On Generic Context Lemmas for Lambda Calculi with Sharing

Manfred Schmidt-Schauß and David Sabel

Institut für Informatik
Johann Wolfgang Goethe-Universität
Postfach 11 19 32
D-60054 Frankfurt, Germany
schauss@cs.uni-frankfurt.de

Technical Report Frank-27
Revised Version[1]

August 10, 2007

Abstract. This paper proves several generic variants of context lemmas and thus contributes to improving the tools to develop observational semantics that is based on a reduction semantics for a language. The context lemmas are provided for may- as well as two variants of must-convergence and a wide class of extended lambda calculi, which satisfy certain abstract conditions. The calculi must have a form of node sharing, e.g. plain beta reduction is not permitted. There are two variants, weakly sharing calculi, where the beta-reduction is only permitted for arguments that are variables, and strongly sharing calculi, which roughly correspond to call-by-need calculi, where beta-reduction is completely replaced by a sharing variant. The calculi must obey three abstract assumptions, which are in general easily recognizable given the syntax and the reduction rules. The generic context lemmas have as instances several context lemmas already proved in the literature for specific lambda calculi with sharing. The scope of the generic context lemmas comprises not only call-by-need calculi, but also call-by-value calculi with a form of built-in sharing. Investigations in other, new variants of extended lambda-calculi with sharing, where the language or the reduction rules and/or strategy varies, will be simplified by our result, since specific context lemmas are immediately derivable from the generic context lemma, provided our abstract conditions are met.

Keywords: lambda calculus, observational semantics, context lemma, functional programming languages

[1] This document is a revised and extended version of an earlier version originally published on the web in January 2007.

1 Introduction

A workable semantics is indispensable for every formal modelling language, in particular for all kinds of programming languages and process calculi. This paper will make a contribution to the tools, in particular so-called context-lemmas, which support operational reasoning about semantical properties of higher-order functional programming languages, process calculi and lambda-calculi on the basis of an observational semantics. A semantics is very useful to obtain safe knowledge about the evaluation and optimizations of programs, correctness of program transformations, and correctness of implementations in other calculi.
For various higher-order calculi a widely used observational semantics is contextual equivalence based on a (small-step) reduction semantics in the style of [Mor68], i.e. two expressions are equal if their termination behavior is always the same when they are plugged into an arbitrary program context. We assume that a calculus is given consisting of a language of terms, a small step reduction relation $\rightarrow$ on terms, and a set of answer terms. A term t is called may-convergent if there exists a finite sequence of $\rightarrow$-reductions starting with t and reaching an answer. Usually answers are weak head normal forms for call-by-need and call-by-name calculi, weak normal forms for call-by-value calculi, and irreducible (or successful) processes in process calculi. For non-deterministic calculi, contextual equivalence must be based on the conjunction of two termination behaviors (see e.g. [Ong93]): May-convergence and must-convergence, where the latter takes all reduction possibilities into account. There are two different definitions of must-convergence in the literature:

1. iff every term t' reachable from t by a sequence of $\rightarrow$-reductions is may-convergent.
2. iff every maximal sequence of reductions starting with t ends in an answer, in particular, there are no infinite reductions. We will call this form of must-convergence also total must-convergence.

The first definition of must-convergence also includes terms that may evaluate infinitely but the chance of finding an answer is never lost. These terms are called weakly divergent in [CHS05]. Note that a similar combination of may- and must-convergence is also known from the use of convex powerdomains in domain-theoretic models (see [Plo76]).
In this paper we will consider several variants of contextual approximations and equivalences based on may-, must- and total must-convergence. Usually, a first step and a strong tool for further proof techniques is to prove a context lemma that reduces the test for convergence (may- and/or must-) to a subclass of contexts, the reduction contexts (also called evaluation contexts), instead of all contexts. This technique dates back to [Mil77] for showing full abstractness of denotational models of lambda-calculi.
We formulate natural conditions on extended lambda-calculi and process calculi and their reduction semantics, and then prove generic context lemmas for the three types of convergences for calculi satisfying these conditions. An informal account of our results is as follows: We assume that a calculus in a higher-order

abstract syntax is given together with a small-step reduction relation, and a a set of answers; Also an algorithm to determine reduction positions is given. The assumptions are as follows:

1. The set of reduction positions of a term is determined top-down, and does not depend on non-reduction positions.
2. The property of being an answer-term does not depend on non-reduction positions.
3. The small-step reduction relation has rather limited abilities to modify subterms at non-reduction positions: it is permitted to remove, transport or duplicate them; also to apply renaming of bound variables. It is not permitted to modify subterms at non-reduction positions, with the exception of a restricted form of variable-variable substitution in weakly sharing calculi.
4. All properties are invariant under renaming of bound variables, and also under permutation of free variables, as long as no type conditions are violated.

The obtained results are six context lemmas for the combinations of strongly and weakly sharing and the three types of convergencies: For strongly sharing calculi, a context lemma allows to restrict the observation of the convergence to reduction contexts instead of all contexts, where reduction contexts are exactly the contexts where the hole is a reduction position. In the case of weakly sharing calculi, we have to observe the behavior of $R[\sigma(s)]$, where $R[]$ is a reduction context, and σ a perhaps non-injective substitution replacing variables by variables.
Our technique works for process calculi like the π-calculus, and program calculi with sharing variants of β-reduction, which is not a restriction in our opinion, since programming languages or their respective abstract machines almost always exploit sharing mechanisms. We consider two forms of sharing lambda calculi: strongly sharing and weakly sharing calculi. In strongly sharing calculi, the (normal-order) reduction may only modify non-reduction positions through renamings of bound variables. E.g. the full beta-rule $(\lambda x.s)\ t \to s[t/x]$ violates our assumptions, since there may be non-reduction occurrences of x in s that are replaced by the beta-rule. The restricted beta-rule $(\lambda x.s)\ y \to s[y/x]$ may be allowed in weakly sharing calculi, but only if the argument position in applications is syntactically restricted to be a variable. In this case, there may be a substitution of variables by variables. The rules $(\lambda x.s)\ t \to (\texttt{let}\ x = t\ \texttt{in}\ s)$ and $(\texttt{let}\ x = v\ \texttt{in}\ R[x]) \to (\texttt{let}\ x = v\ \texttt{in}\ R[v])$ are the sharing variants of beta-reduction and permitted in strongly sharing calculi. The corresponding rules in explicit substitution calculi (see [ACCL91]) are compatible, though in connection with non-deterministic operators the set of rules has to be adapted. Similar considerations hold for other rules like case-rules. Examples for calculi, where the context-lemma for may-convergence is immediately applicable are the deterministic calculi in [AFM$^+$95,AF97,MOW98,AS98,SS06]. Non-deterministic calculi where also the must-context lemmas are applicable, are in [KSS98,Man05,SSS07,NSSSS07], the latter is the calculus in [NSS06] with some adaptations. The context lemmas also hold in process calculi like variants of the π-calculus (see [Mil99,SW01]) and the join-calculus (see [FG96,Lan96]),

which from our point of view are weakly sharing, since a full replacement of names by names is performed by reduction rules.
The context lemma is an important tool for further investigations into correctness of program transformations and optimizations, for example, the diagram methods in [SSSS05,SSS07,NSSSS07] demonstrate their strength only if the context lemma holds. There is no context lemma used in [KSS98], which severely complicates the diagram-proofs.
There is also related work on context lemmas for calculi not satisfying our conditions. For call-by-value languages with beta-reduction, there is a weaker form of a context lemma, the so-called CIU-theorem, which was first proved in [FH92], which also holds for a class of languages, and was even formally checked by an automated reasoner (see [FM01,FM03]). For PCF-like languages, also with full beta-reduction, there is also a context lemma proved for a class of languages extending PCF (see [JM97]).
Another related generic tool is bisimilarity for extended lambda-calculi (see [How89,How96]), and for typed languages (see [Gor99]). An extension for calculi with sharing w.r.t. may-convergence is done for a non-deterministic calculus in [Man05] and for a class of calculi in [MSS06].
The structure of this paper is as follows. After presenting the abstract syntax for higher-order calculi, the assumptions on the calculi are presented and discussed. Section 4 presents the different convergence relations and contextual approximations, section 5 contains the proofs of the various generic context lemmas, and the final section 6 contains a discussion on the range of calculi where the instances of the generic context lemmas hold.

2 Abstract Syntax and Language

In the following we provide generic mechanisms to describe the language of expressions, the renamings and the reduction relation of a calculus *calc*. For the generic formulation of the language we use higher-order abstract syntax, (see e.g. [How89,How96]), which is extended by a system of simple types. The construction of terms of the language requires variables, operators (i.e. symbols with arity), and variable-binding primitives. We allow the extension by a recursive `letrec` which is used as an extra operator with its own binding rules. The main purpose of the types is to allow different syntactic categories in the respective languages, for example, channel names and processes, or lambda-expressions and processes, but also enables to model untyped calculi. It may also be used to model forms of simple typing.

Definition 2.1. *A* signature $\mathcal{L}$ *of a* higher order computation language *is a 5-tuple* $(O, T_0, \alpha, \beta, \beta_\tau)$ *where*

- O *is a (possibly infinite) set of* operators*, which may contain* `letrec`*,*
- T_0 *is the set of* basic types*, which defines the set of types* $\mathcal{TYP}$ *inductively as* $T_0 \subseteq \mathcal{TYP}$ *and* $t_1 \to t_2 \in \mathcal{TYP}$ *if* $t_1, t_2 \in \mathcal{TYP}$[2]*.*

[2] We use the convention of right-association: $\tau_1 \to \tau_2 \to \tau_3$ means $\tau_1 \to (\tau_2 \to \tau_3)$

- $\alpha : (O \setminus \{\texttt{letrec}\}) \to \mathbb{N}_0$ *defines the* arity *for every operator except for*`letrec`*.*
- *For every operator* $f \in O \setminus \{\texttt{letrec}\}$*,* $\beta(f)$ *is an* $\alpha(f)$*-tuple with components in* $\mathbb{N} \cup \{\text{“}V\text{”}, \text{“}T\text{”}\}$*, indicating the number of possible variables that may be bound at the corresponding argument position, or that there are no binders and that only variables (“V”) or that any term (“T”) is permitted.*
- *Let* $f \in O \setminus \{\texttt{letrec}\}$ *with* $\alpha(f) = n$ *and* $\beta(f) = (b_1, \ldots, b_n)$*. Then* f *has a type* $\beta_\tau(f) \in \mathcal{TYP}$ *satisfying the following conditions:*
  - $\beta_\tau(f) = \tau_1 \to \tau_2 \to \ldots \to \tau_{n+1}$
  - *if* $b_i \in \mathbb{N}$*, then* τ_i *must be of the form* $\tau_{i,1} \to \tau_{i,2} \to \ldots \to \tau_{i,b_i+1}$

Note that we do not insist on τ_{n+1} and τ_{i,b_i+1} being base types in the definition of $\beta_\tau(f)$.
Given a signature $\mathcal{L}$ the terms of the higher order computation language are defined as follows, where we assume that there is a subset $\mathcal{TYP}_V \subseteq \mathcal{TYP}$, such that for every $\tau \in \mathcal{TYP}_V$, there is an infinite set of variables V_τ of type τ.

Definition 2.2. *Let* $\mathcal{L} = (O, T_0, \alpha, \beta, \beta_\tau)$ *be a signature of a higher order computation language, then* terms $\mathcal{T}(\mathcal{L})$ *are inductively defined as follows:*

- *Every* $x \in V_\tau$ *for* $\tau \in \mathcal{TYP}_V$ *is a term of type* τ*.*
- *If* $f \in O \setminus \{\texttt{letrec}\}$ *with* $\alpha(f) = 0$*, then* f *is a term with type* $\beta_\tau(f)$*.*
- *If* $f \in O \setminus \{\texttt{letrec}\}$*, then* $f(a_1, \ldots, a_n)$ *is a term provided that* $n = \alpha(f) \geq 1$*, and for every* $i = 1, \ldots, n$ *the following holds:*
  - *if* $\beta(f)_i =$ *“V”, then* a_i *is a variable,*
  - *if* $\beta(f)_i =$ *“T”, then* a_i *is a term, and*
  - *if* $\beta(f)_i = m \in \mathbb{N}$*, then* a_i *is of the form* $x_1, \ldots, x_m \,.\, t_i$*, where* $x_1, \ldots, x_m$ *are different variables, and* t_i *is a term*[3]*.*

  The typing must be as follows: if f *has type* $\tau_1 \to \ldots \to \tau_n \to \tau_{n+1}$*, then* a_i *has type* τ_i *for all* i*, and the operands* $a_i = x_1, \ldots, x_m \,.\, t_i$ *are defined to have type* $\tau_i = (\tau_{i,1} \to \ldots \tau_{i,m} \to \tau_{i,m+1})$*, where* x_j *has type* $\tau_{i,j}$ *for all* j *and* t_i *has type* $\tau_{i,m+1}$*.*
- *If* $\texttt{letrec} \in O$*,* $n \geq 0$*,* $x_1, \ldots, x_n$ *are different variables, and if* $t_1, \ldots, t_n, s$ *are terms, then* $(\texttt{letrec}\ x_1 = t_1, \ldots, x_n = t_n\ \texttt{in}\ s)$ *is a term of type* τ *which is the type of* s*, where the for all* i*: the terms* x_i, t_i *must be equally typed.*

As usual, the scope of every variable $x_i, i = 1, \ldots, n$ *in* $x_1, \ldots, x_m \,.\, t$ *is the term* t*, and the scope of every variable* x_i *in* $(\texttt{letrec}\ x_1 = t_1, \ldots, x_n = t_n\ \texttt{in}\ s)$ *is the set of terms* $t_1, \ldots, t_n, s$*. Variable occurrences that are in the scope of a binder that binds them, are called* bound *occurrences of variables, others are* free occurrences *of variables.*

The set of *free variables* of a term t is denoted as $\mathcal{FV}(t)$. As usual, a term t is *closed* if all of its variables are bound, i.e. $\mathcal{FV}(t) = \emptyset$, otherwise it is called *open*. Since we have to deal in depth with different kinds of renamings in later sections,

[3] The expressions $x_1, \ldots, x_m \,.\, t_i$ are called operands in [How96]

we do not assume anything about implicit renamings, though it is known how to correctly rename terms (cf. [Bar84]).
Concerning the typing, we assume in the following that only correctly typed terms are syntactically acceptable, that every correct term has exactly one type, and that modifications, renamings, reductions do not change the type of a term. At places, where it is important, we emphasize the typing, whereas we often do not mention the typing, if it is clear from the context.

Example 2.3. With $O = \{\lambda, \texttt{letrec}, @, \texttt{Cons}, \texttt{Nil}, \texttt{case}\}$ a language with `let(rec)`, application, abstractions, lists and a `case` is defined. We obtain an untyped lambda-calculus, if *Term* is the only base type and the result type of every operator is just *Term*. The description is $\alpha(\lambda) = 1$, $\beta(\lambda) = (1)$, $\alpha(@) = 2$, $\beta(@) = (\text{“}T\text{”}, \text{“}T\text{”})$, `Cons` is specified like @, $\alpha(\texttt{Nil}) = 0$, and $\alpha(\texttt{case}) = 3, \beta(\texttt{case}) = (\text{“}T\text{”}, 2, \text{“}T\text{”})$. The lambda-term $\lambda x.x$ is represented as $\lambda(x.x)$). A term like
`let x = (Cons x Nil) in case x of (Cons z1 z2) -> y; Nil -> Nil`
would be expressed as $(\texttt{letrec}\ x = \texttt{Cons}(x, \texttt{Nil})\ \texttt{in}\ \texttt{case}(x, z_1 z_2.y, \texttt{Nil}))$.

Examples of languages where “V” is necessary, i.e., with the variable restriction are: [MSC99] where arguments of applications are only variables, and the language in [NSSSS07], where e.g. the first argument in cell-expressions $(x\ \texttt{c}\ t)$ must be a variable.

Example 2.4. The π-calculus with the syntax

$$P, Q ::= (P \mid Q) \mid \nu x.P \mid 0 \mid P + Q \mid x(y_1, \ldots, y_n).P \mid \overline{x}\langle y_1, \ldots, y_n\rangle.P \ \mid\ !P$$

can be represented. We have to use at least two types for an appropriate encoding: “*process*” and “*channel*”, and moreover, we assume that there are variables of type *channel*, but no variables of type *process*. The common scoping policy in $x(y_1, \ldots, y_n).P$ is also easily representable as $\texttt{in}(x, (y_1, \ldots, y_n.P))$, where the first argument of `in` is restricted to variables. The process $\overline{x}\langle y_1, \ldots, y_n\rangle.P$ can be represented as $\texttt{out}_n(x, y_1, \ldots, y_n, P)$ with perhaps different operators, where all arguments but the last one are marked “V” in the arity-tuple.

We will use *positions* to address subterms and variables in binders using a slightly extended Dewey decimal notation. The addressing is such that prefixes of addresses of term positions are again term positions. We write $s_{|p}$ for the subterm of s at position p, and $s(p)$ for the head-symbol of the subterm $s_{|p}$. For example, the term $t = \texttt{Cons}(\lambda(x.@(x, @(y, x)), z)$ has e.g. the following term positions: y is at position 1.1.2.1, x occurs at positions 1.1.1 and 1.1.2.2, and $t(1) = \lambda$ and $t(1.1.1) = x$. The binders are addressed using a “B”, such that the binding position of x above is $1.1.B.1$. For the description of “renamings” of free variables we also assume that there is a virtual binder for free variables at the top of the term.
A *context* C is like a term, where the hole $[\cdot]$ is allowed at a single term-position, but not at a “V”-position, and the hole must be typed according to the abstract

syntax. If the type τ of the hole is important, then we denote this as $C[]_\tau$. The expression $C[s]$ denotes the result of plugging in a (correctly typed) term s into C, where capture of variables is permitted. We will also use *multi-contexts* M that may have more holes, which are contexts with multiple, distinguishable holes, where the holes are in left-to right-order. If $t_1, \ldots, t_n$ are terms, and M is a multicontext with n holes, then $M[t_1, \ldots, t_n]$ is the term after plugging in the n terms, in left-to-right order.

Definition 2.5 (Distinct Variable Convention). *A term t satisfies the* distinct variable convention (DVC)*, iff all bound variables in t are distinct, and moreover, all bound variables in t are distinct from all the free variables in t.*

2.1 Renamings and Substitutions

We introduce notions around renamings, substitutions and mappings, since we have to separate reduction-steps and renaming-steps later on. Usually, a renaming renames bound variables in a term. For a uniform treatment, we will also use the notion "renaming" for a bijective replacement of free variables. We will use the word "substitution" if we mean a perhaps non-bijective mapping on free variables. In this and the following sections we tacitly assume that replacements of variables by variables is only performed if the variables have equal type.
A *modification* (function) of variables of the term s is described by a finite set S of pairs $(p, x \mapsto y)$, where p is a binding position of x. Modifying a term means to apply all the replacements $x \mapsto y$ to all occurrences of the variable x in the scope of the binder at p, and also to the binder, where all the replacements have to be done "in parallel". Note that also free variables may be subject to modification. E.g. for $s = \lambda x.@(x, (\lambda x.x))$, the set $S_1 = \{(B.1, x \mapsto y), (2.B.1, x \mapsto z)\}$ represents the modification of s resulting in $\lambda y.@(y, (\lambda z.z))$. A modification of t is called *capture-free*, iff the relation between occurrence of a variable and its binding position remains unchanged, for all variable occurrences in the term t. Note that the modifications are always meant w.r.t. a given term. Modifications do not make sense without mentioning the term to which they are applied. If it is unambiguous, we represent a modification by a set $S = \{(p_1, x_1 \mapsto y_1), \ldots, (p_n, x_n \mapsto y_n)\}$ by omitting the positions as $\{x_1 \mapsto y_1, \ldots, x_n \mapsto y_n\}$.
Now we define renamings and variable-substitutions as specialized modifications. A *bv-renaming* of s is defined as a modification that is capture-free and renames only bound variables (the virtual binder is not allowed in this case). The relation $s =_\alpha t$ denotes that t can be reached from s by a (perhaps empty) sequence of bv-renamings. An *fvbv-renaming* σ is defined as a capture-free modification that renames free and bound variables of a term s, and that is injective on $\mathcal{FV}(s)$. With $\mathrm{fvp}(\sigma)$ we denote the induced mapping of a fvbv-renaming σ of s on $\mathcal{FV}(s)$. If $\{x_1 \mapsto y_1, \ldots, x_n \mapsto y_n\}$ is the representation of the mapping on variables, then $\{x_1, \ldots, x_n\}$ is the *domain*, and $\{y_1, \ldots, y_n\}$ the *codomain*. A *vvbv-substitution* γ of a term s is a capture-free modification like an fvbv-renaming, but it may

be not injective on $\mathcal{FV}(s)$. We also use $\mathrm{fvp}(\gamma)$, which we call in this case a *vv-substitution*, denoted as ν. Given a term s, a set W of variables with $\mathcal{FV}(s) \subseteq W$, a vv-substitution ν on W, and a vvbv-substitution γ. Then we say γ is *compatible with* ν on W, iff $\mathrm{fvp}(\gamma)(x) = \nu(x)$ for all $x \in W$. This notion is also used for fvbv-renamings.

Example 2.6. The term $\lambda y.x$ can be modified into $\lambda x.y$ by an fvbv-renaming (or a vvbv-substitution). However, this cannot be represented as $\sigma\mu_1$ for a bv-renaming μ_1 and a substitution $\sigma = \{x \mapsto y\}$, nor as $\mu_1\sigma$. It can only be represented as $\sigma_1\mu_1\sigma_2$ with two vv-substitutions $\sigma_2 = \{x \mapsto z\}$, and $\sigma_1 = \{z \mapsto y\}$ and a bv-renaming $\mu_1 = \{y \mapsto x\}$.

Lemma 2.7.

- *If* $t_1 \xrightarrow{\sigma_1} t_2 \xrightarrow{\sigma_2} t_3$ *by vvbv-substitutions* σ_1, σ_2*, then the composition* $\sigma_3 = \sigma_2 \circ \sigma_1$ *with* $t_1 \xrightarrow{\sigma_3} t_3$ *is a vvbv-substitution with* $\mathrm{fvp}(\sigma_3)(x) = \mathrm{fvp}(\sigma_2)\mathrm{fvp}(\sigma_1)(x)$ *for all* $x \in \mathcal{FV}(t_1)$*. This also holds for appropriate restrictions to fvbv-renamings and bv-renamings.*
- $s =_\alpha t$ *iff* $s = t$ *or* $s \xrightarrow{\sigma} t$ *by a single bv-renaming* σ*.*
- *If* $s \xrightarrow{\sigma} t$ *by an fvbv-renaming, then the reverse renaming is also capture-free, i.e. a fvbv-renaming. Again this holds also for bv-renamings.*
- *If* s *is a term not satisfying the DVC, then there is a term* s' *satisfying the DVC, and a bv-renaming* σ *with* $\sigma(s) = s'$*. This can be accomplished by renaming bound variables with fresh variables.*
- *If* $s \xrightarrow{\sigma} t$ *is a vvbv-substitution, and* $\rho = \mathrm{fvp}(\sigma)$*, then there are bv-renamings* σ_1, σ_2*, and a vv-substitution* ρ*, such that* $s \xrightarrow{\sigma_1} \xrightarrow{\rho} \xrightarrow{\sigma_2} t$*, i.e.* $\sigma = \sigma_2 \circ \rho \circ \sigma_1$*.*

Proof. Easy computations.

It is interesting to note that terms and vvbv-substitutions form a category with terms as objects and vvbv-substitutions as arrows; the same holds for fvbv-renamings and bv-renamings. We will also use fvbv-renamings and bv-renamings for contexts and multicontexts.

Definition 2.8. *Let* C *be a (one-hole) context. Then* $BP_{\mathrm{hole}}(C)$ *is defined as the set of binder-positions in* C *that have the hole of* C *in their scope,* $BP_{\overline{\mathrm{hole}}}(C)$ *is the complement, i.e. the set of binder-positions that do not have the hole in their scope, and* $V_{\mathrm{hole}}(C)$ *is the set of variables that are bound by the binders in* $BP_{\mathrm{hole}}(C)$*. For a multi-context* M *with any number of holes, the notation* $BP_{\mathrm{hole}}(M, i)$ *and* $V_{\mathrm{hole}}(M, i)$ *mean the corresponding notions for the* i*th hole.*

It is obvious that all variables bound by binders in $BP_{\mathrm{hole}}(C)$ are different, i.e. there are no binders with equal variables in this set.

Lemma 2.9. *Let* C *be a context,* s *be a term, and* σ *be an fvbv-renaming of* $C[s]$*. Then* σ *can be splitted into an fvbv-renaming* σ_C *of* C*, and an fvbv-renaming* σ_s *of* s*, where* $\sigma(C[s]) = \sigma_C(C)[\sigma_s(s)]$ *and the mapping* $\mathrm{fvp}(\sigma_s)$ *is injective on* $V_{\mathrm{hole}}(C)$ *as well as on* $\mathcal{FV}(s)$*. Also the mapping induced by* σ *on* $V_{\mathrm{hole}}(C)$ *is injective.*

Example 2.10. Let $C = \lambda y_1.(x\ \lambda z.z\ \lambda y_2.[])$ and $s = (x\ y_2\ \lambda u.u)$. Then $V_{\text{hole}}(C) = \{y_1, y_2\}$. Let σ be the fvbv-renaming of $C[s]$ with $\{x \mapsto x', y_1 \mapsto y_1', y_2 \mapsto y_2', z \mapsto z', u \mapsto u'\}$. Then $\sigma_C = \{x \mapsto x', y_1 \mapsto y_1', y_2 \mapsto y_2', z \mapsto z'\}$, and $\sigma_s = \{x \mapsto x', y_1 \mapsto y_1', y_2 \mapsto y_2', u \mapsto u'\}$. Note that y_2 is a bound variable in C, but free in s.

3 Generic Lambda Calculi with Sharing

We assume that a calculus CALC is given and describe the required notions and properties that are required such that the context lemmas hold. We assume that for the calculus CALC the following is given:

- A language of expressions in the higher-order abstract syntax, according to Definition 2.1.
- An algorithm UNWIND, detecting all the potential reduction positions (which must be term positions). We assume that the algorithm has a term or a multi-context as input and non-deterministically produces a sequence of positions, all of which are reduction positions.
- A small-step reduction relation $\to_0$ on terms, where $s \to_0 t_0$ is defined only for terms s satisfying the DVC, but the term t_0 is not further restricted. The *small-step reduction* $s \to t$ is then defined as $s \to_0 t_0 \to_1 t$, where $\to_1$ is a bv-renaming (see Assumption 3.4).
- A set of answers ANS, which are accepted as successful results of reductions.

We distinguish *weakly* and *strongly sharing calculi* in the following, where strongly sharing calculi do not make use of vv-substitutions in the definition of equivalence. To ease notation we use a set VV of vv-substitutions, with $VV = \{Id\}$ for strongly sharing calculi and VV the set of all vv-substitutions for weakly sharing calculi.
In the example calculi (see Examples 3.5, 3.6, 3.7, and 3.8) the reduction $\to$ is either the (call-by-need) normal-order reduction, and the answers are the WHNFs, or the reduction is the process-reduction, and answers are processes without any further communication in 3.7, or successful processes in 3.8.
The algorithm UNWIND has a term or a multicontext t as input and (perhaps non-deterministically) produces a sequence of term-positions, starting with $p_1 = \varepsilon$. Given t, the possible sequences $p_1, p_2, \ldots$ produced by UNWIND are called the *valid* UNWIND-*runs* of t. We do not enforce the sequences to be maximal. The following conditions must hold:

*Assumption 3.1 (*UNWIND-*Assumptions).*

1. If $p_1, \ldots, p_n$ is a valid UNWIND-run of t, then for all $1 \le i \le n$, $p_1, \ldots, p_i$ is also a valid UNWIND-run of t, the set $\{p_1, \ldots, p_n\}$ is a prefix-closed set of positions, and $p_n \notin \{p_1, \ldots, p_{n-1}\}$.
2. If t, t' are terms or multicontexts, $p_1, \ldots, p_n$ is a valid UNWIND-run for t, and for all $i < n$, we have $t(p_i) = t'(p_i)$, then $p_1, \ldots, p_n$ is also a valid UNWIND-run for t'.

3. If t, t' are terms or multicontexts with $t = \sigma(t')$ for an fvbv-renaming σ, and $p_1, \ldots, p_n$ is a valid run for t, then $p_1, \ldots, p_n$ is also a valid UNWIND-run for t'.
4. For weakly sharing calculi, the following additional assumption is required: If t, t' are terms or multicontexts, $\nu \in VV$, γ a vvbv-substitution compatible with ν with $t' = \gamma(t)$, and $p_1, \ldots, p_n$ is a valid UNWIND-run for t, then $p_1, \ldots, p_n$ is also a valid UNWIND-run for t'.

Given a term or a multicontext t. Then the position p of t is called a *reduction position* in t, iff p is contained in some valid UNWIND-run of t. The set of all reduction positions is defined as $\mathrm{RP}(t) = \{p \mid p \text{ is contained in some valid UNWIND-run of } t\}$. Note that every reduction position is a term-position by definition. Since we also apply the formalism to multicontexts, we can speak of reduction positions of multicontexts as well as of terms. A single-hole context $C[]$ is defined as a *a reduction context*, if the hole $[]$ of C is a reduction position in $C[]$: We denote reduction contexts as $R[]$.

Lemma 3.2. *Let M be a multicontext with n holes, and $s_j, j = 1, \ldots, n$ be terms, such that for some i: $M[s_1, \ldots, s_{i-1}, [], s_{i+1}, \ldots s_n]$ is a reduction context. Then there is some $j \in \{1, \ldots, n\}$, such that for all terms $t_k, k = 1, \ldots, n$, $M[t_1, \ldots, t_{j-1}, [], t_{j+1}, \ldots t_n]$ is a reduction context.*

Proof. Let p be the position of the hole in $t := M[s_1, \ldots, s_{i-1}, [], s_{i+1}, \ldots s_n]$. By the assumption on UNWIND, there is a valid UNWIND-run $p_1, \ldots, p_m$ of t, such that $p_m = p$. Let Q be the set of the positions of the n holes of M. Then there is a least k such that $p_1, \ldots, p_k$ is a valid UNWIND-run of t, $p_k \in Q$ and p_k is the position of some hole. Minimality of k implies that $p_1, \ldots, p_{k-1}$ are positions within M, but not the position of any hole of M. Now we can apply the conditions on UNWIND, in particular condition (2): UNWIND produces the valid UNWIND-run $p_1, \ldots, p_k$, irrespective of the terms in the holes of M. Hence the claim of the lemma holds. □

Assumption 3.3 (Answer-Assumption). There is a set ANS of *answer terms*. We assume that the following conditions are satisfied:

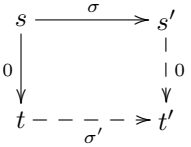
1. If $t \xrightarrow{\sigma} t'$ for terms t, t' by a fvbv-renaming σ, and $t \in \mathrm{ANS}$, then $t' \in \mathrm{ANS}$.
2. If $t = M[t_1, \ldots, t_n]$ is an answer for some multicontext M, and no hole of M is a reduction position, then $M[t'_1, \ldots, t'_n]$ is also an answer.

Note that answers are allowed to be reducible.
The essence of the following assumption is that reduction commutes with renaming, and that reduction in strongly sharing calculi does not modify non-reduction positions up to renamings, and in weakly sharing calculi a replacement of variables by variables is permitted under further restrictions.

Assumption 3.4 (Reduction-Assumption). It is assumed that CALC only defines a (small-step) relation $\to_0$ that is applicable to terms satisfying the DVC, and that the full small-step relation $\to$ is derived from $\to_0$ and a subsequent renaming

of variables. The relation $\to$ is defined such that $s \to t$ holds whenever s, t satisfy the DVC, and $s \to_0 t_0 \xrightarrow{\sigma} t$ for some t_0 and some bv-renaming σ of t_0. We assume that the following conditions are satisfied for $\to$ and $\to_0$:

1. If $s \to t$, then s, t have the same type.
2. Let $t = M[t_1, \ldots, t_n]$ be a term that satisfies the DVC, where M is a multicontext with n holes that are at non-reduction positions, and let t' be a term with $t \to_0 t'$. Then there is a multicontext M' with n' holes, a mapping $\pi : \{1, \ldots, n'\} \to \{1, \ldots, n\}$, vv-substitutions $\nu_i \in VV, i \in \{1, \ldots, n'\}$, where the domain and codomain-variables of all ν_i already occur in M, such that for all terms $s_1, \ldots, s_n$: If $M[s_1, \ldots, s_n]$ satisfies the DVC, then $M[s_1, \ldots, s_n] \to_0 M'[\nu_1(s_{\pi(1)}), \ldots, \nu_{n'}(s_{\pi(n')})]$. In particular, $t' = M'[\nu_1(t_{\pi(1)}), \ldots, \nu_{n'}(t_{\pi(n')})]$.
   Note that ν is capture-free since the DVC holds before reduction and that the assumption enforces that ν is independent of the terms $s_1, \ldots, s_n$.
3. If s is a term satisfying the DVC, $s \to_0 t$, $s \xrightarrow{\sigma} s'$ an fvbv-renaming of s, such that $s' := \sigma(s)$ satisfies the DVC. Then there is a term t' and an fvbv-renaming σ' of t, where $\mathrm{fvp}(\sigma')$ is a restriction of $\mathrm{fvp}(\sigma)$, such that $s' \to_0 t'$ and $t' = \sigma'(t)$.



Note that in (2) the terms s_i satisfy the DVC, but there may be multiple occurrences of some s_i; also the multicontext M' may violate the DVC.

We give examples of calculi and illustrate the assumptions:

Example 3.5. Let the call-by-need λ-calculus[4] be given with syntax $E ::= V \mid (E\ E) \mid \lambda V.E \mid (\texttt{let}\ V = E\ \texttt{in}\ E)$ (see [AFM+95]). Then the search for a normal-order redex is deterministic and can be specified by a label-shift that starts with t^L and has the rules $(t_1\ t_2)^L \to (t_1^L\ t_2)$; $(\texttt{let}\ x = t_1\ \texttt{in}\ t_2)^L \to (\texttt{let}\ x = t_1\ \texttt{in}\ t_2^L)$ and $(\texttt{let}\ x = t\ \texttt{in}\ C[x^L]) \to (\texttt{let}\ x = t^L\ \texttt{in}\ C[x])$. It is clear that the UNWIND-assumptions are satisfied, since the search for the redex does not depend on former non-L-positions.
The answers are defined to be of the form $A ::= \lambda V.E \mid (\texttt{let}\ V = E\ \texttt{in}\ A)$. The Answer-Assumptions are satisfied.
The small-step reduction is defined at reduction positions and makes local changes at reduction positions, but non-reduction positions are never substituted. The full specification is beyond the scope of this paper, but it is easy to see (cf. [AFM+95]) that the reduction assumptions for a strongly sharing calculus are satisfied.

Example 3.6. Let a fragment of the untyped non-deterministic call-by-need λ-calculus with `amb` be given with syntax $E ::= V \mid (E\ E) \mid \lambda V.E \mid (\texttt{amb}\ E\ E) \mid (\texttt{letrec}\ x_1 = E_1, \ldots, x_n = E_n\ \texttt{in}\ E)$ (see [SSS07]). This calculus is strongly sharing in our sense. The search for a normal-order redex specified by a label-shift is non-deterministic. The labels are "T" for top-term, "S" for subterm,

[4] There may be minor variations in the normal-order redex in the cited calculi

and "L" standing for "S" or "T". Let UNWIND start with t^T and let the rules be $(t_1\ t_2)^L \to (t_1^S\ t_2)$; $(\texttt{letrec}\ Env\ \texttt{in}\ r)^T \to (\texttt{letrec}\ Env\ \texttt{in}\ r^S)$, $(\texttt{letrec}\ x = t, Env\ \texttt{in}\ C[x^S]) \to (\texttt{let}\ x = t^S, Env\ \texttt{in}\ C[x])$, $(\texttt{letrec}\ x = t, y = C[x^S], Env\ \texttt{in}\ r) \to (\texttt{letrec}\ x = t^S, y = C[x], Env\ \texttt{in}\ r)$, and the non-deterministic rules for `amb` be: $(\texttt{amb}\ t_1\ t_2)^L \to (\texttt{amb}\ t_1^S\ t_2)$ and $(\texttt{amb}\ t_1\ t_2)^L \to (\texttt{amb}\ t_1\ t_2^S)$. If a position is visited twice, then the algorithm stops. Again it is clear that the UNWIND-assumptions are satisfied, since the search for the redex does not depend on non-labeled positions.
The answers are abstractions $\lambda V.E$ as well as abstractions with an enclosing `letrec`-expression $(\texttt{letrec}\ Env\ \texttt{in}\ \lambda V.E)$. The Answer-Assumptions are satisfied.
The small-step reduction is defined at reduction positions and makes local changes, but again non-reduction positions are not modified by substitution. It is not hard to check that the reduction assumptions are satisfied. We give an illustration of one rule: $(\texttt{letrec}\ x = \lambda y.s, Env\ \texttt{in}\ C[x]) \to (\texttt{letrec}\ x = \lambda y.s, Env\ \texttt{in}\ C[\lambda y.s])$, provided there is an UNWIND-run with intermediate labeling $\ldots C[x^L]$. The reduction assumption (2) is satisfied, since e.g. $M[s_1, \ldots, s_n] = (\texttt{letrec}\ x = \lambda y.M_1[s_1, \ldots, s_n], Env\ \texttt{in}\ C[x])$ is reduced to $(\texttt{letrec}\ x = \lambda y.M_1[s_1, \ldots, s_n], Env\ \texttt{in}\ C[\lambda y.M_1[s_1, \ldots, s_n]])$, which can be written as $M'[s_1, \ldots, s_n, s_1, \ldots, s_n]$, where $M'[\ldots] = (\texttt{letrec}\ x = \lambda y.M_1[\ldots], Env\ \texttt{in}\ C[\lambda y.M_2[\ldots]])$.
The non-deterministic call-by-need λ-calculus with `choice` in [MS99] is a bit different insofar as only variables are permitted as arguments in applications, and that beta-reduction is always a replacement of variables for variables, the rule being $(\lambda x.r)\ y \to r[y/x]$. This calculus is weakly sharing. For the case-rule (see [MS99]) a joint replacement of several variables will take place, which is not enforced to act as an injective mapping. Nevertheless, all our assumptions are satisfied.

Example 3.7. The π-calculus as already mentioned in Example 2.4 can also be checked for an appropriate representation and for the validity of our assumptions. In the literature there are different variants of the π-calculus, which are usually equipped with a theory based on bisimilarity. We will add a variant that is operationally admissible, though a full analysis is left for future work. Instead of equivalence axioms for concurrent processes and new name-binders, we view these as reduction rules, which will turn out to be completely adequate for our may- and must-convergence definitions, but are not compatible with the notion of total must convergence, since this encoding will introduce infinite reduction sequences.
The presentation of UNWIND as a label-shift algorithm has the following rules, where we also add the non-deterministic possibilities: $(P \mid Q)^L \to (P^L \mid Q)$, or $(P \mid Q)^L \to (P \mid Q^L)$; $(\nu x.P)^L \to \nu x.P^L$; and $(!P)^L \to (!\ P^L)$. This algorithm for finding reduction positions satisfies our unwind-assumptions. The reduction rules adapted to our view of calculi are the rules that correspond to "structural equivalences", which are replaced by the corresponding rules, like e.g. $(P \mid Q) \to (Q \mid P)$; $(P_1 \mid (P_2 \mid P_3)) \to ((P_1 \mid P_2) \mid P_3)$; $(\nu x.P) \mid Q \to (\nu x.(P \mid Q))$, if

$x \notin \mathcal{FV}(Q)$, $\nu x.\nu y.P \to \nu y.\nu x.P$, and the rule $!P \to P \mid !P$. The important rules are $(P + Q) \to P$, $(P + Q) \to Q$ and the communication rule (COM): $x(y_1, \ldots, y_n).P \mid \overline{x}\langle z_1, \ldots, z_n\rangle.Q \to P[z_1/y_1, \ldots, z_n/y_n] \mid Q$. The rules can only be applied if the redex is a reduction position and not below an !-operator. The latter restriction shows that the notion of reduction position and the notion of redex may be different.
An appropriate definition of answers (or successful processes) that satisfies our assumptions is as follows: A process P is an answer, iff the rules for $P + Q$ and the communication rules are not applicable, even after a finite number of "equivalence reductions". This set of definitions satisfies all our assumptions for a weakly sharing calculus.
To use the structural equivalences and the definition of reduction modulo this equivalence is not covered by our framework, where the associative-commutative rules can be encoded, but the ν-shifting rules have to be made explicit.
Note that the (COM)-reduction rule from the π-calculus is the only rule among all other considered rules that has a non-linear left hand side, however, the rule application is severely restricted, insofar as its applicability depends only on equality of two variable names.

Example 3.8. The calculus $\lambda(\mathsf{fut})$ in [NSSSS07] is an extension of the lambda-calculus with features of the π-calculus, however, there are cells and futures instead of channels. The representation in our syntax requires, similar as in the π-calculus, the two types process and term, where only variables of type term are permitted. The structural congruences can be represented as reduction rules as for the π-calculus. The lambda-calculus evaluation follows a call-by-value strategy, though there are futures (i.e. variables) used for sharing results instead of a replacing beta-rule. The answers are so called successful processes, which have no pending (non-equivalence) reduction possibilities. The calculus $\lambda(\mathsf{fut})$ satisfies our assumptions and is a strongly sharing calculus.

Remark 3.9. We give examples of constellations in calculi that do not fall into the scope of our method to prove context lemmas:

- If reduction within abstractions is not permitted, then the beta-reduction rule violates our reduction assumption, since non-reduction positions may be modified: an example being $(\lambda x.C[x])(\lambda y.y) \to C[(\lambda y.y)]$. Using the multicontext $M = (\lambda x.[])$ and $s_1 = x$, the only possibility for an M' is $M' = ([])$, however, there is a replacement of x, and hence this is impossible.
- It is not permitted by our assumptions to have a beta-rule replacing only variables by variables like $(\lambda x.C[x])\ y \to C[y]$, if y is at a potential position of a hole: With $M = ((\lambda x.[])\ [])$, it is not possible to find an appropriate M', since the vv-substitution must only depend on M, not on the contents of the holes. If the argument position of applications is restricted to variables then the reduction assumptions are not violated.
- If we try to extend the assumptions and proof method such that $M[s_1, \ldots, s_n]$ may have a more general result $M'[\sigma_1(s_1), \ldots, \sigma_n(s_n)]$ after a single reduction, where σ_i is a substitution of terms for variables, then we

run into technical trouble in the proofs of the context lemma(s) since the substitutions may depend on the terms s_i and not only on the multicontext M.
- Similar arguments hold for pattern match or case-rules that reduce $(\texttt{case}\ (c\ s_1 \dots s_n)\ \texttt{of}(c\ x_1 \dots x_n) \to s, \dots)$ to e.g. $s[t_1/x_1, \dots, t_n/x_n]$, which cannot be covered.
- A rule of the form $f\ (g\ a) \to b$, where f, g are unary and a, b are constants and the subterm $(g\ a)$ is a non-reduction position can not be represented, since this would imply by our assumptions that $f\ s \to b$ for all subterms s.

4 Contextual Preorder and Equivalence for May- and Must-Convergence

In this section we define different kinds of convergence properties of terms, and the corresponding notions of contextual preorder and equivalence. There are three main notions of convergence of a term t: may-convergence, which means that t may reduce to an answer, must-convergence, which means that every term reachable by reduction from t is may-convergent, and total must-convergence, which means that t has no reduction to a must divergent term (failure term) and no infinite reduction.

Definition 4.1. *A term t is called*

- may-convergent *iff there is some answer t' with $t \xrightarrow{*} t'$, denoted as $t\downarrow$.*
- must-divergent *iff t is not may-convergent, denoted as $t\Uparrow$.*
- may-divergent *iff there is some term $t' \Uparrow$ with $t \xrightarrow{*} t'$, denoted as $t\uparrow$.*
- must-convergent *iff $t \xrightarrow{*} t'$ implies $t'\downarrow$, denoted as $t\Downarrow$.*
- totally must-convergent *iff $t\Downarrow$ and there is no infinite reduction starting with t, denoted as $t\Downarrow\!\Downarrow$.*
- totally may-divergent *iff $t\uparrow$, or there is an infinite reduction starting with t, denoted as $t\uparrow\!\uparrow$.*

Note that t is not may-divergent iff it is must-convergent.
In calculi with a deterministic $\to_0$-reduction the may- and must-predicates are identical. As a generalization, we call a calculus *deterministic* iff the may- and must-convergence predicates are identical for all terms. Terms t with $t\Downarrow$, but not $t\Downarrow\!\Downarrow$, are called *weakly divergent* in [CHS05]. Must-convergence is interesting because it is linked to fairness (see e.g. [CHS05,SSS07,NSSSS07]); further justification for non-total may-divergence is in [SS03].

Definition 4.2. *Let s, t be two terms of the same type τ, and $\mathcal{M} \in \{\downarrow, \Downarrow, \Downarrow\!\Downarrow\}$. Then $s \leq_{\mathcal{M},\tau} t$ iff for all $C[]_\tau : C[s]\mathcal{M} \implies C[t]\mathcal{M}$, and*
$s \leq_{\mathcal{M}\nu,\tau} t$, iff for all $C[]_\tau$, for all vv-substitutions $\nu \in VV$ and for all vvbv-substitutions γ_s, γ_t compatible with ν on $\mathcal{FV}(s) \cup \mathcal{FV}(t)$: $C[\gamma_s(s)]\mathcal{M} \implies C[\gamma_t(t)]\mathcal{M}$.

In the following we omit mention of τ in the suffix of the relations, if this is not ambiguous.
Easy consequences are that for all terms s: $s\Downarrow\!\Downarrow \implies s \Downarrow \implies s \downarrow$, that for $\mathcal{M} \in \{\downarrow, \Downarrow, \Downarrow\!\Downarrow\}$, the relations $\leq_{\mathcal{M}}$ are compatible with contexts, and reflexive and transitive, and that $\leq_{\mathcal{M}\nu} \subseteq \leq_{\mathcal{M}}$.
Note that for a general proof of transitivity of $\leq_{\mathcal{M}}$ it is unavoidable that $C[s], C[t]$ may contain free variables. The corresponding proof w.r.t. a definition of $\leq_{\mathcal{M}}$ that restricts $C[s], C[t]$ to be closed is in general not applicable, since the middle term $C[s_2]$ is not necessarily closed, if $C[s_1], C[s_3]$ are closed.
Contextual equivalence is defined as $\sim_{\downarrow} := \leq_{\downarrow} \cap \geq_{\downarrow}$ for deterministic calculi and for nondeterministic calculi as $\sim_{\downarrow\Downarrow} := \sim_{\downarrow} \cap \leq_{\Downarrow} \cap \geq_{\Downarrow}$ or $\sim_{\downarrow\Downarrow\!\Downarrow} := \sim_{\downarrow} \cap \leq_{\Downarrow\!\Downarrow} \cap \geq_{\Downarrow\!\Downarrow}$ depending on the used must-convergence predicate. The relations $\sim_{\downarrow\nu}$ are defined analogously using the respective $\leq$-relations.

Example 4.3. The relation $\leq_{\downarrow\nu}$ may be different from $\leq_{\downarrow}$ (in exotic calculi): Consider the calculus with one binary constructor c and a constant d, and the reduction rule: $c\ x\ x \to d$, let d be the only answer, and let all positions be reduction positions. Then $c\ x\ y \leq_{\downarrow} c\ x\ z$, but $c\ x\ y \not\leq_{\downarrow\nu} c\ x\ z$.

For simplifying several proofs in the following sections, we introduce a *0-1-labelled variant* of $\to$-reduction sequences, which is nothing else but a reduction of the form $s_{1,0} \to_0 s_{1,1} \to_1 s_{2,0} \to_0 s_{2,1} \to_1 \ldots$. I.e., a reduction, where $\to_0$-reductions and bv-renamings $\to_1$ are alternating, and the terms $s_{i,0}$ satisfy the DVC.

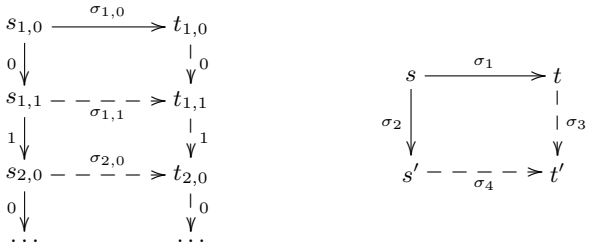


Fig. 1. Reduction diagrams for Lemma 4.4

Lemma 4.4. *Let s, t be terms satisfying the DVC with $s \xrightarrow{\sigma} t$ by an fvbv-renaming σ, and Red be a 0-1-labelled reduction of s as follows: $s = s_{1,0} \to_0 s_{1,1} \to_1 s_{2,0} \to_0 s_{2,1} \ldots \to_0 s_{n,1} \to_1 s_{n+1,0}$. Then there is also a reduction of t of the form $t = t_{1,0} \to_0 t_{1,1} \to_1 t_{2,0} \to_0 t_{2,1} \ldots \to_0 t_{n,1} \to_1 t_{n+1,0}$ with terms $t_{i,k}$, such that for all i, k: $s_{i,k} \xrightarrow{\sigma_{i,k}} t_{i.k}$ by fvbv-renamings $\sigma_{i,k}$, $\mathrm{fvp}(\sigma_{i,k})$ is a restriction of $\mathrm{fvp}(\sigma)$, the terms $t_{i,0}$ satisfy the DVC, and $s_{n+1,0}$ is an answer iff $t_{n+1,0}$ is an answer (see left diagram in figure 1).*
The lemma holds also for bv-renamings instead of fvbv-renamings. The lemma

also holds, if the 0-1-labelled reduction of s starts with a $\to_1$-reduction, in which case the reduction of t also starts with a $\to_1$-reduction.

Proof. We show the claim by induction on the number of reductions. The base case holds using the answer assumption 3.3. Let s, t be terms satisfying the DVC with $s \xrightarrow{\sigma} t$ by a fvbv-renaming σ, and $s \to_0 s'$. Then by the reduction assumption 3.4, there is some t' with $t \to_0 t'$, and $s' \xrightarrow{\sigma'} t'$, where $\mathrm{fvp}(\sigma')$ is a restriction of $\mathrm{fvp}(\sigma)$. Let s, t be terms with $s \xrightarrow{\sigma_1} t$ by an fvbv-renaming, and $s \xrightarrow{\sigma_2} s'$ by a bv-renaming, and s' satisfies the DVC. If s already satisfies the DVC, then σ_2 is the identity. The reverse σ_2^{-1} of σ_2 is also a bv-renaming. Moreover, by Lemma 2.7, there is a bv-renaming σ_3, such that $t \xrightarrow{\sigma_3} t'$ and t' satisfies the DVC. With $\sigma_4 = \sigma_2^{-1} \circ \sigma_1 \circ \sigma_3$, we have $s' \xrightarrow{\sigma_4} t'$ by Lemma 2.7, since composition of fvbv-renamings is also a fvbv-renaming. Moreover, the fv-parts of σ_1 and σ_4 are the same. (see right diagram in figure 1. □

Proposition 4.5. *Let s, s' be terms with $s' = \sigma(s)$, where σ is a fvbv-renaming. Then $s\mathcal{M} \Leftrightarrow s'\mathcal{M}$ for all $\mathcal{M} \in \{\downarrow, \Downarrow, \uparrow, \Uparrow, \Downarrow, \Uparrow\}$.*

Proof. Let s, t be terms with $t = \sigma(s)$, where σ is an fvbv-renaming.
If $s \downarrow$, then Lemma 4.4 and the condition on answer-terms 3.3 shows that a reduction from s to an answer can be translated to a reduction of t to an answer. Hence $t \downarrow$. Since the reverse of σ is a fvbv-renaming by Lemma 2.7, the converse also holds. This immediately also shows that $s \Uparrow \iff t \Uparrow$.
Now let $s \uparrow$. Then Lemma 4.4 and the first part of the proof shows that a reduction from s to a must-divergent term can be translated to a reduction of t to a must-divergent term. Hence $t \uparrow$. Since the reverse of σ is a fvbv-renaming by Lemma 2.7, the converse implication also holds. An immediate consequence is that $s \Downarrow \iff t \Downarrow$.
Let $s \Uparrow$. Then a reduction from s to a must-divergent term or an infinite $\to$-reduction starting from s can be transferred using Lemma 4.4 to a reduction from t to a must-divergent term or to an infinite reduction. Hence $t \Uparrow$. The reverse implication also holds. Again, an immediate consequence is $s \Downarrow \iff t \Downarrow$. □

5 Context Lemmas

We define the preorders restricted to reduction contexts and show the context lemmas for all the combinations of the different notions of convergences and for the two kinds of calculi.

Definition 5.1. *For all terms s, t of equal type τ and $\mathcal{M} \in \{\downarrow, \Downarrow, \Downarrow\}$:*
$s \leq_{\mathcal{M},R\nu,\tau} t$ iff for all reduction contexts $R[]_\tau$, all vv-substitutions $\nu \in VV$, all vvbv-substitutions γ_s, γ_t of s, t compatible with ν on $\mathcal{FV}(s) \cup \mathcal{FV}(t)$, we have $R[\gamma_s(s)]\mathcal{M} \implies R[\gamma_t(t)]\mathcal{M}$.
The relation $s \leq_{\mathcal{M},R,\tau} t$ holds iff for all reduction contexts $R[]_\tau$, we have $R[s]\mathcal{M} \implies R[t]\mathcal{M}$.

Note that in the following, we will drop the type-suffix τ.
We can slightly restrict the necessary reduction contexts for $\leq_{\mathcal{M},R}$, by using renamings and Proposition 4.5:

Lemma 5.2. *Let* CALC *be strongly sharing and* $\mathcal{M} \in \{\downarrow, \Downarrow, \Downarrow\}$. *Then* $\leq_{\mathcal{M},R\nu} = \leq_{\mathcal{M},R}$.

Proof. This follows from Proposition 4.5, since for a reduction context R, σ_s, σ_t (with adapted positions) are also bv-renamings of $R[s], R[t]$, respectively. □

We separate the proofs for weakly and strongly sharing calculi, since there is a different treatment of renamings, and the weakly sharing part requires more arguments w.r.t. vvbv-substitutions.

Lemma 5.3 (May-Convergence and Weakly Sharing). *Let* CALC *be weakly sharing. Then* $\leq_{\downarrow,R\nu} = \leq_{\downarrow\nu}$.

Proof. We show the following generalized claim:
For all n, all multicontexts M with n holes, all $i = 1, \ldots, n$ and all vv-substitutions $\nu_i \in VV$, and compatible vvbv-substitutions $\gamma_{s,i}, \gamma_{t,i}$ on $\mathcal{FV}(s_i) \cup \mathcal{FV}(t_i)$: If for terms s_i, t_i: $s_i \leq_{\downarrow,R\nu} t_i$, then $M[\gamma_{s,1}(s_1), \ldots, \gamma_{s,n}(s_n)] \downarrow \Longrightarrow M[\gamma_{t,1}(t_1), \ldots, \gamma_{t,n}(t_n)] \downarrow$. For convenience let $s'_i := \gamma_{s,i}(s_i), t'_i := \gamma_{t,i}(t_i)$. Note that in the inductive proof below we will only use the weakened precondition $s'_i \leq_{\downarrow,R\nu} t'_i$. Proposition 4.5 permits us to assume, by applying bv-renamings, that the bound variables in s'_i and t'_i are distinct.
The claim is shown by induction on the length l of 0-1-labelled-reductions of $M[s'_1, \ldots, s'_n]$ to an answer, and second on the number of holes of M.
As a base case, the claim is obviously true, if the number n of holes is equal to 0, since then $M[s'_1, \ldots, s'_n] = M[t'_1, \ldots, t'_n]$.
There are two cases:

1. In $M[s'_1, \ldots, s'_n]$ some s'_i is in a reduction position.
This means that at least one of the contexts $M_i = M[s'_1, \ldots, s'_{i-1}, [], s'_{i+1}, \ldots, s'_n]$ is a reduction context. Then Lemma 3.2 shows that there is some j, such that $M[s'_1, \ldots, s'_{j-1}, [], s'_{j+1}, \ldots, s'_n]$ as well as $M[t'_1, \ldots, t'_{j-1}, [], t'_{j+1}, \ldots, t'_n]$ is a reduction context. Using the induction hypothesis for the context $M' := M[[], \ldots, [], s'_j, [], \ldots, []]$, which has $n-1$ holes, it follows that $M[s'_1, \ldots, s'_n] \downarrow \Longrightarrow M[t'_1, \ldots, t'_{j-1}, s'_j, t'_{j+1}, \ldots, t'_n] \downarrow$.
Since $M[t'_1, \ldots, t'_{j-1}, [], t'_{j+1}, \ldots, t'_n]$ is a reduction context, the assumption and $M[t'_1, \ldots, t'_{j-1}, s'_j, t'_{j+1}, \ldots, t'_n] \downarrow$ imply that $M[t'_1, \ldots, t'_{j-1}, t'_j, t'_{j+1}, \ldots, t'_n] \downarrow$.

2. For all i: None of the contexts $M_i = M[s'_1, \ldots, s'_{i-1}, [], s'_{i+1}, \ldots, s'_n]$ is a reduction context. Lemma 3.2 implies that none of the holes of M is at a reduction position. If $l = 0$, then $M[s'_1, \ldots, s'_n]$ is an answer-term, and by Assumption 3.3, the expression $M[t'_1, \ldots, t'_n]$ is also an answer term. Now assume that $l > 0$:

2a. First we consider the case that $M[s'_1, \ldots, s'_n]$ satisfies the DVC and that the reduction on $M[s'_1, \ldots, s'_n]$ is a $\to_0$-reduction. Let $M[s'_1, \ldots, s'_n] \to_0 s'$ be the start of the 0-1-labelled reduction of length l to an answer. By the assumption

3.4 (2) on reductions, there is a multicontext M' with n' holes, $\nu_i \in VV$ for $i \in \{1,\ldots,n'\}$, and a mapping $\pi : \{1,\ldots,n'\} \to \{1,\ldots,n\}$, such that $s' = M'[\nu_1(s'_{\pi(1)}),\ldots,\nu_{n'}(s'_{\pi(n')})]$.
The same holds by assumption 3.4(2) for $M[t'_1,\ldots,t'_n]$: There is a reduction $M[t'_1,\ldots,t'_n] \to_0 M'[\nu_1(t'_{\pi(1)}),\ldots,\nu_{n'}(t'_{\pi(n')})]$. Now we can apply the induction hypothesis, since the number of reductions to an answer of s' is $l-1$, and the required preconditions hold: For all R,ν_R and if $\gamma_{Rs,i},\gamma_{Rt,i}$ are compatible with ν_R on $\mathcal{FV}(s'_i) \cup \mathcal{FV}(t'_i)$, then for all $i = 1,\ldots,n'$: $R[\gamma_{Rs,i}\nu_i(s'_{\pi(i)})] \downarrow \Longrightarrow R[\gamma_{Rt,i}\nu_i(t'_{\pi(i)})] \downarrow$ holds, since $\gamma_{Rs,i}\nu_i$ and $\gamma_{Rt,i}\nu_i$ are also vvbv-substitutions compatible with a common vv-substitution (see Lemma 2.7).

2b. The other case is that $M[s'_1,\ldots,s'_n]$ is the result of a $\to_0$-reduction and the next reduction step in the 0-1-labelled reduction is a bv-renaming. Then the reduction consists of applying some bv-renaming $M[s'_1,\ldots,s'_n] \xrightarrow{\sigma} M'[s''_1,\ldots,s''_n]$, such that $M'[s''_1,\ldots,s''_n]$ satisfies the DVC.
Using Lemma 2.9, let σ_M be the part of the renaming σ for the binder positions that are in M. Let $W_i, i = 1,\ldots,n$ be the set of variables that may be potentially bound in hole i, i.e. $W_i = V_{\text{hole}}(M,i)$, and let $\rho_i, i = 1,\ldots,n$ be the mappings on W_i induced by σ. Note that ρ_i is injective on W_i. The effect of the bv-renaming σ can be modelled as follows:
It induces a bv-renaming $M \xrightarrow{\sigma_M} M'$, and fvbv-renamings μ_i with $s'_i \xrightarrow{\mu_i} s''_i$, where μ_i is compatible with ρ_i for all i. We construct an appropriate bv-renaming σ' for $M[t'_1,\ldots,t'_n]$ by using σ_M again for M, and for every i fvbv-renamings μ'_i for t'_i, where for all i: μ'_i is compatible with ρ_i. For the bv-part of μ'_i fresh variables must be used, which ensures that $\sigma'(M[t'_1,\ldots,t'_n])$ satisfies the DVC. By construction, we have $\sigma'(M[t'_1,\ldots,t'_n]) = M'[\mu'_1(t'_1),\ldots,\mu'_n(t'_n)]$.
It remains to show that the preconditions $\mu_i(s'_i) \leq_{\downarrow,R\nu} \mu'_i(t'_i))$ hold for all pairs $(\mu_i(s'_i),\mu'_i(t'_i))$: Let i be fixed in the following, and let R be a reduction context, let ν' be a vv-substitution, and $\gamma'_{s,i}$ be vvbv-substitutions of $\mu_i(s'_i)$ compatible with ν' such that $R[\gamma'_{s,i}\mu_i(s'_i)] \downarrow$. Now let $\gamma'_{t,i}$ be a vvbv-substitution of $\mu'_i(t'_i)$ compatible with ν'. The ssubstitutions $\gamma'_{s,i}\mu_i$ and $\gamma'_{t,i}\mu'_i$ are compatible with the same vv-substitution $\nu\rho_{i|\rho_i(W_i)}$ by Lemma 2.7. Thus we obtain from $s_i \leq_{\downarrow,R\nu} t_i$ that $R[\gamma'_{t,i}\mu'_i(t'_i)] \downarrow$.
Since the preconditions are satisfied for all i, the multicontext is the same M' for $s_i,t_i,i = 1,\ldots,n$, and the reduction length has been reduced, we can apply the induction hypothesis. □

For a finite set of variables W, a context C is called *fresh for* W, iff for all variables $x \in W$, x is not bound by a binder in $BP_{\overline{\text{hole}}}(C)$.

Lemma 5.4. *Let s,t be terms, $\mathcal{M} \in \{\downarrow,\Downarrow,\Downarrow\}$ and W be a finite set of variables that contains all variables occurring in s,t. Then $s \leq_{\mathcal{M},R} t$ holds, iff for all reduction contexts R that are fresh for W, we have $R[s]\mathcal{M} \Longrightarrow R[t]\mathcal{M}$.*

Proof. Let R be an arbitrary reduction context with $R[s] \downarrow$. We have to show that $R[t] \downarrow$. Let σ be a bv-renaming that renames all the binders in $BP_{\overline{\text{hole}}}(R)$ by fresh variables that are not in W. Then $\sigma(R)$ is a reduction context due

to the unwind-conditions and satisfies the preconditions. Hence $\sigma(R)[s] \downarrow$ by Proposition 4.5. The preconditions imply that $\sigma(R)[t] \downarrow$. Since the reverse of σ is also a bv-renaming, we have also $R[t] \downarrow$, again by Proposition 4.5. The same arguments, but using the other claims of Proposition 4.5, show the other parts of the lemma. □

Lemma 5.5 (May-Convergence and Strongly Sharing). *Let* CALC *be strongly sharing. Then* $\leq_{\downarrow,R} \;=\; \leq_{\downarrow}$.

Proof. We show the following generalized claim:
For all n and all multicontexts M with n holes: If for terms $s_i, t_i, i = 1, \ldots, n$, and for all $i = 1, \ldots, n$: $s_i \leq_{\downarrow,R} t_i$, then $M[s_1, \ldots, s_n] \downarrow \implies M[t_1, \ldots, t_n] \downarrow$.
Note that the induction and the cases are instances of the cases in the proof of Lemma 5.3, where vv-substitutions can be omitted (see also Lemma 5.2); only the final part (2b) is different. We will present this part of the proof in detail:
2b. Consider the case that $M[s_1, \ldots, s_n]$ is the result of a $\to_0$-reduction, that it does not satisfy the DVC, and the next reduction step in the 0-1-labelled reduction is a renaming. Then the reduction consists of applying some bv-renaming $M[s_1, \ldots, s_n] \xrightarrow{\sigma} M'[s'_1, \ldots, s'_n]$, such that $M'[s'_1, \ldots, s'_n]$ satisfies the DVC. Note that every term s_i satisfies the DVC, but there may be double occurrences of the same term.
Using Lemma 2.9, let σ_M be the part of the renaming σ for the binder positions that are in M. Let $W_i = V_{\text{hole}}(M, i), i = 1, \ldots, n$, and let $\rho_i, i = 1, \ldots, n$ be the mappings on W_i induced by σ. Note that ρ_i is injective on W_i. The bv-renaming σ induces fvbv-renamings μ_i with $s_i \xrightarrow{\mu_i} s'_i$, where μ_i is compatible with ρ_i. We construct an appropriate bv-renaming σ' for $M[t_1, \ldots, t_n]$ by using σ_M again for M, and for every i fvbv-renamings μ'_i for t_i, where for all i: μ'_i is compatible with ρ_i. For the bv-part of μ'_i fresh variables must be used, which ensures that $\sigma'(M[t_1, \ldots, t_n])$ satisfies the DVC. By construction, we have $\sigma'(M[t_1, \ldots, t_n]) = M'[\mu'_1(t_1), \ldots, \mu'_n(t_n)]$.
We have to show that the precondition $\mu_i(s_i) \leq_{\downarrow,R} \mu'_i(t_i))$ holds for all pairs $(\mu_i(s_i), \mu'_i(t_i)), i = 1, \ldots, n$: Let i be fixed in the following, and let R be a reduction context with $R[\mu_i(s_i)] \downarrow$, where we assume using Lemma 5.4 that the binders $BP_{\overline{\text{hole}}}(R)$ use fresh variables. We have to show that $R[\mu'_i(t_i)] \downarrow$.
We construct an fvbv-renaming σ_2 of $R[\mu_i(s_i)]$ as follows, such that σ_2 acts as the inverse of μ_i on s_i; moreover, the mapping of σ_2 on $V_{\text{hole}}(R)$ is a restriction of ρ_i^{-1}. Note that σ_2 may also act on free variables in $R[\mu_i(s_i)]$, since R may have too few binders. The construction of σ_2 is possible due to the freshness assumption on R and since s_i satisfies the DVC. Then $\sigma_2(R[\mu_i(s_i)]) = \sigma_2(R)[s_i]$, and $\sigma_2(R)$ is a reduction context by Assumption 3.1 on UNWIND. Proposition 4.5 shows that $\sigma_2(R)[s_i] \downarrow$. The assumptions $s_i \leq_{\downarrow,R} t_i$ now imply that $\sigma_2(R)[t_i] \downarrow$.
Now starting with $R[\mu'_i(t_i)]$, we also construct an fvbv-renaming σ_3 of $R[\mu'_i(t_i)]$ that acts as a reverse of μ'_i on t_i, such that the induced mapping on $V_{\text{hole}}(R)$ is a restriction of ρ_i^{-1}. We also assume that σ_3 renames binders in $BP_{\overline{\text{hole}}}(R)$ exactly as σ_2. Since t_i satisfies the DVC, and due to the freshness assumptions for R, there is no conflict between variables from t_i, bound variables in R and

variables in $\mu'_i(t_i)$, hence σ_3 can be constructed and is an fvbv-renaming. Then $\sigma_3(R[\mu'_i(t_i)] = \sigma_3(R)[t_i] = \sigma_2(R)[t_i]$. Now $\sigma_2(R)[t_i]\downarrow$ and Proposition 4.5 imply $R[\mu'_i(t_i)]\downarrow$. □

Lemma 5.6 (Must-Convergence and Strongly Sharing). *Let* CALC *be strongly sharing. Then* $\leq_{\downarrow,R} \cap \leq_{\Downarrow,R} \;\subseteq\; \leq_{\Downarrow}$.

Proof. We show the following generalized claim, using may-divergence.
For all n and all multicontexts M with n holes: If for all for terms $s_i, t_i, i = 1,\ldots,n$, and for all $i = 1,\ldots,n$: $s_i \leq_{\downarrow,R} t_i \wedge s_i \leq_{\Downarrow,R} t_i$, then $M[t_1,\ldots,t_n]\uparrow \implies M[s_1,\ldots,s_n]\uparrow$.
The claim is shown by induction on the number l of $\to_0$ and $\to_1$-reductions of 0-1-labeled reductions of $M[t_1,\ldots,t_n]$ to a must-divergent term, and second on the number of holes of M. The proof is almost a copy of the proof of the context lemma 5.5 for may-divergence; we give a sketch and emphasize the differences:
If some term t_i is in a reduction position in $M[t_1,\ldots,t_n]$, then the arguments are the same as in the proof of Lemma 5.5.
If no hole of $M[t_1,\ldots,t_n]$ is a reduction position, $l > 0$, and the reduction is a $\to_0$-reduction, then the same arguments as in the in proof of Lemma 5.5 show that we can use induction on l.
The base case $l = 0$ is that $M[t_1,\ldots,t_n]$ is must-divergent: Suppose that $M[s_1,\ldots,s_n]$ is not must-divergent. Then it is may-convergent, which by the assumption $\forall i : s_i \leq_{\downarrow,R} t_i$ and the context lemma 5.5 implies that $M[t_1,\ldots,t_n]\downarrow$, which is a contradiction. Hence $M[s_1,\ldots,s_n]\uparrow$, and the base case is proved.
If no hole of $M[t_1,\ldots,t_n]$ is a reduction position, $l > 0$, and the reduction is a renaming $\to_1$, then the same arguments as in the proof of the may-context lemma 5.5 apply. □

An immediate consequence is:

Corollary 5.7. *Let* CALC *be strongly sharing. Then* $\leq_{\downarrow} \cap \leq_{\Downarrow,R} \;\subseteq\; \leq_{\Downarrow}$.

Lemma 5.8 (Must-Convergence and Weakly Sharing). *Let* CALC *be weakly sharing. Then* $\leq_{\downarrow,R\nu} \cap \leq_{\Downarrow,R\nu} \;\subseteq\; \leq_{\Downarrow\nu}$.

Proof. The proof can be done along the argumentation of the proof of Lemma 5.6 with analogous extensions as done in the proof of Lemma 5.3. □

Lemma 5.9 (Total Must-Convergence and Strongly Sharing). *Let* CALC *be a strongly sharing calculus. Then* $\leq_{\Downarrow\Downarrow,R} t \;=\; \leq_{\Downarrow\Downarrow}$.

Proof. We show the following generalized claim:
For all n and all multicontexts M with n holes: If for all terms s_i, t_i and for all $i = 1,\ldots,n$: $s_i \leq_{\Downarrow\Downarrow,R} t_i$, then $M[s_1,\ldots,s_n]\Downarrow\Downarrow \implies M[t_1,\ldots,t_n]\Downarrow\Downarrow$.
Thus, let us assume that M, s_i, t_i are given, that $M[s_1,\ldots,s_n]\Downarrow\Downarrow$, and that the claim holds for all terms that can be reached from $M[s_1,\ldots,s_n]$ by a 0-1-labelled $\to$-reduction sequence, where at least one $\to_0$ reduction is included. Proposition 4.5 permits us to assume, by applying bv-renamings, that the bound variables in s_i and t_i are distinct. The claim is shown by well-founded induction on

the order $\xrightarrow{+}$ defined by the reduction $\rightarrow$ for all the descendents of $M[s_1, \ldots, s_n]$, and second on the number of holes of M. As a base case, the claim is obviously true, if $n = 0$.
There are several cases: We give a sketch for every case:

1. Some s_i or t_i is in a reduction position in $M[s_1, \ldots, s_n]$ or $M[t_1, \ldots, t_n]$, respectively. Then some hole of $M[., \ldots, .]$ is in a reduction context, and the arguments in case (1) of the proof of Lemma 5.5 (resp. Lemma 5.3) apply using induction on the number of holes.
2. None of the contexts $M_i = M[s_1, \ldots, s_{i-1}, [], s_{i+1}, \ldots, s_n]$ is a reduction context. Then no hole of M is a reduction position. We have to show that all reduction sequences of $M[t_1, \ldots, t_n]$ terminate. If $M[t_1, \ldots, t_n]$ is an answer-term, then we are finished. If $M[t_1, \ldots, t_n]$ is irreducible, then by the same arguments as in in the proof of Lemma 5.5, $M[s_1, \ldots, s_n]$ is also irreducible, which implies that $M[s_1, \ldots, s_n]$ is an answer, and hence $M[t_1, \ldots, t_n]$ is an answer, too. If $M[t_1, \ldots, t_n]$ has a reduction, then using the reduction assumptions 3.4 for $\rightarrow_0$ and the same arguments as in Lemma 5.5, we can apply the induction hypothesis. □

The already demonstrated techniques suffice to prove:

Lemma 5.10 (Total Must-Convergence for Weakly Sharing). *Let* CALC *be weakly sharing. Then* $\leq_{\Downarrow,R\nu} \;=\; \leq_{\Downarrow\nu}$.

Theorem 5.11 (Generic Context Lemma). *Let* CALC *be a calculus such that our assumptions 3.1, 3.3 3.4 are satisfied.*

- *If* CALC *is strongly sharing calculi, then:*
  $\leq_{\downarrow,R} = \leq_{\downarrow}$, $\leq_{\Downarrow,R} = \leq_{\Downarrow}$, *and* $\leq_{\downarrow,R} \cap \leq_{\Downarrow,R} \subseteq \leq_{\Downarrow}$.
- *If* CALC *is weakly sharing, then*
  $\leq_{\downarrow,R\nu} = \leq_{\downarrow\nu} \subseteq \leq_{\downarrow}$, $\leq_{\Downarrow,R\nu} = \leq_{\Downarrow,\nu} \subseteq \leq_{\Downarrow}$, *and*
  $\leq_{\downarrow,R\nu} \cap \leq_{\Downarrow,R\nu} \subseteq \leq_{\Downarrow\nu} \subseteq \leq_{\Downarrow}$.

6 Examples for Calculi and Context Lemmas

6.1 Strengthening the Context Lemma for Weakly Sharing Calculi

The context lemma for weakly sharing calculi has the slight disadvantage that in addition to all reduction contexts, also all vv-substitutions have to be checked, and also that the resulting contextual preorder is differently defined.
This difference can be avoided in most calculi by simulating $s[y_1/x_1, \ldots, y_n/x_n]$ by ($\texttt{letrec}\ x_1 = y_1, \ldots, x_n = y_n\ \texttt{in}\ s$) or a similar context, which is usually of the form $R[s]$. Note, however, that the relation $\forall \ldots : s[y_1/x_1, \ldots, y_n/x_n] \sim$ ($\texttt{letrec}\ x_1 = y_1, \ldots, x_n = y_n\ \texttt{in}\ s$), which is sufficient to drop all the ν's, may require an extra proof in the respective calculus.
In the π-calculus, which is weakly sharing, it is also possible to avoid this extra test using the context: $\nu z.(z(x_1, \ldots, x_n).[\cdot] \mid \overline{z}\langle y_1, \ldots, y_n\rangle.0)$, where the variables

$y_1, \ldots, y_n$ are not necessarily different. The communication operation for channel z will simulate the vv-substitution $[y_1/x_1, \ldots, y_n/x_n]$.
The obtained result in weakly sharing calculi is that the check in the context lemma can be done by ignoring the vv-substitutions, and in addition the contextual preorder definition can also ignore the vv-substitutions. This is the case for the weakly sharing calculus in [Mor98,MSC99].

6.2 Examples for Higher-Order Calculi

Our method to derive context lemmas is applicable in lambda-calculi and other higher-order calculi, even with `letrec`, with strict and non-strict reduction provided there are no substituting rules, which is usually only possible, if a form of sharing is permitted by e.g. `let`, `letrec` or explicit substitutions. As a general guideline, note that the beta-rule (or similar rules like the case-rules) in general violates our assumptions. The restricted beta-rule $(\lambda x.s)\ y \to s[y/x]$ may be allowed in weakly sharing calculi, provided y is a variable-only position. The rules $(\lambda x.s)\ t \to (\texttt{let}\ x = t\ \texttt{in}\ s)$, $(\texttt{let}\ x = v\ \texttt{in}\ R[x]) \to (\texttt{let}\ x = v\ \texttt{in}\ R[v])$ are permitted in strongly sharing calculi, if the replaced position of x is in a reduction position. Our result can be used for may- as well as must-convergence in its two forms, with or without taking infinite reductions into account.
We mention several calculi, where the result is applicable:
The call-by-need-calculi in [AFM$^+$95,AF97,MOW98] are deterministic, use a `let` to represent sharing, and use a sharing variant of beta-reduction. All the assumptions are satisfied, where the answers according to our definition are of the form `let x1 = t1 in let x2 = t2 in ...  in` λ`x.s`. The context lemma for may-termination holds for these strongly sharing calculi.
The `letrec`-calculi in [AS98,SS06] are deterministic and provide `letrec` for expressing sharing. The context lemma for may-convergence holds for these strongly sharing calculi. The non-deterministic call-by-need calculi in [KSS98,Man05] provide a `let` and a non-deterministic choice. The assumptions are satisfied, where UNWIND is deterministic. Context lemmas for may-termination as well as must-termination for these strongly sharing calculi hold. Note that [KSS98] uses total must-divergence, and makes no use of a context lemma, whereas the calculus in [Man05] did not treat must-divergence.
The call-by-need calculus in [MSC99] with letrec, choice, case and constructors uses may- and total must-convergence, and satisfies our assumptions. The calculus is weakly sharing since the beta-rule-variant and the case-rule use vv-substitutions, and since the arguments in applications as well as the arguments in constructor expressions $(c\ x_1 \ldots x_n)$ may only be occupied by variables. The context lemmas for may- , must and total must-convergence hold in this calculus, though only the may- and total must-context lemmas are used.
The call-by-need calculi in [SSS07,Mor98] provide amb, letrec, case and constructors. They satisfy our criteria, where the first is strongly, and the second is weakly sharing. UNWIND and normal-order reduction are non-deterministic. Our results confirms the respective context lemmas, and also shows a new one for the call-by-need variant in [Mor98], since there is no proof of a context lemma

for total must-convergence in [Mor98]. Our method to derive context lemmas is also applicable for the fair (i.e. using resources by annotations) variant of the amb-calculi in [Mor98,SSS06], where the encoding of $(\mathtt{amb}_{m,n}\ s\ t)$ can be done by using the infinitely many operators $\mathtt{amb}_{m,n}$.
Process calculi like the π-calculus are in the scope of our method, the result is that process contexts (no vv-substitutions required) are sufficient to check observational equivalence of processes w.r.t. may- and must-convergence. The call-by-value concurrent process calculus $\lambda(\mathsf{fut})$ in [NSSSS07] has a sharing variant of beta-reduction, and is derived from a calculus with beta-reduction [NSS06]. The sharing variant in [NSSSS07] has mutable cells, and a non-deterministic reduction. It satisfies our assumptions for a strongly sharing calculus, and requires two basic types in our type system. Note that UNWIND is nondeterministic. After some preprocessing is done, the context lemmas for may- and must-convergence for expressions can be derived from our results. Note that the context lemmas for processes in $\lambda(\mathsf{fut})$ is trivial, since all process contexts are reduction contexts in $\lambda(\mathsf{fut})$.
If the calculus permits substituting rules like beta-reduction, then Theorem 5.11 is not applicable, since then Assumption 3.4.(2) does not hold.

7 Conclusion

We have exhibited a broad class of higher-order calculi including untyped, extended non-deterministic lambda-calculi with a form of sharing and higher-order process calculi with a rudimentary type system, where context lemmas for may- as well as must-convergence can be derived. Three natural assumptions must hold to validate the context lemmas. This not only paves the way for analyzing calculi modelling programming languages and communicating processes, but also hints at common properties of calculi that were more or less unrelated in purpose, goal and syntax.

References

ACCL91. M. Abadi, L. Cardelli, P.-L. Curien, and J.-J Lévy. Explicit substitutions. *J. Funct. Programming*, 1(4):375–416, 1991.

AF97. Z. M. Ariola and M Felleisen. The call-by-need lambda calculus. *J. Funct. Programming*, 7(3):265–301, 1997.

AFM+95. Z. M. Ariola, M. Felleisen, J. Maraist, M. Odersky, and P. Wadler. A call-by-need lambda calculus. In *Principles of Programming Languages*, pages 233–246, San Francisco, California, 1995. ACM Press.

AS98. Zena M. Ariola and Amr Sabry. Correctness of monadic state: An imperative call-by-need calculus. In *POPL 98*, pages 62–74, 1998.

Bar84. H. P. Barendregt. *The Lambda Calculus. Its Syntax and Semantics*. North-Holland, Amsterdam, New York, 1984.

CHS05. Arnaud Carayol, Daniel Hirschkoff, and Davide Sangiorgi. On the representation of McCarthy's amb in the pi-calculus. *Theoret. Comput. Sci.*, 330(3):439–473, 2005.

FG96. Cédric Fournet and Georges Gonthier. The reflexive chemical abstract machine and the join-calculus. In *POPL'96*, pages 372–385. ACM Press, 1996.

FH92. Matthias Felleisen and R. Hieb. The revised report on the syntactic theories of sequential control and state. *Theoret. Comput. Sci.*, 103:235–271, 1992.

FM01. Jonathan Ford and Ian A. Mason. Operational techniques in PVS - a preliminary evaluation. *Electron. Notes Theor. Comput. Sci.*, 42, 2001.

FM03. Jonathan Ford and Ian A. Mason. Formal foundations of operational semantics. *Higher-Order and Symbolic Computation*, 16(3):161–202, 2003.

Gor99. Andrew D. Gordon. Bisimilarity as a theory of functional programming. *Theoret. Comput. Sci.*, 228(1-2):5–47, October 1999.

How89. D. Howe. Equality in lazy computation systems. In *4th IEEE Symp. on Logic in Computer Science*, pages 198–203, 1989.

How96. D. Howe. Proving congruence of bisimulation in functional programming languages. *Inform. and Comput.*, 124(2):103–112, 1996.

JM97. T. Jim and A.R. Meyer. Full abstraction and the context lemma. *SIAM J. Comput.*, 25(3):663–696, 1997.

KSS98. Arne Kutzner and Manfred Schmidt-Schauß. A nondeterministic call-by-need lambda calculus. In *International Conference on Functional Programming 1998*, pages 324–335. ACM Press, 1998.

Lan96. C. Laneve. On testing equivalence: May and must testing in the join-calculus. Technical Report Technical Report UBLCS 96-04, University of Bologna, 1996.

Man05. Matthias Mann. Congruence of bisimulation in a non-deterministic call-by-need lambda calculus. *Electron. Notes Theor. Comput. Sci.*, 128(1):81–101, 2005.

Mil77. R. Milner. Fully abstract models of typed λ-calculi. *Theoret. Comput. Sci.*, 4:1–22, 1977.

Mil99. Robin Milner. *Communicating and Mobile Systems: the π-calculus.* Cambridge university press, 1999.

Mor68. J.H. Morris. *Lambda-Calculus Models of Programming Languages.* PhD thesis, MIT, 1968.

Mor98. A. K. D. Moran. *Call-by-name, call-by-need, and McCarthys Amb.* PhD thesis, Dept. of Comp. Science, Chalmers university, Sweden, 1998.

MOW98. John Maraist, Martin Odersky, and Philip Wadler. The call-by-need lambda calculus. *J. Funct. Programming*, 8:275–317, 1998.

MS99. A. K. D. Moran and D. Sands. Improvement in a lazy context: An operational theory for call-by-need. In *POPL 1999*, pages 43–56. ACM Press, 1999.

MSC99. Andrew K. D. Moran, David Sands, and Magnus Carlsson. Erratic fudgets: A semantic theory for an embedded coordination language. In *Coordination '99*, volume 1594 of *Lecture Notes in Comput. Sci.*, pages 85–102. Springer-Verlag, 1999.

MSS06. Matthias Mann and Manfred Schmidt-Schauß. How to prove similarity a precongruence in non-deterministic call-by-need lambda calculi. Frank report 22, Inst. f. Informatik, J.W.Goethe-University, Frankfurt, January 2006.

NSS06. Joachim Niehren, Jan Schwinghammer, and Gert Smolka. A concurrent lambda calculus with futures. *Theoret. Comput. Sci.*, 364(3):338–356, November 2006.

NSSSS07. Joachim Niehren, David Sabel, Manfred Schmidt-Schauß, and Jan Schwinghammer. Observational semantics for a concurrent lambda calculus with reference cells and futures. *Electronic Notes in Theoretical Computer Science*, 173:313–337, 2007.

Ong93. C.-H. L. Ong. Non-determinism in a functional setting. In *Proc. 8th IEEE Symposium on Logic in Computer Science (LICS '93)*, pages 275–286. IEEE Computer Society Press, 1993.

Plo76. G.D. Plotkin. A powerdomain construction. *SIAM Journal of Computing*, 5(3):452–487, 1976.

SS03. Manfred Schmidt-Schauß. FUNDIO: A lambda-calculus with a `letrec`, `case`, constructors, and an IO-interface: Approaching a theory of `unsafePerformIO`. Frank report 16, Inst. f. Informatik, J.W.Goethe-University, Frankfurt, 2003.

SS06. Manfred Schmidt-Schauß. Equivalence of call-by-name and call-by-need for lambda-calculi with letrec. Frank report 25, Inst. f. Informatik, J.W.Goethe-University, Frankfurt, September 2006. accepted for publication RTA'07.

SSS06. David Sabel and Manfred Schmidt-Schauß. A call-by-need lambda-calculus with locally bottom-avoiding choice: Context lemma and correctness of transformations. Frank report 24, Inst. f. Informatik, J.W.Goethe-University, Frankfurt, January 2006. submitted for publication.

SSS07. David Sabel and Manfred Schmidt-Schauß. A call-by-need lambda-calculus with locally bottom-avoiding choice: Context lemma and correctness of transformations. *Math. Structures Comput. Sci.*, 2007. accepted for publication.

SSSS05. Manfred Schmidt-Schauß, Marko Schütz, and David Sabel. A complete proof of the safety of Nöcker's strictness analysis. Frank report 20, Inst. f. Informatik, J.W.Goethe-University, Frankfurt, 2005. submitted for publication.

SW01. D. Sangiorgi and D. Walker. *The π-calculus: a theory of mobile processes*. Cambridge university press, 2001.

A Full Encoding of $\lambda(\mathsf{fut})$ in the General Framework

We recall the original syntax of $\lambda(\mathsf{fut})$ from [NSSSS07]: programs are constructed using a two-layer syntax.

$$p ::= p_1 \,|\, p_2 \mid \nu x(p) \mid x \Leftarrow e \mid x \overset{susp}{\Leftarrow} e \mid x \,\mathsf{c}\, v \mid y \,\mathsf{h}\, x \mid y \,\mathsf{h}\, \bullet$$

$$e ::= x \mid c \mid \lambda x.e \mid e_1\ e_2 \mid \mathbf{exch}(e_1, e_2) \qquad c ::= \mathbf{unit} \mid \mathbf{cell} \mid \mathbf{thread} \mid \mathbf{handle} \mid \mathbf{lazy}$$

On the top layer there are processes p. Inside eager threads ($x \Leftarrow e$) as well as suspended threads ($x \overset{susp}{\Leftarrow} e$) enriched lambda terms are used which are constructed by the non-terminal e, where c are higher-order constants. The content of a cell ($x \,\mathsf{c}\, v$) may only be a value v, i.e. a variable, a constant or a lambda abstraction.

The Signature We now give an encoding of the calculus into our general framework, where most of the encoding is canonical with two exceptions: Used handle components $y \,\mathsf{h}\, \bullet$ are represented by the operator $\overline{\mathsf{h}}$ and more importantly cell components are encoded into seven different operators, depending on the content of the cell: a cell with a variable as content $x \,\mathsf{c}\, y$ is encoded as binary operator $\mathsf{c}_{Var}(x, y)$, for any of the 5 different kind of cells there is an

unary operator for the cell, e.g. $x\,\mathsf{c}\,\mathbf{unit}$ is encoded as $\mathsf{c_u}(x)$, cells containing lambda abstractions are encoded using the operator c_λ, e.g $x\,\mathsf{c}\,\lambda y.e$ is encoded as $\mathsf{c}_\lambda(x, y.e)$. The reason is that the reduction rule CELL.EXCH(ev) (see below) may modify the cell content (which itself is not a reduction position), but with the new constants this modification is either in a reduction position, or is an injective renaming of variables at a variable position (e.g if $\mathsf{c}_{Var}(x)$ is replaced by $\mathsf{c}_{Var}(y)$), or if a cell containing an abstraction is replaced by another abstraction as content, then the appropriate multi-context M' according to M in the reduction assumption can be found. I.e. if we consider the reduction $\mathsf{I}(E[\mathbf{exch}(x, \lambda(y.e_1))], \mathsf{c}_\lambda(x, z.e_2)) \to \mathsf{I}(E[\mathbf{exch}(x, \lambda(z.e_2))], \mathsf{c}_\lambda(x, y.e_1))$ and M is the multi-context $\mathsf{I}(E[\mathbf{exch}(x, \lambda(y.[\cdot]))], \mathsf{c}_\lambda(x, z.[\cdot]))$, then the context M' can be defined as $\mathsf{I}(E[\mathbf{exch}(x, \lambda(z.[\cdot]))], \mathsf{c}_\lambda(x, y[\cdot]))$.

According to the two-layer syntax we use two types to distinguish between process (τ_p) and expressions (τ_e). Thus, $T_0 := \{\tau_e, \tau_p\}$. The following table defines the remaining parts of the signature.

$f \in O$	$\alpha(f)$	$\beta(f)$	$\beta_\tau(f)$	$f \in O$	$\alpha(f)$	$\beta(f)$	$\beta_\tau(f)$
@	2	(T, T)	$\tau_e \to \tau_e \to \tau_e$	$\mathsf{c_u}$	1	(V)	$\tau_e \to \tau_p$
λ	1	(1)	$(\tau_e \to \tau_e) \to \tau_e$	$\mathsf{c_c}$	1	(V)	$\tau_e \to \tau_p$
exch	2	(T, T)	$\tau_e \to \tau_e \to \tau_e$	$\mathsf{c_t}$	1	(V)	$\tau_e \to \tau_p$
unit	0	()	τ_e	$\mathsf{c_I}$	1	(V)	$\tau_e \to \tau_p$
cell	0	()	τ_e	$\mathsf{c_h}$	1	(V)	$\tau_e \to \tau_p$
thread	0	()	τ_e	c_{Var}	2	(V, V)	$\tau_e \to \tau_e \to \tau_p$
handle	0	()	τ_e	c_λ	2	(V, 1)	$\tau_e \to (\tau_e \to \tau_e) \to \tau_p$
lazy	0	()	τ_e	$\overset{susp}{\Leftarrow}$	2	(V, T)	$\tau_e \to \tau_e \to \tau_p$
I	2	(T, T)	$\tau_p \to \tau_p \to \tau_p$	h	2	(V, V)	$\tau_e \to \tau_e \to \tau_p$
ν	1	(1)	$(\tau_e \to \tau_p) \to \tau_p$	$\overline{\mathsf{h}}$	1	(V)	$\tau_e \to \tau_p$
$\Leftarrow$	2	(V, T)	$\tau_e \to \tau_e \to \tau_p$				

Unwinding and Small-Step Reduction The small-step reduction of $\lambda(\mathsf{fut})$ used structural congruence of processes as well as three different kinds of contexts (process evaluation contexts D, call-by-value evaluation contexts E and future evaluation contexts F:

$$
\begin{array}{rl}
p_1 \,\mathsf{I}\, p_2 \equiv p_2 \,\mathsf{I}\, p_1 & \\
(p_1 \,\mathsf{I}\, p_2) \,\mathsf{I}\, p_3 \equiv p_1 \,\mathsf{I}\, (p_2 \,\mathsf{I}\, p_3) & \\
\nu x \nu y p \equiv \nu y \nu x p & \\
\nu x(p_1) \,\mathsf{I}\, p_2 \equiv \nu x(p_1 \,\mathsf{I}\, p_2) \text{ if } x \notin \mathsf{fv}(p_2) &
\end{array}
\qquad
\begin{array}{l}
E ::= x \Leftarrow \widetilde{E} \\
\widetilde{E} ::= [\,] \mid \widetilde{E}\, e \mid v\, \widetilde{E} \mid \mathbf{exch}(\widetilde{E}, e) \mid \mathbf{exch}(v, \widetilde{E}) \\
F ::= x \Leftarrow \widetilde{F} \\
\widetilde{F} ::= \widetilde{E}[[\,]\, v] \mid \widetilde{E}[\mathbf{exch}([\,], v)] \\
D ::= [\,] \mid p \,\mathsf{I}\, D \mid D \,\mathsf{I}\, p \mid \nu x D
\end{array}
$$

Small-step reduction ev is then defined using nine rules:

$$
\begin{array}{ll}
(1)\ D[E[(\lambda y.e)\, v]] \xrightarrow{\mathsf{ev}} D[\nu y(E[e] \,\mathsf{I}\, y \Leftarrow v)] & \\
(2)\ D[E[\mathbf{thread}\, v]] \xrightarrow{\mathsf{ev}} D[\nu z(E[z] \,\mathsf{I}\, z \Leftarrow v\, z)] & \\
(3)\ D[F[x] \,\mathsf{I}\, x \Leftarrow v] \xrightarrow{\mathsf{ev}} D[F[v] \,\mathsf{I}\, x \Leftarrow v] & \\
(4)\ D[E[\mathbf{lazy}\, v]] \xrightarrow{\mathsf{ev}} D[\nu z(E[z] \,\mathsf{I}\, z \overset{susp}{\Leftarrow} v\, z)] & \\
(5)\ D[F[x] \,\mathsf{I}\, x \overset{susp}{\Leftarrow} e] \xrightarrow{\mathsf{ev}} D[F[x] \,\mathsf{I}\, x \Leftarrow e] & \\
(6)\ D[E[\mathbf{handle}\, v]] \xrightarrow{\mathsf{ev}} D[\nu z \nu z'(E[v\, z\, z'] \,\mathsf{I}\, z'\, \mathsf{h}\, z)] & \\
(7)\ D[E[x\, v] \,\mathsf{I}\, x\, \mathsf{h}\, y] \xrightarrow{\mathsf{ev}} D[E[\mathbf{unit}] \,\mathsf{I}\, y \Leftarrow v \,\mathsf{I}\, x\, \mathsf{h}\, \bullet] & \\
(8)\ D[E[\mathbf{cell}\, v]] \xrightarrow{\mathsf{ev}} D[\nu z(E[z] \,\mathsf{I}\, z\, \mathsf{c}\, v)] & \\
(9)\ D[E[\mathbf{exch}(z, v_1)] \,\mathsf{I}\, z\, \mathsf{c}\, v_2] \xrightarrow{\mathsf{ev}} D[E[v_2] \,\mathsf{I}\, z\, \mathsf{c}\, v_1] &
\end{array}
$$

For the translation in the general framework we first define UNWIND-algorithm which operates on terms over the signature. It uses a label l that marks a $D[E]$-contexts.

$$\begin{array}{ccc} |(p_1,p_2)^L \to |(p_1^L,p_2) & |(p_1,p_2)^L \to |(p_1,p_2^L) & \nu(x.p)^L \to \nu(x.p^L) \\ \Leftarrow(x,p)^L \to \Leftarrow(x,p^L) & @(e_1,e_2)^L \to @(e_1^L,e_2) & @(v^L,e_2) \to @(v,e_2^L) \\ \multicolumn{3}{c}{\mathbf{exch}(v^L,e_2) \to \mathbf{exch}(v,e_2)^L \qquad \mathbf{exch}(e_1,e_2)^L \to \mathbf{exch}(e_1^L,e_2)} \end{array}$$

Obviously the UNWIND-Assumption is fulfilled. Structural congruence is encoded in the small-step relation

$$\begin{array}{rl} |(p_1,p_2) & \to |(p_2,p_1) \\ |(p_1,|(p_2,p_3)) & \to |(|(p_1,p_2),p_3) \\ \nu(x.\nu(y.p)) & \to \nu(y.\nu(x.p)) \\ |(\nu(x.p_1),p_2) & \to \nu(x.|(p_1,p_2)) \text{ if } x \notin \mathcal{FV}(p_2) \end{array}$$

The encoding from the small step reduction $\xrightarrow{\mathsf{ev}}$ into the small step reduction $\to$ of the general framework is canonical, except for the rule (8) and (9):

$$\begin{array}{rl} (1) & D[E[@(\lambda(y.e),v)]] \to D[\nu(y.(|(E[e],\Leftarrow(y,v))))] \text{ if } x \neq y \\ (2) & D[E[@(\mathbf{thread},v)]] \to D[\nu(z.(|(E[z],\Leftarrow(z,@(v,z)))))] \\ (3) & D[|(F[x],\Leftarrow(x,v))] \to D[|(F[v],\Leftarrow(x,v))] \\ (4) & D[E[@(\mathbf{lazy},v)]] \to D[\nu(z.(\overset{susp}{\Leftarrow}(z,@(v,z))))] \\ (5) & D[|(F[x],\overset{susp}{\Leftarrow}(x,e))] \to D[|(F[x],\Leftarrow(x,e))] \\ (6) & D[E[@(\mathbf{handle},v)]] \to D[\nu(z.\nu(z'.|(E[@(@(v,z),z')],\mathsf{h}(z',z))))] \\ (7) & D[|(E[@(x,v)],\mathsf{h}(x,y))] \to D[|(E[\mathbf{unit}],|(\Leftarrow(y,v),\overline{\mathsf{h}}(x)))] \end{array}$$

The rule (8) for cell creation is splitted into 7 rules for $\to$ depending of the kind of content of newly created cell.

$$\begin{array}{rl} D[E[@(\mathbf{cell},\lambda x.e)]] & \to D[\nu(z.(|(E[z],\mathsf{c}_\lambda(z,x.e))))] \\ D[E[@(\mathbf{cell},\mathbf{unit})]] & \to D[\nu(z.(|(E[z],\mathsf{c}_\mathsf{u}(z))))] \\ D[E[@(\mathbf{cell},\mathbf{handle})]] & \to D[\nu(z.(|(E[z],\mathsf{c}_\mathsf{h}(z))))] \\ D[E[@(\mathbf{cell},\mathbf{lazy})]] & \to D[\nu(z.(|(E[z],\mathsf{c}_\mathsf{l}(z))))] \\ D[E[@(\mathbf{cell},\mathbf{thread})]] & \to D[\nu(z.(|(E[z],\mathsf{c}_\mathsf{t}(z))))] \\ D[E[@(\mathbf{cell},\mathbf{cell})]] & \to D[\nu(z.(|(E[z],\mathsf{c}_\mathsf{c}(z))))] \\ D[E[@(\mathbf{cell},x)]] & \to D[\nu(z.(|(E[z],\mathsf{c}_{Var}(z,x))))] \end{array}$$

The rule (9) for atomic cell exchange is replaced by 49 rules, since the kind of the cell may change because of the exchange operation. We only show some of them:

$$\begin{array}{rl} D[|(E[\mathbf{exch}(z,\lambda(x.e_1))],\mathsf{c}_\lambda(z,y.e_2))] & \to D[|(E[\lambda(y.e_2)],\mathsf{c}_\lambda(z,x.e_1))] \\ D[|(E[\mathbf{exch}(z,\lambda(x.e_1))],\mathsf{c}_\mathsf{t}(z))] & \to D[|(E[\mathbf{thread}],\mathsf{c}_\lambda(z,x.e_1))] \\ D[|(E[\mathbf{exch}(z,\lambda(x.e_1))],\mathsf{c}_{Var}(z,y))] & \to D[|(E[y],\mathsf{c}_\lambda(z,x.e_1))] \\ D[|(E[\mathbf{exch}(z,\mathbf{unit})],\mathsf{c}_\lambda(z,y.e_2))] & \to D[|(E[\lambda(y.e_2)],\mathsf{c}_\mathsf{u}(z))] \\ \multicolumn{2}{c}{\ldots} \end{array}$$

It is easy to verify that the reduction assumption holds.

Answers Answers are processes where for all eager threads $y \Leftarrow e$ the future y is bound (possibly via a chain of variable-to-variable threads $y \Leftarrow x_1 \,|\, x_1 \Leftarrow x_2 \,|\, \ldots$) to an abstraction, a constant, a cell component, a suspended thread, a handle or a used handle. Using this definition for answers, it not hard to verify that the answer assumption holds.