SUPPLEMENTARY MATERIAL

This supplementary material is structured in three sections. Section SM-1 discusses the second-order transport coefficients from the Shakhov model. Section SM-2 presents the entropy production, while Section SM-3 summarizes the details of the numerical scheme used to solve the Shakhov model equation.

SM-1. Second-order transport coefficients of the relativistic Shakhov model

In this section we employ the method of moments of Refs. [1, 2] to derive the first- and second-order transport coefficients corresponding to the relativistic Shakhov model. These transport coefficients arise at first- and second-order with respect to the Knudsen number Kn, being the ratio of the particle mean free path and a characteristic macroscopic scale, and the inverse Reynolds number Re^{-1}, being the ratio of an out-of-equilibrium and a local-equilibrium macroscopic field.

Irreducible moments and orthogonal basis.– The irreducible moments from Eq. (6) are expressed as [1],

$$\delta f_{\mathbf{k}} = f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_n^{\mu_1\cdots\mu_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \mathcal{H}_{\mathbf{k}n}^{(\ell)}, \qquad \text{(SM-1)}$$

where $N_\ell \to \infty$ is an expansion order. The functions $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ are polynomials of order N_ℓ with respect to $E_{\mathbf{k}}$, defined in full generality in Eq. (29) of Ref. [1], and are constructed such that Eq. (6) is satisfied for $0 \le r \le N_\ell$. We remark that, while Eq. (SM-1) employs an irreducible basis, the expansion does not account explicitly for the negative-order moments $\rho_r^{\mu_1\cdots\mu_\ell}$ with $r < 0$, but these must be reconstructed from those with $0 \le r \le N_\ell$ in a manner which becomes exact only in the limit $N_\ell \to \infty$. The simple structure of the RTA model allows us to circumvent such construction in Eq. (SM-1) by employing a basis-free approach, as discussed in Ref. [3].

We note that the functions $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$, related to the representation of $\delta f_{\mathbf{k}}$ are also useful in the context of the Shakhov model. However, for the Shakhov distribution, N_ℓ is not the expansion order of $\delta f_{\mathbf{k}}$, but the order of the $\mathcal{H}_{\mathbf{k}0}^{(\ell)}$ polynomials satisfying the constraints in Eq. (20), namely $N_0 = 2$, $N_1 = 1$, and $N_2 = 0$.

The Shakhov collision term from Eq. (12) is

$$C_{r-1}^{\mu_1\cdots\mu_\ell} = -\frac{1}{\tau_R}\rho_r^{\mu_1\cdots\mu_\ell} + \frac{1}{\tau_R}\rho_{\mathrm{S},r}^{\mu_1\cdots\mu_\ell}, \qquad \text{(SM-2)}$$

where the second term involves the irreducible moments of $\delta f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\mathbb{S}_{\mathbf{k}}$ defined in Eq. (14). Now, using the Shakhov distribution from Eq. (22), leads to

$$\rho_{\mathrm{S},r} = -\frac{3\Pi}{m_0^2}\left(1 - \frac{\tau_R}{\tau_\Pi}\right)\mathcal{F}_{-r,0}^{(0)}, \quad \rho_{\mathrm{S},r}^{\mu} = V^\mu\left(1 - \frac{\tau_R}{\tau_V}\right)\mathcal{F}_{-r,0}^{(1)},$$
$$\rho_{\mathrm{S},r}^{\mu\nu} = \pi^{\mu\nu}\left(1 - \frac{\tau_R}{\tau_\pi}\right)\mathcal{F}_{-r,0}^{(2)}, \qquad \text{(SM-3)}$$

while the higher-rank moments are set to vanish, i.e., $\rho_{\mathrm{S},r}^{\mu_1\cdots\mu_\ell} = 0$ with $\ell > 2$. Now, using Eq. (28) for polynomial orders $N_0 = 2$, $N_1 = 1$ and $N_2 = 0$ ensures that $\mathcal{F}_{0,0}^{(0)} = \mathcal{F}_{0,0}^{(1)} = \mathcal{F}_{0,0}^{(2)} = 1$ and $\mathcal{F}_{-1,0}^{(0)} = \mathcal{F}_{-2,0}^{(0)} = \mathcal{F}_{-1,0}^{(1)} = 0$.

The second-order transport coefficients also require the knowledge of various other moments $\rho_{r\neq0}^{\mu_1\cdots\mu_\ell}$. Here we recall the first-order approximation to such irreducible moments in the so-called basis-free approach of Ref. [3]:

$$\rho_{r\neq0} \simeq -\frac{3}{m_0^2}\mathcal{R}_{r0}^{(0)}\Pi, \quad \rho_{r\neq0}^{\mu} \simeq \mathcal{R}_{r0}^{(1)}V^\mu, \quad \rho_{r\neq0}^{\mu\nu} \simeq \mathcal{R}_{r0}^{(2)}\pi^{\mu\nu}, \qquad \text{(SM-4)}$$

where

$$\mathcal{R}_{r0}^{(0)} = \frac{\zeta_r}{\zeta}, \quad \mathcal{R}_{r0}^{(1)} = \frac{\kappa_r}{\kappa}, \quad \mathcal{R}_{r0}^{(2)} = \frac{\eta_r}{\eta}. \qquad \text{(SM-5)}$$

Now, substituting the expressions for the first-order transport coefficients from Eqs. (31) into Eq. (SM-5) gives

$$\mathcal{R}_{-r,0}^{(\ell)} = \frac{\tau_R}{\tau_{\mathrm{S}}^{(\ell)}}\frac{\alpha_{-r}^{(\ell)}}{\alpha_0^{(\ell)}} + \left(1 - \frac{\tau_R}{\tau_{\mathrm{S}}^{(\ell)}}\right)\mathcal{F}_{r0}^{(0)}. \qquad \text{(SM-6)}$$

Using these results, the relaxation times can be computed using Eqs. (38) of Ref. [4]:

$$\tau_\Pi = \sum_{r\neq1,2}\tau_{0r}^{(0)}\mathcal{R}_{r0}^{(0)}, \quad \tau_V = \sum_{r\neq1}\tau_{0r}^{(1)}\mathcal{R}_{r0}^{(1)}, \quad \tau_\pi = \sum_{r}\tau_{0r}^{(2)}\mathcal{R}_{r0}^{(2)}. \qquad \text{(SM-7)}$$

Recalling the expression for $\tau_{nr}^{(\ell)}$ from Eqs. (29) together with Eq. (SM-6), the above definitions leads to $\tau_\Pi = \tau_{\mathrm{S}}^{(0)}$, $\tau_V = \tau_{\mathrm{S}}^{(1)}$ and $\tau_\pi = \tau_{\mathrm{S}}^{(2)}$, as expected.

As discussed in Ref. [3], the second-order transport coefficients involve only the coefficients $\mathcal{R}_{-1,0}^{(\ell)}$ and $\mathcal{R}_{-2,0}^{(\ell)}$. These coefficients also require the expressions for $\mathcal{F}_{r0}^{(\ell)}$, computed using the functions $\mathcal{H}_{\mathbf{k}0}^{(\ell)}$ in Eq. (23), as shown below:

$$\mathcal{F}_{r0}^{(0)} = \frac{J_{-r,0}G_{33} - J_{1-r,0}G_{23} + J_{2-r,0}G_{22}}{J_{00}G_{33} - J_{10}G_{23} + J_{20}G_{22}},$$
$$\mathcal{F}_{r0}^{(1)} = \frac{J_{2-r,1}J_{41} - J_{3-r,1}J_{31}}{J_{21}J_{41} - J_{31}^2}, \quad \mathcal{F}_{r0}^{(2)} = \frac{J_{4-r,2}}{J_{42}}. \qquad \text{(SM-8)}$$

Equations of motion.– The relaxation equations for $\Pi = -m_0^2\rho_0/3$, $V^\mu = \rho_0^\mu$, and $\pi^{\mu\nu} = \rho_0^{\mu\nu}$ are obtained by setting $n = 0$ in Eqs. (30). Up to second order with respect to Kn and Re^{-1}, these equations read, see Eqs. (88-93) in Ref. [3],

$$\tau_\Pi\dot{\Pi} + \Pi = -\zeta\theta - \ell_{\Pi V}\nabla_\mu V^\mu - \tau_{\Pi V}V_\mu\dot{u}^\mu - \delta_{\Pi\Pi}\Pi\theta$$
$$- \lambda_{\Pi V}V_\mu\nabla^\mu\alpha + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}, \qquad \text{(SM-9a)}$$
$$\tau_V\dot{V}^{\langle\mu\rangle} + V^\mu = \kappa\nabla^\mu\alpha - \tau_V V_\nu\omega^{\nu\mu} - \delta_{VV}V^\mu\theta$$
$$- \ell_{V\Pi}\nabla^\mu\Pi + \ell_{V\pi}\Delta^{\mu\nu}\nabla_\lambda\pi_\nu^\lambda + \tau_{V\Pi}\Pi\dot{u}^\mu - \tau_{V\pi}\pi^{\mu\nu}\dot{u}_\nu$$
$$- \lambda_{VV}V_\nu\sigma^{\mu\nu} + \lambda_{V\Pi}\Pi\nabla^\mu\alpha - \lambda_{V\pi}\pi^{\mu\nu}\nabla_\nu\alpha, \qquad \text{(SM-9b)}$$
$$\tau_\pi\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + 2\tau_\pi\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta$$
$$- \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} - \tau_{\pi V}V^{\langle\nu}\dot{u}^{\mu\rangle}$$
$$+ \ell_{\pi V}\nabla^{\langle\mu}V^{\nu\rangle} + \lambda_{\pi V}V^{\langle\mu}\nabla^{\nu\rangle}\alpha. \qquad \text{(SM-9c)}$$

Shakhov model for the Bjorken flow.– In the case of the Bjorken expansion, we considered a massive, ideal, uncharged gas, such that $\alpha_r^{(0)}$ is given by Eq. (10). The first-order transport coefficients ζ and η are listed in Eqs. (38). The second-order transport coefficients appearing in Eq. (37) are listed here from Ref. [2]:

$$\delta_{\Pi\Pi} = \tau_\Pi \left(\frac{2}{3} + \frac{m_0^2}{3} \frac{J_{10}}{J_{30}} + \frac{m_0^2}{3} \mathcal{R}_{-2,0}^{(0)} \right), \tag{SM-10}$$

$$\lambda_{\Pi\pi} = \tau_\Pi \frac{m_0^2}{3} \left(\frac{J_{10}}{J_{30}} + \mathcal{R}_{-2,0}^{(2)} \right), \quad \delta_{\pi\pi} = \tau_\pi \left(\frac{4}{3} + \frac{m_0^2}{3} \mathcal{R}_{-2,0}^{(2)} \right),$$

$$\tau_{\pi\pi} = \tau_\pi \left(\frac{10}{7} + \frac{4m_0^2}{7} \mathcal{R}_{-2,0}^{(2)} \right), \quad \lambda_{\pi\Pi} = \tau_\pi \left(\frac{6}{5} + \frac{2m_0^2}{5} \mathcal{R}_{-2,0}^{(0)} \right).$$

Since the Shakhov distribution employed in Eq. (34) uses $\tau_\Pi = \tau_R$, the coefficients $\mathcal{R}_{-r,0}^{(0)}$ reduce to their corresponding values for the AW model, namely

$$\mathcal{R}_{-r,0}^{(0)} \equiv \frac{\alpha_{-r}^{(0)}}{\alpha_0^{(0)}} = \frac{J_{1-r,0}J_{31} - J_{1-r,1}J_{30}}{J_{10}J_{31} - J_{11}J_{30}}, \tag{SM-11}$$

where Eq. (10) was employed to replace $\alpha_r^{(0)}$. On the other hand, $\mathcal{R}_{-r,0}^{(2)}$ becomes

$$\mathcal{R}_{-r,0}^{(2)} = \frac{\tau_\Pi}{\tau_\pi} \frac{J_{3-r,2}}{J_{32}} + \left(1 - \frac{\tau_\Pi}{\tau_\pi} \right) \frac{J_{4-r,2}}{J_{42}}, \tag{SM-12}$$

which, in the limit of $\tau_\Pi = \tau_\pi$, recovers the analogous coefficient appearing in the AW model, $\alpha_{-r}^{(2)}/\alpha_0^{(2)} = J_{3-r,2}/J_{32}$. Therefore, the transport coefficients $\lambda_{\Pi\pi}$, $\delta_{\pi\pi}$, and $\tau_{\pi\pi}$ involving $\mathcal{R}_{-2,0}^{(2)}$ are modified with respect to their AW expressions, while $\delta_{\Pi\Pi}$ and $\lambda_{\pi\Pi}$ remain unchanged.

Shakhov model for longitudinal waves.– In the case of the longitudinal waves concerning an ultrarelativistic classical ideal gas, we have

$$J_{nq} = \frac{P\beta^{2-n}(n+1)!}{2(2q+1)!!}, \qquad P = \frac{ge^\alpha}{\pi^2\beta^4},$$

$$\alpha_r^{(1)} = \frac{P(r+2)!(1-r)}{24\beta^{r-1}}, \qquad \alpha_r^{(2)} = \frac{P(r+4)!}{30\beta^r}. \tag{SM-13}$$

The transport coefficients from Eq. (SM-9b) reduce to [3]:

$$\delta_{VV} = \tau_V, \quad \ell_{V\pi} = \frac{\tau_V}{h}\left(1 - h\mathcal{R}_{-1,0}^{(2)}\right), \quad \tau_{V\pi} = \frac{\tau_V}{h}\left(1 - h\frac{\partial \mathcal{R}_{-1,0}^{(2)}}{\partial \ln\beta}\right),$$

$$\lambda_{VV} = \frac{3}{5}\tau_V, \quad \lambda_{V\pi} = \tau_V \left(\frac{\partial \mathcal{R}_{-1,0}^{(2)}}{\partial\alpha} + \frac{1}{h}\frac{\partial \mathcal{R}_{-1,0}^{(2)}}{\partial\beta} \right), \tag{SM-14}$$

where $h = (e + P)/n$ is the enthalpy per particle. Noting that

$$\mathcal{F}_{10}^{(2)} = \frac{\beta}{5}, \quad \mathcal{R}_{-1,0}^{(2)} = \frac{\beta}{4}\left(1 + \frac{\tau_R - \tau_\pi}{5\tau_\pi}\right), \tag{SM-15}$$

the Shakhov model alters only the following coefficients:

$$\ell_{V\pi} \equiv \tau_{V\pi} = \frac{\beta}{20}\left(1 - \frac{\tau_R}{\tau_\pi}\right)\tau_V, \tag{SM-16a}$$

$$\lambda_{V\pi} = \frac{\beta}{16}\left(1 + \frac{\tau_R - \tau_\pi}{5\tau_\pi}\right)\tau_V. \tag{SM-16b}$$

Similarly, the coefficients appearing in Eq. (SM-9c) are

$$\delta_{\pi\pi} = \frac{4}{3}\tau_\pi, \quad \tau_{\pi\pi} = \frac{10}{7}\tau_\pi, \quad \ell_{\pi V} = \tau_{\pi V} = \lambda_{\pi V} = 0, \tag{SM-16c}$$

while $\kappa = \frac{\beta P}{12}\tau_V$ and $\eta = \frac{4P}{5}\tau_\pi$. The Shakhov collision term considered in Eq. (39) employs $\tau_\pi = \tau_R$, hence the dependence on τ_π disappears in Eqs. (SM-16) and all transport coefficients reduce to the AW ones (with τ_R replaced by τ_V or τ_π, as appropriate), see for comparison Eqs. (168) and (169) in Ref. [3].

SM-2. Entropy production

We now discuss the thermodynamic consistency of the Shakhov model by considering the entropy production

$$\partial_\mu S^\mu = -\int dK\, C_S[f] \ln(f_{\mathbf{k}}/\tilde{f}_{\mathbf{k}}), \tag{SM-17}$$

where $S^\mu = -\int dK\, k^\mu (f_{\mathbf{k}} \ln f_{\mathbf{k}} + a\tilde{f}_{\mathbf{k}} \ln \tilde{f}_{\mathbf{k}})$ is the entropy four-current. As originally pointed out by Shakhov [5], asserting the sign of $\partial_\mu S^\mu$ for arbitrary distributions $f_{\mathbf{k}}$ is difficult, but if the fluid is not far from equilibrium, quadratic terms in $\delta f_{\mathbf{k}}$ or $\delta f_{S\mathbf{k}}$ can be neglected and the logarithm in Eq. (SM-17) can be approximated as:

$$\ln\frac{f_{\mathbf{k}}}{\tilde{f}_{\mathbf{k}}} = \ln\frac{f_{0\mathbf{k}}(1 + \tilde{f}_{0\mathbf{k}}\phi_{\mathbf{k}})}{\tilde{f}_{0\mathbf{k}}(1 - af_{0\mathbf{k}}\phi_{\mathbf{k}})} \simeq \ln\frac{f_{0\mathbf{k}}}{\tilde{f}_{0\mathbf{k}}} + \phi_{\mathbf{k}} + O(\phi_{\mathbf{k}}^2), \tag{SM-18}$$

where $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}}/(f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}})$. Thus, Eq. (SM-17) becomes

$$\partial_\mu S^\mu \simeq \frac{1}{\tau_R}\int dK E_{\mathbf{k}}(\delta f_{\mathbf{k}} - \delta f_{S\mathbf{k}}) \ln\frac{f_{0\mathbf{k}}}{\tilde{f}_{0\mathbf{k}}} + \frac{1}{\tau_R}\int dK E_{\mathbf{k}} \delta f_{\mathbf{k}} \left(\phi_{\mathbf{k}} - \mathbb{S}_{\mathbf{k}}\right), \tag{SM-19}$$

where on the second line, we have used the relation $(\delta f_{\mathbf{k}} - \delta f_{S\mathbf{k}})\phi_{\mathbf{k}} = \delta f_{\mathbf{k}}(\phi_{\mathbf{k}} - \mathbb{S}_{\mathbf{k}})$ with $\mathbb{S}_{\mathbf{k}} = \delta f_{S\mathbf{k}}/\widetilde{f_{0\mathbf{k}} f_{0\mathbf{k}}}$. Since $\ln(f_{0\mathbf{k}}/\tilde{f}_{0\mathbf{k}}) = \alpha - \beta E_{\mathbf{k}}$, the first term on the right-hand side of the equation vanishes due to the matching conditions in Eq. (19). The second term can be estimated using Eq. (13), leading to

$$\phi_{\mathbf{k}} - \mathbb{S}_{\mathbf{k}} \simeq -\frac{\tau_R}{E_{\mathbf{k}}} k^\mu \partial_\mu(\alpha - \beta E_{\mathbf{k}}), \tag{SM-20}$$

and with Eq. (17) confirms the second law of thermodynamics,

$$\partial_\mu S^\mu \simeq \frac{\beta}{\zeta}\Pi^2 - \frac{1}{\kappa}V_\mu V^\mu + \frac{\beta}{2\eta}\pi_{\mu\nu}\pi^{\mu\nu} \geq 0. \tag{SM-21}$$

SM-3. Numerical method for the Shakhov model

To solve the Shakhov kinetic model $k^\mu \partial_\mu f_{\mathbf{k}} = C_S[f]$, we employ a discrete velocity method inspired by the Relativistic Lattice Boltzmann algorithm of Refs. [6, 7, 8, 9, 10]. We consider the rapidity-based moments of $f_{\mathbf{k}}$ introduced in Ref. [2], which eliminates two out of the three dimensions of the momentum space for the particular case of the $(1+1)$-dimensional longitudinal waves SM-3.1, and the $(0+1)$-dimensional boost invariant expansion SM-3.2, respectively.

SM-3.1. Longitudinal wave damping problem

In the application of Sec. 6, the fluid is homogeneous with respect to the x and y directions. Parameterizing the momentum space using $(m_\perp, \varphi_\perp, v^z)$ as in Ref. [2],

$$\begin{pmatrix} k^t \\ k^z \end{pmatrix} = \frac{m_\perp}{\sqrt{1-v_z^2}} \begin{pmatrix} 1 \\ v^z \end{pmatrix}, \quad \begin{pmatrix} k^x \\ k^y \end{pmatrix} = \sqrt{m_\perp^2 - m_0^2} \begin{pmatrix} \cos\varphi_\perp \\ \sin\varphi_\perp \end{pmatrix}, \tag{SM-22}$$

the Boltzmann equation with the Shakhov model for the collision term reduces to

$$\partial_t f_{\mathbf{k}} + v^z \partial_z f_{\mathbf{k}} = -\frac{u\cdot v}{\tau_R}(f_{\mathbf{k}} - f_{\mathrm{S}\mathbf{k}}), \tag{SM-23}$$

where $v^\mu = k^\mu/k^t$ and $u \cdot v = \gamma(1 - \beta^z v^z)$, with β^z being the fluid three-velocity along the z direction and $\gamma = 1/\sqrt{1-\beta_z^2}$. Introducing the rapidity-based moments [2]

$$F_n = \frac{g}{(2\pi)^3} \int_0^{2\pi} d\varphi_\perp \int_{m_0}^\infty \frac{dm_\perp\, m_\perp^{n+1}}{(1-v_z^2)^{(n+2)/2}} f_{\mathbf{k}}, \tag{SM-24}$$

Eq. (SM-23) becomes

$$\partial_t F_n + v^z \partial_z F_n = -\frac{u\cdot v}{\tau_R}(F_n - F_n^{\mathrm{S}}). \tag{SM-25}$$

It can be shown [11] that the macroscopic quantities N^t, N^z, T^{tt}, T^{tz} and T^{zz} can be obtained from F_1 and F_2 via

$$\begin{pmatrix} N^t \\ N^z \end{pmatrix} = \int_{-1}^1 dv^z \begin{pmatrix} 1 \\ v^z \end{pmatrix} F_1, \quad \begin{pmatrix} T^{tt} \\ T^{tz} \\ T^{zz} \end{pmatrix} = \int_{-1}^1 dv^z \begin{pmatrix} 1 \\ v^z \\ v_z^2 \end{pmatrix} F_2. \tag{SM-26}$$

For the case of massless particles considered in Sec. 6, $T^\mu{}_\mu = 0$, such that $T^{xx} = T^{yy} = (T^{tt} - T^{zz})/2$. From the above, it is clear that the time evolution of both N^μ and $T^{\mu\nu}$ is fully determined by the functions F_1 and F_2. In order to solve Eq. (SM-25), the functions F_n^{S} must be obtained by integrating Eq. (39), yielding:

$$F_1^{\mathrm{S}} = \frac{n}{2(u\cdot v)^3} - \frac{3V(\beta^z - v^z)}{2(u\cdot v)^4}\left(1 - \frac{\tau_\pi}{\tau_V}\right), \quad F_2^{\mathrm{S}} = \frac{3P}{2(u\cdot v)^4}. \tag{SM-27}$$

The time discretization is performed using equal time steps $\delta t = 10^{-3}\,\mathrm{fm}/c$ and the time stepping is performed using the third-order total variation diminishing (TVD) Runge-Kutta scheme [12, 13]. The spatial domain $[-L/2, L/2]$ is discretized using $S = 100$ cells of size $\delta s = L/S$, centred on $z_s = (s - \frac{1}{2})\delta s - \frac{L}{2}$, $1 \le s \le S$. The spatial derivative $v^z \partial_z F_n$ is approximated using finite differences:

$$\left(v_z \frac{\partial F_n}{\partial z}\right)_s = \frac{\mathbf{F}_{n;s+1/2} - \mathbf{F}_{n;s-1/2}}{\delta s}, \tag{SM-28}$$

where $\mathbf{F}_{n;s+1/2}$ represents the flux at the interface between cells s and $s+1$. For definiteness, we compute this flux using the upwind-biased fifth-order weighted essentially non-oscillatory (WENO-5) scheme introduced in Ref. [14, 15]. Finally, the v^z momentum space coordinate is discretized via the Gauss-Legendre quadrature with $K = 20$ points, such that $P_K(v_j^z) = 0$, with $1 \le j \le K$ and $P_K(z)$ being the Legendre polynomial of order K. Then, integrals with respect to v^z of a function $g(v^z)$ are approximated via

$$\int_{-1}^1 dv^z\, g(v^z) \simeq \sum_{j=1}^K g_j, \qquad g_j \equiv w_j g(v_j^z), \tag{SM-29}$$

with w_j being the Gauss-Legendre quadrature weights [16].

SM-3.2. Algorithm for the Bjorken expansion

The algorithm for the Bjorken expansion is identical to that described in Ref. [2], hence we only recall the main method here. The spatial rapidity is $\eta_s = \mathrm{artanh}(z/t)$, and the parametrization of the momentum space is as in Eq. (SM-22), with (k^t, k^z) replaced by $(k^\tau, \tau k^{\eta_s})$, where

$$k^\tau = \frac{1}{\tau}(tk^t - zk^z), \quad \tau k^{\eta_s} = \frac{1}{\tau}(tk^z - zk^t). \tag{SM-30}$$

Retaining the definition (SM-24) of the rapidity-based moments, the non-vanishing components of $T^{\mu\nu} = \mathrm{diag}(e, P_\perp, P_\perp, \tau^{-2}P_l)$ are given by [2]

$$\begin{pmatrix} e \\ P_l \end{pmatrix} = \int_{-1}^1 dv^z \begin{pmatrix} 1 \\ v_z^2 \end{pmatrix} F_2, \quad P_\perp = \int_{-1}^1 dv^z \left(\frac{1-v_z^2}{2} F_2 + \frac{m_0^2}{2} F_0\right). \tag{SM-31}$$

The Boltzmann equation becomes

$$\partial_\tau F_n + \frac{1 + (n-1)v_z^2}{\tau} F_n - \frac{1}{\tau}\frac{\partial[v^z(1-v_z^2)F_n]}{\partial v^z} = -\frac{1}{\tau_R}(F_n - F_n^{\mathrm{S}}), \tag{SM-32}$$

where F_n^{S} can be obtained by integrating Eq. (34):

$$F_n^{\mathrm{S}} = F_n^{\mathrm{eq}} - \frac{\beta^2\pi(1 - \frac{\tau_\Pi}{\tau_\pi})}{4(e+P)}\left[m_0^2 F_n^{\mathrm{eq}} - (1 - 3v_z^2)F_{n+2}^{\mathrm{eq}}\right], \tag{SM-33}$$

where $F_n^{\mathrm{eq}} = \frac{g}{4\pi^2}\Gamma(n+2, \zeta)/\beta^{n+2}$, $\Gamma(n,\zeta) = \int_\zeta^\infty dx\, x^{n-1}e^{-x}$ is the incomplete Gamma function and $\zeta = \beta m_0/\sqrt{1-v_z^2}$. Since $\pi = \frac{2}{3}(P_\perp - P_l)$, the system is closed in terms of F_0 and F_2.

The time integration and v^z discretization proceed as in the previous subsection, however now the time step $\delta\tau_n = \tau_{n+1} - \tau_n$ is determined adaptively via

$$\delta\tau_n = \min(\alpha_\tau \tau_n, \alpha_R \tau_R), \tag{SM-34}$$

with $\alpha_\tau = 10^{-3}$ and $\alpha_R = 1/2$. Furthermore, the derivative with respect to v^z is performed by projecting F_n onto the space of Legendre polynomials, as described in Ref. [8]. Considering the discretization with $K = 20$ discrete velocities $1 \le j \le K$, we have

$$\left\{\frac{\partial[v^z(1-v_z^2)F_n]}{\partial v^z}\right\}_j = \sum_{j'=1}^K \mathcal{K}_{j,j'} F_{n;j'}, \tag{SM-35}$$

where the kernel $\mathcal{K}_{j,j'}$ is given in Eq. (3.54) of Ref. [8].

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