Supplementary Material: a novel fast Monte-Carlo method to estimate the statistical uncertainty of a ratio observable used in this letter

In this measurement, a data driven Monte Carlo (MC) method has been used to quantify the correlated uncertainties in the ratio quantity $\mathbb{R} = \frac{\mathrm{R}(\Psi_1)}{\mathrm{R}(\Psi_2)}$, where $\mathrm{R} = \Delta\gamma/v_2 = \langle\cos(\phi^\alpha + \phi^\beta - 2\Psi)\rangle/\langle\cos(\phi - \Psi)\rangle$ as described in the main text. In this quantity, although the event planes (Ψ) are estimated with different acceptance in $\mathrm{R}(\Psi_1)$ and $\mathrm{R}(\Psi_2)$, the particles of interest for $\Delta\gamma$ (ϕ_α, ϕ_β) and v_2 (ϕ) measurements are from the same TPC acceptance. Thus the possible anti-correlation/correlation in the variance of the ratio needs to be examined.

In high energy physics, to study the statistical uncertainties, the most widely used Monte Carlo method is called the "Bootstrap method" [34]. In this study, we designed a new method specific for ratio quantities $\langle x\rangle / \langle y\rangle$. We call this new approach as the "AB method" which is computationally economical. We have also checked the consistency of our approach with the classical Bootstrap method using experimental data. To perform this consistency check we have used about one third of the whole statistics.

For the Bootstrap method we follow the approach described in Ref [34]. The Bootstrap approach requires creating copies of the data sample through Monte Carlo sampling in which some of the events will be duplicated while some will be absent, by construction. We perform this sampling procedure N times to get a distribution of the ratio observable $P^{Bootstrap}(\mathbb{R})$. From the distribution we estimate the mean $\mu^{Bootstrap}$ and width $\sigma^{Bootstrap}$ of the ratio $\mathbb{R}$.

For the AB method, we divide the entire data sample into two halves. We call the two halves "group A" and "group B". For the ratio $\mathbb{R} = \mathrm{R}(\Psi_1)/\mathrm{R}(\Psi_2) = \langle \mathrm{x}\rangle / \langle \mathrm{y}\rangle$, we estimate x and y from the two groups and label them as $\langle x(A)\rangle$, $\langle x(B)\rangle$, $\langle y(A)\rangle$, and $\langle y(B)\rangle$. Thus we can estimate the ratios $\langle x(A)\rangle / \langle y(A)\rangle$, $\langle x(A)\rangle / \langle y(B)\rangle$, $\langle x(B)\rangle / \langle y(A)\rangle$, and $\langle x(B)\rangle / \langle y(B)\rangle$. When the $\langle x\rangle$ and $\langle y\rangle$ come from the same half ($\langle x(A)\rangle / \langle y(A)\rangle$ and $\langle x(B)\rangle / \langle y(B)\rangle$), we call the ratios "AB-same" and when they are came from the different halves ($\langle x(A)\rangle / \langle y(B)\rangle$ and $\langle x(B)\rangle / \langle y(A)\rangle$), we call them "AB-cross". Note, in this case each sample gives us two entries for both AB-same and AB-cross. We repeat the sampling procedure N times to get the probability distributions for AB-same ($P^{AB-same}(\mathbb{R})$) and AB-cross ($P^{AB-cross}(\mathbb{R})$. For the AB-same we can estimate the mean $\mu^{AB-same}$ and width $\sigma^{AB-same}$. Similarly, for AB-cross we can estimate the mean $\mu^{AB-cross}$ and the width $\sigma^{AB-cross}$.

The Bootstrap method is expected to lead to a variance of ratio similar to the analytical expression of variance (σ^2) including correlated fluctuations:

$$\begin{aligned}\sigma^{Bootstrap} \approx\ & (\sigma_x^2(\partial\mathbb{R}/\partial\langle y\rangle)^2 + \sigma_y^2(\partial\mathbb{R}/\partial\langle x\rangle)^2 \\ & + 2\rho\sigma_y\sigma_x(\partial^2 R/\partial\langle y\rangle\partial\langle x\rangle))^{\frac{1}{2}},\end{aligned} \tag{16}$$

where σ_x and σ_y are the widths of the distributions of the numerator and denominator, respectively. ρ is the correlation coefficient. The $\sigma^{AB-same}$ should have the expression as Eq. 16. For AB-cross sample, it should be:

$$\sigma^{AB-cross} = (\sigma_x^2(\partial\mathbb{R}/\partial\langle y\rangle)^2 + \sigma_y^2(\partial\mathbb{R}/\partial\langle x\rangle)^2))^{\frac{1}{2}}, \tag{17}$$

there is no correlation term in contrast to Eq. 16 because these samples of x and y are uncorrelated.

Our expectations are the following:

1. All the three cases should give rise to the same value of mean.

$$\mu^{AB-same} = \mu^{AB-cross} = \mu^{Bootstrap} \tag{18}$$

2. If there is an anti-correlation, we should get:

$$\sigma^{AB-same} \approx \sigma^{Bootstrap} > \sigma^{AB-cross} \tag{19}$$

3. If there is a correlation, we should get:

$$\sigma^{AB-cross} > \sigma^{Bootstrap} \approx \sigma^{AB-same} \tag{20}$$

The expectations of Eq. 19,20 for $\sigma^{AB \quad cross}$ are very easy to understand. Since $\sigma^{AB \quad cross}$ is estimated from two independent data sets, there should be no co-variance between the numerator and the denominator. Therefore, in the presence of correlations ($\rho > 0$) and anti-correlations ($\rho < 0$), the variance of ratio of the terms from two independent data sets will be over and underestimated, respectively. The expectations that $\sigma^{AB \quad same}$ and $\sigma^{Bootstrap}$ are approximately equal is not straightforward but can be easily demonstrated by Monte Carlo simulations as follows.

The results from our exercise are shown in Fig. 6 in terms of the distributions of the ratio observable $P(\mathbb{R})$ in different centralities and acceptance after sampling 3000 times. The left side panels are for the measurements on $\mathrm{R}(\Psi_1)/\mathrm{R}(\Psi_{2,\ 1<|\eta|})$, the right side panels are the measurements on $\mathrm{R}(\Psi_1)/\mathrm{R}(\Psi_{2,\ 2.1<|\eta|<\mathrm{Y_{beam}}})$. The histograms are fitted with Gaussian distributions. The AB-same and Bootstrap give very similar results as expected (see Eq. 19 and 20). The relative differences between the widths obtained from these two methods are consistent within 1%. From our exercise we observe a slightly wider width for the AB-cross case, which indicates the presence of correlated fluctuations as per Eq 20. From the AB-cross results, we find the width difference is less than 5% compared to the AB-same case in 10 50% centrality.

The width of the distribution is proportional to the statistical uncertainty in the measurements of the ratio. We have established the consistency between AB-same and Bootstrap. Therefore, according to Eq.16 and Eq.17 the difference in the widths between AB-same and AB-cross method is an estimate between the true statistical uncertainty and the one ignoring the co-variance term in error propagation. Our exercise indicate the presence of correlated fluctuations and that as a result, the analytical method of error propagation ignoring co-variance overestimates the statistical uncertainty in the quantity $\mathbb{R}$ by 5%.

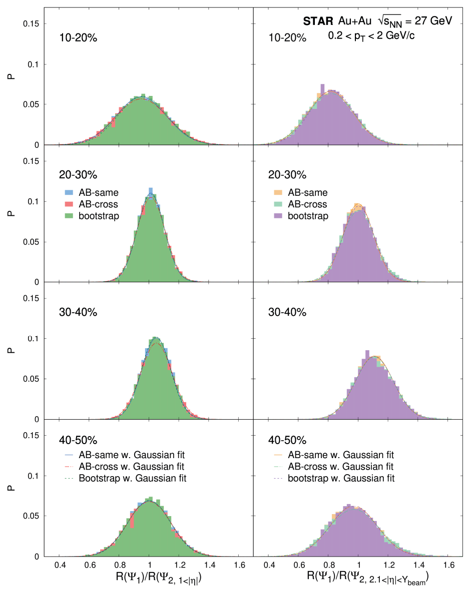

Figure 6: The distribution for the AB-same, AB-cross, and Bootstrap method after 3000 times sampling for 10 50% centrality. The histograms are fitted with Gaussian distributions as shown by lines with different colors. All the distributions have a similar mean. The AB-same and Bootstrap distributions correspond to the correct variance. The wider distributions for AB-cross include correlations.