Supplemental Material:A Brief Description of NREFT amplitude of $\eta' \to \eta\pi^0\pi^0$ Decay

This supplemental material is based on Ref. [1]. In the $\eta' \to \eta\pi^0\pi^0$ decay

$$\eta'(P_{\eta'}) \to \pi^0(p_1)\pi^0(p_2)\eta(p_3), \tag{1}$$

the kinematical variables s_i are defined as $s_i = (P_{\eta'} - p_i)^2, i = 1, 2, 3,$ and $s_1 + s_2 + s_3 = M_{\eta'}^2 + M_\eta^2 + 2M_{\pi^0}^2$. The Dalitz plot distribution of this decay can also described by kinematical variables X and Y

$$\begin{aligned} X &= \frac{\sqrt{3}|s_1 - s_2|}{2M_{\eta'}Q_{\eta'}} = \frac{\sqrt{3}|T_{\pi_1^0} - T_{\pi_2^0}|}{Q_{\eta'}}, \\ Y &= \frac{(M_\eta + 2M_{\pi^0})[(M_{\eta'} - M_\eta)^2 - s_3]}{2M_{\eta'}M_{\pi^0}Q_{\eta'}} - 1 \\ &= \frac{(M_\eta + 2M_{\pi^0})T_\eta}{M_{\pi^0}Q_{\eta'}} - 1, \end{aligned} \tag{2}$$

where T_i denote kinetic energy of mesons in the rest frame of η', and $Q_{\eta'} = M_{\eta'} - M_\eta - 2M_{\pi^0}$. The Dalitz plot distribution can be expanded by X and Y around the center of the Dalitz plot

$$|\mathcal{M}(X,Y)|^2 = |\mathcal{N}|^2(1 + aY + bY^2 + cX + dX^2 + \cdots), \tag{3}$$

which is known as general parameterization. Here $\mathcal{N}$ is a normalization factor and parameter c is fixed at 0 since two π^0s are identical bosons. The general parameterization can be also expressed as

$$\mathcal{M}(X,Y) = \mathcal{N}\{1 + \frac{a}{2}Y + \frac{1}{2}(b - \frac{a^2}{4})Y^2 + \frac{d}{2}X^2 + \cdots\}. \tag{4}$$

The NREFT amplitude of $\eta' \to \eta\pi\pi$ can be decomposed to

$$\begin{aligned} \mathcal{M}_{\eta'\to\eta\pi^0\pi^0} &= \mathcal{M}_N^{tree} + \mathcal{M}_N^{one-loop} + \mathcal{M}_N^{two-loop} + \cdots, \\ \mathcal{M}_{\eta'\to\eta\pi^+\pi^-} &= \mathcal{M}_C^{tree} + \mathcal{M}_C^{one-loop} + \mathcal{M}_C^{two-loop} + \cdots. \end{aligned} \tag{5}$$

The tree level amplitudes are

$$\begin{aligned} \mathcal{M}_N^{tree}(s_1, s_2, s_3) &= \sum_{i=0}^{2} G_i X_3^i + G_3(X_1 - X_2)^2, \\ \mathcal{M}_C^{tree}(s_1, s_2, s_3) &= \sum_{i=0}^{2} H_i X_3^i + H_3(X_1 - X_2)^2, \end{aligned} \tag{6}$$

where $X_k = p_k^0 - M_\eta, k = 1, 2, 3,$ p_i^0 is the energy of particle i in the η' rest frame, and parameters G_i are the low-energy coupling coefficients of $\eta' \to \eta\pi^0\pi^0$ decay and H_i for $\eta' \to \eta\pi^+\pi^-$ decay. The charged decay mode is introduced for the further description of loop level amplitude and we assume $H_i = -G_i$ according to the isospin limit [1–3]. G_i can be evaulated by matching to the general parameterization

$$\begin{aligned} G_0 &= \mathcal{N}\{1 - \frac{a}{2} + \frac{1}{2}(b - \frac{a^2}{4})\}, \\ G_1 &= \mathcal{N}\{\frac{a}{2} - (b - \frac{a^2}{4})\}\frac{M_\eta + 2M_{\pi^0}}{M_{\pi^0}Q_{\eta'}}, \\ G_2 &= \mathcal{N}(b - \frac{a^2}{4})\frac{(M_\eta + 2M_{\pi^0})^2}{2M_{\pi^0}^2 Q_{\eta'}^2}, \\ G_3 &= \mathcal{N}\frac{3d}{2Q_{\eta'}^2}. \end{aligned} \tag{7}$$

The loop level amplitude of $\eta' \to \eta\pi^0\pi^0$ decay are

$$\begin{aligned}
\mathcal{M}_N^{one-loop}(s_1,s_2,s_3) &= \mathcal{B}_{N1}(s_3)J_{+-}(s_3) \\
&+ \mathcal{B}_{N2}(s_3)J_{00}(s_3), \\
\mathcal{M}_N^{two-loop}(s_1,s_2,s_3) &= C_{00}(s_3)\mathcal{B}_{N2}(s_3)J_{00}^2(s_3) \\
&+ C_{00}(s_3)\mathcal{B}_{N1}(s_3)J_{00}(s_3)J_{+-}(s_3) \\
&+ 2C_x(s_3)\mathcal{B}_{C2}(s_3)J_{00}(s_3)J_{+-}(s_3) \\
&+ 2C_x(s_3)\mathcal{B}_{C1}(s_3)J_{+-}^2(s_3),
\end{aligned} \tag{8}$$

with one-loop function

$$\begin{aligned}
J_{ab} &= \frac{iq_{ab}(s_k)}{8\pi\sqrt{s_k}}, \\
q_{ab}^2(s) &= \frac{\lambda(s,M_a^2,M_b^2)}{4s}, \\
\lambda(a,b,c) &= a^2+b^2+c^2-2(ab+ac+bc),
\end{aligned} \tag{9}$$

and neutral channel polynomials

$$\begin{aligned}
\mathcal{B}_{N1}(s_3) &= 2C_x(s_3)[\sum_{i=0}^{2} H_i X_3^i + H_3\frac{4Q_3^2}{3s_3}q_{+-}^2(s_3)], \\
\mathcal{B}_{N2}(s_3) &= C_{00}(s_3)[\sum_{i=0}^{2} G_i X_3^i + G_3\frac{4Q_3^2}{3s_3}q_{00}^2(s_3)],
\end{aligned} \tag{10}$$

and charged channel polynominals

$$\begin{aligned}
\mathcal{B}_{C1}(s_3) &= 2C_{+-}(s_3)[\sum_{i=0}^{2} H_i X_3^i + H_3\frac{4Q_3^2}{3s_3}q_{+-}^2(s_3)], \\
\mathcal{B}_{C2}(s_3) &= C_x(s_3)[\sum_{i=0}^{2} G_i X_3^i + G_3\frac{4Q_3^2}{3s_3}q_{00}^2(s_3)],
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
C_{bc}(s_a) &= C_{bc} + 4D_{bc}q_{bc}^2(s_a) + 16F_{bc}q_{bc}^4(s_a), \\
Q_a^2 &= \frac{\lambda(M_{\eta'}^2, M_a^2, s_a)}{4M_{\eta'}^2}.
\end{aligned} \tag{12}$$

The parameters C_i, D_i and F_i are coupling coefficients of $\pi\pi$ interaction and are evaulated by matching to the effective range expansion of $\pi\pi$ scattering, where i represent differnet $\pi\pi$ rescattering channels: $(00)\pi^0\pi^0 \to \pi^0\pi^0$;$(x)\pi^+\pi^- \to$

$\pi^0\pi^0$;$(+-)\pi^+\pi^- \to \pi^+\pi^-$,

$$\begin{aligned}
C_{00} &= \frac{16\pi}{3}(a_0+2a_2)(1-\xi),\\
C_x &= \frac{16\pi}{3}(a_2-a_0)(1+\frac{\xi}{3}),\\
C_{+-} &= \frac{8\pi}{3}(2a_0+a_2)(1+\xi),\\
\xi &= \frac{M^2_{\pi^\pm}-M^2_{\pi^0}}{M^2_{\pi^\pm}},\\
D_{00} &= \frac{4\pi}{3}(b_0+2b_2),\\
D_x &= \frac{4\pi}{3}(b_2-b_0),\\
D_{+-} &= \frac{2\pi}{3}(2b_0+b_2),\\
F_{00} &= \frac{\pi}{3}(f_0+2f_2),\\
F_x &= \frac{\pi}{3}(f_2-f_0),\\
F_{+-} &= \frac{\pi}{6}(2f_0+f_2),
\end{aligned} \tag{13}$$

where a_i, b_i and f_i are S-wave scattering length, effective ranges and shape parameters of isospin 0 and 2, respectively. a_0 and a_2 are taken as free or fixed parameters in differnet cases in our study, and b_i are fixed to theoretical value $b_0 = (0.276 \pm 0.006) \times M_\pi^{-2}$, $b_2 = (-0.0803 \pm 0.0012) \times M_\pi^{-2}$, and f_i are fixed to 0. The $\pi\eta$ scattering terms are ignored because the $\pi\eta$ scattering is much weaker than the $\pi\pi$ scattering.

[1] B. Kubis and S. P. Schneider, Eur. Phys. J. C **62**, 511 (2009).
[2] P. Adlarson *et al.* (A2 Collaboration), Phys. Rev. D **98**, 012001 (2018).
[3] M. Ablikim *et al.* (BESIII Collaboration), Phys. Rev. D **97**, 012003 (2018).