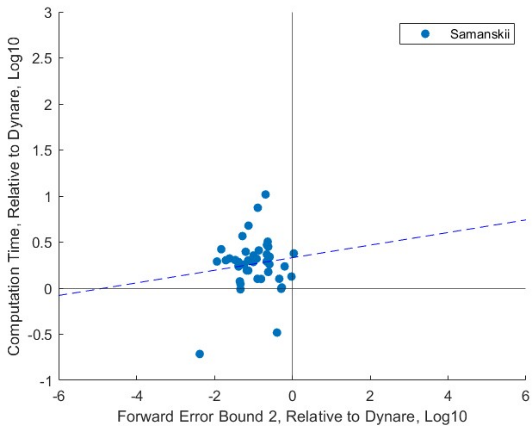
(C) Forward Error 1, Modified Relative to Dynare



(D) Forward Error 2, Modified Relative to Dynare

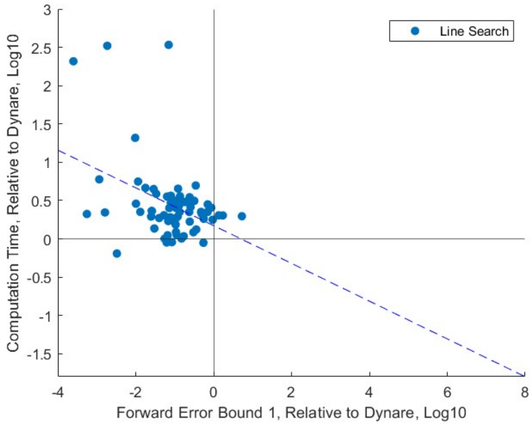


(E) Forward Error 1, Šamanskii Relative to Dynare

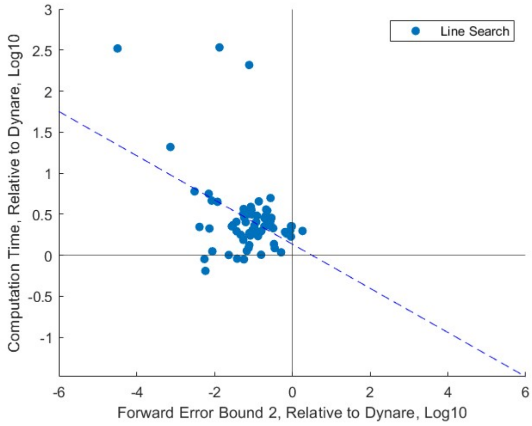


(F) Forward Error 2, Šamanskii Relative to Dynare

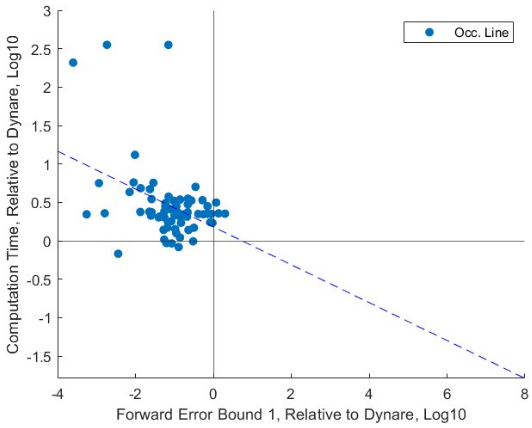
FIGURE 7. Forward Errors and Computation Time for the Macroeconomic Model Data Base (MMB)



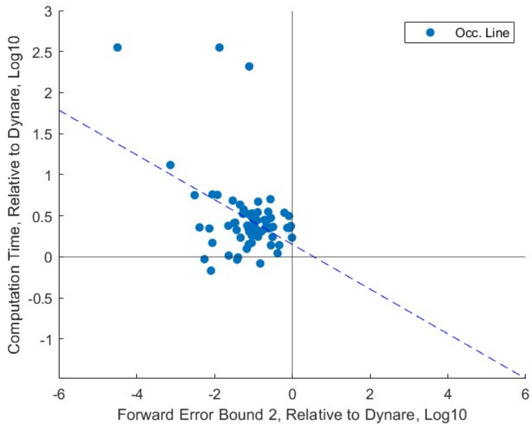
(A) Forward Error 1, Line Searches Relative to Dynare



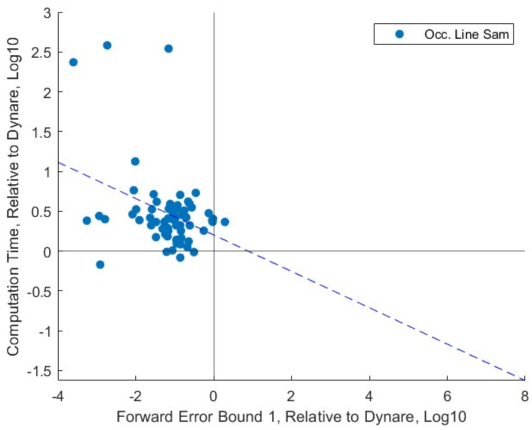
(B) Forward Error 2, Line Searches Relative to Dynare



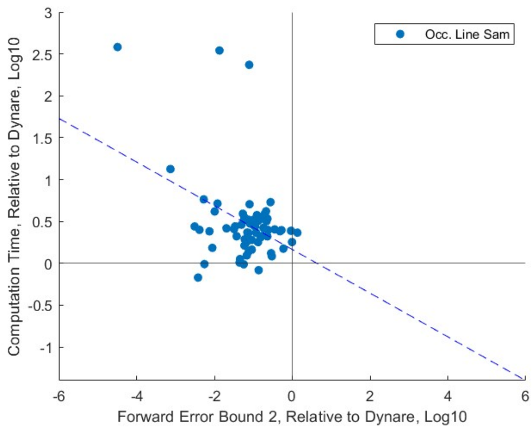
(C) Forward Error 1, Occ. Line Searches Relative to Dynare



(D) Forward Error 2, Occ. Line Searches to Dynare



(E) Forward Error 1, Occ. LS & Šamanskii Relative to Dynare



(F) Forward Error 2, Occ. LS & Šamanskii Relative to Dynare

FIGURE 8. Forward Errors and Computation Time for the Macroeconomic Model Data Base (MMB), Continued

6.6. den Haan and Marcet (1994) Tests

Following Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006), Meyer-Gohde (2023) presents the den Haan and Marcet (1994) error in simulation statistic explicitly in a canonical multivariate DSGE setting. For the model (3)

$$0 = AE_t[y_{t+1}] + By_t + Cy_{t-1} + D\varepsilon_t \tag{A2}$$

define the residuals

$$u_{t+1} = Ay_{t+1} + By_t + Cy_{t-1} + D\varepsilon_t \tag{A3}$$

Clearly $E_t[u_{t+1}] = 0$ and likewise $E_t[u_{t+1} \otimes z_t] = 0$ for any n_z set of t measurable instruments. Given the solution of the model in the form of (4)

$$y_t = P\, y_{t-1} + Q\, \varepsilon_t \tag{A4}$$

the residuals can be expressed as

$$u_{t+1} = A\hat{Q}\varepsilon_{t+1} + \left(A\hat{P}^2 + B\hat{P} + C\right)\hat{y}_{t-1} + \left(A\hat{P}\hat{Q} + B\hat{Q} + D\right)\varepsilon_t \tag{A5}$$

$$= A\hat{Q}\varepsilon_{t+1} + \left(A\hat{P}\hat{Q} + B\hat{Q} + D\right)\varepsilon_t + \left(A\hat{P}^2 + B\hat{P} + C\right)\left(\hat{Q}\varepsilon_{t-1} + \hat{P}\hat{Q}\varepsilon_{t-2} + \dots\right. \tag{A6}$$

We calculate the sample analogs to $E_t[u_{t+1}] = 0$ and likewise $E_t[u_{t+1} \otimes z_t] = 0$ by generating a number N of simulations of length $T + B$ for each of the different methods considered here. We fix the random number seed across methods so that there is no sampling variation across methods. Dynare's QZ and the different Newton models here provide different solutions in the form of P and Q. Accordingly, we label the different linear solutions y_t^j with j corresponding to method j associated with P_j and Q_j given by

$$y_t^j = P_j\, y_{t-1}^j + Q_j\, \varepsilon_t \tag{A7}$$

Starting at $y_0^j = 0$ for all j and for a given sequence of $\{\varepsilon_t\}_{t=1}^{T+B}$, gives the final simulation $\{y_t^j\}_{t=B+1}^{T+B}$, with B being burn-in that we set at 500 periods. The simulated counterpart of $E_t[u_{t+1} \otimes z_t] = 0$ is[18]

$$M_j = \frac{1}{T}\sum_{t=1}^{T}\left(Ay_{t+1}^j + By_t^j + Cy_{t-1}^j + D\varepsilon_t\right) \otimes z_t^j \tag{A8}$$

and an estimate Ω_j of the variance of $\left(Ay_{t+1}^j + By_t^j + Cy_{t-1}^j + D\varepsilon_t\right) \otimes z_t^j$, den Haan and Marcet (1994) give the test statistic

$$J_j = T\, M_j' \Omega_j^{-1} M_j \tag{A9}$$

that is asymptotically χ^2 distributed with potentially $n_y n_z$ degrees of freedom. Examining (A5) the statistic for $z_t = 1$ will be distributed χ^2 with $rank(A\hat{Q})$ degrees of freedom, as not all equations are linearly independent in their expectation components (or contain expectations at all, e.g., market clearing constraints, identities, etc.) and are linear combinations of the underlying shocks in ε.[19]

[18] The measure $E_t[u_{t+1}] = 0$ is of course the simple special case when $z_t = 1$.

[19] Similar conditions can be discerned for different instruments z_t.

T	Residuals		Instrument	
	<5%	>95%	<5%	>95%
500	3.8	5.3	0	86.9
1000	3.9	4.6	0.1	57.4
2000	5.1	4.4	0.8	31.9
5000	4.5	4.6	1.8	14.1
10,000	5.9	5.5	3.6	10.5
100,000	5	5.7	4.2	6.3

TABLE 6. den Haan and Marcet (1994) Tests: Smets-Wouters Model

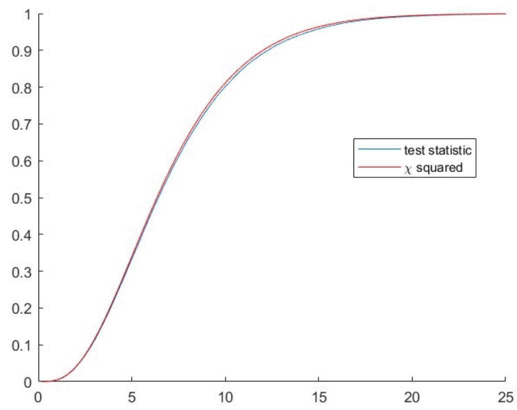
- T is the length of the simulations, the columns report the percentage of the 1000 simulations that fall outside the 90% range of the χ squared distribution.
- Residuals and instrument refer to the tests run with the residuals directly and instrumented against seven time t endogenous variables.
- Dynare's QZ and all Newton methods, initialized at the QZ solutions, deliver identical values for the test statistic.

Table 6 contains the percentages of the $N = 1000$ simulations of different lengths T that we calculated that are in the upper and lower 5% of the reference χ^2 distribution. For accurate solutions, these both should be close to 5% and indeed we see that for the residuals roughly independent of the simulation length and when using the instruments - output, the nominal interest rate, capital, consumption, investment, inflation, and the nominal wage, all at time t - likewise for longer simulations.[20] The results for the simulation lengths of $T = 100,000$ are particularly interesting, as den Haan and Marcet (1994) consider $T = 20,000$ an enormous sample and hence the resulting examination of a model's simulation a very stringent test. This is perhaps not surprising as the numerical error for the methods starting at the QZ solution of the posterior parameterization of Smets and Wouters (2007) is very well behaved numerically - the errors using our favored forward errors in the main text were very small, and it is unlikely that any remaining numerical error will be able to shine through the blinding light of sampling variation.

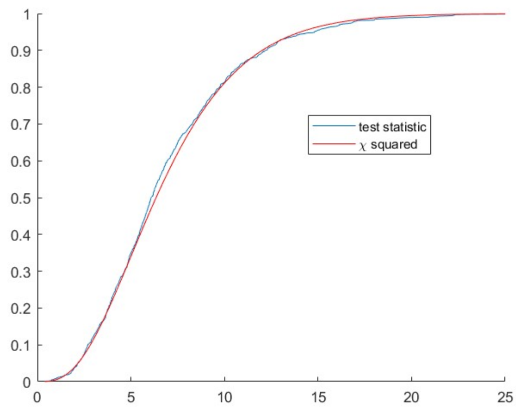
Crucially, all of the different Newton and the QZ method produce the same percentage of simulations above and below the 95% and 5% thresholds. That is the den Haan and Marcet (1994) statistic is unable to discern between the different methods. This is perhaps not surprising as previous studies have noted the test's low power in settings where even different nonlinear methods are being compared, see Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) and also Juillard and Villemot (2011), so numerical inaccuracies in different linear methods are unlikely to be easily detected by this method.

Figure 9 depicts the results graphically. Clearly the null that the approximated solutions' residuals are mean zero is difficult to reject for long or many simulations. For the instrumented statistic however, we see that the instruments perform poorly for shorter simulations, as the errors in simulation likely become

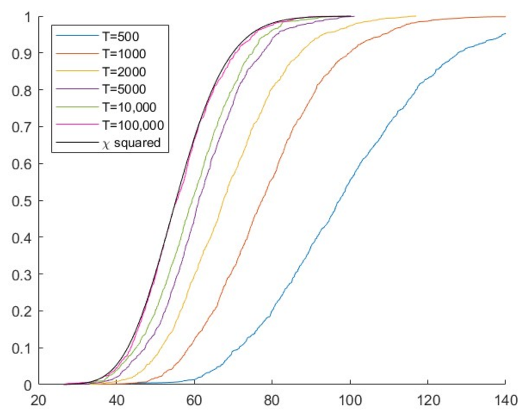
[20]To calculate Ω_j in this case, we followed Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) and used the Newey and West (1987) estimator with the rule of thumb lag truncation in, e.g., Stock and Watson (2020).



(A) Residuals, T=1000 and N=100,000



(B) Residuals, T=100,000 and N=1000



(C) Instrument, Varying T

FIGURE 9. den Haan and Marcet (1994) Tests: Smets-Wouters Model

T is the length of the simulations, N the number of different simulations. Residuals and instrument refer to the tests run with the residuals directly and instrumented against seven time t endogenous variables. Dynare's QZ and all Newton methods, initialized at the QZ solutions, deliver identical values for the test statistic.

serially correlated with the instruments via the approximation errors. Again this is also interesting - here the den Haan and Marcet (1994) statistic *improves* in T, again highlighting that another statistic is needed to discriminate between different numerical solutions to linear DSGE models - and our favored method is the forward error analysis of Meyer-Gohde (2023) in the main text.

REFERENCES

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