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ON NUMBERS WHICH CONTAIN NO FACTORS OF THE FORM $p(kp+1)$*

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"If p^a is the highest power of a prime p which divides the order of a group G, the subgroups of G of order p^a form a single conjugate set and their number is congruent to unity, mod p."

The above well-known theorem, which was first established by Sylow,† suggests the segregation of numbers into two distinct classes: the one class to contain all numbers which have factors of the form $p(kp+1)$ where p may be any prime except unity and k may have any integral value greater than zero; the other class to contain all remaining numbers. Numbers of the second class will be found to have certain properties analogous to the properties of primes and will be denoted by P, numbers of the first class will be found to have certain properties analogous to the properties of composite numbers and will be denoted by C. If a C should be of the form $p(kp+1)$, and not merely a multiple of a number of that form, we will call it a "*fundamental*" C. Certain C's are fundamental for certain p's and not for others; *e.g.*,

$$12 = 2\{2(1\cdot 2+1)\} = 3(1\cdot 3+1),$$

and 12 is therefore fundamental for $p=3$, but not for $p=2$; whether a given C is to be considered a fundamental C or not will be made clear by the context. Two fundamental C's will be called "*different*" if the values of the p's and k's respectively are not both the same in the two C's. The present investigation is intended to develop some of the more fundamental properties of the P's and C's and to establish certain formulae for the number of P's within a given limit.

Corollaries from the Definitions

It follows directly from the definitions above, that:

1. *Any number which is a positive integral power, including the first power, of a prime p is a* P.

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† Sylow, *Théorèmes sur les groupes de substitutions.* Math. Ann. (1872). For the particular statement of the theorem used above, compare Burnside, *The Theory of Groups of Finite Order*, § 78 et seq.

2. *Every even number not a positive integral power of* 2 *is a* C. Hence the only even P's are the positive integral powers of 2.

3. *Every number whose last two digits are* 05 *or* 55 *is a* C.
Such numbers are evidently of the form $5(5n+1)$.

THEOREM

Every number which is divisible by any power of 3 *and contains at least two other factors not multiples of* 3 *is a* C.

Let

$$N = 3^n p^\alpha q^\beta \cdots\cdots, \quad n \geqq 1, \quad \alpha + \beta + \cdots\cdots \geqq 2.$$

Every odd prime is of the form, $3m+1$, or $3m+2$.
Hence, N is a C, if any prime is 2, or of the form $3m+1$.
Also, since

$$(3m_1+2)(3m_2+2) = 3m+1,$$

the theorem follows in any case.
It is evident that in the expression

$$\alpha + \beta + \cdots\cdots \geqq 2$$

all the terms, except one, may be zero, provided that term is not less than 2.

THEOREM

The product of two consecutive odd primes cannot be a C.
Suppose

$$p_1 p_2 = lp(kp+1),$$

where p_1 and p_2 are consecutive odd primes and $p_2 > p_1 > 2$.
Then, since we have only two factors, p_1 and p_2, and $p_2 > p_1$, $l = 1$, and $p = p_1$.
Hence

$$p_1 p_2 = p_1(kp_1+1),$$

and

$$p_2 = kp_1 + 1.$$

But, since p_1 and p_2 are both odd, k must be even.
Therefore

$$p_2 = 2np_1 + 1,$$

or

$$p_2 \geqq 2p_1 + 1.$$

Now, by Bertrand's Postulate,* between x and $2x-2$, $(x > 7/2)$, there is at least one prime, whence p_2 is not consecutive to p_1, as was assumed. If $p_1 = 3$, $p_2 = 5$, we have $p_1 p_2 = 15$, which is not a C, and the theorem holds for all odd primes.

* This theorem was first conjectured by Bertrand (Journal de l'École Polyt., cah. 30) and afterwards proved by Tchébicheff in 1850 in his *Mémoire sur les nombres premiers* presented to the Academy of St. Petersburg and reprinted in *Liouville,* vol. XVII (1852).

Corollary

It follows directly from the proof of the theorem that the product of two odd primes, p_1 and p_2, so related that $p_2 < 2p_1$, cannot be a C.

For, by the theorem

$$p_2 \gtreqless 2p_1 + 1,$$

or

$$p_2 > 2p_1,$$

which is contrary to hypothesis.

Further, the *necessary* and *sufficient* condition that the product of two odd primes, p_1 and p_2, $p_2 > p_1$, should be a C is that $p_2 = 2np_1 + 1$.

Theorem

No rational integral algebraic polynomial can represent only numbers which are P's.

For, suppose

$$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdot\cdot\cdot\cdot\cdot + a_{n-1}x + a_n,$$

should be such a polynomial, then for some value of x, say ζ, the formula would represent a P.

Whence

$$f(\zeta) \equiv a_0\zeta^n + a_1\zeta^{n-1} + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot + a_{n-1}\zeta + a_n = \mathrm{P}.$$

But

$$f(\zeta + \mathrm{P}^2) = a_0(\zeta + \mathrm{P}^2)^n + a_1(\zeta + \mathrm{P}^2)^{n-1} + \cdot\cdot\cdot\cdot\cdot\cdot + a_{n-1}(\zeta + \mathrm{P}^2) + a_n,$$
$$= M\mathrm{P}^2 + \mathrm{P} = \mathrm{P}(M\mathrm{P} + 1).$$

The right member is a C whether P be a prime or a composite number.

Theorem

If x and y are both odd or both even numbers, provided they are not both equal to 1 *or* 2^κ, *then* $x^n + y^n$ *is a* C, *for* $n > 1$.

Since any odd number is of the form $(2k + 1)$, it is only necessary to show that the sum contains at least one odd and one even factor to prove the theorem.

1° Consider

$$S = x^n + y^n, \text{ where } x \text{ and } y \text{ are odd and } x \neq y = 1.$$

(a). Let

$$n = 2\nu + 1.$$

Then

$$S = x^{2\nu+1} + y^{2\nu+1},$$
$$= (x + y)(x^{2\nu} - x^{2\nu-1}y + \cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot + y^{2\nu}).$$

Now, the first factor is the sum of two odd numbers and therefore is even, and the second factor is odd since it has an odd number of odd terms.

Therefore the sum S is a C.

(*b*). Let

$$n = 2\nu.$$

Then

$$S = x^{2\nu} + y^{2\nu},$$
$$= (2x' + 1)^{2\nu} + (2y' + 1)^{2\nu}.$$

If we expand the parentheses, there will be $(2\nu + 1)$ terms in each expansion, of which the last term is unity and all the others contain the factor 2^2. Therefore, we may write

$$S = 2^2X + 1 + 2^2Y + 1,$$
$$= 2\{2(X + Y) + 1\}.$$

Hence S is a C.

It is necessary to exclude the case where x and y are both unity, for in that case

$$S = 1^n + 1^n = 2$$

and S is a P.

2° Next consider

$S = x^n + y^n$, where x and y are both even and $x \neq y = 2^\kappa$.

(*a*). Let

$$x = 2^\alpha x' \text{ and } y = 2^\beta y',$$

where x' and y' are both odd and $\beta > \alpha$.

Then

$$S = 2^{\alpha n}(x'^n + 2^{(\beta-\alpha)n}y'^n).$$

The first factor is even and the second is odd. Therefore S is a C.

(*b*). Let

$x = 2^\alpha x'$ and $y = 2^\alpha y'$, where x' and y' are both odd.

Then

$$S = 2^{\alpha n}(x'^n + y'^n).$$

But the second factor has already been proved a C by Case 1, therefore S is a C.

The case excluded by the theorem, $x = y = 2^\kappa$, gives

$$S = 2^{\kappa n} + 2^{\kappa n} = 2^{\kappa n + 1};$$

and S is a P in this case.

In connection with this theorem, it is to be noted that if x is odd and y is even, then $x^n + y^n$ may be either a P or a C. Two illustrations will serve to prove this statement.

If $x = 11$, $y = 4$, then $x^2 + y^2 = 121 + 16 = 137 \equiv$ P.
$x = 17$, $y = 20$, then $x^2 + y^2 = 289 + 400 = 689 = 13(4 \cdot 13 + 1) \equiv$ C.

Theorem

If p, q are odd primes, and α, β are positive integers, $p^\alpha q^\beta$ is a C, provided $\alpha \overline{\gtrless} \frac{\phi(q)}{\delta}$, where δ is the greatest common factor of $\phi(q)$ and ind p with respect to q.

Let

$$N = p^{\alpha} q^{\beta},$$

then must

$$p^{\alpha} q^{\beta} = p^{\alpha'} q^{\beta-1}\{q(kq+1)\}, \text{ where } \alpha' < \alpha,$$

whence

$$p^{\alpha-\alpha'} = kq + 1,$$

and

$$p^{\alpha-\alpha'} \equiv 1 \ (mod\ q).$$

But, by Fermat's theorem,

$$p^{\phi(p)} \equiv 1 \ (mod\ q),$$

$$\therefore p^{\alpha-\alpha'} \equiv p^{\phi(q)} \ (mod\ q),$$

whence

$$(\alpha - \alpha') \text{ ind } p \equiv 0 \ (mod\ \overline{q-1}),$$

or

$$(\alpha - \alpha') \equiv 0 \ \left(mod\ \frac{q-1}{\delta}\right), \text{ where } (q-1, \text{ ind } p) = \delta.$$

Hence, it follows that the least value of $(\alpha - \alpha')$, and consequently of α, is $\frac{q-1}{\delta}$, a quantity independent of the particular primitive root selected.

Having discussed some of the fundamental properties of the P's and C's, we will now take up the problem of deducing certain formulae of enumeration. It is very evident from the nature of the C's that, if we attempt to carry out the analogy between P's and prime numbers, C's and composite numbers, a new interpretation must be placed upon many of the ordinary operations with numbers, in particular those operations which involve the processes of multiplication. This is due to the fact that the very important theorem, that a number may be resolved into prime factors in only one way, does not apply when we consider fundamental C's as our fundamental factors; thus

$$105 = 5(4 \cdot 5 + 1) = 7(2 \cdot 7 + 1).$$

While these two divisors are entirely different C's in the sense of our definition, they are each equal to 105 and their product in the ordinary sense is equal to 105^2. It seems most convenient for many purposes to define the product of two or more different C's as their least common multiple and to designate the operation by the usual symbols of multiplication; the powers of a C will have the ordinary interpretation. In case of division, in which divisor or dividend or both are products of C's, all multiplications must be performed first; thus, if $C_1 = 2(1 \cdot 2 + 1)$ and $C_2 = 3(1 \cdot 3 + 1)$, $C_1C_2 = 12$, $\frac{C_1C_2}{C_1} = \frac{12}{6} = 2$, and not, $\frac{C_1C_2}{C_1} = C_2 = 12$. The operation of cancellation of C's before multiplying will be indicated by the symbol $=\!=$; thus, $\frac{C_1C_2}{C_1} = C_2$. The divisors of a product of C's will consist, unless otherwise stated, of unity, the number itself, and any product of the C's therein contained.

THEOREM

The number of P's *not greater than n is obtained by expanding the double product*

$$[n]\left\{\begin{array}{l}\left(1-\frac{1}{C_{1,1}}\right)\left(1-\frac{1}{C_{1,2}}\right)\cdots\cdots\left(1-\frac{1}{C_{1,K_1}}\right)\\ \left(1-\frac{1}{C_{2,1}}\right)\left(1-\frac{1}{C_{2,2}}\right)\cdots\cdots\left(1-\frac{1}{C_{2,K_2}}\right)\\ \qquad\vdots\\ \left(1-\frac{1}{C_{i,1}}\right)\left(1-\frac{1}{C_{i,2}}\right)\cdots\cdots\left(1-\frac{1}{C_{i,K_i}}\right)\\ \qquad\vdots\\ \left(1-\frac{1}{C_{\lambda,1}}\right)\left(1-\frac{1}{C_{\lambda,2}}\right)\cdots\cdots\left(1-\frac{1}{C_{\lambda,K_\lambda}}\right)\end{array}\right\}$$

in which $C_{i,k}\equiv p_i(kp_i+1)$, *and then forming the sum by taking the integral part of each term with its proper sign.*

It is evident, in the first place, that for a given p and k, the number of integers not greater than n which contain the factor $p_1(jp_1+1)$, where p_1 is a given prime, is given by

$$\left[\frac{n}{p_1(jp_1+1)}\right],$$

where $[x]$ has its usual interpretation, the greatest integer in x. If we define $\phi_P(n, j, p_1)$ as the number of integers not greater than n, which do not contain the factor $p_1(jp_1+1)$, where p_1 and j have given values, then

$$\phi_P(n, j, p_1)=n-\left[\frac{n}{p_1(jp_1+1)}\right].$$

Now, let p remain fixed, but let k have a sequence of integral values, 1, 2, 3 . . . Many numbers may contain two or more factors of the form $p(kp+1)$, as for instance, $48=4[3(1\cdot 3+1)]=3(5\cdot 3+1)$, and, moreover, the largest factor cannot exceed $[n]$. It will, therefore, be necessary to find the largest value of k for a given prime p and a given number n. It is evident that this maximum value of k will be obtained when, if possible, the factor is exactly equal to n, or

$$\begin{aligned} n &= p_1(kp_1+1),\\ &= kp_1^2+p_1,\end{aligned}$$

whence

$$k=\frac{n-p_1}{p_1^2}.$$

Therefore, the maximum value of k for a given p and n will be given by

$$k=\left[\frac{n-p}{p^2}\right].$$

$\phi_P(n, p_1)$ will now be defined as the number of integers not greater than n which contain no factors of the form $p_1(kp_1+1)$, where p_1 is a given prime and $k=1$ to $\left[\frac{n-p_1}{p_1^2}\right]$, and therefore $\phi_P(n, p_1)$ will necessarily consist of the sum

$$\sum_{k=1}^{\left[\frac{n-p_1}{p_1^2}\right]} \left[\frac{n}{p_1(kp_1+1)}\right]$$

plus some function which will provide for such cases as have two or more different factors of the given form.

Let

$$n_1 = l_1p_1(k_1p_1+1) = l_2p_1(k_2p_1+1), \text{ where } n_1 > n,\ k_1 \neq k_2.$$

Then

$$l_1(k_1p_1+1) = l_2(k_2p_1+1).$$

And since $k_1 \neq k_2$, the two sets l_1, k_1p_1+1 and l_2, k_2p_1+1 have factors common to each other. If δ is the least common multiple of (k_1p_1+1) and (k_2p_1+1), any number of the form $mp_1\delta$ will contain factors of the two forms $p_1(k_1p_1+1)$ and $p_1(k_2p_1+1)$. A similar result holds for any number of factors of the form $p(kp+1)$, and also for the case in which both the p's and k's differ.

We have already defined the product of two or more different C's as their least common multiple. If then we use the notation

$$C_{1,k} \equiv p_1(kp_1+1),$$

the sequence

$$C_{1,1},\ C_{1,2},\ \cdots\cdots\ C_{1,K}$$

will correspond exactly with the sequence

$$p_1(p_1+1),\ p_1(2p_1+1), \ldots\ldots p_1(Kp_1+1)$$

where k and consequently the second subscript k of the C's has the values

$$1, 2, \ldots\ldots \left[\frac{n-p_1}{p_1^2}\right].$$

It follows then from analogy with the manner of forming the function $\phi(n;\ p, q, r)$ of ordinary primes that

$$\phi_P(n, p_1) = n - \left\{\left[\frac{n}{C_{1,1}}\right] + \left[\frac{n}{C_{1,2}}\right] \cdots\cdots + \left[\frac{n}{C_{1,K}}\right]\right\}$$
$$+ \left\{\left[\frac{n}{C_{1,1}C_{1,2}}\right] + \left[\frac{n}{\text{Product of different C's two at a time}}\right]\right\}$$
$$- \left\{\left[\frac{n}{C_{1,1}C_{1,2}C_{1,3}}\right] + \left[\frac{n}{\text{Product of different C's three at a time}}\right]\right\}$$
$$+ \text{ etc.}$$

We may write the formula in the neater form

$$\phi_P(n,\ p_1) = [n]\left(1-\frac{1}{C_{1,1}}\right)\left(1-\frac{1}{C_{1,2}}\right)\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \left(1-\frac{1}{C_{1,K}}\right),$$

if we interpret this form to mean that we shall first expand the product and then take the sum of the integral parts of all the terms with their proper signs. Thus,

$$\phi_P(50,\ 2) = 50 - 27 + 9 - 2 = 30,$$

which is easily verified as the number of integers not greater than 50 which contain no factors of the form $2(2k+1)$.

We will next define $\phi_P(n, \lambda)$ as the number of integers not greater than n which contain no factors of the form $p_i(kp_i+1)$, where p_i is any one of the first first λ primes, $p_1 = 2$, $p_2 = 3, \ldots\ldots p_i \ldots\ldots p_\lambda$, with a limit depending on n, and k runs independently for each prime, being limited as already shown. To find the limit of p for any n we assume $k=1$, since this will give us the maximum value of p, such that $p(kp+1) \leqq [n]$.

Then

$$p\ (p+1) \leqq n,$$

$$p^2 + p \leqq n,$$

$$(p+\tfrac{1}{2})^2 \leqq n + \tfrac{1}{4},$$

$$p \leqq \frac{\sqrt{4n+1}\ -1}{2},$$

whence the maximum value of p will be that of the largest prime not greater than $\left[\frac{\sqrt{4n+1}\ -1}{2}\right]$. If, then, we consider the double sequence of C's,

$$C_{i,k} \equiv p_i(kp_i+1),$$

where p_i is any one of the first λ primes, $p_1 = 2$, $p_2 = 3$, ... p_i . . . . p_λ; $p_\lambda \leqq \frac{\sqrt{4n+1}\ -1}{2}$, and where k runs independently for each p from 1 to $\left[\frac{n-p}{p^2}\right]$, we can write down the desired formula at once, in that it must have the same general form as that for $\phi_P(n,\ p_1)$. Therefore,

$$\phi_P(n,\ \lambda) = n - \sum\left\{\left[\frac{n}{C_{i,K}}\right]\right\}$$

$$+\sum\left\{\left[\frac{n}{\text{Product of different C's two at a time}}\right]\right\}$$

$$-\sum\left\{\left[\frac{n}{\text{Product of different C's three at a time}}\right]\right\}$$

$$+\text{ etc.}$$

Or, as before, we may write

$$\phi_P(n,\ \lambda) = [n]\left\{\begin{matrix} \left(1-\frac{1}{C_{1,1}}\right)\left(1-\frac{1}{C_{1,2}}\right) & \cdots\cdots & \left(1-\frac{1}{C_{1,K_1}}\right) \\ \left(1-\frac{1}{C_{2,1}}\right)\left(1-\frac{1}{C_{2,2}}\right) & \cdots\cdots & \left(1-\frac{1}{C_{2,K_2}}\right) \\ \vdots & & \\ \left(1-\frac{1}{C_{i,1}}\right)\left(1-\frac{1}{C_{i,2}}\right) & \cdots\cdots & \left(1-\frac{1}{C_{i,K_i}}\right) \\ \vdots & & \\ \left(1-\frac{1}{C_{\lambda,1}}\right)\left(1-\frac{1}{C_{\lambda,2}}\right) & \cdots\cdots & \left(1-\frac{1}{C_{\lambda,K_\lambda}}\right) \end{matrix}\right\},$$

if we interpret this form to mean that we are to expand the double product and then form the sum by taking the integral part of each term with its proper sign. If p_λ has the maximum value of the limit already found, the formula will give the number of P's not greater than n.

We will now define an important function $\mu_C(N)$, analogous to Merten's function.* Let $N \equiv C_1 C_2 \ldots\ldots C_n$ be the product of any number of fundamental C's, not necessarily different, then we will define $\mu_C(N)$ as follows:

$\mu_C(N) = 0$, if two or more of the C's are equal.

$\mu_C(N) = +1$, if all the C's are different and their number is even, also if $N = 1$.

$\mu_C(N) = -1$, if all the C's are different and their number is odd.

Lemma

$\sum \mu_C(D) = 0$, *if the sum is extended over all divisors* D (1 *and* N *included*) *of a number*, N, *which is a product of fundamental* C's.

Let us assume $N' = C_1 C_2 \ldots\ldots C_n$, a product of n fundamental C's, then the total number of their products with an even number of factors (among this number we include the case of zero factors) is equal to the total number of their products with an odd number of factors. For the number of products with an even number of factors is

$$1 + \frac{n(n-1)}{1\cdot 2} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4} + \ldots\ldots\ldots\ldots,$$

and with an odd number of factors is

$$\frac{n}{1} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} + \ldots\ldots\ldots\ldots$$

Their difference is zero, being equal to $(1-1)^n$. It follows, then, that in the expanded product

$$(1-C_1)(1-C_2)\ldots\ldots(1-C_n)$$

the number of positive terms is equal to the number of negative terms.

* See Crelle, vol. 77, p. 289.

Whence

$$\sum \mu_c(D) = 0,$$

when this sum is extended over all the divisors D, a product of C's, of N'. Finally, let

$$N = C_1^{\alpha_1} C_2^{\alpha_2} \; . \; . \; . \; . \; . \; . \; C_n^{\alpha_n}.$$

Then all the divisors of N' are divisors of N, while all the remaining divisors of N contain at least the second power of a C, for which $\mu_c(D) = 0$. It follows, therefore, that

$$\sum \mu_c(D) = 0,$$

if the sum is extended over all divisors D (1 and N included) of N.

Theorem

The number of P's not greater than x is given by the formula

$$\sum_{D} \mu_c(D) \left[\frac{x}{D} \right],$$

where D *is a divisor of* Π, *the product of all possible fundamental C's not greater than x.*

Let $f(n)$ and $F(n)$ be two "*zahlentheoretische Functionen*"* which are only different from zero, when n is a divisor of a given number Π, the product of any number of different fundamental C's. Thus, assume

$$f(D) = \sum F(KD)$$

for every divisor D of Π, the sum being formed for all divisors K of $\frac{\Pi}{D}$, where all numbers are subject to our definitions of products, divisors, etc., of C's.

Then we may form the sum

$$\sum_{D} \mu_c(D) f(D) = \sum_{K \cdot D} \mu_c(D) F(KD).$$

The left member is to be summed for all divisors D of Π.

The right member is a double sum which is to be extended over all divisors D of Π and also all divisors K of $\frac{\Pi}{D}$.

Now KD will become equal to every divisor of Π and if the terms of the double sum are collected in which KD is equal to the same divisor Δ of Π, then the entire double sum becomes $\sum_{\Delta} G_\Delta F(\Delta)$, where $G_\Delta = \sum \mu_c(D)$, and is summed for all divisors of Δ. But, by the lemma just proved, $G_\Delta = 0$ for all values of Δ, except $\Delta = 1$, when $G_\Delta = 1$. It follows, then, that

$$\sum_{D} \mu_c(D) f(D) = \sum_{\Delta} G_\Delta F(\Delta) = F(1).$$

* Compare Bachmann, *Zahlentheorie*, Part II, p. 308, for the definition of this term.

Now, let x be any positive quantity and let $\Pi \equiv C_1C_2 \ldots\ldots C_n$ be the product of n different fundamental C's. We may arrange the integers 1, 2, $3 \cdots\cdots [x]$ in groups such that each group contains all the integers which have the same greatest common factor* with Π, and no integer will appear in more than one group.

Defining $\psi(x, \Pi, K)$ as the number of integers not greater than x which have with Π the greatest common factor, K, as already defined, we shall have

$$[x] = \sum_K \psi(x,\ \Pi,\ K),$$

the summation being taken over all divisors K of Π. Let D be a given divisor of Π and K a divisor of $\frac{\Pi}{D}$, then KD is also a divisor of Π. But all the numbers not greater than x which have the greatest common factor KD† with Π will be found among the following numbers:

$$1 \cdot D,\ 2 \cdot D,\ 3 \cdot D \cdots\cdots \left[\frac{x}{D}\right] \cdot D;$$

and there will be just as many such numbers as there are integers not greater than $\frac{x}{D}$ which have the greatest common factor, $\frac{KD}{D}$‡, with $\frac{\Pi}{D}$. It follows that

$$\psi(x,\ \Pi,\ KD) = \psi\left(\frac{x}{D},\ \frac{\Pi}{D},\ \frac{KD}{D}\right).$$

But

$$\left[\frac{x}{D}\right] = \sum_K \psi\left(\frac{x}{D},\ \frac{\Pi}{D},\ \frac{KD}{D}\right),$$

where K is a divisor of $\frac{\Pi}{D}$.

Whence

$$\left[\frac{x}{D}\right] = \sum_K \psi(x,\ \Pi,\ KD),$$

where K is a divisor of $\frac{\Pi}{D}$.

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* In determining the greatest common factor of two numbers, either or both of which are C's or products of C's, the greatest common factor may be any rational function of the C's, provided that in numerical calculations it is integral and a divisor in the ordinary sense of the quantities involved. In the particular case of any number n and Π, the greatest common factor may be only a divisor of Π as already defined, and, moreover, since two or more divisors of Π, while different as products of C's, may be numerically equal, the greatest common factor may be only some one of the numerically distinct divisors of Π. To illustrate this paragraph, let $\Pi \equiv C_1C_2C_3C_4 = 6 \cdot 10 \cdot 12 \cdot 18$. Then the greatest common factor of any number n and Π may be only some one of the numbers 1, 6, 10, 12, 18, 30, 36, 60, 90, 180; thus, the greatest common factor of 50 and Π is 10. Again of the numbers 1, 2, 3, 6, only 3 has $\frac{C_1C_3C_4}{C_1C_3} = \frac{6 \cdot 12 \cdot 18}{6 \cdot 12} = \frac{36}{12} = 3$ as greatest common factor with $C_2C_4 = 10 \cdot 18 = 90$.

† We only consider the distinct divisors KD.

‡ $\frac{KD}{D}$ does not necessarily equal K.

We have here an equation between two "*zahlentheoretische Functionen*" of the type already defined and so we can apply the results obtained.

Hence it follows that

$$\psi(x, \Pi, 1) = \sum_{D} \mu_c(D)\left[\frac{x}{D}\right],$$

where the sum is extended over all divisors D of Π, a product of C's, *i.e.*, the number of integers not greater than x which contain no one of the C's which form the product Π is given by

$$\sum_{D} \mu_0(D)\left[\frac{x}{D}\right].$$

If, now, we take Π as the product of all C's,

$$\prod_{i,k} C_{i,k} \equiv \prod_{p_i,k} \{p_i(kp_i+1)\},$$

where p_i is any one of of the first λ primes, $p_1=2,\ p_2=3 \ldots\ldots p_i \ldots\ldots p_\lambda$, and $k=1$ to $\left[\frac{x-p_i}{p_i^2}\right]$, running independently for each prime, then

$$\phi_P(x,\lambda) = \sum_{D} \mu_0(D)\left[\frac{x}{D}\right],$$

where D is a divisor of Π.

Finally, if p_λ is the largest prime not greater than $\left[\frac{\sqrt{4x+1}-1}{2}\right]$ and the k's are determined as before, then $\phi_P(x,\lambda)$ will denote the number of P's not greater than x. Therefore, the number of P's not greater than x is given by the formula

$$\sum_{D} \mu_c(D)\left[\frac{x}{D}\right],$$

where D is a divisor of Π, the product of all C's not greater than x.

As an illustration of the method, let us find

$$\psi(x, \Pi, 1),$$

where $x=50$ and $\Pi = \{2(2+1)\}\{2(2\cdot 2+1)\}\{3(3+1)\}\{2(4\cdot 2+1)\}$
$= 6\cdot 10\cdot 12\cdot 18.$

$$\psi(50, \Pi, 1) = \sum_{D} \mu_0(D)\left[\frac{50}{D}\right], \text{ where D is a divisor of } \Pi = 6\cdot 10\cdot 12\cdot 18,$$

$$= \left[\frac{50}{1}\right] - \left[\frac{50}{6}\right] - \left[\frac{50}{10}\right] + \left[\frac{50}{30}\right] \text{ (All the other terms destroyed each other.)}$$

$$= 50 - 8 - 5 + 1 = 38.$$

Whence 38 of the first 50 integers are not divisible by 6, 10, 12, or 18.

COMPUTATION FORMULAE FOR $\phi_P(n, \kappa, p_1)$ AND $\phi_P(n, \lambda)$

While the formulae already obtained furnish the solution of the problem in question, the actual computations necessary to obtain numerical results when n increases soon becomes too extended and complicated for practical use. Therefore in the succeeding paragraphs we will obtain formulae which lend themselves more readily to numerical computation.

In accordance with our previous definitions, we will define $\phi_P(n, \kappa, p_1)$ as the number of integers not greater than n which contain no factor of the form $p_1(kp_1+1)$ where p_1 is a given prime and $k = 1, 2, 3 \cdots\cdots \kappa$. If we form the sequence of natural numbers in order from 1 to $[n]$ and then exclude all those numbers which are divisible by factors of the form $p_1(kp_1+1)$ where

$$k = 1, 2, 3 \cdots\cdots (\kappa - 1),$$

the number of integers remaining will be equal to $\phi_P(n, \overline{\kappa-1}, p_1)$. It is still necessary to exclude the numbers which are divisible by $p_1(\kappa p_1+1)$, which are the following:

$$p_1(\kappa p_1+1),\ 2p_1(\kappa p_1+1) \cdots\cdots \left[\frac{n}{p_1(\kappa p_1+1)}\right]\{p_1(\kappa p_1+1)\}.$$

However, we have already excluded all the numbers of this sequence which are divisible by other factors of the given form, so that for our purpose it is only necessary to consider those numbers of this sequence which are divisible by no other factors of the given form than $p_1(\kappa p_1+1)$. We may obtain this number by determining how many integers of the sequence

$$(\kappa p_1+1),\ 2(\kappa p_1+1),\ 3(\kappa p_1+1) \cdots\cdots \left[\frac{n}{p_1(\kappa p_1+1)}\right](\kappa p_1+1)$$

are divisible by any factor of the form (kp_1+1), where $k = 1, 2 \ldots\ldots (\kappa-1)$, a result which is readily obtainable by a sieve process. For convenience we will define

$$\phi_P\{\overset{\nu}{\underset{1}{A}}(\kappa p_1+1),\ (\kappa-1),\ p_1\}$$

as the number of terms of the arithmetical sequence, whose common difference is (kp_1+1) and the number of whose terms is $\nu = \left[\frac{n}{p_1(\kappa p_1+1)}\right]$, which have no factors of the form (κp_1+1), where $k = 1, 2, 3 \cdots\cdots\cdots (\kappa-1)$.

Then we shall have

$$\phi_P(n, \kappa, p_1) = \phi_P(n, \overline{\kappa-1}, p_1) - \phi_P\{\overset{\nu}{\underset{1}{A}}(\kappa p_1+1),\ (\kappa-1),\ p_1\}.$$

If we put

$$\kappa = \left[\frac{n-p_1}{p_1^2}\right],$$

then

$$\phi_P(n, \kappa, p_1) = \phi_P(n, p_1) = \phi_P(n, \overline{\kappa-1}, p_1) - \phi_P\{\overset{\nu}{\underset{1}{A}}(\kappa p_1+1),\ (\kappa-1),\ p_1\}.$$

As an illustration, we will apply the method to determine

$$\phi_P(100, 3) = 81,$$

Now

$$\kappa = \left[\frac{100 - 3}{9}\right] = 10.$$

$$\therefore \phi_P(100, 3) = \phi_P(100, 10, 3).$$

$$\phi_P(100, 10, 3) = \phi_P(100, 9, 3) - \phi_P\{\overset{1}{\underset{1}{A}}(31), 9, 3\} = 82 - 1 = 81.$$

$$\phi_P(100, 9, 3) = \phi_P(100, 8, 3) - \phi_P\{\overset{1}{\underset{1}{A}}(28), 8, 3\} = 82 - 0 = 82.$$

$$\phi_P(100, 8, 3) = \phi_P(100, 7, 3) - \phi_P\{\overset{1}{\underset{1}{A}}(25), 7, 3\} = 83 - 1 = 82.$$

$$\phi_P(100, 7, 3) = \phi_P(100, 6, 3) - \phi_P\{\overset{1}{\underset{1}{A}}(22), 6, 3\} = 84 - 1 = 83.$$

$$\phi_P(100, 6, 3) = \phi_P(100, 5, 3) - \phi_P\{\overset{1}{\underset{1}{A}}(19), 5, 3\} = 85 - 1 = 84.$$

$$\phi_P(100, 5, 3) = \phi_P(100, 4, 3) - \phi_P\{\overset{0}{\underset{1}{A}}(16), 4, 3\} = 85 - 0 = 85.$$

$$\phi_P(100, 4, 3) = \phi_P(100, 3, 3) - \phi_P\{\overset{2}{\underset{1}{A}}(13), 3, 3\} = 87 - 2 = 85.$$

$$\phi_P(100, 3, 3) = \phi_P(100, 2, 3) - \phi_P\{\overset{3}{\underset{1}{A}}(10), 2, 3\} = 89 - 2 = 87.$$

$$\phi_P(100, 2, 3) = \phi_P(100, 1, 3) - \phi_P\{\overset{4}{\underset{1}{A}}(7), 1, 3\} = 92 - 3 = 89.$$

$$\phi_P(100, 1, 3) = 92.$$

The numerical calculations must be made in reverse order, beginning with the last line instead of with the first.

Interpreting $\phi_P(n, \lambda)$ as already defined, we will proceed to find a formula for it in exactly the same manner as employed in the previous case. If we exclude from the sequence of integers, $1, 2 \cdots\cdots [n]$ all those integers which are divisible by factors of the form $p(kp+1)$, where p is any one of the first $(\lambda - 1)$ primes and k is determined in the usual way, there will be left just $\phi_P(n, \lambda - 1)$ numbers. The integers divisible by $p_\lambda(kp_\lambda + 1)$ are the following:

$$p_\lambda(p_\lambda + 1),\ 2p_\lambda(p_\lambda + 1) \cdots\cdots \left[\frac{n}{p_\lambda(p_\lambda + 1)}\right]\left\{p_\lambda(p_\lambda + 1)\right\},$$

$$\vdots$$

$$p_\lambda(\kappa p_\lambda + 1),\ 2p_\lambda(\kappa p_\lambda + 1) \cdots\cdots \left[\frac{n}{p_\lambda(\kappa p_\lambda + 1)}\right]\left\{p_\lambda(\kappa p_\lambda + 1)\right\}.$$

From this double sequence of numbers, by a sieve process, we can exclude all those numbers which contain factors of the form $p(kp+1)$, p being any one of the first $(\lambda-1)$ primes. We will then define

$$\phi_P\left\{\mathop{A}_{\nu=1\ \ k=1}^{\nu=\nu_k\ \ k=\kappa} p_\lambda(kp_\lambda+1),\ \lambda-1\right\},$$

where p_λ is the λth prime, $k=1, 2\cdot\cdot\cdot\left[\frac{n-p_\lambda}{p_\lambda^{\,2}}\right]$, and $\nu=1, 2\cdot\cdot\cdot\left[\frac{n}{p_\lambda(kp_\lambda+1)}\right]$, determined independently for each value of k, as the number of integers of the double sequence which contain no factors of the given form, where p is any one of the first $(\lambda-1)$ primes. We may then write our formula, which is analogous to Meissel's computation formula for primes,*

$$\phi_P(n,\lambda)=\phi_P(n,\lambda-1)-\phi_P\left\{\mathop{A}_{\nu=1\ \ k=1}^{\nu=\nu_k\ \ k=\kappa} p_\lambda(kp_\lambda+1),\ \lambda-1\right\}.$$

If p_λ is the largest prime not greater than $\left[\frac{\sqrt{4n+1}-1}{2}\right]$, then $\phi_P(n,\lambda)$ will give the number of P's not greater than n.

As an illustration, we will calculate the number of P's not greater than 100. In this case

$$p_\lambda \lesseqgtr \left[\frac{\sqrt{400+1}-1}{2}\right] \lesseqgtr 9,\ i.e.,\ p_\lambda=7 \text{ and } \lambda=4.$$

$$\therefore\ \phi_P(100,4)=\phi_P(100,3)-\phi_P\left\{\mathop{A}_{\nu=1\ \ k=1}^{\nu=\nu_1\ \ k=1} 7(7+1),\ 3\right\}=49-0=49.$$

$$\phi_P(100,3)=\phi_P(100,2)-\phi_P\left\{\mathop{A}_{\nu=1\ \ k=1}^{\nu=\nu_3\ \ k=3} 5(k_j5+1),\ 2\right\}=50-1=49.$$

$$\phi_P(100,2)=\phi_P(100,1)-\phi_P\left\{\mathop{A}_{\nu=1\ \ k=1}^{\nu=\nu_{10}\ \ k=10} 3(k_j3+1),\ 1\right\}=56-6=50.$$

$$\phi_P(100,1)=56.$$

The numerical calculations were made in reverse order, beginning with the last line instead of with the first. It is to be noted that

$$\phi_P(n,1)=\left[\frac{n+1}{2}\right]+\left[\frac{\log n}{\log 2}\right],$$

since $\phi_P(n,1)$ is the number of integers less than n which contain no factors of the form $2(2n+1)$ and, therefore, consists of all the odd integers not greater than n plus the powers of 2 which are not greater than n.

Therefore, the number of P's not greater than 100 is 49, a result which is easily verified.

While the enumeration of the P's within a given limit, either by computation or by an asymptotic formula, is a matter of much theoretical interest, in many

* Meissel: *Math. Annalen* II and III.

cases, especially in applications to the theory of groups, to know whether a given number is a P or a C is most important. This information will necessarily be best supplied by a table showing for each integer all its factors of the form $p(kp+1)$. The tabulation of the P's could be made mechanically by a process similar to that of Eratosthenes's sieve, which consists in writing down the integers in their natural order and then cutting out the successive primes, 2, 3, 5 . . . . . . ; but, as in the construction of factor-tables, various devices are available which permit of simpler and more ready computation.* Such a table may be constructed in the following manner. Arrange the numbers by tens, as in a table of common logarithms of numbers, on a sheet of paper sufficiently long to contain the entire list and ruled in rows and columns. In the proper rectangle write the prime factors of each number, the p's of the C's. It then remains only to write in the various values of k for its corresponding p and the table is complete. If the paper has been accurately ruled, the k's may be written in very readily by measurement, for a given C repeats itself every 5C or 10C lines in the same column, according as the C is an even or odd number. Then, in order, take each C of the double sequence of C's, $C_{i,k} \equiv p_i(kp_i+1)$ where the p's and k's are limited as in the preceding paragraphs. Compute its first appearance in each column, and finally enter it by measurement in the remainder of the column, writing the value of k after its corresponding prime. Many devices for checking the results will suggest themselves. On page 17 we give a sample of such a table.†

* For an account of these devices, see the introduction to Professor Lehmer's *Factor Table for the First Ten Millions.*

† For further details, the reader is referred to the writer's *A Sylow Factor Table of the First Twelve Thousand Numbers* which will be published by the Carnegie Institution of Washington and is now in press.

TABLE OF NUMBERS FROM 1 TO 99, SHOWING FACTORS OF THE FORM $p(kp+1)$

	0	1	2	3	4	5	6	7	8	9
0					2^2		2 1 3		2^3	3^2
1	2 2 5		2^2 3 1		2 3 7	3 5	2^4		2 4 3^2	
2	2^2 5	3 2 7	2 5 11		2^3 3 1	5^2	2 6 13	3^3	2^2 7	
3	2 7 3 3 5 1		2^5	3 11	2 8 17	5 7	2^2 3^2 1		2 9 19	3 4 13
4	2^3 5		2 10 3 2 7		2^2 11	3^2 5	2 11 23		2^4 3 5 1	7^2
5	2 12 5^2	3 17	2^2 13		2 13 3^3	5 2 11	2^3 7 1	3 6 19	2 14 29	
6	2^2 3 1, 3 5 1		2 15 31	3^2 2 7	2^6	5 13	2 3 7 11		2^2 17	3 23
7	2 17 5 7		2^3 3^2 1		2 18 37	3 8 5^2	2^2 19	7 11	2 19 3 4 13	
8	2^4 5 3	3^4	2 20 41		2^2 3 9 1, 2 7	5 17	2 21 43	3 29	2^3 11	
9	2 22 3^2 3 5 1	7 13	2^2 23	3 10 31	2 23 47	5 19	2^5 3 1, 5		2 24 7^2	3^2 11

For each number in the above table the following data are given:

1. In the first column of each rectangle, the representation of the number as a product of powers of primes, thus $90 = 2 \cdot 3^2 \cdot 5$.

2. In the second column the value of k greater than zero for each prime greater than unity such that $N = p(kp+1)$, thus $90 = 2(22 \cdot 2 + 1)$, but there are no values of k greater than zero, for which $90 = 3(3k+1)$ or

$$90 = 5(5k+1).$$

3. In the remaining columns, the values of k which give divisors of N of the form $p(pk+1)$, $k > 0$, $p > 2$; thus 90 is divisible by $3(3 \cdot 3 + 1)$, also by

$$5(1 \cdot 5 + 1).$$

Since there is a close analogy between ordinary primes and P's, a comparison between the number of primes and the number of P's within a given limit naturally suggests itself. In the following table (page 19), the number of primes and the number of P's for each century and for each thousand are listed in adjacent columns. The arrangement of the table is so simple that no explanation is necessary. In the appendix a list of all the P's less than 12,229 is given, so arranged that the number of P's between any two limits less than 12,229 may be easily obtained. The cut, facing the table, shows the number of P's and the number of primes by centuries plotted side by side. The abscissas give the century and the ordinates give the number of primes, or the number of P's, for that century. These two curves are approximate curves of frequency for the P's and primes.

APPROXIMATE CURVES OF FREQUENCY FOR PRIMES AND FOR P'S.

Number of Primes and P's for Each Century and for Each Thousand

0-3999

Century	Primes*	P's†	Century	Primes	P's	Century	Primes	P's	Century	Primes	P's
1‡	26	49	11	16	32	21	14	33	31	12	34
2	21	40	12	12	35	22	10	31	32	10	33
3	16	37	13	15	35	23	15	34	33	11	31
4	16	34	14	11	36	24	15	34	34	15	33
5	17	37	15	17	32	25	10	34	35	11	30
6	14	38	16	12	35	26	11	35	36	14	34
7	16	34	17	15	35	27	15	30	37	13	33
8	14	36	18	12	33	28	14	33	38	12	33
9	15	36	19	12	33	29	12	37	39	11	32
10	14	33	20	13	36	30	11	31	40	11	33
	169	374		135	342		127	332		120	326

4000-7999

Century	Primes	P's	Century	Primes	P's	Century	Primes	P's	Century	Primes	P's
41	15	33	51	12	33	61	12	29	71	9	33
42	9	34	52	11	31	62	11	36	72	10	34
43	16	35	53	10	28	63	13	33	73	11	34
44	9	31	54	10	35	64	15	30	74	9	33
45	11	33	55	13	30	65	8	35	75	11	32
46	12	33	56	13	32	66	11	30	76	15	24
47	12	32	57	12	32	67	10	31	77	12	32
48	12	31	58	10	31	68	12	31	78	10	33
49	8	33	59	16	33	69	12	33	79	10	31
50	15	33	60	7	33	70	13	31	80	10	34
	119	328		114	318		117	319		107	320

8000-11999

Century	Primes	P's	Century	Primes	P's	Century	Primes	P's	Century	Primes	P's
81	11	35	91	11	31	101	11	31	111	10	36
82	10	31	92	12	33	102	12	28	112	11	27
83	14	30	93	11	32	103	10	28	113	10	31
84	9	34	94	11	30	104	12	33	114	10	35
85	8	32	95	15	31	105	10	31	115	11	33
86	12	35	96	7	31	106	8	31	116	9	32
87	13	31	97	13	29	107	12	35	117	8	31
88	11	32	98	11	32	108	11	31	118	9	32
89	13	32	99	12	32	109	10	28	119	12	30
90	9	32	100	9	35	110	10	31	120	13	30
	110	324		112	316		106	307		103	317

* The number of primes was obtained from similar tables in the introduction of Glaisher's *Factor Table of the Sixth Million.*

† The number of P's was obtained from the list of P's given in the appendix.

‡ The first century, 0-99, includes 1 as a prime and a P.

Addenda

A careful study of the curves of approximate frequency and the tables from which they were obtained, together with a study, as n increases, of the increase of the following four different types of P's; namely, (1) primes; (2) powers of primes; (3) products of two consecutive odd primes; (4) all other P's; brings out very clearly what appears to be an important relation between the number of P's and the number of primes within a given limit. Various methods of empirically noting the increase of the last three types of P's within the limits of the list of P's given in the appendix, especially of the logarithms of the prime factors involved, have brought out very interesting data, but so far it has been impossible to correlate these data into definite theorems. However, the data obtained and a study of the curve suggest the following theorem, no proof of which has been obtained:

The number of primes and the number of P's within a given limit differ asymptotically by a constant.

APPENDIX

List of P's*

In the following pages is given a list of P's in their order from 1 to 12,229 and so arranged for the ready computation of the number of P's between any two limits less than 12,229. The columns are numbered consecutively from 1 to the last column of the list in the first line at the top of the pages, and the lines are numbered from 1 to 50 in the first column on the left-hand side of each page. The last three digits of the P's are given in the columns, the hundredth's digit only being given for the first P in the hundred, the remaining digits are to be found at the top of the column; *i.e.*, in the second line at the top of the page. In case the entry for any P is in heavy type, preceded by an asterisk, the other digits for the rest of the P's in that column are to be found at the top of the next column to the right. Thus the P at the intersection of column 35 and row 43 is 5127.

* The P's for this list were obtained from the writer's *A Sylow Factor Table of the First Twelve Thousand Numbers*, already referred to (p. 16). I am indebted to Professor D. N. Lehmer for the method of listing the P's, which is identical with the arrangement to be used by him in a *List of Prime Numbers from One to Ten Millions*, of which he kindly showed me the manuscript.

List of P's, 1–2257

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
									1	1	1	1	1	1	1	2
1	1	103	235	371	511	643	787	929	079	229	367	517	663	813	957	103
2	2	07	39	73	12	47	89	31	87	31	69	19	67	17	59	11
3	3	09	41	77	15	49	93	33	91	33	73	23	69	19	61	13
4	4	13	43	79	17	53	97	37	93	35	77	27	71	23	63	17
5	5	15	45	83	19	59	99	41	97	37	81	29	75	25	69	19
6	7	19	47	89	21	61	801	43	99	41	83	31	79	29	73	23
7	8	21	49	91	23	65	03	47	103	43	85	35	81	31	75	25
8	9	23	51	93	27	67	07	49	07	47	87	37	85	35	77	29
9	11	25	56	95	29	71	09	51	09	49	91	41	87	37	79	31
10	13	27	57	97	31	73	11	53	11	53	93	43	89	41	81	37
11	15	28	59	401	33	77	15	59	15	57	97	47	91	43	85	41
12	16	31	61	03	35	79	17	61	17	59	99	49	93	47	87	43
13	17	33	63	07	37	81	21	63	21	61	401	53	97	49	91	47
14	19	35	65	09	39	83	23	65	23	67	03	57	99	51	03	49
15	23	37	67	11	41	85	27	67	27	69	09	59	707	53	97	51
16	25	39	69	13	45	91	29	71	29	71	11	61	09	59	99	53
17	27	41	71	15	47	95	33	73	33	73	15	63	15	61	*003	57
18	29	43	77	19	51	97	35	77	35	77	17	65	17	63	09	59
19	31	45	81	21	53	99	39	83	39	79	23	67	19	65	11	61
20	32	49	83	23	57	701	41	85	41	83	27	71	21	67	17	65
21	33	51	87	25	59	03	43	89	45	85	29	73	23	71	21	67
22	35	53	89	27	63	07	45	91	47	89	31	77	27	73	23	71
23	37	57	93	31	65	09	47	95	49	91	33	79	29	77	27	73
24	41	59	95	33	69	13	51	97	51	93	37	83	33	79	29	77
25	43	61	97	37	71	17	53	*001	53	95	39	85	35	83	31	79
26	45	63	99	39	73	19	57	03	57	97	41	89	39	85	33	83
27	47	67	303	43	75	21	59	07	59	301	45	91	41	89	39	87
28	49	69	07	45	77	25	63	09	63	03	47	93	45	91	43	91
29	51	73	11	47	81	27	65	13	65	07	51	97	47	95	45	95
30	53	75	13	49	83	29	69	17	67	09	53	601	53	97	47	97
31	59	77	17	51	87	31	71	19	69	13	57	03	57	901	48	201
32	61	79	19	57	89	33	75	21	71	15	59	07	59	03	49	03
33	64	81	21	59	91	39	77	24	75	19	65	09	61	07	51	07
34	65	85	23	61	93	43	79	31	77	21	69	11	63	09	53	09
35	67	87	25	63	99	45	81	33	79	25	71	13	65	13	57	13
36	69	91	29	67	601	47	83	37	81	27	73	15	69	15	59	15
37	71	93	31	69	07	49	87	39	87	29	75	19	73	17	63	19
38	73	97	35	73	11	51	91	41	89	31	81	21	77	19	69	21
39	77	99	37	75	13	53	93	43	93	33	83	25	79	21	71	25
40	79	207	39	77	17	57	95	49	95	37	87	27	81	23	75	27
41	81	09	41	79	19	61	99	51	99	39	89	31	83	27	77	29
42	83	11	43	81	21	63	901	57	201	41	93	33	87	31	81	31
43	85	13	47	85	23	67	07	59	03	43	95	37	89	33	83	37
44	87	15	49	87	25	69	09	61	07	45	99	39	93	37	87	39
45	89	17	53	91	29	71	11	63	11	47	501	43	95	39	89	41
46	91	21	59	93	31	73	13	67	13	49	03	45	97	41	93	43
47	95	23	61	99	35	79	17	69	17	51	07	49	99	43	05	45
48	97	27	65	501	37	81	19	73	19	57	09	51	801	45	97	49
49	99	29	67	03	39	83	23	75	23	61	11	57	07	49	99	51
50	101	33	69	09	41	85	25	77	25	63	13	61	11	51	101	57

LIST OF P'S, 2259–4687

	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
	2	2	2	2	2	3	3	3	3	3	3	3	4	4	4	4
1	259	407	551	713	857	011	151	317	473	623	777	929	085	227	381	531
2	63	11	57	19	59	13	61	19	75	25	79	31	87	29	85	33
3	67	13	61	23	61	17	63	21	81	31	81	35	91	31	87	35
4	69	17	63	25	63	19	67	23	87	35	85	37	93	37	91	37
5	73	19	67	27	67	23	69	25	89	37	87	41	96	41	93	41
6	79	21	69	29	69	29	73	27	91	43	91	43	97	43	97	43
7	81	23	71	31	73	31	75	29	93	47	93	47	99	47	99	47
8	83	25	73	33	75	35	77	31	97	49	97	49	101	49	403	49
9	85	27	75	35	79	37	79	35	99	51	99	53	03	53	09	53
10	87	29	79	37	81	39	81	37	501	53	803	57	09	59	11	59
11	91	35	81	41	85	41	83	41	03	59	07	59	11	61	15	61
12	93	37	87	43	87	43	87	43	09	61	09	61	15	65	19	67
13	97	41	89	47	89	47	91	47	11	65	11	67	17	67	21	69
14	99	43	91	49	91	49	93	49	15	67	15	71	19	71	23	71
15	303	47	93	53	93	51	99	53	17	69	17	73	21	73	27	73
16	09	49	97	59	97	53	203	59	21	71	21	77	27	77	29	77
17	11	53	99	61	99	57	09	61	23	73	23	79	29	79	33	79
18	13	59	603	65	903	59	11	65	27	77	27	83	31	81	35	81
19	15	61	09	67	09	61	15	67	29	79	31	85	33	83	39	83
20	17	63	11	71	11	65	17	71	33	83	33	87	35	85	41	89
21	19	65	15	73	13	67	21	73	37	87	35	89	39	89	43	91
22	21	67	17	77	17	71	23	77	39	89	39	91	41	91	47	95
23	23	71	21	79	21	73	27	79	41	91	41	95	45	93	51	97
24	27	73	23	85	23	77	29	83	43	95	45	97	47	95	53	601
25	33	77	27	87	27	79	31	85	45	97	47	*001	49	97	57	03
26	35	79	29	89	29	83	33	89	47	99	49	03	51	99	59	07
27	39	81	33	91	31	85	35	91	51	701	51	07	53	303	61	19
28	41	83	37	95	33	89	39	95	53	03	53	09	57	07	63	21
29	45	89	41	97	35	91	47	97	57	07	59	13	59	09	69	25
30	47	91	43	99	39	93	51	401	59	09	63	19	63	11	71	27
31	49	95	45	801	41	95	53	07	61	13	65	21	69	13	75	31
32	51	97	47	03	49	97	57	09	63	15	67	23	71	15	77	33
33	53	501	51	07	51	101	59	13	69	19	69	27	75	17	79	37
34	57	03	57	09	53	03	63	15	71	21	71	31	77	21	81	39
35	63	07	59	13	57	07	65	19	77	25	77	33	81	25	83	43
36	67	09	61	15	59	09	69	21	79	27	79	37	83	27	87	45
37	69	13	63	19	63	13	71	25	81	33	81	39	87	31	89	49
38	71	15	69	21	65	15	73	27	83	37	87	41	89	33	93	51
39	75	17	71	23	69	19	77	31	87	39	89	43	93	37	97	57
40	77	19	75	25	71	21	81	33	89	43	93	45	95	39	99	59
41	81	21	77	27	77	23	83	39	93	45	99	49	99	43	501	61
42	83	27	81	31	81	25	87	43	95	49	901	51	201	49	07	63
43	87	29	83	33	83	27	91	47	99	51	03	57	03	53	09	67
44	89	31	87	37	87	31	93	49	601	57	07	61	07	57	11	71
45	91	33	89	39	89	33	95	53	07	61	11	67	11	61	13	73
46	93	37	93	41	93	37	99	57	09	63	17	73	13	63	17	77
47	95	39	99	43	95	39	801	61	11	67	19	75	17	69	19	79
48	99	43	701	45	99	45	07	63	13	69	21	79	19	73	23	81
49	401	45	07	51	*001	47	09	67	17	71	23	81	23	75	27	85
50	03	49	11	53	07	49	13	69	19	73	25	83	25	79	29	87

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	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
	4	4	4	5	5	5	5	5	5	6	6	6	6	6	6	7
1	689	841	997	153	821	471	629	783	939	103	251	407	561	715	877	039
2	91	43	99	57	23	73	31	85	41	07	53	09	63	19	81	43
3	93	47	*001	61	27	77	33	91	47	09	57	13	69	21	83	45
4	97	49	03	63	29	79	39	93	51	13	59	15	71	23	87	47
5	99	53	09	65	33	83	41	801	53	15	61	19	75	25	89	49
6	703	57	11	67	37	85	45	03	57	17	63	21	77	27	91	51
7	09	59	13	71	39	89	47	07	59	19	65	23	81	29	93	53
8	13	61	15	73	41	91	51	09	63	21	69	25	83	31	99	57
9	17	65	17	77	45	97	53	13	65	25	71	27	87	33	901	61
10	21	67	21	79	47	501	57	15	69	27	77	31	89	37	07	67
11	23	71	23	83	51	03	59	21	71	29	81	33	93	39	11	69
12	27	73	27	89	53	07	63	23	75	31	83	37	95	43	13	71
13	29	77	29	91	57	09	67	25	77	33	87	39	99	49	17	73
14	33	79	33	95	59	15	69	27	81	37	89	43	601	51	19	75
15	35	83	35	97	61	19	75	31	83	39	91	45	07	61	25	79
16	39	85	39	99	63	21	77	33	87	43	95	49	11	63	27	81
17	41	89	41	203	65	27	81	37	89	45	97	51	13	67	29	85
18	47	91	45	07	69	31	83	39	91	47	99	53	17	77	31	87
19	49	97	47	09	71	33	87	43	93	49	301	57	19	79	35	91
20	51	901	51	13	75	39	89	45	99	51	07	59	21	81	37	93
21	53	03	53	15	77	41	93	47	*001	57	09	63	23	85	39	97
22	57	09	57	21	81	43	95	49	07	61	11	67	25	91	41	99
23	59	11	59	27	83	45	99	51	09	63	13	69	29	93	43	101
24	63	13	63	31	87	49	701	57	11	67	17	71	31	97	47	03
25	65	15	65	33	89	51	03	61	13	69	19	73	35	99	49	09
26	69	19	67	37	91	53	07	63	17	73	23	79	37	801	57	15
27	71	21	69	39	93	57	11	67	19	75	29	81	39	03	59	17
28	77	25	71	43	95	61	13	69	23	79	31	85	41	09	61	21
29	79	27	77	45	99	63	17	73	29	85	33	87	43	11	67	23
30	81	31	81	49	401	67	21	75	31	87	37	91	47	15	71	27
31	83	33	87	51	07	69	23	77	37	89	41	93	49	17	73	29
32	87	37	89	57	11	73	25	79	41	91	43	97	53	19	77	33
33	89	39	91	61	13	75	29	81	43	93	47	99	59	21	79	35
34	93	43	95	63	17	79	31	91	47	97	49	503	61	23	83	39
35	95	45	99	67	19	81	37	93	49	99	53	09	67	27	89	41
36	99	49	101	69	23	85	39	97	53	203	59	11	73	29	91	43
37	801	51	07	73	29	87	41	99	59	07	61	15	77	33	95	45
38	03	57	11	79	31	89	43	903	65	09	65	21	79	35	97	47
39	11	61	13	81	33	91	47	09	67	11	67	23	83	39	99	51
40	13	63	19	83	35	97	49	11	71	17	73	27	87	41	*001	53
41	17	67	21	87	37	99	51	17	73	21	79	29	89	45	03	57
42	19	69	23	91	41	603	53	19	77	27	83	33	91	47	07	59
43	21	73	27	93	43	09	61	21	79	29	85	35	95	49	09	63
44	25	75	29	97	47	11	65	23	81	33	87	39	97	51	13	65
45	29	79	31	303	49	13	67	27	85	35	89	41	701	57	17	69
46	31	81	37	09	53	15	69	29	89	39	91	47	03	59	19	71
47	33	85	41	11	59	17	71	31	91	41	95	51	07	63	27	73
48	35	87	43	15	61	21	73	33	93	43	97	53	09	65	31	77
49	37	91	47	17	65	23	77	35	95	45	401	57	11	69	33	79
50	39	93	49	19	69	27	79	37	101	47	03	59	13	71	37	81

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	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64
	7	7	7	7	7	7	8	8	8	8	8	8	9	9	9	9
1	183	331	487	669	823	979	129	287	437	583	743	899	061	215	373	533
2	87	33	89	73	25	81	31	91	41	87	47	903	65	17	77	35
3	93	39	93	75	27	83	33	93	43	91	49	07	67	21	79	39
4	95	41	95	79	29	85	35	97	47	93	53	09	69	23	83	47
5	97	43	99	81	31	87	37	99	53	97	59	11	71	27	85	49
6	99	45	501	85	35	89	43	303	57	99	61	15	73	29	89	51
7	201	49	07	87	37	91	47	09	59	603	65	17	77	33	91	53
8	07	51	17	91	41	93	49	11	61	09	67	21	79	35	95	57
9	09	61	19	93	49	99	53	15	65	11	71	23	83	39	97	59
10	11	63	23	97	53	*003	59	17	67	15	73	27	85	41	401	63
11	13	67	29	99	59	09	61	21	69	17	77	29	89	45	03	65
12	17	69	37	703	61	11	67	27	71	21	79	33	91	47	07	69
13	19	73	41	09	63	15	71	29	73	23	81	35	95	49	09	71
14	23	75	43	11	67	17	73	31	77	27	83	39	97	51	11	73
15	25	77	47	13	71	21	77	33	79	29	89	41	101	53	13	75
16	29	79	49	15	73	23	79	35	83	33	91	45	03	57	19	77
17	31	81	51	17	77	27	81	39	85	37	93	47	07	59	21	83
18	33	87	59	21	79	29	83	41	89	39	95	51	09	63	25	87
19	35	89	61	23	83	33	85	45	91	41	97	53	13	67	27	89
20	37	91	65	27	85	35	87	47	95	47	803	59	15	69	31	93
21	41	93	71	29	89	39	89	51	97	51	07	63	17	71	33	99
22	43	97	73	33	91	41	91	53	501	53	09	69	19	77	37	601
23	47	99	77	37	95	45	92	57	03	57	13	71	23	79	39	07
24	49	403	83	39	97	47	97	59	07	59	17	75	25	81	41	09
25	51	09	87	41	99	51	99	61	09	63	19	77	27	83	45	13
26	53	11	89	45	901	53	201	63	11	65	21	81	31	87	51	17
27	61	15	91	47	03	57	03	67	13	67	25	83	33	89	57	19
28	63	17	93	51	07	59	07	69	19	69	31	87	37	93	61	23
29	65	21	97	53	09	61	09	71	21	71	33	89	43	99	63	27
30	67	23	601	57	11	63	13	75	23	75	37	93	49	301	67	29
31	69	27	03	59	13	65	19	77	27	77	39	95	51	07	69	31
32	73	31	07	63	15	69	21	81	29	81	43	97	53	11	73	37
33	77	33	09	65	19	71	23	83	31	89	47	99	57	13	75	43
34	79	35	13	67	21	75	25	87	37	91	49	*001	61	19	79	47
35	81	41	15	69	25	77	27	89	39	93	51	07	63	23	81	49
36	83	43	19	71	27	79	31	93	43	95	57	11	67	25	87	59
37	89	45	21	73	29	81	33	95	45	99	61	13	69	29	91	61
38	91	47	27	77	31	83	37	97	49	707	63	17	71	35	93	63
39	95	51	29	81	33	87	43	99	51	09	67	19	73	37	97	65
40	97	53	31	83	37	89	45	401	57	11	69	29	75	41	99	67
41	301	57	33	89	39	93	49	03	59	13	71	31	79	43	501	71
42	03	59	37	93	43	95	57	07	61	17	73	33	81	47	03	73
43	07	63	39	95	49	97	59	11	63	19	79	35	87	49	09	77
44	09	65	43	99	51	99	61	13	67	25	81	37	91	53	11	79
45	13	69	47	801	57	101	63	17	69	27	85	41	93	57	17	83
46	19	71	49	07	63	11	67	19	73	31	87	43	97	59	21	85
47	21	75	57	11	67	13	69	23	75	35	89	47	99	65	23	89
48	23	77	61	13	69	17	73	25	77	37	91	49	203	67	27	93
49	25	81	63	17	71	19	79	29	79	39	93	53	09	69	29	95
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	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
	9	9	10	10	10	10	10	10	10	11	11	11	11	11	11	12
1	701	865	007	171	345	501	657	813	981	125	301	447	601	761	929	079
2	03	69	09	77	47	03	61	17	85	29	03	49	03	63	33	83
3	07	71	13	81	49	07	63	19	87	31	09	53	09	69	39	85
4	13	73	15	83	51	11	67	23	91	41	11	59	11	71	41	89
5	15	75	19	87	57	13	69	25	93	47	13	61	17	73	45	91
6	19	77	21	89	59	17	71	27	97	49	17	63	21	77	47	97
7	21	81	27	93	61	19	73	31	*003	53	21	65	23	79	51	101
8	25	83	31	95	63	23	79	37	07	57	23	67	29	83	53	03
9	27	87	33	201	67	29	81	41	09	59	27	71	33	85	57	07
10	31	89	37	07	69	31	83	43	11	61	29	77	35	87	59	09
11	33	93	39	11	75	37	85	47	13	71	31	79	37	89	61	13
12	37	95	41	13	79	41	87	51	15	73	33	83	39	91	65	15
13	39	97	43	17	81	43	91	53	17	77	35	85	41	97	67	19
14	43	99	49	19	83	47	93	59	21	83	37	89	43	801	69	21
15	45	901	51	21	87	49	97	61	23	85	39	91	47	07	71	23
16	49	07	57	23	91	53	99	67	27	89	41	93	51	09	75	25
17	51	11	61	25	93	59	703	69	29	91	45	97	53	13	81	27
18	53	13	63	29	97	61	09	71	31	95	47	99	57	15	83	31
19	57	17	67	31	99	65	11	73	33	97	51	501	59	19	87	33
20	61	19	69	35	401	67	15	77	35	201	53	03	63	21	89	37
21	63	23	73	37	03	71	17	83	39	03	57	07	65	27	93	39
22	67	25	77	39	09	73	21	89	41	07	59	09	71	31	95	43
23	69	27	79	43	11	77	23	91	45	09	63	13	75	33	99	45
24	71	29	81	47	15	79	27	95	47	13	65	15	77	37	*001	47
25	73	31	85	49	23	81	29	97	51	15	69	19	81	39	03	49
26	81	35	87	53	27	83	33	903	53	19	71	21	83	41	07	51
27	85	37	91	59	29	85	35	07	57	21	77	27	87	45	11	57
28	87	41	93	61	33	89	37	09	59	25	81	31	89	49	13	61
29	91	43	97	65	35	95	39	13	61	27	83	33	93	51	17	63
30	93	49	99	67	39	97	41	15	63	31	87	37	95	57	19	67
31	97	53	103	71	41	99	43	19	65	33	89	39	99	61	21	69
32	99	59	09	73	45	601	45	21	69	37	91	43	701	63	23	73
33	803	61	11	77	47	07	47	27	71	39	93	45	07	67	29	75
34	09	63	13	79	51	09	51	31	75	43	95	47	09	71	31	79
35	11	65	17	89	53	11	53	33	77	45	99	49	17	73	35	85
36	17	67	23	91	57	13	57	37	83	49	401	51	19	75	37	87
37	19	69	29	301	59	17	63	39	87	51	07	53	23	79	39	91
38	21	71	33	03	63	19	65	43	89	57	09	61	25	81	41	93
39	23	73	35	07	67	21	71	49	93	61	11	67	29	85	43	97
40	27	77	39	09	69	25	77	51	95	63	13	69	31	87	47	99
41	29	79	41	13	71	27	79	53	97	67	17	73	33	93	49	203
42	33	81	45	15	73	29	81	57	99	69	19	75	35	97	53	09
43	39	83	47	19	75	31	83	61	101	73	21	79	37	99	57	11
44	41	85	51	21	77	33	89	63	03	79	23	81	41	903	59	15
45	47	87	57	27	81	37	93	67	07	81	25	87	43	09	65	17
46	51	89	59	31	83	39	95	69	11	83	29	89	47	11	67	19
47	53	91	63	33	87	43	99	73	13	87	35	91	49	15	69	21
48	57	95	65	37	89	45	801	75	17	91	37	93	51	17	71	23
49	59	97	67	41	95	49	07	77	19	93	41	97	53	23	73	27
50	63	*001	69	43	99	51	11	79	23	99	43	99	59	27	77	29

VITA

I, Henry Walter Stager, was born in the year 1879 at Nutley, New Jersey, the son of John Willis and Bertha Stager. My elementary education was obtained in the public schools of Nutley and the neighboring town of Montclair. I was prepared for college at the Montclair High School and graduated from the Leland Stanford Junior University in 1902 with the degree of A.B. In 1905, I returned to Stanford for graduate study, pursuing work under Professors Allardice and G. A. Miller, and received the degree of A.M. in 1906. In September, 1907, I entered the University of California and continued graduate work under the direction of Professors Haskell, Lehmer, MacDonald, and Schilling.

I wish to express my appreciation for the kindly interest of all my instructors. My thanks are especially due to the members of my Committee, Professors Lehmer, Haskell, and Schilling; and more especially to Professor Lehmer, whose encouragement and sympathetic advice have been a constant source of help in preparing my dissertation.